THEORY OF RELATIVITY

UNIT 1

Frame of References (FoR)

- The position of any object is defined with respect to a reference.

- Simplest FoR is the cartesian coordinates (x,y,z)

- Other examples are – polar, cylindrical, spherical, centre of mass, etc.

$$x = r\cos\phi,$$
$$y = r\sin\phi.$$

$$r = \sqrt{x^2 + y^2}$$
$$\phi = \operatorname{atan2}(y, x),$$

$$x = r\sin\theta\,\cos\varphi,$$
$$y = r\sin\theta\,\sin\varphi,$$
$$z = r\cos\theta.$$

Dynamics of a system – say a particle

- Position (x)
- Change in position with time – velocity (v)
- Momentum (p) = m*v
- Acceleration (a)
- Force (F) --- Newton’s second law of motion
- Energy

Types of Frame of References

- Inertial – In which the Newton's laws of motion remains the same. The FoR is either at rest or moving with constant velocity.

- Non-inertial – Accelerated FoR – the basic laws of physics change (2^{nd} law).
  - Uniform linear acceleration
  - Uniformly rotating
  - Earth is non-inertial FoR

Transformation equations

- Equations relating two sets of coordinates of an event in two different FoR.
- If the two FoRs are inertial, then the transformations are called <u>Galilean Transformations</u>

  - These transforms are easy!
    - Just try to apply logic.

Transformation of position

Transformation of distance or length

Invariance

Transformation of velocity

Transformation of acceleration

Newton’s law of motion

Law of conservation of momentum

is invariant under G.T.

S $\qquad$ S'

m_1 $\qquad$ m_2

Before collision vel $= u_1 \quad u_2$ $\qquad u_1' \quad u_2'$

After " " $= \underbrace{w_1 \quad w_2}_{S}$ $\qquad w_1' \quad w_2'$

$S \Rightarrow m_1u_1 + m_2u_2 = m_1w_1 + m_2w_2$

$\underline{u_1 = u_1' + v}$ and so on...

$m_1(u_1'+v) + m_2(u_2'+v)$

$= m_1(w_1'+v) + m_2(w_2'+v)$

$$\boxed{\Rightarrow m_1u_1' + m_2u_2' = m_1w_1' + m_2w_2'}$$

$\Rightarrow S'$

$\cancel{mv}$

$\underline{mu = m(u'+v)}$

Law of conservation of energy

is invariant under G. T.

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1w_1^2 + \frac{1}{2}m_2w_2^2 \Big\} S$$

$$\frac{1}{2}m_1(u_1'+v)^2 + \frac{1}{2}m_2(u_2'+v)^2 = - - - - - - - -$$

$$\frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2 + v\,\cancel{\underline{(m_1u_1'+m_2u_2')}} = \frac{1}{2}m_1w_1'^2 + \frac{1}{2}m_2w_2'^2 + v\,\cancel{\underline{[m_1w_1'+m_2w_2']}}$$

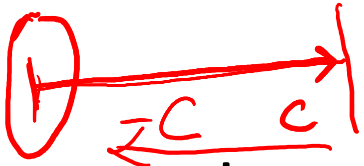
$$\frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2 = \frac{1}{2}m_1w_1'^2 + \frac{1}{2}m_2w_2'^2 \Big\} S'$$

Special Theory of Relativity

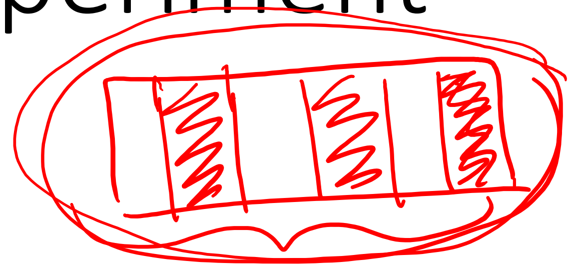
Search for an absolute frame of reference

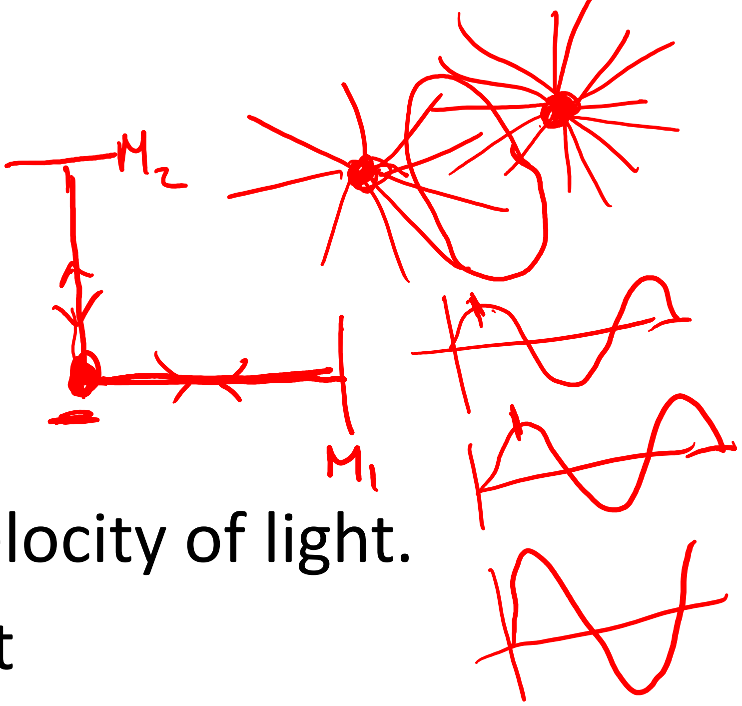
- Sound waves need a medium to travel.
- It was discovered that light is a wave, so what medium does it travel through?
- It was proposed that an invisible material surrounds the universe and space – called ether.
- Ether is an absolute frame of reference.
- However, when experiments were conducted to search for this material and absolute frame of reference, it could not be found/detected.
- One famous experiment is Michelson-Morley experiment.

$u' = u - v$



Michelson-Morley Experiment





- Based on interference of light.
- Results could not find the effect of ether on the velocity of light.
- Final explanation was given by Einstein stating that

<u>Velocity of light is a universal constant.</u> = fixed.

<u>It remains the same in all frames of reference</u>

<u>u'=u-v is not valid in case of speed of light</u>

X

$v \approx c$

$S \rightarrow u = c$

$S' \rightarrow u' \approx c$

Basic Postulates of special theory of relativity

1. The laws of physics all take the same identical form for all frames of reference in uniform relative motion. inertial FoR
2. The velocity of light in free space is the same relative to any inertial frame of reference, i.e, it is invariant to transformation from one inertial frame to another, and thus, has the same value for all observers irrespective of their state of motion.

$c = 3 \times 10^8$ m/s

$\cancel{u' = u - v}$

$u' = u - v$

$u = 0$

$u' = -v$

$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$

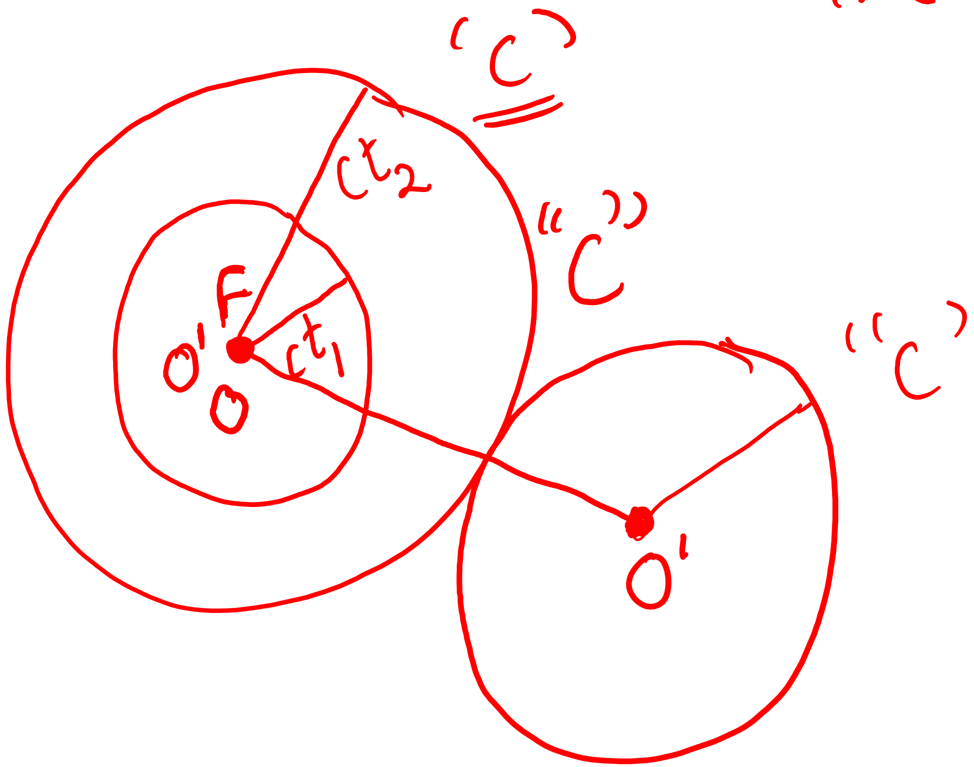
$m \to \infty$

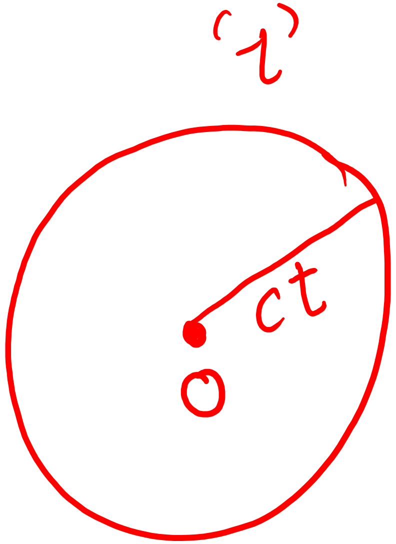
Implications

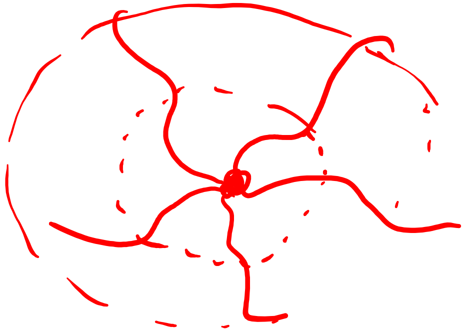
If vel. of light in $S = c$

The " " " in $\underline{\underline{S'}} = c' = c$ ✓

$\neq c - v$







Implications

In S $\rightarrow$ $x^2+y^2+z^2=c^2t^2$ —— 1

In S' $\rightarrow$ $x'^2+y'^2+z'^2=c'^2t'^2$ —— 2

Subs. ~~But~~ in 2 $\boxed{c=c'}$; $t=t'$; $x'=x-vt$; $y'=y$; $z'=z$

$$x^2-2xvt+v^2t^2+y^2+z^2=c^2t^2$$

$\therefore$ $1 \neq 2$ when we use G. T.

$\boxed{v \ll c}$ $\boxed{v \approx c}$

$t' = t$

Lorentz Transformation

Let $x' = \Gamma(x - vt)$ — ①

Γ should be such that -

1) $x = \Gamma(x' + vt')$ Γ = same → ②

2) Γ should not be func of x & t

3) Γ may depend on v

Substitute ① in ②

$$x = \Gamma(\Gamma(x - vt) + vt') = \Gamma^2(x - vt) + \Gamma v t'$$

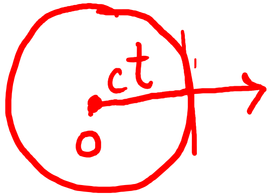
$$\Gamma v t' = x - \Gamma^2(x - vt) = x - \Gamma^2 x + \Gamma^2 vt$$

$$= \Gamma^2 vt + x(1 - \Gamma^2)$$

$$t' = \Gamma t + x\left(\frac{1 - \Gamma^2}{\Gamma v}\right)$$

$$\boxed{t' \neq t}$$

Lorentz Transformation



To find Γ :-

In S, $x = ct$

In S' $\Rightarrow$ $x' = ct'$

$$\Gamma(x - vt) = c\left[\Gamma t + x\left(\frac{1-\Gamma^2}{\Gamma v}\right)\right]$$

$$\Gamma x - \Gamma v t = c\Gamma t + cx\left(\frac{1-\Gamma^2}{\Gamma v}\right)$$

$$x\left[\Gamma - c\left(\frac{1-\Gamma^2}{\Gamma v}\right)\right] = c\Gamma t + \Gamma v t$$

$$x = \frac{(c\Gamma t + \Gamma v t)}{\left[\Gamma - c\left(\frac{1-\Gamma^2}{\Gamma v}\right)\right]}$$

$$x = \cancel{ct}\left[\frac{1 + v/c}{1 - \frac{c}{v}\left(\frac{1}{\Gamma^2} - 1\right)}\right]$$

$$\downarrow$$

$$\cancel{ct}$$

$$1 + \frac{v}{c} = 1 - \frac{c}{v}\left(\frac{1}{\Gamma^2} - 1\right)$$

$$\boxed{\Gamma = \frac{1}{\sqrt{1 - v^2/c^2}}}$$

$$\frac{v}{c} = \beta$$

$$\boxed{\Gamma = \frac{1}{\sqrt{1 - \beta^2}}}$$

Lorentz Transformation

✓ $$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \Gamma(x - vt)$$

$$y' = y \qquad z' = z$$

$$t' = \Gamma t + x\left[\frac{1 - \Gamma^2}{\Gamma v}\right] = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}}$$

✓ $$t' = \Gamma\left(t - \frac{vx}{c^2}\right)$$

If $v << c$, $v^2 << c^2$, $\frac{v^2}{c^2}$ neglected 1

Inverse

$$x = \Gamma(x' + vt') = \frac{x + vt'}{\sqrt{1 - v^2/c^2}}$$

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - v^2/c^2}}$$

Lorentz Transformation

For you to do →

$$x'^2 + y'^2 + z'^2 = ct'^2$$

$$x^2 + y^2 + z^2 = ct^2$$

Put. L.T. in it and prove it equal.

Q: Lorentz Transformation

- As measured by O, an event occurs at x=100 km, y=10 km, z=1 km at $t=5\times10^{-4}$ s. What are the coordinates x', y', z', t' of this event for observer O', moving at -0.8c along x-axis.

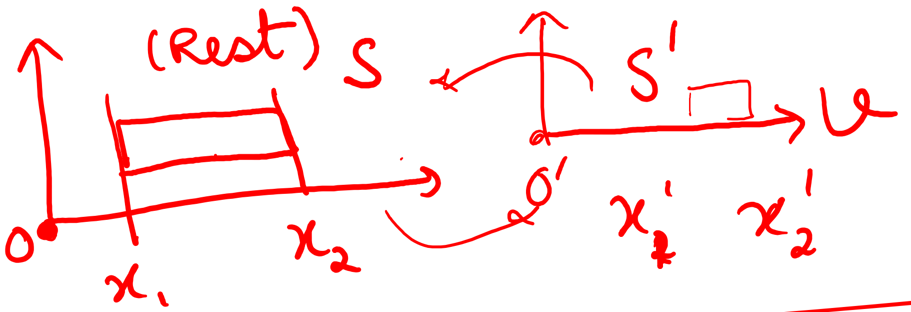
$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \frac{100 - (-0.8)(3\times10^5)(5\times10^{-4})}{\sqrt{1 - \frac{(0.8)^2\cancel{c^2}}{\cancel{c^2}}}} = 367\text{ km}$$

$$y' = y = 10\text{ Km}$$

$$z' = z = 1\text{ Km}$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\beta^2}} = \frac{5\times10^{-4} + \frac{0.8\times100}{3\times10^5}}{\sqrt{1 - 0.64}} = \underline{\underline{12.8\times10^{-4}\text{ s}}}$$

Length Contraction



$S \Rightarrow L_0 = x_2 - x_1$ (P.L)

$S' \Rightarrow L = x_2' - x_1'$ [provided time t']

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}} \qquad x_2 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$\therefore L_0 = \frac{x_2' - x_1'}{\sqrt{1 - v^2/c^2}} = \frac{L}{\sqrt{1 - v^2/c^2}}$$

or

$$\boxed{L = L_0 \sqrt{1 - v^2/c^2} = \frac{L_0}{\Gamma}}$$

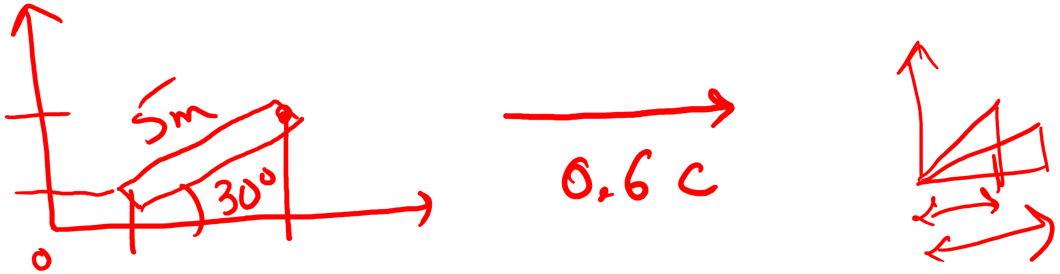
Shows $L < L_0$

Proper length → w.r.t. to the object being at rest.

If $v = c$, then $L = 0$

Same is true if rod is moving & observer is at rest.

Q: Length contraction



- Calculate the length contraction and orientation of a rod of length 5 m making an angle 30° with the x-axis, in a frame of reference which is moving with a vel 0.6c in x- direction.

$$l_x = 5\cos 30^\circ = \frac{5\sqrt{3}}{2} \qquad l_y = 5\sin 30^\circ = \frac{5}{2}$$

$$l_x' = l_x\sqrt{1-\frac{v^2}{c^2}} = \frac{5\sqrt{3}}{2}\sqrt{1-(0.6)^2} = 2\sqrt{3} \qquad l_y' = l_y = \frac{5}{2}$$

$$L = \sqrt{l_x'^2 + l_y'^2} = \sqrt{(2\sqrt{3})^2 + \left(\frac{5}{2}\right)^2} = \underline{4.2\ m}$$

$$\tan\theta = \frac{5/2}{2\sqrt{3}} \Rightarrow \theta = 35.8^\circ \qquad 5 - 4.2 = \underline{\underline{0.8\ m}}$$

Time dilation

Two identical clocks

S $\qquad\qquad$ S'

Time interval

$\Delta t = t_2 - t_1 \qquad \Delta t' = t_2' - t_1'$

$$t_1' = \Gamma\left(t_1 - \frac{vx}{c^2}\right) \qquad t_2' = \Gamma\left(t_2 - \frac{vx}{c^2}\right)$$

$$\Delta t' = \Gamma(t_2 - t_1) = \frac{\Delta t}{\sqrt{1-\beta^2}} \checkmark$$

$\Delta t' > \Delta t$

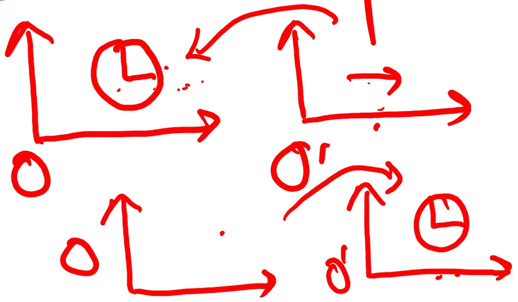
↳ moving clock is slower

It works in both ways.

↳ reverse is also true.

Proper time → time measured in stationary frame.

If $v = c$, clock will ~~stop~~ running.



slow ① $\Delta t = t_2 - t_1$

slow. $O' \Rightarrow \Delta t'$

Q: Time dilation

$$\frac{14.14-3.2}{3.2}\times 100 = \boxed{341\%}$$

- A clock is moving with a speed 0.95c relative to an observer on earth. If the speed is increased by 5%, by what % does time dilation increase?

$v = 0.95c$

$v' = (1.05)(0.95)c$

$= 0.9975c$

$$\Delta t' = \frac{\Delta t}{\sqrt{1-v^2/c^2}} \Rightarrow \frac{\Delta t'}{\Delta t} = \frac{1}{\sqrt{1-(0.95)^2}} = 3.2$$

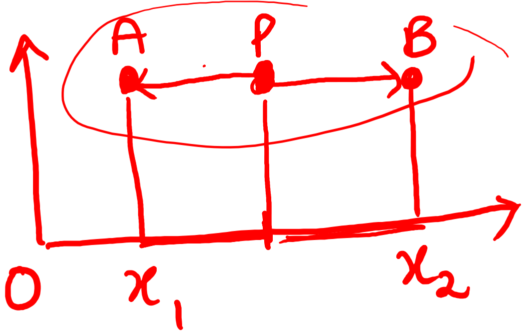
$\times 4.5$

$$\left(\frac{\Delta t'}{\Delta t}\right)_{new} = \frac{1}{\sqrt{1-(0.9975)^2}} = 14.14$$

$$\therefore \% \text{ change} = \frac{14.14-3.2}{14.14}\times 100 = 77.3\%$$

Simultaneity

$\underline{\underline{S}}$



$t_1 = t_2 = $ 't' ✓

$S' \xrightarrow[v]{} (t_1' \,;\, t_2')$

$$t_2' = \Gamma\left(t - \frac{v x_2}{c^2}\right)$$

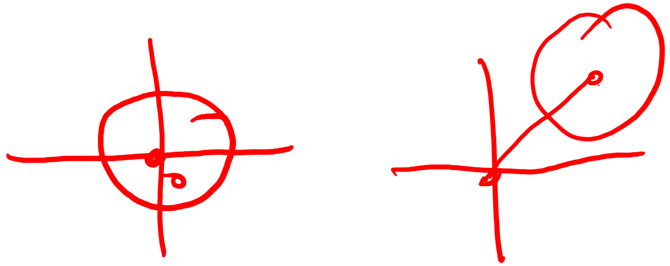
$$t_1' = \Gamma\left(t - \frac{v x_1}{c^2}\right)$$

$$\Delta t' = t_2' - t_1' \neq 0 = \Gamma v \frac{(x_1 - x_2)}{c^2}$$

∴ Events which are simultaneous in S are <u>NOT</u> simultaneous in S'.

Reverse is also <u>true</u> →

Invariance of space-time interval



$\underbrace{x,\ y,\ z}_{\text{space coord}}$

't' → separate

New system → $\underbrace{(x,\ y,\ z,\ ct)}_{\text{space-time system.}}$

$$\boxed{c^2t^2 - x^2 - y^2 - z^2} \rightarrow \text{invariant under L.T.}$$

↓

space-time interval $= S_{12}^2$; $S_{12}'^2$

$$c^2(t_2-t_1)^2 - (x_2-x_1)^2 - (y_2-y_1)^2 - (z_2-z_1)^2$$

$$c^2(t_2'-t_1')^2 - (x_2'-x_1')^2 - (y_2'-y_1')^2 - (z_2'-z_1')^2$$

Prove it.

$x' = \Gamma(x - vt)$ $\quad t' = \Gamma\left(t - \frac{vx}{c^2}\right)$

Transformation of velocity

In frame S, let 'u' be vel of particle

$u \rightarrow \quad u_x = \frac{dx}{dt} \quad u_y = \frac{dy}{dt} \quad u_z = \frac{dz}{dt}$

In frame S', it will be u'

$u' \rightarrow \quad u'_x = \frac{dx'}{dt'}, \quad u'_y = \frac{dy'}{dt'} \quad u'_z = \frac{dz'}{dt'}$

$$u'_x = \frac{dx'}{dt'} = \frac{\Gamma(dx - v\,dt)}{\Gamma\left(dt - \frac{v\,dx}{c^2}\right)} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

$y' = y$ L.T. reduce to G.T. for $v \ll c$

Transformation of velocity

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{\Gamma\left(dt - \frac{v\,dx}{c^2}\right)} = \frac{dy/dt}{\Gamma\left(1 - \frac{v\,dx}{c^2\,dt}\right)} = \frac{u_y}{\Gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

$$u'_z = \frac{u_z}{\Gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

Law for addition of relativistic velocities.

$v \rightarrow$ x-dir^n $\rightarrow$ u is also only in x-dir^n. Then $u_y = u_z = 0$

$u = u_x$

$$\boxed{u' = \frac{u - v}{1 - \frac{uv}{c^2}}} \rightarrow u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Transformation of velocity - example

1). If $u' = c$, then G.T. $\Rightarrow$ $\underline{u = u' + v = c + v > c}$ ✓

Not correct.

$\Rightarrow$ in L.T. $u = \dfrac{u' + v}{1 + \frac{vu'}{c^2}} = \dfrac{c + v}{1 + \frac{vc}{c^2}} = c$

2) let $v = \dfrac{3c}{4}$ $\quad u' = \dfrac{c}{2}$ $\Rightarrow$ $\underline{u = u' + v} = \dfrac{5}{4}c = \underline{\underline{1.25c}}$ G.T.

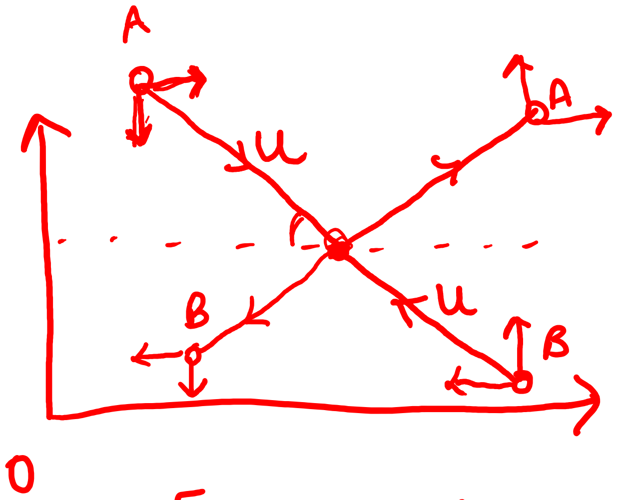
L.T. $\Rightarrow$ $u = \dfrac{10}{11}c < c$

No two vel. can add up to be greater than 'c'.

Relativity of Mass – conservation of momentum

Suppose, FoR at rest, Σ, collision of two particles 'A' and 'B' of mass 'm', glancing angle.

It is an elastic collision,



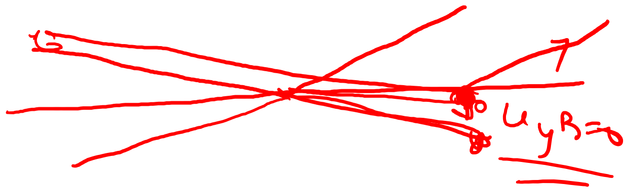
Frame Σ

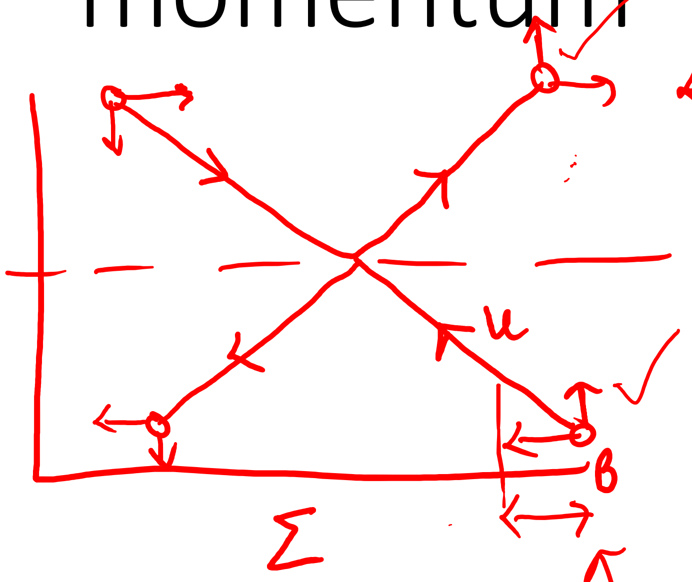
Momentum is conserved.

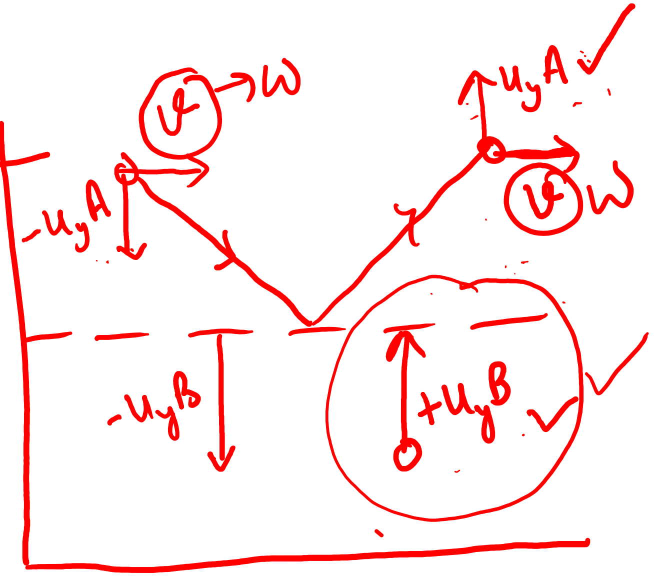
Observe this event in two other FoR.

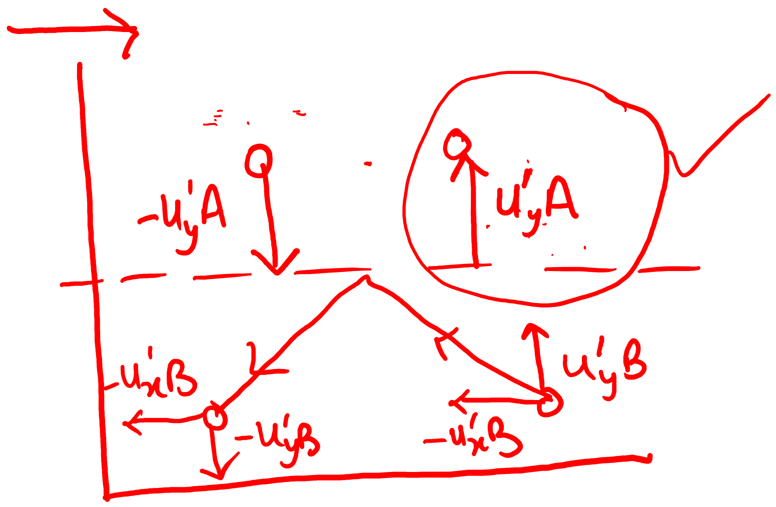
one is S and other is S'

Relativity of Mass – conservation of momentum









S ⇒ moving with ← vel = x comp of B

S' moving with vel = x - comp of A = v

$$\therefore \; u'_yA = u_yB \; ; \; \text{Use L.T.} \Rightarrow \boxed{u_yA = \frac{u'_yA}{\Gamma\left(1 + \frac{v u'_xA}{c^2}\right)} = \frac{u'_yA}{\Gamma}} \quad \left[\because u'_xA = 0\right]$$

Relativity of Mass – conservation of momentum

$\underline{u_y A} = \dfrac{u_y B}{\Gamma} \Rightarrow \boxed{u_y B = \Gamma u_y A}$

Change in vertical vel. comp of A

$u_y A - (-u_y A) = 2u_y A$

For B $\Rightarrow$ $2u_y B$

Change in mom $\rightarrow$

$2mu_y A - 2mu_y B \neq 0$

We have to vary mass to make abv. eqn $= 0$

Let m_A and m_B = mass observed by moving observer.

$2m_A u_y A - 2m_B u_y B = 0$

$m_A u_y A = m_B u_y B$

$m_A u_y A = m_B \Gamma u_y A$

$\underline{m_A = m_B \Gamma}$

$$\boxed{m_A = \frac{m_B}{\sqrt{1-v^2/c^2}}} \Rightarrow \frac{m_A}{m_B} = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\boxed{m = \frac{m_0}{\sqrt{1-v^2/c^2}}} \quad \boxed{m > m_0}$$

* mass-velocity relation

Another method for mass expression

VARIATION OF MASS WITH VELOCITY

Let us consider two frame of references s and s' . s' is moving with a constant velocity v relative to s in the positive X-direction (Fig. 1.9). Suppose in s′, two exactly similar elastic balls A and B each of mass m approach each other at equal speeds (*i.e.* u and $-u$). They collide with each other and coalesce into one body. Considering conservation of momentum

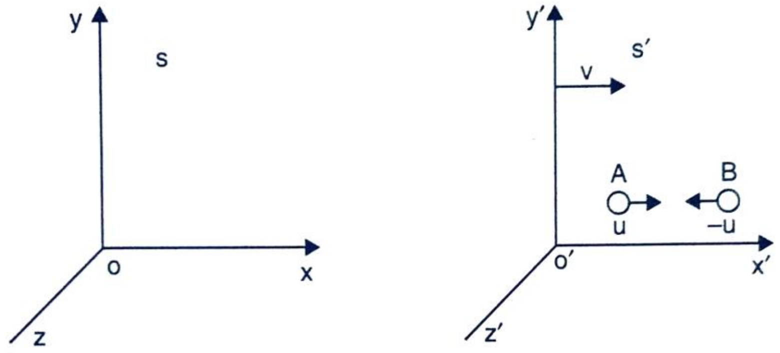


Fig. 1.9

Momentum of ball A + momentum of ball B = momentum of coalesced mass *i.e.* $mu + (-mu)$ = momentum of coalesced mass = 0.

Thus, the coalesced mass must be at rest in s' frame. Now, consider the collision with respect to the frame of reference s. Let u_1 and u_2 be the velocities of the balls relative to s. Then

$$u_1 = \frac{u+v}{1+\frac{uv}{c^2}} \qquad \text{...(1)}$$

$$u_2 = \frac{-u+v}{1-\frac{uv}{c^2}}. \qquad \text{...(2)}$$

After collision velocity of the coalesced mass is v relative to s. Let the mass of the ball A moving with velocity u_1 be m_1 and that of ball B moving with velocity u_2 be m_2 in the frame of reference s. Considering conservation of momentum, we have

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)v. \quad ...(3)$$

You should know why at this step masses are considered to be different.

Because, try to conserve momentum in S frame by keeping both masses as 'm'. It will not come out to be conserved

Substituting for u_1 and u_2 from equations (1) and (2)

$$m_1\left[\frac{u+v}{1+\frac{uv}{c^2}}\right] + m_2\left[\frac{-u+v}{1-\frac{uv}{c^2}}\right] = (m_1 + m_2)v$$

$$m_1\left[\frac{u+v}{1+\frac{uv}{c^2}} - v\right] = m_2\left[v - \frac{-u+v}{1-\frac{uv}{c^2}}\right]$$

$$m_1\left[\frac{u\left(1-\frac{v^2}{c^2}\right)}{1+\frac{uv}{c^2}}\right] = m_2\left[\frac{u\left(1-\frac{v^2}{c^2}\right)}{1-\frac{uv}{c^2}}\right]$$

or
$$\frac{m_1}{m_2} = \frac{1+\frac{uv}{c^2}}{1-\frac{uv}{c^2}}. \quad ...(4)$$

Let us consider the value of the term

$$1-\frac{u_1^2}{c^2} = 1-\frac{\left\{\frac{u+v}{c}\right\}^2}{\left(1+\frac{uv}{c^2}\right)^2} = \frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{uv}{c^2}\right)^2} \quad ...(5)$$

and
$$1 - \frac{u_2^2}{c^2} = \frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{uv}{c^2}\right)^2} \quad ...(6)$$

Dividing eq. (6) by eq. (5)

$$\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}} = \frac{\left(1+\frac{uv}{c^2}\right)^2}{\left(1-\frac{uv}{c^2}\right)^2}$$

$$\frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}} = \frac{\left(1+\frac{uv}{c^2}\right)}{\left(1-\frac{uv}{c^2}\right)} \quad ...(7)$$

Thus, from equations (4) and (7)

$$\frac{m_1}{m_2} = \frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

$$m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}}. \qquad ...(8)$$

From equation (8), it is clear that LHS and RHS are independent of one another and this result may be true only if each is a constant.

Therefore $$m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}} = m_0$$

where m_0 is the rest mass of the body and corresponds to zero velocity.

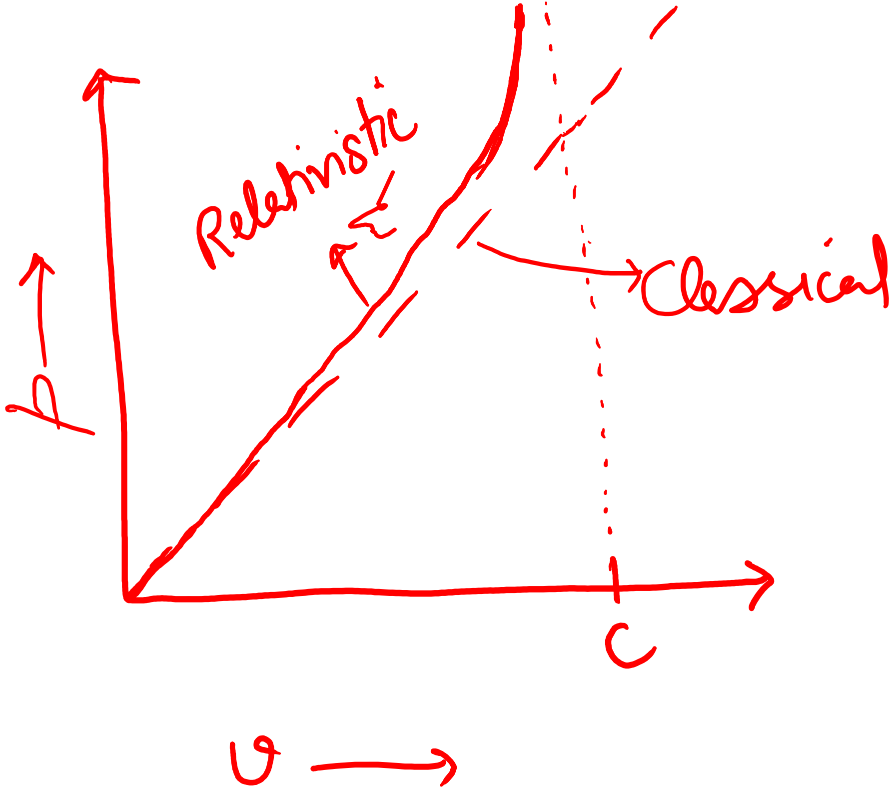
Thus $$m_1 = \frac{m_0}{\sqrt{1-\frac{u_1^2}{c^2}}}.$$

If m be the mass of the body when it is moving with a velocity v then

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}. \qquad ...(9)$$

Equation (9) is the relativistic formula for the variation of mass with velocity. When $v \to c$, $m \to \infty$, *i.e.*, an object travelling at the speed of light would have infinite mass. Thus, no material particle can have a velocity equal to or greater than the velocity of light.

At ordinary velocity, *i.e.*, when $v << c$, $\frac{v^2}{c^2}$ may be neglected. Thus, $m = m_0$.



p and v

$\underbrace{p}$ $= mv$, $m = \text{const}$ in classical mech.

$\underline{y = mx}$

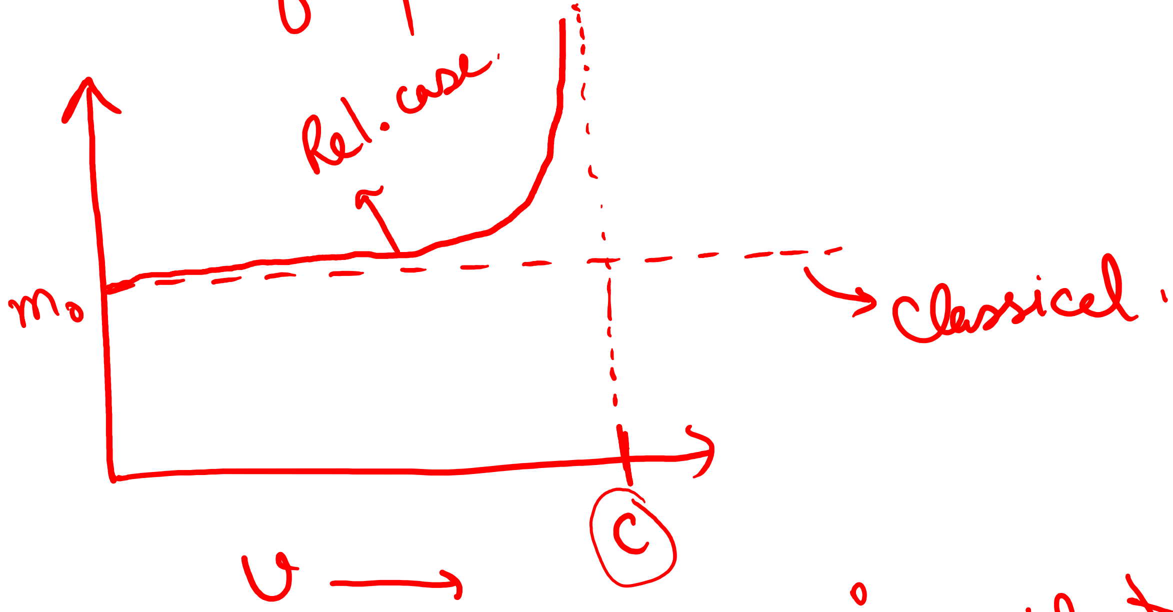
$$p = mv = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

$v \rightarrow c \qquad p \rightarrow \infty$

$\therefore m \rightarrow \infty$

Ultimate speed of Material Particle

Classical mech → mass = const. and no restriction on vel. of particle. Rest mass, m_0



In relativity

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$v \to c \qquad m \to \infty$

$\therefore v \not\geq c$

No material can have speed equal to or greater than 'c'.

$$\boxed{E = mc^2} \rightarrow \text{Einstein's mass-energy relation}$$

Mass-Energy Equivalence

Methods :- (1)

$$m = \frac{m_0}{(1 - v^2/c^2)^{1/2}} = m_0 \left[1 - v^2/c^2\right]^{-1/2}$$

$$m = m_0 \left[1 + \frac{1}{2} v^2/c^2 + \ldots\right] = m_0 + \frac{m_0}{2} v^2/c^2 \quad [\text{neglect higher terms}]$$

Multiply $c^2 \Rightarrow$

$$mc^2 = \underbrace{m_0 c^2}_{\text{Rest energy}} + \underbrace{\frac{m_0 v^2}{2}}_{K.E.}$$

$$E = m_0 c^2 + K.E.$$

$$K.E = mc^2 - m_0 c^2 = \Delta m c^2$$

Rest energy $\Rightarrow$ energy associated with the existence of matter.

$\frac{m\,dv}{dt}$

Mass-Energy Equivalence

Method 2 :-

$$F = ma = \frac{d}{dt}(mv) = v\frac{dm}{dt} + \frac{m\,dv}{dt}$$

If F is applied on a particle so that it gets displaced by dx, the inc. in K.E →

$$dK = F\,dx = v\frac{dm}{dt}dx + \frac{m\,dv}{dt}dx = v^2 dm + mv\,dv$$

But $m = \dfrac{m_0}{\sqrt{1-v^2/c^2}} \Rightarrow m^2 = \dfrac{m_0^2}{1-v^2/c^2} \Rightarrow m^2c^2 - m^2v^2 = m_0^2c^2$

Differentiate ⇒ $c^2\,2m\,dm - v^2\,2m\,dm - 2m^2v\,dv = 0$

$$c^2dm - v^2dm - mv\,dv = 0 \Rightarrow c^2dm = v^2dm + mv\,dv$$

Mass-Energy Equivalence

$\therefore \; dK = c^2 dm$

$$\int dK = \int_{m_0}^{m} c^2 dm$$

$$K = c^2 (m - m_0) = \Delta m c^2$$

$$K = mc^2 - m_0 c^2$$

or $\underbrace{mc^2} = m_0 c^2 + K$.

$$E = R.E + K.E$$

$$\boxed{E = mc^2}$$

1) $e^- + e^+ \longrightarrow 2\,\gamma\text{-rays}$

Annihilation

mass $\longrightarrow$ Energy.

2) Pair production

$2\,\gamma\text{-rays} \longrightarrow e^- + e^+$

Energy $\longrightarrow$ mass

$E = mc^2 = m_0c^2 + E_K$

$F = \boxed{m}\,a$ const

$F \propto a$

Importance of mass-energy relation

$m = m_0 + \frac{E_K}{c^2}$

$m = m_0 + E_K/c^2$

- Total inertial mass of a particle moving with velocity v relative to an observer is the sum of its rest mass and an additional mass (or inertia) equal to E_k/c^2. This suggests that addition of energy to a system increases the inertial mass of the system. OR
- Mass may appear as energy, and energy may appear as mass.
- So law of conservation of energy may be widened to law of conservation of mass-energy.
- Meaning: it is neither mass alone nor energy, that remains conserved, but mass inclusive of energy (in terms of mass), or energy inclusive of mass (in terms of energy), that is really conserved.

Relation between rel. momentum and energy

$\vec{p} = m\vec{v} = \Gamma m_0 \vec{v}$ $\quad E = \Gamma m_0 c^2$

$\therefore \boxed{\vec{p} = \frac{E\vec{v}}{c^2}}$ — ①

(2) $p^2 = \Gamma^2 m_0^2 v^2$

But $\Gamma^2 = \frac{1}{1 - v^2/c^2} = \frac{c^2}{c^2 - v^2}$

$\Gamma^2 v^2 = \Gamma^2 c^2 - c^2$

$p^2 = (\Gamma^2 c^2 - c^2) m_0^2$

Mult. by c^2

$p^2 c^2 = m_0^2 \Gamma^2 c^4 - m_0^2 c^4$

$\therefore \boxed{E^2 = p^2 c^2 + m_0^2 c^4}$

$\underbrace{E^2 - p^2 c^2}_{\text{Quant. is Lorentz invariant}} = m_0^2 c^4 = \text{const.}$

~~Transformation of Rel. momentum and energy~~

Velocity, momentum, Energy of a zero rest mass particle

$$E = (p^2c^2 + m_0^2c^4)^{1/2}$$

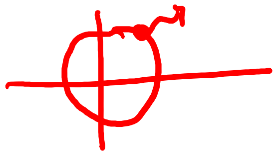
For $m_0 = 0$ $\quad E = pc$ or $p = \dfrac{E}{c}$

But $\quad p = \dfrac{Ev}{c^2} \Rightarrow \dfrac{\cancel{E}}{c} = \dfrac{Ev}{c^2} \Rightarrow \boxed{v = c}$

mass equivalent of energy $= \dfrac{E}{c^2}$

But $\quad E = h\nu \quad m = \dfrac{E}{c^2} = \dfrac{h\nu}{c^2} \Rightarrow \boxed{p = \dfrac{E}{c} = \dfrac{h\nu}{c}}$ ✓

Transformation of Rel. momentum and energy



$S \rightarrow \quad x^2 + y^2 + z^2 = c^2 t^2$

$S' \rightarrow \quad x'^2 + y'^2 + z'^2 = c^2 t'^2$

(*)' $\quad (x, y, z, ct)$

let the momentum of photon be $\vec{p} = \frac{E}{c}$

$p^2 = p_x^2 + p_y^2 + p_z^2 = \frac{E^2}{c^2}$

$p_x'^2 + p_y'^2 + p_z'^2 = \frac{E'^2}{c^2}$

(*) $\quad \left(p_x, p_y, p_z, \frac{E}{c}\right)$

$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \Rightarrow p_x' = \frac{p_x - \frac{vE}{c^2}}{\sqrt{1 - v^2/c^2}}$

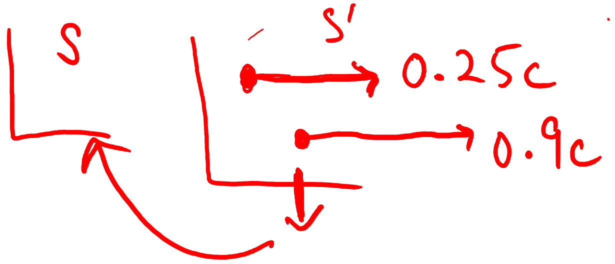
$y' = y \Rightarrow p_y' = p_y$

$z' = z \Rightarrow p_z' = p_z$

$t' = \Gamma\left(t - \frac{vx}{c^2}\right) \Rightarrow \frac{E'}{c^2} = \left(\frac{E}{c^2} - \frac{vp_x}{c^2}\right)\Gamma$

$E' = (E - vp_x)\Gamma$

An experimenter observes a radioactive atom moving a velocity of 0.25c. The atom then emits a beta particle which has a velocity of 0.9c relative to the atom in the direction of its motion. What is the velocity of the beta particle as observed by the experimenter?



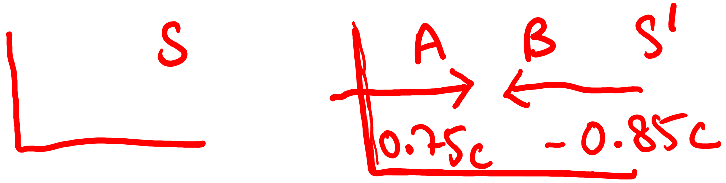
$u = ??$

$v = 0.25c$

$u' = 0.9c$

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{0.9c + 0.25c}{1 + 0.9 \times 0.25} = +0.94c$$

Two spaceships A and B are moving in opposite directions. An observer on the earth measures the speed of A to be 0.75c and the speed of B to be 0.85c. Find the velocity of B wrt A.



$\therefore$

$$u = -0.85c$$

$$v = 0.75c$$

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} = \underline{-0.9771c}$$

A particle has a velocity $\vec{u}'$ = 3i + 4j + 12k m/sec in a coordinate system moving with vel. 0.8c relative to the laboratory along +ve x direction. Find $\vec{u}$ in the laboratory frame.

$$u' = 3\hat{i} + 4\hat{j} + 12\hat{k} = u_x'\hat{i} + u_y'\hat{j} + u_z'\hat{k}$$

$$v = 0.8c$$

$$u = ??$$

$$\therefore \boxed{u = (2.4\times10^8)\hat{i} + 2.4\hat{j} + 7.2\hat{k} \ \text{m/s}}$$

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} = \frac{3 + 0.8c}{1 + \frac{3\times 0.8c}{c^2}} =$$

$$u_y = \frac{u_y'}{\Gamma\left(1 + \frac{u_x' v}{c^2}\right)} = \frac{\sqrt{1 - \left(\frac{0.8c}{c}\right)^2}\cdot 4}{1 + \frac{3\times 0.8c}{c^2}} =$$

$$u_z = \frac{u_z'}{\Gamma\left(1 + \frac{u_x' v}{c^2}\right)} =$$

If the total energy of a particle is thrice its rest energy, what is the velocity of the particle.?

$$E = mc^2 = 3 m_0 c^2$$

$$m = 3 m_0$$

$$\frac{\cancel{m_0}}{\sqrt{1 - v^2/c^2}} = 3 \cancel{m_0}$$

$$\sqrt{1 - v^2/c^2} = \frac{1}{3}$$

$$\boxed{1 - v^2/c^2 = \frac{1}{9}}$$

$$\boxed{v = \frac{2\sqrt{2}}{3} c}$$

Calculate the amount of work to be done to increase the speed of an electron from 0.6c to 0.8c.
Given: rest energy of electron = 0.5 MeV.

$$K = (m - m_0)c^2 = \left(\frac{m_0}{\sqrt{1 - v^2/c^2}} - m_0 \right) c^2 =$$

Find

K_1 (using 0.6c) =

K_2 (using 0.8c) =

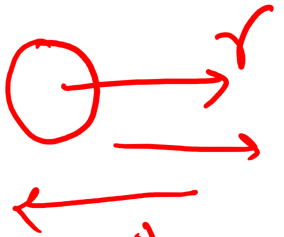
$$\therefore \; W = K_2 - K_1 = 2.1 \times 10^5 \, eV$$
$$= 0.21 \, MeV$$

A nucleus of mass m emits a gamma ray photon of frequency ν_o. Show that the loss of internal energy suffered by the nucleus is not $h\nu_o$ but $h\nu_o(1+ h\nu_o/2mc^2)$.

Solve on your own.

$$E_\gamma = h\nu_0$$

$$p = \frac{h\nu_0}{c} = mv$$



recoils, v

$$K = \boxed{\frac{1}{2}mv^2}$$

$$\text{Total } E_{lost} = h\nu_0 + \frac{1}{2}mv^2$$

$$= h\nu_0 + \frac{(mv)^2}{2m}$$

$$= h\nu_0 + \left(\frac{h\nu_0}{c}\right)^2 \Big/ 2m =$$