ASSIGNMENT-2

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(1) $u = e^{r\cos\theta}\cdot\cos(r\sin\theta)$. $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$

$$\Rightarrow \frac{\partial u}{\partial r} = \cos\theta\cdot e^{r\cos\theta}\cdot\cos(r\sin\theta) + e^{r\cos\theta}\cdot(-\sin(r\sin\theta))\cdot\sin\theta$$

$$= \cos\theta\cdot e^{r\cos\theta}\cdot\cos(r\sin\theta) - e^{r\cos\theta}\cdot\sin(r\sin\theta)\cdot\sin\theta$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = e^{r\cos\theta}\cdot r(-\sin\theta)\cdot\cos(r\sin\theta) + e^{r\cos\theta}\cdot(-\sin(r\sin\theta))\cdot r\cos\theta$$

$$= -e^{r\cos\theta}\cdot r\left[\sin\theta\cdot\cos(r\sin\theta) + \cos\theta\cdot\sin(r\cos\theta)\right]$$

$$= -e^{r\cos\theta}\cdot r\cdot\sin(\theta + r\sin\theta)$$

(2) $z = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$; T.P : $\frac{\partial^2 z}{\partial y\cdot\partial x} = \frac{x^2-y^2}{x^2+y^2}$

$$\Rightarrow \frac{\partial z}{\partial x} = 2x\cdot\tan^{-1}\left(\frac{y}{x}\right) + \frac{x^2\times\cancel{x^2}}{x^2+y^2}\times\left(\frac{-1}{\cancel{x^2}}\right) - \frac{y^2\times y^2}{y^2+x^2}\times\frac{1}{y}$$

$$= 2x\cdot\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2}{x^2+y^2} - \frac{y^3}{x^2+y^2} = 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{(x^2+y^3)}{x^2+y^2}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{2x\times x^2}{x^2+y^2}\times\frac{1}{x} - \left[\frac{3y^2(x^2+y^2) - (x^2+y^3)\cdot 2y}{(x^2+y^2)^2}\right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial x\cdot\partial y} = \frac{2x^2}{x^2+y^2} - \left[\frac{3y^2x^2 + 3y^4 - 2x^2y - 2y^4}{(x^2+y^2)^2}\right]$$

$$= \frac{2x^2}{x^2+y^2} - \left[\frac{3y^2x^2 - 2x^2y + y^4}{(x^2+y^2)^2}\right] = \frac{x^2-y^2}{x^2+y^2}$$

(3) $z = \underbrace{x^n\cdot f\left(\frac{y}{x}\right)}_{u} + \underbrace{y^{-n}\phi\left(\frac{x}{y}\right)}_{v}$ $\Rightarrow x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = nu \rightarrow (1)$

$x^2\cdot\frac{\partial^2 u}{\partial x^2} + 2xy\cdot\frac{\partial^2 u}{\partial x\partial y} + y^2\cdot\frac{\partial^2 u}{\partial y^2} = n(n-1)u \rightarrow (2)$; $z = u + v$

↳ similarly for v. (Eqn (3) and (4))

$\Rightarrow$ (1) + (2) + (3) + (4) and substituting these

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial^2 y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x\partial y}$$

we get, $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + x^2\left(\frac{\partial^2 z}{\partial x^2}\right)$

$+ 2xy\left(\frac{\partial^2 z}{\partial x\partial y}\right) + y^2\left(\frac{\partial^2 z}{\partial y^2}\right) = x^2(u+v)$

$= x^2(z)$.

(4) $u = \sin^{-1}\left[\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$; ST : $x\cdot\dfrac{\partial u}{\partial x} + y\cdot\dfrac{\partial u}{\partial y} + z\cdot\dfrac{\partial u}{\partial z} + 3\tan u = 0$ ②

$\Rightarrow \phi = \sin u = \dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$ Here ϕ is hom. function of order -3.

$\hookrightarrow f(u)$

Applying Euler's Theorem: $x\cdot\dfrac{\partial u}{\partial x} + y\cdot\dfrac{\partial u}{\partial y} + z\cdot\dfrac{\partial u}{\partial y} = -3\cdot\dfrac{f(u)}{f'(u)}$

$\Rightarrow x\cdot\dfrac{\partial u}{\partial x} + y\cdot\dfrac{\partial u}{\partial y} + z\cdot\dfrac{\partial u}{\partial y} + 3\tan(u) = 0$ Hence Proved.

(5) (a) $f(x,y)=0$, then $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{f_x}{f_y} \rightarrow$ (iii)

(b) $u = \dfrac{x^2+y^2}{x^2-y^2} + 4$ then $x\cdot\dfrac{\partial u}{\partial x} + y\cdot\dfrac{\partial u}{\partial y} = n\cdot u$ Here $n=0$

$\therefore x\cdot\dfrac{\partial u}{\partial x} + y\cdot\dfrac{\partial u}{\partial y} = 0$ $\therefore$ False $\longrightarrow$ (ii)

(c) $f(x,y) = \dfrac{1}{x^3} + \dfrac{1}{x^2y} + \dfrac{1}{x^3+5y^3}$, then $x\cdot\dfrac{\partial f}{\partial x} + y\cdot\dfrac{\partial f}{\partial y} = -3f \longrightarrow$ (i)

$\therefore$ True

(6) $u = u\left(\dfrac{y-x}{xy}, \dfrac{z-x}{xz}\right)$; S.T.: $x^2u_x + y^2u_y + z^2u_z = 0$

Let $\dfrac{y-x}{xy} = r$ and $\dfrac{z-x}{xz} = s$. $\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial x}$

$\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\cdot\left[\dfrac{-xy-(y-x)y}{x^2y^2}\right] + \dfrac{\partial u}{\partial s}\left[\dfrac{-xz-(z-x)z}{x^2z^2}\right]$

$\Rightarrow \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\cdot\left[\dfrac{xy-(y-x)x}{x^2y^2}\right] + \dfrac{\partial u}{\partial s}\left[0\right]$

$\Rightarrow \dfrac{\partial u}{\partial z} = \dfrac{\partial u}{\partial r}\times 0 + \dfrac{\partial u}{\partial s}\left[\dfrac{xz-(z-x)\cdot x}{xz}\right]$

$\Rightarrow \therefore x^2u_x + y^2u_y + z^2u_z = 0$ Hence Proved.

(7) $u = x\log(xy)$ where $x^3+y^3+3xy=1$ Find $\dfrac{du}{dx}$

$\dfrac{du}{dx} = \dfrac{\partial u}{\partial x}\cdot\dfrac{dx}{dx} + \dfrac{\partial y}{\partial y}\cdot\dfrac{dy}{dx}$; $3x^2+3y^2\dfrac{dy}{dx} + 3y + 3x\cdot\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{(3x^2+3y)}{3y^2+3x}$

$\Rightarrow \dfrac{du}{dx} = \log(xy) + \dfrac{1}{y}\times y + \dfrac{x}{xy}\cdot x\cdot\dfrac{dy}{dx}$

$\Rightarrow \dfrac{du}{dx} = \log(xy) + 1 - \dfrac{x}{y}\left(\dfrac{3x^2+3y}{3y^2+3x}\right)$

(8) If z is an implicit func. of x and y. (3)

$\frac{\partial z}{\partial x} \neq \partial$Eq shown in next question.

Let $F(x, y, z)$:

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$

Similarly. $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

(9) $u = x^2 y + y^2 z + z^2 x \rightarrow \frac{\partial u}{\partial x} = 2xy + y^2 \frac{\partial z}{\partial x} + z^2 + u(2z) \cdot \frac{\partial z}{\partial x}$

Also, $x^2 + yz + z^3 = 0 \rightarrow \frac{\partial(x^2 + yz + z^3)}{\partial x} = 0$

$\rightarrow 2x + y\frac{\partial z}{\partial x} + 3z^2 \cdot \frac{\partial z}{\partial x} = 0 \rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{2x}{y + 3z^2}} \rightarrow$ (1)

Substituting $\frac{\partial z}{\partial x}$ in $\frac{\partial u}{\partial x}$

$$\rightarrow \frac{\partial u}{\partial x} = 2xy + (y^2 + 2zu)\left(\frac{-2u}{y + 3z^2}\right) + z^2$$

(10) $u = \frac{x}{y - z}$, $v = \frac{y}{z - x}$, $w = \frac{z}{x - y}$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{1}{y-z} & \frac{-x}{(y-z)^2} & \frac{x}{(y-z)^2} \\ \frac{y}{(z-x)^2} & \frac{1}{z-x} & \frac{-y}{(z-x)^2} \\ \frac{-z}{(x-y)^2} & \frac{z}{(x-y)^2} & \frac{1}{x-y} \end{vmatrix}$$

$$\Rightarrow \frac{1}{y-z}\left[\frac{1}{(z-x)} \times \frac{1}{(x-y)} - \frac{zy}{(z-x)^2 \ldots}\right] + \frac{x}{(y-z)^2}\left[\frac{y}{(z-x)^2(x-y)} - \frac{yz}{(z-x)^2(x-y)}\right] + \frac{x}{(y-z)^2}\left[\frac{yz}{(z-x)^2(x-y)^2} + \frac{z}{(x-y)^2(z-x)}\right]$$

on solving. $\rightarrow \underline{0}$ Hence Proved.

(11) $u = x^2 - y^2$, $v = 2xy \rightarrow \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\partial(u, v) / \partial(x, y)}$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2 - 4y^2$$

$$\therefore \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{4(x^2 - y^2)}$$

(12) $u^3 + v^3 = x + y \; ; \; u^2 + v^2 = x^3 + y^3$ ④

S.T : $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{2}\left(\dfrac{(y^2 - x^2)}{uv(u-v)}\right)$; $\dfrac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \dfrac{\partial(f_1,f_2)/\partial(x,y)}{\partial(f_1,f_2)/\partial(u,v)}$

$\rightarrow \cancel{\dfrac{\partial(u,v)}{\partial(x,y)}} \Rightarrow \dfrac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 3x^2 & 3y^2 \end{vmatrix} = 3y^2 - 3x^2$

$\dfrac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} -3u^2 & -3v^2 \\ -2u & -2v \end{vmatrix} = 3u^2v - 3v^2u = 2uv(3u - 3v)$

$\rightarrow \dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{y^2 - x^2}{2uv(u-v)}$ Hence Proved.

(13) $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$

$\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2(y-z) & 2(z-y) \end{vmatrix} \Rightarrow 1(\cancel{2y} - \cancel{2z} - \cancel{2y} + \cancel{2z})$
$-1(2z - 2y - 2x) - 1(2y - 2z + 2x)$
$= 0$

∴ They are functionally dependent.

$\rightarrow u^2 = x^2 + y^2 + z^2 - 2zx - 2zy + 2xy$ | $w = x^2 + y^2 + z^2 - 2yz$
$v^2 = x^2 + y^2 + z^2 - 2yx - 2yz + 2xz$ |

$\rightarrow \boxed{u^2 + v^2 = 2w}$

(14) $\dfrac{\partial(x_1,x_2,x_3,x_4)}{\partial(u_1,u_2,u_3,u_4)} = \dfrac{\partial(f_1,f_2,f_3,f_4)/\partial(u_1,u_2,u_3,u_4)}{\partial(f_1,f_2,f_3,f_4)/\partial(x_1,x_2,x_3,x_4)} = \dfrac{I_1}{I_2}$

$I_1 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ u_2 & u_1 & 0 & 0 \\ u_2 \cdot u_3 & u_1 u_3 & u_1 u_2 & 0 \\ u_2 \cdot u_3 \cdot u_4 & u_3 \cdot u_4 u_1 & u_1 \cdot u_2 u_4 & 0 \end{vmatrix}$; $I_2 = \begin{vmatrix} -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$

$\Rightarrow u_1^3 \cdot u_2^2 \cdot u_3$ $\rightarrow 1$

$\therefore \dfrac{\partial(x_1,x_2,x_3,x_4)}{\partial(u_1,u_2,u_3,u_4)} = \dfrac{I_1}{I_2} = \dfrac{u_1^3 \cdot u_2^2 \cdot u_3}{1}$

$= u_1^2 \cdot u_2^2 \cdot u_3$