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Associative Memory in Asymmetric Diluted Network with Low Level of Activity.

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Abstract. – We extend the analysis of asymmetric diluted networks to the case of low-activity level. The same learning algorithm which was used for the symmetric model turns out to be successful. The use of «V-variables» ($V = 0$; 1) leads to significant enhancing of the storage capacity. The overloading phase transition is found to be of the first order, which means good retrieval quality in all associative memory phases. The intensity of time-dependent nonthermal noise can be diminished considerably by the appropriate choice of the neural threshold. Some sort of «universality» of the performance of the networks with low-activity level can be noted.

1. – A large number of works concerning the spin glass models for associative memory have appeared in the physical literature in the last years. As first proposed by Little [1] and Hopfield [2] (see also [3]), these models were based on many unrealistic assumptions. In the subsequent papers some of these assumptions were overcome. Amit *et al.* [4] and Tsodyks and Feigelman [5] considered a neural network with low level of activity. In [5] it was shown that the storage capacity can be significantly enhanced by the use of «V-variables» ($V = 0$; 1) characterizing the states of neurons and modified connection matrix proposed in [4]. At the limit of very low level of neural activity this learning algorithm leads to the maximal storage capacity which can be obtained in such networks as was estimated by Gardner [6]. A very important step has been made in the works [7, 8], where a diluted and nonsymmetric version of the Hopfield model was solved.

In this letter we extend [8] to the asymmetric diluted network with low level of neural activity p. It was found that the learning algorithm of the previous work [5] in this case leads to a remarkable result: the mean-field equations for both situations coincide at the limit $p \ll 1$. Hence, the asymmetric network possesses all features of the symmetric one: the first-order phase transition (*i.e.* good retrieving quality up to the overloading point) and enhanced storage capacity.

2. – In the following we consider a network of N neurons which are represented by two-value variables $V = 0; 1$. $V = 1$ corresponds to an active neuron, and $V = 0$ to a passive one. The stored images are encoded by sets $\{\eta_i^\mu\}$, where the subscript i numbers different neurons, whereas μ different images. All η_i^μ are independent random variables with the probability distribution

$$p(\eta_i^\mu) = p\delta(\eta_i^\mu - 1) + (1-p)\,\delta(\eta_i^\mu) \tag{1}$$

and we shall be particularly interested in the case $p \ll 1$. Neurons interact with each other via synaptic couplings $c_{ij}J_{ij}$. The value J_{ij} depends on L stored images $\{\eta_i^\mu\}$ according to the rule [5]

$$J_{ij} = \frac{1}{K}\sum_{\mu=1}^{L}(\eta_i^\mu - p)(\eta_j^\mu - p) \tag{2}$$

and factors c_{ij} represent the dilution and the asymmetry. For each pair (i,j) they are assumed to be statistically independent random parameters with the distribution

$$\rho(c_{ij}) = \frac{K}{N}\delta(c_{ij}-1) + \left(1 - \frac{K}{N}\right)\delta(c_{ij}), \tag{3}$$

K is the mean number of synapses per neuron and the model can be solved if $1 \ll K \ll N$ [8]. Following [8] we start from the dynamic equation for soft spin variables V_i

$$\Gamma^{-1}\partial_t V_i(t) = -U_\lambda(V_i(t)) + \sum_{j\neq i} J_{ij}c_{ij}V_j(t) - \theta + f_i(t), \tag{4}$$

where θ is the neural threshold, $f_i(t)$ is the thermal white noise and $U_\lambda(V_i)$ restricts the fluctuations of spin so that for $\lambda \to \infty$ the limit $V_i = 0; 1$ is reached. The value of the threshold θ will be chosen in order to optimize the performance of the (network cf. [5]). The standard way for studying such equations is the dynamic functional method [9] which is especially convenient when the randomness exists. It can be written in the form

$$Z[l,\tilde{l}] = \int \mathcal{D}(V,\tilde{V})\exp[-L] \tag{5}$$

and the weight L is determined by

$$L = -\int \mathrm{d}t \sum_{i\neq j} c_{ij}J_{ij}\tilde{V}_i(t)V_j(t) + L_0, \tag{6}$$

L_0 is the local part of L,

$$L_0 = \int \mathrm{d}t \sum_i \left\{\tilde{V}_i(t)\left[\Gamma^{-1}\partial_t V_i(t) + U_\lambda(V_i) + \theta - \frac{T}{\Gamma}\tilde{V}_i(t)\right] - \tilde{l}_i(t)\tilde{V}_i(t) - l_i(t)V_i(t)\right\}. \tag{7}$$

Here T is the parameter which determines the intensity of the noise f_i and may be called «temperature».

The task is to average the functional (5) over c_{ij}, η_i^μ and to reduce it to a single-spin problem. It can be done in full analogy with [8], so we write the result

$$\begin{cases} L = L_0 + \overline{L}, \\ \overline{L} = \sum_i \int \mathrm{d}t \left[\sum_{\nu=1}^{s} m^\nu(\eta_i^\nu - p)\tilde{V}_i(t) + \frac{1}{2}\alpha\tilde{p}^2\tilde{V}_i(t)C(t-t')\tilde{V}_i(t')\right], \end{cases} \tag{8}$$

where $\alpha = L/K$, $\tilde{p} = p(1-p)$; the quantities m^ν and $C(t-t')$ have to be calculated self-consistently:

$$m^\nu = \frac{1}{N}\sum_i (\eta_i^\nu - p)\langle V_i\rangle_L, \tag{9}$$

$$C(t-t') = \frac{1}{N}\sum_i \langle V_i(t)V_i(t')\rangle_L. \tag{10}$$

As in [8] the contributions of random overlaps with most images vanish as L/N, whereas finite numbers of overlaps $m^1, \ldots, m^s$ could be of the order of one. The last term in (8) manifests the existence of additional time-dependent noise $\sqrt{\alpha}\tilde{p}y(t)$ which has a Gaussian distribution with zero mean and correlation function

$$\langle y(t)y(t')\rangle_y = C(t-t'). \tag{11}$$

3. – So we obtained the dynamics of a single spin $V(t)$ in a time-dependent field

$$h(t) = \sum_{\nu=1}^{s} m^\nu(\eta_i^\nu - p) + \tilde{p}\sqrt{\alpha}\,y(t) - \theta, \tag{12}$$

which has to be calculated self-consistently

$$m^\nu = \langle(\eta^\nu - p)\langle V(t)\rangle_V\rangle_{y,\eta}, \tag{13}$$

$$\langle y(t)y(t')\rangle = C(t-t') = \langle\langle V(t)V(t')\rangle_V\rangle_{y,\eta}. \tag{14}$$

Here $\langle\ \rangle_V$ denotes dynamic averaging at fixed values of $y(t)$ and η_i^ν. At this step we return to the case $\lambda \gg 1$ (see (4)) which leads to the analog of Glauber dynamics [10]

$$\begin{cases} \Gamma^{-1}\dfrac{\mathrm{d}p_+}{\mathrm{d}t} = \dfrac{-p_+\exp[-\beta h(t)] + p_0}{1+\exp[-\beta h(t)]}, \\[2ex] \Gamma^{-1}\dfrac{\mathrm{d}p_0}{\mathrm{d}t} = \dfrac{-p_0 + p_+\exp[-\beta h(t)]}{1+\exp[-\beta h(t)]}, \end{cases} \tag{15}$$

where $p_+(t)$ and $p_0(t)$ are the probabilities to find the spin in the state $V=1$ and $V=0$, respectively, at the time t. Bearing in mind that $p_+ + p_0 = 1$, we get from (15) the following equation for magnetization $\bar{\mu}(t) = \langle V(t)\rangle_V = p_+(t)$:

$$\Gamma^{-1}\frac{\mathrm{d}\bar{\mu}}{\mathrm{d}t} = -\bar{\mu} + K[\beta h(t)], \tag{16}$$

where $K(x) = \exp[x]/(1+\exp[x])$. The solution of eq. (16) is straightforward:

$$\bar{\mu}(t) = \Gamma\int_{-\infty}^{t} \mathrm{d}t'\exp[-\Gamma(t-t')]K[\beta h(t')]\,\mathrm{d}t'. \tag{17}$$

Inserting it into (13), we get the following equations for m^ν and $\mu = \langle \bar{\mu}(t)\rangle_{y,\eta}$:

$$m^\nu = \left\langle (\eta^\nu - p)\int \frac{dy}{\sqrt{2\pi}} \exp\left[-\frac{y^2}{2}\right] K\left[\beta\left(\sum_{\nu=1}^{s} m^\nu(\eta^\nu - p) + \tilde{p}\sqrt{\alpha\mu}\,y - \theta\right)\right]\right\rangle_\eta, \tag{18}$$

$$\mu = \left\langle \int \frac{dy}{\sqrt{2\pi}} \exp\left[-\frac{y^2}{2}\right] K\left[\beta\left(\sum_{\nu=1}^{s} m^\nu(\eta^\nu - p) + \tilde{p}\sqrt{\alpha\mu}\,y - \theta\right)\right]\right\rangle_\eta. \tag{19}$$

Note that the Gaussian functional integration over $y(t)$ in (13) is reduced to a single Gaussian integral with variance $\langle y(t)y(t')\rangle = C(t=0) = \mu$. Recall that in [7, 8] one mean-field equation for m^ν was derived. Now we turn to the most important case $m^\nu = m\delta^{1\nu}$ when one image is retrieving. At the zero temperature limit $\beta \to \infty$ eqs. (18), (19) become

$$m_0 = \varphi\left(\frac{\theta_0 - m_0(1-p)}{\sqrt{\alpha\mu}}\right) - \varphi\left(\frac{\theta_0 + m_0 p}{\sqrt{\alpha\mu}}\right); \qquad \varphi(x) = \int_x^\infty \frac{dz}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right], \tag{20}$$

$$\mu = p\,\varphi\left(\frac{\theta_0 - m_0(1-p)}{\sqrt{\alpha\mu}}\right) + (1-p)\,\varphi\left(\frac{\theta_0 + m_0 p}{\sqrt{\alpha\mu}}\right). \tag{21}$$

Here $\theta_0 = \theta/p$ and $m_0 = m/p$. The value m_0 characterizes the retrieving quality of the stored image. For $p \ll 1$ eqs. (20), (21) coincide with the mean-field equations for nondiluted symmetric model [5]. There is a critical line $\alpha_c(p)$ below which the nontrivial solution $m_0 \neq 0$ of (20) exists. At $p \ll 1$

$$\alpha_c(p) \simeq \frac{\theta_0^2}{2p\ln(1/p)}; \qquad m_c \simeq 1 - \left(\varepsilon^2 \ln\frac{1}{p}\right)^{-1/2} p^{\varepsilon^2}. \tag{22}$$

The value of θ_0 has to be chosen not very close to unity: $\varepsilon = (1-\theta_0)/\theta_0 \gg |\ln p|^{-1/2}$. With the further θ_0 increase the value of $\alpha_c(p)$ decreases

$$\alpha_c(p) \simeq \frac{(1-\theta_0)^2}{p\ln(1/p)}, \qquad \varepsilon \ll |\ln p|^{-1/2}. \tag{23}$$

Hence the maximal storage capacity is achieved at $\rho \sim |\ln p|^{-1/2}$

$$\alpha_c \simeq \frac{1}{2p\ln(1/p)}. \tag{24}$$

The full line $\alpha_c(p)$ can be easily obtained from (20), (21) numerically. Equations (18) and (19) determine the transition temperature $T_c(\alpha)$ below which the system remembers stored patterns. We obtained $T_c(0)$:

$$T_c(\alpha = 0) \simeq p\frac{(1-\theta_0)}{|\ln(1-\theta_0)|}. \tag{25}$$

The retrieving quality at the temperature $T = T_c$ is close to unity:

$$m_0 \simeq 1 - \frac{\varepsilon}{\ln(1/\varepsilon)}. \tag{26}$$

In order to estimate the intensity of time-dependent noise $y(t)$, we calculate a time-persistent part of the spin autocorrelation

$$q = \lim_{t\to\infty} C(t). \tag{27}$$

In analogy with [8] we decompose $y(t)$ in two terms:

$$y(t) = z + r(t), \tag{28}$$

where z and $r(t)$ are independent random Gaussian variables with zero mean and variances

$$\langle z^2 \rangle = q, \tag{29}$$

$$\langle r(t) r(t') \rangle = C(t-t') - q. \tag{30}$$

The equation for q can be easily derived from (27):

$$q = \lim_{t-t'\to\infty} \langle \langle V(t) \rangle_{V,r} \langle V(t') \rangle_{V,r} \rangle_{z,\eta} = \left\langle \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2q}\right] \cdot \right.$$
$$\left. \cdot \left\{ \int_{-\infty}^{\infty} \frac{dr}{\sqrt{2\pi(\mu-q)}} \exp\left[-\frac{r^2}{2(\mu-q)}\right] K\left[\beta\left(m(\eta-p) + \tilde{p}\sqrt{\alpha}\, r + \tilde{p}\sqrt{\alpha}\, z - \theta\right)\right]\right\}^2 \right\rangle_\eta . \tag{31}$$

At $T=0$ (31) has the solution $q(p) < \mu$, which indicates the existence of time-dependent noise. At $p \ll 1$ this solution can be found

$$q = \mu\left(1 - \exp\left[-\frac{\varepsilon^2}{2\alpha p}\right]\right). \tag{32}$$

Formula (32) is valid if $\mu - q \ll \mu$. We see that the time-dependent nonthermal noise $r(t)$ can be made very weak at all $\alpha < \alpha_c$ by the appropriate choice of the threshold: $\varepsilon = (1-\theta_0) \gg |\ln p|^{-1/2}$.

In this letter we have extended the analysis of the diluted and asymmetric Hopfield model to the case of low neural activity level. We have shown that an appropriate choice of the learning algorithm gives some advantages to such networks: the storage capacity is enhanced significantly and the retrieving quality is close to unity up to the critical temperature. It is interesting to note some sort of «universality» of the performance of the network with the low activity level.

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