Damped Oscillations

UNIT 2

Simple Harmonic Oscillation

Damped Oscillations

Consider a body of mass, m, attached to a spring of force const. 'k'. Let x and $\frac{dx}{dt}$ be the displacement and instantaneous velocity of the body. Forces acting on the body are :-

1. Restoring force $-kx$
2. Damping force $-b\frac{dx}{dt}$, b = const.

∴ Total instantaneous force acting on the body is

$$F = -kx - b\frac{dx}{dt} = ma = m\frac{d^2x}{dt^2}$$

$$\therefore \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Put $\frac{b}{m} = 2r$, (r = damping const.)

$\frac{k}{m} = \omega_0^2$, (ω_0 = natural freq. of the oscillator)

$$\therefore \frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = 0$$

Eq. is the differential eqn. of a damped osc.

Soln :-

Let the soln. be $x = Ae^{\alpha t}$

where A and α = const.

$$\frac{dx}{dt} = A\alpha e^{\alpha t}$$

$$\frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$$

Substitute in the differential eqn -

$$A\alpha^2 e^{\alpha t} + 2\gamma A\alpha e^{\alpha t} + \omega_0^2 A e^{\alpha t} = 0$$

$$A e^{\alpha t}(\alpha^2 + 2\gamma\alpha + \omega_0^2) = 0$$

$$\alpha^2 + 2\gamma\alpha + \omega_0^2 = 0$$

This has two solutions :-

$$\alpha_1 = -r + \sqrt{r^2 - \omega_0^2}$$

$$\alpha_2 = -r - \sqrt{r^2 - \omega_0^2}$$

∴ General soln. is a linear combination

$$x = A_1 e^{[-r + \sqrt{r^2 - \omega_0^2}]t} + A_2 e^{[-r - \sqrt{r^2 - \omega_0^2}]t}$$

where $A_1 = \frac{1}{2} A_0 \left[1 + \frac{r}{\sqrt{r^2 - \omega_0^2}}\right]$

$$A_2 = \frac{1}{2} A_0 \left[1 - \frac{r}{\sqrt{r^2 - \omega_0^2}}\right]$$

Case I : When $r^2 > \omega_0^2$ (heavy damping)

Then $\sqrt{r^2 - \omega_0^2}$ is real and less than 'r'. Hence both exponential terms are -ve. This implies that 'x' continuously decreases with time. There is no oscillation and the amplitude decrease exp. with time. The motion is said to be heavily damped or over damped.

CASE II :- When $r^2 = \omega_0^2$ (critical damping).

Then both terms become infinite. So, let $\sqrt{r^2 - \omega_0^2}$ ~~be~~ instead be a very small quantity β.

$$x = A_1 e^{(-r+\beta)t} + A_2 e^{(-r-\beta)t}$$

$$= e^{-rt}\left(A_1 e^{\beta t} + A_2 e^{-\beta t}\right)$$

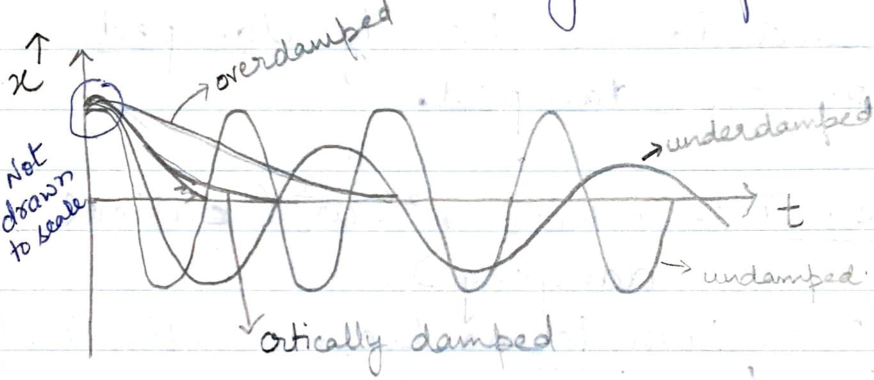
$$= e^{-rt}\left\{A_1\left(1 + \beta t + \frac{\beta^2 t^2}{2!} + \cdots\right) + A_2\left(1 - \beta t + \frac{\beta^2 t^2}{2!} - \cdots\right)\right\}$$

$$= e^{-rt}\left[(A_1 + A_2) + \beta t\,(A_1 - A_2)\right]$$

$$x = e^{-rt}(A + Bt)$$

where $A = A_1 + A_2$ $\quad B = \beta(A_1 - A_2)$

So, even now x falls off exponent. but ~~an~~ amplitude is different. The oscillator returns to $\equiv$brium position very fast. This is k/a critical damping and osc. is said to be critically damped.



CASE III :- $r^2 < \omega_0^2$ (underdamped)

Actual case of damped harmonic osc. where the oscillation's amp. keeps on decreasing with time.

Here $\sqrt{r^2 - \omega_0^2}$ is imaginary.

So, let $\sqrt{r^2 - \omega_0^2} = j\sqrt{\omega_0^2 - r^2} = j\omega$

where $j = \sqrt{-1}$ and

$$\omega = \sqrt{\omega_0^2 - r^2}$$

So,

$$x = A_1 e^{(-r+j\omega)t} + A_2 e^{(-r-j\omega)t}$$

$$= e^{-rt}(A_1 e^{j\omega t} + A_2 e^{-j\omega t})$$

$$= e^{-rt}\{A_1(\cos\omega t + j\sin\omega t) + A_2(\cos\omega t - j\sin\omega t)\}$$

$$= e^{-rt}\{(A_1+A_2)\cos\omega t + j(A_1-A_2)\sin\omega t\}$$

Let $A_1+A_2 = a\sin\phi$ $\quad j(A_1-A_2) = a\cos\phi$

Then,

$$x = e^{-rt}(a\sin\phi\cos\omega t + a\cos\phi\sin\omega t)$$

$$\boxed{x = ae^{-rt}\sin(\omega t + \phi)}$$

This is eqn. of a damped oscillator.

Properties :-

1) Amplitude $= ae^{-rt}$ = decays with time.

e^{-rt} = damping factor.

But the term $\sin(\omega t + \phi)$ means that oscillatory motion is there.

2) Mean lifetime (τ_m) = time interval in which the amp falls by $1/e$ of its initial value.

$$ae^{-\gamma \tau_m} = \frac{1}{e}a = e^{-1}a$$

or $\quad \gamma \tau_m = 1 \Rightarrow \boxed{\tau_m = \frac{1}{\gamma} = \frac{2m}{b}}$

3. <u>Time period</u> :-

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}}$$

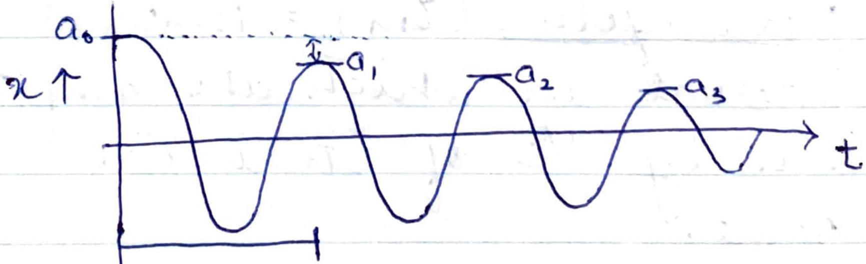
4). <u>Frequency</u> -

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\omega_0^2 - \gamma^2}$$

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

From (3) & (4) it is clear that the time period increases (or freq. decreases) as compared to undamped S.H.O.

5). <u>Logarithmic decrement</u> :-



$$x = ae^{-rt}\sin(\omega t + \phi)$$

Let $\phi = \frac{\pi}{2} \Rightarrow \quad x = ae^{-rt}\cos\omega t.$

At $t = 0$, let $x = a_0$

At time $T, 2T, 3T \ldots\ldots$

$$a_1 = a_0 e^{-rT}$$

$$a_2 = a_0 e^{-2rT}$$

$$a_3 = a_0 e^{-3rT} \quad \ldots\ldots$$

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots\ldots = e^{rt} = e^{\lambda d}$$

where $\boxed{\lambda d = rT = \frac{bT}{2m}}$

is k/a logarithmic decrement.

or

$$\lambda d = \ln\frac{a_0}{a_1} = \ln\frac{a_1}{a_2} = \ldots\ldots$$

$\therefore$ log. dec. is defined as the log. of the ratio of two successive amplitudes of the damped oscillator.

Energy and power dissipation :-

The energy continuously dissipates due to friction. To find the exp:-

Let $r^2 << \omega_0^2$; $\phi = 0$

then $x = ae^{-rt} \sin(\omega t + \phi)$

is $x = ae^{-rt} \sin \omega_0 t$

$$\frac{dx}{dt} = -are^{-rt} \sin \omega_0 t + ae^{-rt} \omega_0 \cos \omega_0 t$$

$$K.E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \frac{1}{2} ma^2 \left[\omega_0 e^{-rt} \cos \omega_0 t - re^{-rt} \sin \omega_0 t\right]^2$$

$$= \frac{1}{2} ma^2 e^{-2rt} \left[\omega_0^2 \cos^2 \omega_0 t + r^2 \sin^2 \omega_0 t - 2r\omega_0 \sin \omega_0 t \cos \omega_0 t\right]$$

To find avg. K.E.

$$K_{av} = \frac{\int_0^T \text{ of above expression}}{T}$$

$$= \frac{1}{2} ma^2 e^{-2rt} \left[\frac{1}{2} \omega_0^2 + \frac{r^2}{2}\right]$$

Here e^{-2rt} was taken out of the integral since we had assumed the damping to be very less, so for one time period, it may be same.

$\therefore \quad K_{av} = \frac{1}{4} m a^2 e^{-2\gamma t} (\omega_0^2 + \gamma^2)$

$$= \frac{1}{4} m a^2 \omega_0^2 e^{-2\gamma t}$$

Similarly, $\quad P.E = U = \frac{1}{2} k x^2$

$$U_{av} = \frac{1}{4} m a^2 \omega_0^2 e^{-2\gamma t}$$

$$E = K_{av} + U_{av} = \frac{1}{2} m a^2 \omega_0^2 e^{-2\gamma t}$$

or $\quad \boxed{E = E_0 e^{-2\gamma t}}$

<u>Relaxation Time</u> - Time required for the decay of mechanical energy to 1/e times its initial value.

$$\frac{E_0}{e} = E_0 e^{-2\gamma \tau}$$

or $\quad \boxed{\tau = \frac{1}{2\gamma}} = \frac{m}{b}$

E can ~~at~~ also be written as -

$$E = E_0^{-t/\tau}$$

<u>Power</u>: $\quad |P| = 2\gamma E = \frac{E}{\tau}$
(dissipated)

Quality factor → Q-factor

measure of damping or the rate of energy decay of the oscillator. Lesser the damping, better the quality or higher the Q-factor of the harmonic oscillator.

Mathematically, it is defined as 2π times the ratio of the energy stored to energy lost per period of the oscillator.

$$Q = 2\pi \frac{E}{PT} = \frac{E}{E/\tau\, T} \cdot 2\pi$$

$$\boxed{Q = \frac{\tau}{T} \cdot 2\pi = \omega\tau}$$