# Longitudinal Waves

The particles of the medium oscillate in the direction of wave propagation.
Ex - Sound waves.
Eqn. for gaseous medium :-
As sound waves propagate, variation in pressure and density of the medium take place.
Consider a given mass of gas at equilibrium pressure $P_0$, volume $V_0$ and density $\rho_0$. Suppose when the waves propagate the pressure changes to $P = P_0 + p$, vol. changes to $V = V_0 + v$, and $\rho = \rho_0 + \rho_d$
Let max. pressure amplitude be denoted by $P_m$ and $dP = p$ be the fluctuating components superimposed on the $\equiv$brium $P_0$.
The change in medium are very small.
Fractional vol change $\frac{v}{V_0} = \delta$
is called dilation.

Fractional change in density $\frac{\rho_d}{\rho_0} = s$ is condensation.

Since mass is constant.

$\therefore \rho_0 V_0 = \rho V = \rho_0 V_0 (1+\delta)(1+s)$

or $(1+\delta)(1+s) = 1$

giving $\delta = -s$ (approximated).

which means dilation is equal and opposite to condensation.

The bulk modulus is a measure of gas compressibility

$$B = -\frac{dP}{dV/V} = -V\frac{dP}{dV}$$

ie ratio of change in pressure for a fractional change in volume. ~~per unit volume -ve sign~~

When sound wave propagates through gas, the total heat content of the system remains const. which means the process is adiabatic. for which

$$PV^{\gamma} = \text{const.}$$

$\gamma$ = ratio of specific heats at constant pressure & volume.

Differentiating we get:

$$V^{\gamma} dP + \gamma P V^{\gamma-1} dV = 0$$

$$V^{\gamma} dP = -\gamma P V^{\gamma-1} dV$$

$$\frac{V^{\gamma}\, dP}{V^{\gamma-1}\, dV} = -\gamma P$$

$$-\frac{V\, dP}{dV} = \gamma P = B_a$$

$\therefore$ $\gamma P$ = const, because $B_a$ is const for a material within limit

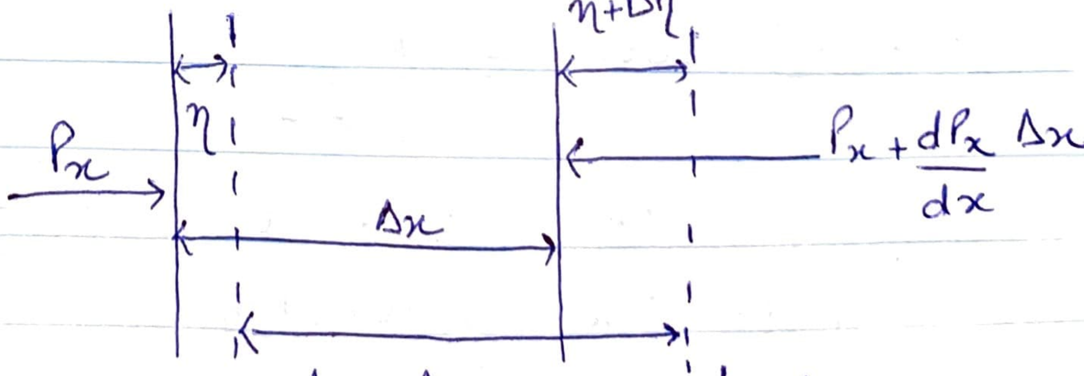
$$B_a = -\frac{p}{\delta}$$

or $$p = -B_a \delta = B_a s$$

<u>Wave equation</u>:

Let the displacement of wave along x-dirⁿ be $\eta(x, t)$

Consider the motion of an element of gaseous medium of thickness $\Delta x$ and unit cross-section.




$$\Delta x + \Delta \eta = \Delta x + \frac{d\eta}{dx} \Delta x$$

The particles in the layer $x$ are displaced by a dist. $\eta$ while those at $x+\Delta x$ are displaced by $\eta + \Delta\eta$. Thus the increase in thickness by $\Delta x$ equals the increase in volume :-

$$\Delta\eta = \frac{d\eta}{dx}\Delta x$$

$$\delta = \frac{v}{V_0} = \frac{d\eta}{dx}\frac{\Delta x}{\Delta x} = \frac{d\eta}{dx} = -s$$

where $\frac{d\eta}{dx}$ is the strain.

Net ~~pos~~ force acting on the element :-

$$P_x - P_{x+\Delta x} = \left[P_x - \left(P_x + \frac{dP_x}{dx}\Delta x\right)\right]$$

$$= -\frac{dP_x}{dx}\Delta x = -\frac{d}{dx}(P_0 + p)\Delta x = -\frac{dp}{dx}\Delta x$$

Mass of element $= \rho_0 \Delta x$

Acceleration $= \frac{d^2\eta}{dt^2}$

Using Newton's second law :-

$$-\frac{dp}{dx}\Delta x = \rho_0 \Delta x \frac{d^2\eta}{dt^2}$$

Using $p = -B_a \delta = -B_a \frac{d\eta}{dx}$.

we get $-\frac{dp}{dx} = B_a \frac{d^2\eta}{dx^2}$

$\therefore$ Comparing we get.

$$B_a \frac{d^2\eta}{dx^2} = \rho_0 \frac{d^2\eta}{dt^2}$$

As $\frac{B_a}{\rho_0} = \frac{\gamma P}{\rho_0}$, it has the

folloing dimension :-

$$\frac{\text{force}}{\text{area}} \; \frac{\text{volume}}{\text{mass}} = (\text{velocity})^2$$

$\therefore$ $\frac{B_a}{\rho_0} = v^2 = \frac{\gamma P}{\rho_0}$

$$\boxed{\frac{d^2\eta}{dx^2} = \frac{1}{v^2} \frac{d^2\eta}{dt^2}}$$

is the wave eqn. of propagating sound wave.

If $\eta_m$ = max amp of displacement

$$\eta = \eta_m e^{i(\omega t - kx)}$$

Particle velocity $\frac{d\eta}{dt} = i\omega\eta$

Condensation $\delta = \frac{d\eta}{dx} = -ik\eta = -s$

$\therefore$ dilation $s = ik\eta$

Excess pressure $p = B_a s = iB_a k\eta$

<u>Acoustic impedance</u>

Impedance (resistance) offered by a medium to a wave is given as :-

specific acoustic impedance $= \frac{p}{d\eta/dt}$

We will state denoting $\frac{d\eta}{dt}$ as $\dot{\eta}$

$= \frac{\text{excess pressure}}{\text{particle velocity}}$

or ratio of force per unit area to the velocity.

$p = B_a \delta = i B_a k\eta$

$\therefore \frac{p}{\dot{\eta}} = \frac{iB_a k\eta}{i\omega\eta} = \frac{B_a k}{\omega} = \frac{B_a}{v} = \rho_0 v$

$\therefore$ Acoustic impedance offered by a medium to the wave is equal to the product of density and the wave velocity.

# Reflection and Transmission of Sound waves:

When a sound wave propagating in medium encounters a boundary separating two media of different acoustic impedances, the following two conditions should be met for reflection & transmission of waves:-

1) The particle velocity, $\dot{\eta}$
2) Acoustic excess pressure, $p$

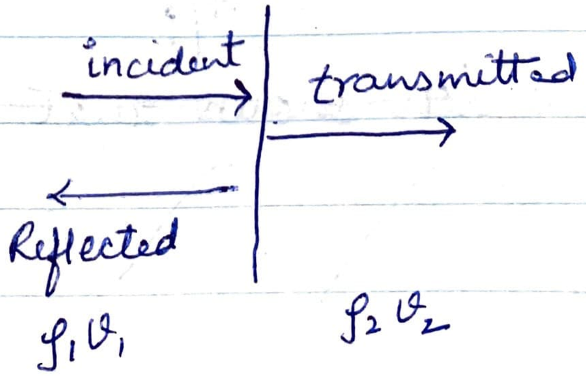
should be continuous across boundary

Suppose a plane sound wave is propagating through a medium of $Z_1 = \rho_1 v_1$. The wave is incident normally at an infinite plane boundary of second medium with impedance $= Z_2 = \rho_2 v_2$

Above boundary cond$^n$ required

$$\dot{\eta}_i + \dot{\eta}_r = \dot{\eta}_t \quad \text{and} \qquad (1)$$

$$p_i + p_r = p_t \qquad (2)$$



For incident wave :-

$$p_i = \rho_1 v_1 \dot{\eta}_i$$

For reflected wave :-

$$p_r = -\rho_1 v_1 \dot{\eta}_r$$

For transmitted wave :-

$$p_t = \rho_2 v_2 \dot{\eta}_t$$

So,

$$\rho_1 v_1 \dot{\eta}_i - \rho_1 v_1 \dot{\eta}_r = \rho_2 v_2 \dot{\eta}_t$$

or $$Z_1 \dot{\eta}_i - Z_1 \dot{\eta}_r = Z_2 \dot{\eta}_t \quad \text{--- (3)}$$

Eliminating $\dot{\eta}_t$ from (1) and (3)

$$Z_1 \dot{\eta}_i - Z_1 \dot{\eta}_r = Z_2 (\dot{\eta}_i + \dot{\eta}_r)$$

Simplify :-

$$\frac{\dot{\eta}_r}{\dot{\eta}_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

But $$\frac{\dot{\eta}_r}{\dot{\eta}_i} = \frac{ik\eta_r}{ik\eta_i} = \frac{\eta_r}{\eta_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \quad \text{--- (4)}$$

Similarly eliminate $\dot{\eta}_r$ :-

$$\frac{\dot{\eta}_t}{\dot{\eta}_i} = \frac{\eta_t}{\eta_i} = \frac{2Z_1}{Z_1 + Z_2}$$

In terms of excess pressure :-

$$\frac{p_r}{p_i} = -\frac{Z_1 \dot{\eta}_r}{Z_1 \dot{\eta}_i} = \frac{Z_2 - Z_1}{Z_1 + Z_2} = -\frac{\dot{\eta}_r}{\dot{\eta}_i} \quad (5)$$

and

$$\frac{p_t}{p_i} = \frac{Z_2 \dot{\eta}_t}{Z_1 \dot{\eta}_i} = \frac{2Z_2}{Z_1 + Z_2}$$

From eqn. (4) & (5) it is clear that if $Z_1 > Z_2$, the incident and reflected particle velocities are in phase, and the incident & reflected acoustic pressures are in opposite phase.

If $Z_1 < Z_2$, the pressures are in one phase, velocities are out of phase.

The transmitted particle velocity and acoustic pressure are always in phase with incident counterparts

## Reflection and transmission of sound intensity

Intensity of sound waves is a measure of the energy flux, ie, rate at which energy crosses unit area.

It is equal to the product of the energy density and velocity.
Intensity is thus :-

$$I = \frac{1}{2} \rho_0 v \dot{\eta}_m^2$$

$$= \frac{1}{2} \rho_0 v \omega^2 \eta_m^2 = \rho_0 v \dot{\eta}_{rms}^2$$

$$= \frac{P_{rms}^2}{\rho_0 v} = P_{rsm} \dot{\eta}_{rms}$$

To find the intensity coefficients of reflection and transmission of sound waves at the boundary b/w two media is :

$$\frac{I_r}{I_i} = \frac{Z_1 (\dot{\eta}_r^2)_{rms}}{Z_1 (\dot{\eta}_i^2)_{rms}} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

$$\frac{I_t}{I_i} = \frac{Z_2 (\dot{\eta}_t^2)_{rms}}{Z_1 (\dot{\eta}_i^2)_{rms}} = \frac{Z_2}{Z_1}\left(\frac{Z_1}{Z_1 + Z_2}\right)^2 = \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2}$$

Using above two expressions it can be shown that-

$$\frac{I_r}{I_i} + \frac{I_t}{I_i} = 1$$

or $\boxed{I_i = I_r + I_t}$