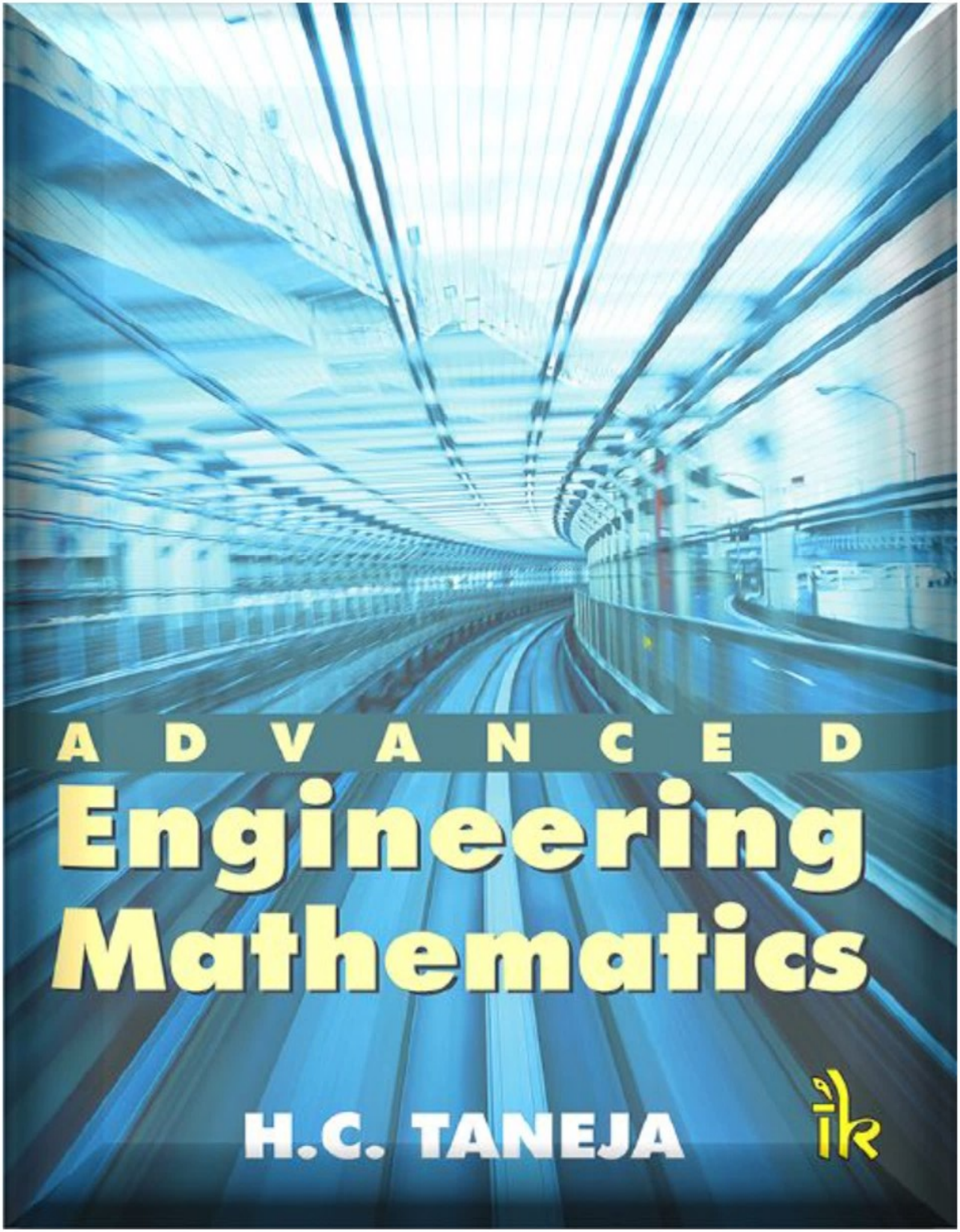

ADVANCED
Engineering
Mathematics
H.C. TANEJA

ADVANCED ENGINEERING MATHEMATICS

H.C. Taneja
Professor & Head
Dept. of Applied Mathematics
Delhi Technological University, Delhi

I.K. International Publishing House Pvt. Ltd.
NEW DELHI • BANGALORE

Published by

I.K. International Publishing House Pvt. Ltd.
S-25, Green Park Extension, Uphaar Cinema Market
New Delhi – 110 016 (India)
E-mail: info@ikinternational.com

ISBN 978-93-82332-64-0

10 9 8 7 6 5 4 3 2 1

Published by Krishan Makhijani for I.K. International Publishing House Pvt. Ltd., S-25, Green Park Extension, Uphaar Cinema Market, New Delhi – 110 016 and Printed by Rekha Printers Pvt. Ltd., Okhla Industrial Area, Phase II, New Delhi – 110 020.

Preface

Both the earlier volumes of *Advanced Engineering Mathematics* have been well received by teachers and students. To give it a broader acceptance, in the present volume a new chapter on "Linear Difference Equations and Z-Transforms" has been added. The text comprising 26 chapters has been divided into 8 parts:

A. Vector Algebra and Matrices (Ch. 1-2)
B. Differential and Integral Calculus (Ch. 3-7)
C. Vector Calculus (Ch. 8-9)
D. Ordinary Differential Equations and Laplace Transforms (Ch. 10-13)
E. Fourier Analysis and Partial Differential Equations (Ch. 14-17)
F. Complex Analysis (Ch. 18-20)
G. Numerical Methods (Ch. 21-23)
H. Statistical Methods and Linear Programming (24-26)

I hope that the book, in its present form will serve the readers better and look forward to their suggestions for further improvement in text.

H.C. Taneja
hctaneja@redifmail.com

Contents

PART B: DIFFERENTIAL AND INTEGRAL CALCULUS

PART A

Vector Algebra and Matrices

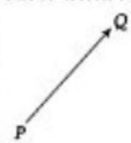
1 Solid Geometry

CHAPTER

Solid geometry is the traditional name for the geometry of three dimensional Euclidean space, the kind of space we live in. Solids have properties, such as volume, surface area. There are two main types of solids, polyhedral, and non-polyhedral; polyhedral must have flat surfaces, and non-polyhedral don't have flat surfaces. The solid models are useful for visualization and analysis in engineering design for numerous applications. Stereometry deals with the measurement of volumes of various solid figures including spheres, cylinders and cones.

1.1 VECTORS IN SPACE

A quantity which is completely specified by its magnitude only is called a scalar, for example, length, time, mass, etc., while *a quantity which is completely specified by its magnitude and direction both is called a vector,* for example, displacement, velocity, force etc. are vector quantities. Vector quantities, like scalar quantities, arise in a natural way while studying the physical systems and so, the knowledge of vector algebra is almost indispensable for engineers and scientists.



A vector can be described geometrically by a directed line segment $\overrightarrow{PQ}$ with its length $|PQ|$ proportional to the magnitude of the vector, line representing the vector parallel to the line of action of the vector and an arrow on the line showing the direction along which the vector acts, starting from P and ending at Q.

A vector is generally represented by a single letter in capital bold type, or with an arrow on it. For example, **V** or $\vec{V}$, may represent the velocity vector. $|\vec{V}|$ or V represents the magnitude of the velocity vector $\vec{V}$.

A vector of unit magnitude is called a *unit vector*. Unit vector corresponding to the vector $\vec{V}$ is denoted by $\hat{V}$. A vector of zero magnitude is called a *zero,* or *null* vector and is denoted by **0**, a bold type zero. The vector $\overrightarrow{QP}$ is defined as the negative of $\overrightarrow{PQ}$. If $\overrightarrow{PQ} = \vec{V}$, then $\overrightarrow{QP} = -\vec{V}$.

Two vectors $\vec{A}$ and $\vec{B}$ with same magnitude and the same, or parallel, directions are said to be *equal* and we write $\vec{A} = \vec{B}$.

Addition and subtraction of vectors. Vectors are added according to the triangle law of addition. Let $\bar{A}$ and $\bar{B}$ be represented by the two sides OP and PQ taken in order of a triangle OPQ in magnitude as well as direction as shown in Fig. 1.1, then their addition $\bar{C} = \bar{A} + \bar{B}$ is represented by the third side OQ of the triangle OPQ but taken in opposite order.

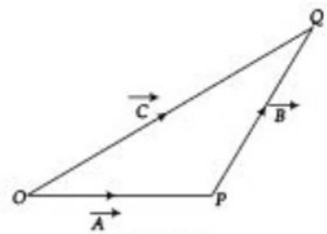


Fig. 1.1

The subtraction of the vector $\bar{B}$ from A is defined as the addition of $-\bar{B}$ to $\bar{A}$ and we write it as $\bar{A} + (-\bar{B}) = \bar{A} - \bar{B}$.

Multiplication of a vector by scalar. The product $\alpha\bar{A}$, of a vector $\bar{A}$ and a scalar α, is defined as a vector whose magnitude is α times that of vector $\bar{A}$ and direction is same or opposite to that of $\bar{A}$ according as α is positive or negative. Thus we can write $\bar{A} = \alpha\hat{A}$, in case $\alpha = |A|$ is the magnitude of vector $\bar{A}$.

The magnitude of a vector has some of the properties of the absolute value of real number as given below:

(i) $|\bar{A}| \geq 0$; $|\bar{A}| = 0$, if, and only if $\bar{A} = 0$

(ii) $|\bar{A} + \bar{B}| \leq |\bar{A}| + |\bar{B}|$

(iii) $|-\bar{A}| = |\bar{A}|$

(vi) $||\bar{A}| - |\bar{B}|| \leq |\bar{A} - \bar{B}|$.

1.1.1 Cartesian Representation of a Vector

We have observed that geometrical interpretation of a vector is easily applied to add and subtract vectors and to multiply them by a scalar. However, to perform more general algebraic operations and extend the theory further, cartesian representation of a vector is much convenient. In the cartesian representation the *position vector* $\bar{r}$ of a point $P(x, y, z)$ from the origin O, as shown in Fig. 1.2, is

$$\bar{r} = \overline{OP} = x\hat{i} + y\hat{j} + z\hat{k}, \qquad \text{...(1.1)}$$

where $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along x-axis, y-axis and z-axis, respectively. Obviously $\hat{i}, \hat{j}, \hat{k}$ are the vectors represented by the directed line segments from the origin to the point (1, 0, 0), (0, 1, 0) and (0, 0, 1) respectively. They form the right-handed system.

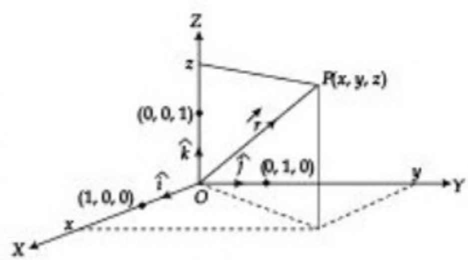


Fig. 1.2

In (1.1) x, y and z are called the *components* of the vector $\bar{r}$ along the coordinate axes. They represent the *projections of the vector* $\bar{r}$ *on x, y, and z axes* respectively.

Let α, β, γ be the angles made by $\overrightarrow{OP}$ with OX, OY and OZ, respectively. These angles are called the *direction angles* of $\overrightarrow{OP}$. Their cosine denoted by l, m, n respectively are called the *direction cosines* of $\overrightarrow{OP}$, which play an important role in the study of three dimensional geometry.

Obviously, $\quad l = x/r, \quad m = y/r,$ and $\quad n = z/r.$

Hence, $\quad \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = r(l\hat{i} + m\hat{j} + n\hat{k}).$

Since, $r = |\bar{r}|$, hence $\hat{r} = li + mj + nk$, a unit vector along the vector $\bar{r}$.

Also, $$l^2 + m^2 + n^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2} = 1. \quad ...(1.2)$$

Thus the sum of the squares of the direction cosines of a straight line is zero.

The numbers $\lambda = kl$, $\mu = km$ and $\nu = kn$ are called the *direction ratios* of $\overrightarrow{QP}$. Obviously $k = \sqrt{\lambda^2 + \mu^2 + \nu^2}$.

We can express the vector $\overrightarrow{P_1P_2}$, from the point $P_1(x, y, z)$ to the point $P_2(x_2, y_2, z_2)$, in terms of the coordinates of P_1 and P_2 as

$$\begin{aligned}\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} &= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) \\ &= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.\end{aligned}$$

Thus, the direction ratios of $\overrightarrow{P_1P_2}$ are $(x_2 - x_1)$, $(y_2 - y_1)$, and $(z_2 - z_1)$ and its direction cosines are

$$\frac{x_2 - x_1}{d}, \quad \frac{y_2 - y_1}{d}, \quad \frac{z_2 - z_1}{d}, \quad ...(1.3)$$

where $d = |\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Example 1.1: (i) Find the distance of a point P from the origin if its position vector is $\bar{r} = 2\hat{i} + 4\hat{j} - 3\hat{k}$.

(ii) If $\bar{A} = 5\hat{i} + \hat{j} - 3\hat{k}$ and $\bar{B} = 2\hat{i} - 2\hat{j} - 7\hat{k}$, find $\bar{A} + \bar{B}$. Also find a vector six units long in the direction of $\bar{A}$.

Solution: (i) The distance of the point P from the origin with position vector $\bar{r} = 2\hat{i} + 4\hat{j} - 3\hat{k}$ is given by

$$|\bar{r}| = \sqrt{2^2+4^2+(-3)^2} = \sqrt{4+16+9} = \sqrt{29}.$$

(ii) $$\bar{A} = 5\hat{i} + \hat{j} - 3\hat{k}, \quad \bar{B} = 2\hat{i} - 2\hat{j} - 7\hat{k},$$

$$\bar{A} + \bar{B} = (5+2)\hat{i} + (1-2)\hat{j} + (-3-7)\hat{k} = 7\hat{i} - \hat{j} - 10\hat{k}.$$

The vector six units long in the direction of $\bar{A}$ is given by

$$\bar{P} = 6\hat{A} = \frac{6\bar{A}}{|\bar{A}|} = \frac{6}{\sqrt{35}}(5\hat{i} + \hat{j} - 3\hat{k}).$$

Example 1.2: Show that the points $P(-4, 9, 6)$, $Q(-1, 6, 6)$ and $R(0, 7, 10)$ form a right angled isosceles triangle. Also find the direction cosines of $\overrightarrow{PQ}$.

Solution: We have, $PQ = \sqrt{(-4+1)^2+(9-6)^2+(6-6)^2} = \sqrt{18} = 3\sqrt{2}$,

$$QR = \sqrt{(-1-0)^2+(6-7)^2+(6-10)^2} = \sqrt{18} = 3\sqrt{2},$$

and, $$RP = \sqrt{(0+4)^2+(7-9)^2+(10-6)^2} = \sqrt{36} = 6.$$

Since, $PQ^2 + QR^2 = RP^2$ and $PQ = QR$, thus the triangle PQR is a right angled isosceles triangle

.

Next, by definition $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (-1+4)\hat{i} + (6-9)\hat{j} + (6-6)\hat{k} = 3\hat{i} - 3\hat{j}.$

Thus, the direction ratios of $\overrightarrow{PQ}$ are 3, −3, 0 and hence the direction cosines are

$$\frac{3}{3\sqrt{2}}, \frac{-3}{3\sqrt{2}}, \frac{0}{3\sqrt{2}}, \text{ or } \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0.$$

Example 1.3: Two points P and Q have position vectors $\bar{A}$ and $\bar{B}$ relative to O as origin. If a point R divides the distance PQ in the ratio $m:n$, find the position vector of R. Hence find the condition of collinearity of three points.

Solution: Let $\bar{C}$ be the position vector of R relative to O as origin as shown in Fig. 1.3.

We have, $$n\overrightarrow{PR} = m\overrightarrow{RQ} \quad \text{or,} \quad n(\overrightarrow{OR} - \overrightarrow{OP}) = m(\overrightarrow{OQ} - \overrightarrow{OR})$$

or, $n(\bar{C}-\bar{A}) = m(\bar{B}-\bar{C})$, which gives, $\bar{C} = \dfrac{n\bar{A}+m\bar{B}}{m+n}$.

Also it gives, $(m+n)\bar{C} - n\bar{A} - m\bar{B} = \bar{0}$.

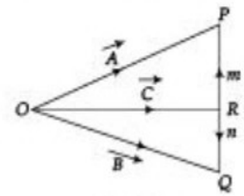


Fig. 1.3

We note that the sum of the coefficients of $\bar{A}$, $\bar{B}$ and $\bar{C}$ in this is zero. Thus the condition of collinearity is, "*The three points P, R and Q with position vectors $\bar{A}$, $\bar{C}$ and $\bar{B}$ respectively are collinear, if their position vectors satisfy a relation of the type*

$$\alpha\bar{A} + \beta\bar{C} + \gamma\bar{B} = \bar{O}, \qquad ...(1.4)$$

where α, β, γ are constants and $\alpha+\beta+\gamma = 0$."

1.1.2 Dot Product of Two Vectors

The *dot product or scalar product* of two vectors $\bar{A}$ and $\bar{B}$, written as $\bar{A}.\bar{B}$, is defined as

$$\bar{A}.\bar{B} = |\bar{A}|\,|\bar{B}|\cos\theta, \qquad ...(1.5)$$

where θ is the angle between $\bar{A}$ and $\bar{B}$.

In case the vectors $\bar{A}$ and $\bar{B}$ are perpendicular, then $\cos\theta = 0$ and hence, $\bar{A}.\bar{B} = 0$.

In case the vectors $\bar{A}$ and $\bar{B}$ are parallel, then $\cos\theta = 1$ and hence, $\bar{A}.\bar{B} = |\bar{A}|\,|\bar{B}|$, which implies that $\bar{A}.\bar{A} = |\bar{A}|^2$. Also we have

$$\left.\begin{array}{l}\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1\\ \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0\end{array}\right\} \qquad ...(1.6)$$

From the definition (1.5) of the dot product it is easy to verify the following properties of the dot products:

(i) *Commutativity:* $\bar{A}.\bar{B} = \bar{B}.\bar{A}$ ...(1.7)

(ii) *Distributivity:* $\bar{A}.(\bar{B}+\bar{C}) = \bar{A}.\bar{B} + \bar{A}.\bar{C}$. ...(1.8)

If $\bar{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$|\bar{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}, \text{ and } \bar{B} = \sqrt{b_1^2 + b_2^2 + b_3^2}.$$

Consider $\bar{A}.\bar{B} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$$= a_1\hat{i}.(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{j}.(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k}.(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1b_1 + a_2b_2 + a_3b_3, \text{ using (1.6).} \qquad ...(1.9)$$

Expression (1.9) is a very convenient way to find the dot product when vectors are in the cartesian form. Also from (1.5), we have

$$\cos\theta = \frac{\bar{A}.\bar{B}}{|\bar{A}||\bar{B}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}. \quad ...(1.10)$$

It gives the angle between the vectors $\bar{A}$ and $\bar{B}$.

1.1.3 Cross Product of Two Vectors

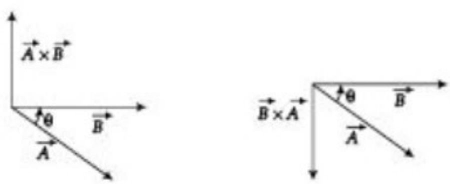
The *cross product*, or *vector product* of two vectors $\bar{A}$ and $\bar{B}$, written as $\bar{A} \times \bar{B}$, is defined as

$$\bar{A} \times \bar{B} = (|\bar{A}|\,|\bar{B}|\,\sin\theta)\hat{N},$$

where $\hat{N}$ is a unit vector with direction perpendicular to both $\bar{A}$ and $\bar{B}$ such that $\bar{A}$, $\bar{B}$ and $\hat{N}$ form a right handed system, as shown in Fig. 1.4a.

Thus the vector $\bar{A} \times \bar{B}$ is both orthogonal to $\bar{A}$ and $\bar{B}$. Also we note that the magnitude of $\bar{A} \times \bar{B}$, that is, $|\bar{A} \times \bar{B}| = |\bar{A}||\bar{B}|\sin\theta$, is the area of the parallelogram formed with $\bar{A}$ and $\bar{B}$ as adjacent sides.

In case the vectors $\bar{A}$ and $\bar{B}$ are parallel, then $\theta = 0$ or π, and hence, $\bar{A} \times \bar{B} = 0$ when $\bar{A}$ and $\bar{B}$ are parallel.



(a) (b)

Fig. 1.4

We note that the vector $\bar{B} \times \bar{A}$ has the same magnitude as $\bar{A} \times \bar{B}$ but opposite direction, as shown in Fig. 1.4b, and hence

$$\bar{B} \times \bar{A} = -(\bar{A} \times \bar{B}). \quad ...(1.11)$$

Thus, commutativity property does not hold in case of cross product of two vectors. But we can verify that distributivity property holds for the cross product, that is

$$\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C}. \quad ...(1.12)$$

Further we can verify that

$$\left.\begin{aligned} \hat{i}\times\hat{j} &= -(\hat{j}\times\hat{i}) &= \hat{k} \\ \hat{j}\times\hat{k} &= -(\hat{k}\times\hat{j}) &= \hat{i} \\ \hat{k}\times\hat{i} &= -(\hat{i}\times\hat{k}) &= \hat{j} \end{aligned}\right\} \qquad ...(1.13)$$

and, $$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \hat{0}. \qquad ...(1.14)$$

Similarly, as in case of dot product, a convenient expression for the cross product of two vectors $\bar{A}$ and $\bar{B}$ is obtained when vectors are given in cartesian forms.

Let $\quad \bar{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then

$$\bar{A}\times\bar{B} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\times(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k} \qquad ...(1.15)$$

using (1.13) and (1.14).

The expression (1.15) can be further put in a convenient form by making use of determinants, as

$$\bar{A}\times\bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad ...(1.16)$$

Example 1.4: For vectors $\bar{A} = \hat{i} - 2\hat{j} - 2\hat{k}$ and $\bar{B} = 6\hat{i} + 3\hat{j} + 2\hat{k}$. Find

(i) $\bar{A}.\bar{B}$ and the angle between $\bar{A}$ and $\bar{B}$.

(ii) The vector projection of $\bar{B}$ on $\bar{A}$ and give its physical interpretation.

(iii) The scalar component of $\bar{B}$ in the direction of $\bar{A}$.

Solution: (i) By definition

$$\bar{A}.\bar{B} = (\hat{i} - 2\hat{j} - 2\hat{k}).(6\hat{i} + 3\hat{j} + 2\hat{k}) = 6 - 6 - 4 = -4,$$

$$|\bar{A}| = \sqrt{1+4+4} = 3, \quad |\bar{B}| = \sqrt{36+9+4} = 7.$$

Thus, $$\theta = \cos^{-1}\left(\frac{\bar{A}.\bar{B}}{|\bar{A}||\bar{B}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) = \cos^{-1}\left(\frac{4}{21}\right).$$

(ii) The vector projection of $\bar{B}$ on $\bar{A}$, denoted by $\text{proj}_{\bar{A}}\bar{B}$, is given by

$$\text{proj}_{\bar{A}}\bar{B} = (|\bar{B}|\cos\theta)\hat{A} \qquad ...(1.17)$$

$$= \left(\frac{\bar{A}.\bar{B}}{|\bar{A}|}\right)\frac{\bar{A}}{|\bar{A}|} = \frac{-4}{9}(\hat{i} - 2\hat{j} - 2\hat{k}).$$

If $\bar{B}$ represents a force, then vector projection of $\bar{B}$ on $\bar{A}$ represents the effective force in the direction of $\bar{A}$.

(iii) The scalar component of $\bar{B}$ in the direction of $\bar{A}$ is given by, $|\bar{B}| \cos\theta = \frac{\bar{B}.\bar{A}}{|\bar{A}|} = \frac{-4}{3}$.

Example 1.5: Express the vector $\bar{B} = 2\hat{i} + 3\hat{j} - 3\hat{k}$ as the sum of a vector parallel to $\bar{A} = 3\hat{i} + \hat{j}$ and a vector orthogonal to $\bar{A}$.

Solution: From Example 1.4 (ii), we write $\bar{B}$ in the desired form as

$$\bar{B} = \text{proj}_{\bar{A}}\bar{B} + (\bar{B} - \text{proj}_{A}B), \quad ...(1.18)$$

since, $\bar{A}.(\bar{B} - \text{proj}_A \bar{B}) = \bar{A}.\bar{B} - \bar{A}.(|\bar{B}| \cos\theta)\hat{A}$

$$= \bar{A}.\bar{B} - (|\bar{B}| \cos\theta)\frac{\bar{A}.\bar{A}}{|\bar{A}|} = \bar{A}.\bar{B} - |\bar{A}|\ |\bar{B}| \cos\theta = 0.$$

Now, $\text{proj}_{\bar{A}}\bar{B} = (|\bar{B}| \cos\theta)\bar{A} = \frac{\bar{A}.\bar{B}}{|\bar{A}|}.\frac{\bar{A}}{|\bar{A}|} = \frac{9}{10}(3\hat{i} + \hat{j}).$

Also, $\bar{B} - \text{proj}_{\bar{A}}\bar{B} = (2\hat{i} + 3\hat{j} - \hat{k}) - \frac{9}{10}(3\hat{i} + \hat{j}) = -\frac{7}{10}\hat{i} + \frac{21}{10}\hat{j} - \hat{k}.$

Thus, $\bar{B} = \frac{9}{10}(3\hat{i} + \hat{j}) + \left(-\frac{7}{10}\hat{i} + \frac{21}{10}\hat{j} - \hat{k}\right).$

Example 1.6: Find a unit vector $\hat{N}$ normal to the plane containing the vector $\bar{A} = 3\hat{i} - 2\hat{j} - \hat{k}$ and $\bar{B} = \hat{i} + 4\hat{j} - 2\hat{k}$ such that they follow the right-handed rule.

Solution:

We have,
$$\bar{A}\times\bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & -1 \\ 1 & 4 & 2 \end{vmatrix} = (-4+4)\hat{i} - (6+1)\hat{j} + (12+2)\hat{k} = -7\hat{i} + 14\hat{k}.$$

Then the desired vector is,

$$\hat{N} = \frac{\bar{A}\times\bar{B}}{|\bar{A}\times\bar{B}|} = \frac{1}{7\sqrt{5}}(-7\hat{i}+14\hat{k}) = -\frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{k}.$$

Work done by a force: The work W done by a force $\bar{F}$ in moving an object through a displacement $\bar{D}$ is given as the product of the scalar component of $\bar{F}$ in the direction of $\bar{D}$ with the magnitude of $\bar{D}$, that is, if θ is the angle between $\bar{F}$ and $\bar{D}$, then

$$W = (|\bar{F}|\cos\theta)\,|\bar{D}| = \bar{F}.\bar{D} \qquad ...(1.19)$$

Moment of a force about a point: Suppose we want to find the moment of a force $\bar{F}$ acting at a point P about a point O, as shown in Fig. 1.5.

From O draw $OM \perp$ to the line of action of $\bar{F}$. If θ is the angle between $\overline{OP}$ and $\bar{F}$ and let $\hat{N}$ be a unit vector perpendicular to their plane, then

$$\overrightarrow{OP}\times\bar{F} = |\overrightarrow{OP}||\bar{F}|\sin\theta\,\hat{N} = |\bar{F}||\overline{OP}|\sin\theta\,\hat{N} = (\bar{F}.\overline{OM})\hat{N} \qquad ...(1.20)$$

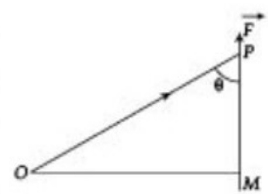


Fig. 1.5

Thus the magnitude of $\overrightarrow{OP}\times\bar{F}$, that is, $|\overrightarrow{OP}\times\bar{F}|$ is numerical measure of the moment of $\bar{F}$ about O and the direction of $\overrightarrow{OP}\times\bar{F}$ is the direction of the moment of $\bar{F}$ about O.

Hence, $\overrightarrow{OP}\times\bar{F}$ is the *vectorial moment* or the *torque* of $\bar{F}$ about O.

The moment of a force $\bar{F}$ *about a line* is the resolved part along that line of the moment of $\bar{F}$ about any point on that line.

Angular velocity of a rigid body: Suppose a rigid body is rotating about an axis OL with angular velocity $\bar{w}$ radian per sec; the direction of $\bar{w}$ being along the line OL as shown in Fig. 1.6.

Let P be any point on the rigid body with position vector $\overrightarrow{OP} = \bar{r}$ and PL be perpendicular from P to the axis OL. If $\hat{N}$ is a unit vector perpendicular to the plane of $\bar{w}$ and $\bar{r}$, then

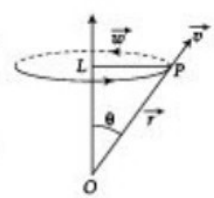


Fig. 1.6

$$\bar{w}\times\bar{r} = |\bar{w}||\bar{r}|\sin\theta\,\hat{N}$$

$= |\vec{w}|\, LP\, \hat{N} = (\text{speed of } P)\, \hat{N}$

$=$ velocity $\vec{v}$ of P in a direction perpendicular to the plane LOP.

Hence, $$\vec{v} = \vec{w} \times \vec{r}. \quad \text{...(1.21)}$$

Example 1.7: A rigid body is rotating with a speed of 3.0 radian per sec about an axis OL, with position vector $\overrightarrow{OL} = 2\hat{i} - 2\hat{j} + \hat{k}$ relative to O. Find the velocity of the body at the point $P(4, 1, 2)$.

Solution: Here $\overrightarrow{OL} = 2\hat{i} - 2\hat{j} + \hat{k}$. Thus a unit vector along $\overrightarrow{OL}$ is $\dfrac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$. Therefore,

$$\vec{w} = 3\left(\frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}\right) = 2\hat{i} - 2\hat{j} + \hat{k}.$$

Let $\vec{r}$ be the position vector of the point $P(4, 1, 2)$, then $\vec{r} = \overrightarrow{OP} = 4\hat{i} + \hat{j} + 2\hat{k}$. Hence,

$$\vec{v} = \vec{w} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 4 & 1 & 2 \end{vmatrix} = (-4-1)\hat{i} - \hat{j}(4-4) + (2+8)\hat{k} = -5\hat{i} + 10\hat{k}.$$

Therefore, the magnitude of the velocity $\vec{v}$ at $P(4, 1, 2)$ is $|\vec{v}| = \sqrt{25 + 100} = 5\sqrt{5}$ units.

Example 1.8: Find the moment about a line through the origin having direction $\hat{i} + \hat{j} + 2\hat{k}$ due to a 20 units force acting at a point $(-3, 1, 5)$ in the direction of $6\hat{i} - \hat{j} + \hat{k}$.

Solution: Let $\overrightarrow{OL}$ be the given line through the origin and $\vec{F}$ the force acting through the point $A(-3, 1, 5)$, as shown in Fig. 1.7. We have

$$\overrightarrow{OA} = -3\hat{i} + \hat{j} + 5\hat{k}$$

$$\vec{F} = 20\left[\frac{6\hat{i} - \hat{j} + \hat{k}}{\sqrt{38}}\right] = \frac{20}{\sqrt{38}}(6\hat{i} - \hat{j} + \hat{k}).$$

Fig. 1.7

Therefore the moment of $\vec{F}$ about $O = \overrightarrow{OA} \times \vec{F}$

$$= \frac{20}{\sqrt{38}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & 5 \\ 6 & -1 & 1 \end{vmatrix} = \frac{20}{\sqrt{38}}(6\hat{i} + 33\hat{j} - 3\hat{k}) = \frac{60}{\sqrt{38}}(2\hat{i} + 11\hat{j} - \hat{k}).$$

The moment of $\bar{F}$ about the line $\overrightarrow{OL}$

$$= \text{resolved part of the moment of } \bar{F} \text{ about } O \text{ along } \overrightarrow{OL}$$

$$= \frac{60}{\sqrt{38}}(2\hat{i} + 11\hat{j} - \hat{k}).\frac{\hat{i} + \hat{j} + 2k}{\sqrt{6}}, \text{ since } \overrightarrow{OL} \text{ is along } \hat{i} + \hat{j} + 2\hat{k}$$

$$= \frac{30}{\sqrt{57}}(2 + 11 - 2) = \frac{330}{\sqrt{57}} \text{ units.}$$

1.1.4 Products of Three or More Vectors

We consider the following products in case of three or more vectors.

Scalar Triple Product: *Let $\bar{A}$, $\bar{B}$ and $\bar{C}$ be any three vectors, then the scalar product of $\bar{A}$ with $\bar{B} \times \bar{C}$ is defined as the scalar triple product of the vectors $\bar{A}$, $\bar{B}$ and $\bar{C}$ and is written as $\bar{A}.(\bar{B} \times \bar{C})$ or $[\bar{A}\,\bar{B}\,\bar{C}]$.*

Numerically, $\bar{A}.(\bar{B} \times \bar{C})$ represents the volume V of a parallelopiped having $\bar{A}$, $\bar{B}$, $\bar{C}$ as its three concurrent edges. We have $[\bar{A}\,\bar{B}\,\bar{C}]$ as V or $-V$ according as $\bar{A}$, $\bar{B}$, $\bar{C}$ form right-handed system or left-handed system. Also $\bar{B}.(\bar{C} \times \bar{A})$ and $\bar{C}.(\bar{A} \times \bar{B})$ represents the same volume as $\bar{A}.(\bar{B} \times \bar{C})$. Hence,

$$\bar{A}.(\bar{B} \times \bar{C}) = \bar{B}.(\bar{C} \times \bar{A}) = \bar{C}.(\bar{A} \times \bar{B}). \qquad ...(1.22)$$

Further *the dot and cross can be interchanged in a scalar product*, that is

$$\bar{A}.(\bar{B} \times \bar{C}) = (\bar{A} \times \bar{B}).\bar{C}, \text{ etc.}$$

In case $\bar{A}$, $\bar{B}$, $\bar{C}$ are mutually perpendicular then $\bar{A}.(\bar{B} \times \bar{C}) = |\bar{A}||\bar{B}||\bar{C}|$ and, in particular, for the three unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$, we have

$$\hat{i}.(\hat{j} \times \hat{k}) = \hat{j}.(\hat{k} \times \hat{i}) = \hat{k}.(\hat{i} \times \hat{j}) = 1. \qquad ...(1.23)$$

If $\bar{A}$, $\bar{B}$, $\bar{C}$ have cartesian representations as

$$\bar{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \bar{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ \bar{C} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k},$$

then we can find very easily that

$$\bar{A}.(\bar{B} \times \bar{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad ...(1.24)$$

which is a convenient expression to find the scalar product of the three vectors.

If the vectors $\bar{A}, \bar{B}, \bar{C}$ are co-planar, then volume of the parallelopiped formed by the vectors $\bar{A}, \bar{B}, \bar{C}$ as concurrent edges is zero, and hence their scalar triple product must be zero, and vice versa. In particular, if any two of the three vectors are equal, then also their scalar triple product is zero.

Further, any three vectors $\bar{A}, \bar{B}$ and $\bar{C}$ are said to be *linearly dependent* if one of them can be expressed as a linear combination of other two, say $\bar{C} = l\bar{A} + m\bar{B}$, where l and m are scalars. Thus, $\bar{C}$ lies in the plane of $\bar{A}$ and $\bar{B}$ and hence $[\bar{A}\ \bar{B}\ \bar{C}] = 0$. Thus, *three vectors are linearly dependent if their scalar triple product is zero otherwise they are independent.*

Vector Triple Product: *Let $\bar{A}, \bar{B}$ and $\bar{C}$ be three vectors then the vector product of $\bar{A} \times \bar{B}$ with $\bar{C}$ is defined as the vector triple product of the three vectors $\bar{A}, \bar{B}, \bar{C}$ and is written as $(\bar{A} \times \bar{B}) \times \bar{C}$.*

We note that vector $(\bar{A} \times \bar{B}) \times \bar{C}$ is perpendicular to vector $\bar{A} \times \bar{B}$ which in turn is perpendicular to the plane containing both $\bar{A}$ and $\bar{B}$, hence $(\bar{A} \times \bar{B}) \times \bar{C}$ lies in the plane containing $\bar{A}$ and $\bar{B}$. Thus, we can write

$$(\bar{A} \times \bar{B}) \times \bar{C} = l\bar{A} + m\bar{B}, \qquad ...(1.25)$$

where l and m are scalars.

Taking the dot product of both sides of (1.25) with $\bar{C}$, we obtain

$$\bar{C}.(\bar{A} \times \bar{B}) \times \bar{C} = l\bar{C}.\bar{A} + m\bar{C}.\bar{B}. \qquad ...(1.26)$$

The scalar triple product on the left side of (1.26) is zero, since two vectors are equal, therefore,

$l(\bar{C}.\bar{A}) + m(\bar{C}.\bar{B}) = 0$, which implies

$$\frac{l}{\bar{C}.\bar{B}} = -\frac{m}{(\bar{C}.\bar{A})} = k \quad \text{(say)}. \qquad ...(1.27)$$

Substituting for l and m from (1.27) in (1.25), we obtain

$$(\bar{A} \times \bar{B}) \times \bar{C} = k(\bar{C}.\bar{B})\bar{A} - k(\bar{C}.\bar{A})\bar{B}. \qquad ...(1.28)$$

To find k, we take in particular $\bar{A} = \hat{i}, \bar{B} = \bar{C} = \hat{j}$, then (1.28) gives

$$\hat{k} \times \hat{j} = k\hat{i}, \text{ or } -\hat{i} = k\hat{i}, \text{ or } k = -1.$$

Hence, (1.28) becomes

$$(\bar{A} \times \bar{B}) \times \bar{C} = (\bar{C}.\bar{A})\bar{B} - (\bar{C}.\bar{B})\bar{A}.$$

Similarly, we can prove that

$$\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A}.\bar{C})\bar{B} - (\bar{A}.\bar{B}).\bar{C}. \qquad ...(1.29)$$

Obviously, $(\bar{A} \times \bar{B}) \times \bar{C} \neq \bar{A} \times (\bar{B} \times \bar{C})$.

From (1.28) there follows another important result given as

$$(\bar{A} \times \bar{B}) \times \bar{C} + (\bar{B} \times \bar{C}) \times \bar{A} + (\bar{C} \times \bar{A}) \times \bar{B} = \bar{0}. \quad ...(1.30)$$

Scalar product of four vectors: *Let $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ be the four vectors, then the scalar product of $\bar{A} \times \bar{B}$ with $\bar{C} \times \bar{D}$ is defined as the scalar product of $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$.*

Consider $(\bar{A} \times \bar{B}).(\bar{C} \times \bar{D}) = \bar{P}.(\bar{C} \times \bar{D})$, where $\bar{P} = \bar{A} \times \bar{B}$

$= (\bar{P} \times \bar{C}).\bar{D}$, since cross and dot can be interchanged

$= \bar{D}.(\bar{P} \times \bar{C})$, since dot product is commutative

$= \bar{D}.\{(\bar{A} \times \bar{B}) \times \bar{C}\} = \bar{D}.\{(\bar{A}.\bar{C})\bar{B} - (\bar{B}.\bar{C})\bar{A}\}$

$$= (\bar{A}.\bar{C})(\bar{B}.\bar{D}) - (\bar{B}.\bar{C})(\bar{A}.\bar{D}) = \begin{vmatrix} \bar{A}.\bar{C} & \bar{B}.\bar{C} \\ \bar{A}.\bar{D} & \bar{B}.\bar{D} \end{vmatrix}, \quad ...(1.31)$$

an expression easy to apply.

Vector product of four vectors: *The vector product of $\bar{A} \times \bar{B}$ with $\bar{C} \times \bar{D}$ is defined as the vector product of the four vectors $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$.*

Obviously the vector $(\bar{A} \times \bar{B}) \times (\bar{C} \times \bar{D})$ is in the plane of $\bar{A}$ and $\bar{B}$. It can be expressed in terms of $\bar{A}$ and $\bar{B}$, and we can show that

$$(\bar{A} \times \bar{B}) \times (\bar{C} \times \bar{D}) = [\bar{A}\,\bar{C}\,\bar{D}]\bar{B} - [\bar{B}\,\bar{C}\,\bar{D}]\bar{A}. \quad ...(1.32)$$

Similarly, it can be expressed in terms of $\bar{C}$ and $\bar{D}$ as

$$(\bar{A} \times \bar{B}) \times (\bar{C} \times \bar{D}) = [\bar{A}\,\bar{B}\,\bar{D}]\bar{C} - [\bar{A}\,\bar{B}\,\bar{C}]\bar{D}. \quad ...(1.33)$$

Example 1.9: Find the volume of the tetrahedron whose vertices are (2, 1, 1), (1, –1, 2), (0, 1, –1) and (1, –2, 1).

Solution: The vertices are $A(2, 1, 1)$, $B(1, -1, 2)$, $C(0, 1, -1)$ and $D(1, -2, 1)$ as shown in Fig. 1.8.

Volume of the tetrahedron $ABCD$

$$= \frac{1}{3}(\text{Area of the base } ABC) \times \text{height}$$

$$= \frac{1}{6}[2\Delta ABC]h,$$

where h is the height of the vertex D above the plane of ΔABC.

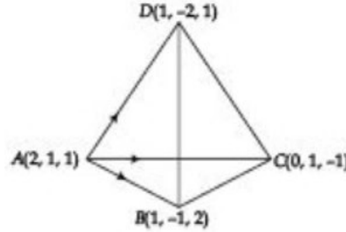


Fig. 1.8

$$= \frac{1}{6}\text{(volume of the parallelopiped with } \overrightarrow{AB}, \overrightarrow{AC} \text{ and } \overrightarrow{AD} \text{ as concurrent edges)}$$

$$= \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \frac{1}{6}[-\hat{i} - 2\hat{j} + \hat{k}, -2\hat{i} - 2\hat{k}\ \ -\hat{i} - 3\hat{j}]$$

$$= \frac{1}{6}\begin{vmatrix} -1 & -2 & 1 \\ -2 & 0 & -2 \\ -1 & -3 & 0 \end{vmatrix} = \frac{1}{6}(-6 + 4 + 6) = \frac{2}{3} \text{ units.}$$

Example 1.10: Show that

$$[\bar{A} + \bar{B}\ \ \bar{B} + \bar{C}\ \ \bar{C} + \bar{A}] = 2[\bar{A}\ \bar{B}\ \bar{C}]$$

Solution: We have,

$$[\bar{A} + \bar{B}\ \ \bar{B} + \bar{C}\ \ \bar{C} + \bar{A}] = (\bar{A} + \bar{B}).[(\bar{B} + \bar{C}) \times (\bar{C} + \bar{A})]$$

$$= (\bar{A} + \bar{B}).[\bar{B} \times \bar{C} + \bar{B} \times \bar{A} + \bar{C} \times \bar{C} + \bar{C} \times \bar{A}]$$

$$= (\bar{A} + \bar{B}).[\bar{B} \times \bar{C} + \bar{B} \times \bar{A} + \bar{C} \times \bar{A}]$$

$$= [\bar{A}\ \bar{B}\ \bar{C}] + [\bar{A}\ \bar{B}\ \bar{A}] + [\bar{A}\ \bar{C}\ \bar{A}] + [\bar{B}\ \bar{B}\ \bar{C}] + [\bar{B}\ \bar{B}\ \bar{A}] + [\bar{B}\ \bar{C}\ \bar{A}]$$

$$= [\bar{A}\ \bar{B}\ \bar{C}] + [\bar{B}\ \bar{C}\ \bar{A}] = 2[\bar{A}\ \bar{B}\ \bar{C}].$$

Example 1.11: Show that the vector

$$(\bar{A} \times \bar{B}) \times (\bar{C} \times \bar{D}) + (\bar{A} \times \bar{C}) \times (\bar{D} \times \bar{B}) + (\bar{A} \times \bar{D}) \times (\bar{B} \times \bar{C})$$

is parallel to the vector $\bar{A}$.

Solution: Let $\bar{A} \times \bar{B} = \bar{P}$, then

$$(\bar{A} \times \bar{B}) \times (\bar{C} \times \bar{D}) = \bar{P} \times (\bar{C} \times \bar{D})$$

$$= (\bar{P}.\bar{D})\bar{C} - (\bar{P}.\bar{C})\bar{D} = [\bar{A}\ \bar{B}\ \bar{D}]\bar{C} - [\bar{A}\ \bar{B}\ \bar{C}]\bar{D}. \quad ...(1.34)$$

Similarly, we have

$$(\bar{A} \times \bar{C}) \times (\bar{D} \times \bar{B}) = -(\bar{D} \times \bar{B}) \times (\bar{A} \times \bar{C}) = -[\bar{D}\ \bar{B}\ \bar{C}]\bar{A} + [\bar{D}\ \bar{B}\ \bar{A}]\bar{C} \quad ...(1.35)$$

and $$(\bar{A} \times \bar{D}) \times (\bar{B} \times \bar{C}) = -(\bar{B} \times \bar{C}) \times (\bar{A} \times \bar{D}) = -[\bar{B}\ \bar{C}\ \bar{D}]\bar{A} + [\bar{B}\ \bar{C}\ \bar{A}]\bar{D}. \quad ...(1.36)$$

Adding (1.34), (1.35) and (1.36) and using the fact that

$$[\bar{B}\,\bar{C}\,\bar{A}]\bar{D} = [\bar{A}\,\bar{B}\bar{C}]\bar{D}, \text{ and } [\bar{D}\,\bar{B}\,\bar{A}]\bar{C} = -[\bar{A}\,\bar{B}\,\bar{D}]\bar{D}$$

we obtain, the given expression

$$= -[\bar{D}\,\bar{B}\,\bar{C}]\bar{A} - [\bar{B}\,\bar{C}\bar{D}]\bar{A} = -2[\bar{B}\,\bar{C}\,\bar{D}]\bar{A} = k\bar{A}$$

for some scalar $k = -2[\bar{B}\,\bar{C}\,\bar{D}]$.

Thus the given vector is parallel to $\bar{A}$.

EXERCISE 1.1

1. Find the vector $\overrightarrow{OC}$ which bisects the angle between two vectors $\overrightarrow{OA} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\overrightarrow{OB} = 3\hat{i} + \hat{j} - 2\hat{k}$, where C is the point on the line joining the points A and B.
2. Five forces act at one vertex A of a regular hexagon in the direction of the other vertices and are proportional to the distances of these vertices from A. Find their resultant.
3. Determine whether the three points $\hat{i} - 2\hat{j} + 3\hat{k}, 2\hat{i} + 3\hat{j} - 4\hat{k}$ and $-7\hat{j} + 10\hat{k}$ are collinear.
4. How much does it take to slide a crate 20 m along a loading dock by pulling on it with a 200 N force at an angle of 30° from the horizontal?
5. Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$. Also find the area of the triangle PQR.
6. Write vector $\bar{B} = 8\hat{i} + 4\hat{j} - 12\hat{k}$ as the sum of a vector parallel to $\bar{A} = \hat{i} + 2\hat{j} - \hat{k}$ and a vector orthogonal to it.
7. Find the volume of the tetrahedron having the vertices as $(0, -1, -1)$, $(4, 5, k)$, $(3, 9, 4)$ and $(-4, 4, 4)$.
   Also find the value of k for which these four points are coplanar.
8. Show that if vectors $\bar{A}, \bar{B}, \bar{C}$ are non-coplanar then $\bar{A} + \bar{B}, \bar{B} + \bar{C}, \bar{C} + \bar{A}$ are also non-coplanar.
9. Find $\bar{A} \times (\bar{B} \times \bar{C})$ and $(\bar{A} \times \bar{B}) \times \bar{C}$ given that $\bar{A} = 3\hat{i} + \hat{j} - 4\hat{k}$, $\bar{B} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\bar{C} = \hat{i} + 5\hat{j} - \hat{k}$.
10. If $\bar{A}, \bar{B}, \bar{C}$ and $\bar{D}$ are vectors and λ, μ, υ are scalars satisfying the equation $\lambda(\bar{B} \times \bar{C}) + \mu(\bar{C} \times \bar{A}) + \upsilon(\bar{A} \times \bar{B}) + \bar{D} = 0$ and if $\bar{A}, \bar{B}$ and $\bar{C}$ are linearly independent, then
    $\lambda = -(\bar{A}.\bar{D})/[\bar{A}, \bar{B}, \bar{C}]$, $\mu = -(\bar{B}.\bar{D})/[\bar{A}, \bar{B}, \bar{C}]$, $\upsilon = -(\bar{C}.\bar{D})/[\bar{A}, \bar{B}, \bar{C}]$.
11. Show that $\bar{A} = \hat{i} + 2\hat{j} + \hat{k}$, $\bar{B} = 2\hat{i} - \hat{j} - \hat{k}$ and $\bar{C} = 4\hat{i} + 3\hat{j} + \hat{k}$ are linearly independent.

    Consider a vector $\bar{D}$ of your choice and then verify the result of the preceding problem.

12. For any vector $\bar{r}$ prove that $\hat{i} \times (\bar{r} \times \hat{i}) + \hat{j} \times (\bar{r} \times \hat{j}) + \hat{k} \times (\bar{r} \times \hat{k}) = 2\bar{r}$.
13. Find the moment of the force $\bar{F} = 2\hat{i} + 2\hat{j} - \hat{k}$ acting at the point $(1, -2, 1)$ about z-axis.
14. A rigid body is rotating with angular velocity 27 radians per second about an axis parallel to $2\hat{i} + \hat{j} - 2\hat{k}$ passing through the point $\hat{i} + 3\hat{j} - \hat{k}$. Find the velocity of the point of the body whose position vector is $4\hat{i} + 8\hat{j} + \hat{k}$.
15. The vector triad $\bar{A}', \bar{B}', \bar{C}'$ defined as

$$\bar{A}' = \frac{\bar{B} \times \bar{C}}{[\bar{A}, \bar{B}, \bar{C}]}, \quad \bar{B}' = \frac{\bar{C} \times \bar{A}}{[\bar{A}, \bar{B}, \bar{C}]}, \quad \bar{C}' = \frac{\bar{A} \times \bar{B}}{[\bar{A}, \bar{B}, \bar{C}]}, \quad [\bar{A}\,\bar{B}\bar{C}] \neq 0.$$

is called the '*reciprocal triad*' of the non-coplanar vectors $\bar{A}, \bar{B}, \bar{C}$. Show that

(i) $\bar{A}.\bar{A}' = \bar{B}.\bar{B}' = \bar{C}.\bar{C}' = 1$ (ii) $[A'B'C'] \neq 0$

(iii) $\bar{A} \times \bar{A}' + \bar{B} \times \bar{B}' + \bar{C} \times \bar{C}' = 0.$

1.2 LINES AND PLANES IN SPACE

In this section we shall use vectors to study the line segments and planes in space.

1.2.1 Equations for Lines in Space

Let l be a line in space which passes through a given point $P_1(x_1, y_1, z_1)$ and is parallel to a given non-zero vector $\bar{v} = a\hat{i} + b\hat{j} + c\hat{k}$. To find the equation of the line l, let $\bar{R}$ be the position vector of an arbitrary point $P(x, y, z)$ on it, as shown in Fig. 1.9.

By vector addition

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P},$$

that is, $$\bar{R} = \bar{A} + \lambda\bar{v}, \qquad ...(1.37)$$

where λ is a scalar quantity.

The Eq. (1.37) is called the *vector form* equation of the line l.

Also, $\bar{R} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad \bar{A} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$$\bar{v} = a\hat{i} + b\hat{j} + c\hat{k}.$$

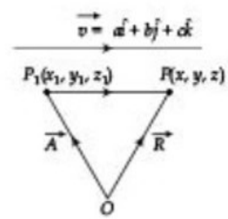


Fig. 1.9

Substituting these in (1.37) and comparing the coefficient of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on both sides, we obtain

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \quad ...(1.38)$$

called the *standard form*, or *symmetric form* of the equation of line l.

The equations

$$x=x_1+a\lambda,\quad y=y_1+b\lambda,\quad z=z_1+c\lambda \quad ...(1.39)$$

are the *parametric form*.

Example 1.12: Represent the equation of the straight line

$$\frac{2x-1}{3}=\frac{y-4}{3}=\frac{-z+5}{2}$$

in vector form.

Solution: The equation can be rewritten as, $\dfrac{x-(1/2)}{3/2}=\dfrac{y-(-4)}{3}=\dfrac{z-5}{-2}$.

Comparing it with Eq. (1.38), we obtain

$$\bar{A}=(1/2)\hat{i}-4\hat{j}+5\hat{k}, \text{ and } \quad \bar{v}=(3/2)\hat{i}+3\hat{j}-2\hat{k}.$$

Therefore, $\bar{R}=\frac{1}{2}(1+3\lambda)-(4-3\lambda)\hat{j}+(5-2\lambda)\hat{k}$ is the vector form of the given equation.

Example 1.13: Find the value of λ so that the lines

$$\frac{x-1}{-3}=\frac{y-1}{2\lambda}=\frac{z-9}{2}, \text{ and } \quad \frac{x-1}{3\lambda}=\frac{y-2}{1}=\frac{z-3}{-5}$$

are perpendicular to each other.

Solution: From the standard form (1.38), the two given lines are perpendicular if $\bar{v}_1=-3\hat{i}+2\lambda\hat{j}+2\hat{k}$ and $\bar{v}_2=3\lambda\hat{i}+\hat{j}-5\hat{k}$ are perpendicular, that is, if

$$(-3)(3\lambda)+(2\lambda)(1)+2(-5)=0, \quad \text{or} \quad -9\lambda+2\lambda-10=0, \text{ or } \quad \lambda=-10/7.$$

Example 1.14: Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution: We have, $\overrightarrow{PQ}=4\hat{i}-3\hat{j}+7\hat{k}$.

Equation of a line passing through $P(-3, 2, -3)$ and parallel to $\overrightarrow{PQ}$ is

$$\frac{x+3}{4} = \frac{y-2}{-3} = \frac{z+3}{7} = \lambda \quad \text{(say)}.$$

In parametric form it is, $x = -3 + 4\lambda, \quad y = 2 - 3\lambda, \quad z = -3 + 7\lambda.$

This passes through P for $\lambda = 0$ and through Q for $\lambda = 1$. Hence, to parametrize this line to line segment PQ we have $0 \le \lambda \le 1$. Thus, the segment PQ in parametric form is

$$x = -3 + 4\lambda, \quad y = 2 - 3\lambda, \quad z = -3 + 7\lambda, \quad 0 \le \lambda \le 1.$$

The distance from a point to a line in space: The distance d from a point A to a line l which passes through a point P and parallel to a vector $\bar{v}$, is the length of the component of $\overrightarrow{PA}$ normal to the line, which is $|\overrightarrow{PA}| \sin\theta$, as shown in Fig. 1.10.

Thus, $$d = |\overrightarrow{PA}| \sin\theta = \frac{|\overrightarrow{PA} \times \bar{v}|}{|\bar{v}|}. \qquad \text{...(1.40)}$$

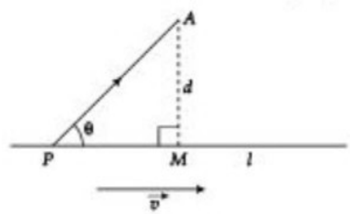


Fig. 1.10

Example 1.15: Find the distance from the point $A(1, 1, 5)$ to the line $\frac{x-1}{1} = \frac{-y+1}{1} = \frac{z}{2}$.

Solution: The given line passes through a point $P(1, 1, 0)$ and is parallel to vector $\bar{v} = i - j + 2k$.

Also, $\overrightarrow{PA} = 5\hat{k}$, and $\overrightarrow{PA} \times \bar{v} = \begin{vmatrix} i & j & k \\ 0 & 0 & 5 \\ 1 & -1 & 2 \end{vmatrix} = 5i + 5\hat{j}$.

Thus, $$d = \frac{|\overrightarrow{PA} \times \bar{v}|}{|\bar{v}|} = \frac{5\sqrt{2}}{\sqrt{6}} \text{ units.}$$

1.2.2 Equations for Planes in Space

Let the plane L pass through a point $P_1(x_1, y_1, z_1)$ and is normal to the non-zero vector $\bar{N} = a\hat{i} + b\hat{j} + c\hat{k}$, as shown in Fig. 1.11.

Consider any point $P(x, y, z)$ in the plane L, then

$$\hat{N}.\overrightarrow{P_1P} = 0, \quad ...(1.41)$$

which gives

$$(a\hat{i} + b\hat{j} + c\hat{k}).[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}] = 0$$

or,

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0. \quad ...(1.42)$$

Normally it is written as

$$ax + by + cz + d = 0, \quad ...(1.43)$$

where $d = -(ax_1 + by_1 + cz_1)$.

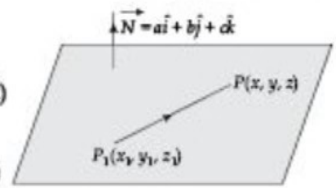


Fig. 1.11

The Eq. (1.43) is called the *general equation* of the plane, Eq. (1.41) is called the *vector equation* of the plane passing through P_1 and normal to $\hat{N}$, and Eq. (1.42) is the equation of any plane passing through the point $P\,(x_1, y_1, z_1)$ and normal to $\vec{N} = a\hat{i} + b\hat{j} + c\hat{k}$.

The equation

$$lx + my + nz = p \quad ...(1.44)$$

is the *normal equation of the plane*, when l, m, n are the direction cosines of the normal to the plane and p is the perpendicular distance from the origin.

Example 1.16: Find the equation of the plane passing through the points $A(1, 0, 0)$, $B(0, 2, 0)$ and $C(0, 0, 3)$.

Solution: We need to find a vector normal to the requisite plane. It is given by the cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\hat{i} + 3\hat{j} + 2\hat{k}.$$

The equation of the plane passing through $A(1, 0, 0)$ (we can take any one of the points given) and normal to $6\hat{i} + 3\hat{j} + 2\hat{k}$ is $6(x - 1) + 3y + 2z = 0$ or, $6x + 3y + 2z = 6$.

Alternatively, the equation of the plane passing through the points $A(1, 0, 0)$, $B(0, 2, 0)$ and $C(0, 0, 3)$ may be found using the fact that the points must satisfy the equation $ax + by + cz = d$, which gives $a = d$, $b = d/2$ and $c = d/3$. Substituting in the equation and simplifying we obtain $6x + 3y + 2z = 6$ as the requisite equation of the plane.

Example 1.17: Find the angles between the two planes $L_1: a_1x + b_1y + c_1z + d_1 = 0$ and $L_2: a_2x + b_2y + c_2z + d_2 = 0$. Also find the condition when the planes will be, (i) parallel, (ii) perpendicular.

Solution: The angle between the two planes is the angle between their normals.

The direction ratios of the normal to L_1 are a_1, b_1, c_1 and the direction ratios of the normal to L_2 are a_2, b_2, c_2. If θ is the angle between L_1 and L_2, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Thus the planes will be

(i) parallel, if their normals will be parallel, that is, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

(ii) perpendicular, if their normals will be perpendicular, that is, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Remark Equation of any plane parallel to the plane $ax + by + cz = d$ is $ax + by + cz = k$.

Example 1.18: Find the perpendicular distance of the point $P_1(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$.

Solution: Let PM be the perpendicular distance from $P(x_1, y_1, z_1)$ to the given plane

$$ax + by + cz + d = 0 \quad ...(1.45)$$

as shown in Fig. 1.12.

Let $A(\alpha, \beta, \gamma)$ be a point on this plane, then

$$a\alpha + b\beta + c\gamma + d = 0 \quad ...(1.46)$$

PM= projection of $\overrightarrow{AP}$ on $\overrightarrow{MP} = \overrightarrow{AP}.\overrightarrow{MP}$

$$= [(x_1 - \alpha)\hat{i} + (y_1 - \beta)\hat{j} + (z_1 - \gamma)\hat{k}].[l\hat{i} + m\hat{j} + n\hat{k}], \quad ...(1.47)$$

where l, m, n are the direction consines of the MP given by

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

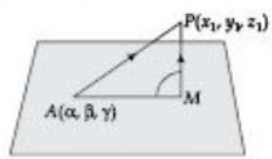


Fig. 1.12

Using these in (1.47) and then using (1.46), we obtain

$$PM = \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}},$$

as the requisit perpendicular distance.

Example 1.19: Find the equation of the plane passing through the point (1, 2, –1) and perpendicular to the planes $x + y - 2z = 5$ and $3x - y + 4z = 12$.

Solution: Let $ax + by + cz + d = 0$ be the requisite plane. Then we have

$$a + b - 2c = 0 \text{ and } 3a - b + 4c = 0.$$

These two give, $\frac{a}{2} = \frac{b}{-10} = \frac{c}{-4} = \lambda$, say.

That is, $a = 2\lambda,\quad b = -10\lambda,\quad c = -4\lambda.$

Also equation of any plane passing through the given point $(1, 2, -1)$ is

$$a(x-1) + b(y-2) + c(z+1) = 0.$$

Substituting for a, b and c, we obtain

$$2(x-1) - 10(y-2) - 4(z+1) = 0$$

or, $$2x - 10y - 4z + 14 = 0$$

or, $$x - 5y - 2z + 7 = 0.$$

Example 1.20: Find the equation of the planes bisecting the angle between two planes

$$3x - 4y + 5z = 3 \text{ and } 5x + 3y - 4z = 9.$$

Solution: The planes are

$$3x - 4y + 5z - 3 = 0 \qquad \text{...(1.48)}$$

and $$5x + 3y - 4z - 9 = 0. \qquad \text{...(1.49)}$$

Let $P(x, y, z)$ be any point on the either plane bisecting the angle between the planes (1.48) and (1.49). Then perpendicular distances of P from (1.48) and (1.49) must be equals. Thus,

$$\frac{|3x - 4y + 5z - 3|}{\sqrt{9 + 16 + 25}} = \frac{|5x + 3y - 4z - 9|}{\sqrt{25 + 9 + 16}}$$

or, $$3x - 4y + 5z - 3 = \pm(5x + 3y - 4z - 9),$$

which gives $$2x + 7y - 9z - 6 = 0, \text{ and } 8x - y + z - 12 = 0$$

as the requisite equations.

Example 1.21: Find the parametric equation of the line of intersection of the two planes $3x - 6y - 2z = 15$, and $2x + y - 2z = 5$.

Solution: Any normal to the plane $3x - 6y - 2z = 15$ is, $\bar{N}_1 = 3\hat{i} - 6\hat{j} - 2\hat{k}$.

Similarly, any normal to the plane $2x + y - 2z = 5$ is, $\bar{N}_2 = 2\hat{i} + \hat{j} - 2\hat{k}$.

The line of intersection of the given plane lies in the both planes so it should be normal to both $\bar{N}_1$ and $\bar{N}_2$, thus it should be parallel to $\bar{N}_1 \times \bar{N}_2$. We have

$$\bar{N}_1 \times \bar{N}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\hat{i} + 2\hat{j} + 15\hat{k}.$$

Any point P_1 on the line of intersection of the given plane may be obtained by setting one variable say $x = 0$ in the equations of the plane and then solving for y and z, we obtain

$$x = 0, \quad y = \frac{-10}{7}, \quad z = \frac{-45}{14}.$$

Hence, if $P(x, y, z)$ is any arbitrary point on the line. Then

$$\overrightarrow{P_1P} = x\hat{i} + \left(y + \frac{10}{7}\right)\hat{j} + \left(z + \frac{45}{14}\right)\hat{k}.$$

Since, $\overrightarrow{P_1P}$ is parallel to $\bar{N}_1 \times \bar{N}_2$, thus $x\hat{i} + \left(y + \frac{10}{7}\right)\hat{j} + \left(z + \frac{45}{14}\right)\hat{k} = (14\hat{i} + 2\hat{j} + 15\hat{k})\lambda$

for some parameter λ. Comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on both sides, we obtain

$$\frac{x}{14} = \frac{y + 10/7}{2} = \frac{z + 45/14}{15} = \lambda,$$

which gives $$x = 14\lambda, \quad y = \frac{-10}{7} + 2\lambda, \quad z = \frac{-45}{14} + 15\lambda$$

as the parametric equations.

Remarks

1. Two linear equations in x, y and z e.g., $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, taken together represent a straight line which is the line of intersection of these two planes.
2. Two lines which do not lie in one plane are called *skew lines*. The shortest distance between two skew lines is the length of the common perpendicular between the two.

Example 1.22: Find the points on the lines $\frac{x-1}{2} = \frac{y}{-1} = \frac{z-1}{1}$ and $\frac{x-1}{3} = \frac{y-2}{1} = \frac{z}{2}$

which are nearest to each other. Hence find the shortest distance between these two lines and its equation.

Solution: The lines are

$$\frac{x-1}{2} = \frac{y}{-1} = \frac{z-1}{1} \qquad ...(1.50)$$

and, $$\frac{x-1}{3} = \frac{y-2}{1} = \frac{z}{2}. \qquad ...(1.51)$$

Any point on the line (1.50) is $P_1(1 + 2\lambda_1, -\lambda_1, 1 + \lambda_1)$ and any point on the line (1.51) is $P_2(1 + 3\lambda_2, 2 + \lambda_2, 2\lambda_2)$. The direction ratios for the line P_1P_2 are $(3\lambda_2 - 2\lambda_1, 2 + \lambda_2 + \lambda_1, 2\lambda_2 - 1 - \lambda_1)$.

The line P_1P_2 will be perpendicular to the line (1.50) when

$$2(3\lambda_2 - 2\lambda_1) - 1(2 + \lambda_2 + \lambda_1) + (2\lambda_2 - 1 - \lambda_1) = 0$$

or,
$$-6\lambda_1 + 7\lambda_2 = 3. \qquad ...(1.52)$$

Similarly, the line P_1P_2 will be perpendicular to the line (1.51), when

$$3(3\lambda_2 - 2\lambda_1) + (2 + \lambda_2 + \lambda_1) + 2(2\lambda_2 - 1 - \lambda_1) = 0$$

or,
$$\lambda_1 - 2\lambda_2 = 0. \qquad ...(1.53)$$

Solving (1.52) and (1.53) for λ_1 and λ_2, we obtain, $\lambda_1 = -6/5$ and $\lambda_2 = -3/5$. Thus, the nearest points are $P_1(-7/5, 6/5, -1/5)$, and $P_2(-4/5, 7/5, -6/5)$, and hence, the shortest distance is

$$\sqrt{(-3/5)^2 + (-1/5)^2 + (1)^2} = \sqrt{\frac{9}{25} + \frac{1}{25} + 1} = \sqrt{7/5}.$$

The direction ratios of $\overrightarrow{P_1P_2}$ are $-3/5, -1/5, 1$ and it passes through P_1 and hence the equation of the line P_1P_2 is

$$\frac{x + 7/5}{-3/5} = \frac{y - 6/5}{-1/5} = \frac{z + 1/5}{1} \quad \text{or,} \quad \frac{5x + 7}{-3} = \frac{5y - 6}{-1} = \frac{5z + 1}{5}.$$

EXERCISE 1.2

1. Determine whether the lines

$$\frac{x - 3}{-1} = \frac{y - 7}{2} = \frac{z - 1}{3} \text{ and } \frac{x - 9}{4} = \frac{y - 4}{1} = \frac{z - 3}{8} \text{ intersect.}$$

2. Find parametric equations for the line through $A(-3, 2, -1)$ and $B(1, -1, 4)$.
3. Find the line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$.
4. Find in symmetrical form the equation of the line $x + y + z + 1 = 0, 4x + y - 2z + 2 = 0$.
5. Find the equations of two straight lines through the origin, each of which intersects the straight line $\frac{1}{2}(x - 3) = y - 3 = z$ and is inclined at an angle of $60°$ to it.
6. Find the distance of the point $(2, 3, 4)$ from the plane $3x - 6y + 2z + 11 = 0$.
7. Find the line of the intersection of the two planes $x - 4y + 2z + 7 = 0$ and $3x + 3y - z - 2 = 0$.
8. Show that the two lines $\frac{x - 5}{4} = \frac{y - 7}{4} = \frac{z + 3}{-5}$; $\frac{x - 8}{7} = \frac{y - 4}{1} = \frac{z - 5}{3}$

   are coplanar, find their common point and equation of the plane in which they lie.

9. Find the equation of the straight line perpendicular to both the lines

$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z+2}{3} \text{ and } \frac{x+2}{2}=\frac{y-5}{-1}=\frac{z+3}{2}$$

and passing through their point of intersection.

10. Find the equation of the plane through the line $\frac{x-1}{3}=\frac{y-4}{2}=\frac{z-4}{-2}$

and parallel to the line $\frac{x+1}{2}=\frac{y-1}{-4}=\frac{z+2}{1}$.

Hence, find the shortest distance between them.

11. Find the image of the point (1, 2, 5) in the plane $2x-y-z+3=0$.

12. A variable plane passes through a fixed point (a, b, c) and meets the coordinate axes in A, B, C. Show that the locus of the point common to the planes through the points A, B and C parallel to the respective coordinate planes is $a/x+b/y+c/z=1$.

13. If $2d$ is the shortest distance between the lines $y/b+z/c=1, x=0$ and $x/a-z/c=1, y=0$ show that, $\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

14. Two control cables in the form of straight lines AB and CD are laid such that the coordinates of A, B, C and D are (1, 4, 4), (4, 6, 2), (− 1, 1 − 2) and (3, − 7, 0). Determine the amount of clearance between the cables.

15. Assuming the plane $4x-3y+7z=0$ to be horizontal find the equations of the line of greatest slope through the point (2, 1, 1) in the plane $2x+y-5z=0$.

1.3 SPHERE

A sphere is the locus of a point in space which moves in such a way that its distance from a fixed point remains constant.

The fixed point is called the *centre* and the constant distance the *radius* of the sphere.

1.3.1 Different Forms of Equation of a Sphere

If $C(x_0, y_0, z_0)$ is the centre of the sphere and $P(x, y, z)$ is any point on it and $CP=r$, then the equation of the sphere is

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2 \quad ...(1.54)$$

In particular, the equation of the sphere with centre (0, 0, 0) and radius r is

$$x^2+y^2+z^2=r^2 \quad ...(1.55)$$

Eq. (1.54) is called the *standard form of the equation* of the sphere.

Consider any equation of the form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \qquad \text{...(1.56)}$$

It can be rewritten as

$$(x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d. \qquad \text{...(1.57)}$$

Comparing Eqs. (1.54) and (1.57) we interpret that Eq. (1.56) represents the sphere with centre $(-u, -v, -w)$ and radius $\sqrt{u^2 + v^2 + w^2 - d}$, provided $u^2 + v^2 + w^2 - d > 0$. The Eq. (1.56) is called the *general form of the equation* of the sphere.

Another form of the equation of the sphere is the *diameter form*. It is the equation of the sphere with $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as the extremities of a diameter.

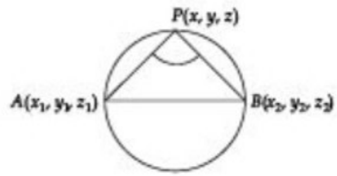


Fig. 1.13

For any point $P(x, y, z)$ on the sphere as shown in Fig. 1.13 the direction ratios of AP and BP are $\langle x - x_1, y - y_1, z - z_1 \rangle$ and $\langle x - x_2, y - y_2, z - z_2 \rangle$ respectively. Also AP and BP are perpendicular at P and hence

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0, \qquad \text{...(1.58)}$$

which is the required equation of the sphere in the *diametric form*.

Example 1.23: Obtain the equation of the sphere which passes through the points (1, 0, 0), (0, 1, 0), (0, 0, 1) and has its

(a) centre on the plane $x + y + z = 6$, (b) radius as small as possible.

Solution: (a) Let the required equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \qquad \text{...(1.58a)}$$

It passes through (1, 0, 0), (0, 1, 0) and (0, 0, 1), therefore,

$$1 + 2u + d = 0, \qquad \text{...(1.59)}$$

$$1 + 2v + d = 0, \qquad \text{...(1.60)}$$

and

$$1 + 2w + d = 0. \qquad \text{...(1.61)}$$

Also the centre $(-u, -v, -w)$ lies on the plane $x + y + z = 6$, therefore,

$$-u - v - w = 6. \qquad \text{...(1.62)}$$

From Eq. (1.59), (1.60) and (1.61), we obtain

$$u = v = w = -\frac{1 + d}{2}. \qquad \text{...(1.63)}$$

Substituting this in Eq. (1.62), we obtain $\dfrac{3(1 + d)}{2} = 6$, or $d = 3$, and hence from (1.63) $u = v = w = -2$.

Substituting for u, v, w and d in (1.58a), the required equation of the sphere is

$$x^2 + y^2 + z^2 - 4x - 4y - 4z + 3 = 0$$

(b) We have, $r^2 = u^2 + v^2 + w^2 - d = \dfrac{3(1+d)^2}{4} - d = \dfrac{3}{4}\left[\left(d + \dfrac{1}{3}\right)^2 + \dfrac{8}{9}\right]$.

Thus the minimum value of r^2, and hence of r, occurs at $d = -1/3$. Using in (1.63), it gives

$$u = v = w = -1/3.$$

Hence, the required sphere is, $x^2 + y^2 + z^2 - 2/3x - 2/3\,y - 2/3z - 1/3 = 0$.

Example 1.24: A sphere of radius 4 units touches the coordinate axes. Find the equation of the sphere if its centre lies in the positive octant.

Solution: Let the required sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad ...(1.64)$$

Its radius is 4 unit, hence

$$u^2 + v^2 + w^2 - d = 16. \qquad ...(1.65)$$

The sphere touches the x-axis. At the point of contact $y = 0, z = 0$, thus Eq. (1.64) becomes

$$x^2 + 2ux + d = 0.$$

This must have equal roots, therefore, $4u^2 = 4.1.d$, or $u^2 = d$.

Similarly, $v^2 = d$ and $w^2 = d$, and thus (1.65) gives $3d - d = 16$, or $d = 8$, and hence $u^2 = v^2 = w^2 = d = 8$, which gives, $u = v = w = \pm 2\sqrt{2}$.

Also the centre $(-u, -v, -w)$ is given to be in the positive octant, therefore, $u = v = w = -2\sqrt{2}$, rejecting the positive value, and thus the equation of the sphere is

$$x^2 + y^2 + z^2 - 4\sqrt{2}(x + y + z) + 8 = 0.$$

Example 1.25: A plane passes through a fixed point (a, b, c). Show that the locus of the foot of perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$.

Solution: Let $A(a, b, c)$ be the fixed point of the variable plane α and let $M(x, y, z)$ be the foot of $\perp$ from the origin $O(0, 0, 0)$ to the plane as shown in Fig. 1.14.

The direction ratios of OM are $\langle x-0, y-0, z-0\rangle$, that is, $\langle x, y, z\rangle$ and that of MA are $\langle x-a, y-b, z-c\rangle$. Since $OM \perp MA$, we have

$$x(x-a)+y(y-b)+z(z-c)=0$$

or,
$$x^2+y^2+z^2-ax-by-cz=0,$$

which is the required locus and represents a sphere.

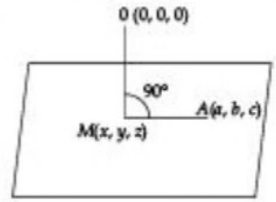


Fig. 1.14

Example 1.26: A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that

(a) the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$,

(b) the locus of the foot of the perpendicular from O to the plane ABC is given by

$$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4k^2.$$

Solution: (a) Let $OA=a$, $OB=b$, $OC=c$. Then the sphere passes through the points $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ and thus the equation of the sphere can be taken as

$$x^2+y^2+z^2-ax-by-cz=0. \quad \text{...(1.66)}$$

Radius of the sphere $=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2+\left(\frac{c}{2}\right)^2}=k$, given.

Squaring, we have
$$\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}=k^2,$$

or,
$$a^2+b^2+c^2=4k^2. \quad \text{...(1.67)}$$

Let $P(x_1, y_1, z_1)$ be the centroid of the triangle ABC, then

$$x_1=\frac{a+0+0}{3}=\frac{a}{3},\quad y_1=\frac{0+b+0}{3}=\frac{b}{3},\quad z_1=\frac{0+0+c}{3}=\frac{c}{3}.$$

Therefore, $a=3x_1$, $b=3y_1$, $c=3z_1$. Substituting these values of a, b, c in (1.67), we get $9x_1^2+9y_1^2+9z_1^2=4k^2$, and therefore, the locus of the centroid $P(x_1, y_1, z_1)$ is $9(x^2+y^2+z^2)=4k^2$, which is a sphere.

(b) Equation of the plane ABC is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \quad \text{...(1.68)}$$

The direction ratios of the line perpendicular to this plane are $\langle 1/a, 1/b, 1/c\rangle$. Therefore, the equation of the line through $O(0, 0, 0)$ and perpendicular to the plane (1.68) is

$$\frac{x-0}{1/a}=\frac{y-0}{1/b}=\frac{z-0}{1/c},\ \text{or}\quad ax=by=cz. \quad \text{...(1.69)}$$

To find the locus of the point of intersection of the plane (1.68) and line (1.69), we eliminate the unknown constants a, b and c from (1.67), (1.68) and (1.69). Taking (1.69) as $ax=by=cz=\lambda$, say. This gives $a=\frac{\lambda}{x}$, $b=\frac{\lambda}{y}$, $c=\frac{\lambda}{z}$. Substituting these values in (1.67) to obtain

$$\lambda^2(x^{-2}+y^{-2}+z^{-2})=4k^2. \quad ...(1.70)$$

Next substituting the values of a, b, and c in (1.68), we get

$$\frac{1}{\lambda}(x^2+y^2+z^2)=1, \text{ or } \frac{1}{\lambda^2}(x^2+y^2+z^2)^2=1. \quad ...(1.71)$$

Multiplying (1.70) and (1.71), we obtain $(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4k^2$, as the required locus.

EXERCISE 1.3

1. Find the sphere through four points (0, −2, 3) (2, 0, 1), (1, −5, −1) and (4, −1, 2). Also find its centre and the radius.
2. Find equation of the sphere which passes through the points (1, −3, 4), (1, −5, 2), (1, −3, 0) and whose centre lies on the plane $x+y+z=0$.
3. A plane passes through a fixed point (p, q, r) and cuts the axes in P, Q and R. Show that the locus of the centre of the sphere $OPQR$ is $\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=2$.
4. A point moves so that the sum of squares of its distances from the six faces of a cube is constant; show that the locus is a sphere.
5. Find the equation of the sphere circumscribing the tetrahedron whose faces are

$$\frac{y}{b}+\frac{z}{c}=0, \quad \frac{z}{c}+\frac{x}{a}=0, \quad \frac{x}{a}+\frac{y}{b}=0, \text{ and } \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

6. A sphere of radius $2k$ passes through the origin and meets the axes in A, B and C. Find the locus of the centroid of the tetrahedron $OABC$.
7. Find the equation of the sphere with centre at the point (2, 3, −1) and touching the line $\frac{x+1}{5}=\frac{y-8}{-3}=\frac{z-4}{-4}$.
8. Show that the sphere $x^2+y^2+z^2-2x+6y+14z+3=0$ divides the line joining the points (2, −1, −4) and (5, 5, 5) internally and externally in the ratio 1:2.
9. A is the point (1, 3, 4) and B the point (1, −2, −1). A point P moves so that $3PA=2PB$. Prove that the locus of P is the sphere $x^2+y^2+z^2-2x-14y-16z+42=0$.
10. Find the shortest and the longest distance from the point (3, 5, 6) to the sphere $x^2+y^2+z^2-4y-9z-3=0$.

1.3.2 Section of a Sphere by a Plane

It can be observed very easily that *the intersection of a sphere with a plane is a circle.* If the plane of intersection passes through the centre of the sphere it is called a *great circle.* Clearly the centre and the radius of the great circle are the same as that of the sphere.

Let the equation of the sphere be

$$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad ...(1.72)$$

and that of the plane be

$$P = Ax + By + Cz + D = 0. \qquad ...(1.73)$$

Then the Eqs. (1.72) and (1.73) taken together represent a circle having center L and radius $LP = \sqrt{r^2 - p^2}$, where r is the radius of the sphere (1.72) given by $r^2 = \sqrt{u^2 + v^2 + w^2 - d}$ and p is the perpendicular distance from the centre $(-u, -v, -w)$ to the plane (1.73) given by $p = \dfrac{|-Au - Bv - Cw + D|}{\sqrt{A^2 + B^2 + C^2}}$.

If for a given plane, $p = r$, then the plane (1.73) touches the sphere (1.72) and is called the *tangent plane* to the sphere, and, if $p > r$, then the plane (1.73) does not intersect the sphere (1.72) at all.

Next, we give the following result:

The equation of any sphere through the circle of intersection of the sphere $S = 0$ and the plane $P = 0$ is given by $S + kP = 0$, where k is arbitrary constant.

Obviously, the equation $S + kP = 0$ represents a sphere and the points of intersection of the sphere $S = 0$ and the plane $P = 0$ satisfy the equation $S + kP = 0$.

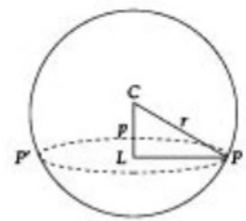


Fig. 1.15

1.3.3 Equation of the Tangent Plane

The equation of the tangent plane at any point (x_1, y_1, z_1) of the sphere $x^2 + y^2 + z^2 = a^2$ is $xx_1 + yy_1 + zz_1 = a^2$.

Let $P(x, y, z)$ be any point on the tangent plane at $P_1(x_1, y_1, z_1)$ to the given sphere with centre $O(0, 0, 0)$ and radius r, then the direction ratios of P_1P are $\langle x - x_1, y - y_1, z - z_1 \rangle$.

Also the direction ratios of the radius OP_1 are $\langle x_1 - 0, y_1 - 0, z_1 - 0 \rangle$, or $\langle x_1, y_1, z_1 \rangle$.

Since, OP_1 is normal to the tangent plane at P_1, thus $OP_1 \perp P_1P$. Therefore,

$$x_1(x - x_1) + y_1(y - y_1) + z_1(z - z_1) = 0$$

or,

$$xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2 = a^2,$$

since, the point $P_1(x_1, y_1, z_1)$ lies on the sphere.

Thus, the desired equation of the tangent plane is $xx_1 + yy_1 + zz_1 = a^2$.

Proceeding on the similar lines we can show that *the equation of the tangent plane*, at any point (x_1, y_1, z_1) of the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$.

Example 1.27: Find the centre and radius of the circle given by

$$x^2+y^2+z^2-2x-4y-6z-2=0,\ x+2y+2z=20.$$

Also find the sphere with this circle as great circle.

Solution: The centre of the given sphere is $C(1, 2, 3)$ and its radius is

$$CP=\sqrt{1^2+2^2+3^2-(-2)}=4.$$

The equation of the perpendicular from the point $C(1, 2, 3)$ on the plane $x+2y+2z=20$ is

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{2}=\lambda,\ \text{say}.$$

Any point on this line is $(\lambda+1, 2\lambda+2, 2\lambda+3)$. It lies on the plane $x+2y+2z=20$, if

$$(\lambda+1)+2(2\lambda+2)+2(2\lambda+3)=20,\ \text{or}\ \ \lambda=1.$$

Therefore, the coordinate of L, the centre of the circle is shown in Fig. 1.14 are $(2, 4, 5)$.

Radius of the circle, $LP=\sqrt{CP^2-CL^2}=\sqrt{4^2-(2-1)^2-(4-2)^2-(5-3)^3}=\sqrt{7}$.

Further, to find the sphere with the given circle as great circle, we know that the centre and the radius of the great circle are the same as that of the sphere, therefore, the required equation of the sphere is

$$(x-2)^2+(y-4)^2+(z-5)^2=(\sqrt{7})^2$$

or,
$$x^2+y^2+z^2-4x-8y-10z+38=0.$$

Alternatively; the equation of any sphere through the given circle is

$$x^2+y^2+z^2-2x-4y-6z-2+k(x+2y+2z-20)=0$$

or, $x^2+y^2+z^2-(2-k)x-2(2-k)y-2(3-k)z-2(1+10k)=0.$

In order that this may have the given circle as its great circle, its centre $\left(1-\frac{k}{2};2-k,3-k\right)$ must lie on the plane $x+2y+2z=20$, thus $1-\frac{k}{2}+4-2k+6-2k=20$, or $k=-2$.

Therefore, the desired sphere is obtained by substituting $k=-2$, we have

$$x^2+y^2+z^2-4x-8y-10z+38=0.$$

Example 1.28: Prove that the circle

$$x^2+y^2+z^2-2x+3y+4z-5=0,\quad 5y+6z+1=0$$
$$x^2+y^2+z^2-3x-4y+5z-6=0,\quad x+2y-7z=0$$

lie on the same sphere and find its equation.

Solution: Circles are:

$$x^2+y^2+z^2-2x+3y+4z-5=0, \quad 5y+6z+1=0, \qquad ...(1.74)$$

$$x^2+y^2+z^2-3x-4y+5z-6=0, \quad x+2y-7z=0. \qquad ...(1.75)$$

Any sphere through circle (1.74) is

$$x^2+y^2+z^2-2x+3y+4z-5+k_1(5y+6z+1)=0 \qquad ...(1.76)$$

and through circle (1.75) is

$$x^2+y^2+z^2-3x-4y+5z-6+k_2(x+2y-7z)=0. \qquad ...(1.77)$$

The circles (1.74) and (1.75) will lie on the same sphere if (1.76) and (1.77) represent the same sphere for some values of k_1 and k_2.

Equating the coefficients of the like terms in (1.76) and (1.77), we obtain

$$-2=-3+k_2, \quad 3+5k_1=-4+2k_2, 4+6k_1=5-7k_2, \quad \text{and} \quad -5+k_1=-6.$$

All these are satisfied for $k_1=-1$ and $k_2=1$.

Hence both the circles lie on the common sphere

$$x^2+y^2+z^2-2x-2y-2z-6=0.$$

Example 1.29: Find the equation of the two spheres which pass through the circle

$$x^2+y^2+z^2-2x+2y-4z-3=0, \quad 2x+y+z=4 \qquad ...(1.78)$$

and touch the plane

$$3x+4y=14 \qquad ...(1.79)$$

Solution: The equation of any sphere through the circle (1.78) is

$$x^2+y^2+z^2-2x+2y-4z-3+k(2x+y+z-4)=0. \qquad ...(1.80)$$

Its centre and radius are

$$C\left(1-k,-1-\frac{k}{2},2-\frac{k}{2}\right), \text{ and } r=\sqrt{(1-k)^2+\left(-1-\frac{k}{2}\right)^2+\left(2-\frac{k}{2}\right)^2-(-3-4k)}$$

$$=\sqrt{\frac{3k^2}{2}+k+9}.$$

The sphere (1.80) touches the plane (1.79), if the perpendicular distance from its centre $\sqrt{1-k,-1-\frac{k}{2},2-\frac{k}{2}}$ to the plane $3x+4y=14$ is equal to r, that is, if

$$\sqrt{\frac{3k^2}{2}+k+9}=\frac{\left|3(1-k)+4\left(-1-\frac{k}{2}\right)-14\right|}{5}$$

or,

$$\frac{3k^2}{2}+k+9=\frac{(5k+15)^2}{25}$$

which after simplification gives $k = 0$ and 10.

Thus the required spheres are

$x^2 + y^2 + z^2 - 2x + 2y - 4z - 3 = 0$, and $x^2 + y^2 + z^2 + 18x + 12y - 44z - 43 = 0$.

Example 1.30: Find the equation of the sphere which touches the sphere $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$ at the point $(1, 2, -2)$ and passes through the origin.

Solution: The given sphere is

$$x^2 + y^2 + z^2 + 2x - 6y + 1 = 0. \quad \text{...(1.81)}$$

The equation of the tangent plane at $(1, 2, -2)$ to the sphere (1.81) is

$$x(1) + y(2) + z(-2) + (x + 1) - 3(y + 2) + 1 = 0$$

or,
$$2x - y - 2z - 4 = 0. \quad \text{...(1.82)}$$

The required sphere touching the sphere (1.81) at $(1, 2, -2)$ is the sphere through the point circle of intersection of (1.81) and the tangent plane (1.82). This is

$$x^2 + y^2 + z^2 + 2x - 6y + 1 + k(2x - y - 2z - 4) = 0. \quad \text{...(1.83)}$$

It passes through $(0, 0, 0)$, if $1 - 4k = 0$, or $k = \dfrac{1}{4}$. Hence, the sphere is

$$4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0.$$

Example 1.31: Find the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 = 9$ which are parallel to the plane $3x + 4y + 5z = 7$.

Solution: The given sphere is

$$x^2 + y^2 + z^2 = 9. \quad \text{...(1.84)}$$

Its centre is $(0, 0, 0)$ and radius is 3.

The equation of any plane parallel to the plane $3x + 4y + 5z = 7$ is given by

$$3x + 4y + 5z = k, \quad \text{...(1.85)}$$

where k is arbitrary constant.

The plane (1.85) will be tangent plane to the sphere (1.84) if the perpendicular distance from the centre $(0, 0, 0)$ to this plane is equal to the radius of the sphere, that is, if

$$\frac{|3(0) + 4(0) + 5(0) - k|}{\sqrt{9 + 16 + 25}} = 3, \quad \text{or} \quad |k| = 15\sqrt{2}, \quad \text{or} \quad k = \pm 15\sqrt{2}.$$

The required planes are $3x + 4y + 5z = \pm 15\sqrt{2}$.

Example 1.32: Find the equation of the tangent line to the circle $x^2 + y^2 + z^2 = 3$, $3x - 2y + 4z + 3 = 0$ at the point $(1, 1, -1)$.

Solution: Equation of the sphere is

$$x^2+y^2+z^2=3 \quad ...(1.86)$$

and the plane of the circle is

$$3x-2y+4z+3=0. \quad ...(1.87)$$

The tangent line to the circle

$$x^2+y^2+z^2=3, \quad 3x-2y+4z+3=0 \quad ...(1.88)$$

is the line of intersection of the tangent plane to the sphere (1.86) at $(1, 1, -1)$ and the plane of the circle (1.87). The equation of the tangent plane the sphere (1.86) at $(1, 1, -1)$ is

$$x(1)+y(1)+z(-1)=3, \text{ or } x+y-z=0. \quad ...(1.89)$$

Therefore, the equation of the tangent line is

$$3x-2y+4z+3=0, \text{ and } x+y-z-3=0. \quad ...(1.90)$$

The Eq. (1.90) can be put in *symmetrical form* as follows.

Let $\langle a, b, c\rangle$ be the direction ratios of the tangent line (1.90), then

$$a+b-c=0, \text{ and } 3a-2b+4c=0. \text{ Which give } \frac{a}{2}=\frac{b}{-7}=\frac{c}{-5}.$$

Hence, the *d.r's* of the tangent line are $\langle 2, -7, -5\rangle$. Also the tangent line passes through the point $(1, 1, -1)$, therefore, its equation is $\frac{x-1}{2}=\frac{y-1}{-7}=\frac{z-1}{-5}$.

EXERCISE 1.4

1. Find the centre and the radius of the circle

$$x^2+y^2+z^2+2x-2y-4z-19=0, \quad x+2y+2z+7=0$$

2. Find the equations of the spheres which pass through the circle $x^2+y^2+z^2=4$, $2x+2y+5z-6=0$ and touch the plane $z=0$.
3. A sphere of radius 4 units touches the coordinate axes. Find the equation of the sphere if its centre lies in the positive octant.
4. Determine the equation of the sphere which touches the plane $x-2y-2z=7$ at the point $(3, -1, -1)$ and passes through the point $(1, 1, -3)$.
5. Show that the points $(5, 0, 2)$, $(2, -6, 0)$, $(7, -3, 8)$, $(4, -9, 6)$ are concyclic.
6. Prove that the plane $x+2y-z=4$ cuts the spheres $x^2+y^2+z^2-x+z-2=0$ in a circle of radius unity. Also find the equation of the sphere which has this circle for one of its great circles.

7. Find equation of the sphere which passes through the circle $x^2+y^2+z^2=5$, $x+2y+3z=3$, and touch the plane $4x+3y=15$.
8. Obtain the equations of the tangent planes to the sphere $x^2+y^2+z^2+6x-2z+1=0$ which pass through the line $3(16-x)=2y+30=3z$.
9. Find the equation of the sphere of radius r which touches the three coordinate axes. How many such spheres can be drawn?
10. A sphere is inscribed in the tetrahedron whose faces are $x=0$, $y=0$, $z=0$, $2x+6y+3z=14$. Find its equation.
11. Determine the centre and radius of the section of the sphere $x^2+y^2+z^2=16$ by the plane $2x+y+2z=9$. Also find the area of the projection of this circle on the x-y plane.

1.3.4 Angle of Intersection of Two Spheres

The angle of intersection of two spheres is the angle between their tangent planes at a common point.

Since, the radii of the spheres through a common point are perpendicular to the tangent planes at the point, so the angle between the radii of the sphere at the common point is equal to the angle between their tangent planes, that is, the angle of intersection of the spheres.

As is clear from the Fig. 1.16, we have

$$\theta = \cos^{-1}\left(\frac{r_1^2+r_2^2-d}{2r_1 r_2}\right), \qquad ...(1.91)$$

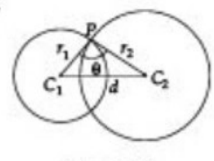


Fig. 1.16

where the letters have their usual meanings.

Orthogonal Spheres: *Two spheres are said to be orthogonal spheres if they cut each other orthogonally, that is, if their tangent planes at a point of intersection are at right angles.*

When $\theta = 90°$, then from (1.91)

$$r_1^2 + r_2^2 = d^2. \qquad ...(1.92)$$

Thus two spheres cut orthogonally if the square of the distance between their centres is equal to the sum of the square of their radii.

We can derive easily using (1.92) that the condition for the orthogonality of the two spheres

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$

and

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0,$$

is

$$2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2. \qquad ...(1.93)$$

Remark. It should be noted that the intersection of two spheres

$$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$

and

$$S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$$

is also a circle. The equation of any sphere through this circle is $S_1 + kS_2 = 0$, where k is an arbitrary constant.

The equation of the plane of the circle through the two spheres $S_1 = 0, S_2 = 0$ is

$$S_1 - S_2 = 2(u_1 - u_2)x + 2(v_1 - v_2)y + 2(w_1 - w_2)z + (d_1 - d_2) = 0.$$

Thus the equation of any sphere through S_1 and S_2 may also be taken as $S_1 + k(S_1 - S_2) = 0$.

Example 1.33: Show that the spheres $x^2 + y^2 + z^2 = 100$, and $x^2 + y^2 + z^2 - 24x - 30y - 32z + 400 = 0$ touch externally and find the point of contact.

Solution: Equation of the first sphere is $x^2 + y^2 + z^2 = 100$. Its centre is $C_1(0, 0, 0)$ and radius is 10.

Equation of the second sphere is $x^2 + y^2 + z^2 - 24x - 30y - 32z + 400 = 0$. Its centre is $C_2(12, 15, 16)$ and radius is $\sqrt{144 + 225 + 256 - 400} = 15$.

If the two spheres touch externally, then the distance between their centres is equal to sum of their radii. The distance between the centres is, $\sqrt{(12-0)^2 + (15-0)^2 + (16-0)^2} = \sqrt{625} = 25$.

Also sum of the radii is $15 + 10 = 25$.

The condition is satisfied and hence the two spheres touch externally.

Next, if A is the point of contact of these two spheres, then A lies on the line C_1C_2 and divides C_1C_2 internally in the ratio 10:15, that is, 2:3, in the ratio of their radii.

If (x_1, y_1, z_1) are the coordinates of A, then

$$x_1 = \frac{3(0) + 2(12)}{2+3} = \frac{24}{5}, \quad y_1 = \frac{3(0) + 2(15)}{2+3} = 6, \quad z_1 = \frac{3(0) + 2(16)}{2+3} = \frac{32}{5}.$$

Thus, the point of contact is $A\left(\frac{24}{5}, 6, \frac{32}{5}\right)$.

Example 1.34: Find the equation of the sphere which cuts orthogonally each of the four spheres

$$x^2 + y^2 + z^2 + 2ax = a^2, \qquad x^2 + y^2 + z^2 + 2by = b^2,$$
$$x^2 + y^2 + z^2 + 2cy = c^2, \text{ and} \qquad x^2 + y^2 + z^2 = a^2 + b^2 + c^2.$$

Solution: The given spheres are

$$x^2 + y^2 + z^2 + 2ax - a^2 = 0, \quad \text{...(1.94)}$$

$$x^2 + y^2 + z^2 + 2by - b^2 = 0, \quad \text{...(1.95)}$$

$$x^2 + y^2 + z^2 + 2cy - c^2 = 0, \quad \text{...(1.96)}$$

and

$$x^2 + y^2 + z^2 - a^2 - b^2 - c^2 = 0. \quad \text{...(1.97)}$$

Let the required sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \quad \text{...(1.98)}$$

If the sphere (1.98) cuts the sphere (1.94) orthogonally, then using the condition (1.93) of orthogonality, we have

$$2u(a) + 2v(0) + 2w(0) = d - a^2 \Rightarrow 2u = \frac{d}{a} - a.$$

Similarly, we have $2v = \frac{d}{b} - b$, and $2w = \frac{d}{c} - c$.

Also using the condition of orthogonality of (1.98) and (1.97), we have $d = a^2 + b^2 + c^2$.

Using this value of d, we have

$$2u = \frac{b^2 + c^2}{a}, \quad 2v = \frac{c^2 + a^2}{b}, \quad 2w = \frac{a^2 + b^2}{c} \qquad \text{...(1.99)}$$

Hence, the required equation of the sphere (1.98) becomes

$$x^2 + y^2 + z^2 + \frac{b^2 + c^2}{a}x + \frac{c^2 + a^2}{b}y + \frac{a^2 + b^2}{c}z + a^2 + b^2 + c^2 = 0$$

Example 1.35: Prove that the spheres that can be drawn through the origin and each set of points where the planes, parallel to the plane, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ cut the coordinate axes, form a system of spheres which are cut orthogonally by the spheres $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$, if $au + bv + cw = 0$.

Solution: The equation of any plane parallel to the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = k. \qquad \text{...(1.100)}$$

The plane (1.100) cuts the coordinate axes at the points $A(ka, 0, 0)$, $B(0, kb, 0)$, $(0, 0, kc)$.

We can see very easily that the equation of the sphere passing through the origin and the points A, B, C is

$$x^2 + y^2 + z^2 - kax - kby - kcz = 0. \qquad \text{...(1.101)}$$

The sphere (1.101) cuts the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$ orthogonally, if

$$2u\left(-\frac{1}{2}ka\right) + 2v\left(-\frac{1}{2}kb\right) + 2w\left(-\frac{1}{2}kc\right) = 0 + 0, \quad \text{or} \quad au + bv + cw = 0,$$

which is the required condition.

Example 1.36: Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, 2, -1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

Solution: Let the equation of the required sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \qquad \text{...(1.102)}$$

Equation of the tangent plane to this sphere at (1, –2, 1) is

$$x(1) + y(-2) + z(1) + u(x+1) + v(y-2) + w(z+1) + d = 0,$$

or
$$(1+u)x + (v-2)y + (w+1)z + (u - 2v + w + d) = 0.$$

If this is the same plane as the given one, that is, $3x + 2y - z + 2 = 0$, then comparing these two planes, we get

$$\frac{1+u}{3} = \frac{v-2}{2} = \frac{w+1}{-1} = \frac{u-2v+w+d}{2}. \qquad ...(1.103)$$

The first and last terms in (1.103) give

$$2(1+u) = 3(u - 2v + w + d), \quad \text{or} \quad u - 6v + 3w + 3d = 2.$$

Similarly, comparing second and last, and then third and last terms in (1.103), we get respectively ;

$$u - 3v + w + d = -2, \quad \text{and} \quad -u + 2v - 3w - d = 2.$$

Also the condition that the sphere (1.102) cuts the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ orthogonally gives

$$2u(-2) + 2v(3) + 2w(0) = d + 4, \quad \text{or} \quad -4u + 6v - d = 4.$$

Solving these equations for u, v, w and d, we obtain $u = \frac{7}{2}$, $v = 5$, $w = \frac{-5}{2}$ and $d = 12$.

Substituting these values in (1.102) we get the required equation as

$$x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0.$$

EXERCISE 1.5

1. Show that the spheres $x^2 + y^2 + z^2 + 6y + 2z + 8 = 0$ and $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$ are orthogonal. Also find their plane of intersection.
2. At what angle does the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z + 10 = 0$ intersects the sphere which has points (1, 2, –3) and (5, 0, 1) as the extremities of a diameter.
3. Find the equation of the sphere that passes through the two points (0, 3, 0) and (–2, –1, –4) and cuts orthogonally the two spheres $x^2 + y^2 + z^2 + x - 3z - 2 = 0$ and $2(x^2 + y^2 + z^2) + x + 3y + 4 = 0$.
4. Show that every sphere through the circle $x^2 + y^2 - 2ax + r^2 = 0, z = 0$ cuts orthogonally every sphere through the circle $x^2 + z^2 = r^2, y = 0$.
5. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 + 3x + y + 4z - 3 = 0, x^2 + y^2 + 2x + 3y + 6 = 0$ and the point (1, –2, 3).

6. Find the equation of the sphere having its centre on the plane $4x - 5y - z = 3$ and passing through the circle $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0, x^2 + y^2 + z^2 + 4x + 5y - 6z + 2 = 0$.
7. Show that the locus of the centres of spheres which pass through the points (1, 0, 0), (0, 2, 0) and (2, 3, 4) is a straight line. Express its equation in symmetrical form.
8. Find the two tangent planes to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$ which are parallel to the plane $2x + 2y = z$. Find their points of contact.

1.4 CONE

A cone is a surface generated by a straight line which passes through a fixed point and satisfies one more condition. For example, it may intersect a given curve, called the guiding curve or touch a given surface.

The fixed point is called the *vertex* and the straight line in any position is called the *generator*.

For example, the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$,

as shown in Fig. 1.17, represents an elliptic cone with vertex at the origin O and the guiding curve, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the plane $z = c$.

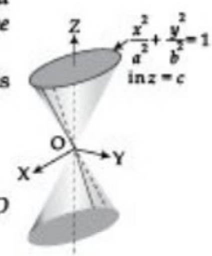


Fig. 1.17

1.4.1 Homogeneous Equation Representation of a Cone

An equation $f(x, y, z) = 0$ is said to be homogeneous in x, y, z if the sum of powers of x, y, z in each term is the same.

For example, $2x^3 + 3x^2y + xy^2 + 3yz^2 = 0$ is homogeneous equation of degree 3 in x, y, z.

In case of homogeneous equation $f(x, y, z) = 0$, we have $f(rx, ry, rz) = 0$ for all values of r.

We have the following result.

Theorem 1.1: *Every homogeneous equation in x, y, z represents a cone with vertex at the origin and conversely.*

Proof. Let the homogeneous equation be

$$f(x, y, z) = 0. \qquad ...(1.104)$$

If $P(x_1, y_1, z_1)$ be any point on the surface represented by (1.104), then $f(x_1, y_1, z_1) = 0$. Also since Eq. (1.104) is homogeneous in x, y, z, therefore $f(rx_1, ry_1, rz_1) = 0$ for all values of r. But the point $Q(rx_1, ry_1, rz_1)$ is any point on the line OP. Thus, every point on the line OP lies on the surface (1.104), hence the surface $f(x, y, z) = 0$ is being generated by the line through $O(0, 0, 0)$, and therefore represents a cone with vertex at the origin.

Conversely, let the equation of the cone with vertex at the origin be $f(x, y, z) = 0$, and let $P(x_1, y_1, z_1)$ be any point on the cone, then $f(x_1, y_1, z_1) = 0$.

The equation of any generator OP is

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1} = r, \quad \text{say.}$$

Therefore, any point on OP is $Q(rx_1, ry_1, rz_1)$. Since, the generator completely lies on the cone, thus the point $Q(rx_1, ry_1, rz_1)$ lies on the cone for all values of r, and therefore $f(rx_1, ry_1, rz_1) = 0$ for all values of r.

The equation $f(x_1, y_1, z_1) = 0$ and $f(rx_1, ry_1, rz_1) = 0$ for all values of r, simultaneously implies that $f(x_1, y_1, z_1) = 0$ and, hence, $f(x, y, z) = 0$ is homogeneous in x, y, z.

Remarks

1. The degree of the equation of a cone depends upon the nature of its guiding curve. In case the guiding curve is a conic, the equation of the cone will be of the second degree. Such cones are called *quadric cones*.

2. The second degree homogeneous equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a cone with vertex at the origin.

3. The general equation of a cone which passes through the coordinate axes is of the form $fyz + gzx + hxy = 0$. For, the cone which passes through the axes will have origin as its vertex, therefore, the equation of the cone will be $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$. Since, it passes through the x-axis whose direction cosines are (1, 0, 0) and which should satisfy the equation of the cone, hence $a = 0$. Similarly, $b = 0$ and $c = 0$, thus we get the required equation.

Example 1.37: Find the equation of the cone whose vertex is (1, 2, 3) and the guiding curve is the circle $x^2 + y^2 + z^2 = 4, x + y + z = 1$.

Solution: Let the generator be

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = \lambda, \text{ say.} \qquad ...(1.105)$$

Any point on it is $P(1 + l\lambda, 2 + m\lambda, 3 + n\lambda)$. If (1.105) intersects the given curve, the coordinate of P should satisfy

$$(1 + l\lambda)^2 + (2 + m\lambda)^2 + (3 + n\lambda)^2 = 4 \qquad ...(1.106)$$

and,

$$(1 + l\lambda) + (2 + m\lambda) + (3 + n\lambda) = 1 \qquad ...(1.107)$$

Eliminating λ from those two equations we obtain

$$\left(1 - \frac{5l}{l+m+n}\right)^2 + \left(2 - \frac{5m}{l+m+n}\right)^2 + \left(3 - \frac{5n}{l+m+n}\right)^2 = 4$$

or

$$(-4l + m + n)^2 + (2l - 3m + 2n)^2 + (3l + 3l - 2n)^2 = 4(l + m + n)^2$$

or,

$$25l^2 + 15m^2 + 5n^2 - 30mn - 20nl - 10lm = 0$$

or,

$$5l^2 + 3m^2 + n^2 - 6mn - 4nl - 2lm = 0. \qquad ...(1.108)$$

Eliminating l, m, n from (1.105) and (1.108), we obtain

$$5(x-1)^2 + 3(y-2)^2 + (z-3)^2 - 6(y-2)(z-3) - 4(z-3)(x-1) - 2(x-1)(y-2) = 0,$$

or, $\quad 5x^2 + 3y^2 + z^2 - 6yz - 4zx - 2xy + 6x + 8y + 10z - 26 = 0,$

as the required equation.

Example 1.38: Find the equation of the cone with vertex (α, β, γ) and base $y^2 - 4ax = 0, z = 0$.

Solution: Any line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}. \qquad ...(1.109)$$

If it intersects the given curve, then $z = 0$, we have

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}, \quad \text{which gives} \quad x = \alpha - \gamma\frac{l}{n}, \quad y = \beta - \gamma\frac{m}{n}.$$

Substituting these values in $y^2 - 4ax = 0$, we have

$$\left(\beta - \gamma\frac{m}{n}\right)^2 - 4a\left(\alpha - \gamma\frac{l}{n}\right)^2 = 0. \qquad ...(1.110)$$

Eliminating l, m, n, from (1.109) and (1.110) we obtain

$$\left(\beta - \gamma\frac{y-\beta}{z-\gamma}\right)^2 - 4a\left(\alpha - \gamma\frac{x-\alpha}{z-\gamma}\right)^2 = 0, \quad \text{or} \quad (\beta z - \gamma y)^2 - 4a(\alpha z - \gamma x)^2 = 0.$$

as the required equation.

Example 1.39: Find the equation of the cone whose vertex is at the origin and the base is the circle $y^2 + z^2 = b^2, x = a$.

Solution: Any line through (0, 0, 0) is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r, \quad \text{say}. \qquad ...(1.111)$$

Any point on (1.111) is $P(lr, mr, nr)$. If (1.111) intersects the given curve $y^2 + z^2 = b^2, x = a$, then the point P should satisfy its equation. Therefore,

$$m^2r^2 + n^2r^2 = b^2 \text{ and } lr = a. \qquad ...(1.112)$$

Eliminating r from (1.112), we have

$$\frac{(m^2+n^2)a^2}{l^2} = b^2. \qquad ...(1.113)$$

Eliminating l, m, n from (1.111) and (1.113), we obtain

$$\frac{a^2(y^2+z^2)}{x^2} = b^2, \quad \text{or} \quad a^2(y^2 + z^2) - b^2x^2 = 0$$

as the required equation

Example 1.40: Find the equation of a cone whose vertex is the origin and base is the curve $f(x, y) = 0, z = k$.

Solution: Any line through (0, 0, 0) is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad ...(1.114)$$

At $z = k$, we have $x = k\frac{l}{n}, \quad y = k\frac{m}{n}$.

Substituting in $f(x, y) = 0$, we have

$$f\left(k\frac{l}{n}, k\frac{m}{n}\right) = 0. \qquad ...(1.115)$$

Eliminating l, m, n from (1.114) and (1.115), we obtain $f\left(k\frac{x}{z}, k\frac{y}{z}\right) = 0$, as the required equation.

1.4.2 Right Circular Cone

A right circular cone is a surface generated by a straight line which passes through a fixed point and makes a constant angle with a fixed line.

The constant angle $\angle AVC$ as shown in Fig. 1.18 is called its *semi-vertical angle* and the fixed line VC is called the *axis*. The section of a right circular cone by a plane perpendicular to its axis is a circle.

Equation of the Right Circular Cone Let $V(\alpha, \beta, \gamma)$ be the vertex, VC the axis, with direction ratios l, m, n and $P(x, y, z)$ be any point on the cone. Obviously VP is the generator of the cone and let $\angle CVP = \theta$ (say) be the semi-vertical angle.

The direction ratios of the generator VP are $x - \alpha, y - \beta, z - \gamma$. Also direction ratios of the axis VC are l, m, n. Therefore,

$$\cos\theta = \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{l^2+m^2+n^2}\cdot\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

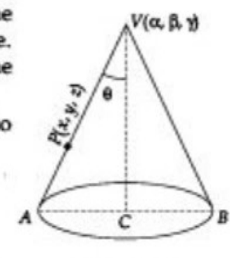


Fig. 1.18

or, $[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2 = (l^2 + m^2 + n^2)[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]\cos^2\theta \qquad ...(1.116)$

which is the required equation of the right circular cone in the *general form*.

If vertex is at the origin, and l, m, n are the direction cosines of the axis, then the Eq. (1.116) becomes

$$(lx + my + nz)^2 = (x^2 + y^2 + z^2)\cos^2\theta \qquad ...(1.117)$$

since $l^2 + m^2 + n^2 = 1$.

If vertex is at the origin and z-axis is the axis of the cone then Eq. (1.116) becomes

$$z^2 = (x^2 + y^2 + z^2)\cos^2\theta \text{ (since } dc's \text{ are } 0, 0, 1)$$

or, $$x^2+y^2+z^2=z^2\sec^2\theta=z^2(1+\tan^2\theta)$$

or, $$x^2+y^2=z^2\tan^2\theta, \quad ...(1.118)$$

the equation of the cone in the *standard form.*

Example 1.41: Find the equation of the cone whose vertical angle is $\pi/2$, which has its vertex at the origin and its axis along the line $x=-2y=z$.

Solution: Let $P(x, y, z)$ be any point on the cone with vertex O and axis OC, which is given as

$$x=-2y=z \text{ or, } \frac{x}{-2}=\frac{y}{1}=\frac{z}{-2}.$$

The direction ratios of OP are x, y, z and that of the axis OC are $-2, 1, -2$. Since $\angle POC=\pi/4$, therefore,

$$\cos\frac{\pi}{4}=\frac{-2x+y-2z}{\sqrt{4+1+4}.\sqrt{x^2+y^2+z^2}} \quad \text{or,} \quad \frac{1}{\sqrt{2}}=\frac{-2x+y-2z}{3\sqrt{x^2+y^2+z^2}}$$

Squaring and simplifying, we obtain $x^2+7y^2+z^2+8xy+8yz-16zx=0$, as the required equation.

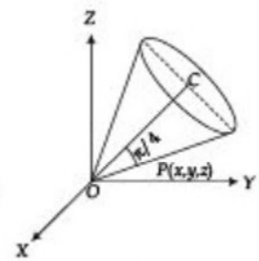


Fig. I.19

Example 1.42: Find the equation of the right circular cone generated by rotating the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ about the line $\frac{x}{-1}=y=\frac{z}{2}$.

Solution: Equation of the generator and the axis of the cone are respectively

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3} \text{ and } \frac{x}{-1}=\frac{y}{1}=\frac{z}{2}.$$

The vertex of the cone is the point of intersection of these two lines which is obviously $O(0, 0, 0)$. Let α be the semi-vertical angle of the cone, then

$$\cos\alpha=\frac{(-1)(1)+(1)(2)+(2)(3)}{\sqrt{1^2+1^2+2^2}.\sqrt{1^2+2^2+3^2}}=\frac{7}{\sqrt{84}}.$$

If $P(x, y, z)$ be any point on the cone, then direction ratios of OP are x, y, z. Therefore,

$$\frac{7}{\sqrt{84}}=\frac{x(-1)+y(1)+z(2)}{\sqrt{x^2+y^2+z^2}.\sqrt{1+1+4}}$$

Squaring and simplifying, we obtain $5x^2+5y^2-z^2+4xy+8xz-8yz=0$ as the required equation.

Example 1.43: Find the semi-vertical angle and the equation of the right circular cone having its vertex at the origin and passing through the circle $y^2+z^2=4,\ x=4$.

Solution: Since x-axis is the axis of the cone, therefore, its direction cosines are $<1, 0, 0>$. Let $P(x, y, z)$ be any point on the cone, then direction ratios of OP are $<x, y, z>$.

It α be the semi-vertical angle, then

$$\cos\alpha = \frac{1.x + 0.y + 0.z}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}.$$

Also from Fig. 1.20, $\tan\alpha = \frac{2}{4}$, which gives, $\cos\alpha = \frac{4}{\sqrt{20}} = \frac{2}{\sqrt{5}}$ and, therefore, $\frac{2}{\sqrt{5}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$ or, $4(y^2+z^2) = x^2$ is the required equation of the cone.

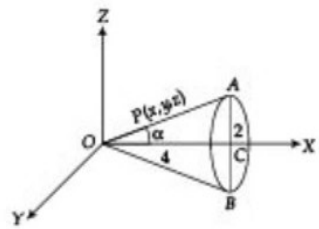


Fig. 1.20

Example 1.44: Prove that the plane $ax + by + cz = 0$ meets the cone $yz + zx + xy = 0$ in perpendicular lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.

Solution: The given plane is

$$ax + by + cz = 0 \quad \text{...(1.119)}$$

and the cone is

$$yz + zx + xy = 0. \quad \text{...(1.120)}$$

Let one of the lines of the section be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \quad \text{...(1.121)}$$

From (1.119) and (1.121), we obtain

$$al + bm + cn = 0. \quad \text{...(1.122)}$$

Similarly, from (1.118) and (1.121) we obtain

$$mn + nl + lm = 0. \quad \text{...(1.123)}$$

From (1.122), $n = \frac{-(al+bm)}{c}$. Substituting this in (1.123), we obtain

$$-m\left[\frac{al+bm}{c}\right] - l\left[\frac{al+bm}{c}\right] + lm = 0$$

or,
$$m(al+bm) + l(al+bm) - clm = 0$$

or,
$$al^2 + lm(a+b-c) + bm^2 = 0.$$

Dividing throughout by m^2, it becomes $a\left(\frac{l}{m}\right)^2 + (a+b-c)\frac{l}{m} + b = 0$, a quadratic in $\frac{l}{m}$.

Let $\frac{l_1}{m_1}, \frac{l_2}{m_2}$ be its two roots, then $\frac{l_1}{m_1}\frac{l_2}{m_2} = \frac{b}{a}$, or $\frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b}$.

Similarly, we have $\frac{l_1l_2}{1/a}=\frac{n_1n_2}{1/c}$. Therefore, $\frac{l_1l_2}{1/a}=\frac{m_1m_2}{1/b}=\frac{n_1n_2}{1/c}=\frac{l_1l_2+m_1m_2+n_1n_2}{1/a+1/b+1/c}$.

The two lines will be perpendicular if $l_1l_2+m_1m_2+n_1n_2=0$, or if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$, this proves the result.

Example 1.45: Find the equation to the right circular cone whose vertex is $(2,-3,5)$ and the axis makes equal angles with the coordinate axes, and which passes through $(1,-2,3)$.

Solution: Since, the axis of the cone makes equal angles with the coordinate axes, so its direction cosines are $\left\langle \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle$.

Let $P(x,y,z)$ be any point on the cone, then direction ratios of the line VP, where $V(2,-3,5)$ is the vertex, are given by $<x-2, y+3, z-5>$. If θ is the semi-vertical angle of the cone, then

$$\cos\theta=\frac{(x-2)\frac{1}{\sqrt{3}}+(y+3)\frac{1}{\sqrt{3}}+(z-5)\frac{1}{\sqrt{3}}}{\sqrt{(x-2)^2+(y+3)^2+(z-5)^2}}. \quad ...(1.124)$$

Since, the point $(1,-2,3)$ lies on the cone, therefore it must satisfy (1.140), which gives

$$\cos\theta=\frac{-2}{\sqrt{3}\sqrt{(1+1+4)}}=\frac{-\sqrt{2}}{3}.$$

Thus, (1.124) becomes

$$\frac{-\sqrt{2}}{3}=\frac{x+y+z-4}{\sqrt{3}.\sqrt{(x-2)^2+(y+3)^2+(z-5)^2}}$$

Squaring and simplifying, we obtain

$x^2+y^2+z^2+6(xy+yz+zx)-16x-36y-4z-76=0$, as the required cone.

1.4.3 Enveloping Cone

A surface generated by tangent lines drawn from a given point to a given surface is known as the tangent cone or the enveloping cone of that surface with the given point as its vertex.

Equation of the enveloping cone of the sphere $x^2+y^2+z^2=a^2$, with vertex at the point (α,β,γ).

The equation of the sphere is

$$x^2+y^2+z^2=a^2. \quad ...(1.125)$$

Equation of any line through the vertex (α,β,γ) is

$$\frac{x-\alpha}{l}+\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r, \quad \text{say.} \quad ...(1.126)$$

The coordinates of any point on the line (1.126) are $(\alpha + lr, \ \beta + mr, \ \gamma + nr)$. This point will lie on the sphere (1.125), if

$$(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 = a^2$$

or, $$r^2(l^2 + m^2 + n^2) + 2r(\alpha l + \beta m + \gamma n) + \alpha^2 + \beta^2 + \gamma^2 - a^2 = 0. \quad ...(1.127)$$

This gives two values of r. The line (1.126) will be tangent to (1.125) if both the roots of the Eq. (1.127) are equal, that is, if

$$(\alpha l + \beta m + \gamma n)^2 - (l^2 + m^2 + n^2)(\alpha^2 + \beta^2 + \gamma^2 - a^2) = 0. \quad ...(1.128)$$

Eliminating l, m, n between (1.126) and (1.128), we get

$$[\alpha(x - \alpha) + \beta(y - \beta) + \gamma(z - \gamma)]^2 - [(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2]\,(\alpha^2 + \beta^2 + \gamma^2 - a^2) = 0$$

which is the equation of the enveloping cone. This equation can be put in the form

$$(x^2 + y^2 + z^2 - a^2)(\alpha^2 + \beta^2 + \gamma^2 - a^2) = (\alpha x + \beta y + \gamma z - a^2)^2. \quad ...(1.129)$$

Symbolically (1.129) can be written in the form $SS_{11} = S_1^2$, where

$$S = x^2 + y^2 + z^2 - a^2, \quad S_1 = \alpha x + \beta y + \gamma z - a^2 \text{ and } S_{11} = \alpha^2 + \beta^2 + \gamma^2 - a^2.$$

Example 1.46: Find the equation of the enveloping cone with vertex at (1, 2, 3) to the sphere $x^2 + y^2 + z^2 = 4$.

Solution: The equation of the sphere is $x^2 + y^2 + z^2 - 4 = 0$ and the vertex is (1, 2, 3).

Here, $S = x^2 + y^2 + z^2 - 4, \quad \alpha = 1, \quad \beta = 2, \quad \gamma = 3$. Thus

$$S_1 = (1)(x) + 2(y) + 3(z) - 4 = x + 2y + 3z - 4 \text{ and } S_{11} = 1 + 4 + 9 - 4 = 10.$$

Therefore, equation of the enveloping cone is, $(x^2 + y^2 + z^2 - a^2)(10) = (x + 2y + 3z - 4)^2$.

Simplifying, we get $9x^2 + 6y^2 + z^2 - 4xy - 12yz - 6zx + 8x + 16y + 24z - 56 = 0$, as the required equation.

EXERCISE 1.6

1. Find the equation of the cone whose vertex is at the origin and the guiding curve is $x^2/4 + y^2/9 + z^2/1 = 1, x + y + z = 1$.
2. The generators of a cone pass through the point (1, 1, 1) and their direction cosines $<l, m, n>$ satisfy the relation $l^2 + m^2 = 3n^2$. Obtain the equation of the cone.
3. Find the equation of the cone which passes through the three coordinate axes and the three mutually perpendicular lines $x/1 = y/-2 = z/3, x/1 = y/-1 = z/-1, x/5 = y/4 = z/1$.
4. Find the equation of the cone whose vertex is (1, 2, 3) and guiding curve is the circle $x^2 + y^2 + z^2 = 4, x + y + z = 1$.
5. Find the equation of the right circular cone generated when the straight line $2y + 3z = 6$, $x = 0$ revolves about z-axis.
6. Find the semi-vertical angle and the equation of the right circular cone having its vertex at the origin and passing through the circle $y^2 + z^2 = 25, x = 4$.

7. Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$.
8. Find the equation to the right circular cone with vertex at (1, 0, 1) and axis equally inclined to the co-ordinate axes, and also cone passes through the point (1, 1, 1).
9. Find the equation of the cone generated by rotating the line $x/1 = y/m = z/n$ about the line $x/a = y/b = z/c$ as axis.
10. Find the equation of the right circular cone generated by the straight lines drawn from the origin to cut the circle through the three points (1, 2, 2), (2, 1, −2) and (2, −2, 1).
11. Lines are drawn from the origin with direction cosines proportional to (1, 2, 2), (2, 3, 6) (3, 4, 12); find the direction cosines of the axis of right circular cone through them, and prove that the semi-vertical angle of the cone is $\cos^{-1}(1/\sqrt{3})$. Also find the equation of the right circular cone.
12. Find the condition that the plane $ux + vy + wz = 0$ may cut the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular lines.
13. Find the semi-vertical angle, the axis and the equation of the right circular cone with vertex at the origin and passing through the straight lines $x/3 = y/6 = z/-2, x/2 = y/2 = z/-1$ and $x/-1 = y/2 = z/2$.
14. Find the equation of the cone generated by the straight lines drawn from the origin to cut the circle through the points $A(1, 0, 0)$, $B(0, 2, 0)$, $C(2, 1, 1)$ and prove that the acute angle between two straight lines in which the plane $x = 2y$ cuts the cone is $\cos^{-1}\sqrt{(5/14)}$.
15. Find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 = 11$ which has its vertex at (2, 4, 1). Further show that the plane $z = 0$ cuts this enveloping cone in a rectangular hyperbola.

1.5 CYLINDER

A cylinder is a surface generated by a straight line which is parallel to a fixed straight line and satisfies one more condition; for instance, it may intersect a fixed curve or touch a given surface.

The straight line in any position is called the *generator*, the fixed line is called the *axis* of the cylinder, and if the generator intersects a fixed curve, then the curve is called the *guiding curve*.

For example, the surface $x^2 + 4z^2 = 4$, as shown in Fig. 1.21, is an elliptic cylinder parallel to the y-axis. Here y-axis is the axis of the cylinder and the guiding curve is the ellipse $x^2 + 4z^2 = 4$, $y = 0$ in the xz-plane. The cylinder extends along the entire y-axis.

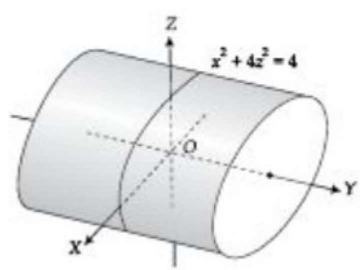


Fig. 1.21

Equation of the cylinder with axis as line $x/l = y/m = z/n$ and the guiding curve as the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0$.

The equation of the axis is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \text{...(1.130)}$$

and of the guiding curve is

$$ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0. \qquad \text{...(1.131)}$$

Let $P_1(x, y, z)$ be any point on the cylinder, then equations of the generator, parallel to the line (1.130) through $P_1(x_1, y_1, z_1)$ are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}. \qquad \text{...(1.132)}$$

Since, it meets the plane $z=0$, we have $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{0-z_1}{n}$ which gives,

$$x=x_1-\frac{l}{n}z_1,\quad y=y_1-\frac{m}{n}z.$$

This point lies on the conic (1.131). Thus,

$$a\left(x-\frac{l}{n}z_1\right)^2+2h\left(x_1-\frac{l}{n}z_1\right)\left(y_1-\frac{m}{n}z_1\right)+b\left(y_1-\frac{m}{n}z_1\right)^2+2g\left(x_1-\frac{l}{n}z_1\right)2f\left(y_1-\frac{m}{n}z_1\right)+c=0.$$

Locus of the point P_1, as obtained by changing $x_1\to x$, $y_1\to y$, $z_1\to z$, is

$$a\left(x-\frac{l}{n}z\right)^2+2h\left(x-\frac{l}{n}z\right)\left(y-\frac{m}{n}z\right)+b\left(y-\frac{m}{n}z\right)^2+2g\left(x-\frac{l}{n}z\right)+2f\left(y-\frac{m}{n}z\right)+c=0$$

or, $a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2+2ng(nx-lz)+2nf(ny-mz)+cn^2=0,$ ...(1.133)
which is the required equation of the cylinder.

If the generators of the cylinder are parallel to z-axis, so that, $l=0$, $m=0$ and $n=1$, the Eq. (1.133) reduces to $ax^2+2hxy+by^2+2gx+2fy+c=0$, an already known equation free from z.

Thus to find the equation of the cylinder whose generators are parallel to z-axis and intersect a given conic, eliminate z from the equation of the conic. Similarly, if the generators are parallel to x-axis, then eliminate x and if the generators are parallel to y-axis, then eliminate y from the equations of the conic to get the equation of the cylinder.

Example 1.47: Find the equation of the cylinder whose generators are parallel to the line $x=\frac{y}{-2}=\frac{z}{3}$ and whose guiding curve is the ellipse $x^2+2y^2=1$, $z=3$.

Solution: Let $P(x_1, y_1, z_1)$ be any point on the cylinder, then the equations of the generator through P are $\frac{x-x_1}{1}=\frac{y-y_1}{-2}=\frac{z-z_1}{3}$.

It meets the plane $z = 3$ in $\frac{x - x_1}{1} = \frac{y - y_1}{-2} = \frac{3 - z_1}{3}$, which gives

$$x = x_1 + \frac{3 - z_1}{3},\ y = y_1 - \frac{2}{3}(3 - z_1).$$

This point lies on the guiding curve, $x^2 + 2y^2 = 1$, therefore,

$$\left(x_1 + \frac{3 - z_1}{3}\right)^2 + 2\left(y_1 - \frac{2(3 - z_1)}{3}\right)^2 = 1.$$

Simplifying, we have $3x_1^2 + 6y_1^2 + 3z_1^2 + 8y_1z_1 - 2z_1x_1 + 6x_1 - 24y_1 - 18z_1 + 24 = 0$.

Replacing $x_1 \to x,\ y_1 \to y,\ z_1 \to z,$ we obtain $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$, as the required equation.

Example 1.48: Obtain the equation of the cylinder whose generators are parallel to x-axis and guiding curve is $x^2 + y^2 + 2z^2 = 12,\ x - y + z = 1$.

Solution: Let any point on the cylinder be $P(x_1, y_1, z_1)$ then the equations of the generator parallel to x-axis, through the point P, are $\frac{x - x_1}{1} = \frac{y - y_1}{0} = \frac{z - z_1}{0} = r$, say.

Which gives $x = x_1 + r,\quad y = y_1,\quad z = z_1$. This lies on the guiding curve $x^2 + y^2 + 2z^2 = 12,\ x - y + z = 1$, hence

$$(x_1 + r)^2 + y_1^2 + 2z_1^2 = 12 \text{ and } (x_1 + r) - y_1 + z_1 = 1.$$

Eliminating r from the above two equations, we get

$$(y_1 - z_1 + 1)^2 + y_1^2 + 2z_1^2 = 12, \quad \text{or} \quad 2y_1^2 + 3z_1^2 - 2y_1z_1 + 2y_1 - 2z_1 - 11 = 0.$$

Replacing $x_1 \to x, y \to y_1, z \to z_1$, we obtain $2y^2 + 3z^2 - 2yz + 2y - 2z - 11 = 0$, as the required equation.

Alternatively, since the generators are parallel to the x-axis, the equation of the cylinder can be obtained by eliminating x from the equation of the guiding curve and hence the cylinder is

$$(y - z + 1)^2 + y^2 + 2z^2 = 12, \quad \text{or} \quad 2y^2 + 3z^2 - 2yz + 2y - 2z - 11 = 0,$$ as obtained above.

Example 1.49: Find the equation of the cylinder whose generators are parallel to z-axis and which pass through the curve of intersection of $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$.

Solution: The guiding curve is $x^2 + y^2 + z^2 = 1,\ x + y + z = 1$.

Since, the generators of the cylinder are parallel to z-axis, the equation of the cylinder will be free from z. This can be obtained by eliminating z from the equation of the guiding curve.

From $x + y + z = 1$, we have $z = 1 - x - y$; substituting in $x^2 + y^2 + z^2 = 1$, we obtain

$$x^2 + y^2 + (1 - x - y)^2 = 1$$

or,
$$x^2 + y^2 + x^2 + y^2 + 1 - 2x - 2y + 2xy = 1$$

or, $x^2 + y^2 + xy - x - y = 0$, as the required equation of the cylinder.

1.5.1 Right Circular Cylinder

A right circular cylinder is a surface generated by a straight line which remains parallel to a fixed straight line and is at a constant distance from it.

The fixed line is called the *axis* and the constant distance is called the *radius* of the right circular cylinder. The section of a right circular cylinder by any plane perpendicular to its axis is a circle.

Equation of the right circular cylinder with axis $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ ***and radius r.***

Let $P(x, y, z)$ be any point on the surface of the cylinder. Draw PM perpendicular to the axis. The $PM = r$, the radius of the cylinder. If A is the point (α, β, γ) on the axis, then, from Fig. 1.22,

$$AP^2 - AM^2 = r^2 \qquad ...(1.134)$$

Also AM = Projection of AP on the line whose direction ratios are l, m, n

$$= \frac{(x-\alpha)l + (y-\beta)m + (z-\gamma)n}{\sqrt{l^2+m^2+n^2}}$$

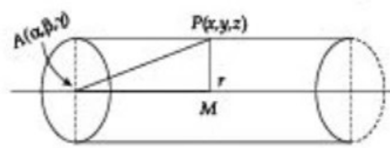


Fig. 1.22

and, $\quad AP^2 = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2.$

Using in (1.134), we obtain

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - \frac{[(x-\alpha)l + (y-\beta)m + (z-\gamma)n]^2}{l^2+m^2+n^2} = r^2$$

or, $$[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2](l^2+m^2+n^2) - [(x-\alpha)l + (y-\beta)m + (z-\gamma)n]^2 = r^2(l^2+m^2+n^2) \qquad ...(1.135)$$

as the equation of the right circular cylinder in the *general form*.

If z-axis is the axis of the cylinder, then the desired equation is obtained from Eq. (1.135) by substituting $l = 0, m = 0, n = 1$ and $\alpha = \beta = \gamma = 0$, we obtain

$$x^2 + y^2 = r^2 \qquad ...(1.136)$$

as the equation of the right circular cylinder in the *standard form*.

Example 1.50: Obtain the equation of the right circular cylinder of radius 2 and axis $\frac{x-1}{2} = y = 3 - z$.

Solution: The axis is $\frac{x-1}{2} = y = 3 - z.$

Let $P(x, y, z)$ be any point on the cylinder and let PM be perpendicular on the axis, then $PM = 2$ is the radius of the cylinder, as shown in Fig. 1.23.

AM = Projection of AP on the axis with direction ratios $<2, 1, -1>$,

$$= \frac{2(x-1)+1(y-0)-1(z-3)}{\sqrt{2^2+1^2+(-1)^2}} = \frac{1}{\sqrt{6}}(2x+y-z+1)$$

Also $\quad AP^2 - AM^2 = PM^2$, thus we have

$$(x-1)^2+(y-0)^2+(z-3)^2-\frac{1}{6}(2x+y-z+1)^2 = 4.$$

Simplifying, we obtain

$$2x^2+5y^2+5z^2-4xy+2yz+4zx-16x-2y-34z+35=0,$$

as the required equation of the right circular cylinder.

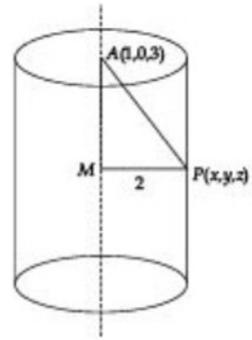


Fig. 1.23

Exercise 1.51: Find the equation of the right circular cylinder described on the circle through the points $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$ as the guiding curve.

Solution: The equation of the plane through the points $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$ is $x/a + y/a + z/a = 1$, or $x + y + z = a$.

It is obvious that the given points A, B, C are equidistant from the origin $O(0, 0, 0)$. Therefore, a sphere with a centre O and radius $OA = 'a'$, passes through A, B and C. Hence the guiding curve is a circle given by

$$x^2+y^2+z^2=a^2 \text{ and } x+y+z=a. \qquad ...(1.137)$$

The perpendicular from the centre $O(0, 0, 0)$ of the sphere on the plane $x + y + z = a$ passes through the centre of this circle. Therefore, it is the axis of the cylinder with direction ratios $<1, 1, 1>$. Also the length of the perpendicular from $O(0, 0, 0)$ to the plane $x + y + z = a$ is $\frac{a}{\sqrt{3}}$.

From Fig. 1.24, radius of the circle (1.137) = radius of the cylinder,

which is given by $\qquad r^2 = a^2 - \frac{a^2}{3} = \frac{2a^2}{3}$ or, $r = a\sqrt{2/3}$.

Now if $P(x, y, z)$ is any point on the cylinder, then

$OP^2 = ON^2 + NP^2$, which gives

$$x^2+y^2+z^2 = \left(\frac{1.x+1.y+1.z}{\sqrt{1+1+1}}\right)^2 + \frac{2}{3}a^2,$$

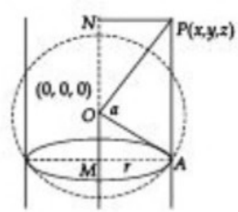


Fig. 1.24

since ON is the projection of OP on the axis with $d.r's$ $<1, 1, 1>$, of the cylinder.

Simplifying we obtain $x^2+y^2+z^2-yz-zx-xy=a^2$, as the required equation of the cylinder.

1.5.2 Enveloping Cylinder

A surface generated by the tangent lines to a surface which are parallel to a given line is known as the enveloping cylinder of that surface.

Equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = a^2$, with generators parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

The given sphere is

$$x^2 + y^2 + z^2 = a^2 \quad ...(1.138)$$

and the line is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \quad ...(1.139)$$

Let $P(x_1, y_1, z_1)$ be any point on a tangent to the sphere (1.138) parallel to line (1.139) then the equation of the tangent is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r, \quad \text{say.} \quad ...(1.140)$$

Any point on the line (1.140) is $(x_1 + lr,\ y_1 + mr,\ z_1 + nr)$. If it lies on the sphere (1.138), then we have

$$(x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 = a^2$$

or, $(l^2 + m^2 + n^2)\, r^2 + 2(lx_1 + my_1 + nz_1)r + (x_1^2 + y_1^2 + x_1^2 - a^2) = 0.$ ...(1.141)

This being quadratic in r gives two roots. The line (1.139) will be tangent to the sphere (1.140) if both the roots of the equation (1.141) are equal, that is, if

$$(lx_1 + my_1 + nz_1)^2 - (l^2 + m^2 + n^2)(x_1^2 + y_1^2 + z_1^2 - a^2) = 0. \quad ...(1.142)$$

The locus of $P(x_1, y_1, z_1)$ is obtained by replacing $x_1 \to x$, $y_1 \to y$ and $z_1 \to z$. Thus, we obtain $(lx^2 + my^2 + nz^2) - (l^2 + m^2 + n^2)(x^2 + y^2 + z^2 - a^2) = 0$, as the required equation of the enveloping cylinder.

Remarks

1. If z-axis is the axis of the cylinder then its d.c. are $< 0, 0, 1 >$. The corresponding equation of the enveloping cylinder is obtained from (1.142) by substituting $l = m = 0$ and $n = 1$. We get

$$z^2 - (x^2 + y^2 + z^2 - a^2) = 0 \text{ or, } x^2 + y^2 = a^2.$$

2. A cylinder is a limiting form of a cone, if the vertex of a cone is taken to infinity, it becomes a cylinder.

Example 1.52: Find equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$ having its generators parallel to the line $x = -2y = 2z$.

Solution: The given sphere is

$$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0 \quad ...(1.143)$$

and the line is

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{1}. \qquad ...(1.144)$$

Let $P(x_1, y_1, z_1)$ be any point on a tangent line to (1.143), parallel to (1.144). Therefore, equations of this tangent are

$$\frac{x-x_1}{-2} = \frac{y-y_1}{1} = \frac{z-z_1}{1} = r. \qquad ...(1.145)$$

Any point on the line (1.144) is (x_1-2r, y_1+r, z_1+r). If it lies on the sphere (1.143), then

$$(x_1-2r)^2 + (y_1+r)^2 + (z_1+r)^2 - 2(y_1+r) - 4(z_1+r) - 11 = 0$$

or, $$6r^2 + 2r(y_1 + z_1 - 2x_1 - 3) + (x_1^2 + y_1^2 + z_1^2 - 2y_1 - 4z_1 - 11) = 0 \qquad ...(1.146)$$

which is quadratic in r.

Since the line (1.145) touches the sphere (1.143) so it meets the sphere only in one point, thus, the two values of r in (1.146) must be equal for which the discriminate must be equal to zero, that is,

$$4(y_1 + z_1 - 2x_1 - 3)^2 - 24\,(x_1^2 + y_1^2 + z_1^2 - 2y_1 - 4z_1 - 11) = 0$$

or, $$6(x_1^2 + y_1^2 + z_1^2 - 2y_1 - 4z_1 - 11) - (y_1 + z_1 - 2x_1 - 3)^2 = 0$$

or, $$2x_1^2 + 5y_1^2 + 5z_1^2 + 4x_1y_1 + 4x_1z_1 - 2y_1z_1 - 12x_1 - 6y_1 - 18z_1 - 75 = 0.$$

Thus locus of $P(x_1, y_1, z_1)$ is, $2x^2 + 5y^2 + 5z^2 + 4xy + 4xz - 2yz - 12x - 6y - 18z - 75 = 0$ which is the required equation of the enveloping cylinder.

EXERCISE 1.7

1. Find the equation of a cylinder whose generating lines have the direction cosines l, m, n and which pass through the circumference of the fixed circle $x^2 + z^2 = a^2$ in the xz-plane.
2. Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{3} = \frac{y}{1} = \frac{z}{2}$ and guiding curve is $x^2 + y^2 = 16$, $z = 0$.
3. Find the equation of the quadratic cylinder whose generators intersect the curve $ax^2 + by^2 = 2z$, $lx + my + nz = p$ and are parallel to z-axis.
4. Find the equation of the cylinder whose guiding curve is $x^2 + z^2 - 4x - 2z + 4 = 0$, $y = 0$ and whose axis contains the point (0, 3, 0). Find also the area of the section of the cylinder by a plane parallel to xz-plane.
5. Find the equation of the cylinder having generators parallel to the line $x = y = z$ and having guiding curve as $x^2 + 2y^2 + 6xy - 2z + 8 = 0$, $x - 2y + 3 = 0$.
6. The radius of a normal section of a right circular cylinder is 2 units; the axis lies along the straight line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}$, find its equation.

7. Find the equation to the right circular cylinder of radius a, whose axis passes through the origin and makes equal angles with the co-ordinate axis.
8. Obtain the equation of the right circular cylinder described on the circle through three points (1, 0, 0), (0, 1, 0), (0, 0, 1) as guiding curve.
9. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = 16$ having generators parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$.
10. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$ being its generators parallel to the line $x = y = z$.
11. Find the equation of the enveloping cylinder of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ whose generators are parallel to the line $x/l = y/m = z/n$.

1.6 QUADRIC SURFACES

A *quadric surface or conicoid is the surface represented by a second degree equation in x, y and z*. The most general form of the second degree equation in x, y and z is

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy + 2Ux + 2Vy + 2Wz + D = 0,$$

which can be simplified by translation and rotation. We will study only some simpler forms of this resulting in surfaces which we normally come across in applications. We have already studied sphere, cylinder and cone which are some examples of quadric surfaces.

1. *Ellipsoid*: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ...(1.147)

It cuts the coordinate axes at $(\pm a, 0, 0)$, $(0, \pm b, 0)$ and $(0, 0, \pm c)$ and lies within the rectangular box defined by $|x| \le a$, $|y| \le b$, $|z| \le c$. The surface is symmetric with respect to each co-ordinate plane, axis and origin, as shown in Fig. 1.25.

The curves in which the three coordinate planes cut the surface are ellipse, for example the yz-plane cuts the surface in

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ when } x = 0.$$

The section cut from the surface by the plane $z = k$, $|k| < c$, is the ellipse $\frac{x^2}{a^2[1-(k/c)^2]} + \frac{y^2}{b^2[1-(k/c)^2]} = 1.$

If any two of the semi axes a, b and c are equal, the surface is an ellipsoid of revolution. For example, when $a = b$, the surface is

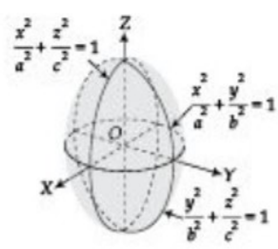


Fig. 1.25

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1, \text{ or } c^2(x^2 + y^2) + a^2(z^2) = a^2c^2.$$

It is obtained by revolving the ellipse $\frac{y^2}{a^2}+\frac{z^2}{c^2}=1$ about the z-axis. When all the three semi-axes are equal, that is, if $a=b=c$, then (1.147) become $x^2+y^2+z^2=a^2$, a sphere with centre (0, 0, 0) and radius a.

2. *Elliptic Paraboloid*: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$ ...(1.148)

This surface is symmetric with respect to the planes $x=0$ and $y=0$ and axis of z. The vertex is at the origin $O(0, 0, 0)$ and it is the only intercept on the axes. Except for this point, the surface lies entirely above or below the xy-plane depending whether c is positive or negative. The section cut by the co-ordinate plane $x=0$ is the parabola $z=\frac{c}{b^2}y^2$; by $y=0$ is again parabola $z=\frac{c}{a^2}x^2$; and by $z=0$ is the point (0, 0, 0), the vertex O, refer to Fig. 1.26.

The section by a plane $z=k$ is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k}{c}.$$

The paraboloid extends upwards or downwards indefinitely. When $a=b$, the surface is *paraboloid of revolution* or *circular paraboloid* given as

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=\frac{z}{c}. \quad ...(1.149)$$

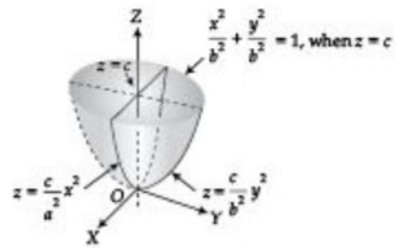


Fig. 1.26

It is obtained by revolving the parabola $\frac{y^2}{a^2}=\frac{z}{c}$ about the z-axis. The cross-section of the surface (1.149) by planes perpendicular to z-axis are circles centered on the z-axis.

Antennas in radio telescopes and microwave radio links are circular paraboloid in shape.

3. *Hyperbolic Paraboloid*: $\frac{y^2}{b^2}-\frac{x^2}{a^2}=\frac{z}{c}$ ...(1.150)

This surface is symmetric with respect to the planes $x=0$ and $y=0$, and the axis of z. The vertex is at the origin $O(0, 0, 0)$.

The section cut by the co-ordinate plane $x=0$ is the parabola $z=\frac{c}{b^2}y^2$ and by the co-ordinate plane $y=0$ it is again parabola $z=-\frac{c}{a^2}x^2$. Both these parabolas are with vertex $O(0, 0, 0)$, and for $c>0$ the first one opens upward and the second one opens downward, refer to Fig. 1.27.

The section of the surface by the plane $z=k$ for $k \lessgtr 0$ is a hyperbola and for $z=0$, the surface is a pair of intersecting lines. Near the origin surface is shaped like a saddle and origin is *saddle point* of the surface.

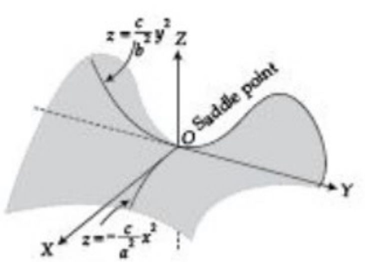


Fig. 1.27

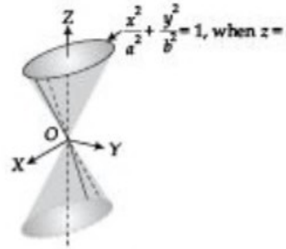


Fig. 1.28

4. ***Elliptic Cone:*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ ...(1.151)

The surface is symmetric with respect to each of the coordinate planes and meets the axis at the origin $O(0, 0, 0)$. The section cut by the coordinate plane $x = 0$ is the pair of straight lines $z=\pm\frac{c}{b}y$ and by the plane $y = 0$ again is the pair of lines $z=\pm\frac{c}{a}x$. But section by the plane $z = 0$ is the point $(0, 0, 0)$, the vertex O, refer to Fig. 1.28.

The sections cut by the planes $z = k \gtrless 0$ are ellipses with centre on the z-axis. In case $a = b$, cone is a right circular cone.

5. ***Hyperboloid of One Sheet:***

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad ...(1.152)$$

The surface, refer to Fig. 1.29, is symmetric with respect to each of the three coordinate planes. The section cut out by the coordinate plane, $x = 0$ is hyperbola $\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$; $y = 0$ is hyperbola $\frac{x^2}{a^2}-\frac{z^2}{c^2}$ $= 1$; and $z = 0$ is ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The sections cut be the planes $z = k$ are ellipse with centre on the z-axis. When $a = b$, Eq. (1.152) gives the surface

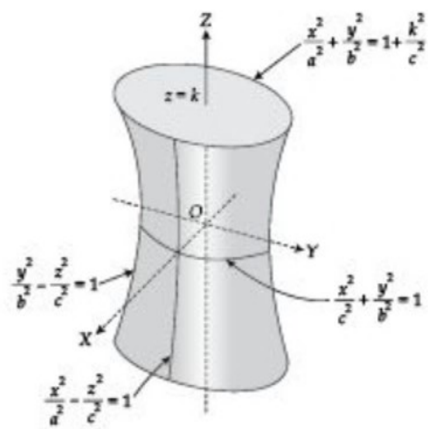


Fig. 1.29

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=1, \qquad ...(1.153)$$

the *hyperboloid of revolution* obtained by revolving the hyperbola $\frac{y^2}{a^2}-\frac{z^2}{c^2}=1$ about the z-axis.

The surface (1.152) is connected in the sense that it is possible to move on the surface from one point to any other point without leaving the surface and that is why it is called hyperboloid of one sheet.

6. Hyperboloid of Two Sheets:

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad ...(1.154)$$

This surface is symmetric with respect to each of the three coordinate planes. Sections cut by the planes parallel to $x = 0$ and $y = 0$ are hyperbolas while the section by the horizontal planes $z = k$ for $|k| \geq c$ are ellipses, refer to Fig. 1.30.

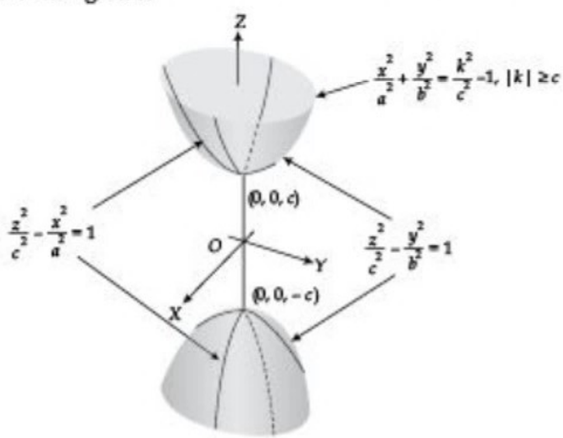


Fig. 1.30

The surface is separated into two portions (sheets) one above the plane $z = c$ and other below the plane $z = -c$ and that justifies the name.

EXERCISE 1.8

1. What surface is represented by $36x^2 + 9y^2 + 4z^2 = 36$? Find the area of the cross-section cut from this surface by the plane $z = k$ as a function of k.
2. What surface is represented by $(y^2/b^2) - (x^2/a^2) = z/c$? Find the vertex and focus of the section of the surface cut by the plane $y = k$.

3. Find the locus of a point whose distance from the line $(x-\alpha)/l = (y-\beta)/m = (z-\lambda)/n$ is constant.
4. Discuss and sketch the following surfaces:
   (a) $x^2+4z^2=9$
   (b) $4x^2+y^2+z^2=9$
   (c) $y^2/2+z^2/3-x^2/2=1$
   (d) $y^2-x^2=2z.$
   (e) $2(x^2+y^2)=3z^2$
   (f) $x^2/2-y^2/2-z^2/4=1$
   (g) $z=x^2+6y^2$
   (h) $y=-2(x^2+z^2).$

ANSWERS

Exercise 1.1 (p. 17)

1. $2(\hat{i}+\hat{j}-\hat{k})$
2. $6\,\overrightarrow{AO}$
3. Collinear
4. 3464.10 J
5. $\frac{i+j}{\sqrt{2}},\ 3\sqrt{2}$
6. $\frac{14}{3}(i+2j-k)+\frac{2}{3}(5\hat{i}-8\hat{j}-11\hat{k})$
7. $\frac{55}{6}(k-1),\ k=1$
9. $29\,\hat{i}+37\hat{j}+31\hat{k},\ 12\hat{i}+8\hat{j}+52\hat{k}$
13. 6
14. $9\sqrt{293}$

Exercise 1.2 (p. 25)

1. No,
2. $x=1+4\lambda, y=-1-3\lambda, z=4+7\lambda$
3. $x=\lambda, y=-7+2\lambda, z=2\lambda$
4. $\frac{3x+1}{-3}=\frac{3y+2}{6}=\frac{z}{-1}$
5. $2x=y=-2z, 2x=-y=2z$ 6. 1
7. $\frac{x+13/15}{-2}=\frac{y-23/15}{7}=\frac{z}{15}$
8. The point of intersection is $(1,3,2)$; $17x-47y-24z+172=0$
9. $\frac{x-2}{7}=\frac{y-3}{4}=\frac{x-1}{5}$
10. $6x+7y+16z=98,\ \frac{120}{\sqrt{341}}$
11. $(7/3, 4/3, 13/3)$
12. $\frac{120}{\sqrt{341}}$
15. $\frac{x-2}{3}=\frac{y-1}{-1}=\frac{z-1}{1}.$

Exercise 1.3 (p. 30)

1. $x^2+y^2+z^2-4x+6y-2z+5=0$; $(2,-3,1)$; 3
2. $x^2+y^2+z^2-2x+6y-4z+10=0$

5. $\dfrac{x^2+y^2+z^2}{a^2+b^2+c^2}-\dfrac{x}{a}-\dfrac{y}{b}-\dfrac{z}{c}=0$

6. $x^2+y^2+z^2=k^2$

7. $x^2+y^2+z^2-4x-6y+2z+5=0$

10. $3\sqrt{3}-4,\quad 3\sqrt{3}+4$

Exercise 1.4 (p. 35)

1. $\left(\dfrac{-7}{3},\dfrac{-5}{2},\dfrac{-2}{3}\right)$; 3

2. $x^2+y^2+z^2-2x-2y-5z+2=0,\quad x^2+y^2+z^2-4x-4y-10z+8=0$

3. $x^2+y^2+z^2-2\sqrt{2}\,(x+y+z)+8=0$

4. $x^2+y^2+z^2-10y-10z-31=0$

6. $x^2+y^2+z^2-2x-2y+2z+2=0$

7. $x^2+y^2+z^2+2x+4y+6z-11=0,\ 5(x^2+y^2+z^2)-4x-8y-12z-13=0$

8. $2x+2y-z-2=0,\quad x+2y-2z+14=0$

9. $2(x^2+y^2+z^2)+2\sqrt{2}\,r\,(\pm x\pm y\pm z)+r^2=0,\ 8$

10. $81(x^2+y^2+z^2)-126(x+y+z)+98=0$

11. $(2,1,2),\ \sqrt{7},\ \dfrac{14}{3}\pi$ sq. units.

Exercise 1.5 (p. 39)

1. $3x+y+z+6=0$

2. $\cos^{-1}\left(\dfrac{2}{3}\right)$

3. $x^2+y^2+z^2+2x-2y+4z-3=0$

5. $x^2+y^2+z^2+7y-8z+24=0$

6. $x^2+y^2+z^2+7x+9y-11z-1=0$

7. $\dfrac{x-47/10}{8}=\dfrac{y-31/10}{4}=\dfrac{z}{-5}$

8. $2x+2y-z+10=0$ and $2x+2y-z-8=0$; $(0,-3,4)$; $(4,1,2)$.

Exercise 1.6 (p. 47)

1. $27x^2+32y^2+72(xy+yz+zx)=0$

2. $x^2+y^2-3z^2-2x-2y+6z-1=0$

3. $5yz+8zx-3xy=0$

4. $5x^2+3y^2+z^2-6yz-4zx-2xy+6x+8y+10z-26=0$

5. $4x^2+4y^2-9z^2+36z-36=0$

6. $\cos^{-1}4/\sqrt{41},\quad 25x^2-16y^2-16z^2=0$

7. $x/-1=y/4=z/2;\quad x/-1=y/2=z/0$

8. $yz+zx+xy-x-2y-z+1=0$

9. $(al + bm + cn)^2(x^2 + y^2 + z^2) = (ax + by + cz)^2(l^2 + m^2 + n^2)$
10. $8x^2 - 4y^2 - 4z^2 + 5xy + 5zx + yz = 0$
11. $xy - yz + zx = 0$
12. $(b + c)u^2 + (c + a)v^2 + (a + b)\,w^2 = 0$
13. $\cos^{-1}\dfrac{1}{\sqrt{3}};\quad x = y = z;\quad xy + yz + zx = 0$
14. $8z^2 + 4yz - zx - 5xy = 0$
15. $6x^2 - 6y^2 + 9z^2 - 4xz - 8zy - 16yx + 44x + 88y + 22z - 231 = 0;$
    $6x^2 - 6y^2 - 16yx + 44x + 88y - 231 = 0.$

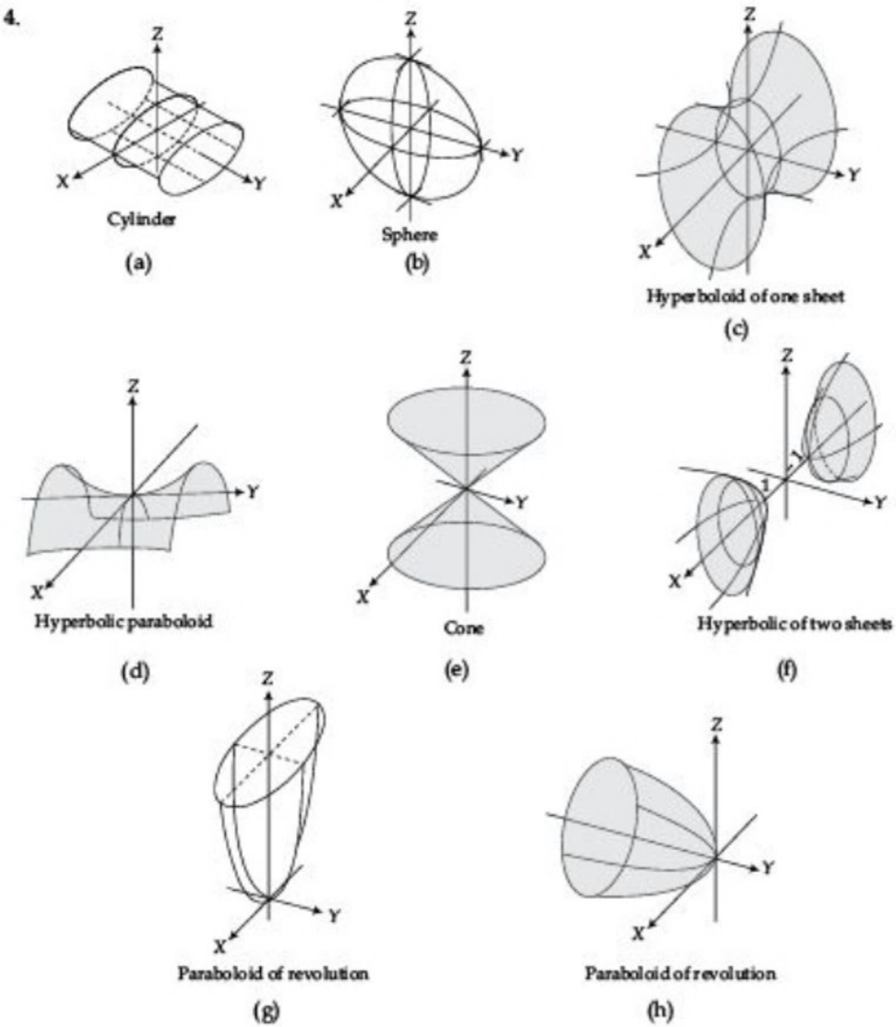
Exercise 1.7 (p. 54)

1. $(mx - ly)^2 + (mz - ny)^2 = a^2m^2$
2. $2x^2 + 2y^2 + 5z^2 - 6xz - 2yz - 32 = 0$
3. $n(ax^2 + by^2) + 2lx + 2\,my - 2p = 0$
4. $9x^2 + 5y^2 + 9z^2 + 12xy + 6yz - 36x - 30y - 18z + 36 = 0$; π units
5. $18x^2 + 18y^2 - 36xy + 76x - 74y - 2z + 83 = 0$
6. $26x^2 + 29y^2 + 5z^2 + 4xy + 10yz - 20zx + 150y + 30z + 75 = 0$
7. $2(x^2 + y^2 + z^2 - xy - yz - zx) = 3a^2$
8. $x^2 + y^2 + z^2 - xy - yz - zx - 1 = 0$
9. $13x^2 + 10y^2 + 5z^2 - 4xy - 12yz - 6zx - 224 = 0$
10. $x^2 + y^2 + z^2 - xy - yz - zx - 2x + 7y + z - 2 = 0$
11. $\left(\dfrac{lx}{a^2} + \dfrac{my}{b^2} + \dfrac{nz}{c^2}\right)^2 = \left(\dfrac{l^2}{a^2} + \dfrac{m^2}{b^2} + \dfrac{n^2}{c^2}\right)\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1\right).$

Exercise 1.8 (p. 58)

1. ellipsoid, $\dfrac{2\pi(9 - k^2)}{9}$
2. hyperbolic paraboloid. $(0, k, ck^2/b^2)$, $(0, k, c\,(k^2/b^2) - a^2/4c)$
3. A right circular cylinder.

4.



Cylinder
(a)

Sphere
(b)

Hyperboloid of one sheet
(c)

Hyperbolic paraboloid
(d)

Cone
(e)

Hyperbolic of two sheets
(f)

Paraboloid of revolution
(g)

Paraboloid of revolution
(h)

2 Matrices, Determinants and Eigenvalue Problems

CHAPTER

There are many problems of interest in science and engineering where the solution oftenly leads to a system of linear algebraic simultaneous equations. Matrices are very elegant and powerful tool to analyze such a system as a single entity. However, the field of applications of matrices and determinants comprises much more than the solution of linear algebraic equations. The study of matrices includes linear transformations and eigenvalue problems also. With the advancement of digital computers, matrices find applications in almost all branches of science and engineering like electrical networks, graph theory, computer graphics, optimization problems, system of differential equations and stochastic processes just to name a few.

2.1 MATRICES: SOME BASIC DEFINITIONS

Matrix. An $m \times n$ matrix is an array of mn entries called elements arranged in m horizontal and n vertical columns in the form

$$\mathbf{A} = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \ldots(2.1)$$

We say that the matrix (2.1) is of order $m \times n$ (m by n) and the element a_{ij} ($i = 1, 2, \ldots m$, $j = 1, 2, \ldots n$), common to the ith row and the jth column, is called the *general element*. Normally, the elements are numbers real or complex, although they may occasionally be other objects such as differential operators or even matrices itself. The matrices are generally denoted by the boldface upper case letters like **A**, **B**, **C**, etc.

If all the elements of a matrix are real it is called a *real matrix*, whereas if one or more of its elements are complex it is called a *complex matrix*.

Square Matrix. If $m = n$, then **A** is called a *square matrix* of order n, and in that case diagonal containing the entries $a_{11}, a_{22}, \ldots, a_{nn}$ is called the *principal diagonal* of **A**. The sum of the diagonal elements of a square matrix is called the *trace* of the matrix **A**. A matrix which is not square is called a *rectangular* matrix.

Triangular matrices. A square matrix $A = [a_{ij}]$ is called a *lower triangular matrix* if $a_{ij} = 0$, whenever $i < j$, that is, if all elements above the principal diagonal are zero, and an *upper triangular matrix* if $a_{ij} = 0$, whenever $i > j$, that is, if all the elements below the principal diagonal are zeros. The matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

respectively are the upper triangular and the lower triangular matrices. If a matrix is upper triangular or lower triangular it is said to be *triangular*.

Row and Column Matrices. A matrix having a single row is called a *row matrix* or *row vector* and a matrix having a single column is called a *column matrix* or *column vector*.

Null matrix. A matrix **A** of order $m \times n$ in which all the elements are zero is called a *null matrix* and is normally denoted by **O**.

Diagonal matrix. A square matrix **A** in which all the off-diagonal elements a_{ij} $(i \neq j)$ are zero is called a *diagonal matrix*, normally denoted by **D**. Thus

$$\mathbf{D} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

is a diagonal matrix of order n. Sometimes it is written as $\mathbf{D} = \text{diag}\,[a_{11}, a_{22}, \ldots, a_{nn}]$.

Scalar matrix. A diagonal matrix of order n with all its diagonal elements equal, that is $a_{ii} = d$, $i = 1, 2, \ldots n$, is called a *scalar matrix* of order n. Thus $\mathbf{D} = \text{diag}\,(d, d, \ldots, d)$ is a scalar matrix.

Unit matrix or Identity matrix. If all the diagonal elements are equal to 1, then the matrix $\mathbf{D} = \text{diag}\,(1, 1, \ldots, 1)$ is called a *unit matrix* or an *identity matrix* of order n, and is denoted by $\mathbf{I}_n$.

For example, $\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Equal matrices. Two matrices $\mathbf{A} = [a_{ij}]$ of order $m \times n$ and $\mathbf{B} = [b_{ij}]$ of order $p \times q$ are *equal*, written as $\mathbf{A} = \mathbf{B}$, if, and only if **A** and **B** are of the same order, so that $m = p$ and $n = q$ and $a_{ij} = b_{ij}$, for each i and j.

2.2 MATRIX ALGEBRA

In case of matrices we define the following basic operations.

I. **Matrix addition and subtraction**
II. **Scalar multiplication**
III. **Matrix multiplication**

I. Matrix addition and subtraction. If $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are any two matrices of the same order, say $m \times n$, then their sum, denoted by $\mathbf{A} + \mathbf{B}$, is defined as

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$$

and is itself an $m \times n$ matrix.

If $\mathbf{A}$ and $\mathbf{B}$ are of the same order, they are said to be *conformable for addition*, otherwise $\mathbf{A} + \mathbf{B}$ is not defined.

Similarly, the difference of two matrices $\mathbf{A}$ and $\mathbf{B}$ each of the same order $m \times n$, denoted by $\mathbf{A} - \mathbf{B}$, is defined as

$$\mathbf{A} - \mathbf{B} = [a_{ij} - b_{ij}]$$

and is again an $m \times n$ matrix.

II. Scalar multiplication. If $\mathbf{A} = [a_{ij}]$ is a matrix of order $m \times n$ and λ is any scalar, then the multiplication of the matrix, $\mathbf{A}$ by a scalar λ, denoted by $\lambda\mathbf{A}$, is defined as

$$\lambda\mathbf{A} = [\lambda a_{ij}]$$

and is itself a matrix of order $m \times n$.

We do not distinguish between $\lambda\mathbf{A}$ and $\mathbf{A}\lambda$ and further, we define $-\mathbf{A} = (-1)\mathbf{A}$ as the *negative of the matrix* $\mathbf{A}$.

For example, if $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 5 & 4 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, then

$$3\mathbf{A} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 6 \\ 3 & 6 & 15 \end{bmatrix}, \mathbf{A} + \mathbf{B} = \begin{bmatrix} 6 & 6 & 4 \\ 2 & 2 & 2 \\ 2 & 4 & 8 \end{bmatrix}, \text{and } \mathbf{A} - \mathbf{B} = \begin{bmatrix} -4 & -2 & 2 \\ -2 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Properties of matrix addition and scalar multiplication. If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $m \times n$ matrices, $\mathbf{O}$ is an $m \times n$ null matrix and α, β are scalars, real or complex, then

(i) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ *(commutativity)*
(ii) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ *(associativity)*
(iii) $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$ *(additive identity)*
(iv) $\mathbf{A} + (-\mathbf{A}) = \mathbf{O}$ *(additive inverse)*
(v) $\alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A}$
(vi) $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$
(vii) $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$

The proofs follow directly from the definitions.

III. Matrix Multiplication. The product of the two matrices **A** and **B**, denoted by **AB**, is defined only when the number of columns in **A** is equal to the number of rows in **B**. Two such matrices are said to be *conformable for multiplication*.

If $\mathbf{A} = [a_{ij}]$ is a matrix of order $m \times n$ and $\mathbf{B} = [b_{ij}]$ is a matrix of order $n \times p$, then their product **AB** is defined as $\mathbf{AB} = \left[\sum_{k=1}^{n} a_{ik}\, b_{kj}\right]$

which itself is a matrix of order $m \times p$ and, if we denote $\mathbf{AB} = \mathbf{C} = [c_{ij}]$, then $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$.

For example, if $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$, then

$$\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}_{2\times 2}$$

In the product **AB**, **B** is said to be *pre-multiplied* by **A**, or **A** is said to be *post-multiplied* by **B**.

If **A** is square matrix, then the product **AA** is defined as $\mathbf{A}^2$. Similarly we can define the higher powers of **A**. If $\mathbf{A}^2 = \mathbf{A}$, then the matrix **A** is called *idempotent*.

Properties of matrix multiplication. If **A**, **B**, **C**, **I** and **O** are matrices of suitable orders and α, β are scalars, then

(a) $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$
(b) $\mathbf{OA} = \mathbf{AO} = \mathbf{O}$
(c) $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
(d) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
(e) $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
(f) $\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\,\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$
(g) $\mathbf{A}(\alpha\mathbf{B} + \beta\mathbf{C}) = \alpha\mathbf{AB} + \beta\mathbf{AC}$

The proofs follow directly from the definitions. In addition to this, we note that

(h) $\mathbf{AB} \neq \mathbf{BA}$, that is, in general, *matrix multiplication is not commutative*.
(i) $\mathbf{AB} = \mathbf{AC}$, does not necessarily imply that $\mathbf{B} = \mathbf{C}$
(j) $\mathbf{AB} = \mathbf{O}$, does not necessarily imply that $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.

For example, if $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then **AB** and **BA** both are defined and we can see easily that $\mathbf{AB} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$ and $\mathbf{BA} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$; and thus $\mathbf{AB} \neq \mathbf{BA}$.

Also, if $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$, then $\mathbf{AB} = \mathbf{O}$ but neither $\mathbf{A} = \mathbf{O}$, nor $\mathbf{B} = \mathbf{O}$.

We note that if for two matrices **A** and **B**, both the product matrices **AB** and **BA** are defined and if matrix **A** is of order $p \times q$, then matrix **B** must be of the order $q \times p$.

2.2.1 Partitioning of Matrices

The matrices encountered in modern applications may be of quite large order and such large matrices create special computational problems. It is often advantageous to work instead with a number of smaller matrices through the use of partitioning.

Any matrix **A** may be partitioned into a number of submatrices called blocks, by vertical lines that extend from bottom to top and horizontals lines that extend from left to right. There is more than one way in which a matrix can be partitioned. For example, consider a matrix **A** of order 3×3 given as

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}.$$

One way in which this matrix can be partitioned is as follows:

$$\mathbf{A} = \left[\begin{array}{cc|c} 3 & -1 & 2 \\ \hline 1 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right]$$

This can now be written in block matrix form as $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, where the submatrices are

$$\mathbf{A}_{11} = [3, -1], \mathbf{A}_{12} = [2], \mathbf{A}_{21} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \text{ and } \mathbf{A}_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The addition and scalar multiplication of block matrices follow the same rules as those for ordinary matrices but care must be exercised when multiplying block matrices. Consider the partition of the matrix **A** defined above and the matrix **B** of order 3×4 given as

$$\mathbf{B} = \left[\begin{array}{c|ccc} 1 & 2 & 2 & 1 \\ 3 & 1 & 1 & 0 \\ \hline 2 & 3 & 0 & 2 \end{array}\right]$$

which are conformable for the product **AB**, which itself is a 3×4 matrix. If **B** is partitioned as indicated above, then $\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$ where the submatrices are

$$\mathbf{B}_{11} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{B}_{12} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \mathbf{B}_{21} = [2], \text{and } \mathbf{B}_{22} = [3, 0\ 2].$$

Using the definition of the matrix multiplication, we may write the matrix product in the condensed form as

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}$$

where the partitioned matrices have been multiplied as though their elements were ordinary numbers. This result holds because of appropriate partitioning, such that each product of submatrices is conformable for multiplication and the matrices sums are conformable for additions. We can check that

$$\mathbf{A}_{11}\,\mathbf{B}_{11} + \mathbf{A}_{12}\,\mathbf{B}_{21} = [4], \mathbf{A}_{11}\,\mathbf{B}_{12} + \mathbf{A}_{12}\,\mathbf{B}_{22} = [11 \quad 5 \quad 7]$$

$$\mathbf{A}_{21}\,\mathbf{B}_{11} + \mathbf{A}_{22}\,\mathbf{B}_{21} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \mathbf{A}_{21}\,\mathbf{B}_{12} + \mathbf{A}_{22}\,\mathbf{B}_{22} = \begin{bmatrix} 4 & 4 & 1 \\ 5 & 5 & 2 \end{bmatrix},$$

and so $$\mathbf{AB} = \begin{bmatrix} [4] & [11 \quad 5 \quad 7] \\ \begin{bmatrix} 7 \\ 5 \end{bmatrix} & \begin{bmatrix} 4 & 4 & 1 \\ 5 & 5 & 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 11 & 5 & 7 \\ 7 & 4 & 4 & 1 \\ 5 & 5 & 5 & 2 \end{bmatrix}$$

The result can be confirmed by direct matrix multiplication.

Matrix partitioning is particularly useful in applying the multiplication of matrices if one of these can be partitioned in such a way that some of its submatrices are null matrices. Then the computational time is drastically reduced.

2.3 SPECIAL MATRICES

In this section we shall discuss some special matrices like *symmetric, skew symmetric, orthogonal* in case of matrices over real, and *Hermitian, skew-Hermitian* and *unitary* in case of matrices over complex number system. Also we shall introduce the *transpose* and *conjugate* of a matrix.

2.3.1 Transpose of a Matrix

If **A** is a matrix of order $m \times n$, then the *transpose* of **A**, denoted by **A**′ or $\mathbf{A}^T$, is obtained by interchanging the rows and columns of the matrix **A**. That is, if $\mathbf{A} = [a_{ij}]_{m \times n}$ is a matrix of order $m \times n$, then the transpose of **A** is the $n \times m$ matrix, $\mathbf{A}^T = [a_{ji}]_{n \times m}$.

For example, if $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 3 \\ 4 & 6 & 7 \end{bmatrix}$, then $\mathbf{A}^T = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 6 \\ 1 & 3 & 7 \end{bmatrix}$

Properties of the Transpose. The basic properties of the transpose are

(a) $(\mathbf{A}^T)^T = \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(c) $(\alpha\mathbf{A})^T = \alpha\mathbf{A}^T$

(d) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$,

The proofs follow directly from the definitions.

In general, if $\mathbf{A}_1 \mathbf{A}_2 \ldots \mathbf{A}_k$ is defined then $(\mathbf{A}_1 \mathbf{A}_2 \ldots \mathbf{A}_k)^T = \mathbf{A}_k^T \mathbf{A}_{k-1}^T \ldots \mathbf{A}_1^T$.

2.3.2 Symmetric and Skew-symmetric Matrices

A matrix $\mathbf{A} = [a_{ij}]$ is

(a) *Symmetric*, if $a_{ij} = a_{ji}$ for all i and j, that is $\mathbf{A} = \mathbf{A}^T$.

(b) *Skew-symmetric* or *antisymmetric*, if $a_{ij} = -a_{ji}$ for all i and j, that is, $\mathbf{A} = -\mathbf{A}^T$.

We note that for either of these properties to apply $\mathbf{A}$ must be square, otherwise $\mathbf{A}$ and $\mathbf{A}^T$ will be of different orders. Further, for $\mathbf{A}$ to be skew-symmetric all of its leading diagonal elements must be zero, since $a_{ij} = -a_{ji}$ for $i = j$, gives $2a_{ii} = 0$, that is, $a_{ii} = 0$ for all i. For example, if

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix}$$

Then, $\mathbf{A}$ is symmetric and $\mathbf{B}$ is skew-symmetric.

Further, *every square matrix A can be expressed as the sum of a symmetric and a skew-symmetric matrix*, for, the square matrix $\mathbf{A}$ can be written as $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$, and we can verify that the matrix $\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ is symmetric and the matrix $\mathbf{C} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$ is skew-symmetric.

2.3.3 Orthogonal Matrix

A square matrix $\mathbf{A} = [a_{ij}]$ of order n is *orthogonal* matrix, if $\mathbf{AA}^T = \mathbf{I}_n$

For example, the matrix $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal, since

$$\mathbf{AA}^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$$

Also we note that *if* **A** *and* **B** *are orthogonal matrices, then the product matrices* **AB** *and* **BA** *are also orthogonal matrices.*

2.3.4 Conjugate of a Matrix

If **A** is a complex matrix of order $m \times n$, then the *conjugate of* **A**, denoted by $\overline{\mathbf{A}}$, is obtained by replacing the elements with their corresponding complex conjugates. That is, if $\mathbf{A} = [a_{ij}]$ is a matrix of order $m \times n$, then the conjugate of **A** is the matrix $\overline{\mathbf{A}} = [\bar{a}_{ij}]$, where $\bar{a}_{ij}$ is the complex conjugate of a_{ij}.

We note that *in case of real matrices,* **A** *and its conjugate* $\overline{\mathbf{A}}$ *are the same.*

2.3.5 Hermitian and Skew-Hermitian Matrices

A complex matrix $\mathbf{A} = [a_{ij}]$ is

(a) *Hermitian*, if $a_{ij} = \bar{a}_{ji}$ for all i and j, that is, if $\mathbf{A} = (\overline{\mathbf{A}})^{\mathbf{T}}$

(b) *Skew-Hermitian*, if $a_{ij} = -\bar{a}_{ji}$ for all i and j, that is, if $\mathbf{A} = -(\overline{\mathbf{A}})^{\mathbf{T}}$.

Sometimes Hermitian matrix is denoted by $\mathbf{A}^{\mathbf{H}}$ or $\mathbf{A}^{\theta}$.

We note that for either of these properties to apply **A** must be square, otherwise **A** and $\mathbf{A}^{\mathbf{H}}$ will be of different orders. Further, for **A** to be Hermitian all of its leading diagonal elements must be real and for **A** to be skew-Hermitian all of its leading diagonal elements must be either zero or purely imaginary.

Also, every complex square matrix **A** *can be expressed as the sum of a Hermitian and a skew-Hermitian matrix*, for, the matrix **A** can be written as $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathbf{H}}) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathbf{H}})$ and we can verify that the matrix $\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathbf{H}})$ is Hermitian and the matrix $\mathbf{C} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathbf{H}})$ is skew-Hermitian.

2.3.6 Unitary Matrix

A complex square matrix $\mathbf{A} = [a_{ij}]$ of order n is *unitary matrix*, if $\mathbf{AA}^{\mathbf{H}} = \mathbf{I}_n$

For example the matrix $\mathbf{A} = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is unitary, since

$$\mathbf{AA}^{\mathbf{H}} = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$$

Also we note that *if* **A** *and* **B** *are unitary matrices, then the product matrices* **AB** *and* **BA** *are also unitary matrices.*

Example 2.1: If $\mathbf{A}=\begin{bmatrix}1 & 3 & 0\\ -1 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$, and $\mathbf{B}=\begin{bmatrix}2 & 3 & 4\\ 1 & 2 & 3\\ -1 & 1 & 2\end{bmatrix}$, compute **AB** and **BA** and show that $\mathbf{AB}\neq\mathbf{BA}$.

Solution: Considering the rows of **A** and columns of **B**, we have

$$\mathbf{AB}=\begin{bmatrix}1.2+3.1+0.-1 & 1.3+3.2+0.1 & 1.4+3.3+0.2\\ -1.2+2.1+1.-1 & -1.3+2.2+1.1 & -1.4+2.3+1.2\\ 0.2+0.1+2.-1 & 0.3+0.3+2.1 & 0.4+0.3+2.2\end{bmatrix}=\begin{bmatrix}5 & 9 & 13\\ -1 & 2 & 4\\ -2 & 2 & 4\end{bmatrix}$$

Again considering the rows of **B** and columns of **A**, we have

$$\mathbf{BA}=\begin{bmatrix}2.1+3.-1+4.0 & 2.3+3.2+4.0 & 2.0+3.1+4.2\\ 1.1+2.-1+3.0 & 1.3+2.2+3.0 & 1.0+2.1+3.2\\ -1.1+1.-1+2.0 & -1.3+1.2+2.0 & -1.0+1.1+2.2\end{bmatrix}=\begin{bmatrix}-1 & 12 & 11\\ -1 & 7 & 8\\ -2 & -1 & 5\end{bmatrix},$$

which is not equal to **AB**.

Example 2.2: If $\mathbf{A}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, show that $\mathbf{A}^2-5\mathbf{A}=2\mathbf{I}$, where **I** is the unit matrix of order 2. Hence, determine $\mathbf{A}^4$.

Solution: We have,

$$\mathbf{A}^2=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}, \text{ and } 5\mathbf{A}=5\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}5 & 10\\ 15 & 20\end{bmatrix}$$

Hence, $\mathbf{A}^2-5\mathbf{A}=\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}-\begin{bmatrix}5 & 10\\ 15 & 20\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=2\mathbf{I}.$

From the above result $\mathbf{A}^2=5\mathbf{A}+2\mathbf{I}$, hence

$$\mathbf{A}^4=\mathbf{A}^2\mathbf{A}^2=(5\mathbf{A}+2\mathbf{I})(5\mathbf{A}+2\mathbf{I})=25\mathbf{A}^2+10\mathbf{AI}+10\mathbf{IA}+4\mathbf{I}^2$$

$$=25\mathbf{A}^2+20\mathbf{A}+4\mathbf{I}, \text{ since } \mathbf{AI}=\mathbf{IA}=\mathbf{A} \text{ and } \mathbf{I}^2=\mathbf{I}$$

$$=25\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}+20\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}175 & 250\\ 375 & 550\end{bmatrix}+\begin{bmatrix}20 & 40\\ 60 & 80\end{bmatrix}+\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}=\begin{bmatrix}199 & 290\\ 405 & 634\end{bmatrix}$$

Example 2.3: If **A**, **B**, **C** are three matrices such that

$$\mathbf{A} = [x\ y\ z],\ \mathbf{B} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ then find } \mathbf{ABC}.$$

Solution: Since associative law holds for matrices multiplication, therefore **ABC** can be written as A(BC), or (AB)C.

$$\text{Now, } \mathbf{BC} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + hy + gz \\ hx + by + fz \\ gx + fy + cz \end{bmatrix}, \text{ therefore,}$$

$$\mathbf{A(BC)} = [x\ y\ z] \begin{bmatrix} ax + hy + gz \\ hx + by + fz \\ gx + fy + cz \end{bmatrix}$$

$$= [x(ax + hy + gz) + y(hx + by + fz) + z(gx + fy + cz)]$$

$$= [ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz].$$

Example 2.4: Express the matrix $\begin{bmatrix} 3 & 1 & -2 \\ 2 & 1 & 7 \\ -4 & 5 & 3 \end{bmatrix}$ as the sum of two matrices, one symmetric and one skew-symmetric.

Solution:

$$\text{Let } \mathbf{A} = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 1 & 7 \\ -4 & 5 & 3 \end{bmatrix}, \text{ then } \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 3 & 2 & -4 \\ 1 & 1 & 5 \\ -2 & 7 & 3 \end{bmatrix},$$

Consider

$$\frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathbf{T}}) = \begin{bmatrix} 3 & 3/2 & -3 \\ 3/2 & 1 & 6 \\ -3 & 6 & 3 \end{bmatrix} = \mathbf{B}, \text{ and } \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathbf{T}}) = \begin{bmatrix} 0 & -1/2 & 1 \\ 1/2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \mathbf{C}, \text{ say.}$$

$$\text{Then,} \quad \mathbf{B} + \mathbf{C} = \begin{bmatrix} 3 & 3/2 & -3 \\ 3/2 & 1 & 6 \\ -3 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & 1 \\ 1/2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 1 & 7 \\ -4 & 5 & 3 \end{bmatrix} = \mathbf{A},$$

here we can check that the matrix **B** is symmetric, that is, $\mathbf{B}^{\mathbf{T}} = \mathbf{B}$ and the matrix **C** is skew-symmetric, that is, $\mathbf{C}^{\mathbf{T}} = -\mathbf{C}$.

Example 2.5: If the matrix $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal, then find the values of a, b and c.

Solution:

Let
$$\mathbf{A} = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}, \text{ then } \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}, \text{ and}$$

$$\mathbf{AA}^{\mathrm{T}} = \begin{bmatrix} 0 & 2b & c \\ 0 & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix}$$

If $\mathbf{A}$ is orthogonal, then $\mathbf{AA}^{\mathrm{T}} = \mathbf{I}$; thus

$$\begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we obtain

$$4b^2+c^2 = 1, \quad 2b^2-c^2 = 0, \quad a^2+b^2+c^2 = 1.$$

Solving for a, b and c we get, $a = \pm\dfrac{1}{\sqrt{2}}$, $b = \pm\dfrac{1}{\sqrt{6}}$, and $c = \pm\dfrac{1}{\sqrt{3}}$.

Example 2.6: Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

Solution: A square matrix $\mathbf{A}$ such that $\mathbf{A}^p = 0$, but $\mathbf{A}^{p-1} \neq 0$, p being positive integer, is called *nilpotent matrix of order p*. We have

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \neq \mathbf{0}, \text{ and}$$

$$\mathbf{A}^3 = \mathbf{A}.\mathbf{A}^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0+3-3 & 0+3-3 & 0+9-9 \\ 0+6-6 & 0+6-6 & 0+18-18 \\ 0-3+3 & 0-3+3 & 0-9+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$$

Thus $\mathbf{A}$ is nilpotent of order 3.

EXERCISE 2.1

1. If $A=\begin{bmatrix}1 & -2 & 1 & 7 & -9\\ 8 & 2 & -5 & 0 & 0\end{bmatrix}$, $B=\begin{bmatrix}-5 & 1 & 8 & 21 & 7\\ 12 & -6 & -2 & -1 & 9\end{bmatrix}$, find $4A+8B$.

2. If $A=\begin{bmatrix}-4 & 6 & 2\\ -2 & -2 & 3\\ 1 & 1 & 8\end{bmatrix}$, $B=\begin{bmatrix}-2 & 4 & 6 & 12 & 5\\ -3 & -3 & 1 & 1 & 4\\ 0 & 0 & 1 & 6 & -9\end{bmatrix}$, determine which of **AB** and **BA** is defined.
   Also carry out that product.

3. If $A=\begin{bmatrix}1 & 3 & 2\\ 2 & 0 & -1\\ 1 & 2 & 3\end{bmatrix}$, prove that $A^3-4A^2-3A+11I=0$

4. Express the matrix $\begin{bmatrix}1 & 2 & 4\\ 6 & 8 & 1\\ 3 & 5 & 7\end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

5. If $A=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$, show that $A^n=\begin{bmatrix}\cos n\alpha & \sin n\alpha\\ -\sin n\alpha & \cos n\alpha\end{bmatrix}$

6. Let **A** and **B** be $n\times n$ symmetric matrices.
   (a) Give an example to show that **AB** need not be symmetric.
   (b) Prove that **AB** is symmetric if, and only if $AB=BA$.

7. If **A** and **B** are square matrices of the same order and **A** is symmetric, then show that $B'AB$ is also symmetric.

8. If **A** and **B** are symmetric matrices, then prove that $AB-BA$ is skew-symmetric.

9. If $A=\begin{bmatrix}1 & 2 & -1\\ 3 & 0 & 2\\ 4 & 5 & 0\end{bmatrix}$, $B=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 1 & 3\end{bmatrix}$, verify that $(AB)^T=B^TA^T$

10. Show that the following matrices are orthogonal

    (a) $\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$ (b) $\begin{bmatrix}-2/3 & 1/3 & 2/3\\ 2/3 & 2/3 & 1/3\\ 1/3 & -2/3 & 2/3\end{bmatrix}$

11. If **A** is Hermitian, is iA also Hermitian? Explain. What about A^2? Is it Hermitian?

12. Check which of the following matrices are Hermitian

(a) $\begin{bmatrix} 3 & 1+4i \\ 1-4i & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2+i & 0 & 3-5i \\ 7 & 1 & 4i \\ 2 & i & 3 \end{bmatrix}$

13. Show that the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ is nilpotent and find its order.

14. Show that the matrix $\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} & 0 \\ \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unitary matrix.

15. Write the matrix $\begin{bmatrix} 1+i & 3+i & 3+2i \\ -1+3i & 2 & 4+i \\ -3-2i & 2+3i & 4+2i \end{bmatrix}$ as the sum of a Hermitian and a skew-Hermitian matrix.

16. Classify the following matrices as orthogonal, Hermitian, skew-Hermitian or unitary.

(a) $\begin{bmatrix} 1 & 2i & -3 \\ -2i & 2 & 1+4i \\ -3 & 1-4i & 3 \end{bmatrix}$ (b) $\frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\theta} & e^{i\left(\frac{\pi}{2}+\theta\right)} \\ e^{i\left(\frac{\pi}{2}-\theta\right)} & -e^{i(\pi-\theta)} \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1+i & 2+i \\ -1+i & 0 & -2+3i \\ -2+i & 2+3i & 0 \end{bmatrix}$

17. If $\mathbf{U}$ is a unitary matrix, then show that $\overline{\mathbf{U}}$, $\mathbf{U}^{\mathrm{T}}$ and $\mathbf{U}^{n}$ (n, a positive integer) are also unitary matrices.

2.4 DETERMINANTS

In this section we introduce a scalar quantity associated with every square matrix, called the *determinant* of the matrix. In addition to their numerous applications, determinants play a key role in the theory of system of linear algebraic equations.

2.4.1 Some Basic Definitions

Determinant. If $\mathbf{A}$ is a square matrix of order n, then *determinant* of $\mathbf{A}$, denoted by $\det \mathbf{A}$, or $|\mathbf{A}|$, is defined as

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad \text{...(2.2)}$$

The determinant has a value which is real, if the matrix is real, and may be real or complex, if the matrix is complex. The vertical bars are used to distinguish **det A**, which is a number, from the matrix **A**, which is an $n \times n$ array of numbers.

Consider a $|\mathbf{A}|$ of order 2×2 given by $|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

Its value is given by $|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$. For example,

$$\begin{vmatrix} 3 & -1 \\ 2 & 6 \end{vmatrix} = (3)(6) - 2(-1) = 18 + 2 = 20, \quad \text{and} \quad \begin{vmatrix} 1+i & i \\ -3i & 2 \end{vmatrix} = (1+i)2 - (-3i)\,i = -1 + 2i.$$

Minors and cofactors. The *minor* $\mathbf{M}_{ij}$, associated with the element a_{ij} in the ith row and jth column of the nth order **det A**, is the determinant of order $n-1$ obtained from **det A** by deleting the elements in the ith row and jth column.

The *cofactor* $\mathbf{A}_{ij}$ associated with the element a_{ij} in **det A** is defined in terms of the minor $\mathbf{M}_{ij}$, as $\mathbf{A}_{ij} = (-1)^{i+j}\mathbf{M}_{ij}$, for $i, j = 1, 2, \ldots, n$. For example, if

$$|\mathbf{A}| = \begin{vmatrix} 4 & 7 & -2 \\ 0 & 3 & 2 \\ 1 & 5 & 6 \end{vmatrix}, \text{ then } \mathbf{M}_{11} = \begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix}, \mathbf{M}_{22} = \begin{vmatrix} 4 & -2 \\ 1 & 6 \end{vmatrix}, \mathbf{M}_{32} = \begin{vmatrix} 4 & -2 \\ 0 & 2 \end{vmatrix}$$

and, $\mathbf{A}_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix}$, $\mathbf{A}_{22} = (-1)^{2+2}\begin{vmatrix} 4 & -2 \\ 1 & 6 \end{vmatrix} = \begin{vmatrix} 4 & -2 \\ 1 & 6 \end{vmatrix}$, $\mathbf{A}_{32} = (-1)^{3+2}\begin{vmatrix} 4 & -2 \\ 0 & 2 \end{vmatrix} = -\begin{vmatrix} 4 & -2 \\ 0 & 2 \end{vmatrix}$

We note that a *determinant of order n has n^2 minors and corresponding number of cofactors.*

2.4.2 Expansion of a Determinant

A determinant of order n can be expanded through the elements of any row or any column and *the value of the determinant is the sum of the products of the element of the ith row (or, the jth column) and their corresponding co-factors*, thus

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{ij}\mathbf{A}_{ij} \quad \left(\text{or, } \sum_{i=1}^{n} a_{ij}\,\mathbf{A}_{ij}\right)$$

In terms of minors it is $|\mathbf{A}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\mathbf{M}_{ij} \left(\text{or, } \sum_{i=1}^{n} (-1)^{i+j} a_{ij}\,\mathbf{M}_{ij}\right)$

Let $\det A = \begin{vmatrix} 0 & 2 & -1 \\ 4 & 3 & 5 \\ 2 & 0 & -4 \end{vmatrix}$, then by definition, $\det A = \sum_{j=1}^{3} a_{ij} \mathbf{A}_{ij}$

For $i = 1$, $\det \mathbf{A} = a_{11}\mathbf{A}_{11} + a_{12}\mathbf{A}_{12} + a_{13}\mathbf{A}_{13} = a_{11}\mathbf{M}_{11} - a_{12}\mathbf{M}_{12} + a_{13}\mathbf{M}_{13}$

$$= (0)\begin{vmatrix} 3 & 5 \\ 0 & -4 \end{vmatrix} - (2)\begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix} = 0 - 2(-16-10) - (0-6) = 58.$$

We can expand the determinant through any row or column, the value remains the same. But, in general, it is convenient to choose that row or column which contains most zeros in it. In fact we note that if all the elements of any row or column are zero, then $\det \mathbf{A} = 0$. If determinant of a square matrix **A** is zero, matrix is said to be *singular*, otherwise *non-singular*.

Also we note that, *the sum of the products of the elements of any row (or, column) with the corresponding cofactors of any other row (or, column) is zero*. Thus, we have the following results:

$$\sum_{k=1}^{n} a_{ik}\mathbf{A}_{jk} = \begin{cases} |\mathbf{A}|, & i = j \\ 0, & i \neq j \end{cases} \quad \text{...(2.3)}$$

$$\sum_{k=1}^{n} a_{ki}\mathbf{A}_{kj} = \begin{cases} |\mathbf{A}|, & i = j \\ 0, & i \neq j \end{cases} \quad \text{...(2.4)}$$

We observe that for large n the cofactor expansion process is very laborious and time-consuming even with the fast computers. However the scientific calculations, in general, involve determinant of higher orders. Next, we study the various properties of determinants which can be used to simplify the evaluation of determinants.

2.4.3 Properties of Determinants

Following are some important properties satisfied by the determinants.

1. *A determinant remains unaltered by changing its rows into columns and columns into rows, that is,* $|A| = |A^T|$.

2. *If any two rows (columns) of a determinant are interchanged, then the numerical value of the determinant remains unchanged but changes in sign.*

In general, if any row (column) is shifted over p rows (column), then the value of the resulting determinant is $(-1)^p$ times the original determinant.

3. *If the corresponding elements of any two rows (columns) of a determinant are the same, then determinant vanishes.*

4. *If each element of a row (column) is multiplied by the same scalar, then the value of the determinant is multiplied by the same scalar.*

As a consequence to this, if α is a common factor of each element of a row (column) then the factor α can be taken out of the determinant. But we must note that when we multiply a matrix by a scalar α, then every element of the matrix is multiplied by α, and therefore $|\alpha \mathbf{A}| = \alpha^n |\mathbf{A}|$, where **A** is a square matrix of order n.

5. If each element of a row (column) can be expressed as sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

6. If a non-zero constant multiple of the elements of any row (column) are added to the corresponding elements of some other row (column), then the value of the determinant remains unchanged.

In case the elements of the jth row are multiplied by a constant $k \neq 0$ and are added to the corresponding element of the ith row, then this operation is symbolized as $R_i \to R_i + kR_j$ and is called the *elementary row operation* Similarly, corresponding column operation is symbolized as $C_i \to C_i + kC_j$, and is called the *elementary column operation*.

7. If the elements of a determinant are functions of x and two rows (columns) become identical when $x = a$, then $x - a$ is a factor of the value of the determinant.

For example, if $|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$, then **det A** vanishes for $x = y$, thus, $(x - y)$ is a factor of the value of the **det A**. Similarly, $(y - z)$, $(z - x)$ are factors of **det A** and, in fact, we can verify that $|\mathbf{A}| = (x - y)(y - z)(z - x)$.

This result is known as *factor theorem on determinants*.

8. The determinants of diagonal, lower triangular and upper triangular matrices are product of their principal diagonal elements.

Addition and multiplication of two determinants The two determinants can be added or multiplied only when they are of the same order and these operations are performed on the lines of matrices addition and multiplication. Since the value of a determinant does not change by interchanging the rows and columns, so multiplication can be carried out row by row multiplication rule, or column by column multiplication rule. Further, in general,

$$\det(\mathbf{A} + \mathbf{B}) \neq \det \mathbf{A} + \det \mathbf{B}, \qquad \text{...(2.5)}$$

but,

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}). \qquad \text{...(2.6)}$$

Example 2.7: Evaluate (a) $\begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$

Solution: (a) Let $\Delta = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \\ 7 & 9 & 11 & 13 \end{vmatrix}$, $[R_2 \to R_2 + (-1)R_1,\ R_3 \to R_3 + (-1)R_2,\ R_4 \to R_4 + (-1)R_3]$

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{vmatrix} \quad \begin{matrix} [R_3 \to R_3 + (-1)R_2, \\ R_4 \to R_4 + (-1)R_3] \end{matrix}$$

$$= 0, \text{ since } R_3 \text{ and } R_4 \text{ are identical.}$$

(b) Since there are two zeros in the second row, therefore, expanding by this row, we get

$$\Delta = -\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} + 0 - 3\begin{vmatrix} 0 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix} + 0$$

$$= -\left[1\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 0\right] - 3\left[0 - 2\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}\right]$$

(Expanding by C_1) (Expanding by C_1)

$$= -[(0-1) - 3(4-3)] - 3[-2(2-0) + 3(1-9)] = 4 + 84 = 88.$$

Example 2.8: Evaluate (a) $\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$ (b) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$

Solution: (a) Let $\Delta = \begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & a+b & a \\ a+b+c & b+c & b \\ a+b+c & c+a & c \end{vmatrix}, \quad [C_1 \to C_1 + C_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & a+b & a \\ 1 & b+c & b \\ 1 & c+a & c \end{vmatrix} \quad \left[C_1 \to \frac{1}{a+b+c}C_1\right]$$

$$= (a+b+c)\begin{vmatrix} 1 & a+b & a \\ 0 & c-a & b-a \\ 0 & c-b & c-a \end{vmatrix}, \quad [R_2 \to R_2 - R_1;\ R_3 \to R_3 - R_1]$$

$$= (a+b+c)\,[(c-a)^2 - (c-b)(b-a)] \quad [\text{expanding by } C_1]$$
$$= (a+b+c)\,[(c^2 + a^2 - 2ca) - (cb - ca - b^2 + ba)]$$
$$= (a+b+c)\,(a^2 + b^2 + c^2 - bc - ca - ab) = a^3 + b^3 + c^3 - 3\,abc.$$

(b) Let
$$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}, \quad [C_1 \to xC_1, C_2 \to yC_2, C_3 \to zC_3]$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}, \quad [R_3 \to \frac{1}{xyz}R_3]$$

$$= (-1)^2\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}, \quad [R_2 \leftrightarrow R_3 \text{ and then } R_1 \leftrightarrow R_2]$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2-x^2 & z^2-x^2 \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix}, \quad [C_2 \to C_2 - C_1; C_3 \to C_3 \to C_1]$$

$$= \begin{vmatrix} y^2-x^2 & z^2-x^2 \\ y^3-x^3 & z^3-x^3 \end{vmatrix}, \quad [\text{Expanding by } R_1]$$

$$= (y-x)(z-x)\begin{vmatrix} y+x & z+x \\ y^2+xy+x^2 & z^2+zx+x^2 \end{vmatrix}, \quad [C_1 \to \frac{1}{y-x}C_1, C_2 \to \frac{1}{z-x}C_2]$$

$$= (y-x)(z-x)\begin{vmatrix} y+x & z-y \\ y^2+xy+x^2 & (z^2-y^2)+(z-y)x \end{vmatrix}, \quad [C_2 \to C_2 - C_1]$$

$$= (y-x)(z-x)(z-y)\begin{vmatrix} y+x & 1 \\ y^2+xy+x^2 & x+y+z \end{vmatrix}, \quad [C_2 \to \frac{1}{z-y}C_2]$$

$$= (y-x)(z-x)(z-y)\,[(y+x)(x+y+z) - (y^2+xy+x^2)]$$
$$= (x-y)(y-z)(z-x)(xy+yz+zx).$$

Example 2.9: Prove that (a) $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x-a)^3(x+3a)$

(b) $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$

Solution:

(a) Let $$\Delta = \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix}, \quad [C_1 \to C_1 + C_2 + C_3 + C_3]$$

$$= (x+3a)\begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix} \quad [C_1 \to \frac{1}{x+3a} C_1]$$

$$= (x+3a)\begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix}, \quad [R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - R_1]$$

$$= (x+3a)\,(x-a)^3,$$

since the determinant of an upper triangular matrix is the product of its principal diagonal elements.

(b) Let $$\Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}, \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}, \quad [C_1 \to \frac{1}{2(a+b+c)} C_1]$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}, \quad [R_2 \to R_2 - R_1;\ R_3 \to R_3 - R_1]$$

$$= 2(a+b+c)^3.$$

Example 2.10: Without actual expansion, show that $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = 0.$

Solution:

Let $\Delta = \begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} = \Delta_1 + \Delta_2$, say.

Consider, $\Delta_2 = \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} = \dfrac{1}{abcd} \begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix}$

$$= \begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix} = (-1)^3 \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = -\Delta_1$$

Therefore, $\Delta = 0$.

Example 2.11: Without actual expansion, show that $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$.

Solution:

Let $$\Delta = \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \begin{vmatrix} \cos A\cos P + \sin A \sin P & \cos A \cos Q + \sin A \sin Q & \cos A \cos R + \sin A \sin R \\ \cos B \cos P + \sin B \sin P & \cos B \cos Q + \sin B \sin Q & \cos B \cos R + \sin B \sin R \\ \cos C \cos P + \sin C \sin P & \cos C \cos Q + \sin C \sin Q & \cos C \cos R + \sin C \sin R \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix} = 0 \times 0 = 0.$$

Example 2.12: Solve the equation $\begin{vmatrix} a+x & b+x & c+x \\ b+x & c+x & a+x \\ c+x & a+x & b+x \end{vmatrix} = 0$, when $a \neq b \neq c$.

Solution: The equation is $\begin{vmatrix} a+x & b+x & c+x \\ b+x & c+x & a+x \\ c+x & a+x & b+x \end{vmatrix} = 0$

or, $$\begin{vmatrix} a+b+c+3x & b+x & c+x \\ a+b+c+3x & c+x & a+x \\ a+b+c+3x & a+x & b+x \end{vmatrix} = 0, \quad [C_1 \to C_1 + C_2 + C_3]$$

or, $$(a+b+c+3x)\begin{vmatrix} 1 & b+x & c+x \\ 1 & c+x & a+x \\ 1 & a+x & b+x \end{vmatrix} = 0, \quad [C_1 \to \frac{1}{a+b+c+3x} C_1]$$

or, $$(a+b+c+3x)\begin{vmatrix} 1 & b+x & c+x \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = 0, \quad [R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

or, $(a+b+c+3x)\,[(c-b)(b-c)-(a-b)(a-c)] = 0$

or, $(a+b+c+3x)\,[(b-c)^2+(a-b)(a-c)] = 0$

or, $(a+b+c+3x)\,[a^2+b^2+c^2-ab-bc-ca] = 0$

or, $\frac{1}{2}(a+b+c+3x)\,[(a-b)^2+(b-c)^2+(c-a)^2] = 0$

or, $x = -\frac{1}{3}(a+b+c)$, since $(a-b)^2+(b-c)^2+(c-a)^2$ is non-negative.

Example 2.13: If a, b, c are all different and $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$, show that $abc = -1$.

Solution:

$$\text{Let } \Delta = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

$$= (-1)^2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (1+abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (1+abc)\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad [R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

$$= (1+abc)\begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix} = (1+abc)(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (1+abc)(b-a)(c-a)(c-b) = (1+abc)(a-b)(b-c)(c-a).$$

Since $\Delta = 0$ and $a \neq b \neq c$, therefore, $1 + abc = 0$, that is, $abc = -1$.

Example 2.14: Show that $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$

Solution: Let $\Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$. Put $a = -b$, it becomes

$$\begin{vmatrix} 2b & 0 & c-b \\ 0 & -2b & b+c \\ c-b & c+b & -2c \end{vmatrix} = \begin{vmatrix} 2b & 0 & c+b \\ 0 & -2b & c+b \\ c-b & c+b & -(c+b) \end{vmatrix} \quad [C_3 \to C_3 + C_1]$$

$$= (c+b)\begin{vmatrix} 2b & 0 & 1 \\ 0 & -2b & 1 \\ c-b & c+b & -1 \end{vmatrix} = (c+b)\begin{vmatrix} c+b & c+b & 0 \\ c-b & c-b & 0 \\ c-b & c+b & -1 \end{vmatrix} \quad [R_1 \to R_1 + R_3, R_2 \to R_2 + R_3]$$

$$= (c+b)^2(c-b)\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ c-b & c+b & -1 \end{vmatrix} = 0$$

Therefore $(a+b)$ is a factor of Δ. Similarly $(b+c)$, $(c+a)$ are factors of Δ. Since Δ is of degree 3 in a, b, c, therefore any other factor of Δ must be independent of a, b and c. Thus,

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(a+b)(b+c)(c+a),$$

where k is a constant. To evaluate k, we put arbitrary values for a, b and c, say $a = b = 1$ and $c = 0$, we get

$$\begin{vmatrix} -2 & 2 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = k(2)\,(1)\,(1).$$

Simplifying, we get $2k = 8$, that is, $k = 4$.

Example 2.15: Show that $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$, where the capital letters denote the cofactor of the corresponding small letters.

Solution: Consider $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_2A_1 + b_2B_1 + c_2C_1 & a_3A_1 + b_3B_1 + c_3C_1 \\ a_1A_2 + b_1B_2 + c_1C_2 & a_2A_2 + b_2B_2 + c_2C_2 & a_3A_2 + b_3B_2 + c_3C_2 \\ a_1A_3 + b_1B_3 + c_1C_3 & a_2A_3 + b_2B_3 + c_2C_3 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & 0 & 0 \\ 0 & a_2A_2 + b_2B_2 + c_2C_3 & 0 \\ 0 & 0 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3, \text{ where } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Thus, $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$

Example 2.16: Show that $\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix} = \lambda^3\,(a^2 + b^2 + c^2 + d^2 + \lambda).$

Solution:

Let $$\Delta = \begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$$

$$= abcd\begin{vmatrix} a+\frac{\lambda}{a} & b & c & d \\ a & b+\frac{\lambda}{b} & c & d \\ a & b & c+\frac{\lambda}{c} & d \\ a & b & c & d+\frac{\lambda}{d} \end{vmatrix} \left[R_1 \to \frac{1}{a}R_1, R_2 \to \frac{1}{b}R_2, R_3 \to \frac{1}{c}R_3, R_4 \to \frac{1}{d}R_4\right]$$

$$= \frac{abcd}{abcd}\begin{vmatrix} a^2+\lambda & b^2 & c^2 & d^2 \\ a^2 & b^2+\lambda & c^2 & d^2 \\ a^2 & b^2 & c^2+\lambda & d^2 \\ a^2 & b^2 & c^2 & d^2+\lambda \end{vmatrix} [C_1 \to aC_1, C_2 \to bC_2, C_3 \to cC_3, C_4 \to dC_4]$$

$$= \begin{vmatrix} a^2+b^2+c^2+d^2+\lambda & b^2 & c^2 & d^2 \\ a^2+b^2+c^2+d^2+\lambda & b^2+\lambda & c^2 & d^2 \\ a^2+b^2+c^2+d^2+\lambda & b^2 & c^2+\lambda & d^2 \\ a^2+b^2+c^2+d^2+\lambda & b^2 & c^2 & d^2+\lambda \end{vmatrix} [C_1 \to C_1 + C_2 + C_3 + C_4]$$

$$= (a^2+b^2+c^2+d^2+\lambda)\begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 1 & b^2+\lambda & c^2 & d^2 \\ 1 & b^2 & c^2+\lambda & d^2 \\ 1 & b^2 & c^2 & d^2+\lambda \end{vmatrix} \left[C_1 \to \frac{1}{a^2+b^2+c^2+\lambda}C_1\right]$$

$$= (a^2+b^2+c^2+d^2+\lambda)\begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} [R_2 \to R_2 - R_1, R_3 - R_1, R_4 \to R_4 - R_1]$$

$= \lambda^3(a^2+b^2+c^2+d^2+\lambda)$, since the value of the triangular determinant is the product of its principal diagonal elements.

Example 2.17: Prove that

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

Solution:

Let $$\Delta = \begin{vmatrix} a^2-2ax+x^2 & b^2-2bx+x^2 & c^2-2cx+x^2 \\ a^2-2ay+y^2 & b^2-2by+y^2 & c^2-2cy+y^2 \\ a^2-2az+z^2 & b^2-2bz+z^2 & c^2-2cz+z^2 \end{vmatrix} = \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} \begin{vmatrix} 1 & -2x & x^2 \\ 1 & -2y & y^2 \\ 1 & -2z & z^2 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} (-2) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 2 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x).$$

Alternatively, this result can be proved using *factor theorem* also as follows:

Substituting $a = b$ in Δ, the first and second columns become equal and hence $\Delta = 0$, thus $(a-b)$ is a factor of Δ. Similarly, $(b-c)$, $(c-a)$, $(x-y)$, $(y-z)$ and $(z-x)$ are factors of Δ. Also since Δ is a determinant of degree 6, thus any other factor must be independent of a, b, c, x, y, and z and therefore,

$$\Delta = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = k(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

where k is a constant. To evaluate k, we put arbitrary values for x, y, z, a, b, c, say $x = 0, y = 1, z = -1$, $a = 0, b = 1, c = -1$, we get

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 0 \end{vmatrix} = k(-1)(2)(-1)(-1)(2)(-1)$$

Simplifying it gives $8 = 4k$, or $k = 2$. Thus,

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x).$$

EXERCISE 2.2

1. For $\mathbf{A} = \begin{bmatrix} -3 & 0 & 4 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, verify $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.

2. Without actual expansion prove that the following determinants vanish.

(a) $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

3. Evaluate

(a) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 3 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} 3 & -2 & 1 & 2 \\ 2 & 3 & -2 & 4 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{vmatrix}$

Prove the results in Problems **(4-10)** without a direct expansion of the determinant by using the properties of the determinants.

4. $\begin{vmatrix} 1+a & a & a \\ b & 1+b & b \\ c & c & 1+c \end{vmatrix} = (1 + a + b + c)$

5. $\begin{vmatrix} x^2+a^2 & ab & ac \\ ab & x^2+b^2 & bc \\ ac & cb & x^2+c^2 \end{vmatrix} = x^4(x^2 + a^2 + b^2 + c^2)$

6. $\begin{vmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \\ 1 & 1 & 1 & k \end{vmatrix} = (k+3)\,(k-1)^3.$

7. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} = (\alpha-\beta)\,(\alpha-\gamma)\,(\alpha-\delta)\,(\beta-\gamma)\,(\beta-\delta)\,(\gamma-\delta).$

8. $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (a+c)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$

9. $\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix} = (x-2y+z)^2.$

10. $\begin{vmatrix} a & b & a & a \\ a & b & b & b \\ b & b & b & a \\ a & a & b & a \end{vmatrix} = -(a-b)^4.$

11. Solve the following equations:

(a) $\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0$

(b) $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$

12. Prove that if a, b, c are all different and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, then $(abc)(bc+ca+ab) = a+b+c.$

13. Show that

(a) $\begin{vmatrix} \sin^2 A & \sin A\cos A & \cos^2 A \\ \sin^2 B & \sin B\cos B & \cos^2 B \\ \sin^2 C & \sin C\cos C & \cos^2 C \end{vmatrix} = -\sin(A-B)\sin(B-C)\sin(C-A)$

(b) $\begin{vmatrix} \sin(a+\alpha) & \sin(b+\alpha) & \sin(c+\alpha) \\ \sin(a+\beta) & \sin(b+\beta) & \sin(c+\beta) \\ \sin(a+\gamma) & \sin(b+\gamma) & \sin(c+\gamma) \end{vmatrix} = 0$

14. Show that

$$\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2$$

15. If $p = ax+by+cz$, $q = ay+bz+cx$, and $r = az+bx+cy$, then show that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = p^3+q^3+r^3-3pqr.$$

2.5 THE INVERSE OF A MATRIX

In this section, we shall exclusively consider the square matrices.

Let **A** *be any matrix. A matrix* **B**, *if it exists, is called the inverse of the matrix* **A** *if the products* **AB** *and* **BA** *are defined and* **AB** = **BA** = *I*. Since in case of the existence of the inverse of **A**, the product **AB** and **BA** both are defined and equal thus, **A** and **B** both are square matrices of the same order. Further, since $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = |\mathbf{I}| = 1$, thus, both $|\mathbf{A}|$ and $|\mathbf{B}|$ must be non-zero *that is both the matrix and its inverse must be non-singular.*

Interchanging the order of **A** and **B** we observe that *if* **B** *is the inverse of* **A**, *then* **A** *must be the inverse of* **B**.

The inverse of a square matrix **A** of order *n*, if exists, is denoted by $\mathbf{A}^{-1}$. Thus

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n,$$

where $\mathbf{I}_n$ is a unit matrix of order *n*.

It is easy to find non-zero square matrices that have no inverse. For example, consider $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$. If **B** is the inverse of **A** and say, $\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\mathbf{AB} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which gives $2a = 1$, $a = 0$, $2b = 0$ and $b = 0$, which are impossible conditions, thus **A** does not has an inverse. We can check that **A** is a *singular* matrix, since $|\mathbf{A}| = 0$.

On the other hand, some matrices do have inverses, for example, we have

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus the matrices $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ are inverse of each other

In fact, we can check in this case that both matrices are non-singular.

2.5.1 Method to Find the Inverse of a Square Matrix *A*

The inverse of a non-singular square matrix **A** *is given by*

$$\mathbf{A}^{-1} = \frac{adj(\mathbf{A})}{|\mathbf{A}|}, \ |A| \neq 0, \qquad \text{...(2.7)}$$

where $adj(\mathbf{A})$ = *adjoint matrix of* **A** = *transpose of the matrix of cofactors of* **A**.

To establish it, consider $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then

$$\mathbf{A}\,adj(\mathbf{A}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}, \qquad \text{...(2.8)}$$

where A_{ij} is the cofactor of a_{ij} in A.

Performing the multiplication on the right side of (2.8) and using (2.3), we obtain

$$\mathbf{A}\,adj(\mathbf{A}) = \begin{bmatrix} |\mathbf{A}| & 0 & \cdots & 0 \\ 0 & |\mathbf{A}| & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & |\mathbf{A}| \end{bmatrix} = |\mathbf{A}|\,\mathbf{I}_n.$$

Since $|\mathbf{A}| \neq 0$, dividing both sides by $|\mathbf{A}|$, we get $\mathbf{A}\dfrac{adj(\mathbf{A})}{|\mathbf{A}|} = \mathbf{I}_n$, which gives (2.7).

Also, *if* **A** *has an inverse, then it is unique.*

If possible let **B** and **C** both be two different inverses of the matrix **A**. Then

$$\mathbf{B} = \mathbf{BI} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}.$$

Hence the inverse of **A** is unique

2.5.2 Properties of Inverse Matrices

1. *The unit matrix* **I** *is its own inverse, that is,* $\mathbf{I} = \mathbf{I}^{-1}$
2. *The inverse of the inverse matrix is the matrix itself, that is,* $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
3. *Inverse of the transpose is the transpose of inverse, that is,* $(\mathbf{A}^{T})^{-1} = (\mathbf{A}^{-1})^{T}$.
4. *If* **A** *and* **B** *are non-singular matrices, then* **AB** *is also non-singular and* $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
5. *If* **A** *is non-singular, then* $(\mathbf{A}^{-1})^{m} = (\mathbf{A}^{m})^{-1}$ *for any positive integral, value of m.*
6. *If* **A** *is invertible, then* $det(\mathbf{A}^{-1}) = \dfrac{1}{\det \mathbf{A}}$.
7. *The inverse of a non-singular upper or lower triangular matrix is respectively an upper or lower triangular matrix.*

8. *If* $\mathbf{D} = diag(d_{11}, d_{22}, \ldots, d_{nn})$, $d_i \neq 0$, *then* $\mathbf{D}^{-1} = diag(1/d_{11}, 1/d_{22}, \ldots, 1/d_{nn})$.
9. *If* $\mathbf{AB} = \mathbf{O}$ *and* $\mathbf{A}$ *is a non-singular matrix, then* $\mathbf{B}$ *must be a null matrix. Similarly, if* $\mathbf{B}$ *is non-singular, then* $\mathbf{A}$ *must be a null matrix.*
10. *If* $\mathbf{AB} = \mathbf{AC}$ *and* $\mathbf{A}$ *is non-singular, then* $\mathbf{B} = \mathbf{C}$.
11. *The inverse of the sum of two matrices is, in general, not equal to the sum of their individual inverses, that is,* $(\mathbf{A}+\mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$

Example 2.18: Find the inverse of $\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \\ 1 & 5 & -2 \end{bmatrix}$, if it exists.

Solution: We have

$$|\mathbf{A}| = \begin{vmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \\ 1 & 5 & -2 \end{vmatrix} = 3\begin{vmatrix} 1 & 4 \\ 5 & -2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 3(-2-20) + (8+1) = -57 \neq 0. \text{ Thus } \mathbf{A}^{-1} \text{ exists.}$$

Since *adj* $\mathbf{A}$ is the transpose of the cofactor matrix, therefore,

$$adj\ \mathbf{A} = \begin{bmatrix} -22 & 4 & -1 \\ -1 & -5 & -13 \\ 9 & -12 & 3 \end{bmatrix}^T = \begin{bmatrix} -22 & -1 & 9 \\ 4 & -5 & -12 \\ -1 & -13 & 3 \end{bmatrix}$$

Thus,
$$\mathbf{A}^{-1} = \frac{adj.\mathbf{A}}{|\mathbf{A}|} = -\frac{1}{57}\begin{bmatrix} -22 & -1 & 9 \\ 4 & -5 & -12 \\ -1 & -13 & 3 \end{bmatrix}$$

We may verify that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_3$.

Example 2.19: Show that

$$\begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan(\theta/2) \\ -\tan(\theta/2) & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Solution: Let $\mathbf{A} = \begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & \tan(\theta/2) \\ -\tan(\theta/2) & 1 \end{bmatrix}$

First we calculate $\mathbf{B}^{-1}$. We have

$$|B| = \begin{bmatrix} 1 & \tan(\theta/2) \\ -\tan(\theta/2) & 1 \end{bmatrix} = 1 + \tan^2(\theta/2) = \sec^2(\theta/2) \neq 0 \text{ for any value of } \theta.$$

Also, $adj\,(\mathbf{B}) = \begin{bmatrix} 1 & \tan(\theta/2) \\ -\tan(\theta/2) & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix}$

Thus $\quad \mathbf{B}^{-1} = \dfrac{adj(\mathbf{B})}{|\mathbf{B}|} = \dfrac{1}{\sec^2(\theta/2)} \begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix}$

Hence $\mathbf{AB}^{-1} = \dfrac{1}{\sec^2(\theta/2)} \begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\theta/2) \\ \tan(\theta/2) & 1 \end{bmatrix}$

$$= \frac{1}{\sec^2(\theta/2)} \begin{bmatrix} 1-\tan^2(\theta/2) & -2\tan(\theta/2) \\ 2\tan(\theta/2) & 1-\tan^2(\theta/2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(\theta/2)-\sin^2(\theta/2) & -2\sin(\theta/2)\cos(\theta/2) \\ 2\sin(\theta/2)\cos(\theta/2) & \cos^2(\theta/2)-\sin^2(\theta/2) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

2.6 SOLUTION OF LINEAR SYSTEM OF EQUATIONS ($n \times n$ FORM)

Consider the system of n equations in n unknowns

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}\right\} \qquad \ldots(2.9)$$

In matrix form the system of Eqs. (2.9) is

$$\mathbf{Ax} = \mathbf{b}, \qquad \ldots(2.10)$$

where $\mathbf{A} = [a_{ij}]_{n\times n}$, $\mathbf{x} = [x_1, \; x_2, \; \cdots, \; x_n]^T$ and $\mathbf{b} = [b_1, \; b_2, \; \cdots, \; b_n]^T$

The matrix $\mathbf{A}$ is called the *coefficient matrix* and $\mathbf{x}$ the *solution vector*. If $\mathbf{b} \neq \mathbf{0}$, then the system of Eqs (2.9) is called *non-homogeneous*; and in case $\mathbf{b} = \mathbf{0}$, the system is said to be *homogeneous*. Further, the system of equations is *consistent* if it has at least one solution and *inconsistent* if it has no solution at all.

Next we discuss the solution of non-homogeneous system of equations.

2.6.1 Method of Determinants: Cramer's Rule

We explain this method by considering the system of three linear equations in three unknowns:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}$$

The determinant of the coefficient matrix $\mathbf{A}$ is, $|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

We have, $x_1|\mathbf{A}| = \begin{vmatrix} x_1a_{11} & a_{12} & a_{13} \\ x_1a_{21} & a_{22} & a_{23} \\ x_1a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & a_{12} & a_{13} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & a_{22} & a_{23} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = |\mathbf{A}_1|$, say.

Thus, $x_1 = \dfrac{|\mathbf{A}_1|}{|\mathbf{A}|}$, provided $|\mathbf{A}| \neq 0$. Similarly, $x_2 = \dfrac{|\mathbf{A}_2|}{|\mathbf{A}|}$, and $x_3 = \dfrac{|\mathbf{A}_3|}{|\mathbf{A}|}$, where $|\mathbf{A}_i|$ is the determinant of the matrix $\mathbf{A}_i$ obtained by replacing the ith column of $\mathbf{A}$ by the right-hand side column vector $\mathbf{b} = [b_1, b_2, b_3]^T$.

This result can be generalized to system of n linear equations in n variables. The following three cases arise:

(a) When $|\mathbf{A}| \neq 0$, the system of equation is *consistent and has the unique solution* given by $x_i = \dfrac{|\mathbf{A}_i|}{|\mathbf{A}|}, i = 1, 2, \ldots n$.

(b) When $|\mathbf{A}| = 0$ and at least one of the $|\mathbf{A}_i| \neq 0$, then the system of equations has no solution, and thus the system is *inconsistent*.

(c) When $|\mathbf{A}| = 0$ and all the $|\mathbf{A}_i| = 0, i = 1, 2, \ldots, n$, then the system of equation is *consistent and has infinite number of solutions*.

2.6.2 The Matrix Method

The system of Eqs. (2.9) in the matrix form is

$$\mathbf{Ax} = \mathbf{b} \qquad \text{...(2.10a)}$$

Let $\mathbf{A}$ be non-singular, pre multiplying (2.10a) by $\mathbf{A}^{-1}$ we obtain

$$\mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}\mathbf{b}, \text{ or } \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b}, \quad \text{or} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \qquad \text{...(2.11)}$$

The unique solution is obtained by equating the values of x_i, $i = 1, 2, \ldots n$ to the corresponding elements in the resultant product matrix on the right side of (2.11).

In case $\mathbf{b} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$; that is, the trivial solution is the only solution.

When the *system of equation is homogeneous* of the form

$$\mathbf{Ax} = \mathbf{0}, \qquad \text{...(2.12)}$$

then trivial solution $\mathbf{x} = \mathbf{0}$ is always a solution of this system, thus *a homogeneous system is always consistent.*

If $\mathbf{A}$ is non-singular, then $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{0}) = \mathbf{0}$ is the only solution of the homogeneous system (2.12).

A non-trivial solution of the homogeneous system (2.12) exists if, and only if $\mathbf{A}$ *is singular and in this case the homogeneous system has infinite number of solutions.*

These solutions are at least *one-parameter family* of solutions.

Example 2.20: Show that the system of equations $3x+y+2z=3, 2x-3y-z=-3, x+2y+z=4$, has a unique solution. Find the solution by (a) matrix method (b) Cramer's rule.

Solution: Here $|\mathbf{A}| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-3+2)-1(2+1)+2(4+3)=8 \neq 0.$

Since $|\mathbf{A}| \neq 0$, thus the system of equations has a unique solution.

(a) *The matrix method.* We have

$$adj(\mathbf{A}) = \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}. \quad \text{Thus } A^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

Therefore, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 \\ 16 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, which gives $x=1, y=2$, and $z=-1$.

(b) *Cramer's rule.* We have

$$|\mathbf{A_1}| = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 3(-3+2)-1(-3+4)+2(-6+12)=8$$

$$|\mathbf{A_2}| = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 3(-3+4)-3(2+1)+2(8+3)=16$$

$$|\mathbf{A_3}| = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = 3(-12+6)-1(8+3)+3(4+3)=-8.$$

Therefore, $\quad x = \frac{|\mathbf{A_1}|}{|\mathbf{A}|} = 1, \quad y = \frac{|\mathbf{A_2}|}{|\mathbf{A}|} = 2$, and $\quad z = \frac{|\mathbf{A_3}|}{|\mathbf{A}|} = -1.$

Example 2.21: Solve the homogeneous system of equations $x+3y-2z=0, 2x-y+4z=0, x-11y+14z=0$.

Solution: Here, $|\mathbf{A}| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix} = 1(-14+44)-3(28-4)-2(-22+1)=30-72+42=0.$

Since $|\mathbf{A}| = 0$, hence the given system has infinite number of solutions. Rewriting the first two equations, in terms of z as $x + 3y = 2z$, $2x - y = -4z$.

Solving these for x and y, we obtain $x = -10z/7, y = 8z/7$. Hence the solution is

$$x = -10\alpha/7, \quad y = 8\alpha/7, \quad z = \alpha$$

where α is arbitrary. This is one parameter family of solution; α being the parameter. If satisfies the third equation also.

Remark. The case when the system of linear equations has infinite number of solutions will be dealt in detail in Section (2.10.2).

EXERCISE 2.3

1. Given the matrix $\mathbf{A} = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$, compute $adj(\mathbf{A})$ and prove that

$$\mathbf{A}(adj(\mathbf{A})) = (adj\mathbf{A})\,\mathbf{A} = |\mathbf{A}|\,\mathbf{I}_3.$$

2. Find the inverse of the following matrices. Verify that in each case $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

(a) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$

3. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, then verify that

(a) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ (b) $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det \mathbf{A}}$. (c) $(\mathbf{A}+\mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$

(d) $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$ (e) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ (f) $[adj(\mathbf{A})]^{-1} = adj(\mathbf{A}^{-1})$.

4. If the matrix $\mathbf{A}$ is nilpotent with $\mathbf{A}^P = 0$, show that

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots + \mathbf{A}^P.$$

Solve the following system of equations by Cramer's rule.

5. $x - y + z = 4, \quad 2x + y - 3z = 0, \quad x + y + x = 2$
6. $x + y + z = 3, \quad x + 2y + 3z = 4, \quad x + 4y + 9z = 6$
7. $x + y + z = 6.6, \ x - y + z = 2.2, \quad x + 2y + 3z = 15.2$
8. $2x + y + z = 0, \ 3x + 2y + 3z = 18, \ x + 4y + 9z = 16$

Solve the following system of equations by matrix method.

9. $2x + 5y + 3z = 1, \ -x + 2y + z = 2, \quad x + y + z = 0$
10. $3x - y + z = 6, \quad 4x - y + 2z = 7, \quad 2x - y + z = 4$

11. $x-y+z=4,\quad 2x+y-3z=0,\quad x+y+z=2$
12. $2x-z=1\quad 5x+y=7,\quad y+3z=5$
13. Solve the system of equations $2yz-zx+xy=3xyz,\quad 3yz+2zx+4xy=19xyz,\quad 6yz+7zx-xy=17xyz.$
14. Solve the system of equations $x^2z^3/y=e^8,\quad y^2z/x=e^4,\quad x^3y/z^4=1.$
15. Determine the values of k for which the system of equations $x-ky+z=0,\quad kx+3y-kz=0,\quad 3x+y-z=0$ has (i) only trivial solution, (ii) non-trivial soloution
16. If the system of equations $x+ay+az=0,\quad bx+y+bz=0,\quad cx+cy+z=0, a, b, c \neq 0, 1$ has a non-trivial solution, then show that $\dfrac{a}{1-a}+\dfrac{b}{1-b}+\dfrac{c}{1-c}=-1.$

2.7 ELEMENTARY TRANSFORMATIONS. RANK, NORMAL AND ECHELONS FORM OF A MATRIX. INVERSE BY GAUSS-JORDEN METHOD

First we discuss elementary transformations on matrices.

2.7.1 Elementary Row and Column Transformations

The three *elementary row transformations* that are performed on a matrix are:

(i) Interchange of two rows, denoted by $R_i \leftrightarrow R_j$

(ii) Multiplication of a row by a non-zero constant, denoted by $R_i \to \alpha R_i$

(iii) Addition of a constant multiplication of one row to another row, denoted by $R_i \to R_i + \alpha R_j$

The corresponding three transformations when performed on columns of a matrix are called *elementary column transformations* and are denoted by $C_i \leftrightarrow C_j$, $C_i \to \alpha C_i$ and $C_i \to C_i + \alpha C_j$, respectively.

For example, consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 6 & 4 & -3 & 2 \\ 2 & 0 & 1 & 7 & 4 \\ 5 & 2 & 8 & 2 & 3 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$, $\mathbf{A}$ gives, $\mathbf{A}_1 = \begin{bmatrix} 5 & 2 & 8 & 2 & 3 \\ 2 & 0 & 1 & 7 & 4 \\ 1 & 6 & 4 & -3 & 2 \end{bmatrix}$

Applying $R_1 \to 2R_1$, $\mathbf{A}_1$ gives, $\mathbf{A}_2 = \begin{bmatrix} 10 & 4 & 16 & 4 & 6 \\ 2 & 0 & 1 & 7 & 4 \\ 1 & 6 & 4 & -3 & 2 \end{bmatrix}$

Applying $R_2 \to R_2 + 2R_3$, $\mathbf{A}_2$ gives, $\mathbf{A}_3 = \begin{bmatrix} 10 & 4 & 16 & 4 & 6 \\ 4 & 12 & 9 & 1 & 8 \\ 1 & 6 & 4 & -3 & 2 \end{bmatrix}$

Similarly, we can apply elementary column transformations.

Equivalents Matrices. Two $m \times n$ matrices **A** and **B** are said to be *equivalent* if one can be obtained from the other by applying a sequence of elementary transformations.

The equivalence between the matrices **A** and **B** is denoted by $\mathbf{A} \sim \mathbf{B}$.

2.7.2 Elementary Matrices

An $n \times n$ *elementary matrix* is a matrix that is obtained from an $n \times n$ unit matrix by performing a single elementary row (column) transformation.

We will denote by $\mathbf{R}_{ij}$, the elementary matrix obtained from the unit matrix **I** by interchanging its ith and jth rows; by $\mathbf{R}_{i(\alpha)}$ the elementary matrix obtained from the unit matrix **I** by multiplying its ith row with the scalar $\alpha \neq 0$; and by $\mathbf{R}_{i+j(\alpha)}$, the elementary matrix obtained from the unit matrix **I** by adding α times the jth row to its ith row.

The notations for elementary matrices obtained from corresponding column operations will be $\mathbf{C}_{ij}$, $\mathbf{C}_{i(\alpha)}$, and $\mathbf{C}_{i+j(\alpha)}$ respectively.

For example, if $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\mathbf{R}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{C}_{1(5)} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$\mathbf{R}_{2+3(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \text{and } \mathbf{C}_{1+2(-1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We note from the definition of the elementary matrices and properties of the determinants, that

$|\mathbf{R}_{ij}| = |\mathbf{C}_{ij}| = -1$, $|\mathbf{R}_{i(\alpha)}| = |\mathbf{C}_{i(\alpha)}| = \alpha$, and $|\mathbf{R}_{i+j(\alpha)}| = |\mathbf{C}_{i+j(\alpha)}| = 1$.

Next we state an important result:

Let **R** *be an $m \times m$ elementary matrix obtained by performing an elementary row operation on the unit matrix* $\mathbf{I}_m$ *and let* **A** *be an $m \times n$ matrix, then the matrix* **RA** *is the matrix that is obtained from the matrix* **A** *by performing the same row operation.*

Similarly, if **C** *is an $n \times n$ elementary matrix obtained by performing an elementary column operation on the unit matrix* $\mathbf{I}_n$, *then the matrix* **AC** *is the matrix that is obtained from the matrix* **A** *by perfoming the same column operation.*

These two results can be stated jointly as follows.

Elementary row (column) transformation of a matrix **A** *can be performed by pre-multiplying (post-multiplying)* **A** *by the corresponding elementary matrix.*

For example, consider $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$. Then

$$\mathbf{R}_{13}\,\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$$

It is the same as obtained from **A** by interchanging Ist and 3rd rows.

$$\text{Also, } \mathbf{AC}_{1(5)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5a_{11} & a_{12} & a_{13} & a_{14} \\ 5a_{21} & a_{22} & a_{23} & a_{24} \\ 5a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

It is the same as obtained from **A** by multiplying the first column by 5.

2.7.3 Rank of a Matrix

A matrix **A**, *not necessarily square, is of rank r if it contains at least one square submatrix of order $r \times r$ with non-zero determinant but no square submatrix of order larger than $r \times r$ with non-zero determinant.*

Thus, *rank is the order of the largest non-zero minor of* **A**. If a matrix **A** has a non-zero minor of order r, then its rank is greater than or equal to r. If all the minors of order $r+1$ of a matrix A are zeros, then its rank is less than or equal to r.

The rank of a matrix **A** is denoted by $\rho(\mathbf{A})$.

We have the following results concerning the rank of a matrix;

1. *For a rectangular matrix* **A** *of order $m \times n$, the rank $(\mathbf{A}) \le \min(m, n)$.*
2. *For a square matrix* **A** *of order n, the rank $(\mathbf{A}) = n$, if $|\mathbf{A}| \ne 0$, otherwise rank $(\mathbf{A}) < n$. If rank $(\mathbf{A}) = n$, then the square matrix is said to be non-singular, otherwise singular.*
3. *Every non-zero matrix* **A** *is of rank greater than or equal to one.*
4. *The rank of a matrix is zero if, and only if it is a null matrix.*
5. *For any matrix* **A**, *the rank of* **A** *is equal to rank of $\mathbf{A}^T$.*
6. *The rank of the product of two matrices cannot exceed the rank of either matrix.*
7. *The rank of a matrix remains unaltered even if it is subjected to elementary row or column transformations, that is, equivalent matrices have the same rank.*

The last result is quite useful in finding the rank of a matrix.

2.7.4 Normal form of a Matrix

Every non-zero matrix **A** of order $m \times n$ with rank r can be reduced by a sequence of elementary transformations to the form

$$\begin{bmatrix} \mathbf{I}_r & \mathbf{O}_{r \times n-r} \\ \mathbf{O}_{m-r \times r} & \mathbf{O}_{m-r \times n-r} \end{bmatrix}_{m \times n}$$

called the *normal form* of **A**, where $\mathbf{I}_r$ is a unit matrix of order $r \times r$; and $\mathbf{O}_{r \times n-r}$, $\mathbf{O}_{m-r \times r}$ and $\mathbf{O}_{m-r \times n-r}$ are null matrices of the orders indicated.

Next, since each elementary row (column) transformation over a matrix **A** can be affected by pre (post) multiplying with a suitable elementary matrix and each elementary matrix is non-singular, therefore, we have the following result:

Corresponding to every matrix **A** *of order* $m \times n$ *with rank* r*, there exist non-singular matrices* **P** *of order* $m \times m$ *and* **Q** *of order* $n \times n$ *such that*

$$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{O}_{r \times n-r} \\ \mathbf{O}_{m-r \times r} & \mathbf{O}_{m-r \times n-r} \end{bmatrix}$$

We should note that for a given matrix **A**, the matrices **P** and **Q** are not necessarily unique.

Example 2.22: Determine the rank of the following matrices using elementary row transformations:

$$\text{(a)} \begin{bmatrix} 2 & 1 & -3 & 4 \\ 2 & 4 & -2 & 5 \\ 0 & 3 & 1 & 3 \\ 2 & 1 & -3 & -2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$$

Solution: (a) Let

$$A = \begin{bmatrix} 2 & 1 & -3 & 4 \\ 2 & 4 & -2 & 5 \\ 0 & 3 & 1 & 3 \\ 2 & 1 & -3 & -2 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_2 \to R_2 - R_1, \\ R_4 \to R_4 - R_1 \end{matrix} \sim \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

$$\begin{matrix} \text{Operate} \\ R_3 \to R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -6 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_4 \to R_4 + 3R_3 \end{matrix} \sim \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the matrix obtained is 3 because a minor of order 3×3, that is $\begin{vmatrix} 2 & 1 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 12 \neq 0$, and the only minor of order 4×4 is zero. Thus, the rank of given matrix is also 3.

(b) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_2 \to R_2 - 2R_1, \\ R_3 \to R_3 - R_1, \\ R_4 \to R_4 - 8R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_3 \to R_3 + R_2, \\ R_4 \to R_4 - 5R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the matrix obtained is two because a minor of order 2×2, e.g., $\begin{vmatrix}1 & 2\\ 0 & -3\end{vmatrix} = -3 \neq 0$ and every minor of order 3×3 is zero. Hence the rank of given matrix is also 2.

Example 2.23: Determine the rank of the following matrix using elementary column transformations only $\begin{bmatrix}1 & 1 & -1 & 1\\ -1 & 1 & -3 & -3\\ 1 & 0 & 1 & 2\\ 1 & -1 & 3 & 3\end{bmatrix}$

Solution: Let

$$A = \begin{bmatrix}1 & 1 & -1 & 1\\ -1 & 1 & -3 & -3\\ 1 & 0 & 1 & 2\\ 1 & -1 & 3 & 3\end{bmatrix} \begin{matrix}\text{Operate}\\ C_2 \to C_2 - C_1,\\ C_3 \to C_3 + C_1,\\ C_4 \to C_4 - C_1\end{matrix} \sim \begin{bmatrix}1 & 0 & 0 & 0\\ -1 & 2 & -4 & -2\\ 1 & -1 & 2 & 1\\ 1 & -2 & 4 & 2\end{bmatrix} \begin{matrix}\text{Operate}\\ C_3 \to C_3 + 2C_2,\\ C_4 \to C_4 + C_2\end{matrix} \sim \begin{bmatrix}1 & 0 & 0 & 0\\ -1 & 2 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 2 & 0 & 0\end{bmatrix}$$

The rank of this matrix is two, since there are non-zero minors of order 2 and every minor of order three is zero. Hence, the rank of given matrix is also 2.

Example 2.24: Find the rank of the matrix $\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$ using elementary transformations.

Solution: Let

$$A = \begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix} \begin{matrix}\text{Operate}\\ R_1 \leftrightarrow R_2\end{matrix} \sim \begin{bmatrix}1 & -1 & -2 & -4\\ 2 & 3 & -1 & -1\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix} \begin{matrix}\text{Operate}\\ C_2 \to C_2 + C_1,\\ C_3 \to C_3 + 2C_1,\\ C_4 \to C_4 + 4C_1\end{matrix} \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 5 & 3 & 7\\ 3 & 4 & 9 & 10\\ 6 & 9 & 12 & 17\end{bmatrix}$$

$$\begin{matrix}\text{Operate}\\ R_2 \to R_2 - 2R_1,\\ R_3 \to R_3 - 3R_1,\\ R_4 \to R_4 - 6R_1\end{matrix} \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 5 & 3 & 7\\ 0 & 4 & 9 & 10\\ 0 & 9 & 12 & 17\end{bmatrix} \begin{matrix}\text{Operate}\\ R_2 \to R_2 - R_3,\\ R_4 \to R_4 - 2R_3\end{matrix} \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -6 & -3\\ 0 & 4 & 9 & 10\\ 0 & 1 & -6 & -3\end{bmatrix}$$

$$\begin{matrix}\text{Operate}\\ C_3 \to C_3 + 6C_2, \\ C_4 \to C_4 + 3C_2\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 33 & 22 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{matrix}\text{Operate}\\ R_3 \to R_3 - 4R_2, \\ R_4 \to R_4 - R_2\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 3, since minor of order 4 is zero and there is a non-zero minor $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 33 \end{vmatrix}$ of order three. Hence the rank of the given matrix is also 3.

Example 2.25: Reduce the following matrix into its normal form and hence find its rank.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

Solution: The matrix is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix}\text{Operate}\\ C_2 \to C_2 - C_1, \\ C_3 \to C_3 - 2C_1\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix}\text{Operate}\\ R_2 \to R_2 - R_1\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\begin{matrix}\text{Operate}\\ C_3 \to C_3 - C_2\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{matrix}\text{Operate}\\ R_3 \to R_3 + R_2\end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is the normal form of A. Thus rank of A is 2.

Example 2.26: For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ find non-singular matrices **P** and **Q** such that **PAQ** is in the normal form. Hence, find the rank of **A**.

Solution: Write **A** = **IAI**, that is,

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We shall perform every elementary row (column) transformation of A by subjecting the prefactor (post factor) of A to the same operation.

Operate $R_2 \to R_2 + (-4)R_1$, we obtain

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $C_2 \to C_2 + C_1$, $C_3 \to C_3 + (-2)C_1$, $C_4 \to C_4 + 3C_1$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 5 & -8 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -4 & 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $C_4 \to C_4 + (-2)C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & -2 \\ 0 & 5 & -8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -4 & 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_3 \to R_3 + (-3)R_2$, $R_4 \to R_4 + (-5)R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ -4 & 1 & 0 & -5 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $C_4 \to C_4 + 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -8 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ -4 & 1 & 0 & -5 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_4 \to R_4 + 8R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ -4 & 1 & 8 & -29 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_4 \to -\frac{1}{12} R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 1/3 & -1/12 & -2/3 & 29/12 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The L.H.S is in its normal form. Hence,

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 1/3 & -1/12 & -2/3 & 29/12 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the rank of $\mathbf{A} = 4$.

2.7.5 The Echelon Form of a Matrix

A matrix of order $m \times n$ is said to be in row (column) echelon form, if

(1) *the entries in a row (column) appear to the right (below) of the first non-zero entry*

(2) *the number of zeros preceding the first non-zero element in the ith row (column) is less than that the number of such zeros in the $(i+1)$th row (column); and*

(3) *all rows (columns) that consist entirely of zeros lie at the bottom (right) of the matrix.*

For example, the matrices

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{and} \begin{bmatrix} 1 & 0 & 5 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are in their row echelon forms and the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}, \text{and} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

are in their column echelon form.

We note that in case of a square matrix the row echelon form is an upper triangular matrix and the column echelon form is a lower triangular matrix.

A matrix **A** is reduced to its row-echelon form by performing a sequence of appropriate row transformations over it, and is reduced to its column echelon form by performing a sequence of appropriate column transformations over it.

The rank of a matrix can also be found from its echelon form. In fact *rank is equal to the number of non-zero rows (columns) in its row (column) echelon form*. In addition to this echelon approach is applied to test whether a given set of vectors are linearly independent or not, which we shall discuss in Section 2.8.2.

2.7.6 Gauss-Jorden Method of Finding the Inverse

The method is stated as follows.

The elementary row operations which reduce a given square matrix **A** *of order* $n \times n$ *to a unit matrix* $\mathbf{I}_n$ *when applied to the unit matrix* $\mathbf{I}_n$ *give the inverse of the matrix* **A**.

Proof. Let the successive row operations which reduce the given square matrix **A** of order $n \times n$ to $\mathbf{I}_n$ result from the pre-multiplication of **A** by the elementary matrices $\mathbf{R}_1, \mathbf{R}_2, ..., \mathbf{R}_k$, so that

$$\mathbf{R}_k \mathbf{R}_{k-1} ... \mathbf{R}_2 \mathbf{R}_1 \mathbf{A} = \mathbf{I}$$

Post-multiplying this by $\mathbf{A}^{-1}$, we obtain

$$(\mathbf{R}_k \mathbf{R}_{k-1} ... \mathbf{R}_2 \mathbf{R}_1 \mathbf{A})\mathbf{A}^{-1} = \mathbf{I}\mathbf{A}^{-1}, \text{ or } \mathbf{R}_k \mathbf{R}_{k-1} ... \mathbf{R}_2 \mathbf{R}_1 (\mathbf{A}\mathbf{A}^{-1}) = \mathbf{A}^{-1}$$

or,

$$\mathbf{R}_k \mathbf{R}_{k-1} ... \mathbf{R}_2 \mathbf{R}_1 \mathbf{I} = \mathbf{A}^{-1}, \text{or } \mathbf{A}^{-1} = \mathbf{R}_k \mathbf{R}_{k-1} ... \mathbf{R}_2 \mathbf{R}_1 \mathbf{I}.$$

To find $\mathbf{A}^{-1}$ we write the augmented matrix $[\mathbf{A}; \mathbf{I}]$, where **I** is the unit matrix of the same order as that of **A**. Then perform the same row operations on both **A** and **I**. As and when **A** reduces to **I**, the other matrix represents $\mathbf{A}^{-1}$.

This method also fails to work when det $\mathbf{A} = 0$. In such a case the elementary row operations applied produce one or more rows of zeros at the bottom so that **A** cannot be reduced to I_n. In fact, in this case **A** reduces to its normal form $\begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ since the rank of **A** in this case is less than n.

Remark: The method can also be applied by subjecting $[\mathbf{A}; \mathbf{I}]$ to a sequence of elementary column operations only.

Example 2.27: Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ by applying Gauss-Jorden method.

Solution: Consider the augmented matrix

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \text{Operate} \\ R_2 \to R_2 + 2R_1 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 9 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \text{Operate} \\ R_2 \leftrightarrow R_3 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 9 & 1 & 2 & 1 & 0 \end{array}\right] \begin{array}{l} \text{Operate} \\ R_1 \to R_1 - 3R_2, \\ R_3 \to R_3 - 9R_2 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & -3 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -8 & 2 & 1 & -9 \end{array}\right]$$

$$\begin{array}{l} \text{Operate} \\ R_3 \to (-1/8)\, R_3 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & -3 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/4 & -1/8 & 9/8 \end{array}\right]$$

$$\begin{array}{l} \text{Operate} \\ R_2 \to R_2 - R_3, \\ R_1 \to R_1 + 3R_3 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -3/8 & 3/8 \\ 0 & 1 & 0 & 1/4 & 1/8 & -1/8 \\ 0 & 0 & 1 & -1/4 & -1/8 & 9/8 \end{array}\right]$$

$$\text{Hence, } A^{-1} = \frac{1}{8}\begin{bmatrix} 2 & -3 & 3 \\ 2 & 1 & -1 \\ -2 & -1 & 9 \end{bmatrix}$$

EXERCISE 2.4

1. If $\mathbf{P} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then

'for the following' matrices verify by direct calculation that, (a) pre-multiplication by **P** multiplies row 1 by 3; (b) pre-multiplication by **Q** interchanges rows 1 and 3; and (c) post-multiplication by **R** adds twice column 2 to column 1.

(a) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \\ 3 & 1 & 2 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix}$

2. Using the elementary row transformations, find the rank of the following matrices:

(a) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$

3. Using the elementary column transformations, find the rank of the following matrices:

(a) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & -1 & 0 \\ 4 & 1 & 0 & 5 \\ 0 & 1 & 3 & 6 \\ 6 & 1 & -2 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$

4. Using elementary row and column operations, find the rank of the following matrices:

(a) $\begin{bmatrix} 2 & 1 & -2 \\ -1 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 5 \end{bmatrix}$

5. Using elementary row and column operations reduce the following matrices to their normal forms and hence find their ranks.

(a) $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 7 & 11 & 15 & 19 \\ 7 & 15 & 21 & 27 \end{bmatrix}$

6. For the following matrices find non-singular matrices **P** and **Q** such that **PAQ** is in normal form. Also find their ranks.

(a) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & -1 & 0 \\ 4 & 1 & 0 & 5 \\ 0 & 1 & 3 & 6 \\ 6 & 1 & -2 & 6 \end{bmatrix}$

7. For the two matrices given in the preceding problem verify that the rank of the product does not exceed the rank of the either.

8. Illustrate with an example that rank (**A**) = rank(**B**) does not necessarily imply that rank($\mathbf{A}^2$) = rank($\mathbf{B}^2$).

9. Reduce the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$ to its row echelon form and find its rank.

10. Reduce the matrix $\begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix}$ to its column echelon form and find its rank.

11. Use Gauss-Jorden method to find the inverse of the following matrices:

(a) $\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 4 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

2.8 VECTOR SPACES

In this section we discuss vector space and other related concepts.

2.8.1 Vector Space

A *non-empty* set V of elements which may be vectors, matrices or functions, etc. denoted by $\boldsymbol{a}$, **b**,... is called a *vector space* and these elements are called *vectors*, if in V are defined two algebraic operations: *Vector addition* and *scalar multiplication* as follows:

I. *Vector addition* associates with every pair of vectors $\boldsymbol{a}$ and $\mathbf{b}$ of V a unique vector of V, called the sum of $\boldsymbol{a}$ and $\mathbf{b}$ denoted by $\boldsymbol{a} + \mathbf{b}$, such that the following axioms are satisfied.

(i) $\boldsymbol{a} + \mathbf{b} = \mathbf{b} + \boldsymbol{a}$	(commutativity)
(ii) $(\boldsymbol{a} + \mathbf{b}) + \mathbf{c} = \boldsymbol{a} + (\mathbf{b} + \mathbf{c})$	(associativity)
(iii) $\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}$	(existence of a unique zero vector in V)
(iv) $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$	(existence of additive inverse in V)

II. *Scalar multiplication* associates with every vector $\boldsymbol{a}$ of V and any scalar α, a unique vector of V, called the scalar-product of α and $\boldsymbol{a}$, denoted by $\alpha\,\boldsymbol{a}$, such that the following axioms are satisfied.

(v) $\alpha(\boldsymbol{a} + \mathbf{b}) = \alpha\,\boldsymbol{a} + \alpha\,\mathbf{b}$	(distributivity)
(vi) $(\alpha + \beta)\boldsymbol{a} = \alpha\,\boldsymbol{a} + \beta\,\boldsymbol{a}$	(distributivity)
(vii) $\alpha(\beta\,\boldsymbol{a}) = (\alpha\,\beta)\boldsymbol{a}$	(associativity)
(viii) $1\boldsymbol{a} = \boldsymbol{a}$	(existence of multiplicative identity)

The vector addition and scalar multiplication defined above are not necessarily the usual addition and multiplication operators. Thus the vector space depends not only on the set V of vectors, but also on the definitions of the algebraic operations on V.

If both the elements of V and the scalars α, β are real, then V is called a *real vector space*, otherwise V is called a *complex vector space*.

Examples of a few vector spaces under the usual operations of vector addition and scalars are:

1. All ordered n-tuples of real numbers as vectors and real numbers as scalars forms $\mathbb{R}^n$, n-dimensional real vector space. In particular, for $n = 3$ we have $\mathbb{R}^3$ consisting of ordered triplets vectors in space.
2. The set of all $m \times n$ matrices under usual operations of matrices addition and scalar multiplication forms a vector space.
3. The set of all constant, linear and quadratic polynomials in x together forms a vector space under the usual addition of two polynomials and multiplication of a polynomial by a real number.

But we must note that set of all quadratic polynomials in x does not form a vector space under the usual addition and scalar multiplication by a real, since sum of the two quadratic polynomial may not be a quadratic polynomial.

2.8.2 Linear Dependence and Independence of Vectors

Let V be a vector space. A finite set of m vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ each with the same number of components of V is said to be linearly dependent, if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero, such that

$$\alpha_1\,\boldsymbol{a}_1 + \alpha_2\,\boldsymbol{a}_2 + \ldots + \alpha_m\,\boldsymbol{a}_m = \mathbf{0} \qquad \ldots(2.13)$$

because then we can express at least one of the vectors as a linear combination of the others. For example, if in (2.13) $\alpha_1 \neq 0$, then we can rewrite (2.13) as

$$\boldsymbol{a}_1 = \left(\frac{-\alpha_2}{\alpha_1}\right)\boldsymbol{a}_2 + \left(\frac{-\alpha_3}{\alpha_1}\right)\boldsymbol{a}_3 + \ldots + \left(\frac{-\alpha_m}{\alpha_1}\right)\boldsymbol{a}_m$$

or, $$\mathbf{a}_1 = \beta_2 \mathbf{a}_2 + \beta_3 \mathbf{a}_3 + \ldots + \beta_m \mathbf{a}_m, \text{ where } \beta_i = \left(\frac{-\alpha_i}{\alpha_1}\right), i = 1, 2, \ldots, m$$

If (2.13) is satisfied only for $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$, then the set of vectors is said to be *linearly independent*.

For example, the three vectors $\mathbf{a}_1 = [1, 1, 1, 3]$, $\mathbf{a}_2 = [1, 2, 3, 4]$, and $\mathbf{a}_3 = [3, 4, 5, 10]$ are linearly dependent since it is easy to check that these vectors satisfy the relation

$$2\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 = 0. \qquad \ldots(2.14)$$

Method to check linear independence (dependence) of vectors. For a given set of vectors, it is easy to verify an equation of the form (2.14) but it is not so obvious to construct it to find out whether the given set of vectors are linearly independent or dependent. A test to check this for m vectors, with m components each, is as follows.

In this case, (2.13) gives a homogeneous system of m algebraic equations in m unknowns $\alpha_1, \alpha_2, \ldots, \alpha_m$. Non-trivial solution exists if the determinant of the coefficient matrix is zero, that is, *vectors are linearly dependent, if the determinant of the coefficient matrix is zero. In case of non-zero determinant the vectors are linearly independent.*

In general, to check the linear dependence or independence of m vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ with n components each, when n may or may not be equal to m, find the rank of the matrix with rows (or columns) as vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. *If the matrix with row (or, column) vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ has rank m, the vectors are linearly independent. However, if the rank is less than m the vectors are linearly dependent.* The rank of the matrix formed so gives the number of linearly independent vectors in the set of m vectors.

In fact, we reduce the matrix of m row (column) vectors to its row (or, column) echelon form, the number of non-zero rows (or columns) gives the number of linearly independent vectors, in the set of m vectors.

Also we have the following result which follows immediately from above. *The m vectors, with n components each $n < m$ are always linearly dependent.* For example, three or more vectors in the plane are always linearly dependent.

Another result of interest is *a set of vectors containing* **0** *as one of its elements is always linearly dependent*, since it can always be expressed as

$$\alpha_1 \mathbf{0} + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3 + \ldots + \alpha_m \mathbf{a}_m = 0$$

for $\alpha_1 \neq 0$, and $\alpha_2 = \alpha_3 = \ldots = \alpha_m = 0$.

2.8.3 Dimension, Basis and Span of a Vector Space

Let V be a vector space. The maximum number of linearly independent vectors in V is called the *dimension* of V and is denoted by *dim* (V).

The set S consisting of the maximum possible number of linearly independent vector of V is called a *basis* of V. Thus the number of vectors in a basis of V equals the *dimension of* V.

The set of all linear combinations of the linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ with the same number of components is called the *span* of these vectors. We must note that *a basis set S spans the vector space V.*

For example, consider the vector space $\mathbb{R}^3$. Following sets of vectors of $\mathbb{R}^3$ are linearly independent and thus each set forms a basis of $\mathbb{R}^3$.

(i) $[1, 0, 0], [0, 1, 0], [0, 0, 1]$ (ii) $[1, -1, 0], [0, 1, -1], [0, 0, 1]$

We observe that *there can be more than one basis for the same vector space.*

As another example, the three different basis for $\mathbb{R}^2$ are:

(i) $[1, 0], [0, 1]$ (ii) $[1, 1], [1, -1]$, and (iii) $[1, 0], [0, -1]$

Obviously, $\dim(\mathbb{R}^2) = 2$, and $\dim(\mathbb{R}^3) = 3$.

If elements of a vector space V are the real 2×2 matrices, then the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

forms a basis of V and spans V, since any element $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of V can be expressed as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The dimension of the vector space V is four, the number of elements in S. Similarly, the real $m \times n$ matrices, with fixed m and n form an mn-dimensional vector space.

2.8.4 Subspaces

Let V be a vector space defined with two algebraic operations of vector addition and scalar multiplication. A non-empty subset W of V that itself forms a vector space under the same two algebraic operations as defined for the vectors of V, is called a *subspace* of the vector space V.

The vector space V is also taken as a subspace of itself.

To show that W is a subspace of V, it is sufficient to show only that (i) W is closed under vector addition and scalar multiplication, and (ii) the existence of the '*zero element*' and the '*additive inverse*' in W.

For example, if V is the set of all $n \times n$ real square matrices with usual algebraic operations of matrix addition and scalar multiplication, then the subset W consisting of all symmetrical matrices of order $n \times n$ forms a subspace of V, but the subset W' consisting of all $n \times n$ matrices having real positve elements does not form a subspace of V since W' is not closed under scalar multiplication and also the null matrix of order $n \times n$ in V does not belong to W'.

Example 2.28: Test whether the following set of vectors in $\mathbb{R}^4$ is linearly dependent or independent: $\mathbf{x}_1 = (2, 1, 1, 0)$, $\mathbf{x}_2 = (0, 2, 0, 1)$, $\mathbf{x}_3 = (1, 1, 0, 2)$, $\mathbf{x}_4 = (0, 2, 1, 1)$.

Solution: Consider the vector equation, $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \alpha_4\mathbf{x}_4 = \mathbf{0}$.

Substituting for $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, we obtain

$$\alpha_1 (2, 1, 1, 0) + \alpha_2(0, 2, 0, 1) + \alpha_3(1, 1, 0, 2) + \alpha_4 (0, 2, 1, 1) = \mathbf{0}.$$

Comparing, we get

$$\left.\begin{aligned} 2\alpha_1 + 0.\alpha_2 + \alpha_3 + 0.\alpha_4 &= 0 \\ \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 &= 0 \\ \alpha_1 + 0.\alpha_2 + 0.\alpha_3 + \alpha_4 &= 0 \\ 0.\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 &= 0 \end{aligned}\right\} \quad \text{...(2.15)}$$

Determinant of the coefficient matrix of the system of Eqs. (2.15) is

$$|\mathbf{A}| = \begin{vmatrix} 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 2\{-(4-1)\} + \{1(-1) - 1(2-2)\} = -7 \neq 0$$

Therefore, the set of equations (2.15) has only a trivial solution $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ and hence the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ are linearly independent.

Example 2.29: Verify that the following set of vectors in $\mathbb{R}^3$ is linearly dependent:

$$\mathbf{x}_1 = (1, 0, 1), \mathbf{x}_2 = (1, 1, 1), \mathbf{x}_3 = (1, 1, 2), \mathbf{x}_4 = (1, 2, 1)$$

Also find the number of linearly independent vectors.

Solution: Consider the matrix **A** with column vectors as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, we have

$$|\mathbf{A}| = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_3 \to R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{matrix} \text{Operate} \\ C_2 \to C_2 - C_1, \\ C_3 \to C_3 - C_1, \\ C_4 \to C_4 - C_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} \text{Operate} \\ C_3 \to C_3 - C_2, \\ C_4 \to C_4 - 2C_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus the rank of **A** is 3, hence the given set of vectors is linearly dependent and the number of linearly independent vectors is three.

Alternatively, it is a set of m (= 4) vectors with n (= 3) components each and $m > n$, hence is linearly dependent.

Example 2.30: Test whether the following sets of vectors are linearly dependent or independent. Find the dimension and the basis of the given set of vectors.

(a) $(1, 3, 5), (2, -1, 4), (-2, 8, 2)$

(b) $(2, 3, 6, -3, 4), (4, 2, 12, -3, 6), (4, 10, 12, -9, 10)$

(c) $(3, 0, 2, 2), (-1, 7, 4, 9), (7, -7, 0, -5)$

Solution: (a) Consider the matrix **A** with row vectors $(1,3,5), (2,-1,4)$ and $(-2,8,2)$, we have

$$\mathbf{A}=\begin{bmatrix}1&3&5\\2&-1&4\\-2&8&2\end{bmatrix}\begin{matrix}\text{Operate}\\R_2\to R_2-2R_1,\\R_3\to R_3+2R_1\end{matrix}\sim\begin{bmatrix}1&3&5\\0&-7&-6\\0&14&12\end{bmatrix}\begin{matrix}\text{Operate}\\R_3\to R_3+2R_2\end{matrix}\sim\begin{bmatrix}1&3&5\\0&-7&-6\\0&0&0\end{bmatrix}$$

This is the row-echelon form of **A** and since all the rows in the row-echelon form of **A** are not non-zero, the given set of vectors is linearly dependent.

The dimension is two equal to the number of non-zero rows and a basis can be taken as the set $\{(1,3,5),(0,-7,-6)\}$.

(b) Consider the matrix **A** with row vectors as the given vectors, we have

$$\mathbf{A}=\begin{bmatrix}2&3&6&-3&4\\4&2&12&-3&6\\4&10&12&-9&10\end{bmatrix}$$

$$\begin{matrix}\text{Operate}\\R_2\to R_2-2R_1,\\R_3\to R_3-2R_1\end{matrix}\sim\begin{bmatrix}2&3&6&-3&4\\0&-4&0&3&-2\\0&4&0&-3&2\end{bmatrix}\begin{matrix}\text{Operate}\\R_3\to R_2+R_3\end{matrix}\sim\begin{bmatrix}2&3&6&-3&4\\0&-4&0&3&-2\\0&0&0&0&0\end{bmatrix}$$

Since all the rows in the row echelon form of **A** are not non-zero, the given set of vectors is linearly dependent with dimension 2, the number of non-zero rows. A basis can be taken as the set $\{(2,3,6,-3,4),(0,-4,0,3,-2)\}$.

(c) Consider the matrix **A** with row vectors $(3,0,2,2), (-1,7,4,9)$ and $(7,-7,0,-5)$, we have

$$\mathbf{A}=\begin{bmatrix}3&0&2&2\\-1&7&4&9\\7&-7&0&-5\end{bmatrix}\begin{matrix}\text{Operate}\\R_2\to 3R_2,\\R_3\to 3R_3\end{matrix}\sim\begin{bmatrix}3&0&2&2\\-3&21&12&27\\21&-21&0&-15\end{bmatrix}$$

$$\begin{matrix}\text{Operate}\\R_2\to R_2+R_1,\\R_3\to R_3-7R_1\end{matrix}\sim\begin{bmatrix}3&0&2&2\\0&21&14&29\\0&-21&-14&-29\end{bmatrix}\begin{matrix}\text{Operate}\\R_3\to R_3+R_2,\end{matrix}\sim\begin{bmatrix}3&0&2&2\\0&21&14&29\\0&0&0&0\end{bmatrix}$$

Since all the rows in the row echelon form of **A** are not non-zero, the given set of vectors are linearly dependent with dimension 2, the number of non-zero rows. A basis can be taken as the set $\{(3,0,2,2),(0,21,14,29)\}$.

Example 2.31: In the following determine a basis for the subspace S of $\mathbb{R}^n$ and determine the dimension of the subspace.

(a) S consists of all vectors on or parallel to the plane $x+y+z=0$ in $\mathbb{R}^3$.

(b) S consists of all vectors $(0, x, y, 0, y)$ in $\mathbb{R}^5$

Solution: (a) A vector (x,y,z) in $\mathbb{R}^3$ is in S when $z=-x-y$, thus vectors in S can be expressed as

$$(x,y,z)=(x,y,-x-y)=x(1,0,-1)+y(0,1,-1)$$

Thus the two vectors $(1, 0, -1)$ and $(0, 1, -1)$ span S and also the vectors are linearly independent, since one can't be expressed as a scalar multiple of the second, and hence form a basis for S and $\dim(S) = 2$.

(b) The set S consists of all vectors of the form $(0, x, y, 0, y)$ in $\mathbb{R}^5$ can be expressed as

$$(0, x, y, 0, y) = x(0, 1, 0, 0, 0) + y(0, 0, 1, 0, 1)$$

Thus, the two vectors $(0, 1, 0, 0, 0)$ and $(0, 0, 1, 0, 1)$ span S and also the two are linearly independent vectors. Hence, form a basis for S and $\dim(S) = 2$.

Example 2.32: Let V be the set of all ordered pairs (x, y), where x, y are real numbers. The two algebraic operations of addition and scalar multiplication are defined as

$$(x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$$
$$\alpha(x_1, y_1) = (\alpha x_1/3, \alpha y_1/3)$$

Show that V is not a vector-space.

Solution: Consider the commutative law for addition

$$(x_2, y_2) + (x_1, y_1) = (2x_2 - 3x_1, y_2 - y_1) \neq (x_1, y_1) + (x_2, y_2).$$

Thus the law does not hold.

Next, consider the associative law for addition

$$\{(x_1, y_1) + (x_2, y_2)\} + (x_3, y_3) = (2x_1 - 3x_2, y_1 - y_2) + (x_3, y_3) = (4x_1 - 6x_2 - 3x_3, y_1 - y_2 - y_3)$$

and, $(x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\} = (x_1, y_1) + (2x_2 - 3x_3, y_2 - y_3) = (2x_1 - 6x_2 - 9x_3, y_1 - y_2 + y_3)$

Hence, $\{(x_1, y_1) + (x_2, y_2)\} + (x_3, y_3) \neq (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}$

Thus, associative law for addition is also not satisfied.

Consider $1.(x_1, y_1) = (x_1/3, y_1/3) \neq (x_1, y_1)$

Thus, the existence of multiplicative identity is not satisfied.

Hence V is not a vector-space.

Example 2.33: Find the span of (a) $\mathbf{x}_1 = (5, 1)$, $\mathbf{x}_2 = (1, 3)$ in $\mathbb{R}^2$ (b) $\mathbf{x}_1 = (1, 2, 2)$, $\mathbf{x}_2 = (-1, 0, 2)$ in $\mathbb{R}^3$

Solution: (a) Let $\mathbf{y} = (y_1, y_2)$ be any given vector in $\mathbb{R}^2$. We try to express $\mathbf{y} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$.

That is, $(y_1, y_2) = \alpha_1 (5, 1) + \alpha_2(1, 3) = (5\alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2)$

Thus $5\alpha_1 + \alpha_2 = y_1, \alpha_1 + 3\alpha_2 = y_2$

It is obvious, that for given y_1, y_2 the above system of equation is solvable for α_1, α_2 and hence $\{x_1, x_2\}$ spans $\mathbb{R}^2$.

(b) Let $\mathbf{y} = (y_1, y_2, y_3)$ be any given vector in $\mathbb{R}^3$. We try to express $\mathbf{y} = \alpha_1\mathbf{x}_1 + \alpha_1\mathbf{x}_2$.

That is, $(y_1, y_2, y_3) = \alpha_1 (1, 2, 2) + \alpha_2(-1, 0, 2) = (\alpha_1 - \alpha_2, 2\alpha_1, 2\alpha_1, + 2\alpha_2)$.

Thus $\alpha_1 - \alpha_2 = y_1, 2\alpha_1 = y_2, 2\alpha_1 + 2\alpha_2 = y_3$ ...(2.16)

Solving first two equations in (2.16) for α_1, α_2 and substituting in the third equation, we get $2y_1 - 2y_2 + y_3 = 0$

Thus, the span of $\{\mathbf{x}_1, \mathbf{x}_2\}$ is the set of all possible vectors $\mathbf{y} = (y_1, y_2, y_3)$ such that $2y_1 - 2y_2 + y_3 = 0$. That is, the span of $\{\mathbf{x}_1, \mathbf{x}_2\}$ is the subspace of $\mathbf{R}^3$ consisting of all vectors in the plane $2y_1 - 2y_2 + y_3 = 0$.

EXERCISE 2.5

1. Is the given set, under usual addition and scalar multiplication a vector space or not? Give reason. In case the answer is yes, find a basis and determine the dimension:
   (a) All 2×2 matrices such that $a_{11} + a_{22} = 0$
   (b) All polynomials in x of degree less than or equal to 4.
   (c) All $m \times n$ matrices with positive enteries
   (d) All vectors (x, y, z) in $\mathbf{R}^3$ such that $2x + 3z = 0$
2. Show that the following sets are linearly dependent by expressing one of the vectors as a linear combination of the others
   (a) $\{(1, 1), (1, 2), (3, 4)\}$
   (b) $\{(1, 2, 3), (3, 2, 1), (5, 5, 5)\}$
   (c) $\{(1, 0, 0,), (0, 1, 0), (3, 3, 0), (2, -7, 9)\}$
3. Determine whether the following set is linearly independent or linearly dependent. If it is linearly dependent, then find a linear combination of the vectors.
   (a) $\{(2, 3, 0), (1, -2, 4)\ (1, 1, 0), (1, 1, 1)\}$
   (b) $\{(4, -1, 1, 2), (3, 0, 2, 5), (0, 0, 0, 0)\}$
   (c) $\{(1, 3, 4, 2), (3, -5, 2, 2), (2, -1, 3, 2)\}$
   (d) $\{(1, -1, 0), (0, 1, -1), (0, 0, 1)\}$
   (e) $\{(1, 9, 9, 8), (2, 0, 0, 3), (2, 0, 0, 8)\}$.
4. Show whether the given sets are identical. Explain
   (a) span $\{(2, -1, -1), (3, 1, 0)\}$; and span $\{(2, -1, -1), (5, 5, 2)\}$
   (b) span $\{(1, -1, 0), (0, 0, 1), (1, 2, 3)\}$; and span $\{(1, -1, 0), (0, 1, -1), (0, 0, 1)\}$
   (c) span $\{(1, 2, -1), (-3, 0, 0)\}$; and span $\{(1, 0, 0), (1, 3, 0)\}$
5. Let V be the set of all ordered triplets (x, y, z) in $\mathbf{R}^3$ with vector addition defined as $(x, y, z) + (u, v, w) = (3x + 4u, y - 2v, z + w)$ and scalar multiplication defined as $\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z/3)$. Discuss whether V is a vector space or not.
6. Let V be the set of all positive real numbers with addition defined as $x + y = xy$ and scalar multiplication defined as $\alpha x = x^{\alpha}$. Discuss whether V is a vector space or not.
7. Prove that the set S of all vectors lying in any plane in $\mathbf{R}^3$ that passes through the origin forms a subspace of $\mathbf{R}^3$. Does the set of all vectors lying in any plane in $\mathbf{R}^3$ that don't pass through origin form a subspace of $\mathbf{R}^3$?
8. Let V be the set of all 2×2 complex matrices with algebraic operations of usual matrix addition and scalar multiplication and S consisting of all matrices of the form: $\begin{bmatrix} z & x+iy \\ x-iy & t \end{bmatrix}$, where x, y, z, t are real numbers. Check whether S forms a subspace of V when (a) scalars are real numbers, (b) scalars are complex numbers.

9. Find a basis of the row space and of the column space of the matrix

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

10. Show that a 'zero vector space', that is, a vector space consisting of the zero vector alone, has no basis.

2.9 MATRICES AS LINEAR TRANSFORMATIONS

The matrices of order $n \times m$ can be viewed as linear transfomations from $\mathbf{R}^n$ to $\mathbf{R}^m$. We discuss this aspect in this section.

2.9.1 Linear Transformation

Let X and Y be two vector spaces both real or complex, over the same field of scalars. If to each vector $\mathbf{x}$ in X is assigned a unique vector $\mathbf{y}$ in Y, then we say that a *mapping* or *transformation* of X into Y is given. Let this transformation be denoted by T. The vector $\mathbf{y}$ in Y assigned to vector $\mathbf{x}$ in X is called the *image* of $\mathbf{x}$ under T and is denoted by $T(\mathbf{x})$ and $\mathbf{x}$ is called the *pre-image* of $\mathbf{y}$.

The transformation T is called a *linear transformation*, if for all vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in X and scalars α,

$$T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2), \text{ and } T(\alpha\mathbf{x}_1) = \alpha T(\mathbf{x}_1).$$

These conditions are equivalent to

$$T(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \alpha T(\mathbf{x}_1) + \beta T(\mathbf{x}_2)$$

for $\mathbf{x}_1$ and $\mathbf{x}_2$ in X and any scalars α and β.

Sum and product of linear transformations. Let T_1 and T_2 be two linear transformations from X into Y. The *sum* $T_1 + T_2$ is defined as

$$(T_1 + T_2)(\mathbf{x}) = T_1(\mathbf{x}) + T_2(\mathbf{x}), \mathbf{x} \in X.$$

It can be verified that $T_1 + T_2$ is also a linear transformation, and $T_1 + T_2 = T_2 + T_1$.

If X, Y and Z are three vector spaces, all real or complex, on the same scalar field and T_1 is defined from X into Y and T_2 is defined from Y into Z, then the *product* T_2T_1 is defined from X into Z as

$$T_2T_1(\mathbf{x}) = T_2(T_1(\mathbf{x})).$$

The transformation T_2T_1 is also called *composite transformation*.

It can be verified that T_2T_1 is also a linear transformation and, in general, $T_2T_1 \neq T_1T_2$.

2.9.2 Matrix Representation of a Linear Transformation

Let $X = \mathbf{R}^n$ and $Y = \mathbf{R}^m$. Then any real $m \times n$ matrix $\mathbf{A} = [a_{ij}]_{m\times n}$ gives a transformation of $\mathbf{R}^n$ into $\mathbf{R}^m$,

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \qquad \ldots(2.17)$$

where $\mathbf{x}$ is in $\mathbf{R}^n$ and $\mathbf{y}$ is in $\mathbf{R}^m$.

The transformation defined by (2.17) is linear transformation, since

$$\mathbf{A}(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \alpha\mathbf{A}\mathbf{x}_1 + \beta\mathbf{A}\mathbf{x}_2.$$

It can also be shown that every linear transformation T of $\mathbf{R}^n$ into $\mathbf{R}^m$ can be given in terms of an $m \times n$ matrix $\mathbf{A}$, after selecting a basis for $\mathbf{R}^n$ and a basis for $\mathbf{R}^m$.

In two-dimensional space $\mathbf{R}^2$ the standard basis is $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

In three-dimensional space $\mathbf{R}^3$ the standard basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

For example, the matrix representing the linear transformation that maps (x_1, x_2) to $(2x_1 - 5x_2, 3x_1 + 4x_2)$ is $\mathbf{A} = \begin{bmatrix}2 & -5\\3 & 4\end{bmatrix}$ since this transformation can be represented as

$$y_1 = 2x_1 - 5x_2$$
$$y_2 = 3x_1 + 4x_2$$

Similarly, the matrix representing the linear transformation (x_1, x_2, x_3) to $(x_2 + x_3, x_2 - x_3)$ is

$$\mathbf{A} = \begin{bmatrix}0 & 1 & 1\\0 & 1 & -1\end{bmatrix}_{2\times 3}$$

If in (2.17), the matrix of transformation $\mathbf{A}$ is square matrix of order $n \times n$, then (2.17) maps $\mathbf{R}^n$ into $\mathbf{R}^n$. If the matrix $\mathbf{A}$ is singular the transformation (2.17) is called *singular transformation*. If $\mathbf{A}$ is *non-singular*, then (2.17) is called *non-singular or regular transformation*. Further if $\mathbf{A}$ is non-singular, then $\mathbf{A}^{-1}$ exists; pre-multiplying (2.17) by $\mathbf{A}^{-1}$, gives the *inverse transformation*

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \qquad \text{...(2.18)}$$

The inverse transformation is also a linear transformation.

Orthogonal transformation The linear transformation $\mathbf{y} = \mathbf{A}\mathbf{x}$ is said to be orthogonal if it transforms

$$y_1^2 + y_2^2 + \ldots + y_n^2 \text{ into } x_1^2 + x_2^2 + \ldots + x_n^2. \qquad \text{...(2.19)}$$

The matrix of orthogonal transformation is an *orthogonal matrix*, (refer to Section 2.3.3), for (2.19) gives

$$\mathbf{x}^T\mathbf{x} = \mathbf{y}^T\mathbf{y} = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x}.$$

It is possible only if, $\mathbf{A}^T\mathbf{A} = \mathbf{I}$. Thus $\mathbf{A}$ is an orthogonal matrix.

For example, the transformation $y_1 = x_1,\ \ y_2 = \cos\theta\, x_2 - \sin\theta\, x_3,\ \ y_3 = \sin\theta\, x_2 + \cos\theta\, x_3$

is orthogonal. We can verify that it transforms $x_1^2+x_2^2+x_3^2$ to $y_1^2+y_2^2+y_3^2$, and the coefficient matrix

$$A=\begin{vmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{vmatrix}$$ is orthogonal.

Example 2.34: Test whether the transformation (x_1, x_2, x_3) to $(2x_1+x_2+x_3, x_1+x_2+2x_3, x_1-2x_3)$ is non-singular. If so write the inverse transformation.

Solution: The given transformation is

$$y_1 = 2x_1 + x_2 + x_3$$
$$y_2 = x_1 + x_2 + 2x_3$$
$$y_3 = x_1 + 0x_2 - 2x_3$$

The matrix of transformation is $\mathbf{A}=\begin{bmatrix}2 & 1 & 1\\ 1 & 1 & 2\\ 1 & 0 & -2\end{bmatrix}$

Here, $|\mathbf{A}| = -1 \neq 0$. Hence the transformation is non-singular. The inverse transformation is given by $\mathbf{x}=\mathbf{A}^{-1}\mathbf{y}$. We can see easily that $\mathbf{A}^{-1}=\begin{bmatrix}2 & -2 & -1\\ -4 & 5 & 3\\ 1 & -1 & -1\end{bmatrix}$. Thus the inverse transformation is

$$x_1 = 2y_1 - 2y_2 - y_3$$
$$x_2 = -4y_1 + 5y_2 + 3y_3$$
$$x_3 = y_1 - y_2 - y_3$$

Example 2.35: Find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2 when $x_1 = 3y_1 + 2y_2$, $x_2 = -y_1 + 4y_2$, and $y_1 = z_1 + 2z_2$, $y_2 = 3z_1$.

Solution: The matrix of transformation from $\mathbf{y}$ to $\mathbf{x}$ is $\mathbf{A}=\begin{bmatrix}3 & 2\\ -1 & 4\end{bmatrix}$

The matrix of transformation from $\mathbf{z}$ to $\mathbf{y}$ is $\mathbf{B}=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$

Hence, the matrix of transformation from $\mathbf{z}$ to $\mathbf{x}$ is

$$\mathbf{C}=\mathbf{AB}$$
$$=\begin{bmatrix}3 & 2\\ -1 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}=\begin{bmatrix}9 & 6\\ 11 & -2\end{bmatrix}$$

Therefore, $x_1 = 9z_1 + 6z_2$, $x_2 = 11z_1 - 2z_2$.

EXERCISE 2.6

1. Find the inverse transformation of the following:

(a) $y_1 = 3x_1 + 2x_2$
$y_2 = 4x_1 + x_2$

(b) $y_1 = (-2x_1 + x_2 + 2x_3)/3$
$y_2 = (2x_1 + 2x_2 + x_3)/3$
$y_3 = (x_1 - 2x_2 + 2x_3)/3$

(c) $y_1 = x_1$
$y_2 = x_2 \cos\theta + x_3 \sin\theta$
$y_3 = -x_2 \sin\theta + x_3 \cos\theta$

2. Obtain the transformation from (x_1, x_2, x_3) to (z_1, z_2, z_3) when

$$\begin{aligned} y_1 &= 2x_1 + x_2 \\ y_2 &= x_2 - 2x_3 \\ y_3 &= -x_1 + 2x_2 + x_3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} z_1 &= y_1 + y_2 + y_3 \\ z_2 &= y_1 + 2y_2 + 3y_3 \\ z_3 &= y_1 + 3y_2 + 5y_3 \end{aligned}$$

3. Find a, b, and c such that the following transformation is orthogonal

$$\begin{aligned} y_1 &= ax_1 + bx_2 + cx_3, \qquad & y_2 &= -x_4, \\ y_3 &= cx_1 + ax_2 - bx_3, \qquad & y_4 &= -bx_1 + cx_2 - ax_3. \end{aligned}$$

4. Prove that the composite transformation of two orthogonal transformations is also orthogonal.

2.10 SOLUTION OF LINEAR SYSTEM OF EQUATIONS: GENERAL FORM

In this section we shall consider the non-homogeneous as well as homogeneous system of equations of the $m \times n$ form.

2.10.1 System of Linear Non-homogeneous Equations

Consider the system of m linear non-homogeneous equations in n unknowns $x_1, x_2, \ldots x_n$ given by

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}\right\} \qquad \ldots(2.20)$$

The system of Eqs. (2.20) in the matrix form is

$$\mathbf{Ax} = \mathbf{b}, \qquad \ldots(2.21)$$

where $\mathbf{A} = [a_{ij}]_{m \times n}$ is the coefficient matrix,
$\mathbf{x} = (x_1, x_2, \ldots x_n)^T$ is the solution vector; and
$\mathbf{b} = (b_1, b_2, \ldots b_n)^T$.

Using the concept of the rank, we discuss first the *consistency of the system of* Eqs. (2.20). Consider the ranks of the *coefficient matrix* $\mathbf{A}$ and the *augmented matrix* $\tilde{\mathbf{A}}$, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \text{ and } \tilde{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}_{m\times(n+1)}$$

We have the following result:

Theorem 2.1 (Fundamental Theorem for Linear System) *The system of Eqs. (2.20) is consistent, if and only if the coefficient matrix* **A** *and the augmented matrix* $\tilde{\mathbf{A}}$ *have the same rank.*

Proof. Let $\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \dots \mathbf{c}_{(n)}$ be the column vectors of the coefficient matrix **A**, then (2.21) can be written as

$$\mathbf{c}_{(1)} x_1 + \mathbf{c}_{(2)} x_2 + \dots + \mathbf{c}_{(n)} x_n = \mathbf{b} \qquad \dots(2.22)$$

Let the rank of **A** be r. Since r is the rank of **A**, so the matrix **A** has r linearly independent columns and without loss of generality, we can take these to be $\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \dots, \mathbf{c}_{(r)}$. Thus, each of the column vectors $\mathbf{c}_{(r+1)}, \mathbf{c}_{(r+2)}, \dots, \mathbf{c}_{(n)}$ will be a linear combination of $\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \dots \mathbf{c}_{(r)}$.

Now if the system (2.22) is consistent, then there must exist scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$\alpha_1 \mathbf{c}_{(1)} + \alpha_2 \mathbf{c}_{(2)} + \dots + \alpha_n \mathbf{c}_{(n)} = \mathbf{b}$$

which shows that **b** is a linear combination of $\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \dots \mathbf{c}_{(n)}$ and hence that of $\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \dots, \mathbf{c}_{(r)}$. Thus, the rank of the matrix $\mathbf{A} = [\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \dots, \mathbf{c}_{(n)}]$ is equal to the rank of the augmented matrix $\tilde{\mathbf{A}} = [\mathbf{c}_{(1)}, \mathbf{c}_{(2)}, \dots, \mathbf{c}_{(n)}, \mathbf{b}]$.

Conversely, if rank of **A** is equal to rank of $\tilde{\mathbf{A}}$, then **b** must be a linear combination of the column vectors of **A**, say

$$\mathbf{b} = \alpha_1 \mathbf{c}_{(1)} + \alpha_2 \mathbf{c}_{(2)} + \dots + \alpha_n \mathbf{c}_{(n)} \qquad \dots(2.23)$$

since otherwise rank of $\tilde{\mathbf{A}}$ = rank $\mathbf{A} + 1$. But (2.23) implies that the given system is consistent and has a solution $x_1 = \alpha_1, x_2 = \alpha_2, \dots, x_n = \alpha_n$ by comparing (2.22) with (2.23).

This completes the proof.

Remarks. 1. The system (2.20) has a unique solution if the common rank r of **A** and $\tilde{\mathbf{A}}$ equals n, the number of unknowns.

2. If r is less than n, then the system has infinitely many solutions and all these solutions are obtained by determining r suitable unknowns in terms of the remaining $(n - r)$ unknowns to which arbitrary values can be assigned. Solutions form the $(n - r)$ parameters family of solutions.

3. The solution(s), if exists, can be obtained by reducing the augmented matrix $\tilde{A}$ in the row-echelon form.

Example 2.36: Test for consistency and solve the system of equations

$$x + y + z = 6, x - y + 2z = 5, 3x + y + z = 8, 2x - 2y + 3z = 7.$$

Solution: Consider the augmented matrix

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_2 - R_1, \\ R_3 - 3R_1, \\ R_4 - 2R_1, \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{bmatrix}$$

$$\begin{matrix}\text{Operate}\\ R_3 - R_2, \\ R_4 - 2R_2\end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & -0 & -1 & -3 \end{bmatrix} \quad \text{Operate } R_4 - \frac{1}{3}R_3 \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is clear that rank of $\mathbf{A}$ = rank of $\widetilde{\mathbf{A}} = 3$. Thus the system of equations is consistent and has a unique solution.

To find the solution, we have the equations

$$x + y + z = 6$$
$$-2y + z = -1$$
$$-3z = -9$$

Using the backward substitution, we obtain $x = 1$, $y = 2$, and $z = 3$.

Example 2.37: Find the values of λ and μ so that the equations

$$2x + 3y + 5z = 9, \quad 7x + 3y - 2z = 8, \quad 2x + 3y + \lambda z = \mu$$

have (i) no solution, (ii) a unique solution, and (iii) an infinite number of solutions.

Solution: The coefficient and the augmented matrices are respectively

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}, \text{ and } \widetilde{\mathbf{A}} = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

The system has the unique solution if, and only if the rank of A is three, and for this

$$|\mathbf{A}| = \begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5 - \lambda) \neq 0.$$

That is, $\lambda \neq 5$, thus for a unique solution $\lambda \neq 5$ and μ can have any value.

If $\lambda = 5$, then system will have no solution for those values of μ for which the matrices

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$$

have different ranks, but first matrix is of rank 2 and second is not of rank 2 unless $\mu = 9$. Thus, the system will have no solution if $\lambda = 5$ and $\mu \neq 9$. Further, if $\lambda = 5$ and $\mu = 9$, the system will have infinite number of solutions.

Example 2.38: Test for the consistency of the following system of equations

$$x+y+2z+w=5,\ 2x+3y-z-2w=2,\ 4x+5y+3z=7.$$

Solution: The augmented matrix is

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_3 - 2R_2 \\ R_2 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & -1 & 5 & 4 & 3 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

Now the rank of $\mathbf{A} = 2$, and rank of $\tilde{\mathbf{A}} = 3$, since $\begin{vmatrix} 2 & 1 & 5 \\ -5 & -4 & -8 \\ 0 & 0 & -5 \end{vmatrix} = (-5)(-8+5) = 15 \neq 0.$

Hence, the given system of equations is inconsistent.

Example 2.39: Test for consistency of the following system of equations

$$2x-3y+5z=1,\ 3x+y-z=2,\ x+4y-6z=1,$$

and, if consistent, solve the system.

Solution: The augmented matrix is

$$\tilde{\mathbf{A}} = \begin{bmatrix} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{bmatrix} \text{Operate } R_1 \leftrightarrow R_3 \sim \begin{bmatrix} 1 & 4 & -6 & 1 \\ 3 & 1 & -1 & 2 \\ 2 & -3 & 5 & 1 \end{bmatrix}$$

$$\begin{matrix} \text{Operate} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & -11 & 17 & -1 \end{bmatrix} \text{Operate } R_3 - R_2 \sim \begin{bmatrix} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, the rank of A = rank $\tilde{A} = 2$. Hence, the system is consistent but since the common value of the rank is less than the number of unknowns, the system has infinite number of solutions.

The two independent equations are: $x+4y-6z=1, -11y+17z=-1.$

Solving these in terms of z, we get $x = \dfrac{7-2z}{11}, y = \dfrac{1+17z}{11}.$

Thus, the infinite solutions are given by $x = \dfrac{7-2t}{11}, y = \dfrac{1+17t}{11}, z = t$, where t is a parameter.

2.10.2 System of Linear Homogeneous Equations

Consider the system of m linear homogeneous equations in n unknowns $x_1, x_2, \ldots, x_n$ given by

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots \quad + \ldots + \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right\} \quad \ldots(2.24)$$

In the matrix form the system of equations (2.24) is

$$\mathbf{Ax} = \mathbf{0} \quad \ldots(2.25)$$

Since the coefficient matrix **A** and the augmented matrix $\tilde{\mathbf{A}}$ in this case have the same rank, say r, hence the system (2.24) is always consistent.

If $r = n$, the number of unknowns, the system (2.24) has only trivial solution

$$x_1 = 0, x_2 = 0, \ldots, x_n = 0.$$

If $r < n$, the system has infinitely many non-zero solutions obtained by determining r suitable unknowns in terms of the remaining $(n - r)$ variables to which arbitrary values can be assigned.

We note that in case the number of equations m is less than the number of variables n, then the homogeneous system (2.24) always has solutions other than the trivial solution $x_1 = x_2 = \ldots = x_n = 0$.

In case the number of equations m is equal to the number of variables then the necessary and sufficient condition for the system to have non-trivial solution is that $|\mathbf{A}| = 0$, as discussed earlier in Section 2.6.

Example 2.40: Solve the equations

$$4x + 2y + z + 3w = 0, \quad 6x + 3y + 4z + 7w = 0, \quad 2x + y + w = 0$$

Solution: The coefficient matrix is

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{array}{c} \text{Operate} \\ R_2 - \frac{3}{2}R_1 \\ R_3 - \frac{1}{2}R_1 \end{array} \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix} \text{Operate } R_3 + \frac{1}{5}R_2 \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the rank of **A** is 2 less than the number of variables, thus there are infinite number of solutions.

The two independent equations are $4x + 2y + z + 3w = 0$, $z + w = 0$.

Solving these for, say z and y in terms of x and w, we have $z = -w$, $y = -2x - w$.

Hence the solutions are: $x = \alpha$, $y = -2\alpha - \beta$, $z = -\beta$, $w = \beta$, where α and β are two parameters.

Example 2.41: Find the values for λ for which the following equations $3x+y-\lambda z=0$, $4x-2y-3z=0$, $2\lambda x+4y+\lambda z=0$ possess a non-trivial solution, and also find the corresponding solutions.

Solution: The system will have non-trivial solution when the determinant of the coefficient matrix is zero, that is $\begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix}=0$ which gives $\lambda^2+8\lambda-9=0$, or $\lambda=1,-9$.

For $\lambda=1$, the system of equations is

$$3x+y-z=0,\ 4x-2y-3z=0,\ 2x+4y+z=0.$$

Taking any two equations, say the first two, we have

$$\frac{x}{-5}=\frac{y}{5}=\frac{z}{-10}, \text{ which gives, } x=\frac{z}{2}, y=\frac{-z}{2}.$$

Hence, the solution is $x=\dfrac{\alpha}{2}, y=\dfrac{-\alpha}{2}, z=\alpha$, where α is a parameter.

Similarly for $\lambda=-9$, we obtain the solution as $x=\dfrac{-3\beta}{2}, y=\dfrac{-9\beta}{2}, z=\beta$, where β is another parameter.

EXERCISE 2.7

Test for consistency the following systems of linear equations and solve, if consistent

1. $5x+3y+7z=4,\quad 3x+26y+2z=9,\quad 7x+2y+10z=5$
2. $x-4y+5z=8,\quad 3x+7y-z=3,\quad x+15y-11z=14.$
3. $x+2y-3z-4u=6,\quad x+3y+z-2u=4,\quad 2x+5y-2z-5u=10$
4. $x+2y+z=3,\quad 2x+3y+2z=5,\quad 3x-5y+5z=2,\quad 3x+9y-z=4$
5. $2x+6y+11=0,\quad 6x+20y-6z+3=0,\quad 6y-18z+1=0$

   Solve the following homogeneous systems of linear equations
6. $x-2y+z-w=0,\quad x+y-2z+3w=0,\quad 4x+y-5z+8w=0,\quad 5x-7y+2z-w=0.$
7. $3x+2y+z=0,\quad 2x+3z=0,\quad y+5z=0,\quad x+2y+3z=0$
8. $x+3y+2z=0,\quad 2x-y+3z=0,\quad 3x-5y+4z=0,\quad x+17y+4z=0.$
9. $x+y-z+w=0,\quad 2x+3y+z+4w=0,\quad 3x+2y-6z+w=0$
10. $x+y+z+w=0,\quad -x+y+z-w=0,\quad -x-y+z+w=0,\quad x+y-z+w=0.$
11. Discuss the consistency of the system of equations
    $2x-3y+6z-5w=3,\quad y-4z+w=1,\quad 4x-5y+8z-9w=\lambda$ for various values of λ, if consistent, find the solution
12. Find the value of λ for which the following equations have non-trivial solution. Find the corresponding families of solutions
    $(\lambda-1)x+(3\lambda+1)y+2\lambda z=0,\quad (\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0, 2x+(3\lambda+1)y+3(\lambda-1)z=0.$

13. Show that the equations $3x+4y+5z=a, 4x+5y+6z=b, 5x+6y+7z=c$ are inconsistent unless $a+c=2b$.
14. Find the value of λ for which the following system of equations possess a non-trivial solution and also find the solution in each case
$$2x-2y+z=\lambda x, \quad 2x-3y+2z=\lambda y, \quad -x+2y=\lambda z.$$

2.11 MATRIX EIGENVALUE PROBLEMS. CAYLEY-HAMILTON THEOREM

Let $\mathbf{A}=[a_{ij}]$ be a square matrix of order $n \times n$. The matrix eigenvalue problem concerns with set of equations of the form
$$\mathbf{Ax}=\lambda \mathbf{x}, \quad \text{...(2.26)}$$
where $\mathbf{x}$ is an unknown vector and λ is an unknown scalar and our aim is to determine both. Obviously, $\mathbf{x}=\mathbf{0}$ is a solution of (2.26) but trivial solution is of no practical interest. We are interested in non-trivial solutions. These are called *eigenvectors* of $\mathbf{A}$ and the values of λ for which such non-trivial solutions exist are called *eigenvalues* or *characteristic values* of $\mathbf{A}$. Geometrically, solving (2.26) implies looking for those vectors $\mathbf{x}$ which are transformed by the mapping $\mathbf{Ax}$ into vector $\lambda\mathbf{x}$ with components proportional to $\mathbf{x}$ itself, λ being the constant of proportionality.

The problem of determining the eigenvalues and eigenvectors of a matrix is called an eigenvalue problem.

Matrix eigenvalue problems find numerous applications in the field of engineering and physical sciences e.g. in the study of Markov processes, stretching of elastic membrane, vibrations of beams, population models, etc.

2.11.1 Eigenvalues and Eigenvectors

The set of equations (2.26) can be written as
$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}. \quad \text{...(2.27)}$$

This homogeneous system of equations has a non-trivial solution if, and only if the determinant of the coefficient matrix $(\mathbf{A}-\lambda\mathbf{I})$ is zero, that is,
$$|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0. \quad \text{...(2.28)}$$

The $|\mathbf{A}-\lambda\mathbf{I}|$ is called the *characteristic determinant* and the equation $|\mathbf{A}-\lambda\mathbf{I}|=0$ is called the *characteristic equation of the matrix* $\mathbf{A}$. Expanding (2.28) we obtain characteristic equation of $\mathbf{A}$ of the form
$$|\mathbf{A}-\lambda\mathbf{I}| = (-1)^n\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n = 0 \quad \text{...(2.29)}$$
a polynomial of degree n in λ, here c's are expressible in terms of the elements a_{ij}. Thus, the eigenvalues of a square matrix $\mathbf{A}$ are the roots of its characteristic equation (2.28).

The characteristic equation (2.28) has n roots which may be real or complex, simple or repeated, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ called the *eigenvalues*. The set of the eigenvalues is called the *spectrum of* **A** and the largest of absolute values of the eigenvalues of **A** is called the *spectral radius* of **A**.

After determining the eigenvalues λ_i's we solve the homogeneous equations $(\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{x} = 0$ for each λ_i to get the corresponding *eigenvector* $\mathbf{x}_i$. Corresponding to n distinct eigenvalues we get n independent eigenvectors. But corresponding to repeated eigenvalues, it may or may not be possible to get linearly independent eigenvectors.

Also, *if* $\mathbf{x}$ *is an eigenvector of a matrix* **A** *corresponding to an eigenvalues* λ, *then so is* $k\mathbf{x}$ *for any* $k \neq 0$, *since* $\mathbf{Ax} = \lambda\mathbf{x} \Rightarrow \mathbf{A}\,(k\mathbf{x}) = \lambda(k\mathbf{x})$.

Thus the eigenvector corresponding to an eigenvalue is not unique in this sense.

Example 2.42: Find the eigenvalues and eigenvectors of the following matrices:

(a) $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Solution: (a) The characteristic equation of the matrix **A** is $|\mathbf{A} - \lambda\mathbf{I}| = 0$, that is, $\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$ or, $\lambda^2 - 5\lambda - 6 = 0$, or $\lambda = 6, -1$. Thus the eigenvalues of **A** are 6 and -1.

Corresponding to $\lambda = 6$, the eigenvector **x** is given by, $(\mathbf{A} - 6\mathbf{I})\mathbf{x} = \mathbf{O}$,

or, $$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives only one independent equation $5x_1 + 2x_2 = 0$. From this, we obtain $\dfrac{x_1}{2} = \dfrac{x_2}{-5}$ and thus the eigenvector is $[2, -5]^T$.

Corresponding to $\lambda = -1$, the eigenvector **x** is given by, $(\mathbf{A} + \mathbf{I})\mathbf{x} = \mathbf{O}$

or, $$\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives only one independent solution $x_1 - x_2 = 0$, or $x_1 = x_2$, and thus the eigenvector is $[1, 1]^T$.

(b) The characteristic equation for the matrix **A** is $|\mathbf{A} - \lambda\mathbf{I}| = 0$, which gives

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0.$$

On expansion, we obtain $\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$, or $(\lambda - 1)(\lambda - 1)(\lambda - 7) = 0$.

Therefore, the eigenvalues of A are $\lambda = 1, 1, 7$.

For $\lambda = 1$, if $\mathbf{x}$ is the eigenvector, then we have $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{O}$, which gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives only one independent equation $x_1 + x_2 + x_3 = 0$.

Taking $x_1 = 0, x_2 = 1$, and $x_3 = -1$; $x_1 = 1, x_2 = 0, x_3 = -1$.

We can have two linearly independent eigenvectors as $[0, 1, -1]^T$ and $[1, 0, -1]^T$.

For $\lambda = 7$, if $\mathbf{x}$ is the eigenvector, then we have $(\mathbf{A} - 7\mathbf{I})\mathbf{x} = \mathbf{O}$, which gives

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -5x_1 + x_2 + x_3 &= 0 \\ 2x_1 - 4x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 - 3x_3 &= 0 \end{aligned}$$

From the first two equations we have

$$\frac{x_1}{6} = \frac{x_2}{2+10} = \frac{x_3}{20-2}, \text{ or } \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}.$$ Therefore the eigenvector is $[1, 2, 3]^T$.

Example 2.43: Find the eigenvalues and the eigenvectors of the following matrices:

$$\text{(a) } \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(b) } \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(c) } \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: (a) The characteristic equation for the matrix $\mathbf{A}$ is $|\mathbf{A} - \lambda\mathbf{I}| = 0$. It gives

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0, \text{ or } (1-\lambda)^3 = 0.$$

Therefore the eigenvalues for $\mathbf{A}$ are $\lambda = 1, 1, 1$.

Since eigenvalues are repeated, we will be interested to know whether the matrix $\mathbf{A}$, which is of order 3×3, has three independent eigenvectors or it has less than three independent eigenvectors.

For $\lambda = 1$, the eigenvector $\mathbf{x}$ is given by $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$, that is, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

which gives $x_2 = 0, x_3 = 0$ and x_1 arbitrary. Choosing $x_1 = 1$, hence the only eigenvector is $[1, 0, 0]^T$.

(b) The eigenvalues are $\lambda = 1, 1, 1$. The eigenvector $\mathbf{x}$ corresponding to $\lambda = 1$ in this case is given by $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $x_2 = 0$, and x_1 and x_3 arbitrary. Taking $x_1 = 0$, $x_3 = 1$ and $x_1 = 1$, $x_3 = 0$, the two independent eigenvectors are $[0\ \ 0\ \ 1]^T$ and $[1\ \ 0\ \ 0]$.

(c) The eigenvalues are $\lambda = 1, 1, 1$. The eigenvector $\mathbf{x}$ corresponding to $\lambda = 1$ in this case is given by $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

which gives x_1, x_2, x_3 arbitrary. Taking $x_1 = 1, x_2 = x_3 = 0$; $x_2 = 1, x_1 = x_3 = 0$; $x_1 = x_2 = 0, x_3 = 1$. The three independent eigenvectors are $[1\ \ 0\ \ 0]^T$, $[0\ \ 1\ \ 0]^T$, $[0\ \ 0\ \ 1]^T$.

Remark. The order M_λ of an eigenvalue λ as a root of the characteristic equation is called the *algebraic multiplicity* of λ. The number m_λ of linear independent eigenvectors corresponding to λ is called the *geometric multiplicity* of λ. The difference $M_\lambda - m_\lambda$ is called the *defect* of λ. In the above example the algebraic multiplicity of $\lambda = 1$ is 3, but, defect of λ is 2 in (a), 1 in (b) and 0 in (c).

2.11.2 Properties of Eigenvalues and Eigenvectors

1. *The sum of the eigenvalues of a matrix* $\mathbf{A}$ *is the sum of the elements of the principal diagonal, that is, equal to trace* $(\mathbf{A})$.

We prove this for a square matrix $\mathbf{A}$ of order 3×3. We have

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = -\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 - \ldots \qquad \ldots(2.30)$$

If λ_1, λ_2 and λ_3 are the eigenvalues of the matrix $\mathbf{A}$, then

$$|\mathbf{A} - \lambda\mathbf{I}| = (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = -\lambda^3 + (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - \ldots \qquad \ldots(2.31)$$

Comparing the right-hand sides of (2.30) and (2.31), we obtain

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = \text{trace}\,(\mathbf{A}).$$

2. *The product of the eigenvalue of a matrix* $\mathbf{A}$ *is equal to its determinant.*

Set $\lambda = 0$ in (2.31), we obtain $|\mathbf{A}| = (-1)^6 \lambda_1\lambda_2\lambda_3$, that is, $\lambda_1\lambda_2\lambda_3 = |\mathbf{A}|$.

From this result, we observe that *if* $\mathbf{A}$ *is singular then at least one of its eigenvalue is zero and conversely.*

3. *If* λ *is an eigenvalue of a matrix* $\mathbf{A}$, *then* $1/\lambda$ *is the eigenvalue of* $\mathbf{A}^{-1}$, *provided the inverse exists.*

Let $\mathbf{x}$ be the eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Pre-multiplying both sides of this by $\mathbf{A}^{-1}$, we get $\mathbf{A}^{-1}(\mathbf{A}\mathbf{x}) = \mathbf{A}^{-1}(\lambda\mathbf{x})$, or $\mathbf{I}\mathbf{x} = \lambda(\mathbf{A}^{-1}\mathbf{x})$, or $\mathbf{A}^{-1}\mathbf{x} = \dfrac{1}{\lambda}\mathbf{x}$.

Thus $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$.

4. *The matrices* $\mathbf{A}$ *and* $\mathbf{A}^T$ *has the same eigenvalues.*

This result follows from the fact that determinant $|\mathbf{A}-\lambda\mathbf{I}| = |\mathbf{A}^T-\lambda\mathbf{I}|$.

5. *If* λ *is an eigenvalue of an orthogonal matrix, then* $1/\lambda$ *is also its eigenvalue.*

Since, a matrix is orthogonal if $\mathbf{A}\mathbf{A}^T = \mathbf{I}_n$, that is, if $\mathbf{A}^T = \mathbf{A}^{-1}$. Thus the result follows from properties 3 and 4 above.

6. *If* λ *is an eigenvalue of a matrix* $\mathbf{A}$, *then* λ^m *is eigenvalue for* $\mathbf{A}^m$, *m being a positive integer.*

For, if $\mathbf{x}$ is the corresponding eigenvector of $\mathbf{A}$, then $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.

Pre-multiplying by $\mathbf{A}$ on both sides, we obtain $\mathbf{A}^2\mathbf{x} = \mathbf{A}(\lambda\mathbf{x}) = \lambda(\mathbf{A}\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$, and so on. In general, we have $\mathbf{A}^m\mathbf{x} = \lambda^m\mathbf{x}$

Thus λ^m is an eigenvalue of $\mathbf{A}^m$.

7. *If* λ *is an eigenvalue for* $\mathbf{A}$, *then* $\lambda - k$ *is an eigenvalue for* $\mathbf{A} - k\mathbf{I}$ *for any scalar k.*

For, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ implies $\mathbf{A}\mathbf{x} - k\mathbf{I}\mathbf{x} = \lambda\mathbf{x} - k\mathbf{x}$, or $(\mathbf{A} - k\mathbf{I})\mathbf{x} = (\lambda - k)\mathbf{x}$

Thus $(\lambda - k)$ is an eigenvalue of $\mathbf{A} - k\mathbf{I}$. This property is called *spectral shift*.

8. *If* λ *is an eigenvalue of* $\mathbf{A}$, *then* $k\lambda$ *is an eigenvalue of* $k\mathbf{A}$ *for any non-zero scalar k.*

For, $\quad \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ implies $k\mathbf{A}\mathbf{x} = k\lambda\mathbf{x}$, or $(k\mathbf{A})\mathbf{x} = (k\lambda)\mathbf{x}$

Thus $k\lambda$ is an eigenvalue of $k\mathbf{A}$.

9. *Corresponding to two distinct eigenvalues, eigenvectors are also distinct (linearly independent).*

Let $\mathbf{x}$ be the common eigenvector corresponding to the two distinct eigenvalues λ_1 and λ_2, then

$$\mathbf{A}\mathbf{x} = \lambda_1\mathbf{x} \text{ and } \mathbf{A}\mathbf{x} = \lambda_2\mathbf{x}, \text{ which gives } \lambda_1\mathbf{x} = \lambda_2\mathbf{x}, \text{ or } (\lambda_1 - \lambda_2)\mathbf{x} = \mathbf{0}$$

Since $\lambda_1 \neq \lambda_2$, thus $\mathbf{x} = \mathbf{0}$, a contradiction since eigenvectors are non-zero.

But the result is not true otherwise, refer to Example 2.43.

In fact we have the following result which we state without proof. *If* λ *is an eigenvalue of multiplicity of a square matrix* $\mathbf{A}$ *of order n, then the number of linearly independent eigenvectors associated with* λ *is* $k = n - r$, *where* $r = rank(\mathbf{A} - \lambda\mathbf{I})$, $1 \le k \le m$.

10. *Eigenvalues of diagonal and triangular matrices, upper and lower both, are same as the diagonal elements.*

11. *For a real matrix* $\mathbf{A}$, *if* $\alpha + i\beta$ *is an eigenvalue, then its conjugate* $\alpha - i\beta$ *is also an eigenvalue of* $\mathbf{A}$. *This result does not hold in case* $\mathbf{A}$ *is a complex matrix.*

Theorem 2.2 (Cayley-Hamilton Theorem). *Every square matrix satisfies its own characteristic equation.*

Proof. Let $\mathbf{A}$ be a square matrix of order $n \times n$, then its characteristic equation is

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \text{ or, } (-1)^n\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n = 0.$$

Let $\mathbf{P}$ be the adjoint of the matrix $\mathbf{A} - \lambda\mathbf{I}$, then elements of $\mathbf{P}$ will be the polynomials of degree $(n-1)$ in λ. Thus the matrix $\mathbf{P}$ can be expressed in the form of a matrix polynomial

$$\mathbf{P} = \mathbf{P}_1\lambda^{n-1} + \mathbf{P}_2\lambda^{n-2} + \ldots + \mathbf{P}_{n-1}\lambda + \mathbf{P}_n,$$

where $\mathbf{P}_i$'s are all the square matrices of order n, whose elements are functions of the elements of $\mathbf{A}$.

Also we know that, $[\mathbf{A} - \lambda\mathbf{I}]\ adj.(\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}|\ \mathbf{I}$; here $\mathbf{I}$ is a unit matrix of order $n \times n$.

Therefore, $[\mathbf{A}-\lambda\mathbf{I}][\mathbf{P}_1\lambda^{n-1}+\mathbf{P}_2\lambda^{n-2}+\ldots+\mathbf{P}_{n-1}\lambda+\mathbf{P}_n]=[(-1)^n\lambda^n+c_1\lambda^{n-1}+\ldots+c_{n-1}\lambda+c_n]\,\mathbf{I}$

Equating the coefficients of various powers of λ, we get

$$\begin{aligned}
-\mathbf{P}_1 &= (-1)^n\mathbf{I}\\
\mathbf{AP}_1-\mathbf{P}_2 &= c_1\mathbf{I}\\
\mathbf{AP}_2-\mathbf{P}_3 &= c_2\mathbf{I}\\
&\ldots\\
\mathbf{AP}_{n-1}-\mathbf{P}_n &= c_{n-1}\,\mathbf{I}\\
\mathbf{AP}_n &= c_n\mathbf{I}.
\end{aligned}$$

Pre-multiply these equations from top to bottom, respectively by $\mathbf{A}^n, \mathbf{A}^{n-1}, \ldots, \mathbf{A}, \mathbf{I}$ and add the terms on the left get cancelled in pair and we obtain

$$\mathbf{O}=(-1)^n\mathbf{A}^n+c_1\mathbf{A}^{n-1}+\ldots+c_{n-1}\,\mathbf{A}+c_n\mathbf{I}, \qquad \ldots(2.32)$$

the desired result.

The result (2.32) gives an alternate method to find the inverse of a matrix **A**, pre-multiplying (2.32) by $\mathbf{A}^{-1}$ and rearranging the terms, we obtain

$$\mathbf{A}^{-1}=-\frac{1}{c_n}[(-1)^n\mathbf{A}^{n-1}+c_1\mathbf{A}^{n-2}+\ldots+c_{n-1}I] \qquad \ldots(2.33)$$

Also (2.32) can be used to obtain $\mathbf{A}^n$ in terms of lower powers of n as

$$\mathbf{A}^n=(-1)^{n+1}[c_1\mathbf{A}^{n-1}+c_2\mathbf{A}^{n-2}+\ldots+c_{n-1}\mathbf{A}+c_nI]. \qquad \ldots(2.34)$$

Example 2.44: Verify Cayley Hamilton theorem for the matrix $\mathbf{A}=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$ and hence find $\mathbf{A}^{-1}$.

Solution: The characteristic equation of the matrix **A** is $|\mathbf{A}-\lambda\mathbf{I}|=0$, which gives

$$\begin{vmatrix}1-\lambda & 2 & -2\\ -1 & 3-\lambda & 0\\ 0 & -2 & 1-\lambda\end{vmatrix}=0, \text{ or } \lambda^3-5\lambda^2+9\lambda-1=0.$$

To verify Cayley-Hamilton theorem we are to verify that

$$\mathbf{A}^3-5\mathbf{A}^2+9\mathbf{A}-I=0 \qquad \ldots(2.35)$$

We have $\mathbf{A}^2=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}=\begin{bmatrix}-1 & 12 & -4\\ -4 & 7 & 2\\ 2 & -8 & 1\end{bmatrix}$,

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

Therefore, $\mathbf{A}^3 - 5\mathbf{A}^2 + 9\mathbf{A} - \mathbf{I}$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -13+5+9-1 & 42-60+18-0 & -2+20-18-0 \\ -11+20-9-0 & 9-35+27-1 & 10-10+0-0 \\ 10-10+0-0 & -22+40-18-0 & -3-5+9-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This verifies Cayley Hamiltan theorem.

Pre-multiplying (2.35) by $\mathbf{A}^{-1}$ and rearranging the terms, we obtain

$$\mathbf{A}^{-1} = \mathbf{A}^2 - 5\mathbf{A} + 9\mathbf{I} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Example 2.45: For $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show $\mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}$, for $n \geq 3$, and hence find $\mathbf{A}^n$.

Solution: The characteristic equation of the matrix $\mathbf{A}$ is $|\mathbf{A} - \lambda\mathbf{I}| = 0$, which gives

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0, \text{ or } \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

Using Cayley-Hamilton theorem, we have $\mathbf{A}^3 - \mathbf{A}^2 - \mathbf{A} + \mathbf{I} = \mathbf{0}$

$$\mathbf{A}^3 - \mathbf{A}^2 = \mathbf{A} - \mathbf{I} \qquad ...(2.36)$$

Pre-multiplying both sides successively by $\mathbf{A}$, we obtain

$$\mathbf{A}^4 - \mathbf{A}^3 = \mathbf{A}^2 - \mathbf{A}$$

$$\mathbf{A}^5 - \mathbf{A}^4 = \mathbf{A}^3 - \mathbf{A}^2$$

$$\cdots\cdots\cdots \quad \cdots\cdots\cdots$$

$$\mathbf{A}^{n-1} - \mathbf{A}^{n-2} = \mathbf{A}^{n-3} - \mathbf{A}^{n-4}$$

$$\mathbf{A}^n - \mathbf{A}^{n-1} = \mathbf{A}^{n-2} - \mathbf{A}^{n-3}$$

Adding these equations along with (2.36), we obtain

$$\mathbf{A}^n - \mathbf{A}^2 = \mathbf{A}^{n-2} - \mathbf{I}$$

$$\text{or,} \quad \mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I},\ n \geq 3. \qquad \text{...(2.37)}$$

Using the Eq. (2.37) recursively, we obtain

$$\begin{aligned} \mathbf{A}^n &= (\mathbf{A}^{n-4} + \mathbf{A}^2 - \mathbf{I}) + \mathbf{A}^2 - \mathbf{I} \\ &= \mathbf{A}^{n-4} + 2(\mathbf{A}^2 - \mathbf{I}) \\ &= \mathbf{A}^{n-6} + 3(\mathbf{A}^2 - \mathbf{I}) \\ &\dots\dots\dots\dots \\ &= \mathbf{A}^{n-(n-2)} + \frac{n-2}{2}(\mathbf{A}^2 - \mathbf{I}) \\ &= \mathbf{A}^2 + \frac{n-2}{2}(\mathbf{A}^2 - \mathbf{I}) = \frac{n}{2}\mathbf{A}^2 - \frac{1}{2}(n-2)\mathbf{I}. \end{aligned} \qquad \text{...(2.38)}$$

We have,

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Thus from (2.38)

$$\mathbf{A}^n = \frac{n}{2}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \frac{1}{2}(n-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ n/2 & 1 & 0 \\ n/2 & 0 & 1 \end{bmatrix}$$

EXERCISE 2.8

1. For the matrix, $\mathbf{A} = \begin{bmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ verify that if λ_i's are the eigenvalues of $\mathbf{A}$, then

   (a) $\lambda_i + 2$ are the eigenvalues of $\mathbf{A} + 2\mathbf{I}$,
   (b) $3\lambda_i$ are the eigenvalues of $3\mathbf{A}$
   (c) $\dfrac{1}{\lambda_i}$ are the eigenvalues of $\mathbf{A}^{-1}$,
   (d) λ^2 are the eigenvalues of $\mathbf{A}^2$.

2. Find the sum and product of the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{bmatrix}$

Find the eigenvalues and the eigenvectors of the matrices (3–5)

3. $\begin{bmatrix} 4 & -6 & -6 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ 5. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Verify Cayley-Hamilton theorem for the matrices (6–8). Find inverse, if it exists.

6. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ 7. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ 8. $\begin{bmatrix} 2 & -1 & +1 \\ -1 & 2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$

9. If $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, then show $\mathbf{A}^8 - 5\mathbf{A}^7 + 7\mathbf{A}^6 - 3\mathbf{A}^5 + \mathbf{A}^4 - 5\mathbf{A}^3 + 8\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} = \mathbf{0}$.

10. Find $\mathbf{A}^8$, if $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

2.12 SIMILAR MATRICES. DIAGONALIZATION

In this section we explore the properties of eigenvectors further. If an $n \times n$ square matrix **A** has n linearly independent eigenvectors they can be used for transforming **A** into diagonal form, an aspect important from application point of view.

2.12.1 Similar Matrices

A square matrix $\hat{\mathbf{A}}$ of order $n \times n$ is called *similar* to a square matrix **A** of order $n \times n$ if

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \qquad \text{...(2.39)}$$

for some non-singular matrix **P**.

The transformation (2.39) which gives $\hat{\mathbf{A}}$ from **A**, is called a *similarity transformation* and **P** *similarity matrix*.

Pre-multiplying both sides of (2.39) by **P** and post-multiplying by $\mathbf{P}^{-1}$, we get

$$\mathbf{P}\,\hat{\mathbf{A}}\mathbf{P}^{-1} = \mathbf{A} \qquad \text{...(2.40)}$$

Therefore, **A** is similar to $\hat{\mathbf{A}}$, if, and only if $\hat{\mathbf{A}}$ is similar to **A**.

The similarity transformations are important since they preserve eigenvalues.

We have the following result:

Theorem 2.3. *If $\hat{\mathbf{A}}$ is similar to* **A**, *then $\hat{\mathbf{A}}$ has the same eigenvalue as* **A**, *and further if* **x** *is an eigenvector of* **A** *corresponding to the eigenvalue λ, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue λ, where* **P** *is the similarity matrix.*

Proof. Since λ is an eigenvalue and $\mathbf{x}$ is the corresponding eigenvector of $\mathbf{A}$, thus $\mathbf{Ax} = \lambda\mathbf{x}$ Pre-multiplying both sides by $\mathbf{P}^{-1}$, we get

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\lambda\mathbf{x} = \lambda\mathbf{P}^{-1}\mathbf{x} = \lambda\mathbf{y} \qquad \text{...(2.41)}$$

Consider
$$\begin{aligned}\mathbf{P}^{-1}\mathbf{Ax} &= \mathbf{P}^{-1}\mathbf{AIx}\\ &= \mathbf{P}^{-1}\mathbf{APP}^{-1}\mathbf{x}, \quad \text{since } \mathbf{I} = \mathbf{PP}^{-1}\\ &= \hat{\mathbf{A}}\,(\mathbf{P}^{-1}\mathbf{x}) = \hat{\mathbf{A}}\mathbf{y}.\end{aligned}$$

Hence from (2.41),

$$\hat{\mathbf{A}}\mathbf{y} = \lambda\mathbf{y}$$

Thus λ is an eigenvalue of $\hat{\mathbf{A}}$ and the corresponding eigenvector is $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$.

Further, $\mathbf{P}^{-1}\mathbf{x} \neq \mathbf{0}$, since $\mathbf{P}^{-1}\mathbf{x} = \mathbf{0}$ would give $\mathbf{PP}^{-1}\mathbf{x} = \mathbf{P0} = \mathbf{0}$, or $\mathbf{Ix} = \mathbf{0}$, or $\mathbf{x} = \mathbf{0}$, which is a contradiction.

We must note that the converse of the above result is not true. *Two matrices which have the same eigenvalues may not be similar.*

Also, if $\mathbf{A}$ is similar to $\hat{\mathbf{A}}$ and $\hat{\mathbf{A}}$ is similar to $\hat{\mathbf{A}}_1$, then $\mathbf{A}$ is also similar to $\hat{\mathbf{A}}_1$, since

$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP}$, $\hat{\mathbf{A}}_1 = \mathbf{Q}^{-1}\hat{\mathbf{A}}\mathbf{Q}$, for some non-singular matrices $\mathbf{P}$ and $\mathbf{Q}$, gives

$$\hat{\mathbf{A}}_1 = \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{AP})\,\mathbf{Q} = (\mathbf{PQ})^{-1}\mathbf{A}(\mathbf{PQ}) = \mathbf{R}^{-1}\mathbf{AR},$$

where $\mathbf{R} = \mathbf{PQ}$ is non-singular.

2.12.2 Diagonalization of a Matrix

Generally, it is difficult to find a non-singular matrix $\mathbf{P}$ which satisfies the equation $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP}$ for any two matrices $\hat{\mathbf{A}}$ and $\mathbf{A}$.

However, it is possible to find $\mathbf{P}$ when $\hat{\mathbf{A}}$ or $\mathbf{A}$ is a diagonal matrix. Thus, for a given matrix $\mathbf{A}$ our interest is to find a non-singular matrix $\mathbf{P}$ such that

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}, \qquad \text{...(2.42)}$$

where $\mathbf{D}$ is a diagonal matrix. If such a matrix exists we say that $\mathbf{A}$ is *diagonalizable matrix* and the transformation (2.42) is called the *diagonalization transformation*.

Further, since similar matrices have the same eigenvalues thus the diagonal elements of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$.

We have the following result:

Theorem 2.4. *If a square matrix* $\mathbf{A}$ *of order* $n \times n$ *has* n *linearly independent eigenvectors, then*

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} \qquad \text{...(2.43)}$$

is diagonal matrix, with the eigenvalues of $\mathbf{A}$ *as the elements on the principal diagonal, and the similarity matrix* $\mathbf{P}$ *is the matrix with eigenvectors of* $\mathbf{A}$ *as column vectors.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent eigenvectors corresponding respectively to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix $\mathbf{A}$. Further let

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n] \text{ and } \mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

We have, $\mathbf{AP} = \mathbf{A}[\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n] = [\mathbf{Ax}_1, \mathbf{Ax}_2, \ldots, \mathbf{Ax}_n]$

$$= [\lambda_1\mathbf{x}_1, \lambda_2\mathbf{x}_2, \ldots, \lambda_n\mathbf{x}_n] = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]\,\mathbf{D} = \mathbf{PD}. \quad \ldots(2.44)$$

Since, the columns of $\mathbf{P}$ are linearly independent vectors, the matrix $\mathbf{P}$ is of rank n and, therefore, is invertible. Pre-multiplying both sides of (2.44) by $\mathbf{P}^{-1}$, we obtain $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, which is (2.43).

The matrix $\mathbf{P}$ here is called the *modal matrix* of $\mathbf{A}$ and the diagonal matrix $\mathbf{D}$ is called the *spectral matrix* of $\mathbf{A}$.

We must note that in the above result the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ not necessarily be distinct since even if eigenvalues are repeated the corresponding eigenvectors may be independent.

2.12.3 Calculation of Power of a Matrix

Diagonalization of a matrix is quite useful for obtaining powers of a matrix.

Let $\mathbf{A}$ be a square matrix of order n with n linearly independent eigenvectors, then $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$. It gives $\mathbf{A} = \mathbf{PDP}^{-1}$. Thus,

$$\mathbf{A}^2 = \mathbf{AA} = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}^2\mathbf{P}^{-1}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = (\mathbf{PD}^2\mathbf{P}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}^3\mathbf{P}^{-1}$$

In general, we have $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$ for positive integers n.

Example 2.46: Examine whether the following matrices are diagonizable or not.

(a) $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Solution: (a) The matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ has eigenvalues $\lambda = 6, -1$, and the corresponding linearly independent eigenvectors are $[2, -5]^T$ and $[1, 1]^T$, refer to Example 2.42a. Hence, the given matrix is diagonizable.

(b) The matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ has eigenvalues $\lambda = 1, 1, 7$, refer to Example 2.42b.

Corresponding to $\lambda = 1$, there are two linearly independent eigenvectors $[0, 1, -1]^T$ and $[1, 0, -1]^T$. Corresponding to $\lambda = 7$, the eigenvector is $[1, 2, 3]^T$. Since, the matrix has three linearly independent eigenvectors, hence it is diagonizable.

Example 2.47: Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is digonalizable. Find $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix and hence find $\mathbf{A}^2$.

Solution: The characteristic equation of the matrix **A** is $\begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0.$

It gives $(3-\lambda)[(1-\lambda)(2-\lambda)-6]=0$, or $(\lambda+1)(\lambda-3)(\lambda-4)=0$. Thus, the eigenvalues of A are $\lambda=-1$, 3, 4.

For $\lambda=-1$, the eigenvector $\mathbf{x}_1$ is given by

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 2x_1 + 6x_2 + x_3 &= 0 \\ x_1 + 3x_2 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

This gives $x_3=0, x_1=-3x_2$. Take $x_2=1$, the eigenvector is $\mathbf{x}_1=[-3,\ 1,\ 0]^T$

For $\lambda=3$, the eigenvector $\mathbf{x}_2$ is given by

$$\begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -2x_1 + 6x_2 + x_3 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

This gives $x_1=x_2, x_3=-4x_2$. Take $x_2=1$, the eigenvector is $\mathbf{x}_2=[1,\ 1,\ -4]^T$.

For $\lambda=4$, the eigenvector $\mathbf{x}_3$ is given by

$$\begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -3x_1 + 6x_2 + x_3 &= 0 \\ x_1 - 2x_2 &= 0 \\ -x_3 &= 0 \end{aligned}$$

This gives $x_3=0, x_1=2x_2$. Take $x_2=1$, the eigenvector $\mathbf{x}_3=[2,\ 1,\ 0]^T$

Since **A** has three linearly independent eigenvectors, thus it is diagonizable. The modal matrix **P** of **A** is $\mathbf{P} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$

To find $\mathbf{P}^{-1}$, we have $|\mathbf{P}| = -20$, and $adj\ \mathbf{P} = \begin{bmatrix} 4 & -8 & 1 \\ 0 & 0 & 5 \\ -4 & -12 & -4 \end{bmatrix}$

Thus, $\mathbf{P}^{-1} = \dfrac{adj\ \mathbf{P}}{|\mathbf{P}|} = \dfrac{1}{20}\begin{bmatrix} -4 & 8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$, and hence the spectral matrix **D** is

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \frac{1}{20}\begin{bmatrix} -4 & 8 & -1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

To find $\mathbf{A}^2$, we have $\mathbf{A}^2 = \mathbf{PD}^2\mathbf{P}^{-1}$

$$= \frac{1}{20}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} (-1)^2 & 0 & 0 \\ 0 & (3)^2 & 0 \\ 0 & 0 & (4)^2 \end{bmatrix}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}\begin{bmatrix} -4 & 8 & 1 \\ 0 & 0 & -5 \\ 4 & 12 & 4 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 140 & 360 & 80 \\ 60 & 200 & 20 \\ 0 & 0 & 180 \end{bmatrix} = \begin{bmatrix} 7 & 18 & 4 \\ 3 & 10 & 1 \\ 0 & 0 & 9 \end{bmatrix}$$

Example 2.48: The eigenvectors of a 3×3 matrix $\mathbf{A}$ corresponding to the eigenvalues 1, 2, 3 are $[-1, -1, 1]^T$, $[0, 1, 0]^T$ and $[0, -1, 1]^T$ respectively. Find the matrix $\mathbf{A}$.

Solution: The modal and spectral matrices of $\mathbf{A}$ are, respectively

$$\mathbf{P} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

We have $\mathbf{P}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and, therefore,

$$\mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix}$$

EXERCISE 2.9

1. For the matrices $\mathbf{A} = \begin{bmatrix} 10 & -3 & 5 \\ 0 & 1 & 0 \\ -15 & 9 & -10 \end{bmatrix}$ and $\mathbf{P} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$, find $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP}$. Also find the eigenvectors $\mathbf{y}$ of $\hat{\mathbf{A}}$ and show that $\mathbf{x} = \mathbf{Py}$ are the eigenvectors of $\mathbf{A}$.

Test the following matrices (2-4) for diagonalization. If diagonalizable, find $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

2. $\begin{bmatrix} 5 & 10 & -10 \\ 10 & 5 & -20 \\ 5 & -5 & -10 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ 4. $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

Find the matrix **A** whose eigenvalues and the corresponding eigenvectors are given as below.

5. Eigenvalues 1, –1, 2; Eigenvectors $[1,\ 1,\ 0]^T, [1,\ 0,\ 1]^T, [3,\ 1,\ 1]^T$
6. Eigenvalues 0, 0, 3; Eigenvectors $[1,\ 2,\ -1]^T, [-2,\ 1,\ 0]^T, [3,\ 0,\ 1]^T$
7. Eigenvalues 3, –4, 0; Eigenvectors $[-1,\ 3,\ -1]^T, [1,\ -1,\ 3]^T, [2,\ 1,\ 4]$
8. Find a matrix **P** which transforms the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to diagonal form. Hence, calculate $\mathbf{A}^4$.
9. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$, then show that $\mathbf{A}^8 = \begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$
10. Show that the matrix $\mathbf{A} = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, $a \neq b$ is transformed to a diagonal form $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, when $\mathbf{P} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where $\theta = \dfrac{1}{2}\tan^{-1}\left(\dfrac{2h}{a-b}\right)$.

2.13 SPECIAL MATRICES EIGENVALUES

In case of *real square matrices* we have already considered three special types of matrices namely, *symmetric*, *skew-symmetric* and *orthogonal*, refer to Section 2.3. These matrices occur quite frequently in applications.

In case of *complex square matrices*, the three special types of matrices are the generalization of the matrices considered in case of reals. These are *Hermitian*, *skew-Hermitian* and *unitary matrices*, refer to Section 2.3.

It is quite interesting to note that the spectra, that is the set of eigenvalues of these special matrices can be located as follows, refer to Fig. 2.1.

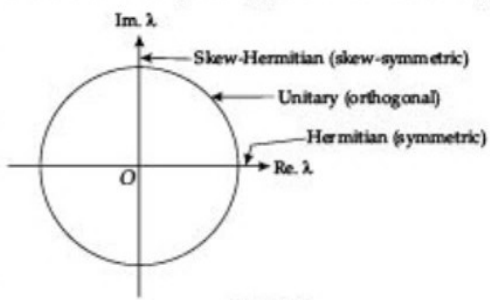


Fig. 2.1

We have the following result concerning the eigenvalues of special matrices:

Theorem (2.5) (Eigenvalues of Special Matrices)

(a) *The eigenvalues of a Hermitian matrix (and thus, of a symmetric matrix) are real.*

(b) *The eigenvalues of a skew-Hermitian matrix (and thus, of a skew-symmetric matrix) are pure imaginary or zero.*

(c) *The eigenvalues of a unitary matrix (and thus, of an orthogonal matrix) have absolute value 1.*

Proof. Let λ be an eigenvalue of a matrix **A** and **x** be the corresponding eigenvector. Then $\mathbf{Ax} = \lambda\mathbf{x}$. Pre-multiplying both sides of this by $\bar{\mathbf{x}}^T$, we get

$$\bar{\mathbf{x}}^T \mathbf{Ax} = \lambda \bar{\mathbf{x}}^T \mathbf{x}, \quad \text{or} \quad \lambda = \frac{\bar{\mathbf{x}}^T \mathbf{Ax}}{\bar{\mathbf{x}}^T \mathbf{x}}. \qquad \text{...(2.45)}$$

We note that $\bar{\mathbf{x}}^T \mathbf{Ax}$ and $\bar{\mathbf{x}}^T \mathbf{x}$ are scalars and the denominator $\bar{\mathbf{x}}^T \mathbf{x}$ is always real and positive. Therefore, the nature of λ depends upon the nature of the numerator $\bar{\mathbf{x}}^T \mathbf{Ax}$.

(a) *The matrix* **A** *is Hermitian, that is,* $\mathbf{A}^T = \bar{\mathbf{A}}$.

Since $\bar{\mathbf{x}}^T \mathbf{Ax}$ is scalar, we have

$$\bar{\mathbf{x}}^T \mathbf{Ax} = (\bar{\mathbf{x}}^T \mathbf{Ax})^T = \mathbf{x}^T \mathbf{A}^T \bar{\mathbf{x}} = \mathbf{x}^T \bar{\mathbf{A}} \bar{\mathbf{x}} = \overline{(\bar{\mathbf{x}}^T \mathbf{Ax})} \qquad \text{...(2.46)}$$

Hence $\bar{\mathbf{x}}^T \mathbf{Ax}$ is real, and thus from (2.45) λ is real.

(b) *The matrix* **A** *is skew-Hermitian, that is* $\mathbf{A}^T = -\bar{\mathbf{A}}$

Proceeding as in (2.46), in this case, we obtain $\bar{\mathbf{x}}^T \mathbf{Ax} = -\overline{(\bar{\mathbf{x}}^T \mathbf{Ax})}$ so $\bar{\mathbf{x}}^T \mathbf{Ax}$ is purely imaginary or zero, therefore, from (2.45), λ is purely imaginary or $\lambda = 0$.

(c) *The matrix A is unitary, that is,* $\bar{\mathbf{A}}^T = \mathbf{A}^{-1}$.

Consider $\qquad \mathbf{Ax} = \lambda\mathbf{x} \qquad$ ...(2.47)

and its conjugate transpose

$$(\bar{\mathbf{A}}\bar{\mathbf{x}})^T = (\bar{\lambda}\bar{\mathbf{x}})^T \qquad \text{...(2.48)}$$

From (2.47) and (2.48), we have

$$(\bar{\mathbf{A}}\bar{\mathbf{x}})^T (\mathbf{Ax}) = (\bar{\lambda}\bar{\mathbf{x}})^T (\lambda\mathbf{x}) = |\lambda|^2 \bar{\mathbf{x}}^T \mathbf{x}.$$

or, $$\bar{\mathbf{x}}^T \bar{\mathbf{A}}^T \mathbf{Ax} = |\lambda|^2 \bar{\mathbf{x}}^T \mathbf{x}, \text{ or } \bar{\mathbf{x}}^T \mathbf{x} = |\lambda|^2 \bar{\mathbf{x}}^T \mathbf{x}, \text{ since } \bar{\mathbf{A}}^T = \bar{\mathbf{A}}^{-1}.$$

Since $\bar{\mathbf{x}}^T \mathbf{x} \neq 0$, we have $|\lambda|^2 = 1$, and hence, $|\lambda| = 1$.

This completes the proof.

Remark. The numerator $\bar{\mathbf{x}}^T \mathbf{Ax}$ in (2.45) is called a *form* in the components $x_1, x_2, \ldots, x_n$; and **A** is called its *coefficient matrix*.

Example 2.49: For the Hermitian, skew-Hermitian and unitary matrices given respectively as

$$\mathbf{A}=\begin{bmatrix}4 & 1-3i\\ 1+3i & 7\end{bmatrix}, \mathbf{B}=\begin{bmatrix}3i & 2+i\\ -2+i & -i\end{bmatrix}, \text{ and } \mathbf{C}=\begin{bmatrix}\frac{1}{2}i & \frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i\end{bmatrix}$$

find the eigenvalues.

Solution: For the *Hermitian matrix* **A**, the characteristic equation is $(4-\lambda)(7-\lambda)-10=0$, or $\lambda^2-11\lambda+18=0$, which gives eigenvalues as $\lambda = 9, 2$.

For the *skew-Hermitian matrix* **B**, the characteristic equation is $(3i-\lambda)(-i-\lambda)-5=0$, or $\lambda^2-2i\lambda+8=0$, which gives eigenvalues as $\lambda = 4i, -2i$.

Similarly, for the *unitary matrix* **C**, the characteristic equation is $\lambda^2 - i\lambda - 1 = 0$, which gives the eigenvalues as $\lambda = \frac{1}{2}i \pm \frac{1}{2}\sqrt{3}$. We observe that $|\lambda| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$.

2.14 QUADRATIC FORMS. REDUCTION TO CANONICAL FORM

In this section we discuss quadratic forms and their reduction to canonical form. The quadratic forms find applications in physics and geometry e.g., in conic sections and quadratic surfaces.

2.14.1 Quadratic Form

Let $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ be an arbitrary vector in $\mathbf{R}^n$, then quadratic form is an expression of the form

$$\mathbf{Q} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j.$$

For example, in case of $n = 3$, the quadratic form is

$$\mathbf{Q} = a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + a_{22}x_2^2 + (a_{23}+a_{32})x_2x_3 + a_{33}x_3^2 + (a_{31}+a_{13})x_3^2$$

$$= [x_1, x_2, x_3]\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$= \mathbf{x}^T\mathbf{A}\mathbf{x} \qquad ...(2.49)$$

Using the definition of matrices multiplication, set $c_{ij} = \frac{a_{ij}+a_{ji}}{2}$, then the matrix $\mathbf{C} = [c_{ij}]$ is symmetric, since $c_{ij} = c_{ji}$, and also, $c_{ij} + c_{ji} = a_{ij} + a_{ji}$. The form (2.49) can be expressed as

$$\mathbf{Q} = \mathbf{x}^T\mathbf{C}\mathbf{x}, \qquad ...(2.50)$$

where the matrix $\mathbf{C} = [c_{ij}]$ is symmetric with $c_{ij} = \frac{a_{ij}+a_{ji}}{2}$. For example, for $n = 3$.

$$c_{11} = a_{11},\ c_{22} = a_{22},\ c_{33} = a_{33},\ c_{12}\, c_{21} = \frac{a_{12} + a_{21}}{2},\quad c_{13} = c_{31} = \frac{a_{13} + a_{31}}{2}, \text{ and } c_{23} = c_{32} = \frac{a_{23} + a_{32}}{2}.$$

Example 2.50: Obtain the corresponding symmetric matrix for the quadratic form

(a) $\mathbf{Q} = 3x_1^2 + 10x_1x_2 + 2x_2^2$ (b) $\mathbf{Q} = x_1^2 + 3x_2^2 + 2x_3^2 + 2x_1x_2 + 6x_2\,x_3$.

Solution: (a) Here $a_{11} = 3, a_{12} + a_{21} = 10, a_{22} = 2$. Therefore, $c_{11} = 3, c_{12} = c_{21} = \dfrac{10}{2} = 5, c_{22} = 2$.

The symmetric matrix is $\mathbf{C} = \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}$

(b) Here $a_{11} = 1, a_{22} = 3, a_{33} = 2, a_{12} + a_{21} = 2, a_{23} + a_{22} = 6, a_{13} + a_{31} = 0$

Therefore, $c_{11} = 1, c_{22} = 3, c_{33} = 2, c_{12} = c_{21} = 1, c_{23} = c_{32} = 3, c_{13} = c_{31} = 0$

The symmetric matrix is $\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 2 \end{bmatrix}$

2.14.2 Definiteness

A real quadratic form $\mathbf{Q} = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and its matrix $\mathbf{A} = [a_{ij}]$ are said to be

(a) *positive definite*, if $\mathbf{Q} > 0$ for all $\mathbf{x} \neq \mathbf{0}$
(b) *negative definite*, if $\mathbf{Q} < 0$ for all $\mathbf{x} \neq \mathbf{0}$
(c) *semi-positive definite*, if $\mathbf{Q} \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$
(d) *semi-negative definite*, if $\mathbf{Q} \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$
(e) *indefinite*, if $\mathbf{Q}$ is not of definite sign for all $\mathbf{x} \neq \mathbf{0}$

We state following results (without proofs) in connection with the definiteness of a matrix $\mathbf{A}$:

1. *A necessary and sufficient condition for positive definiteness is that all the leading minors, that is,*

$$A_1 = a_{11}\quad A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\ A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \ldots, A_n = \det \mathbf{A}$$

are positive.

2. *The eigenvalues of a positive definite matrix are real and positive, and the eigenvalues of a negative definite matrix are real and negative. In case of semi-positive definite (semi-negative definite) matrix at least one eigenvalue is zero, and rest are positive (negative). In case of indefiniteness some eigenvalues are positive and some are negative.*

Example 2.51: Examine the nature of the following matrices

(a) $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution: (a) The quadratic form of the matrix **A** is given by

$$\mathbf{Q} = \mathbf{x}^T\mathbf{A}\mathbf{x} = 3x_1^2 + 3x_1x_2 + 4x_2^2 = 3\left[x_1^2 + x_1x_2 + \frac{1}{4}x_2^2\right] + \frac{13}{4}x_2^2 = 3\left[x_1 + \frac{1}{2}x_2\right]^2 + \frac{13}{4}x_2^2 > 0$$

unless $\mathbf{x} = 0$. Hence **A** is *positive definite.*

Remark. We can verify that eigenvalues of **A** are 2 and 5 both positive; and the leading minors of **A** are 3 and 10, again both positive. We must note it is sufficient to verify only one of these three aspects to verify the nature of a matrix.

(b) The characteristic equation of the matrix **A** is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$, which gives eigenvalues as $\lambda = 5, -3, -3$. Since, the eigenvalues are positive as well as negative. The matrix **A** is *indefinite.*

Example 2.52: Find the nature of the quadratic form:

(a) $\mathbf{Q} = x^2 + 5y^2 + z^2 + 2xy + 2yz + 6zx$ (b) $\mathbf{Q} = 2x^2 + 2y^2 + 3z^2 + 2xy - 4yz - 4zx$

Solution: (a) The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Its characteristic equation gives

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0, \text{ or } (\lambda + 2)(\lambda - 3)(\lambda - 6) = 0.$$

Hence its eigenvalues are $\lambda = -2, 3, 6$. Since the two of these being positive and one is negative, thus the given quadratic form is indefinite.

(b) The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$. Its characteristic equation gives

$$\begin{vmatrix} 2-\lambda & 1 & -2 \\ 1 & 2-\lambda & -2 \\ -2 & -2 & 3-\lambda \end{vmatrix} = 0, \text{ or } \lambda^3 - 7\lambda^2 + 7\lambda - 1 = 0$$

Solving we obtain, $\lambda = 1, \lambda = 3 \pm \sqrt{8}$ as its eigenvalues which are all positive. So the given quadratic form is positive definite.

2.14.3 Reduction of Quadratic Form to Canonical Form

If by any real non-singular linear transformation, a real quadratic form can be expressed as a sum and difference of the squares of the new variables, then this latter expression is called the canonical form of the given quadratic form.

For example, consider the quadratic form for the three components x_1, x_2, x_3. It is

$$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} = a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + (a_{12} + a_{21})x_1x_2 + (a_{23} + a_{32}) x_2x_3 + (a_{31} + a_{13}) x_3x_1,$$

where the coefficient matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is symmetric.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the matrix $\mathbf{A}$ (not necessarily all different) and $\mathbf{x}_1 = [x_{11}, x_{21}, x_{31}]^T, \mathbf{x}_2 = [x_{12}, x_{22}, x_{32}]^T, \mathbf{x}_3 = [x_{13}, x_{23}, x_{33}]^T$ be the corresponding *linearly independent eigenvectors* in the normalized form, that is, each element in an eigenvectors is divided by square root of sum of the squares of all three elements in that eigenvector.

The modal matrix $\mathbf{P}$ given by $\mathbf{P} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ is orthogonal matrix and the transformation

defined by

$$\mathbf{x} = \mathbf{P}\mathbf{y} \qquad \ldots(2.51)$$

is the *orthogonal transformation*.

This transforms the quadratic form $\mathbf{Q}$ to the canonical form as follows:

We have, $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, or $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ is the spectral matrix.

Consider
$$\begin{aligned}
\mathbf{Q} &= \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{x} \\
&= \mathbf{x}^T(\mathbf{P}^T)^T\mathbf{D}\mathbf{P}^{-1}\mathbf{x}, \\
&= \mathbf{x}^T(\mathbf{P}^{-1})^T\mathbf{D}\mathbf{P}^{-1}\mathbf{x}, \text{ since } \mathbf{P} \text{ is orthogonal, that is, } \mathbf{P}^T = \mathbf{P}^{-1} \\
&= (\mathbf{P}^{-1}\mathbf{x})^T\mathbf{D}\mathbf{P}^{-1}\mathbf{x} \\
&= \mathbf{y}^T\mathbf{D}\mathbf{y}, \text{ since } \mathbf{P}^{-1}\mathbf{x} = \mathbf{y}, \text{ refer to (2.51)} \\
&= [y_1, y_2, y_3]\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2, \qquad \ldots(2.52)
\end{aligned}$$

which is the desired canonical form.

The form (2.52) is also known as *sum of the squares form* or *the principal axes form*.

The number of positive square terms in the canonical form is called the *index* of the quadratic form and the difference between the number of positive and negative square terms is called the *signature* of the form.

Example 2.53: Transform the quadratic form $\mathbf{Q} = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$ to the canonical form. What type of conic section does it represent? Also find the matrix of transformation and the transformation.

Solution: Here the coefficient matrix is $\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The characteristic equation of $\mathbf{A}$ is $(17-\lambda)^2 - 225 = 0$, which gives $\lambda = 2$ and $\lambda = 32$ as its two eigenvalues.

Hence the given quadratic form becomes $\mathbf{Q} = 2y_1^2 + 32y_2^2$. Thus $\mathbf{Q} = 128$ represents the ellipse

$$2y_1^2 + 32y_2^2 = 128, \quad \text{or} \quad \frac{y_1^2}{64} + \frac{y_2^2}{4} = 1.$$

To find $\mathbf{P}$, the matrix of transformation, we find eigenvectors of $\mathbf{A}$.

For $\lambda = 2$, $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$, gives $\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 15x_1 - 15x_2 = 0$, or $x_1 = x_2$

Taking $x_2 = 1$, the normalized eigenvector is $[1/\sqrt{2}, 1/\sqrt{2}]^T$.

Similarly for $\lambda_2 = 32$, the normalized eigenvector is $[-1/\sqrt{2}, 1/\sqrt{2}]^T$.

Hence, the matrix of transformation is $\mathbf{P} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ and the transformation is

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, which gives $x_1 = \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{2}}y_2$, and $x_2 = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2$.

It represents 45° rotation.

Example 2.54: Reduce the quadratic form $\mathbf{Q} = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2 = 144$ to the canonical form. What does it represent? Find the matrix of transformation.

Solution: For the given quadratic form $\mathbf{Q}$ the coefficient matrix is

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic equation, $|\mathbf{A} - \lambda\mathbf{I}| = 0$, is $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0.$

It gives eigenvalues as $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 6$. Hence the quadratic form $\mathbf{Q}$ reduces to the canonical form $2y_1^2 + 3y_2^2 + 6y_3^2$ and thus $\mathbf{Q} = 144$ gives

$$2y_1^2 + 3y_2^2 + 6y_3^2 = 144, \quad \text{or} \quad \frac{y_1^2}{72} + \frac{y_2^2}{48} + \frac{y_3^2}{24} = 1$$

which is an ellipsoid.

To find the matrix of transformation P, we find eigenvectors of the matrix A, which, we can verify are $[1,\ 0,\ -1]^T$, $[1,\ 1,\ 1]^T$, and $[1,\ -2,\ 1]^T$. In the normalized form these are

$$[1/\sqrt{2},\ 0,\ -1/\sqrt{2}]^T,\ [1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3}]^T,\ [1/\sqrt{6},\ -2/\sqrt{6},\ 1/\sqrt{6}]^T.$$

Hence the matrix of transformation **P** is

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

EXERCISE 2.10

1. Are the following matrices Hermitian, skew-Hermitian or unitary? Find their eigenvalues and eigenvectors.

   (a) $\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$ (b) $\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$

2. Find the coefficient matrix for the following quadratic forms and hence find the nature of the form

   (a) $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ (b) $2x_1x_2 + 2x_2x_3 + 2x_3x_1$

   (c) $|x_1|^2 + |x_2|^2 + 3|x_3|^2 + i\bar{x}_1x_3 - ix_1\bar{x}_3$ (d) $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$

3. Examine the nature of the following matrices

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

4. Find $\bar{\mathbf{x}}^T\mathbf{A}\mathbf{x}$ when

   (a) $\mathbf{A} = \begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} i \\ 4 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} -i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

5. Reduce the following quadratic forms to canonical forms by an orthogonal transformation and give the matrix of transformation

   (a) $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ (b) $4x_1^2 + 3x_2^2 + x_3^2 - 8x_1x_2 - 6x_2x_3 + 4x_3x_1$

6. Find out what type of conic section is represented by following quadratic forms by transforming to canonical forms. Also find the transformation in each case.

   (a) $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$ (b) $32x_1^2 - 60x_1x_2 + 7x_2^2 = -52$

   (c) $9x_1^2 - 6x_1x_2 + x_2^2 = 40$

ANSWERS

Exercise 2.1 (p. 74)

1. $\begin{pmatrix} -36 & 0 & 68 & 196 & 20 \\ 128 & -40 & -36 & -8 & 72 \end{pmatrix}$

2. $AB = \begin{bmatrix} -10 & -34 & -16 & -30 & -14 \\ 10 & -2 & -11 & -8 & -45 \\ -5 & 1 & 15 & 61 & -63 \end{bmatrix}$; BA is not defined.

4. $\begin{bmatrix} 1 & 4 & 7/2 \\ 4 & 8 & 3 \\ 7/2 & 3 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 1/2 \\ 2 & 0 & -2 \\ -1/2 & 2 & 0 \end{bmatrix}$

12. (a) Hermitian (b) not Hermitian. 13. $n = 3$

15. $\begin{bmatrix} 1 & 1-i & 2i \\ 1+i & 2 & 3-i \\ -2i & 3+i & 4 \end{bmatrix} + \begin{bmatrix} i & 2+2i & 3 \\ -2+2i & 0 & 1+2i \\ -3 & -1+2i & 2i \end{bmatrix}$

16. (a) Hermitian (b) Unitary (c) Skew-Hermitian.

Exercise 2.2 (p. 88)

3. (a) 1 (b) 286 11. (a) $0, 0, -1/2$ (b) $-1, -1, 2$

Exercise 2.3 (p. 96)

2. (a) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 31/2 & -17/2 & -11 \\ 9/2 & -5/2 & -3 \\ -7 & 4 & 5 \end{bmatrix}$ 5. $x = 2, y = -1, z = 1$

6. $x = 2, y = 1, z = 0$ 7. $x = 1.2, y = 2.2, z = 3.2$ 8. $x = 7, y = -9, z = 5$

9. $x = -1, y = 0, z = 1$ 10. $x = 2, y = z = 1$ 11. $x = 2, y = -1, z = 1$

12. $x = 1, y = 2, z = 1$ 13. $x = 1, y = 1/2, z = 1/3$ 14. $x = y = z = e^2$

15. (i) $k \neq 2, -3$ (ii) $k = 2$, or -3

Exercise 2.4 (p. 106)

2. (a) 2 (b) 3 (c) 3 (d) 2. 3. (a) 2 (b) 2 (c) 4 (d) 2.

4. (a) 3 (b) 3 (c) 3 (d) 2. 5. (a) 2 (b) 2

6. (a) 3 (b) 4.

9. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; Rank is 2. 10. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$; Rank is 2.

11. (a) $\frac{1}{10}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$ (b) $\frac{1}{21}\begin{bmatrix} 1 & 10 & -7 \\ 1 & -11 & 14 \\ -3 & 12 & 0 \end{bmatrix}$

(c) $\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & -1/3 & 1/3 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & 1/3 & -1/3 & 0 \end{bmatrix}$

Exercise 2.5 (p. 115)

1. (a) yes!; $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; 3 1. (b) yes!; $[1, x, x^2, x^3, x^4]$; 5.
1. (c) No; set is not closed under matrices addition, scalar multiplication also additive identity, that is, null matrix is not in the set.
1. (d) Yes!; $(3, 0, -2), (0, 1, 0)$; 2.
3. (a) linearly dependent; $(2, 3, 0) = -\frac{1}{2}(1, -2, 4) + (1, 1, 0) + \frac{4}{3}(1, 1, 1)$
   (b) linearly dependent; $(0,0,0,0) = 0(3, 0, 2, 5) + 0(4, -1, 1, 2)$
   (c) linearly dependent $(2, -1, 3, 2) = \frac{1}{2}(1, 3, 4, 2) + \frac{1}{2}(3, -5, 2, 2)$
   (d) linearly independent (e) linearly independent
4. (a) Yes! (b) Yes (c) No. 5. No; 6. Yes! 7. No
8. (a) Yes (b) No. 9. e.g., $(3, 1, 4), (0, 5, 8)$; and $\left(1, 0, -1, \frac{1}{3}\right)^T (1, 2, 1, 1,)^T$.

Exercise 2.6 (p. 119)

1. (a) $x_1 = -0.2y_1 + 0.4y_2$
   $x_2 = 0.8y_1 - 0.6y_2$
   (b) $x_1 = (-2y_1 + 2y_2 + y_3)/3$
   $x_2 = (y_1 + 2y_2 - 2y_3)/3$
   $x_3 = (2y_1 + y_2 + 2y_3)/3$
   (c) $x_1 = y_1$
   $x_2 = \cos\theta y_2 - \sin\theta y_3$
   $x_3 = \sin\theta y_2 + \cos\theta y_3$.

2. $z_1 = x_1 + 4x_2 - x_3$
   $z_2 = -x_1 + 9x_2 - x_3$
   $z_3 = -3x_1 + 14x_2 - x_3$
3. $a = 2/7, b = 3/7, c = 6/7$

Exercise 2.7 (p. 124)

1. $x = 7/11, y = 3/11, z = 0$
2. Inconsistent.
3. $x = 11\alpha + 10, y = -4\alpha - 2, z = \alpha, u = 0$; α arbitrary.
4. $x = -1, y = 1, z = 2$
5. Inconsistent
6. $x = \alpha - \frac{5}{3}\beta, y = \alpha - \frac{4}{3}\beta, z = \alpha, w = \beta$; α, β are arbitrary
7. $x = y = z = 0$; the only solution
8. $x = 11\alpha, y = \alpha, z = -7\alpha$; α arbitrary.
9. $x = 4\alpha + \beta, y = -3\alpha - 2\beta, z = \alpha, w = \beta$
10. $x = y = z = w = 0$; the only solution
11. Consistent for $\lambda = 7, x = 3\alpha + \beta + 3, y = 4\alpha - \beta + 1, z = \alpha, w = \beta$.
12. Consistent for $\lambda = 0, 3$. For $\lambda = 0$; $x = y = z = \alpha$; α being parameter.
    For $\lambda = 3$; $x = -5\alpha - 3\beta, y = \alpha, z = \beta$; α, β being parameters.
14. For $\lambda = 1$; $x = 2\alpha - \beta, y = \alpha, z = \beta$. For $\lambda = -3, x = -\alpha, y = -2\alpha, z = \alpha$, α arbitrary.

Exercise 2.8 (p. 132)

1. Eigenvalues of A: $1, -3, 7$.
2. 10, 15
3. $2, [3\ 0\ 1]^T; -2, [1\ 1\ 0]^T; 1, [2\ 0\ 1]^T$
4. $1, 1, 1; [0\ 3\ -2]^T$
5. $0, [1\ 2\ 2]^T; 3, [2\ 1\ -2]^T; 15, [2\ -2\ 1]^T$
6. $\frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$
7. $\frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$
8. $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$
9. $\begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$
10. $625\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 2.9 (p. 137)

1. $\hat{\mathbf{A}} = \begin{bmatrix} 355 & -42 & 560 \\ 0 & 1 & 0 \\ -225 & 27 & -355 \end{bmatrix}$ Eigenvalues are: $-5, 1, 5$

   $\mathbf{y} = [-14\ 0\ 9]^T, [-14\ 2\ 9]^T, [-8\ 0\ 5]^T$ $\quad$ $\mathbf{x} = [-1\ 0\ 3]^T, [-1\ 2\ 3]^T, [-1\ 0\ 1]^T$

2. $\mathbf{P}=\begin{bmatrix}1&2&0\\1&0&1\\0&1&1\end{bmatrix}\mathbf{D}=\begin{bmatrix}15&0&0\\0&0&0\\0&0&-15\end{bmatrix}$

3. Not diagonalizable.

4. $\mathbf{P}=\begin{bmatrix}3&2&2\\-1&0&1\\3&1&0\end{bmatrix}\mathbf{D}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&2\end{bmatrix}$

5. $\begin{bmatrix}6&-5&-7\\1&0&-1\\3&-3&-4\end{bmatrix}$

6. $\frac{3}{8}\begin{bmatrix}3&6&15\\0&0&0\\1&2&3\end{bmatrix}$

7. $\frac{1}{10}\begin{bmatrix}73&2&-37\\-115&10&55\\177&18&-93\end{bmatrix}$

8. $\mathbf{P}=\frac{1}{6}\begin{bmatrix}-3&0&3\\2&-2&2\\1&2&1\end{bmatrix}\mathbf{A}^4=\begin{bmatrix}251&485&235\\485&1051&485\\235&485&251\end{bmatrix}$

Exercise 2.10 (p. 145)

1. (a) Hermitian, $\lambda=-3,7;\ \mathbf{x}=[-3-4i\quad 5]^T,[3+4i\quad 5]^T$

   (b) Skew-Hermitian, $\lambda=-i,i\ \mathbf{x}=[0\quad -1\quad 1]^T,[0\quad 1\quad 1]^T,[1\quad 0\quad 0]^T$

2. (a) $\begin{bmatrix}3&1&1\\1&3&-1\\1&-1&3\end{bmatrix}$; positive definite (b) $\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$; indefinite

   (c) $\begin{bmatrix}1&0&i\\0&1&0\\-i&0&3\end{bmatrix}$; positive definite. (d) $\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}$; positive semi-definite

3. (a) positive definite (b) positive definite (c) positive definite 4. (a) $17i$ (b) $16i$

5. (a) $y_1^2+2y_2^2+4y_3^2$, $\mathbf{P}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0\\0&1&1\\0&1&-1\end{bmatrix}$ (b) $4y_1^2-y_2^2+y_3^2$, $\mathbf{P}=\begin{bmatrix}1&1&-3\\0&1&-1\\0&0&1\end{bmatrix}$

6. (a) Ellipse $\frac{y_1^2}{50}+\frac{y_2^2}{20}=1;\ x_1=(y_1+y_2)/\sqrt{2},\ x_2=(-y_1+y_2)/\sqrt{2}$

   (b) Hyperbola $\frac{y_1^2}{4}-\frac{y_2^2}{1}=1;\ x_1=(2y_1+3y_2)/\sqrt{13},\ x_2=(3y_1-2y_2)/\sqrt{13}$

   (c) Straight lines $y_2=\pm 2;\ x_1=(y_1+3y_2)/\sqrt{10},\ x_2=(3y_1-y_2)/\sqrt{10}$

PART B

Differential and Integral Calculus

3 Infinite Series

CHAPTER

Most of the functions which are encountered in mathematical applications can be represented in terms of an infinite series. There we need to study their convergence or divergence because unless a series employed in an investigation is convergent, it may lead to absurd conclusions. This makes the concept of convergence and divergence of an infinite series of vital importance to the students of engineering and technology.

3.1 SEQUENCES AND SERIES

We first introduce the concept of sequences which is a prerequisite for the study of infinite series. After this, we will discuss infinite series and their convergence and divergence.

3.1.1 Sequences

A *sequence* is an ordered set of numbers $u_1, u_2, u_3, \ldots$, and is denoted by $\{u_n\}$; here u_n is called the *nth* term, or the *general term* of the sequence $\{u_n\}$. If the number of terms are infinite, then the sequence is said to be an *infinite sequence*. If the number of terms are finite, it is said to be a *finite sequence*.

For example, the set of numbers $2, 4, 8, 10, \ldots, 2n, \ldots$ is an infinite sequence and is represented by $\{2n\}$. Here $u_n = 2n$ is the *nth term*, or the *general term*.

Similarly, $\{(-1)^n\}$ represents the sequence $-1, 1, -1, 1, -1, \ldots$ with $u_n = (-1)^n$ as its general term.

Limit of a sequence: A number l is called the *limit* of an infinite sequence $u_1, u_2, u_3, \ldots$ if, for given $\epsilon > 0$, no matter how small, we can find a positive integer N, depending upon ϵ, such that $|u_n - l| < \epsilon$ for all $n \geq N$. In such a case we write $\lim\limits_{n \to \infty} u_n = l$, or simply $\{u_n\} \to l$, as $n \to \infty$.

Convergence or divergence of a sequence: If the limit of a sequence exists the sequence is called *convergent*, otherwise, it is called *divergent*, or *oscillating*. For example, the sequence $\left\{\dfrac{1}{2n-1}\right\}$ is convergent but the sequence $\{2n\}$ is divergent. For any convergent sequence, the limit is always unique. The sequence $\{(-1)^{n-1}\}$ is not convergent, it is oscillating. It oscillates between -1 and $+1$.

Bounded sequence: If $u_n \leq M$ for $n = 1, 2, 3, \ldots$, where M is a constant (not depending on n) we say that the sequence $\{u_n\}$ is *bounded above* and M is called an *upper bound* of $\{u_n\}$. If $u_n \geq m$ for $n = 1, 2, 3, \ldots$, where m is a constant, then we say that the sequence $\{u_n\}$ is *bounded below* and m is called a *lower bound* of $\{u_n\}$.

If $m \leq u_n \leq M$ for all $n \in N$, the sequence $\{u_n\}$ is called *bounded*. We have an important result to state that *every convergent sequence is bounded but its converse is not necessarily true.*

For example the sequence $\{(-1)^n\}$ is bounded but not convergent.

Monotonic sequence: If $u_{n+1} \geq u_n$, the sequence $\{u_n\}$ is said to be *monotonically increasing* and if $u_{n+1} \leq u_n$, the sequence is called *monotonically decreasing*. Both increasing and decreasing sequences are called *monotonic sequences.*

A monotonic sequence always tends to a definite limit, finite or infinite. Thus *a sequence which is monotonic and bounded is always convergent.*

3.1.2 Series

If $u_1, u_2, u_3, \ldots, u_n, \ldots$ is an infinite sequence of real numbers, then

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

is called an *infinite series*. It is denoted by $\sum_{n=1}^{\infty} u_n$, or $\sum_n u_n$, or simply Σu_n. The sum of its first n terms, denoted by $S_n = u_1 + u_2 + \ldots + u_n$, is called the *nth partial sum* of Σu_n. The sequence $\{S_n\}$ is called the *Sequence of partial sums* of the infinite series Σu_n. Thus to every infinite series Σu_n, there corresponds a sequence $\{S_n\}$ of its partial sums.

Convergence, divergence or oscillation of a series: The convergence, divergence or oscillation of an infinite series Σu_n is defined according to the sequence $\{S_n\}$ of its partial sum converges, diverges or oscillates.

Consider an infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \ldots$ and let $S_n = u_1 + u_2 + \ldots + u_n$ be the sum of its first n terms. Three cases arise:

(a) If S_n tends to a unique finite limit as $n \to \infty$, then the series Σu_n is said to be *convergent.*

(b) If S_n tends to an infinite limit as $n \to \infty$, then the series Σu_n is said to be *divergent*.

(c) If S_n does not tend to a unique limit, finite or infinite as $n \to \infty$, then the series Σu_n is said to be *oscillatory*.

We must note the following in context with the convergence or divergence of an infinite series.

(a) The nature of an infinite series does not change by the addition or removal of a finite number of terms.

(b) The convergence or divergence of an infinite series remains unaffected by multiplying or dividing each term by a finite non-zero constant.

(c) If two series Σu_n and Σv_n are convergent, then the series $\Sigma (u_n + v_n)$ is also convergent, but the result may not hold otherwise.

Example 3.1: Examine the convergence of the series $1 + 3 + 5 + 7 + \ldots \infty$.

Solution: Here, $S_n = 1 + 3 + 5 + 7 + \ldots + (2n - 1) = \frac{n}{2}\{2 + (n-1)2\} = n^2$.

Consider $\lim_{n\to\infty} S_n = \lim_{n\to\infty} n^2 \to \infty$. Hence the given series is *divergent*.

Example 3.2: Test the convergence of the series $6-10+4+6-10+4+6-10+4+\ldots\infty$.

Solution: Here $S_n = 6-10+4+6-10+4+6-10+4+\ldots = 0, 6, -4$ according as n is of the form $3m, 3m+1, 3m+2$ for any positive integer m. Clearly in this case S_n does not tend to a unique limit. Hence, the given series is *oscillatory*.

Example 3.3: Test the convergence of the series $1^2+2^2+3^2+\ldots+n^2\ldots$

Solution: Here $S_n = 1^2+2^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Consider $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{n(n+1)(2n+1)}{6} \to \infty$. Therefore, the given series is *divergent*.

Example 3.4: Discuss the convergence of the geometric series $a+ar+ar^2+\ldots+ar^{n-1}+\ldots$

Solution: Here, $S_n = a+ar+ar^2+\ldots+ar^{n-1}+\ldots = a\dfrac{(1-r^n)}{1-r}$, provided $r \neq 1$.

We have the following cases.

Case I: $|r|<1$, then $r^n \to 0$ as $n\to\infty$.

Therefore, $\lim_{n\to\infty} S_n = \dfrac{a}{1-r}$, which is finite and unique, thus the given series *converges* to $\dfrac{a}{1-r}$.

Case II: $|r|>1$, that is, $r>1$, or $r<-1$.

When $r>1$, then $r^n\to\infty$ as $n\to\infty$.

Therefore, $\lim_{n\to\infty} S_n = \dfrac{a(r^n-1)}{r-1} \to \infty$, thus the given series is *divergent*.

When $r<-1$, then $r^n \to +\infty$ or $-\infty$ as $n\to\infty$. Hence, $\lim_{n\to\infty} S_n$ oscillates either to ∞ or $-\infty$. Thus, the given series *oscillates infinitely*.

Case III: $|r|=1$, that is, $r=1$, or -1. When $r=1$, the series becomes, $a+a+a+\ldots+a\ldots$

Therefore, $S_n = na$. Thus $S_n\to\infty$ as $n\to\infty$ and the series *diverges*.

When $r=-1$, the series becomes, $a-a+a-a+\ldots$ and, therefore,

$$S_n = \begin{cases} 0, & \text{when } n \text{ is even.} \\ a, & \text{when } n \text{ is odd.} \end{cases}$$

Thus the given series *oscillates finitely* between 0 and a.

EXERCISE 3.1

1. Determine the general term of each of the following sequences and show that each is convergent.

   (a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots$ (b) $\frac{1^2}{1!}, \frac{2^2}{2!}, \frac{3^2}{3!}, \frac{4^2}{4!}, \frac{5^2}{5!} \ldots$

   (c) $\frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}, \frac{1}{4.5}, \ldots$ (d) $\left(\frac{2}{1}\right)^1, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \left(\frac{5}{4}\right)^4 \ldots$

2. Examine the convergence of each of the following sequences, whose general term is

   (a) $u_n = (\sqrt[3]{n^3+1} - n)$ (b) $u_n = \left(1 + \frac{1}{n}\right)^{n+1}$

   (c) $u_n = \frac{\ln(n)}{n+1}$ (d) n^2

3. Examine the following series for their convergence:

   (a) $1 + 2 + 3 + 4 + \ldots$ (b) $\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \frac{1}{4.6} + \ldots$

   (c) $\sum_{n=0}^{\infty} (-1)^n$ (d) $4 - 3 - 1 + 4 - 3 - 1 + 4 - 3 - 1 + \ldots$

   (e) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 + \ldots$ (f) $2 + 3 + \frac{9}{2} + \frac{27}{4} + \frac{81}{8} + \ldots$

3.2 POSITIVE TERMS SERIES

An infinite series in which all the terms, after some finite number of terms, are positive, is called a positive term series. For example, the series $-5 - 4 - 3 - 2 - 1 + 0 + 1 + 2 + 3 + 4 + 5 + \ldots$

is a positive term series, since by omitting a few initial terms it can be considered as a series of only positive terms. Even a series which consists all the terms after some finite number of terms as negative, may be treated as a series of positive terms.

It is easy to see that a series of positive terms either converges or diverges to $+\infty$, for the sum S_n of first n terms in this case, barring the negative terms, tends either to a finite limit or to $+\infty$.

Cauchy's convergence principle for series: *The necessary and sufficient condition for an infinite series Σu_n to converge is that given $\epsilon > 0$, no matter how small, we can find a positive integer $N(\epsilon)$, such that*

$$|u_{n+1} + u_{n+2} + \ldots + u_{n+m}| < \epsilon \text{ for every } n > N \text{ and } m = 1, 2, \ldots \quad \ldots(3.1)$$

In practice, this result becomes difficult to apply to test the convergence of series. We shall study a few specialized tests which are convenient to apply.

A necessary condition for convergence: *A necessary condition for a positive term series Σu_n to be convergent is that*

$$\lim_{n\to\infty} u_n = 0. \qquad \text{...(3.2)}$$

To establish this, consider $S_n = u_1 + u_2 + ... + u_n$. Since the series Σu_n is convergent, therefore $\lim_{n\to\infty} S_n = k$, a finite quantity and also $\lim_{n\to\infty} S_{n-1} = k$. Consider $u_n = S_n - S_{n-1}$. This gives

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} [S_n - S_{n-1}] = 0$$

The result (3.2) can also be derived from (3.1) for $m = 1$.

We must note that the converse of the above result is not true. For example, consider the series

$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+...+\frac{1}{\sqrt{n}}+...$ Here, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. But

$$S_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+...+\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+...+\frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n},$$

which tends to infinity as n tends to infinity.

Thus, the series is divergent. So, *$\lim_{n\to\infty} u_n = 0$ is only a necessary condition but not sufficient one for a series to be convergent.* In fact, the importance of the above result is that it leads to a simple test for divergence of an infinite series, that is

If $\lim_{n\to\infty} u_n \neq 0$, the positive terms series Σu_n must be divergent.

For example, consider the series $1+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+...+\frac{n}{n+1}+...$

Here $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, hence the series is divergent.

3.3 TESTS FOR THE CONVERGENCE OF POSITIVE TERMS SERIES

We have observed that the convergence or divergence of a series can be established by taking the limits of S_n, its nth partial sum, as $n \to \infty$. But in actual practice, sometimes it becomes cumbersome to find the sum of the first n terms of an infinite series. In this section, we outline the various tests employed to test for the convergence of the positive terms series.

3.3.1 Comparison Tests

The three comparison tests are:

A. *If two positive terms series Σu_n and Σv_n are such that*

(i) *Σv_n is convergent,*

(ii) *$u_n \le v_n$ for all values of n, then Σu_n is also convergent.*

Since, Σv_n is convergent, thus $\lim_{n\to\infty} (v_1 + v_2 + \dots + v_n) = S$; a finite quantity. Also, since $u_1 \le v_1$, $u_2 \le v_2, \dots, u_n \le v_n$, thus we have

$$u_1 + u_2 + \dots + u_n \le v_1 + v_2 + \dots + v_n$$

It gives $\lim_{n\to\infty} (u_1 + u_2 + \dots + u_n) \le \lim_{n\to\infty} (v_1 + v_2 + \dots v_n) = S$

Thus, the series Σu_n is also convergent.

B. *If two positive terms series Σu_n and Σv_n be such that:*

(i) *Σv_n is divergent,*

(ii) *$u_n \ge v_n$ for all values of n, then Σu_n is also divergent.*

The result follows on the similar lines as in **A**.

C. *If two positive term series Σu_n and Σv_n be such that $\lim_{n\to\infty} \frac{u_n}{v_n}$ = finite quantity ($\neq 0$), then Σu_n and Σv_n converge or diverge together.*

Since $\lim_{n\to\infty} \frac{u_n}{v_n} = l (\neq 0)$, then by definition of limits there exists a positive number $\in > 0$, no matter how small, such that

$$\left|\frac{u_n}{v_n} - l\right| < \in, \text{ for } \quad n \ge m \quad \text{or,} \quad l - \in < \frac{u_n}{v_n} < l + \in, \text{ for } \quad n \ge m.$$

Ignoring the first m terms of both the series, we have

$$l - \in < \frac{u_n}{v_n} < l + \in \text{ for all } \quad n \ge 1 \qquad \dots(3.3)$$

Case I: Σv_n is convergent, then

$$\lim_{n\to\infty} (v_1 + v_2 + \dots + v_n) = k, \text{ a finite quantity.}$$

Now from (3.3), $\frac{u_n}{v_n} < l + \in$, for all n, which gives $u_n < v_n(l + \in)$

or, $\sum_{n=1}^{n} u_n < (l + \in) \sum_{n=1}^{n} v_n$, or $\lim_{n\to\infty} \sum_{n=1}^{n} u_n < (l + \in) \lim_{n\to\infty} \sum_{n=1}^{n} v_n = (l + \in)k$.

Hence Σu_n is also convergent.

Case II: Σv_n is divergent, then $\lim_{n\to\infty} (v_1 + v_2 + ... + v_n) \to \infty$.

Now from (3.3), $l - \epsilon < \frac{u_n}{v_n}$, for all n, which gives $u_n > (l - \epsilon)v_n$,

or, $\sum_{n=1}^{n} u_n > (l-\epsilon) \sum_{n=1}^{n} v_n$, or $\lim_{n\to\infty} \sum_{n=1}^{n} u_n > (l-\epsilon) \lim_{n\to\infty} \sum_{n=1}^{n} v_n \to \infty$.

Hence Σu_n is divergent.

It should be noted that because of the property that the nature of an infinite series is unaffected by addition or deletion of a finite number of terms, the comparison tests A and B hold good even for $u_n \le v_n$ (or, $u_n \ge v_n$) for $n \ge m$; where m is a finite positive integer. Further to test the nature of a series, the comparison test C is very useful, we choose the series Σv_n in such a way that its nature is already known and after applying the test we conclude the nature of the series Σu_n.

3.3.2 Integral Test

A positive term series $f(1) + f(2) + ... + f(n) + ...$, where $f(n)$ decreases as n increases, converges or diverges according as the integral $\int_1^\infty f(x)dx$ is finite or infinite.

From Fig. 3.1, it is clear that

$$f(1) + f(2) + ... + f(n) \ge \int_1^{n+1} f(x)dx \ge f(2) + f(3) + ... + f(n+1),$$

or,
$$S_n \ge \int_1^{n+1} f(x)dx \ge S_{n+1} - f(1) \qquad ...(3.4)$$

From the two terms on the right of (3.4), we have

$$S_{n+1} \le \int_1^{n+1} f(x)dx + f(1)$$

Taking limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} S_{n+1} \le \int_1^\infty f(x)dx + f(1).$$

Hence, if the integral $\int_1^\infty f(x)\,dx$ is finite, then so is

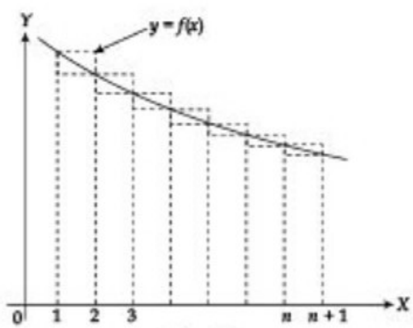


Fig. 3.1

$\lim_{n\to\infty} S_{n+1}$. Similarly, from (3.4), by considering, $S_n \ge \int_1^{n+1} f(x)dx$, we observe that if the integral

$\int_1^\infty f(x)dx$ is infinite, then so is $\lim_{n\to\infty} S_n$. Since the given series, being of positive terms, either converges or diverges, that is, $\lim_{n\to\infty} S_n$ is either finite or infinite, hence the desired result follows from the integral $\int_1^\infty f(x)dx$.

Next, we apply integral test to discuss the convergence of the *harmonic series of order p*, that is, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 0$. This result is oftenly used in comparison tests.

Example 3.5: Show that the harmonic series of order p, $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ is convergent for $p > 1$, and divergent for $p \le 1$.

Solution: By integral test, this series will converge or diverge according as the integral $\int_1^\infty \frac{dx}{x^p}$ is finite or infinite. For $p \neq 1$,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{n\to\infty} \int_1^n \frac{dx}{x^p} = \lim_{n\to\infty} \left[\frac{n^{1-p}-1}{1-p}\right],$$

which is finite and is equal to $1/(p-1)$, for $p > 1$; tends to infinity for $p < 1$.

For $p = 1$, $\int_1^\infty \frac{dx}{x^p} = \lim_{n\to\infty} \ln(n) = \infty$.

Hence the series is convergent for $p > 1$ and divergent for $p \le 1$.

Example 3.6: Test the series $\frac{1}{2} + \frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{8} + \ldots + \frac{\sqrt{n}}{n^2-1} + \ldots$ for its convergence or divergence.

Solution: Here, $u_n = \sqrt{n}/(n^2-1)$. Take $v_n = \sqrt{n}/n^2 = 1/n^{3/2}$. Consider

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{\sqrt{n}}{n^2-1}\right)\left(\frac{n^{3/2}}{1}\right) = \lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \frac{1}{1-(1/n^2)} = 1; \text{ a finite non-zero quantity.}$$

Thus, by comparison test the series Σu_n and Σv_n either both converge, or both diverge. But the series $\Sigma v_n = \Sigma(1/n^{3/2})$ is convergent by integral test, since $p = 3/2 > 1$, hence the series Σu_n is also convergent.

Example 3.7: Test the series $\sum\left(\sqrt[3]{n^3+1}-n\right)$ for its convergence, or divergence.

Solution: Here, $u_n = (n^3+1)^{1/3} - n = n\left[1+\frac{1}{n^3}\right]^{1/3} - n$

$$= n\left[1+\frac{1}{3n^3}-\frac{1}{9n^6}+\ldots\right] - n = \frac{1}{3n^2}-\frac{1}{9n^5}+\ldots$$

Take $v_n = \frac{1}{n^2}$. Consider,

$$\frac{u_n}{v_n} = n^2\left[\frac{1}{3n^2}-\frac{1}{9n^5}+\ldots\right] = \frac{1}{3}-\frac{1}{9n^2}+\ldots$$

It tends to $\frac{1}{3}$, a finite non-zero number, as n tends to infinity. Hence, by comparison test the series $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent. But the series Σv_n is convergent by integral test since $p = 2 > 1$, hence the series $\Sigma\, u_n$ is also convergent.

Example 3.8: Test the series $\sum_{n=2} \frac{1}{n \ln(n)}$ for its convergence or divergence.

Solution: It is easy to check that the series $\sum_{n=2} \frac{1}{n \ln(n)}$ is decreasing one. Here the nth term is

$f(n) = \frac{1}{(n+1)\ln(n+1)}$. Consider, the integral $I = \int_1^{\infty} \frac{1}{(x+1)\ln(x+1)}dx$.

Set $\ln(x+1) = t$, it gives $\frac{dx}{x+1} = dt$, so I becomes

$$I = \int_{\ln 2}^{\infty} \frac{dt}{t} = \lim_{n\to\infty}\int_{\ln 2}^{n} \frac{dt}{t} = \lim_{n\to\infty}\left[\ln(n) - \ln(\ln 2)\right] = \infty.$$

Hence, by integral test the given series is divergent.

EXERCISE 3.2

Test the following series for their convergence or divergence.

1. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots$
2. $\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots$

3. $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\dots$

4. $\sum\left(\sqrt{(n^4+1)}-\sqrt{(n^4-1)}\right)$

5. $\sum\left[1/n^{\left(1+\frac{1}{n}\right)}\right]$

6. $\sum\frac{2n^3+5}{4n^5+1}$

7. $\frac{1}{1+x}+\frac{1}{2+x}+\frac{1}{3+x}+\dots$

8. $\sum\left(\sqrt{n^2+1}-n\right)$

9. $\sum n^{\ln x}$

10. $\sum\frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^3+2n+7}}$

11. $\sum\frac{\sqrt{n+1}-\sqrt{n}}{n^p},\ p>\frac{1}{2}$

12. $\sum\frac{\sqrt{n+1}-1}{(n+2)^3-1}$

13. $\sum\sqrt{\frac{2^n-1}{3^n-1}}$

14. $\sum n\tan^{-1}\left(\frac{1}{n^3}\right)$

15. $\sum n^p/(n+1)^q$

16. Test for the convergence of harmonic series $\sum 1/n^p$ using comparison tests.

3.3.3 D'Alembert's Ratio Test

"If Σu_n is a positive terms series such that $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=k$, then the series Σu_n is convergent, if $k<1$, divergent, if $k>1$, and for $k=1$, the test fails".

Case I: When $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=k<1$. By definition of the limit, we can find a number $r<1$ such that

$$\frac{u_{n+1}}{u_n}<r, \text{for all} \quad n>m.$$

Omitting the first m terms, let the series be $u_1+u_2+u_3+\dots$, so that

$$\frac{u_2}{u_1}<r,\quad \frac{u_3}{u_2}<r,\quad \frac{u_4}{u_3}<r\ \dots \quad \text{and so on.}$$

Then $\quad u_1+u_2+u_3+\dots$

$$= u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}.\frac{u_2}{u_1}+\frac{u_4}{u_3}.\frac{u_3}{u_2}.\frac{u_2}{u_1}+...\right) < u_1(1+r+r^2+r^3+...) = \frac{u_1}{1-r}, \text{ a finite quantity.}$$

Hence Σu_n is convergent.

Case II: When $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = k > 1$. By definition of limit, we can find a number m, such that $\frac{u_{n+1}}{u_n} \geq 1$, for all $n > m$.

Omitting the first m terms let the series be $u_1 + u_2 + u_3 + ...$, so that

$$\frac{u_2}{u_1} \geq 1, \quad \frac{u_3}{u_2} \geq 1, \quad \frac{u_4}{u_3} \geq 1 \text{ ... and so on.}$$

Therefore, $u_1 + u_2 + u_3 + ...$

$$= u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}.\frac{u_2}{u_1}+\frac{u_4}{u_3}.\frac{u_3}{u_2}.\frac{u_2}{u_1}+...\right) \geq u_1(1+1+1+1+ ...),$$

which tends to infinity. Hence, Σu_n is divergent.

Remarks

1. The ratio test fails when $k = 1$. Consider, for instance, the series $\sum\frac{1}{n}$. Here $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n}{n+1} = 1$. But we know that $\sum\frac{1}{n}$ is divergent, being harmonic series with $p = 1$. Similarly, incase of $\sum\frac{1}{n^2}$, the $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1$, but $\sum\frac{1}{n^2}$ is convergent, being harmonic series with $p = 2 > 1$. Hence, when $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 1$, the series Σu_n may be convergent or divergent.

2. The test makes no reference to the magnitude of $\frac{u_{n+1}}{u_n}$ for any finite value of n but concerns only with the limit of this ratio as $n \to \infty$.

3. This test is also used in the following form: *If Σu_n is a positive term series such that $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = k$, then the series Σu_n is convergent if $k > 1$, divergent if $k < 1$, and for $k = 1$ the test fails.*

Example 3.9: Test for convergence of the series

(a) $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+ ... \infty$ (b) $\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}.\frac{2}{5}\right)^2+\left(\frac{1}{3}.\frac{2}{5}.\frac{3}{7}\right)^2 + ... \infty.$

Solution: (a) Here $u_n = \frac{n^p}{n!}$ and $u_{n+1} = \frac{(n+1)^p}{(n+1)!}$.

Consider $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)!} \frac{n!}{n^p} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p \frac{1}{n+1} = 0 < 1.$

Hence by ratio test the series is convergent.

(b) Here, $u_n = \left[\frac{n!}{3.5.7.\ldots(2n+1)}\right]^2$, and $u_{n+1} = \left[\frac{(n+1)!}{3.5.7.\ldots(2n+3)}\right]^2$

Consider $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left[\frac{(n+1)!}{3.5.7.\ldots(2n+3)}\right]^2 \cdot \left[\frac{3.5.7.\ldots(2n+1)}{n!}\right]^2$

$$= \lim_{n\to\infty} \left[\frac{(n+1)}{(2n+3)}\right]^2 = \lim_{n\to\infty} \left[\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right]^2 = \frac{1}{4} < 1.$$

Hence, by ratio test the series is convergent.

Example 3.10: Test for convergence of the series

(a) $\sum \frac{n!2^n}{n^n}$ (b) $\sum \frac{x^n}{n}$ $(x > 0)$

Solution: (a) Here $u_n = \frac{n!2^n}{n^n}$ and $u_{n+1} = \frac{(n+1)!2^{n+1}}{(n+1)^{n+1}}$. Consider

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!\,2^n} = \lim_{n\to\infty} 2.\left(\frac{n}{1+n}\right)^n = 2.\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1.$$

Hence, by ratio test the series is convergent.

(b) Here $u_n = \frac{x^n}{n}$, and $u_{n+1} = \frac{x^{n+1}}{n+1}$. Consider

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} = \lim_{n\to\infty} \frac{n}{n+1} x = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} . x = x.$$

Hence, the given series is convergent for $x < 1$ and divergent for $x > 1$. At $x = 1$; the series is $1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n} + \ldots$ which is divergent, being harmonic series with $p = 1$.

Example 3.11: Discuss the convergence of the series $\frac{x}{1+x}+\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}+\dots,\ (x>0)$

Solution: Here $u_n=\frac{x^n}{1+x^n}$, and $u_{n+1}=\frac{x^{n+1}}{1+x^{n+1}}$. Consider

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{n+1}}{1+x^{n+1}}\,\frac{1+x^n}{x^n}=x\lim_{n\to\infty}\frac{1+x^n}{1+x^{n+1}}.$$

For $x<1$, $\lim_{n\to\infty}x^n=0=\lim_{n\to\infty}x^{n+1}$, and hence, $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=x<1.$

Thus, the series $\Sigma\, u_n$ is convergent for $x<1$.

For $x>1$, $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{x^n}{1+x^n}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{x^n}}=1\neq 0.$

Hence the series $\Sigma\, u_n$ is divergent.

For $x=1$, the series becomes $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots\infty$, which is divergent; since $S_n=\frac{n}{2}$ tends to infinity as n tends to infinity.

Thus the given series is convergent for $x<1$ and divergent for $x\geq 1$.

EXERCISE 3.3

Test for convergence of the following series:

1. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\dots$

2. $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\dots$

3. $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\dots+\frac{x^n}{n^2+1}+\dots$

4. $\frac{1^2.2^2}{1!}+\frac{2^2.3^2}{2!}+\frac{3^2.4^2}{3!}+\frac{4^2.5^2}{4!}+\dots$

5. $\sum\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n$

6. $1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\dots,\ (\alpha,\beta>0)$

7. $1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\dots+\frac{2^n-2}{2^n+1}x^{n-1}+\dots,\ (x>0)$

8. $2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \ldots + \frac{(n+1)x^n}{n^3} + \ldots$

9. $\sum \frac{\sqrt{(n-1)}}{\sqrt{n^2+1}} x^n$

10. $\sum \frac{1}{x^n + x^{-n}}$

11. $\sum \frac{a^n}{x^n + a^n}$

12. $\sum \left(\frac{n^2}{2^n} + \frac{1}{n^2} \right)$

13. $\sum \frac{n^3 - n + 1}{n!}$

14. $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \ldots$

15. $\frac{4}{18} + \frac{4.12}{18.27} + \frac{4.12.20}{18.27.36} + \ldots$

3.3.4 Raabe's Test

We have seen that the ratio test fails when $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$. In that situation in general, we apply the Raabe's test given as follows:

If Σu_n is a positive term series such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$, then the series converges for $k > 1$, diverges for $k < 1$, and the test fails for $k = 1$.

When $k > 1$, let p be a number such that $k > p > 1$. Compare the given series Σu_n with the harmonic series of order p, that is, with $\Sigma 1/n^p$, which is convergent for $p > 1$. The series Σu_n will be convergent, if

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p}.$$

$$\text{or if,} \quad \frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)^p = 1 + \frac{p}{n} + \frac{p(p-1)}{2n^2} + \ldots$$

$$\text{or if,} \quad n\left\{\frac{u_n}{u_{n+1}} - 1\right\} > p + \frac{p(p-1)}{2n} + \ldots$$

$$\text{or if,} \quad \lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}} - 1\right\} > \lim_{n\to\infty} \left\{p + \frac{p(p-1)}{2n} + \ldots\right\} = p$$

that is, if $k > p$, which is true. Hence Σu_n is convergent.

The case $k < 1$ can be proved on the similar lines.

Another test which is generally employed as a substitute to Raabe's test, when the ratio test fails, is the *logarithmic test*. This is employed when either n occurs as an exponent in u_n/u_{n+1}, or evaluation of limit $n \to \infty$ becomes simpler on taking logarithm of $\frac{u_n}{u_{n+1}}$.

3.3.5 Logarithmic Test

If series Σu_n is a positive term series such that $\lim_{n\to\infty}\left(n \ln \frac{u_n}{u_{n+1}}\right) = k$, then the series converges for $k > 1$, diverges for $k < 1$, and the test fails for $k = 1$.

The proof is similar to that of Raabe's test. As in Raabe's test for $k > p > 1$, the series Σu_n will be convergent, if

$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p,$$

$$\text{or if,}\qquad \ln \frac{u_n}{u_{n+1}} > p \ln\left(1+\frac{1}{n}\right) = p\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)$$

$$\text{or if,}\qquad n \ln \frac{u_n}{u_{n+1}} > p\left(1-\frac{1}{2n}+\frac{1}{3n^2}-\ldots\right)$$

$$\text{or if,}\qquad \lim_{n\to\infty}\left(n \ln \frac{u_n}{u_{n+1}}\right) > p, \text{ that is if } \quad k > p,$$

which is true. Hence the series Σu_n is convergent.

The case $k < 1$ can be proved on the similar lines.

Example 3.12: Test for convergence the series $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\frac{x^4}{7.8}+\ldots$

Solution: Here $u_n = \frac{x^n}{(2n-1).2n}$, and $u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$. Consider

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^{n+1}}{(2n+1)(2n+2)}\,\frac{(2n-1)2n}{x^n}$$

$$= \lim_{n\to\infty}\frac{x\left(1-\frac{1}{2n}\right)}{\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)} = x.$$

Thus Σu_n is convergent for $x < 1$, divergent for $x > 1$ and test fails for $x = 1$.
When $x = 1$ consider

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)}{2n(2n-1)} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{8n+2}{2n(2n-1)}\right] = \lim_{n\to\infty} \frac{\left(2+\frac{2}{4n}\right)}{\left(1-\frac{1}{2n}\right)} = 2 > 1.$$

Hence, by Raabe's test the series is convergent for $x = 1$.
Thus, the given series is convergent for $x \le 1$ and divergent for $x > 1$.

Example 3.13: Discuss the convergence of the series $1 + \frac{1!}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \frac{4!}{5^4}x^4 + \ldots$

Solution: Here $u_n = \frac{n!}{(n+1)^n}x^n$, and $u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}}x^{n+1}$. Consider

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+2)^{n+1}} x^{n+1} \frac{(n+1)^n}{n!} \frac{1}{x^n}$$

$$= \lim_{n\to\infty} n^n\left(1+\frac{1}{n}\right)^n \frac{(n+1)}{n^{n+1}\left(1+\frac{2}{n}\right)^{n+1}} x = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)^n\left(1+\frac{2}{n}\right)} x = \frac{ex}{e^2} = \frac{x}{e},$$

since $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$.

Hence, by ratio test the given series is convergent if $x < e$, divergent if $x > e$, and the test fails for $x = e$. At $x = e$,

$$\frac{u_n}{u_{n+1}} = \frac{n!}{(n+1)^n} e^n \frac{(n+2)^{n+1}}{(n+1)!} \frac{1}{e^{n+1}} = \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}} \frac{1}{e}$$

Since the expression involves the number e, we apply *logarithmic test*. We have

$$\ln \frac{u_n}{u_{n+1}} = (n+1)\ln\left(1+\frac{2}{n}\right) - (n+1)\ln\left(1+\frac{1}{n}\right) - \ln e$$

$$= (n+1)\left[\ln\left(1+\frac{2}{n}\right)-\ln\left(1+\frac{1}{n}\right)\right]-1$$

$$= [(n+1)\left[\left(\frac{2}{n}-\frac{1}{2}\cdot\frac{4}{n^2}+\frac{1}{3}\cdot\frac{8}{n^3}\cdots\right)-\left(\frac{1}{n}-\frac{1}{2}\frac{1}{n^2}+\frac{1}{3}\cdot\frac{1}{n^3}\cdots\right)\right]-1$$

$$= (n+1)\left[\frac{1}{n}-\frac{3}{2}\cdot\frac{1}{n^2}+\frac{7}{3}\frac{1}{n^3}\cdots\right]-1 = -\frac{1}{2n}+\frac{5}{6n^2}+\ldots$$

Therefore, $\lim\limits_{n\to\infty} n \ln \dfrac{u_n}{u_{n+1}} = \lim\limits_{n\to\infty}\left(-\dfrac{1}{2}+\dfrac{5}{6n}+\ldots\right) = -1/2 < 1$

Hence by logarithmic test the series is divergent for $x = e$.

Thus the given series $\Sigma\, u_n$ converges for $x < e$ and diverges for $x \geq e$.

Example 3.14: Discuss the convergence of the series $x^2(\ln 2)^q + x^3(\ln 3)^q + x^4(\ln 4)^q + \ldots$

Solution: $u_n = x^{n+1}[\ln (n+1)]^q$ and $u_{n+1} = x^{n+2}[\ln (n+2)]^q$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{x^{n+1}[\ln (n+1)]^q}{x^{n+2}[\ln (n+2)]^q} = \lim_{n\to\infty}\left[\frac{\ln(n)+\ln\left(1+\frac{1}{n}\right)}{\ln(n)+\ln\left(1+\frac{2}{n}\right)}\right]^q\frac{1}{x}$$

$$= \lim_{n\to\infty}\left[\frac{\ln(n)+\frac{1}{n}-\frac{1}{2n^2}+\ldots}{\ln(n)+\frac{2}{n}-\frac{1}{2}\cdot\frac{4}{n^2}+\ldots}\right]^q\frac{1}{x}$$

$$= \lim_{n\to\infty}\left[\frac{\ln(n)\left(1+\frac{1}{n\ln(n)}-\frac{1}{2n^2\ln(n)}+\ldots\right)}{\ln(n)\left(1+\frac{2}{n\ln(n)}-\frac{2}{n^2\ln(n)}+\ldots\right)}\right]^q\frac{1}{x}$$

$$= \lim_{n\to\infty}\left[\frac{1+\frac{1}{n\ln(n)}-\frac{1}{2n^2\ln(n)}+\ldots}{1+\frac{2}{n\ln(n)}-\frac{2}{n^2\ln(n)}+\ldots}\right]^q\frac{1}{x} = \frac{1}{x}.$$

Hence, by ratio test the series is convergent for $\dfrac{1}{x} > 1$, that is, for $x < 1$ and divergent for $x > 1$ and the test fails for $x = 1$. For $x = 1$,

$$\frac{u_n}{u_{n+1}} = \left[\frac{1+\frac{1}{n\ln(n)}-\frac{1}{2n^2\ln(n)}+\ldots}{1+\frac{2}{n\ln(n)}-\frac{2}{n^2\ln(n)}+\ldots}\right]^q$$

$$= \left(1+\frac{1}{n\ln(n)}-\frac{1}{2n^2\ln(n)}+\ldots\right)^q\left(1+\frac{2}{n\ln(n)}-\frac{2}{n^2\ln(n)}+\ldots\right)^{-q}$$

$$= \left[1+q\left(\frac{1}{n\ln(n)}-\frac{1}{2n^2\ln(n)}+\ldots\right)+\ldots\right]\left[1-q\left(\frac{2}{n\ln(n)}-\frac{2}{n^2\ln(n)+\ldots}+\ldots\right)\right]$$

$$= 1+\frac{1}{n\ln(n)}(q-2q)+\ldots$$

This gives, $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty}\left[\frac{-q}{\ln(n)}+\ldots\right] = 0 < 1.$

Hence, by Raabe's test, the series is divergent for $x = 1$. Thus the series Σu_n is convergent for $x < 1$ and divergent for $x \geq 1$.

Example 3.15: Test for convergence of the series

$$1+\frac{a}{b}x+\frac{a(a+1)}{b(b+1)}x^2+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3+\ldots \quad (a, b > 0, x > 0)$$

Solution: Here, $u_n = \frac{a(a+1)(a+2)\ldots(a+n-1)}{b(b+1)(b+2)\ldots(b+n-1)}x^n$ and, $u_{n+1} = \frac{a(a+1)(a+2)\ldots(a+n)}{b(b+1)(b+2)\ldots(b+n)}x^{n+1}$.

Consider $$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{a+n}{b+n}.x = \lim_{n\to\infty}\frac{\left(1+\frac{a}{n}\right)}{\left(1+\frac{b}{n}\right)}x = x.$$

Hence, by ratio test the series is convergent for $x < 1$ and divergent for $x > 1$, and the test fails at $x = 1$. At $x = 1$,

$$\frac{u_n}{u_{n+1}} = \frac{b+n}{a+n}, \text{ which gives}$$

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty} n\left[\frac{b+n}{a+n}-1\right] = \lim_{n\to\infty}\frac{n(b-a)}{n+a} = \lim_{n\to\infty}\frac{b-a}{\left(1+\frac{a}{n}\right)} = b-a.$$

Thus, by Raabe's test at $x=1$, the series is convergent for $b-a>1$ and divergent for $b-a<1$ and the test fails at $b-a=1$.

Further, at $x=1$ and $b=a+1$, the given series becomes

$$1+\frac{a}{a+1}+\frac{a}{a+2}+\frac{a}{a+3}+\frac{a}{a+4}+\ldots$$

Obviously, the series is decreasing one with positive terms, because $a>0$.

Take, $f(x)=\dfrac{a}{a+x}$, and consider

$$\int_0^{\infty} f(x)dx=\int_0^{\infty}\frac{a}{a+x}dx=\lim_{n\to\infty}\int_0^{n}\frac{a}{a+x}dx=a\lim_{n\to\infty}\left[\ln(a+n)-\ln a\right]=\infty.$$

Hence, by integral test the series is divergent for $b=a+1$ (at $x=1$).

Thus, the given series is convergent for $x<1$ and divergent for $x>1$ and at $x=1$, the series is convergent for $b-a>1$ and divergent for $b-a\le 1$.

EXERCISE 3.4

Test the following series for convergence

1. $\dfrac{1}{2}+\dfrac{1.3}{2.4}+\dfrac{1.3.5}{2.4.6}+\ldots$

2. $1+\dfrac{3}{7}x+\dfrac{3.6}{7.10}x^2+\dfrac{3.6.9}{7.10.13}x^3+\ldots$

3. $\dfrac{(a+x)}{1!}+\dfrac{(a+2x)^2}{2!}+\dfrac{(a+3x)^3}{3!}+\ldots$

4. $\dfrac{x}{1}+\dfrac{1.x^3}{2.3}+\dfrac{1.3}{2.4}\dfrac{x^5}{5}+\dfrac{1.3.5}{2.4.6}\dfrac{x^7}{7}+\ldots\ (x>0)$

5. $\sum\dfrac{(n!)^2}{(2n)!}x^{2n}$

6. $x+\dfrac{2^2x^2}{2!}+\dfrac{3^3x^3}{3!}+\dfrac{4^4x^4}{4!}+\ldots$

7. $\dfrac{1^2}{2^2}+\dfrac{1^2.3^2}{2^2.4^2}x+\dfrac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2+\ldots$

8. $\dfrac{1}{1^2}+\dfrac{1+2}{1^2+2^2}+\dfrac{1+2+3}{1^2+2^2+3^2}+\ldots$

9. $1+\dfrac{\alpha.\beta}{1.\gamma}x+\dfrac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\ldots$

10. $\dfrac{\sqrt{1}}{\sqrt{2}}x+\dfrac{\sqrt{2}}{\sqrt{5}}x^2+\dfrac{\sqrt{3}}{\sqrt{10}}x^3+\ldots$

11. $\sum\left(\dfrac{n^2}{2^n}+\dfrac{1}{n^2}\right)$

12. $\sum\left(\sqrt{n^2+1}-n\right)x^{2n}$

13. $\sum \frac{n!}{x(x+1)(x+2)(x+n-1)}$

14. $\frac{a}{b}+\frac{a(a+d)}{b(b+d)}x+\frac{a(a+d)(a+2d)}{b(b+d)(b+2d)}x^2+\ldots$

15. $x \ln x + x^2 \ln(2x) + x^3 \ln(3x) + \ldots$

3.3.6 Cauchy's Root Test

In a positive term series Σu_n, if $\lim_{n\to\infty} (u_n)^{1/n} = k$, then the series converges for $k<1$, diverges for $k>1$, and the test fails for $k=1$.

Case I: When $\lim_{n\to\infty} (u_n)^{1/n} = k < 1$. By definition of limit we can find a positive number p, where $k<p<1$, such that

$$(u_n)^{1/n} < p, \text{ for all } n > m, \quad \text{or} \quad u_n < p^n, \text{ for all } n > m.$$

Since $p<1$, the geometric series Σp^n is convergent, and hence by comparison test, the series Σu_n is also convergent.

Case II: When $\lim_{n\to\infty} (u_n)^{1/n} = k > 1$. By definition of limit we can find a positive number q, where $k>q>1$, such that

$$(u_n)^{1/n} > q, \text{ for all } n > m, \quad \text{or} \quad u_n > q^n, \text{ for all } n > m.$$

Since $q>1$, the geometric series Σq^n is divergent, and hence by comparison test, the series Σu_n is also divergent.

The test fails when $k=1$. For example, consider the series $\sum \frac{1}{n^p}$. Here

$$\lim_{n\to\infty}\left(\frac{1}{n^p}\right)^{1/n} = \lim_{n\to\infty}\left[n^{1/n}\right]^{-p} = 1$$

irrespective of the values of p, whereas we know that series is convergent for $p>1$ and divergent for $p\le 1$.

3.3.7 Gauss Test

In a positive term series Σu_n, if $\frac{u_n}{u_{n+1}}$ can the expanded in the form $\frac{u_n}{u_{n+1}} = 1+\frac{k}{n}+o\left(\frac{1}{n^2}\right)$, then Σu_n converges, if $k>1$ and diverges, if $k\le 1$.

This test is also applied after the failure of the ratio test when it is possible to expand $\frac{u_n}{u_{n+1}}$ in powers of $\frac{1}{n}$. Also we note that this test does not fail even if $k=1$.

Example 3.16: Test for convergence the series $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$

Solution: Here, $u_n=\left(\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right)^{-n}$. Consider

$$\lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n}\right)\right]^{-1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-1}$$

$$=\frac{1}{e-1}<1,\text{ since }\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e.$$

Hence by Cauchy's root test, the given series is convergent.

Example 3.17: Discuss the convergence of the series

$$1+\frac{\alpha\beta}{1.\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3\gamma(\gamma+1)(\gamma+2)}+\ldots$$

Solution: Neglecting the first term, we have

$$u_n=\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{1.2.3\ldots n.\gamma(\gamma+1)\ldots(\gamma+n-1)}\quad u_{n+1}=\frac{\alpha(\alpha+1)\ldots(\alpha+n)\beta(\beta+1)\ldots(\beta+n)}{1.2.3\ldots(n+1)\gamma(\gamma+1)\ldots(\gamma+n)}$$

Consider, $$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}=1$$

Hence, the ratio test fails.

Next consider,

$$\frac{u_n}{u_{n+1}}=\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}=\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1+\frac{\alpha}{n}\right)^{-1}\left(1+\frac{\beta}{n}\right)^{-1}$$

$$=\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1-\frac{\alpha}{n}+\frac{\alpha^2}{n^2}-\ldots\right)\left(1-\frac{\beta}{n}+\frac{\beta^2}{n^2}-\ldots\right)$$

$$= \left(1+\frac{1}{n}+\frac{\gamma}{n}+\frac{\gamma}{n^2}\right)\left(1-\frac{\alpha}{n}-\frac{\beta}{n}+\frac{\alpha\beta}{n^2}+\frac{\alpha^2}{n^2}+\frac{\beta^2}{n^2}+\ldots\right)$$

$$= 1+(1+\gamma-\alpha-\beta)\frac{1}{n} + \text{terms of second and higher order in } \frac{1}{n}.$$

Hence, by Gauss test, the series $\Sigma\, u_n$ converges if $1+\gamma-\alpha-\beta>1$, that is, if $\gamma>\alpha+\beta$ and diverges if $\gamma \le \alpha+\beta$.

EXERCISE 3.5

Examine the convergence of the following series:

1. $\Sigma 3^{-n-(-1)^n}$
2. $\sum \frac{1}{(\ln n)^n}$
3. $\sum \left(\frac{n+1}{n+2}\right)^n x^n, \quad (x>0)$
4. $\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$
5. $\sum \frac{(1+nx)^n}{n^n}$
6. $\sum \frac{(n-\ln n)^n}{2^n-n^n}$
7. $\frac{1}{2}x+\left(\frac{2}{3}\right)^4 x^2+\left(\frac{3}{4}\right)^9 x^3+\ldots, \ (x>0)$
8. $\frac{2^2.4^2}{3^2.5^2}+\frac{2^2.4^2.6^2}{3^2.5^2.7^2}+\frac{2^2.4^2.6^2.8^2}{3^2.5^2.7^2.9^2}+\ldots$
9. $(a+b)+(a^2+b^2)+(a^3+b^3)+\ldots\ (a, b>0)$
10. $\Sigma\,(n \ln n)^{-1}(\ln \ln n)^{-k}$

3.4 ALTERNATING SERIES

A series in which the terms are alternately positive or negative is called an alternating series. For example, the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5} \ldots$ is an alternating series.

The general form of this series is $u_1 - u_2 + u_3 - u_4 + \ldots$

Next we give a test for the convergence of an alternating series called the *Leibnitz's test.*

Leibnitz test: *An alternating series $u_1-u_2+u_3-u_4+\ldots$ is convergent if each term is numerically less than its preceding term, that is if, $u_1>u_2>u_3 \ldots$, and $\lim_{n\to\infty} u_n = 0$.*

To establish this test consider S_{2n}, the sum of the first $2n$ terms, as

$$S_{2n} = (u_1-u_2)+(u_3-u_4)+(u_5-u_6)+\ldots+(u_{2n-1}-u_{2n}) \quad \ldots(3.5)$$

or as,

$$S_{2n} = u_1-(u_2-u_3)-(u_4-u_5)-\ldots-(u_{2n-2}-u_{2n-1})-u_{2n} \quad \ldots(3.6)$$

Each term in the brackets in (3.5) is positive, since $u_1>u_2>u_3 \ldots$, hence S_{2n} is positive and increases as n increases.

Similarly, each term in the brackets in (3.6) is positive and hence S_{2n} is always less than u_1.

Thus sequence $\{S_{2n}\}$ is monotonically increasing and bounded, therefore $\lim\limits_{n\to\infty} S_{2n}$ exists and is finite. Further,

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + u_{2n+1}) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1} = \lim_{n\to\infty} S_{2n},$$

since $\lim\limits_{n\to\infty} u_{2n+1} = 0$. Thus S_n tends to the same finite limit whether n is odd or even.

Hence, the given series is convergent.

When $\lim\limits_{n\to\infty} u_n \neq 0$, then $\lim\limits_{n\to\infty} S_{2n} \neq \lim\limits_{n\to\infty} S_{2n+1}$.

Thus in this case the given alternating series is *oscillatory*.

Example 3.18: Discuss the convergence of the series.

(a) $1 - \dfrac{1}{2!} + \dfrac{1}{4!} - \dfrac{1}{6!} + \ldots$ (b) $\dfrac{1}{6} - \dfrac{2}{11} + \dfrac{3}{16} - \dfrac{4}{21} + \ldots$

Solution: (a) The terms of the given series are alternately positive and negative; each term being numerically less than its preceding term. Also,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2n!} = 0.$$

Hence, by Leibnitz test the series $\sum u_n$ is *convergent*.

(b) Here, $u_n = \dfrac{n}{5n+1}$, and $u_{n+1} = \dfrac{n+1}{5n+6}$. Consider,

$$u_{n+1} - u_n = \frac{n+1}{5n+6} - \frac{n}{5n+1} = \frac{1}{(5n+6)(5n+1)} > 0;$$

thus, the terms are not numerically of decreasing order. Also,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{5n+1} = \lim_{n\to\infty} \frac{1}{5+1/n} = \frac{1}{5} \neq 0.$$

Thus, the given alternating series is not *convergent*.

Example 3.19: Discuss the convergence of the series

(a) $\dfrac{x}{1+x} - \dfrac{x^2}{1+x^2} + \dfrac{x^3}{1+x^3} - \dfrac{x^4}{1+x^4} + \ldots,\quad (0 < x < 1)$

(b) $\left(\dfrac{1}{2} - \dfrac{1}{\ln 2}\right) - \left(\dfrac{1}{2} - \dfrac{1}{\ln 3}\right) + \left(\dfrac{1}{2} - \dfrac{1}{\ln 4}\right) - \ldots$

Solution: (a) The terms of the given series are alternately positive and negative. Here

$$u_n - u_{n-1} = \frac{x^n}{1+x^n} - \frac{x^{n-1}}{1+x^{n-1}}$$

$$= x^{n-1}\left[\frac{x+x^n-1-x^n}{(1+x^n)(1+x^{n-1})}\right] = \frac{x^{n-1}(x-1)}{(1+x^n)(1+x^{n-1})} < 0, \ (0<x<1).$$

Thus, each term is numerically less than its preceding term. Also,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{1+x^n} = 0, \quad (0<x<1).$$

Hence, by Leibnitz test, the given series is *convergent*.

(b) Terms of the given series are alternately positive and negative. Here

$$u_n - u_{n-1} = \left(\frac{1}{2} - \frac{1}{\ln(n+1)}\right) - \left(\frac{1}{2} - \frac{1}{\ln(n)}\right) = \frac{1}{\ln(n)} - \frac{1}{\ln(n+1)} > 0.$$

Thus, each term is not numerically less than its preceding term. Also,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{\ln(n+1)}\right) = \frac{1}{2} \neq 0.$$

Hence, series is not *convergent*.

EXERCISE 3.6

Discuss the convergence of the following series

1. $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \frac{11}{8} + \ldots$

2. $1 - 2x + 3x^2 - 4x^3 + \ldots \ (0<x<1)$

3. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \ (0<x<1)$

4. $\sum \frac{(-1)^{n-1} n}{2n-1}$

5. $\sum \frac{\cos n\pi}{n^2+1}$

6. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1} x^n}{n(n-1)}, \quad 0<x<1$

7. $\sum \frac{(-1)^n}{n \ln n}$

8. $\sum \frac{(-1)^{n-1}}{2^{n-1}} \sin \frac{1}{\sqrt{n}}$

9. $\sum \frac{\sqrt{n} \cos n\pi}{\ln n}$

3.5 ABSOLUTE CONVERGENCE OF A SERIES

If a series of arbitrary terms $u_1 + u_2 + u_3 + ... + u_n + ...$ is such that the series $|u_1| + |u_2| + |u_3| + ... + |u_n| + ...$ is convergent, then the series Σu_n is said to be absolutely convergent.

In case the series $\Sigma |u_n|$ is divergent but the series Σu_n is convergent, then the series Σu_n is said to be conditionally convergent.

For example, the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} ...$ is absolutely convergent since the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ is convergent being geometric series with common ratio $\frac{1}{2} < 1$. But the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$ is only conditionally convergent, since this is convergent by Leibnitz test, however, the corresponding series of absolute terms, that is, the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ...$ is divergent, being harmonic series of order 1.

Next we state an important result concerning the absolute convergence of series.

An absolutely convergent series is necessarily convergent but not conversely.

For if $\Sigma |u_n|$ is convergent, then since

$$u_1 + u_2 + ... + u_n + ... \leq |u_1| + |u_2| + |u_n| + ...$$

thus Σu_n is also convergent.

To show that converse is not true, consider the series $\sum \frac{(-1)^n}{n}$. It is convergent by alternate series test, but, the corresponding series of absolute terms, the series $\sum \frac{|(-1)^n|}{n} = \sum \frac{1}{n}$ is known to be divergent.

Remarks

1. The convergence of $\Sigma |u_n|$ implies the convergence of Σu_n, but divergence of $\Sigma |u_n|$ does not imply the divergence of Σu_n.
2. Since the series $\Sigma |u_n|$ is of positive terms series, the test already established for positive terms series are applicable to examine the absolute convergence of the series Σu_n.

Example 3.20: Test whether the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - ...$ converges or not.

Solution: The corresponding series of absolute terms is

$$\frac{|\sin x|}{1^3} + \frac{|\sin 2x|}{2^3} + \frac{|\sin 3x|}{3^3} + ...$$

Here, $|u_n| = \frac{|\sin nx|}{n^3}$. Consider $v_n = \frac{1}{n^3}$. Clearly, $\frac{|\sin nx|}{n^3} \le \frac{1}{n^3}$.

By comparison test, the series $\sum \frac{|\sin nx|}{n^3}$ is convergent, since the series $\sum \frac{1}{n^3}$ is convergent, being harmonic series of order $3 > 1$.

Thus the given series is absolutely convergent and hence it is convergent also.

Example 3.21: Examine the convergence and absolute convergence of the series $\sum \frac{(-1)^{n+1} n}{n^2+1}$.

Solution: The series is an alternating series of the form $u_1 - u_2 + u_3 - u_4 + \ldots$ with $u_n = \frac{n}{n^2+1} > 0$, numerically for all n. Here,

$$u_{n+1} - u_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{-(n^2+n)+1}{(n^2+1)[(n+1)^2+1]} < 0.$$

Thus, the terms are of decreasing order. Also,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+(1/n)} = 0.$$

Hence, by Leibnitz's test the given series is convergent.

To discuss the absolute convergence, consider the corresponding series with positive terms, that is, the series $\sum \frac{n}{n^2+1}$. Choose $v_n = \frac{1}{n}$. We have,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n^2}} = 1.$$

Hence, by comparison test the series $\sum \frac{n}{n^2+1}$ and $\sum \frac{1}{n}$ both converge or diverge together. But the series $\sum \frac{1}{n}$ is divergent, being harmonic series of order 1, thus, the series $\sum \frac{n}{n^2+1}$ is also divergent. Therefore, the given series is not absolutely convergent; it is only conditionally convergent.

EXERCISE 3.7

Discuss the convergence of the following series:

1. $1 - \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} - \frac{1}{3^5} + \frac{1}{2^3} - \frac{1}{3^7} + \ldots$

2. $1-\frac{2}{3}+\frac{3}{3^2}-\frac{4}{3^3}+\ldots$

3. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$

4. $2\sin\frac{x}{3}+4\sin\frac{x}{9}+8\sin\frac{x}{27}+\ldots$

5. $\sum\frac{(-1)^{n-1}\cos^2 nx}{n\sqrt{n}}$

3.6 POWER SERIES

A series of the form

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots, \qquad \ldots(3.7)$$

where all the a_i's are independent of x, is called a power series in x.

Such a series may converge for some or all values of x. In the power series (3.7) $u_n = a_nx^n$. Therefore,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{a_{n+1}x^{n+1}}{a_nx^n}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}x=lx,$$

where $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=l$. Thus, by ratio test, the series converges, when lx is numerically less than one, that is, when $|x|<\frac{1}{l}$ and diverges, when $|x|>\frac{1}{l}$. Hence, the power series converges when x lies in the interval $-\frac{1}{l}<x<\frac{1}{l}$. This interval is called the *interval of convergence* of the power series (3.7).

3.6.1 A Few Special Power Series

We discuss the convergence of the following power series which occur most frequently in applications.

(a) The exponential series: *The exponential series*

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots \qquad \ldots(3.8)$$

is convergent for all values of x.

Here, $$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^n}{n!}\frac{(n-1)!}{x^{n-1}}=\lim_{n\to\infty}\frac{x}{n}=0<1.$$

Hence, by ratio test the series (3.8) converges for all real values of x.

(b) The logarithmic series: *The logarithmic series*

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^n\frac{x^n}{n}+\ldots \qquad \ldots(3.9)$$

is convergent for $-1<x\le 1$.

Here, $$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(-1)^{n+1}x^{n+1}}{(n+1)}\frac{n}{(-1)^n x^n}$$

$$=\lim_{n\to\infty}\frac{n}{n+1}(-x)=-x.$$

Hence, by ratio test the series is convergent for $|x|<1$ and divergent for $|x|>1$, and the test fails for $|x|=1$. When $x=+1$, the series becomes

$$1-\frac{1}{2}+\frac{1}{3}-\ldots(-1)^n\frac{1}{n}+\ldots$$

which is convergent by Leibnitz test for alternating series. When $x=-1$, the series becomes

$$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots\right),$$

which is divergent, being negative of harmonic series of order 1.

Hence the logarithmic series (3.9) converges for $-1<x\le 1$.

(c) **The binomial series:** *The binomial series*

$$1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots\infty \qquad \ldots(3.10)$$

converges for $|x|<1$.

Here, $$\lim_{r\to\infty}\frac{u_{r+1}}{u_r}=\lim_{r\to\infty}\frac{n(n-1)\ldots(n-r+1)}{r!}x^r\frac{(r-1)!}{n(n-1)\ldots(n-r+2)x^{r-1}}$$

$$=\lim_{r\to\infty}\frac{n-r+1}{r}x=\lim_{r\to\infty}\left(\frac{n+1}{r}-1\right)x=-x.$$

Hence by ratio test the series (3.10) converges for $|x|<1$, diverges for $|x|>1$.

Example 3.22: Examine the convergence of the series $\frac{1}{1-x}+\frac{1}{2(1-x)^2}+\frac{1}{3(1-x)^3}+\ldots(x\ne 1)$.

Solution: Here $$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim_{n\to\infty}\frac{|n(1-x)^n|}{|(n+1)(1-x)^{n+1}|}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)|1-x|}=\frac{1}{|1-x|}.$$

Hence, by ratio test the series is convergent for $|1-x|>1$ and divergent for $|1-x|<1$, except at $|1-x|=0$; and the test fails at $|1-x|=1$.

When $(1-x)=1$, that is $x=0$, the series becomes $1+\frac{1}{2}+\frac{1}{3}+\ldots$ which is divergent, and when $(1-x)=-1$, that is at $x=2$, the series becomes $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\ldots$ which is convergent.

Thus, the given series is convergent for $|1-x|>1$ and for $x=2$; divergent for $|1-x|<1$ and for $x=0$, except at $|1-x|=0$ for which it is not defined.

EXERCISE 3.8

Find the interval of convergence for the following series.

1. $x-\frac{x^2}{\sqrt{2}}+\frac{x^3}{\sqrt{3}}-\frac{x^4}{\sqrt{4}}+\ldots$

2. $x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\ldots$

3. $\frac{a+x}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+\ldots$

4. $\frac{x}{1+\sqrt{1}}+\frac{x^2}{2+\sqrt{2}}+\frac{x^3}{3+\sqrt{3}}+\ldots$

5. $\sum \frac{(x+2)^n}{3^n.n}$

6. $\sum \frac{x^n}{(2n-1)^2 2^n}$

7. $\sum \frac{nx^n}{(n+1)(n+2)}$ $(x>0)$

8. $\sum \frac{2^2.4^2.6^2\ldots(2n)^2}{3^2.5^2.7^2\ldots(2n+1)^2}x^n$

9. $\sum_{n=0}^{\infty} \frac{(3x+5)^n}{(n+1)!}$

10. $\sum \frac{(-1)^n(x-1)^{2n}}{n3^n}$

ANSWERS

Exercise 3.1 (p. 156)

1. (a) $\frac{n}{n+1}$ (b) $\frac{n^2}{n!}$ (c) $\frac{1}{n(n+1)}$ (d) $\left(\frac{n+1}{n}\right)^n$
2. (a) Convergent (b) Convergent (c) Convergent (d) Divergent
3. (a) Divergent (b) Convergent (c) Oscillatory (d) Oscillatory
   (e) Divergent (f) Divergent.

Exercise 3.2 (p. 161)

1. Convergent 2. Divergent 3. Convergent 4. Convergent
5. Divergent 6. Convergent 7. Divergent 8. Divergent
9. Convergent for $x<\frac{1}{e}$ and divergent for $x\geq\frac{1}{e}$ 10. Divergent
11. Convergent 12. Convergent 13. Convergent 14. Divergent
15. Convergent for $q>1+p$, and divergent for $q\leq 1+p$.

Exercise 3.3 (p. 165)

1. Convergent
2. Convergent
3. Convergent for $x \le 1$, and divergent for $x > 1$
4. Convergent
5. Convergent for $x < 1$; and divergent for $x \ge 1$
6. Convergent if $\beta > \alpha > 0$; and divergent if $\alpha \ge \beta > 0$
7. Convergent if $x < 1$; and divergent if $x \ge 1$
8. Convergent for $x \le 1$, and divergent for $x > 1$
9. Convergent for $x < 1$, and divergent for $x \ge 1$
10. Convergent for $x < 1$ or $x > 1$, and divergent for $x = 1$
11. Convergent for $x > a$, and divergent for $x \le a$
12. Convergent
13. Convergent
14. Divergent
15. Convergent.

Exercise 3.4 (p. 171)

1. Divergent
2. Convergent for $x \le 1$, and divergent for $x > 1$
3. Convergent for $x < \dfrac{1}{e}$, and divergent for $x \ge \dfrac{1}{e}$
4. Convergent for $x \le 1$, and divergent for $x > 1$
5. Convergent for $x^2 < 4$, and divergent for $x^2 \ge 4$
6. Convergent for $x < \dfrac{1}{e}$, and divergent for $x \ge \dfrac{1}{e}$
7. Convergent for $x < 1$, and divergent for $x \ge 1$
8. Divergent
9. Convergent for $x < 1$ or at $x = 1$ for $\gamma > \alpha + \beta$, and divergent for $x > 1$ or at $x = 1$ for $\gamma \le \alpha + \beta$
10. Convergent for $x < 1$, and divergent for $x \ge 1$
11. Convergent
12. Convergent for $x^2 < 1$, and divergent for $x^2 \ge 1$
13. Convergent for $x > 2$, and divergent for $x \le 2$
14. Convergent for $x < 1$ or for $x = 1$ and $b > a + d$, and divergent for $x > 1$ or for $x = 1$ and $b \le a + d$
15. Convergent for $x < 1$, and divergent for $x \ge 1$.

Exercise 3.5 (p. 174)

1. Convergent
2. Convergent
3. Convergent for $x < 1$, and divergent for $x \ge 1$
4. Convergent
5. Convergent for $x < 1$, and divergent for $x \ge 1$
6. Convergent
7. Convergent for $x < e$, and divergent for $x \ge e$
8. Divergent
9. Convergent for $a, b, < 1$, and divergent for $a, b \ge 1$
10. Convergent for $k > 1$, and divergent for $k \le 1$.

Exercise 3.6 (p. 176)

1. Oscillatory
2. Convergent
3. Convergent
4. Oscillatory
5. Convergent
6. Convergent
7. Convergent
8. Convergent
9. Not convergent.

Exercise 3.7 (p. 178)

1. Absolutely convergent
2. Absolutely convergent
3. Conditionally convergent
4. Absolutely convergent
5. Absolutely convergent

Exercise 3.8 (p. 181)

1. $-1 < x \leq 1$
2. $-1 < x \leq 1$
3. $x < \frac{1}{e}$
4. $x < 1$
5. $x < 1$
6. $x < 2$
7. $x < 1$
8. $x < 1$
9. $-\infty < x < \infty$
10. $1 - \sqrt{3} < x < 1 + \sqrt{3}$.

4 CHAPTER Differentiation and Its Applications

Calculus is a major area in mathematics with applications in science, engineering, medicine and finance. The concept of derivative, measuring the rate of charge, is at the core of differential calculus. Applications of derivative include computations involving velocity and acceleration, errors and approximations, power series representation of a function, finding tangents, normals and asymptotes to a curve, determining curvature, etc. The interesting mean value theorems and their consequences extend the scope of differential calculus further.

4.1 SINGLE AND HIGHER ORDER DERIVATIVES

In this section, first we discuss the concept of derivative of a function and afterward, we consider successive differentiation of $f(x)$ resulting in higher order derivatives.

4.1.1 Derivative of a Function

Let $y = f(x)$ be a real-valued function defined on an interval I and let x_0 be a point in I. Then $f(x)$ is said to be differentiable at $x = x_0$, if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}, \text{ or } \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists and is finite. The derivative of $f(x)$ at $x = x_0$ is denoted by $f'(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$, or $Df(x_0)$.

If $f(x)$ is differentiable at every point of the interval (a, b), then $f(x)$ is said to be differentiable in (a, b). If the interval $[a, b]$ is closed, then we say that $f(x)$ is differentiable at every point of $[a, b]$, if $f(x)$ is differentiable in (a, b) and right hand derivative of $f(x)$ exists at 'a' and left hand derivative of $f(x)$ exists at 'b'. Geometrically, *the derivative of $f(x)$ at a given point $P(x_0, y_0)$ gives the slope of the tangent line to the curve $y = f(x)$ at that point P, that is, $\tan\theta = f'(x_0)$*, as shown in Fig. 4.1. Also if a function $y = f(x)$ is differentiable at a point P, it is necessarily continuous there. The converse, however is not true, for example, $f(x) = |x|$ is continuous at $x = 0$, but not differentiable.

If f and g are two differentiable functions, then the following properties are satisfied:

(a) $(cf')(x) = cf'(x)$, c is any constant

(b) $(f \pm g)'(x) = f'(x) \pm g'(x)$

(c) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(d) $\left(\dfrac{f}{g}\right)'(x) = \dfrac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2}$, $g(x) \neq 0$

(e) If $h(x) = g[f(x)]$, then $h'(x) = g'[f(x)]f'(x)$

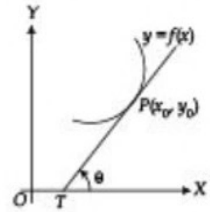


Fig. 4.1

Example 4.1: (i) Investigate the function $f(x) = |\ln x|$ for differentiability at the point $x = 1$.

(ii) Find the derivative of the function $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Solution: (i) At $x = 1$, $\dfrac{\Delta y}{\Delta x} = \dfrac{f(1+\Delta x) - f(1)}{\Delta x} = \dfrac{|\ln(1+\Delta x)|}{\Delta x}$

$$= \begin{cases} \dfrac{\ln(1+\Delta x)}{\Delta x} & \text{at } \Delta x > 0 \\ \dfrac{-\ln(1+\Delta x)}{\Delta x} & \text{at } \Delta x < 0 \end{cases}$$

Hence, $\lim\limits_{\Delta x \to 0^+} \dfrac{\Delta y}{\Delta x} = +1$ and $\lim\limits_{\Delta x \to 0^-} \dfrac{\Delta y}{\Delta x} = -1$.

Since, the left hand and right hand derivatives are different at the point $x = 1$. Hence $f(x)$ is not differentiable at $x = 1$.

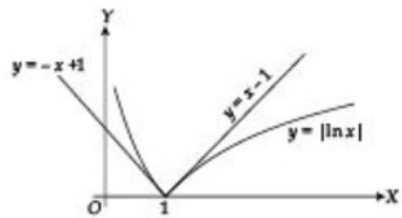


Fig. 4.2

(ii) We have, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\left(\dfrac{2x}{1+x^2}\right)^2}} \dfrac{2(1+x^2) - 4x^2}{(1+x^2)^2} = \dfrac{2(1-x^2)}{|1-x^2|(1+x^2)} = \begin{cases} \dfrac{2}{1+x^2} & \text{at } |x| < 1 \\ -\dfrac{2}{1+x^2} & \text{at } |x| > 1 \end{cases}$

At $|x| = 1$ the derivative does not exist.

4.1.2 Successive Differentiation: Higher Order Derivatives

The derivative of $f(x)$ at any point x, if it exists, is again a function of x say $f'(x) = g(x)$. If the function $g(x)$ is itself differentiable, then the second derivative of $f(x)$ is defined as $f''(x) = g'(x)$. It is also denoted by $f^{(2)}(x)$, or $\dfrac{d^2 f}{dx^2}$, or $D^2 f(x)$. The nth order derivative of $f(x)$ is defined as $f^{(n)}(x) = \dfrac{d}{dx}\left[f_{(x)}^{(n-1)}\right]$.

The nth Derivative of the Product of Two Functions

The nth order derivative of the product of two functions f and g is obtained by *Leibnitz rule*, stated as follows.

Leibnitz's Rule: *If f and g are functions of x possessing derivatives of the nth order, then*

$$(fg)^{(n)} = f^{(n)}g + C_1^n f^{(n-1)}g^{(1)} + C_2^n f^{(n-2)}g^{(2)} + \ldots + C_n^n fg^{(n)} \qquad \ldots(4.1)$$

Proof. The result (4.1) will be proved by induction.

Since, we have $\quad (fg)^{(1)} = f^{(1)}g + fg^{(1)}$, thus, the result is true for $n = 1$.

Let it be true for $n = k$, that is, let

$$(fg)^{(k)} = f^{(k)}g + C_1^k f^{(k-1)}g^{(1)} + C_2^k f^{(k-2)}g^{(2)} + \ldots + C_k^k fg^{(k)}.$$

Differentiating it both sides w.r.t. x, gives

$$\begin{aligned}(fg)^{(k+1)} &= (f^{(k+1)}g + f^{(k)}g^{(1)}) + C_1^k(f^{(k)}g^{(1)} + f^{(k-1)}g^{(2)}) + C_2^k(f^{(k-1)}g^{(2)} + f^{(k-1)}g^{(3)}) + \ldots + C_k^k(f^{(1)}g^{(k)} + fg^{(k+1)}) \\ &= f^{(k+1)}g + (C_0^k + C_1^k)f^{(k)}g^{(1)} + (C_1^k + C_2^k)f^{(k-1)}g^{(2)} + \ldots + C_k^k fg^{(k+1)} \\ &= f^{(k+1)}g + C_1^{k+1}f^{(k)}g^{(1)} + C_2^{k+1}f^{(k-1)}g^{(2)} + \ldots + C_{k+1}^{k+1}fg^{(k+1)}\end{aligned}$$

Thus, the result is true for $n = k + 1$ also.

This proves the induction step and since the result is true for $n = 1$, hence it is true for all positive integral values of n.

Next, we list the nth order derivatives of some frequently used functions.

1. $\dfrac{d^n}{dx^n}(ax+b)^m = \dfrac{m!}{(m-n)!}a^n(ax+b)^{m-n}, \quad n \le m, \quad m > 0$

2. $\dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

3. $\dfrac{d^n}{dx^n}\ln(ax+b) = \dfrac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$

4. $\dfrac{d^n}{dx^n}(a^{mx}) = m^n(\ln a)^n a^{mx}$

5. $\dfrac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(ax+b+n.\dfrac{\pi}{2}\right)$

6. $\dfrac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(ax+b+n.\dfrac{\pi}{2}\right)$

7. $\dfrac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = (a^2+b^2)^{n/2}e^{ax}\sin\left(bx+c+n\tan^{-1}\dfrac{b}{a}\right)$

8. $\frac{d^n}{dx^n}[e^{ax}\cos(bx+c)] \quad = (a^2+b^2)^{n/2}e^{ax}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Example 4.2: Find the nth derivatives of the following functions

(i) $\frac{x^4}{(x-1)(x-2)}$ (ii) $e^{ax}\sin(bx+c)$

Solution: (i) Let $y=\frac{x^4}{(x-1)(x-2)}$. Resolving into partial fractions, we have

$$y = x^2+3x+7-\frac{1}{x-1}+\frac{16}{x-2}$$

Differentiating it n (> 2) times, we obtain

$$y^{(n)} = (-1n)\, n!\left[\frac{16}{(x-2)^{n+1}}-\frac{1}{(x-1)^{n+1}}\right]$$

(ii) Consider, $y=e^{ax}\sin(bx+c)$. We have

$$y^{(1)} = e^{ax}[a\sin(bx+c)+b\cos(bx+c)]$$

Substituting $a=r\cos\phi$, $b=r\sin\phi$, and thus, $r=\sqrt{a^2+b^2}$ and $\phi=\tan^{-1}\frac{b}{a}$, and simplifying we obtain

$$y^{(1)} = (a^2+b^2)^{1/2}\, e^{ax}\sin\left(bx+c+\tan^{-1}\frac{b}{c}\right)$$

Similarly, $$y^{(2)} = (a^2+b^2)^{2/2}\, e^{ax}\sin\left(bx+c+2\tan^{-1}\frac{b}{c}\right)$$

and, in general $$y^{(n)} = (a^2+b^2)^{n/2}\, e^{ax}\sin\left(bx+c+n\tan^{-1}\frac{b}{c}\right)$$

Example 4.3: Find the nth order derivative of the function $y=\frac{3x+2}{x^2-2x-5}$ at the point $x=0$.

Solution: Rewriting the given function as

$$y(x^2-2x+5) = 3x+2. \qquad ...(4.2)$$

Differentiating this n times using Leibnitz rule, we obtain

$$y^{(n)}(x^2-2x+5)+ny^{(n-1)}(2x-2)+\frac{n(n-1)}{2}\,y^{(n-2)}(2)=0.$$

For $x=0$, we have $5y^{(n)}(0)-2ny^{n-1}(0)+n(n-1)y^{n-2}(0)=0$. Hence,

$$y^{(n)}(0) = \frac{2}{5} ny^{n-1}(0) - \frac{n(n-1)}{5} y^{n-2}(0), \quad n \geq 2. \qquad ...(4.3)$$

Also from (4.2), $y(0) = 2/5$, and $y'(x) = \dfrac{-3x^2 - 4x + 19}{(x^2 - 2x + 5)^2}$, which gives $y'(0) = \dfrac{19}{25}$.

For $n = 2, 3, ...$ in (4.3), the values of the derivatives of higher orders can be obtained at $x = 0$.

Example 4.4: If $y = a\cos(\ln x) + b\sin(\ln x)$, show that $x^2y^{(2)} + xy^{(1)} + y = 0$, and

$$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0.$$

Solution: We have, $y = a\cos(\ln x) + b\sin(\ln x)$. Differentiating w.r.t. x,

$$y^{(1)} = -a\sin(\ln x).\frac{1}{x} + b\cos(\ln x).\frac{1}{x}, \text{ or } xy^{(1)} = -a\sin(\ln x) + b\cos(\ln x).$$

Again differentiating w.r.t. x, we get, $xy^{(2)} + y^{(1)} = -a\cos(\ln x)\dfrac{1}{x} - b\sin(\ln x)\dfrac{1}{x}$

or,
$$x^2y^{(2)} + xy^{(1)} + y = 0. \qquad ...(4.4)$$

Differentiating (4.4) n times using Leibnitz's rule, we have

$$x^2y^{(n+2)} + 2nxy^{(n+1)} + n(n-1)y^{(n)} + xy^{(n+1)} + ny^{(n)} + y^{(n)} = 0.$$

or,
$$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0.$$

EXERCISE 4.1

1. Show that function $y = 3|x| + 1$ is not differentiable at $x = 0$.
2. Find the derivative of $f(x) = x|x|, -1 \leq x \leq 1$.
3. Find the derivatives of the nth order of the following functions:

   (a) $y = \ln(x^2 + x - 2)$ (b) $y = \dfrac{ax+b}{cx+d}$ (c) $\dfrac{1}{1+x+x^2}$

4. If $y = (\sin^{-1}x)^2$, prove that $(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2y^{(n)} = 0$.
5. If $y = x^{n-1}\ln x$, prove that $y^{(n)} = [(n-1)!/x]$.
6. If $y^{1/m} + y^{-1/m} = 2x$, prove that $(x^2-1)y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2 - m^2)y^{(n)} = 0$.
7. If $y = \left[x + \sqrt{(1+x^2)}\right]^m$, find $y^{(n)}(0)$.
8. If $y = \tan^{-1}x$, prove that $(1+x^2)\,y^{(n+2)} + 2(n+1)\,xy^{(n+1)} + n(n+1)y^{(n)} = 0$. Find $y^{(n)}(0)$.
9. If $x = \sin t, y = \cos pt$ prove that $(1-x^2)y^{(2)} - xy^{(1)} + p^2y = 0$. Hence prove that $(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - (n^2-p^2)y^{(n)} = 0$.
10. If $y^{(n)} = \dfrac{d^n}{dx^n}(x^n\ln x)$, then $y^{(n)} = ny^{(n-1)} + (n-1)!$ and hence show that

$$y^{(n)} = n!\left(\ln x + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right).$$

4.2 ERRORS AND APPROXIMATIONS

Let $y = f(x)$ be a real valued differentiable function then the increment Δy in the function $y = f(x)$, corresponding to the increment Δx in x can be expressed as

$$\Delta y = f(x + \Delta x) - f(x) = f'(x)\Delta x + \alpha \Delta x,$$

where α is an infinitesimal small quantity dependent on Δx and tends to zero as $\Delta x \to 0$. The principal linear part of this increment $f'(x)\Delta x$ is called the *differential* and is denoted by $df(x)$ or dy.

In the limiting form, the differential is also written as $df(x) = f'(x)dx$. Hence an *approximation* to $f(x + \Delta x)$ can be written as

$$f(x + \Delta x) \approx f(x) + f'(x)dx. \qquad ...(4.5)$$

As we move from x to a nearby point $x + \Delta x$, we define $df = f'dx$, $\frac{df}{f}$, and $\frac{df}{f} \times 100$ as *absolute error*, *relative error* and *percentage error*, respectively.

Example 4.5: About how accurately should we measure the radius r of a sphere to calculate the surface area $S = 4\pi r^2$ within 1% of its true value?

Solution: We require, $|\Delta S| \le \frac{S}{100} = \frac{4\pi r^2}{100}$ and here, $S = 4\pi r^2$, thus $dS = 8\pi r dr$.

Replacing ΔS in this inequality with dS, we have

$$|8\pi r dr| \le \frac{4\pi r^2}{100}, \text{ or } \quad |dr| \le \frac{r}{200}.$$

Thus r should be measured no more than 0.5% of the true value.

Example 4.6: Using the concept of differential, find the approximate value of the function

$$f(x) = \sqrt[5]{\frac{2-x}{2+x}} \text{ at } x = 0.15.$$

Solution: We have

$$f(x + \Delta x) \approx f(x) + f'(x)dx \qquad ...(4.6)$$

Here, $f(x) = \sqrt[5]{\frac{2-x}{2+x}}$. It gives $f'(x) = -\frac{4}{5}\left(\frac{2+x}{2-x}\right)^{4/5} \frac{1}{(2+x)^2}$.

At $x = 0$, $f(0) = 1$ and $f'(0) = -\frac{1}{5}$.

Further taking $x = 0$ and $\Delta x = 0.15$ in (4.6) and approximating dx with Δx, we obtain

$$f(0.15) \approx 1 - \frac{0.15}{5} = 0.97.$$

Example 4.7: The volume V of a fluid flowing through a small pipe in a unit of time at a fixed pressure is a constant times the fourth power of the pipe's radius r. How will a 10% increase in r affect V?

Solution: We have $V = kr^4$, where k is a constant. Thus $dV = 4kr^3 dr$, and hence, $\frac{dV}{V} = 4\frac{dr}{r}$.

A 10% increase in r means $\frac{dr}{r} = \frac{1}{10}$, which gives, $\frac{dV}{V} = 4 \times \frac{1}{10} = \frac{2}{5}$.

Thus a 10% increase in r will produce a 40% increase in V.

EXERCISE 4.2

1. The radius of a sphere is increasing at a variable rate and is equal to 1 cm/sec, when the radius is 3 cm. Find the rate of change in volume at this time.
2. If there is a possible error of 0.02 cm in the measurement of the diameter of a sphere, then find the possible percentage error in its volume, when the radius is 10 cm.
3. A vessel is in the form of an inverted cone of semivertical angle 45°. Water is poured into this vessel at the rate of 100 cc per second. Find the rate of rise in the water level when it is 2 cm deep.
4. Use the concept of differential to find an approximate value of
$$y = 3(4.02)^2 - 2(4.02)^{3/2} + 8(4.02)^{-1/2}.$$
5. All faces of a copper cube with 5 cm sides were uniformly ground down. As a result the weight of the cube was reduced by 0.96 gm. Knowing the specific density of copper as 8, find the reduction in the cube size, that is, the amount by which its side was reduced.

4.3 TANGENTS AND NORMALS

The '*equation of the tangent*' to the curve $y = f(x)$ at a point $P(x_0, y_0)$, as shown in Fig. 4.3 is

$$y - y_0 = f'(x_0)(x - x_0) \qquad ...(4.7)$$

where $f'(x_0) = \tan \psi$ is the slope of the tangent line PT, ψ is the angle which this line makes with the positive direction of the x-axis.

A straight line PN passing through the point of contact $P(x_0, y_0)$ and perpendicular to the tangent line PT is called the *normal to the curve* $y = f(x)$ at $P(x_0, y_0)$. Its equation is given by

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)\ f'(x_0) \neq 0. \quad ...(4.8)$$

The angle of intersection of two curves is the angle between the tangents to the curves at their point of intersection.

If m_1 and m_2 are the slopes of the tangents to the curves $y = f(x)$ and $y = g(x)$ at their point of intersection, say (x_0, y_0), then the angle of intersection 'θ' between these two curves at (x_0, y_0) is

$$\tan\theta = \frac{|m_1 - m_2|}{1 + m_1 m_2} \quad ...(4.9)$$

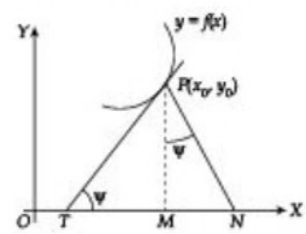


Fig. 4.3

In case $m_1 m_2 = -1$, $\theta = 90°$, that is, the two curves cut orthogonally.

Segments of tangent, normal, subtangent and subnormal

Draw the ordinate PM. Then PT and PN are called the *segment of the tangent* and *segment of the normal* respectively; and TM, MN are called the *sub-tangent* and *sub-normal* respectively.

From Fig. 4.3, for any arbitrary point $P(x, y)$ we have

(a) *Length of tangent segment:* $TP = MP \operatorname{cosec}\psi = |y|\sqrt{1 + (dx/dy)^2}$ ...(4.10)

(b) *Length of normal segment:* $NP = MP \sec\psi = |y|\sqrt{1 + (dy/dx)^2}$ ...(4.11)

(c) *Length of subtangent:* $TM = MP \cot\psi = \left|y\frac{dx}{dy}\right|$ ...(4.12)

(d) *Length of subnormal:* $MN = MP \tan\psi = \left|y\frac{dy}{dx}\right|$. ...(4.13)

Example 4.8: Find the equations of the tangent and normal at the point 'θ' to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. Also find the length of tangent segment, normal segment, and that of the subtangent and the subnormal.

Solution: We have,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1 + \cos\theta)} = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2} = \tan\frac{\theta}{2}$$

Hence, the equations of the tangent and that of the normal at point θ respectively are:

$$y - a(1 - \cos\theta) = \tan\frac{\theta}{2}(x - a(\theta + \sin\theta))$$

and, $$y - a(1 - \cos\theta) = -\cot\frac{\theta}{2}(x - a(\theta + \sin\theta))$$

(a) Length of tangent segment $= |y|\sqrt{1+\left(\frac{dx}{dy}\right)^2} = |a(1-\cos\theta)|\sqrt{1+\cot^2\frac{\theta}{2}}$

$$= \left|2a\sin^2\frac{\theta}{2}\operatorname{cosec}\frac{\theta}{2}\right| = 2\left|a\sin\frac{\theta}{2}\right|.$$

(b) Length of normal segment $= |y|\sqrt{1+\left(\frac{dy}{dx}\right)^2} = |a(1-\cos\theta)\sec\theta/2|$

$$= 2|a\sin\theta/2\tan\theta/2|.$$

(c) Subtangent $= \left|y\frac{dx}{dy}\right| = \left|a(1-\cos\theta)\cot\frac{\theta}{2}\right| = a|\sin\theta|.$

(d) Subnormal $= \left|y\frac{dy}{dx}\right| = \left|a(1-\cos\theta)\tan\frac{\theta}{2}\right| = \left|2a\sin^2\frac{\theta}{2}\tan\frac{\theta}{2}\right|.$

Example 4.9: Show that the condition for the line $x\cos\alpha + y\sin\alpha = p$ to touch the curve $(x/a)^m + (y/b)^m = 1$ is $(a\cos\alpha)^{m/(m-1)} + (b\sin\alpha)^{m/(m-1)} = p^{m/m-1}$

Solution: Equation of the curve is

$$(x/a)^m + (y/b)^m = 1. \qquad \text{...(4.14)}$$

Differentiating (4.14) w.r.t. x, we have

$$\frac{mx^{m-1}}{a^m} + \frac{my^{m-1}}{b^m}\frac{dy}{dx} = 0, \text{ which gives, } \frac{dy}{dx} = -\left(\frac{b}{a}\right)^m\left(\frac{x}{y}\right)^{m-1}.$$

Therefore, the equation of the tangent at point $P(x_0, y_0)$ to the curve (4.14) is

$$y - y_0 = -\left(\frac{b}{a}\right)^m\left(\frac{x_0}{y_0}\right)^{m-1}(x - x_0)$$

or,
$$\frac{x_0^{m-1}}{a^m}x + \frac{y_0^{m-1}}{b^m}y = \frac{x_0^m}{a^m} + \frac{y_0^m}{b^m} = 1. \qquad \text{...(4.15)}$$

If the given line touches the curve (4.14) at (x_0, y_0) then (4.15) must be of the form

$$x\cos\alpha + y\sin\alpha = p \qquad \text{...(4.16)}$$

Comparing the corresponding coefficients of (4.15) and (4.16), we get

$$\frac{x_0^{m-1}}{a^m\cos\alpha} = \frac{y_0^{m-1}}{b^m\sin\alpha} = \frac{1}{p}$$

or, $$\left(\frac{x_0}{a}\right)^{m-1} = \frac{a\cos\alpha}{p}, \quad \left(\frac{y_0}{b}\right)^{m-1} = \frac{b\sin\alpha}{p}$$

or, $$\left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}} + \left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}} = \left(\frac{x_0}{a}\right)^m + \left(\frac{y_0}{b}\right)^m = 1.$$

Hence, $(a\cos\alpha)^{m/(m-1)} + (b\sin\alpha)^{m/(m-1)} = p^{m/(m-1)}$, the desired condition.

Example 4.10: Find the condition that the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ cut orthogonally.

Solution: Given curves are

$$ax^2 + by^2 = 1, \qquad ...(4.17)$$

and $$a'x^2 + b'y^2 = 1 \qquad ...(4.18)$$

Let $P(h, k)$ be a point of intersection of these curves, thus

$$ah^2 + bk^2 = 1 \quad \text{and} \quad a'h^2 + b'k^2 = 1,$$

which give $$\frac{h^2}{-b+b'} = \frac{k^2}{-a'+a} = \frac{1}{ab'-a'b}$$

or, $$h^2 = (b'-b)/(ab'-a'b), \quad k^2 = (a-a')/(ab'-a'b) \qquad ...(4.19)$$

From (4.17), we have

$$2axdx + 2by\frac{dy}{dx} = 0, \text{ or } \quad \frac{dy}{dx} = -ax/by.$$

Similarly from (4.18), we have $\dfrac{dy}{dx} = -a'x/b'y$.

Thus, m = slope of the tangent to (4.17) at $P(h,k) = -\dfrac{ah}{bk}$,

m' = slope of the tangent to (4.18) at $P(h,k) = -\dfrac{a'h}{b'k}$.

The curves (4.17) and (4.18) cut orthogonally, if $mm' = -1$,

or if, $$\left(-\frac{ah}{bk}\right)\left(-\frac{a'h}{b'k}\right) = -1, \text{ or if, } \quad aa'h^2 + bb'k^2 = 0$$

or if, $$\frac{aa'(b'-b)}{(ab'-a'b)} + \frac{bb'(a-a')}{(ab'-a'b)} = 0 \qquad \text{(using 4.19)}$$

or if, $\dfrac{a-a'}{aa'} = \dfrac{b-b'}{bb'}$, or if, $\dfrac{1}{a} - \dfrac{1}{a'} = \dfrac{1}{b} - \dfrac{1}{b'}$, the required condition.

Example 4.11: In the catenary $y = \cosh(x/c)$, prove that the length of the portion of the normal intercepted between the curve and the axis of x is y^2/c.

Solution: The equation of the curve is $y = c\cosh x/c$. Differentiating with respect to x, we obtain

$$\frac{dy}{dx} = \sinh\frac{x}{c}.$$

The length of the normal intercepted between the curve and the axis of x, refer to Eq. (4.11), is

$$|y|\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \left|c\cosh\frac{x}{c}\right|\sqrt{1+\sinh^2 x/c} = c\left|\cosh\frac{x}{c}\right|^2 = y^2/c.$$

4.4 DERIVATIVE OF ARC LENGTH

Before finding the derivative of arc length, we derive an important result concerning the *limit of the ratio of the arc to the chord for any two points P and Q on a curve* to be used to find the derivative. *We prove that,*

$$\lim_{Q \to P} \frac{\text{arc } PQ}{\text{chord } PQ} = 1. \qquad \text{...(4.20)}$$

Let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be any two points on a curve $y = f(x)$ such that the arc PQ, throughout its length, is concave to the chord PQ as shown in Fig. 4.4. Let QS be the perpendicular from Q to the tangent to the curve at P. We have

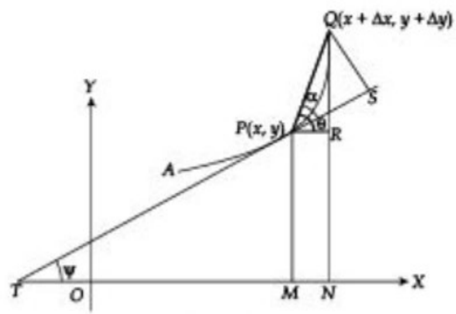


Fig. 4.4

$$\text{Chord } PQ < \text{arc } PQ < PS + SQ.$$

Dividing throughout by PQ, we obtain

$$1 < \frac{\text{arc } PQ}{PQ} < \frac{PS + SQ}{PQ}$$

or,
$$1 < \frac{\text{arc } PQ}{PQ} < \cos\alpha + \sin\alpha, \quad \alpha = \underline{|QPS} \qquad \text{...(4.21)}$$

Let $Q \to P$ so that the chord PQ tends to the tangent PS in its limiting position and $\alpha \to 0$. Thus, from (4.21), we obtain (4.20) since $\cos\alpha + \sin\alpha \to 1$ as $\alpha \to 0$.

4.4.1 Length of Arc as a Function

With reference to a fixed point A, as shown in Fig. 4.4, let $\widehat{AP} = s$, and $\widehat{AQ} = s + \Delta s$.

We note that s is a function of x for the curve $y = f(x)$.

For the curve in parametric form: $x = f(t)$, $y = g(t)$, s is a function of the parameter t; and, s is a function of θ when the curve is in polar form $r = f(\theta)$.

Next we derive the derivative of s in all the three cases.

(A) Cartesian form: For the curve $y = f(x)$, from the right angled ΔPQR, refer to Fig. 4.4, we have

$$(PQ)^2 = (PR)^2 + (RQ)^2 = (\Delta x)^2 + (\Delta y)^2.$$

In the limiting position arc PQ = chord PQ, the above equation may be written as

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

Dividing throughout by $(\Delta x)^2$ and considering the limit, $\Delta x \to 0$ (that is, $Q \to P$), we obtain

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

Thus, we have derivative of arc in cartesian form as

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad ...(4.22)$$

taking the positive sign as per the convention that for the curve $y = f(x)$, s is measured positively in the direction of increasing x.

If the equation of the curve is of the form $x = f(y)$, then s is considered as a function of y and it can be shown that

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}. \qquad ...(4.23)$$

(B) Parametric form: If the curve is of parametric form $x = f(t)$, $y = g(t)$, then the derivative of arc is

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \qquad ...(4.24)$$

where we measure s positively in the increasing direction of t.

The result can be derived from (4.22) or directly also.

Also in Fig. 4.4, assuming $\underline{|RPQ} = \theta$, it is observed that as $Q \to P$, the angle θ tends to coincide with ψ. The angle made by the tangent to the curve at P with the positive direction of the x-axis.

From $\quad \Delta PQR$, $\sin\theta = \dfrac{RQ}{PQ} = \dfrac{\Delta y}{\Delta s}$, $\quad \cos\theta = \dfrac{PR}{PQ} = \dfrac{\Delta x}{\Delta s}$, $\tan\theta = \dfrac{RQ}{PR} = \dfrac{\Delta y}{\Delta x}$.

Hence, as $Q \to P$, we obtain

$$\sin\psi = \frac{dy}{ds}, \quad \cos\psi = \frac{dx}{ds}, \quad \tan\psi = \frac{dy}{dx}. \qquad ...(4.25)$$

We will be using these results later on.

(C) Polar form: Let s denote the arc length of any point $P(r, \theta)$ from some fixed point A on the curve $r = f(\theta)$ as shown in Fig. 4.5. Take a point $Q(r + \Delta r, \theta + \Delta\theta)$ near to P on the curve and let arc $AQ = s + \Delta s$, so that arc $PQ = \Delta s$.

If PR is perpendicular to OQ, then

$$PR = r \sin \Delta\theta. \qquad ...(4.26)$$

Also,

$$RQ = OQ - OR = r + \Delta r - r\cos \Delta\theta$$
$$= r(1 - \cos \Delta\theta) + \Delta r. \qquad ...(4.27)$$

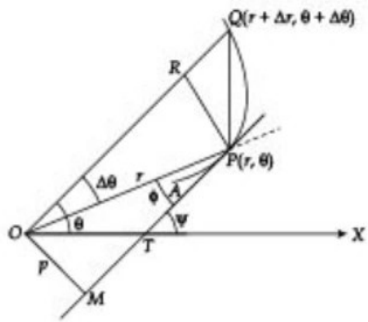


Fig. 4.5

In the limiting position as $Q \to P$, $\Delta\theta \to 0$, so we can take, $\sin \Delta\theta \approx \Delta\theta$ and $\cos \Delta\theta \approx 1$, and hence from (4.26) and (4.27), we have respectively

$$PR \approx r\Delta\theta \text{ and } RQ \approx \Delta r. \qquad ...(4.28)$$

Next, from the rt. angled ΔPQR, refer Fig. 4.5, we have

$$(PQ)^2 = (PR)^2 + (RQ)^2.$$

In the limiting position, arc PQ = chord PQ, the above equation using (4.28) may be written as

$$(\Delta s)^2 = (\Delta r)^2 + (r\Delta\theta)^2,$$

which, as $Q \to P$, gives the *derivative of arc length in polar form* as

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \qquad ...(4.29)$$

and,

$$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \qquad ...(4.30)$$

here s is to be measured in the positive direction of θ, or r, as the case may be.

The formula (4.29) is used when the equation is of the form $r = f(\theta)$, while (4.30) is used when equation is of the form $\theta = f(r)$.

4.4.2 Angle Between Tangent and Radius Vector

From the right angled triangle PQR, refer to Fig. 4.5, we have

$$\tan \angle RQP = \frac{RP}{QR} = \frac{r\sin\Delta\theta}{r(1 - \cos\Delta\theta) + \Delta r}, \qquad \text{(from (4.26) and (4.27))}$$

$$= \frac{r\sin\Delta\theta}{2r\sin^2\dfrac{\Delta\theta}{2} + \Delta r}. \qquad ...(4.31)$$

When $Q \to P$ the angle $\angle OQP = (\angle RQP)$ tends to ϕ, the angle between the positive direction of the tangent and the radius vector at $P(r, \theta)$, and also $\Delta\theta \to 0$, thus (4.31) gives

$$\tan\phi = \lim_{\Delta\theta\to 0}\frac{r\sin\Delta\theta}{2r\sin^2\frac{\Delta\theta}{2}+\Delta r} = \lim_{\Delta\theta\to 0}\frac{r(\sin\Delta\theta/\Delta\theta)}{r\left(\sin\frac{\Delta\theta}{2}\right)\left(\sin\frac{\Delta\theta}{2}\Big/\frac{\Delta\theta}{2}\right)+\frac{\Delta r}{\Delta\theta}}$$

$$= r\frac{d\theta}{dr}, \text{ using } \lim_{\theta\to 0} = \frac{\sin\theta}{\theta} = 1. \quad ...(4.32a)$$

Similarly, we can show that

$$\sin\phi = r\frac{d\theta}{ds} \text{ and } \cos\phi = \frac{dr}{ds} \quad ...(4.32b)$$

Remark: The result (4.32a) is useful in finding the angle of intersection of two curves when the curves are in polar form, since, *if ϕ_1 and ϕ_2 are the angles between the common radius vector and the tangents to the two curves at their point of intersection, then the angle of intersection of these curves is $\phi_1 - \phi_2$.*

4.4.3 Pedal Equation of a Curve

If $OM = p$ is the length of the perpendicular from the pole O to the tangent PT, as shown in Fig. 4.5 then (p, r) are called the *pedal co-ordinates of the point P*.

In right angled triangle MPO, we have

$$\sin\phi = \frac{p}{r}, \text{ or } \quad p = r\sin\phi \quad ...(4.33)$$

From (4.33), $\dfrac{1}{p^2} = \dfrac{1}{r^2\sin^2\phi} = \dfrac{1}{r^2}[1+\cot^2\phi] = \dfrac{1}{r^2}\left[1+\dfrac{1}{r^2}\left(\dfrac{dr}{d\theta}\right)^2\right]$ (using (4.32a))

or,

$$\frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2, \quad ...(4.34)$$

an expression for the *length of the perpendicular from pole on the tangent*, to the curve $r = f(\theta)$ at the point $P(r, \theta)$.

The equation obtained by eliminating θ between (4.34) and the curve $r = f(\theta)$ is called the *pedal equation of the curve.*

4.4.4 Polar subtangent and Polar subnormal. Polar tangent and Polar normal

With reference to Fig. 4.6, let NOT be a straight line through the pole O and perpendicular to the radius vector OP. Let the tangent at P meets this line at T and the normal at P meets the line at N. Then OT and ON at called the *polar subtangent* and *polar subnormal* respectively.

Let $OM \perp PT$ it is obvious that $\underline{|PNO} = \underline{|MOT} = \phi$, the angle between the radius vector and the tangent at $P(r, \theta)$. Thus,

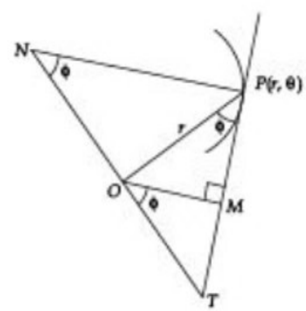


Fig. 4.6

Polar subtangent; $OT = r\tan\phi = r.r\dfrac{d\theta}{dr} = r^2\dfrac{d\theta}{dr};$ ...(4.35)

Polar subnormal; $ON = r\cot\phi = r.\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{dr}{d\theta};$ ...(4.36)

Polar tangent; $PT = r\sqrt{1+\tan^2\phi} = r\sqrt{1+r^2\left(\dfrac{d\theta}{dr}\right)^2}$ ...(4.37)

Polar normal; $PN = r\sqrt{1+\cot^2\phi} = \sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}$ ...(4.38)

Example 4.12: Find $\dfrac{ds}{dx}, \dfrac{ds}{dy}$ for the curve $y = a\ln\sec\dfrac{x}{a}$ and also prove that $x = \psi$, where ψ is the angle which the tangent at an arbitrary point $P(x, y)$ makes with the positive direction of the x-axis.

Solution: Equation of the curve is, $y = a\ln\sec\dfrac{x}{a}$.

This gives, $\dfrac{dy}{dx} = \tan\dfrac{x}{a}$ ...(4.39)

Thus, $\dfrac{ds}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{1+\tan^2\dfrac{x}{a}} = \sec\dfrac{x}{a}$. Similarly, $\dfrac{ds}{dy} = \operatorname{cosec}\dfrac{x}{a}$.

Also, $\dfrac{dy}{dx} = \tan\psi$ ...(4.40)

From (4.39) and (4.40), we obtain, $\dfrac{x}{a} = \psi$, or $x = a\psi$.

Example 4.13: For the curve $r^2 = a^2\cos 2\theta$, find $\dfrac{ds}{d\theta}$ and $\dfrac{ds}{dr}$.

Solution: We have $r^2 = a^2\cos 2\theta$, which gives $2r\dfrac{dr}{d\theta} = -2a^2\sin 2\theta$, or $\dfrac{dr}{d\theta} = -\dfrac{a^2\sin 2\theta}{r}$.

Therefore, $\dfrac{ds}{d\theta} = \sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2} = \sqrt{r^2+\dfrac{a^4\sin^2 2\theta}{r^2}} = \sqrt{a^2\cos 2\theta+\dfrac{a^4\sin^2 2\theta}{a^2\cos 2\theta}}$

$$= a\sqrt{\frac{\cos^2 2\theta+\sin^2 2\theta}{\cos 2\theta}} = \frac{a}{\sqrt{\cos 2\theta}} = \frac{a^2}{r}.$$

Also
$$\frac{ds}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2} = \sqrt{1+r^2\frac{r^2}{a^4\sin^2 2\theta}}$$
$$= \sqrt{1+\frac{a^4\cos^2 2\theta}{a^4\sin^2 2\theta}} = \operatorname{cosec} 2\theta = \frac{a^2}{\sqrt{a^2-r^2}}.$$

Example 4.14: Find the pedal equation of the parabola $y^2 = 4a(x+a)$.

Solution: Differentiating the given equation $y^2 = 4a(x+a)$ with respect to x, we have
$$2y\frac{dy}{dx} = 4a, \text{ or } \frac{dy}{dx} = \frac{2a}{y}.$$

Thus, the equation of the tangent to the given curve at the point (x, y) is
$$Y - y = \frac{2a}{y}(X-x), \text{ or } \frac{2a}{y}X - Y - \left(\frac{2ax}{y} - y\right) = 0$$

If p be the length of the perpendicular from the origin to this tangent, then
$$p = \frac{\left|\frac{2ax}{y} - y\right|}{\sqrt{\left(\frac{2a}{y}\right)^2+1}} = \frac{|2ax - y^2|}{\sqrt{y^2+4a^2}} = \frac{|2ax - 4a(x+a)|}{\sqrt{4a(x+a)+4a^2}} \text{ or, } p = \sqrt{a(x+2a)} \quad ...(4.41)$$

Also, $r^2 = x^2 + y^2 = x^2 + 4a(x+a) = (x+2a)^2$ or, $r = |x+2a|$ ...(4.42)

Eliminating x between (4.41) and (4.42), we obtain $p^2 = ar$ as the required pedal equation.

Example 4.15: Find the polar subtangent and polar subnormal to the curve $r = a(1-\cos\theta)$. Also find the pedal equation of this curve.

Solution: The equation of the given curve is $r = a(1-\cos\theta)$, which gives $\frac{dr}{d\theta} = a\sin\theta$.

Hence, polar subtangent $= r^2\frac{d\theta}{dr} = \frac{a^2(1-\cos\theta)^2}{a\sin\theta} = a\frac{4.\sin^4\theta/2}{2\sin\theta/2\cos\theta/2} = 2a\sin^3\theta/2\sec\theta/2.$

Also, polar subnormal $= \frac{dr}{d\theta} = a\sin\theta.$

Next, if ϕ is the angle between the radius vector and the tangent at a point $P(r,\theta)$ to the given curve $r = a(1-\cos\theta)$, then

$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\sin\theta} = \tan\theta/2.$ Thus, $\phi = \theta/2$, and hence $p = r\sin\phi = r\sin\theta/2.$

Also, $r = a(1-\cos\theta) = 2a\sin^2\theta/2$. Eliminating θ, between these two, we obtain $r = 2a\dfrac{p^2}{r^2}$, or $r^3 = 2ap^2$, as the required pedal equation of the curve.

Example 4.16: Find the angle of intersection of the curves $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$.

Solution: Let $P(r,\theta)$ be a point of intersections of the two curves. Consider the first curve,

$$r = \sin\theta + \cos\theta \qquad \text{...(4.43)}$$

Taking logarithm this gives, $\ln r = \ln(\sin\theta + \cos\theta)$, and hence

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} = \frac{1-\tan\theta}{1+\tan\theta}.$$

Thus, $\tan\phi_1 = r\dfrac{d\theta}{dr} = \dfrac{1+\tan\theta}{1-\tan\theta} = \tan\left(\dfrac{\pi}{4}+\theta\right)$, which implies $\phi_1 = \dfrac{\pi}{4}+\theta$, where ϕ_1 is the angle between the tangent and the radius vector to the curve (4.43) at $P(r,\theta)$.

Similarly, for the second curve

$$r = 2\sin\theta \qquad \text{...(4.44)}$$

we have, $\ln r = \ln(2\sin\theta) = \ln 2 + \ln\sin\theta$ and hence we obtain

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{\sin\theta}. \text{ Thus, } \tan\phi_2 = r\frac{d\theta}{dr} = \tan\theta,$$

which implies $\phi_2 = \theta$ where ϕ_2 is the angle between the tangent and the radius vector at the same point $P(r,\theta)$ on the curve (4.44). Therefore, the angle of intersection between the two curves is

$$\phi_1 - \phi_2 = \left(\frac{\pi}{4}+\theta\right) - \theta = \frac{\pi}{4}$$

EXERCISE 4.3

1. If the tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at any point on it meets the x-axis at P and y-axis at Q, then show that $OP + OQ = a$, O being the origin.
2. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the x-axis, then show that its equation is $y\cos\phi - x\sin\phi = a\cos 2\phi$.
3. Show that the parabolas $y^2 = 4ax$ and $2x^2 = ay$ intersect at an angle $\tan^{-1}(3/5)$.
4. Show that the curves $\dfrac{x^2}{a^2+\alpha} + \dfrac{y^2}{b^2+\alpha} = 1$ and $\dfrac{x^2}{a^2+\beta} + \dfrac{y^2}{b^2+\beta} = 1$ intersect orthogonally irrespective of the values of α and β.
5. Show that in the exponential curve $y = be^{x/a}$, the subtangent is the constant length and that the subnormal varies as the square of the ordinate.

6. For the curve $x = a(\cos t + \ln \tan t/2), y = a \sin t$, prove that the portion of the tangent between the curve and x-axis is constant. Also find its subtangent.
7. Prove that for the curve $ky^2 = (x + a)^3$, the square of the subtangent is proportional to the subnormal.
8. Show that the curves given by $r = \dfrac{a}{1+\cos\theta}$ and $r = \dfrac{b}{1-\cos\theta}$ intersect orthogonally.
9. Find the angle of intersection of the curves given by $r = a$, and $r = 2a \cos \alpha$.
10. Prove that the pedal equation of the asteroid $x = a\cos^3 t, y = a\sin^3 t$ is $r^2 = a^2 - 3p^2$.
11. Find the pedal equation of the following curves:
    (a) $r^2 = a^2 \sin^2\theta$ (b) $r = a\sec t, \quad \theta = \tan t - t$
12. Find $\dfrac{ds}{dx}$ for the curve $3ay^2 = x(x-a)^2$.
13. For the ellipse $x = a \cos t, y = b \sin t$, prove that $\dfrac{ds}{dt} = a(1 - e^2 \cos^2 t)^{1/2}$.
14. Show that for the hyperbolical spiral $r\theta = a$, $\dfrac{ds}{dr} = \dfrac{\sqrt{r^2 + a^2}}{r}$.
15. Show that for any pedal curve $p = f(r)$, $\dfrac{ds}{dr} = \dfrac{r}{\sqrt{r^2 - p^2}}$.
16. Show that the pedal equation of the curve $a^2(x^2 + y^2) = x^2y^2$ is $\dfrac{1}{p^2} + \dfrac{3}{r^3} = \dfrac{1}{a^2}$.

4.5 MEAN VALUE THEOREMS

Mean value theorems play a very important role in the study of differential calculus. Of these, Rolle's theorem is the most fundamental one.

Theorem 4.1: **(Rolle's Theorem)** *If a function $f(x)$ is such that (i) it is continuous in the closed interval $[a, b]$, (ii) derivable in the open interval (a, b), and (iii) $f(a) = f(b)$, then there exists at least one value 'c' of x lying within (a, b) such that $f'(c) = 0$.*

Geometrically, *if the graph of a function is continuous curve from A to B, has a unique tangent at every point between A and B, and the ordinates of its extremities A, B are equal, then there exists at least one point P of the curve other than A and B, the tangent at which is parallel to x-axis*, a result quite evident from Figs. 4.7a, b, & c.

Proof. When $f(x)$ satisfies the above conditions, two cases arise. Either $f(x) = 0$ identically, or $f(x) \neq 0$ at least at some points in $[a, b]$.

If $f(x) = 0$ for all $x \in [a, b]$, then $f'(x) = 0$ at all x, and hence the theorem is proved trivially.

If $f(x) \neq 0$ at least at some points in $[a, b]$, then $f(x)$ being continuous in $[a, b]$, it must be bounded. Let

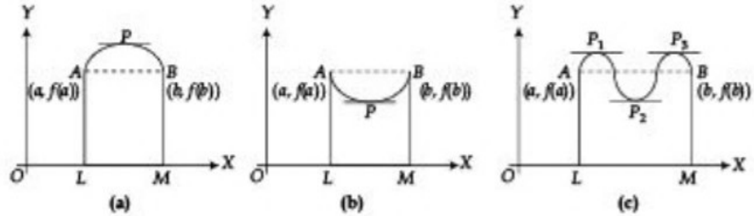


Fig. 4.7

M and m be the upper and lower bounds respectively. Thus, for some $x_1, x_2 \in [a, b]$, we have $f(x_1) = M$ and $f(x_2) = m$.

If $M = m$ then $f(x) =$ constant and hence $f'(x) = 0$; the theorem is proved.

However, if $M \neq m$, then at least one of M or m must be different from $f(a)$ and $f(b)$. Let for some c, $M = f(c) \neq f(a)$ or $f(b)$. Then $c \in (a, b)$ and we prove that $f'(c) = 0$.

Now $f(c) \geq f(c + h)$ for values of h both positive or negative. Then

$$\frac{f(c+h) - f(c)}{h} \leq 0, \text{ for } h > 0 \qquad ...(4.45)$$

and,
$$\frac{f(c+h) - f(c)}{h} \geq 0, \text{ for } h < 0 \qquad ...(4.46)$$

Since, f is differentiable in (a, b), from (4.45) and (4.46) as $h \to 0$, we have $f'(c) \leq 0$ and $f'(c) \geq 0$. Hence, $f'(c) = 0$ for some $c \in (a, b)$.

Similarly, we can prove in case $m = f(d)$ for some $d \in (a, b)$.

This completes the proof.

An other form of the Rolle's theorem is obtained by taking $b = a + h$, $h > 0$. In this case *if $f(x)$ is continuous in $[a, a + h]$, derivable in $(a, a + h)$ and $f(a) = f(a + h)$, then there exists at least one number $\theta \in (0, 1)$ such that $f'(a + \theta h) = 0$, $0 < \theta < 1$.*

The conclusion of Rolle's Theorem may not hold good for a function which does not satisfy any of these conditions.

For example, consider the function $f(x) = |x|$ in the interval $[-1, 1]$. Clearly $f(x)$ is continuous in the interval $[-1, 1]$, and $f(-1) = f(1)$. Its derivative $f'(x)$ is 1 for $0 < x \leq 1$ and is -1 for $-1 \leq x < 0$; and $f'(x)$ does not exist for $x = 0$. Thus, $f'(x)$ vanishes no where in the interval $(-1, 1)$, and hence the Rolle's theorem fails.

The failure is explained by the fact that $f(x)$ is not derivable in $(-1, 1)$. Geometrically, $y = |x|$ does not have a unique tangent at the origin as shown in Fig. 4.8.

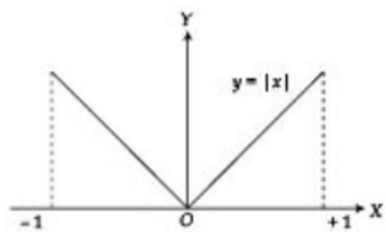


Fig. 4.8

Theorem 4.2: ***(Lagrange's Mean Value Theorem)*** *If a function $f(x)$ is such that (i) it is continuous in the closed interval $[a, b]$, (ii) derivable in the open interval (a, b), then there exists at least one value 'c' of x lying in (a, b) such that*

$$\frac{f(b)-f(a)}{b-a}=f'(c). \quad ...(4.47)$$

Geometrically, *if the graph of a function is a continuous curve from A to B and has a unique tangent at every point between A and B, then there exists at least one point P on the curve such that the tangent at P is parallel to the chord AB joining its extremities*, a result quite evident from Figs. 4.9a & b.

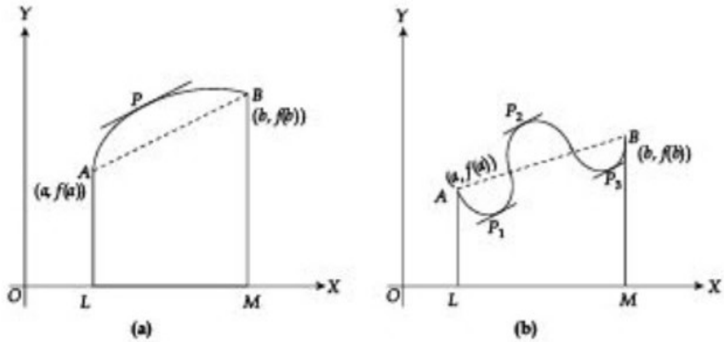


Fig. 4.9

Proof. Define a function

$$\phi(x)=f(x)+Ax \quad ...(4.48)$$

where A is a constant such that $\phi(a)=\phi(b)$. Using this gives

$$A=-\frac{f(b)-f(a)}{(b-a)} \quad ...(4.49)$$

From the form of $\phi(x)$, obviously it is continuous in $[a, b]$, differentiable in (a, b) and since $\phi(a)=\phi(b)$, thus $\phi(x)$ satisfies all the conditions of Rolle's Theorem and hence there exists at least one $c\in(a, b)$ such that $\phi'(c)=0$, which gives from (4.48)

$$f'(c)+A=0, \text{ or } f'(c)=-A, \text{ that is } f'(c)=\frac{f(b)-f(a)}{b-a} \quad \text{using (4.49)}$$

This completes the proof.

Another form of the Lagrange's mean value theorem is obtained by taking $b=a+h$, $h>0$. In this case, *if $f(x)$ is continuous in $[a, a+h]$ and derivable in $(a, a+h)$, then there exists at least one number θ*, $0<\theta<1$, such that

$$f(a+h)=f(a)+hf'(a+\theta h), \quad ...(4.50)$$

A few important results that follow obviously from the Lagrange's mean value theorem are:

1. *If the derivative of a function vanishes for all values of x in an interval, then the function must be a constant.*

2. *If two functions $f(x)$ and $g(x)$ have the same derivative for every value of x in an interval, then they differ only by a constant.*
3. *A function whose derivative is positive for every value of x in an interval is a monotonically increasing function of x in that interval.*
4. *A function whose derivative is negative for every value of x in an interval is a monotonically decreasing function of x in that interval.*

Theorem 4.3: **(Cauchy's Mean Value Theorem)** *If two functions $f(x)$ and $g(x)$ are (i) continuous in the closed interval (a, b), (ii) derivable in the open interval (a, b), and $g'(x) \neq 0$ for any value of x in (a, b), then there exists at least one c in the open interval (a, b) such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}. \qquad ...(4.51)$$

Here we note that $g(b) - g(a)$ can't be equal to zero.

Clearly for $g(x) = x$, Cauchy mean value theorem reduces to Lagrange's mean value theorem. Also the Cauchy's mean value theorem has the same geometrical interpretation as of Lagrange's mean value theorem.

The proof of this theorem follows immediately by defining a function

$$\phi(x) = \{f(b) - f(a)\}g(x) - \{g(b) - g(a)\}f(x)$$

The function $\phi(x)$ satisfies all the conditions of Rolle's theorem; the result (4.51) follows from $\phi'(c) = 0$, for some $c \in (a, b)$.

Example 4.17: Prove that the equation $3x^5 + 15x - 8 = 0$ has only one real root.

Solution: The existence of at least one real root follows from the fact that the polynomial $f(x) = 3x^5 + 15x - 8$ is of an odd power.

To prove its uniqueness let us suppose that there exists two roots a and b, say $a < b$. Then in the interval $[a, b]$, the function $f(x) = 3x^5 + 15x - 8$ satisfies all the conditions of Rolle's theorem, hence there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

But $f'(x) = 15(x^4 + 1) > 0$. This contradiction proves that $f(x) = 3x^5 + 15x - 8$ has only one real root.

Example 4.18: Show that $\frac{h}{1+h^2} < \tan^{-1} h < h$, when $h \neq 0$ and $h > 0$.

Solution: Take $f(x) = \tan^{-1}x$, $0 \leq x \leq h$. The function $f(x)$ satisfies the conditions of Lagrange's mean value theorem in $[0, h]$. Thus,

$$\frac{\tan^{-1} h - \tan^{-1} 0}{h-0} = \frac{1}{1+k^2}, \text{ for some } k \in (0, h), \text{ or } \tan^{-1} h = \frac{h}{1+k^2}.$$

Now $0 < k < h \Rightarrow 0 < k^2 < h^2$, which gives

$$1 < 1 + k^2 < 1 + h^2$$

$$\text{or, } h > \frac{h}{1+k^2} > \frac{h}{1+h^2}, \quad (h > 0), \quad \text{or} \quad h > \tan^{-1} h > \frac{h}{1+h^2}.$$

Example 4.19: Using the mean value theorem show that $3+\frac{1}{28}<\sqrt[3]{28}<3+\frac{1}{27}$.

Solution: If a function $f(x)$ satisfies all the conditions of Lagranges MVT in $[x, x+h]$, then

$$f(x+h) = f(x) + hf'(x+\theta h), \quad 0<\theta<1.$$

Take $f(x)=\sqrt[3]{x}=x^{1/3}$, so that $f'(x)=\frac{1}{3x^{2/3}}$. Set $x=27$, $h=1$, we get

$$f(x+h) = 3+\frac{1}{3(27+\theta)^{2/3}}, \quad 0<\theta<1 \qquad ...(4.52)$$

For $\theta=0$, the right side of (4.52), is

$$3+\frac{1}{3(27)^{2/3}}=3+\frac{1}{27} \qquad ...(4.53)$$

and, for $\theta=1$, it is

$$3+\frac{1}{3(28)^{2/3}}=3+\frac{1}{(27)^{1/3}(28)^{2/3}}>3+\frac{1}{28} \qquad ...(4.54)$$

Since $0<\theta<1$, from (4.52), (4.53) and (4.54), we get $3+\frac{1}{28}<\sqrt[3]{28}<3+\frac{1}{27}$.

Example 4.20: If $1<a<b$, then show that there exists a point c in (a, b) such that

$$\frac{\ln b-\ln a}{b-a}=\frac{b+a}{2c^2}.$$

Solution: Take $f(x)=\ln x$ and $g(x)=x^2$. Functions $f(x)$ and $g(x)$ satisfies the conditions of Cauchy's mean value theorem in the interval $[a, b]$. Hence, for some c in (a, b), we have

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}, \text{or} \quad \frac{\ln b-\ln a}{b^2-a^2}=\frac{1/c}{2c}=\frac{1}{2c^2}, \text{or} \quad \frac{\ln b-\ln a}{b-a}=\frac{b+a}{2c^2}.$$

Example 4.21: Let C be a curve defined parametrically as $x=a\cos^3\theta$, $y=a\sin^3\theta$, $0\le\theta\le\pi/2$. Determine a point P on C, where the tangent to C is parallel to the chord joining the point $(a, 0)$ and $(0, a)$.

Solution: Take $x=a\cos^3\theta=f(\theta)$ and $y=a\sin^3\theta=g(\theta)$. Clearly $f(\theta)$ and $g(\theta)$ satisfy the conditions of Cauchy's mean value theorem over the interval $[0, \pi/2]$.

Thus, using the Cauchy's mean value theorem, at some point θ, the slope of tangent is equal to the slope of the chord joining the points corresponding to $\theta=0$ and $\theta=\pi/2$, that is, the points $(a, 0)$ and $(0, a)$. Thus,

$$\frac{g'(0)}{f'(0)}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=\frac{a-0}{0-a}, \text{or} -\tan\theta=-1, \text{or} \quad \theta=\pi/4.$$

Therefore, the required point is $\left(a/2\sqrt{2},\ a/2\sqrt{2}\right)$.

Example 4.22: Show that $x/(1+x) < \ln(1+x) < x$, for $x > 0$.

Solution: Consider

$$f(x) = \ln(1+x) - \frac{x}{1+x}, \quad \text{...(4.55)}$$

It gives
$$f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2}.$$

Thus, $f'(x) > 0$, for $x > 0$ and $f'(0) = 0$. Hence, $f(x)$ is monotonically increasing in the interval $[0, \infty]$. Also $f(0) = 0$, therefore, $f(x) > f(0) = 0$ for $x > 0$. Thus, (4.55) gives

$$\ln(1+x) > x/(1+x), \text{ for } x > 0 \quad \text{...(4.56)}$$

Similarly by taking $g(x) = x - \ln(1+x)$ we can prove that $g(x)$ is monotonically increasing in the interval $[0, \infty]$ and also $g(0) = 0$. Therefore,

$$x > \ln(1+x), \text{ for } x > 0 \quad \text{...(4.57)}$$

From (4.56) and (4.57), we get the desired result.

EXERCISE 4.4

1. On the curve $y = x^3$ find the point at which the tangent line is parallel to the chord through the points $A(-1, -1)$ and $B(2, 8)$.
2. Without solving, prove that the equation $x^4 - 4x - 1 = 0$ has two different real roots.
3. Prove that all roots of the derivative of the given polynomial $f(x) = (x+1)(x-1)(x-2)(x-3)$ are real.
4. Calculate approximately $\sqrt[5]{245}$ by using Lagrange's mean value theorem.
5. Determine the root of the equation $x^3 + 5x - 10 = 0$ which lies in $(1, 2)$ correct to two decimal places using mean value theorem.
6. Show that for all x, $\sin x$ lies between $x - \frac{x^3}{6}$ and $x - \frac{x^3}{6} + \frac{x^5}{120}$.
7. Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$, if $0 < x < \frac{\pi}{2}$.
8. Determine the intervals in which the function $(x^4 + 6x^3 + 17x^2 + 32x + 32)e^{-x}$ is increasing or decreasing.
9. Show that for all $x > 0$, $1 - x < e^{-x} < 1 - x + \frac{x^2}{2}$.
10. A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $0 < c < b$. Prove that there is at least one value ξ, $a < \xi < b$ for which $f''(\xi) < 0$.

11. Prove the inequality $\tan^{-1}x_2 - \tan^{-1}x_1 < x_2 - x_1$
12. Using Rolle's Theorem prove that the derivative $f'(x)$ of the function

$$f(x) = \begin{cases} x\sin\dfrac{\pi}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$$

vanishes on an infinite set of points of the interval (0, 1).

4.6 TAYLOR'S AND MACLAURIN'S THEOREMS AND SERIES

A useful technique in the analysis of real valued functions is the approximation of continuous functions by polynomial. Taylor's and Maclaurin's theorems are important tools which provide such an approximation of the real valued functions. These theorems are regarded as '*generalized mean value theorems*' in the sense that mean value theorems relate the value of the function and its first order derivative, whereas, Taylor's and Maclaurin's theorems generalize this relation to higher order derivatives.

4.6.1 Taylor's Theorem with Lagrange's form of Remainder

Theorem 4.4: **(*Taylor's Theorem*)** *If a function $f(x)$ is such that*

(a) *$f(x), f'(x), f''(x), \ldots, f^{(n-1)}(x)$ are continuous in the closed interval $[a, a+h]$*

(b) *$f^{(n)}(x)$ exists in the open interval $(a, a+h)$, then there exists at least one number θ, between 0 and 1, such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h),\ 0 < \theta < 1 \quad \ldots(4.58)$$

Proof. We define a function $\phi(x)$ involving $f(x)$ and its derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$, designed so as to satisfy the conditions for Rolle's theorem. Let

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{(a+h-x)^n}{n!}A, \quad \ldots(4.59)$$

where A is a constant to be determined such that

$$\phi(a) = \phi(a+h). \quad \ldots(4.60)$$

From (4.59) and (4.60), we obtain

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}A = f(a+h) \quad \ldots(4.61)$$

The functions $f(x), f'(x), f''(x), \ldots, f^{(n-1)}(x)$ are continuous in the closed interval $[a, a+h]$ and are derivable in the open interval $(a, a+h)$; also $(a+h-x)$, $\dfrac{(a+h-x)^2}{2!}, \ldots, \dfrac{(a+h-x)^n}{n!}$ and A are continuous in the closed interval $[a, a+h]$ and are derivable in the open interval $(a, a+h)$, therefore, $\phi(x)$ as defined in (4.59) is continuous in $[a, a+h]$, derivable in $(a, a+h)$ and also $\phi(a+h) = \phi(a)$.

Thus, $\phi(x)$ satisfies all the conditions for Rolle's theorem. There exists, therefore, at least one number θ between 0 and 1 such that $\phi'(a+\theta h) = 0$. From (4.59),

$$\phi'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - (a+h-x) f''(x)$$

$$+ \frac{(a+h-x)^2}{2!} f'''(x) - \frac{(a+h-x)^2}{2!} f'''(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - \frac{(a+h-x)^{n-1}}{(n-1)!} A$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} [f^{(n)}(x) - A], \text{ other terms cancelling in pairs.}$$

Therefore $\phi'(a+\theta h) = 0$ gives, $0 = \phi'(a+\theta h) = \dfrac{(h-\theta h)^{n-1}}{(n-1)!} [f^{(n)}(a+\theta h) - A]$ which gives

$$A = f^{(n)}(a+\theta h), \text{ since } \quad (1-\theta) \neq 0, \text{ for } \quad 0 < \theta < 1.$$

Substituting this value of A in (4.61), we get (4.58).

The $(n+1)$th term $\dfrac{h^n}{n!} f^{(n)}(a+\theta h)$ in (4.58) is called the Lagrange's form of remainder after n terms in the Taylors expansion of $f(a+h)$ and is denoted by $R_n(x)$.

Taking $n = 1$ in (4.58), we obtain

$$f(a+h) = f(a) + hf'(a+\theta h), \quad 0 < \theta < 1,$$

which is Lagrange's mean value theorem, refer to (4.50)

Thus LMVT is only a paticular case of Taylor's theorem with Lagrange's form of the remainder.

Remark: Another form of Taylor's expansion is with *Cauchy's form of remainder*. In this case the remainder after n terms is

$$R_n = \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(a+\theta h), \quad 0 < \theta < 1. \qquad \ldots(4.62)$$

4.6.2 Maclaurin's Theorem with Lagrange's Form of Remainder

Theorem 4.5: ***(Maclaurin's Theorem)*** *If a function $f(x)$ is such that*

(a) $f(x), f'(x), f''(x), \ldots, f^{(n-1)}(x)$ are continuous in the closed interval $[0, x]$.

(b) $f^{(n)}(x)$ exists in the open interval $(0, x)$, then there exists at least one number θ between 0 and 1 such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\theta x). \qquad \ldots(4.63)$$

This is obtained from Taylor's theorem by considering the interval $[0, x]$ instead of $[a, a+h]$ and changing a to 0 and h to x in (4.58).

4.6.3 Taylor's Infinite Series

Rewrite (4.58) as $f(a+h) = S_n + R_n$, where

$$S_n = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a),$$

and,
$$R_n = \frac{h^n}{n!}f^{(n)}(a+\theta h), \quad 0<\theta<1.$$

Suppose that $R_n \to 0$, as $n \to \infty$, then $\lim\limits_{n\to\infty} S_n = f(a+h)$, so that the series

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \ldots$$

converges and its sum is equal to $f(a+h)$. Thus, we obtain that

(i) *if $f(x)$ possesses derivatives of every order in the interval $[a, a+h]$, and*

(ii) *the remainder $\frac{h^n}{n!}f^{(n)}(a+\theta h)$ tends to zero as n tends to infinity, then*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^{(n)}(a) + \ldots \qquad \ldots(4.64)$$

This series is known as *Taylor's infinite series.*

4.6.4 Maclaurin's Infinite Series

Set 0 for a and x for h in the Taylor's infinite series (4.64), we obtain that

(i) *if $f(x)$ possesses derivatives of every order in the interval $[0, x]$, and*

(ii) *the remainder $\frac{x^n}{n!}f^n(\theta x)$ tends to zero as n tends to infinity, then*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots \qquad \ldots(4.65)$$

This series is known as *Maclaurin's series* for the expansion of function $f(x)$ in powers of x.

Remark: In order to find out if any given function can be expanded as an infinite Taylor series (or, Maclaurin series) or not, it is necessary to examine the behaviour of R_n as n tends to infinity, and for this we need to obtain the general expression for the nth derivative of the function. However, a formal expansion of a function as a power series is obtained under the assumption that as $n \to \infty$, $R_n \to 0$.

Example 4.23: Obtain the fourth degree polynomial approximation to $f(x)=e^{2x}$ about $x=0$. Find the maximum error when $0\le x\le 0.5$.

Solution: The Maclaurin's theorem with Lagrange's form of remainder after the fourth degree term is

$$f(x)=f(0)+xf^{(1)}(0)+\frac{x^2}{2!}f^{(2)}(0)+\frac{x^3}{3!}f^{(3)}(0)+\frac{x^4}{4!}f^{(4)}(0)+\frac{x^5}{5!}f^{(5)}(\theta x),\ 0<\theta<1.$$

For $f(x)=e^{2x}$, we have $f^{(r)}(x)=2^re^{2x}$, and therefore, $f^{(r)}(0)=2^r$.

Also, $$f^{(5)}(\theta x)=2^5e^{2\theta x}=32e^{2\theta x}$$

Therefore, $$f(x)=e^{2x}=1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\frac{16x^4}{4!}+\ldots$$

$$\approx 1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4.$$

The error term is given by $R_5(x)=\dfrac{x^5}{5!}f^{(5)}(\theta x)=\dfrac{32}{5!}x^5e^{2\theta x},\quad 0<\theta<1$

$$=\frac{32}{5!}x^5e^{2c},\quad 0<c<x,\ \text{taking}\quad \theta x=c.$$

Thus, $$|R_5(x)|\le\frac{32}{120}\left[\max_{0\le x\le 0.5}.\,x^5\right]\left[\max_{0<c<0.5}e^{2c}\right]\le\frac{32}{120}(0.5)^5(e^{2(0.5)})=\frac{e}{120}.$$

Thus maximum error is $e/120$, for $0<x<0.5$.

Example 4.24: Prove by Taylor's series that

$$\tan^{-1}(x+h)=\tan^{-1}x+h\sin\alpha\frac{\sin\alpha}{1}-(h\sin\alpha)^2\frac{\sin 2\alpha}{2}+(h\sin\alpha)^3\frac{\sin 3\alpha}{3}-\ldots,$$

where $x=\cot\alpha$.

Solution: Here $x=\cot\alpha$, which gives $\alpha=\cot^{-1}x$. Thus,

$$\frac{d\alpha}{dx}=\frac{-1}{1+x^2}=\frac{-1}{1+\cot^2\alpha}=-\sin^2\alpha.$$

Consider $f(x)=\tan^{-1}x$, therefore,

$$f'(x)=\frac{1}{1+x^2}=\frac{1}{1+\cot^2\alpha}=\frac{1}{\operatorname{cosec}^2\alpha}=\sin^2\alpha$$

$$f''(x)=2\sin\alpha\cos\alpha\frac{d\alpha}{dx}=\sin 2\alpha\,(-\sin^2\alpha)=-\sin^2\alpha\sin 2\alpha$$

$$f'''(x) = -[2\sin\alpha\cos\alpha.\sin 2\alpha + 2\cos 2\alpha\sin^2\alpha]\frac{d\alpha}{dx}$$
$$= -2\sin\alpha[\cos\alpha\sin 2\alpha + \cos 2\alpha\sin\alpha](-\sin^2\alpha)$$
$$= 2\sin^3\alpha\sin 3\alpha.$$

By Taylor's series, $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + ...$ Substituting for $f(x), f'(x), f''(x), ...,$ we obtain

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin^2\alpha + \frac{h^2}{2!}(-\sin^2\alpha\sin 2\alpha) + \frac{h^3}{3!}(2\sin^3\alpha\sin 3\alpha) - ...$$

or, $$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\alpha\frac{\sin\alpha}{1} - (h\sin\alpha)^2\frac{\sin 2\alpha}{2} + (h\sin\alpha)^3\frac{\sin 3\alpha}{3} - ...$$

the required expansion.

Example 4.25: Expand $\ln\sin x$ in powers of $(x-3)$.

Solution: Let $f(x) = \ln\sin x$.

Therefore, $f'(x) = \frac{\cos x}{\sin x} = \cot x,\quad f'(3) = \cot 3$

$$f''(x) = -\text{cosec}^2 x,\quad f''(3) = -\text{cosec}^2 3$$
$$f'''(x) = 2\,\text{cosec}^2 x\cot x,\quad f'''(3) = 2\,\text{cosec}^2 3\cot 3 \text{ etc.}$$

write $f(x) = f(3 + \overline{x-3}) = f(3+h)$, where $h = x-3$.

By Taylor's series $f(x) = f(3+h) = f(3) + hf'(3) + \frac{h^2}{2!}f''(3) + ...$

Substituting for $f(x), f(3), f'(3), f''(3)$ etc. we obtain

$$\ln\sin x = \ln\sin 3 + (x-3)\cot 3 - \frac{(x-3)^2}{2}\text{cosec}^2 3 + \frac{(x-3)^3}{3}\text{cosec}^2 3\cot 3 + ...$$

Example 4.26: Apply Taylor's series to calculate the value of $f(11/10)$, where

$$f(x) = x^3 + 3x^2 + 15x - 10.$$

Solution: By Taylor's series, $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + ...$

Put $x = 1$, $h = 1/10$, this gives

$$f\left(\frac{11}{10}\right) = f(1) + \frac{1}{10}f'(1) + \left(\frac{1}{10}\right)^2\frac{f''(1)}{2!} + \left(\frac{1}{10}\right)^3\frac{f'''(1)}{3!} + ... \quad ...(4.66)$$

Here, $f(x) = x^3 + 3x^2 + 15x - 10,\quad f(1) = 9$

$f'(x) = 3x^2 + 6x + 15,\quad f'(1) = 24$

$f''(x) = 6x + 6,\quad f''(1) = 12$

$f'''(x) = 6.\quad f'''(1) = 6$ etc.

Substituting for $f(1), f'(1), f''(1)$, etc. in (4.66), we obtain

$$f\left(\frac{11}{10}\right) = 9 + \frac{1}{10}(24) + \frac{1}{100}\left(\frac{12}{2}\right) + \frac{1}{1000}\left(\frac{6}{6}\right) = 9 + 2.4 + 0.06 + 0.0001 = 11.461.$$

Example 4.27: Calculate the approximate value of $\sqrt{10}$ to four decimal places using Taylor's series.

Solution: Consider $f(x) = \sqrt{x}$. By Taylor's series $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$

Put $x = 9, h = 1$; we obtain

$$f(10) = f(9) + f'(9) + \frac{f''(9)}{2!} + \frac{f'''(9)}{3!} + \ldots \qquad \ldots(4.67)$$

Now, $f(x) = \sqrt{x},\quad f(9) = 3$

$f'(x) = \frac{1}{2\sqrt{x}},\quad f'(9) = \frac{1}{6}$

$f''(x) = -\frac{1}{4x\sqrt{x}},\quad f''(9) = -\frac{1}{108}$

$f'''(x) = \frac{3}{8x^2\sqrt{x}},\quad f'''(9) = \frac{1}{648}$ etc.

Substituting for $f(9), f'(9), f''(9)$, etc. in (4.67), we obtain

$$\sqrt{10} = 3 + \frac{1}{6} - \frac{1}{2(108)} + \frac{1}{6(648)} - \ldots = 3 + 0.16666 - 0.00463 + 0.00025 + \ldots = 3.1623$$

Example 4.28: Expand $\tan\left(x + \frac{\pi}{4}\right)$ as far as the term x^4, and evaluate tan 44° to four significant digits.

Solution: Let $f(x) = \tan x$. By Taylor's series expansion

$$f\left(x + \frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) + xf'\left(\frac{\pi}{4}\right) + \frac{x^2}{2!}f''\left(\frac{\pi}{4}\right) + \frac{x^3}{3!}f'''\left(\frac{\pi}{4}\right) + \frac{x^4}{4!}f^{iv}\left(\frac{\pi}{4}\right) + \ldots \qquad \ldots(4.68)$$

Now,
$$f(x) = \tan x$$
$$f'(x) = \sec^2 x = 1 + \tan^2 x$$
$$f''(x) = 2\sec^2 x \tan x = 2[\tan x + \tan^3 x]$$
$$f'''(x) = 2[\sec^2 x + 3\tan^2 x \sec^2 x] = 2[1 + 4\tan^2 x + 3\tan^4 x]$$
$$f^{iv}(x) = 2[8\tan x \sec^2 x + 12\tan^3 x \sec^2 x] = 16\tan x + 40\tan^3 x + 24\tan^5 x.$$

Therefore, $f\left(\frac{\pi}{4}\right) = 1,\ f'\left(\frac{\pi}{4}\right) = 2,\ f''\left(\frac{\pi}{4}\right) = 4,\ f'''\left(\frac{\pi}{4}\right) = 16,\ f^{iv}\left(\frac{\pi}{4}\right) = 80.$

Substituting in (4.68), we obtain

$$\tan\left(x + \frac{\pi}{4}\right) = 1 + 2x + \frac{4x^2}{2!} + \frac{16x^3}{3!} + \frac{80x^4}{4!} + \ldots$$

$$= 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \ldots \qquad \ldots(4.69)$$

To evaluate tan 44°, take $x = -1° = -\frac{\pi}{180} = -0.017453$ radians. Substituting in (4.69), and simplifying, we obtain $\tan 44° \simeq 0.9657$, up to four significant digits.

Example 4.29: Show that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ and from this derive the expansion for $\cos x$.

Solution: Here $f(x) = \sin x$, therefore

$$f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x, f^{iv}(x) = \sin x,\ f^{v}(x) = \cos x, \text{ etc.}$$

Maclaurin's series expansion is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \ldots$$

Substituting for $f(x), f(0), f''(0), \ldots$ we get $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

To obtain the expansion for $\cos x$ differentiating it term by term, we obtain $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

Example 4.30: Show that, $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots$

Hence, find the value for π correct upto three decimal places.

Solution: Consider $y(x) = \sin^{-1} x$. It gives $\qquad \ldots(4.70)$

$$y_1(x) = \frac{1}{\sqrt{1-x^2}} \qquad ...(4.71)$$

or, $$y_1^2(1-x^2) = 1$$

Differentiating w.r.t. x, we obtain

$$2y_1y_2(1-x^2) - 2y_1^2x = 0$$

$$y_2(1-x^2) - y_1x = 0 \qquad ...(4.72)$$

Differentiating (4.72) n times using Leibnitz rule, we have

$$[y_{n+2}(1-x^2) + C_1^n y_{n+1}(-2x) + C_2^n y_n(-2)] - [y_{n+1}\,x + C_1^n y_n] = 0$$

or, $$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0 \qquad ...(4.73)$$

Substituting $x = 0$ in (4.70), (4.71), (4.72) and (4.73), we obtain respectively

$$y(0) = 0,\ y_1(0) = 1,\ y_2(0) = 0, \text{ and } y_{n+2}(0) = n^2y_n(0) \qquad ...(4.74)$$

From (4.74), $y_3(0) = 1^2y_1(0) = 1^2$, $y_4(0) = 2^2y_2(0) = 0$

$y_5(0) = 3^2y_3(0) = 1^2.3^2$, $y_6(0) = 4^2y_4(0) = 0$, $y_7(0) = 5^2y_5(0) = 1^2.3^2.5^2$, and so on.

Substituting these in the Maclaurin's series

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + ...$$

we obtain, $$\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + ... \qquad ...(4.75)$$

the required expansion.

Since, the series (4.75) is convergent for $-1 < x < 1$, to find the value for π, put $x = \frac{1}{2}$ in (4.75), we have

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{3}{40}\left(\frac{1}{2}\right)^5 + \frac{5}{112}\left(\frac{1}{2}\right)^7 + ...$$

or, $$\pi = 3 + \frac{1}{8} + \frac{9}{640} + \frac{30}{14336} + ... = 3.141 \text{ approximately.}$$

Example 4.31: Prove that if terms of x^3 and higher orders are neglected, then

$$\tan^{-1}[(1+x)\tan\alpha] = \alpha + x\sin\alpha\cos\alpha - x^2\sin^3\alpha\cos\alpha.$$

Solution: Let $f(x) = \tan^{-1}[(1+x)\tan\alpha]$

It gives $$f'(x) = \frac{\tan\alpha}{1+(1+x)^2\tan^2\alpha},\ f''(x) = -\tan\alpha[1+(1+x)^2\tan^2\alpha]^{-2}[2(1+x)\tan^2\alpha].$$

Therefore, $f(0) = \tan^{-1}(\tan\alpha) = \alpha, f'(0) = \dfrac{\tan\alpha}{1+\tan^2\alpha} = \dfrac{\tan\alpha}{\sec^2\alpha} = \sin\alpha\cos\alpha$

$$f''(0) = -\tan\alpha[1+\tan^2\alpha]^{-2}(2\tan^2\alpha) = -\frac{2\tan^3\alpha}{\sec^4\alpha} = -2\sin^3\alpha\cos\alpha.$$

Maclaurin's series expansion, neglecting the terms of x^3 and higher order, is

$f(x) \approx f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0)$. Substituting for $f(x), f(0), f'(0), f''(0)$, we obtain

$$\tan^{-1}[(1+x)\tan\alpha] = \alpha + x\sin\alpha\cos\alpha - x^2\sin^3\alpha\cos\alpha.$$

Example 4.32: Expand $y = \sin[\ln(x^2+2x+1)]$ in powers of x using Maclaurin's series up to x^5.

Solution: Let $$y(x) = \sin[\ln(x^2+2x+1)]. \qquad ...(4.76)$$

It gives $$y_1(x) = \cos[\ln(x^2+2x+1)].\frac{2x+2}{x^2+2x+1}$$

$$= \frac{2}{x+1}\cos[\ln(x^2+2x+1)] \qquad ...(4.77)$$

or, $$(x+1)y_1 = 2\cos[\ln(x^2+2x+1)].$$

Differentiating it again w.r.t. x, we get

$$(x+1)y_2 + y_1 = -2\sin[\ln(x^2+2x+1)].\frac{2(x+1)}{(x+1)^2} = -\frac{4y}{x+1}$$

or, $$(x+1)^2y_2 + (x+1)y_1 + 4y = 0. \qquad ...(4.78)$$

Differentiating (4.78) n times using Leibnitz's rule, we get

$$[(x+1)^2y_{n+2} + C_1^n 2(x+1)y_{n+1} + C_2^n(2)y_n] + [(x+1)y_{n+1} + C_1^n(1)y_n] + 4y_n = 0$$

or, $$(x+1)^2y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0. \qquad ...(4.79)$$

Substituting $x = 0$, in (4.76), (4.77), (4.78) and (4.79), we obtain

$y(0) = 0,\quad y_1(0) = 2,\quad y_2(0) = -2$, and $y_{n+2}(0) + (2n+1)y_{n+1}(0) + (n^2+4)y_n(0) = 0$

or, $$y_{n+2}(0) = -(2n+1)y_{n+1}(0) - (n^2+4)y_n(0) \qquad ...(4.80)$$

Set $n = 1, 2, 3, ...$ in (4.80), we get

$$y_3(0) = -3y_2(0) - 5y_1(0) = -4,\ y_4(0) = -5y_3(0) - 8y_2(0) = 36$$

$$y_5(0) = -7y_4(0) - 13y_3(0) = -200, \text{ and so on.}$$

Substituting these values in Maclaurin's series expansion,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...,$$

we obtain, $\sin[\ln(x^2+2x+1)] = 2x - x^2 - \frac{2}{3}x^3 + \frac{3}{2}x^4 - \frac{5}{3}x^5 + \dots$

as the required expansion.

EXERCISE 4.5

1. Using Taylor's series prove that

   (i) $\cos\left(x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} - \dots\right)$

   (ii) $\ln\sin(x+h) = \ln\sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\operatorname{cosec}^2 x \cot x.$

   (iii) $a^{x+h} = a^x\left[1 + h\ln a + \frac{h^2}{2!}(\ln a)^2 + \frac{h^3}{3!}(\ln a)^3 + \dots\right]$

   (iv) $\ln(x+h) = \ln x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \dots$

   (v) $\tan^{-1}(x+h) = \tan^{-1}x + \frac{h}{1+x^2} - \frac{xh^2}{(1+x^2)^2} + \dots$

   (vi) $\tan(x+h) = \tan x + h\sec^2 x + h^2\sec^2 x\tan x + \frac{h^3}{3}\sec^2 x(1+3\tan^2 x) + \dots$

   (viii) $\sec^{-1}(x+h) = \sec^{-1}x + \frac{h}{x\sqrt{x^2-1}} - \frac{h^2}{2!}\frac{2x^2-1}{x^2(x^2-1)^{3/2}} + \dots$

2. Using Taylor's series expand

   (i) $\cos x$ in powers of $\left(x - \frac{\pi}{4}\right)$ up to 4 terms.

   (ii) e^x in powers of $(x-2)$

   (iii) $\sin x$ in powers of $\left(x - \frac{\pi}{2}\right)$. Hence, find the value of $\sin 91°$ correct to 4 decimal places.

   (iv) $\tan^{-1}x$ in powers of $(x-1)$

   (v) $2x^3 + 7x^2 + x - 1$ in powers of $(x-2)$.

3. Compute to four decimal places, the value of $\cos 32°$ using Taylor's series.
4. Given $\log_{10}4 = 0.6021$, calculate approximately $\log_{10}404$.

5. If $f(x)=x^3+2x^2-5x+11$, calculate the value of $f\left(\frac{9}{10}\right)$ by the application of Taylor's series.

6. Calculate the approximate value of $\sqrt{17}$ correct to four decimal places using Taylor's series.

7. Prove that: (i) $f(ax)=f(x)+(a-1)xf'(x)+\frac{(a-1)^2x^2}{2!}f''(x)+\ldots$

(ii) $f\left(\frac{x^2}{1+x}\right)=f(x)-\frac{x}{1+x}f'(x)+\left(\frac{x}{1+x}\right)^2\frac{f''(x)}{2!}-\left(\frac{x}{1+x}\right)^3\frac{f'''(x)}{3!}+\ldots$

8. Show that the error terms in the Taylor's polynomial expansion of the function $f(x)=\sin x$ about the point $x=\frac{\pi}{4}$ tends to zero as $n\to\infty$ for any real x.

9. Expand the following functions using Maclaurin's series in powers of x.
(i) e^x (ii) $\cos x$ (iii) $\tan x$ (iv) $\sinh x$ (v) $\tan^{-1}x$ (vi) $\ln(1+x)$ (viii) $\sec x$ (viii) $\tan^{-1}(1+x)$
(ix) a^{x+h}

10. Prove that

(i) $e^x\sec x=1+x+\frac{x^2}{2}+\frac{2x^3}{3}+\ldots$ (ii) $\ln\cosh x=\frac{1}{2}x^2-\frac{x^4}{12}+\frac{x^6}{45}-\ldots$

(iii) $\cos^2 x=1-x^2+\frac{x^4}{3}-\ldots$ (iv) $\frac{x}{\sin x}=1+\frac{x^2}{6}+\frac{7x^4}{360}+\ldots$

(v) $\frac{e^x}{\cos x}=1+x+\frac{2x^2}{2!}+\frac{4x^3}{3!}+\ldots$ (vi) $e^{\sin x}=1+x+\frac{x^2}{2}-\frac{x^4}{8}+\ldots$

11. Prove that

(i) $\ln\tan\left(x+\frac{\pi}{4}\right)=2x+\frac{4}{3}x^3+\frac{4}{2}x^5+\ldots$ (ii) $\ln(1+\sin x)=x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\ldots$

(iii) $\ln\sec x=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\ldots$ (iv) $\ln(\sec x+\tan x)=x+\frac{x^3}{6}+\frac{x^5}{24}+\ldots$

12. By forming a differential equation, prove that

(i) $(\sin^{-1}x)^2=2\frac{x^2}{2!}+2.2^2\frac{x^4}{4!}+2.2^24^2\frac{x^6}{6!}+\ldots$

(ii) $e^{m\tan^{-1}x}=1+mx+\frac{m^2}{2!}x^2+\frac{m(m^2-2)}{3!}x^3+\frac{m^2(m^2-8)}{4!}x^4+\ldots$

(iii) $\ln\left[x+\sqrt{1+x^2}\right] = x - \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \ldots$

(iv) $\left(x+\sqrt{1+x^2}\right)^n = 1 + nx + \frac{n^2x^2}{2!} + \frac{n^2(n^2-1^2)}{3!}x^3 + \frac{n^2(n^2-2^2)}{4!}x^4 + \ldots$

(v) $\frac{\ln(1+x)}{1+x} = x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{25}{12}x^4 + \ldots$

(vi) $\sin(2\sin^{-1}x) = 2x - x^3 - \frac{x^5}{4} - \ldots$ (vii) $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{8}{15}x^5 + \ldots$

13. Using Maclaurin's series show that, $e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \ldots$

Also deduce that $\cos x = 1 - \frac{x^2}{2!} + \ldots$

14. Apply Maclaurin's series to obtain the expansion of the function $e^{ax}\sin bx$ in an infinite series of power of x giving the general term.

15. Obtain by Maclaurin's theorem, the first four terms of the expansion of $e^{x\cos x}$. Using this show that $\lim_{x\to 0}\frac{e^x - e^{x\cos x}}{x - \sin x} = 3$.

16. Find the minimum number of terms that must be retained in the Maclaurin's series expansion of the function $\sin x\cos x$ in the interval $[0, 1]$, such that $|\text{Error}| < 0.0005$.

4.6.5 Use of Some Standard Series

Sometimes it is cumbersome to find the successive derivatives of a function. In such cases the use of Maclaurin's expansion of some standard functions is advisable. The expansions for some standard functions are listed below:

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

2. $a^x = 1 + x(\ln a) + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \ldots$

3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

4. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

5. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots,\ |x| < 1$

6. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

7. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

8. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$ 9. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

10. $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots \ (|x|<1).$

Next we give some examples illustrating the use of some of these series.

Example 4.33: Prove that $\ln(1+x+x^2+x^3+x^4) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \frac{4}{5}x^5 + \frac{x^6}{6} + \ldots$

Solution: We have, $1+x+x^2+x^3+x^4 = \frac{1-x^5}{1-x}$. Therefore,

$$\ln(1+x+x^2+x^3+x^4) = \ln(1-x^5) - \ln(1-x)$$

$$= \left[-x^5 - \frac{(x^5)^2}{2} - \frac{(x^5)^3}{3} - \ldots\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6}\right]$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \frac{4}{5}x^5 + \frac{x^6}{6} + \ldots, \quad |x|<1,$$

the required expansion.

Example 4.34: Prove that $(1+x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \frac{33}{40}x^6 + \ldots$

Solution: We have $(1+x)^x = e^{\ln(1+x)^x} = e^{x\ln(1+x)} = e^t$, where $t = x\ln(1+x)$. Consider

$$t = x\ln(1+x) = x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\right) = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \ldots$$

Therefore, $(1+x)^x = e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$

$$= 1 + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \ldots\right) + \frac{1}{2!}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \ldots\right)^2$$

$$+ \frac{1}{3!}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \ldots\right)^3 + \ldots$$

$$= 1 + x^2 - \frac{x^3}{2} + x^4\left(\frac{1}{3} + \frac{1}{2}\right) + x^5\left(-\frac{1}{4} - \frac{1}{2}\right) + x^6\left(\frac{1}{5} + \frac{1}{8} + \frac{1}{6} + \frac{1}{3}\right) + \ldots$$

$$= 1 + x^2 - \frac{x^3}{2} + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \frac{33}{40}x^6 + \ldots$$

is the required expansion.

Example 4.35: Expand $\ln[1 - \ln(1-x)]$ in powers of x by Maclaurin's series up to the term of x^3 and deduce the expansion of $\ln[1 + \ln(1+x)]$.

Solution: Let $f(x) = \ln[1 - \ln(1-x)]$. We have, $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$

Therefore, $f(x) = \ln[1 - \ln(1-x)]$

$$= \ln\left[1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right] = \ln(1+t), \text{ where } t = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

Further $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + \ldots$ Therefore,

$$\ln[1 - \ln(1-x)] = \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right) - \frac{1}{2}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)^2 + \frac{1}{3}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)^3 + \ldots$$

$$= x + \frac{1}{6}x^3 + \ldots, \qquad \ldots(4.81)$$

on simplification.

To get the expansion of $\ln[1 + \ln(1+x)]$, replace x by $\frac{x}{1+x}$ in (4.81). The left side of (4.81) becomes

$$\ln[1 - \ln(1-x)] = \ln\left[1 - \ln\left(1 - \frac{x}{1+x}\right)\right] = \ln\left[1 - \ln\left(\frac{1}{1+x}\right)\right] = \ln[1 + \ln(1+x)].$$

Therefore, (4.81) becomes

$$\ln[1 + \ln(1+x)] = x(1+x)^{-1} + \frac{1}{6}x^3(1+x)^{-3} + \ldots$$

$$= x\left[1 - x + x^2 + x^3 - \ldots\right] +$$

$$\frac{1}{6}x^3\left[1 - 3x + \frac{(-3)(-4)}{2!}x^2 + \frac{(-3)(-4)(-5)}{3!}x^3 + \ldots\right]$$

$$= x - x^2 + x^3\left(1 + \frac{1}{6}\right) + \ldots = x - x^2 + \frac{7}{6}x^3 + \ldots$$

Example 4.36: Expand $\cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right)$ in ascending powers of x.

Solution: Put $x=\cot\theta$, then

$$\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)=\cos^{-1}\left(\frac{\cot\theta-\tan\theta}{\cot\theta+\tan\theta}\right)=\cos^{-1}\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\right)=\cos^{-1}(\cos 2\theta)$$

$$=2\theta=2\cot^{-1}x=2\left(\frac{\pi}{2}-\tan^{-1}x\right)=\pi-2\tan^{-1}x.$$

Also, $\tan^{-1}x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots$ Therefore,

$$\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)=\pi-2\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right)=\pi-2x+\frac{2x^3}{3}-\frac{2x^5}{5}+\frac{2x^7}{7}-\ldots$$

Example 4.37: Find the first four terms in the expansion of $\ln(1+\tan x)$.

Solution: Write $\ln(1+\tan x)=\ln(1+t)$, where $t=\tan x$. We have,

$$\ln(1+t)=t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\ldots \text{ and, } t=\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots$$

Therefore, $\ln(1+\tan x)=\left(x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\ldots\right)-\dfrac{1}{2}\left(x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\ldots\right)^2$

$$+\frac{1}{3}\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots\right)^3-\frac{1}{4}\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots\right)^4+\ldots$$

$$=x-\frac{1}{2}x^2+x^3\left(\frac{1}{3}+\frac{1}{3}\right)+x^4\left(-\frac{1}{3}-\frac{1}{4}\right)+\ldots$$

$$=x-\frac{1}{2}x^2+\frac{2}{3}x^3-\frac{7}{12}x^4+\ldots$$

is the required expansion.

Example 4.38: Show that $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}=\dfrac{1}{2}\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots\right)$.

Solution: Put $x = \tan\theta$, we obtain

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta} = \tan^{-1}\frac{\sec\theta-1}{\tan\theta}$$

$$= \tan^{-1}\frac{1-\cos\theta}{\sin\theta} = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

The requisite result follows from the expression, $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

4.7 INDETERMINATE FORMS

If functions $f(x)$ and $g(x)$ are both zero at $x = a$, then $\lim_{x\to a}(f(x)/g(x))$ cannot be found by substitution $x = a$. The substitution produces 0/0, a meaningless expression, known as an *indeterminate form*. This does not imply that $\lim_{x\to a}(f(x)/g(x))$ does not exist. In fact in many cases it has a finite value, as we have already noticed that the determination of the differential coefficient dy/dx is itself equivalent to finding the limit of a fraction $\Delta y/\Delta x$ which assumes an indeterminate form as $\Delta x \to 0$.

4.7.1 L' Hospital's Rule

We will assume that $f(x)$ and $g(x)$ possess continuous derivatives of every order that appear in the process of finding the limit in a certain interval enclosing $x = a$.

Since $f(a) = 0 = g(a)$, we can write

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{[f(x)-f(a)]/(x-a)}{[g(x)-g(a)]/(x-a)}$$

$$= \lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} \quad \ldots(4.82)$$

provided the limit on the right exists. This rule is known as *L' Hospital's rule*.

Suppose now that $f'(a) = 0 = g'(a)$. Then we repeat the application of the L' Hospital's rule on $f'(x)/g'(x)$ and obtain

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} = \frac{f''(a)}{g''(a)} \quad \ldots(4.83)$$

provided the limits exist.

The application of this rule can be repeated as long as the indeterminate form is evaluated.

L' Hospital's rule also applies to quotients that lead to the indeterminate form ∞/∞. If $f(x)$ and $g(x)$ both approach infinity as $x \to a$, then it can be shown that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. Here, a may itself, be either finite or infinite.

The L' Hospital's rule is designed to use when the ratio is of the form $0/0$ or ∞/∞. But we can sometimes handle the indeterminate forms $0.\infty$ and $\infty - \infty$ by using algebra to get $0/0$ or ∞/∞. Limits that lead to the indeterminate forms 1^{∞}, 0^0 and ∞^0 can sometimes be handled by taking logarithms first. We use this rule to find the limit of the logarithm and then exponentiate to find the original limit.

We must note that since the functions of the form 0^{∞}, $\infty.\infty$, $\infty + \infty$, ∞^{∞}, $\infty^{-\infty}$ are not of indeterminate form and hence L' Hospital's rule is not applicable in these cases.

Example 4.39: Evaluate $\lim\limits_{x \to 1} \dfrac{x^x - x}{x - 1 - \ln x}$.

Solution: The function is of the form $0/0$, using L' Hospital rule we have

$$\lim_{x \to 1} \frac{x^x - x}{x - 1 - \ln x} = \lim_{x \to 1} \frac{x^x(1 + \ln x) - 1}{1 - 1/x} = \lim_{x \to 1} \frac{x^x(1 + \ln x)^2 + x^x(1/x)}{1/x^2} = \frac{2}{1} = 2.$$

Example 4.40: Find the values of a and b in order that $\lim\limits_{x \to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3} = 1$.

Solution: The function is of the form $0/0$, using L' Hospital's rule, we have

$$\lim_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} = \lim_{x \to 0} \frac{1 + a\cos x - ax\sin x - b\cos x}{3x^2}.$$

The denominator being 0 for $x = 0$, the function will tend to a finite limit if the numerator is also zero for $x = 0$ and for this

$$1 + a - b = 0 \qquad \text{...(4.84)}$$

Suppose that (4.84) is satisfied, we have

$$\lim_{x \to 0} \frac{1 + a\cos x - ax\sin x - b\cos x}{3x^2} = \lim_{x \to 0} \frac{-2a\sin x - ax\cos x + b\sin x}{6x}$$

$$= \lim_{x \to 0} \frac{-3a\cos x + ax\sin x + b\cos x}{6} = \frac{b - 3a}{6}$$

As given, $$\frac{b - 3a}{6} = 1, \text{ or } \; b - 3a = 6 \qquad \text{...(4.85)}$$

Solving (4.84) and (4.85), we obtain $a = -5/2$, $b = -3/2$.

Example 4.41: Find $\lim\limits_{x \to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x} \right)$.

Solution: If $x \to 0^+$, then $\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\infty-\infty$ and if, $x\to 0^-$, then $\lim_{x\to 0^-}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=-\infty+\infty$, in both cases the form is indeterminate.

In case we combine the fraction, then it is of the form 0/0. Applying L' Hospital's rule, we have

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}=\lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x}=\frac{0}{2}=0.$$

Example 4.42: Find $\lim_{x\to 0}(\cos x)^{1/x^2}$.

Solution: Consider $f(x)=(\cos x)^{1/x^2}$. If $x\to 0$, then $f(x)$ is of the form 1^{∞}, which is indeterminate.

Taking logarithm both sides we obtain, $\ln f(x)=\frac{\ln(\cos x)}{x^2}$

It is of the form 0/0. Applying L' Hospital's rule we have

$$\lim_{x\to 0}\ln f(x)=\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}=\lim_{x\to 0}\frac{-\tan x}{2x}=-\frac{1}{2}, \text{ since } \lim_{x\to 0}\frac{\tan x}{x}=1.$$

Thus, $\lim_{x\to 0}(\ln f(x))=-\frac{1}{2}$, or $\lim_{x\to 0}f(x)=e^{-1/2}$, that is, $\lim_{x\to 0}(\cos x)^{1/x^2}=e^{-1/2}$

Example 4.43: Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$.

Solution: Consider

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} &= \lim_{x\to 0}\left[\frac{x+\frac{x^3}{3}+\frac{2}{15}x^5+\dots}{x}\right]^{1/x^2}\\
&= \lim_{x\to 0}\left[1+\frac{x^2}{3}+\frac{2}{15}x^4+\dots\right]^{1/x^2}\\
&= \lim_{x\to 0}[1+x^2f(x)]^{1/x^2}, \text{ where } f(x)=\frac{1}{3}+\frac{2}{5}x^2+\dots\\
&= \lim_{x\to 0}\left[\left(1+x^2f(x)\right)^{\frac{1}{x^2f(x)}}\right]^{f(x)}\\
&= \lim_{x\to 0}e^{f(x)}, \text{ since } \lim_{x\to 0}(1+x)^{1/x}=e \text{ and } x^2f(x)\to 0 \text{ as } x\to 0,\\
&= e^{f(0)}=e^{1/3}.
\end{aligned}$$

EXERCISE 4.6

Evaluate the following limits

1. $\lim_{x\to 0} \dfrac{1-\cos x}{x+x^2}$

2. $\lim_{x\to 0} \dfrac{xe^x - \ln(1+x)}{x^2}$

3. $\lim_{x\to 0} \dfrac{e^x - e^{-x} - 2\ln(1+x)}{x \sin x}$

4. $\lim_{x\to 0} \left\{\dfrac{1}{x} - \dfrac{1}{x^2}\ln(1+x)\right\}$

5. If the limit of $\dfrac{\sin 2x + a\sin x}{x^3}$ as x tends to zero be finite, find the value of a and the limit.

Evaluate the following limits.

6. $\lim_{x\to\infty} x \tan\left(\dfrac{1}{x}\right)$

7. $\lim_{x\to\infty} \displaystyle\int_x^{2x} \dfrac{1}{t}\,dt$

8. $\lim_{x\to a} \dfrac{\ln(x-a)}{\ln(e^x - e^a)}$

9. $\lim_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$

10. $\lim_{x\to 0} (\cot x)^{\sin 2x}$

11. $\lim_{x\to 0} \dfrac{(1+x)^{1/x} - e}{x}$

12. $\lim_{x\to 0} \dfrac{1+\sin x - \cos x + \ln(1-x)}{x \tan^2 x}$

13. $\lim_{x\to\infty} \sqrt{\dfrac{x+\sin x}{x-\cos^2 x}}$

14. If $f(x) = e^{-1/x^2}, x \neq 0$ and $f(0) = 0$, show that the derivative of every order of $f(x)$ vanishes for $x = 0$, i.e., $f^n(0) = 0$ for all n.

15. Discuss the continuity of $f(x)$ at the origin when $f(x) = x \ln \sin x$ for $x \neq 0$ and $f(0) = 0$.

4.8 EXTREME VALUES OF A FUNCTION

Here we shall be interested in determining those values of a continuous function defined over an interval $[a, b]$ which are the greatest or the least in their immediate neighbourhood, technically known as the *local maximum* and the *local minimum* values, or the *extreme values*. The knowledge of these values is helpful in studying the behaviour of a system over an interval and is of great practical importance in engineering and science.

4.8.1 Maxima and Minima

Let c be any interior point of the domain of definition of a function $f(x)$. We say that $f(c)$ is a *maxima* of the function $f(x)$, if there exists some interval $(c-h, c+h), h > 0$, around c such that $f(c) > f(x)$, for all $x \in (c-h, c+h)$ other than c itself. The interval $(c-h, c+h)$ is called a *neighbourhood (nbd.)* of c.

On the other hand, $f(c)$ is a *minima* of the function $f(x)$, if $f(c) < f(x)$, for all $x \in (c-h, c+h)$ other than c itself.

The points at which maximum or minimum values of a function exist are called the *critical points*, or the *stationary points*.

The values of the function at these points are called the *extreme values*.

Thus, if $f(c)$ is an extreme value of a function $f(x)$, then $f(c+h) - f(c)$ keeps the same sign for values of h, sufficiently small numerically. Further it should be clear that a maximum value may not be the greatest and a minimum value may not be the least of all the values of the function in any finite interval. In fact a function can have several maximum and minimum values and a minimum value can even be greater than the maximum value as shown in Fig. 4.10.

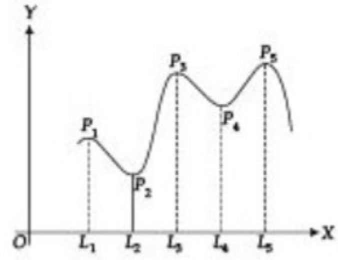


Fig. 4.10

For the function $y = f(x)$ the points P_1, P_3, P_5 are the points of maxima, while the points P_2, P_4 are the points of minima, but the ordinate L_4P_4 at P_4 is greater than that of L_1P_1 at P_1.

4.8.2 A Necessary Condition for Extreme Values

A necessary condition for $f(c)$ to be an extreme value of $f(x)$ is that $f'(c) = 0$.

To prove it, consider $f(x)$ to be maximum at $x = c$, thus $f(c) \geq f(c+h)$, for all h close to zero.

Also, $f'(c) = \lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$. If $h > 0$, then $\dfrac{f(c+h)-f(c)}{h} \leq 0$ and thus $\lim\limits_{h\to 0^+} \dfrac{f(c+h)-f(c)}{h} \leq 0$.

Similarly if $h < 0$, then $\lim\limits_{h\to 0^-} \dfrac{f(c+h)-f(c)}{h} \geq 0$. Since $f'(c)$ exists, thus both of these inequalities will hold simultaneously if, and only if, $f'(c) = 0$.

The case when $f(x)$ is minimum at $x = c$ follows on the same line.

Geometrically interpreted, the necessary condition for extreme values means that the tangent to a curve at a point where the ordinate is maximum or minimum is parallel to x-axis.

Further we note that $f'(c) = 0$ *is only a necessary but not sufficient condition for $f(x)$ to be an extreme value.* For example, consider the function $f(x) = x^3$ at $x = 0$. For $x > 0$, $f(x)$ is positive and is, therefore, greater than $f(0)$ which is zero; and for $x < 0$, $f(x)$ is negative and is, therefore, less than $f(0)$. Thus, $f(0)$ is not an extreme value even though $f'(0) = 0$.

Stationary value: A function $f(x)$ is said to be stationary for $x = c$, if the derivative $f'(x)$ vanishes for $x = c$, and c is said to be a *stationary* value of $f(x)$.

In fact, *a maximum or a minimum value is always a stationary value but a stationary value may neither be a maximum nor a minimum value.*

4.8.3 A Sufficient Criteria for Extreme Values

A function $f(x)$ has a maximum value at $x = c$ if, and only if the sign of $f'(x)$ changes from positive to negative as x passes through c and $f(x)$ has a minimum value at $x = c$, if, and only if $f'(x)$ changes sign from negative to positive as x passes through c.

The second and higher order derivatives test for extreme values. The extreme values of a function can sometimes be found more conveniently by the use of derivatives of the second and higher orders. The result is as follows:

Let $f(x)$ be differentiable at $x = c$, and $f'(c) = 0$. If $f''(x)$ exists and is continuous in a nbd. of c, then

(i) *$f(x)$ has a maximum value at $x = c$, iff $f''(c) < 0$,*

(ii) *$f(x)$ has a minimum value at $x = c$, iff $f''(c) > 0$.*

When $f''(c) = 0$, further investigation is needed to decide whether $x = c$ is a point of maxima, or minima, or neither a maxima nor a minima.

In this situation the following is the general criteria:

Let $f'(c) = f''(c) = ... = f^{(n-1)}(c) = 0$ but $f^{(n)}(c) \neq 0$. Then $x = c$ is

(i) *a point of minima of $f(x)$, if $f^{(n)}(c) > 0$ and n is even;*

(ii) *a point of maxima of $f(x)$, if $f^{(n)}(c) < 0$ and n is even;*

(iii) *neither a point of maxima nor a minima of $f(x)$, if n is odd.*

4.8.4 Extreme Values of a Function Represented Parametrically

Let a function $y = f(x)$ be represented parametrically as $x = \phi(t)$, $y = \psi(t)$, where the functions $\phi(t)$ and $\psi(t)$ have derivatives both of the first and second orders within a certain interval of the argument t, and $\phi'(t) \neq 0$ in that interval.

Further, let, at $t = t_0$, $\psi'(t_0) = 0$. Then

(i) *if $\psi''(t_0) < 0$, the function $y = f(x)$ has a maximum at $x = c = \phi(t_0)$,*

(ii) *if $\psi''(t_0) > 0$, the function $y = f(x)$ has a minimum at $x = c = \phi(t_0)$*

(iii) *if $\psi''(t_0) = 0$, the existence of maximum or minimum remains open.*

The points at which $\phi'(t) = 0$ require a special investigation.

4.8.5 Concavity and Points of Inflexion

Consider a curve $y = f(x)$. Draw the tangents at the points P_1, P_2, and P_3 as shown in Fig. 4.11. The curve is *concave downward* (or, *convex upward*) at the point P_1 and is *concave upward* (or, *convex downward*) at the point P_3.

At the point P_2, however there is a change in the direction of bending of the curve from concavity downwards to concavity upwards. Such a point is called a *point of inflexion* of the curve. At a point of inflexion, the curve changes from concavity downwards to concavity upwards or vice versa. Infact, it crosses the tangent at the point of inflexion.

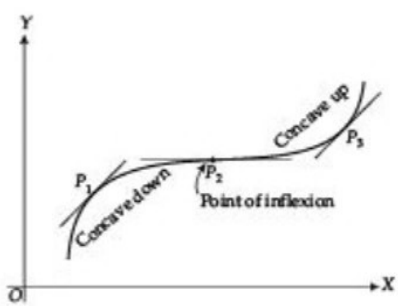


Fig. 4.11

The second order derivative test for concavity

Let $y = f(x)$ be twice differentiable on an interval I. Then

(i) *if $f''(x) > 0$ on I, the graph of $y = f(x)$ is concave upward on I,*

(ii) *if $f''(x) < 0$ on I, the graph of $y = f(x)$ is concave downward on I,*

(iii) *a point where $f''(x)$ changes sign from positive to negative or viceversa is a point of inflexion.*

We observe that at the point of inflexion $\frac{d^2y}{dx^2} = 0$, but $\frac{d^3y}{dx^3} \neq 0$.

Example 4.44: Find the points of maxima and minima values of $x^4 - 8x^3 + 22x^2 - 24x + 1$.

Solution: Let $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 1$. This gives

$$f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x^3 - 6x^2 + 11x - 6) = 4(x-1)(x-2)(x-3).$$

Thus $f'(x) = 0$, for $x = 1, 2, 3$; and hence these are the possible extreme values.

Now, $x < 1 \Rightarrow f'(x) < 0, 1 < x < 2 \Rightarrow f'(x) > 0, 2 < x < 3 \Rightarrow f'(x) < 0, x > 3 \Rightarrow f'(x) > 0$.

Since, $f'(x)$ changes sign from negative to positive as x passes through 1 and 3, thus $x = 1$ and $x = 3$ are the points of minima and $f(1) = -8$ and $f(3) = -8$ are the corresponding minimum values.

Again $f'(x)$ changes sign from positive to negative as x passes through 2, and therefore $x = 2$ is the point of maxima and the corresponding maximum value is $f(2) = -7$.

Example 4.45: Investigate the nature of the extreme value of the function $f(x) = \cosh x + \cos x$ at the point $x = 0$.

Solution: Since the function $f(x) = \cosh x + \cos x$ is an even function of x, thus $x = 0$ is an extreme value of $f(x)$.

Here,

$$\begin{aligned} f'(x) &= \sinh x - \sin x; & f'(0) &= 0 \\ f''(x) &= \cosh x - \cos x; & f''(0) &= 0 \\ f'''(x) &= \sinh x + \sin x; & f'''(0) &= 0 \\ f''''(x) &= \cosh x + \cos x; & f''''(0) &= 2 > 0. \end{aligned}$$

Since the first non-zero derivative at the point $x = 0$ is a derivative of an even order, and is positive, thus $x = 0$ is a minima of $f(x)$ and the minimum value is $f(0) = 2$.

Example 4.46: Find the points of maxima and minima values of the function $\sin x + \cos 2x$.

Solution: Here,

$$\begin{aligned} f(x) &= \sin x + \cos 2x \\ f'(x) &= \cos x - 2\sin 2x = \cos x - 4\sin x\cos x \\ f'(x) &= 0 \Rightarrow \cos x = 0, \text{ or } \sin x = 1/4. \end{aligned}$$

We consider values of x between 0 and 2π only, for the given function is periodic with period 2π.

$\cos x = 0 \Rightarrow x = \pi/2$ and $3\pi/2$, $\sin x = \frac{1}{4} \Rightarrow x = \sin^{-1}\frac{1}{4}$, and $\pi - \sin^{-1}\frac{1}{4}$.

Now,

$$f''(x) = -\sin x - 4\cos 2x.$$

At

$$\begin{aligned} x &= \pi/2, & f''(\pi/2) &= 3 > 0 \\ x &= 3\pi/2, & f''(3\pi/2) &= 5 > 0 \end{aligned}$$

At

$$x = \sin^{-1}\frac{1}{4} \text{ and } \pi - \sin^{-1}\frac{1}{4},$$

$$f''(x) = -\sin x - 4\cos 2x = -\sin x - 4(1 - 2\sin^2 x) = -\frac{1}{4} - 4(1 - 2/16) = -15/4 < 0.$$

Therefore $x = \pi/2, 3\pi/2$ are the points of minima and $x = \sin^{-1}\frac{1}{4}, \pi - \sin^{-1}\frac{1}{4}$ are the points of maxima of $f(x)$. The corresponding minimum values are $f(\pi/2) = 0$ and $f(3\pi/2) = -2$ and the corresponding maximum values are 9/8 and 9/8.

Example 4.47: 20 metre of wire is available for fencing off a flower-bed which should have the form of a circular sector. What must be the radius of the circle bed if we wish the flower-bed of the greatest possible surface area?

Solution: Let x be the radius of the circle and y be the length of the arc, as shown in Fig. 4.12. Then

$$20 = 2x + y. \text{ This gives}$$
$$y = 2(10 - x).$$

The area of the circular sector $S = \frac{1}{2}xy = x(10 - x), 0 \le x \le 10.$

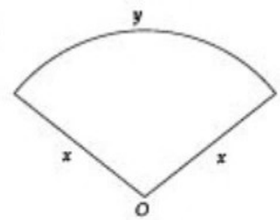


Fig. 4.12

Here $\frac{dS}{dx} = 0$, gives $x = 5$ and $\frac{d^2S}{dx^2} = -2 < 0$. Hence $x = 5$ gives the maximum surface area.

Example 4.48: Assuming that the fuel burnt per hour in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c miles per hour is $3c/2$ miles per hour.

Solution: Let the velocity of the boat be v miles per hour so that its relative velocity is $v - c$ when going against the current. Thus, the time required to cover a distance of d miles $= \frac{d}{v - c}$ hours.

The fuel burnt per hour $= kv^3$, where k is a constant.

If y be the amount of the fuel burnt to cover a distance of d miles, then $y = kd\frac{v^3}{v - c}$.

Therefore, $\frac{dy}{dv} = kd\,\frac{v^2(2v - 3c)}{(v - c)^2}$, and $\frac{dy}{dv} = 0$, gives $v = 0, 3c/2$. Also, $\frac{dy}{dv}$ changes sign from negative to positive as v passes through $3c/2$. Hence minima exists at $v = 3c/2$. Since, $v = 0$ is not desirable, thus, $v = 3c/2$ gives the desired least value of y and is the most economical speed.

Example 4.49: Find the diameter and height of a cylinder of maximum volume which can be cut from a sphere of radius 12 cm.

Solution: A cylinder of radius r and height h is shown enclosed in a sphere of radius 12 cm. in Fig. 4.13.

Volume of cylinder $v = \pi r^2 h$. ...(4.86)

Also from ΔOPQ, we have, $r^2 + \left(\frac{h}{2}\right)^2 = 144$. ...(4.87)

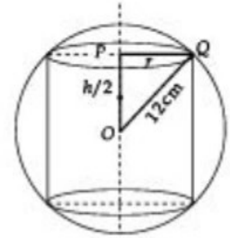


Fig. 4.13

From (4.86) and (4.87), we obtain

$$v = 144\pi h - \frac{\pi h^3}{4}, \text{ which gives } \frac{dv}{dh} = 144\pi - \frac{3\pi h^2}{4}.$$

For maximum or minimum value $\frac{dv}{dh} = 0$. Hence, $h = \frac{\sqrt{(144)(4)}}{3} = 13.86$ cm.

We can verify that $\frac{d^2v}{dh^2} = -\frac{6\pi h}{4}$ is negative for $h = 13.86$. Thus $h = 13.86$ gives a maximum value.

Also from Eq. (4.87)

$$r^2 = 144 - \frac{h^2}{4} = 144 - \frac{(13.86)^2}{4}, \text{ or, } r = 9.80 \text{ cm.}$$

Hence, diameter of cylinder $= 2r = 2(9.80) = 19.60$ cm.

Example 4.50: Find the points of inflexion for the curve $y = \frac{x+1}{x^2+1}$.

Solution: The equation of the curve is $y = \frac{x+1}{x^2+1}$.

The first and the second derivatives are; $y' = \frac{-x^2-2x+1}{(x^2+1)^2}$, $y'' = \frac{2x^3+6x^2-6x-2}{(x^2+1)^3}$

Now, $y'' = 0$ implies that $x^3 + 3x^2 - 3x - 1 = 0$. This gives $x = -2-\sqrt{3},\ -2+\sqrt{3},\ 1$.

Sign of y'' depends upon the sign of the numerator

$$2x^3 + 6x^2 - 6x - 2 = 2(x + 2 + \sqrt{3})(x + 2 - \sqrt{3})(x - 1)$$

We have the following table:

Interval	*Sign of y''*	*Conclusion*
$-\infty < x < -2-\sqrt{3}$	–ve	concave downward
$-2-\sqrt{3} < x < -2+\sqrt{3}$	+ve	concave upward
$-2+\sqrt{3} < x < 1$	–ve	concave downward
$1 < x < \infty$	+ve	concave upward

Hence, $x = -2 \pm \sqrt{3}, 1$ corresponds to the points of inflexion and the corresponding points of inflexion are

$$\left(-2 \pm \sqrt{3}, \frac{1 \pm \sqrt{3}}{4}\right), \text{ and } (1, 1).$$

EXERCISE 4.7

1. Using the first derivative, find the points of maxima and minima of the function $f(x) = x(x+1)^3 (x-3)^2$.
2. Find the extreme value (s) of the function $f(x) = \sqrt{e^{x^2} - 1}$.
3. Investigate $f(x) = x^4 e^{-x^2}$ for its extreme values.
4. For the function $x = \phi(t) = t^5 - 5t^3 - 20t + 7, \quad y = \psi(t) = 4t^3 - 3t^2 - 18t + 3$, find the extreme values.
5. Find the points of maxima and minima of the function $f(x) = 2 \sin x + \cos 2x$.
6. Find the points of maxima and minima of the function $f(x) = \dfrac{40}{3x^4 + 8x^3 - 18x^2 + 60}$.
7. Find the points of maxima and minima of the function $f(x) = \sin x(1 + \cos x), \quad 0 \le x \le 2\pi$.
8. It is required to construct an open cylindrical reservoir of capacity V_0. The thickness of the material is d. What dimensions, that is, the base radius and height should the reservoir have so as to ensure the least possible expenditure on the material?
9. Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone.
10. Find the semi-vertical angle of a right circular cone of maximum volume and a given surface area.
11. Prove that the least parameter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$.
12. A tree trunk L feet long is in the shape of a frustum of a cone the radii of its ends being a and b feet $(a > b)$. It is required to cut from it a beam of uniform square section. Prove that the beam of the greatest volume that can be cut is $aL/3(a-b)$ feet long.
13. Show that the curve $y = \dfrac{x+1}{x^2+1}$ has three points of inflexion lying in a straight line.
14. Show that the points of inflexion of the curve $y = x \sin x$ lie on the curve $y^2(4 + x^2) = 4x^2$.

4.9 CURVATURE AND EVOLUTE

In many practical problems we are concerned with the comparison of bending of two curves or even with the bending of a curve at its different points. For example the problem may be of interest while laying the rail tracks or designing the highways. As another example, in case of parabolic path, bending is different at its different points being maximum at its vertex. The *curvature gives a numerical measure of the sharpness of the bending of a curve.*

4.9.1 Definition and Measure of Curvature

Let A be a fixed point on the curve from which the arc length is measured and let arc lengths of the points P and Q on the curve be s and $s + \delta s$, respectively so that arc $PQ = \delta s$ as shown in Fig. 4.14. Let ψ, $\psi + \delta\psi$ be the angles which the positive directions of the tangents at P and Q make with some fixed line. Thus $\delta\psi$ is the angle through which the tangent turns as a point moves along the curve from P to Q through a distance δs.

The quantity $\delta\psi$ is called the *total curvature* and the ratio $\delta\psi/\delta s$ is called the *average curvature* of the arc PQ.

The curvature of the curve at P is defined as $\lim\limits_{Q\to P} \dfrac{\delta\psi}{\delta s} = \lim\limits_{\delta s\to 0} \dfrac{\delta\psi}{\delta s} = \dfrac{d\psi}{ds}$.

Generally it is denoted by Greek letter k (kappa), thus

$$k = \frac{d\psi}{ds}. \qquad ...(4.88)$$

Curvature of a circle: *The curvature of a circle is constant.*

Consider a circle with radius r and centre O. Let P, Q be any two points on the circle and let arc $PQ = \delta s$, as shown in Fig. 4.15.

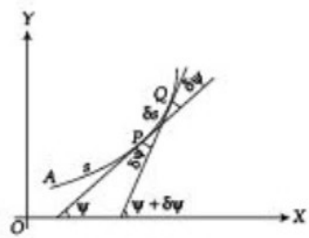


Fig. 4.14

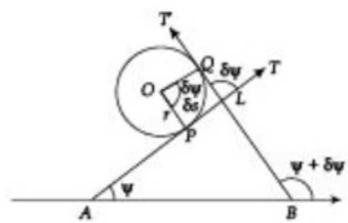


Fig. 4.15

Let L be the point where the tangents PT, QT' at P and Q meet. We have $\angle POQ = \angle TLT' = \delta\psi$.

Also in case of a circle, $\dfrac{\delta s}{r} = \delta\psi$, or $\dfrac{\delta\psi}{\delta s} = \dfrac{1}{r}$.

Taking limit $Q \to P$, we obtain

$$\lim_{\delta s\to 0} \frac{\delta\psi}{\delta s} = \frac{1}{r}, \text{or} \quad \frac{d\psi}{ds} = \frac{1}{r}, \qquad ...(4.89)$$

which is a constant, being the reciprocal of its radius.

We observe that as the radius r increases, the curvature $1/r$ decreases. When r tends to infinity, the arc of the circle approximates a section of a straight line and the curvature tends to zero. Hence, *the curvature of a straight line is zero at any of its point.* The same is expected intuitively also, since a line has no bending from one point to another point.

4.9.2 Radius of Curvature

The reciprocal of the curvature of a curve at any point, in case it is non-zero, is called the radius of curvature at that point. Generally it is denoted by ρ, thus

$$\rho = \frac{ds}{d\psi}. \qquad ...(4.90)$$

We observe that the radius of curvature of a circle at any point is constant and is equal to its radius, refer to (4.89), and also the radius of curvature of a straight line is infinity.

The expression (4.90) for radius of curvature is suitable only for curves with equation of the form $s = \phi(\psi)$, called the *intrinsic form*. We transform (4.90) suitable to cartesian, parametric, and polar curves.

4.9.3 Radius of Curvature for Cartesian Curves

(a) *Explicit equation:* $y = f(x)$

Let ψ be the angle which the tangent at any point $P(x, y)$ makes with the positive direction of the x-axis. Then, $\tan\psi = \dfrac{dy}{dx}$.

Differentiating w.r.t. s, we get, $\sec^2\psi \dfrac{d\psi}{ds} = \dfrac{d^2y}{dx^2}\dfrac{dx}{ds}$, which gives

$$\frac{ds}{d\psi} = \frac{(1+\tan^2\psi)\dfrac{ds}{dx}}{\dfrac{d^2y}{dx^2}}$$

Also, $$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2},$$ refer to (4.22). Therefore,

$$\rho = \frac{ds}{d\psi} = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{(1+y_1^2)^{3/2}}{y_2} \qquad ...(4.91)$$

provided $y_2 \neq 0$.

The radius of curvature ρ is positive or negative according as $\dfrac{d^2y}{dx^2}$ is positive or negative, that is, as the curve is concave upwards or downwards, refer to Section 4.8.5. At a point of inflexion, that is, at a point where $\dfrac{d^2y}{dx^2} = 0$, curvature is zero.

Remark: In case the tangent to the curve at the point under consideration is parallel to y-axis, then dy/dx tends to infinity and thus (4.91) does not hold good. In this case $dx/dy = 0$, so we use the formula

$$\rho = \frac{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]^{3/2}}{\dfrac{d^2x}{dy^2}}, \text{provided } \frac{d^2x}{dy^2} \neq 0. \qquad \text{...(4.92)}$$

Since, the value of ρ is independent of the choice of the axes, (4.92) is obtained simply by interchanging x and y in (4.91).

(b) *Implicit equation:* $f(x, y) = 0$.

Here, $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$, provided $f_y \neq 0$, and $\dfrac{d^2y}{dx^2} = -\dfrac{f_{xx}(f_y)^2 - 2f_xf_yf_{xy} + f_{yy}(f_x)^2}{(f_y)^3}$

Substituting these values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (4.91) and simplifying we obtain

$$\rho = \frac{[(f_x)^2 + (f_y)^2]^{3/2}}{f_{xx}(f_y)^2 - 2f_xf_yf_{xy} + f_{yy}(f_x)^2} \qquad \text{...(4.93)}$$

(c) *Parametric equations:* $x = f(t),\ y = g(t)$

Here, $\dfrac{dy}{dx} = \dfrac{g'(t)}{f'(t)}$, provided $f'(t) \neq 0$, and $\dfrac{d^2y}{dx^2} = \dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^2}\dfrac{1}{f'(t)}$.

Substituting these values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (4.91), we get

$$\rho = \frac{\left[(f'(t))^2 + (g'(t))^2\right]}{f'(t)g''(t) - g'(t)f''(t)}. \qquad \text{...(4.94)}$$

Example 4.51: Find the radius of curvature at any point (s, ψ) on the curve

$$s = a \ln \cot(\pi/4 - \psi/2) + \frac{a \sin\psi}{\cos^2\psi}$$

Solution: The curve is

$$s = a \ln \cot(\pi/4 - \psi/2) + \frac{a \sin\psi}{\cos^2\psi}.$$

The radius of curvature is

$$\rho = \frac{ds}{d\psi} = a\frac{\frac{1}{2}\operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{\psi}{2}\right)}{\cot(\pi/4-\psi/2)} + a\frac{\cos^2\psi\cos\psi - \sin\psi(-2\sin\psi\cos\psi)}{\cos^4\psi}$$

$$= a\left[\frac{1}{\sin\left(\frac{\pi}{2}-\psi\right)} + \frac{\cos^2\psi + 2\sin^2\psi}{\cos^3\psi}\right] = a\left[\frac{1}{\cos\psi} + \frac{1+\sin^2\psi}{\cos^3\psi}\right] = 2a\sec^3\psi.$$

Example 4.52: Find the radius of curvature at any point (x, y) of the rectangular hyperbola $xy = c^2$.

Solution: The curve is $y = c^2/x$.

It gives, $\frac{dy}{dx} = -c^2/x^2$ and $\frac{d^2y}{dx^2} = \frac{2c^2}{x^3}$. Therefore,

$$\rho = \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{d^2y/dx^2}\right]^{3/2} = \frac{\left(1+\frac{c^4}{x^4}\right)^{3/2}}{2c^2/x^3} = \frac{(x^4+c^4)^{3/2}}{2x^3c^2} = \frac{(x^2+y^2)^{3/2}}{2c^2}.$$

Example 4.53: Find the radius of curvature at any point on the curve

$$x = c\ln\left[s+\sqrt{s^2+c}\right], \quad y = \sqrt{s^2+c}.$$

Solution: The curve is $x = c\ln\left[s+\sqrt{s^2+c}\right]$, $y = \sqrt{s^2+c}$.

Differentiating w.r.t. s, we obtain $\frac{dx}{ds} = \frac{c}{\sqrt{s^2+c^2}}$, $\frac{dy}{ds} = \frac{s}{\sqrt{s^2+c^2}}$.

Therefore, $\frac{dy}{dx} = \frac{dy}{ds}\Big/\frac{dx}{ds} = s/c$, which gives $\tan\psi = s/c$, or $s = c\tan\psi$,

where ψ is the angle which the tangent at $P(x, y)$ to the given curve makes with the positive direction of x-axis. Therefore, the radius of curvature

$$\rho = \frac{ds}{d\psi} = c\sec^2\psi = c(1+\tan^2\psi) = c\left(1+\frac{s^2}{c^2}\right) = \frac{s^2+c^2}{c}.$$

Example 4.54: If ρ is the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S is its focus, then show that ρ^2 varies as $(SP)^3$; also show that the radius of curvature at the vertex is equal to the length of the semi-latus rectum.

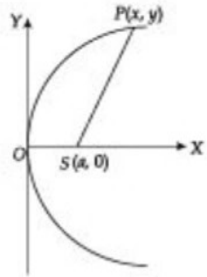


Fig. 4.16

Solution: The parabola is $y^2 = 4ax$ with $S(a, 0)$ as its focus and $P(x, y)$ any point on it as shown in Fig. 4.16.

We have, $2yy_1 = 4a$, or $y_1 = \frac{2a}{y}$. It gives $y_2 = -\frac{2a}{y^2}y_1 = -\frac{2a}{y^2}\frac{2a}{y} = -\frac{4a^2}{y^3}$.

The radius of curvature is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{4a^2}{y^2}\right)^{3/2}}{-\frac{4a^2}{y^3}} = -\frac{(y^2+4a^2)^{3/2}}{4a^2}$$

$$= \frac{-(4ax+4a^2)^{3/2}}{4a^2} = -\frac{2}{\sqrt{a}}(x+a)^{3/2}. \quad ...(4.95)$$

Also, $SP = \sqrt{(x-a)^2+y^2} = \sqrt{(x-a)^2+4ax} = |x+a|$. From (4.95), we have

$$\rho^2 = \frac{4}{a}(x+a)^3 = \frac{4}{a}(SP)^3,$$

which implies that $\rho^2 \alpha (SP)^3$.

Next, to find ρ at vertex (0, 0) we note that $y_1 \to \infty$ as $y \to 0$. Therefore, we apply (4.92) to find ρ, given by $\rho = \frac{\left[1+(dx/dy)^2\right]^{3/2}}{d^2x/dy^2} = \frac{1}{d^2x/dy^2}$.

Now $\frac{dx}{dy} = \frac{y}{2a}$ gives $\frac{d^2x}{dy^2} = \frac{1}{2a}$ and, therefore, $\rho = 2a$, the length of the semi-latus rectum of the parabola $y^2 = 4ax$.

Example 4.55: Find the radius of curvature at any point θ of the cycloid

$$x = a(\theta+\sin\theta), \quad y = a(1-\cos\theta).$$

Solution: The curve is $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$. It gives $\frac{dx}{d\theta} = a(1+\cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$.

Therefore, $\dfrac{dy}{dx} = \dfrac{a \sin \theta}{a(1 + \cos \theta)} = \dfrac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \theta/2} = \tan \theta/2.$

and, $$\frac{d^2y}{dx^2} = \frac{1}{2} \sec^2 \frac{\theta}{2} \frac{d\theta}{dx} = \frac{1}{2} \sec^2 \frac{\theta}{2} \frac{1}{2a \cos^2 \frac{\theta}{2}} = \frac{1}{4a} \frac{1}{\cos^4 \frac{\theta}{2}}$$

The radius of curvature $\rho = \dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \dfrac{\left(1 + \tan^2 \dfrac{\theta}{2}\right)^{3/2}}{\dfrac{1}{4a \cos^4 \dfrac{\theta}{2}}} = 4a \cos \dfrac{\theta}{2}.$

Example 4.56: Show that the curvature at the point $(3a/2, 3a/2)$ on the folium $x^3 + y^3 = 3axy$ is $-8\sqrt{2}/3a$.

Solution: The curve is $x^3 + y^3 = 3axy$.

Differentiating w.r.t. x and cancelling 3 from both sides, we obtain

$$x^2 + y^2 \frac{dy}{dx} = ay + ax \frac{dy}{dx} \qquad ...(4.96)$$

or, $$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}. \text{ Therefore, } \left[\frac{dy}{dx}\right]_{\left(\frac{3}{2}a, \frac{3}{2}a\right)} = -1.$$

Again differentiating (4.96) w.r.t. x, we get

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2 \frac{d^2y}{dx^2} = a\frac{dy}{dx} + a\frac{dy}{dx} + ax\frac{d^2y}{dx^2}$$

Substituting $x = y = \dfrac{3a}{2}$ and $\dfrac{dy}{dx} = -1$ in this, we get $\left[\dfrac{d^2y}{dx^2}\right]_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = -\dfrac{32}{3a}.$

Hence, the curvature at $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right) = \dfrac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} = \dfrac{-8\sqrt{2}}{3a}.$

Example 4.57: If ρ_1 and ρ_2 be the radii of curvature at the extremities of two conjugate semi diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, prove that $\left(\rho_1^{2/3} + \rho_2^{2/3}\right)(ab)^{2/3} = a^2 + b^2$.

Solution: Let P and Q be the extremities of the conjugate semi-diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, as shown in Fig. 4.17.

Now, if P is $(a\cos\theta, b\sin\theta)$, then Q is $\left[a\cos\left(\theta+\frac{\pi}{2}\right), b\sin\left(\theta+\frac{\pi}{2}\right)\right]$.

The parametric equations of the ellipse are $x = a\cos\theta$, $y = b\sin\theta$.

Thus, $$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = b\cos\theta$$

and, $$\frac{d^2x}{d\theta^2} = -a\cos\theta, \quad \frac{d^2y}{d\theta^2} = -b\sin\theta.$$

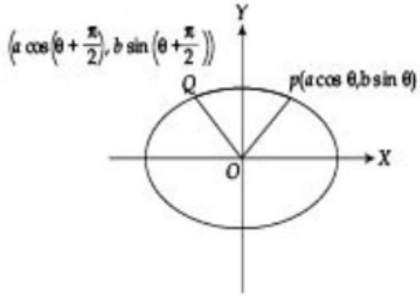


Fig. 4.17

Therefore, $$\rho \text{ at } \theta = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab\sin^2\theta + ab\cos^2\theta}$$

That is, $$\rho_1 = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}. \qquad ...(4.97)$$

The ρ at $\left(\theta + \frac{\pi}{2}\right)$, that is, ρ_2 is obtained by replacing θ with $\theta + \frac{\pi}{2}$ in (4.97) and, therefore,

$$\rho_2 = \frac{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}{ab} \qquad ...(4.98)$$

From (4.97) and (4.98), we have

$$\rho_1^{2/3} + \rho_2^{2/3} = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{(ab)^{2/3}} + \frac{a^2\cos^2\theta + b^2\sin^2\theta}{(ab)^{2/3}} = \frac{a^2+b^2}{(ab)^{2/3}}$$

or, $(\rho_1^{2/3} + \rho_2^{2/3})(ab)^{2/3} = a^2 + b^2$.

EXERCISE 4.8

1. Find the radius of curvature at any point of the following curves

   (i) $s = 4a\sin\psi$ (ii) $s = 4a\sin\frac{1}{3}\psi$

   (iii) $s = a\ln(\tan\psi + \sec\psi) + a\tan\psi\sec\psi$

2. Find the radius of curvature for the curves

 (i) $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $\left(\frac{a}{4},\frac{a}{4}\right)$ (ii) $y=4\sin x-\sin 2x$ at $x=\frac{\pi}{2}$

 (iii) $y^2=\frac{a^2(a-x)}{x}$ at $(a, 0)$

3. Show that for the parabola $(y-y_1)^2=4a(x-x_1)$, ρ^2 varies as $(SP)^3$, where ρ is the radius of curvature at any point P of the parabola, S is the focus of the parabola and (x_1, y_1) are the co-ordinates of vertex of the parabola.
4. Find the radius of curvature of the curve $y=e^x$ at the point where it crosses the y-axis.
5. Prove that the radius of curvature at any point of the asteroid $x^{2/3}+y^{2/3}=a^{2/3}$ is three times the length of the perpendicular from the origin to the tangent at that point.
6. If ρ_1 and ρ_2 be the radii of curvature at the extremities of a focal chord of a parabola whose latus rectum is $2l$, then prove that $\rho_1^{-2/3}+\rho_2^{-2/3}=l^{-2/3}$.
7. Show that for the curve $x=a\cos\theta(1+\cos\theta)$, $y=a\sin\theta(1+\cos\theta)$, the radius of curvature at the point $\theta=-\pi/4$ is a.
8. Show that the radius of curvature of the curve given by $x^2y=a\left(x^2+a^2/\sqrt{5}\right)$ is least for the point $x=a$ and its least value is $9a/10$.
9. Show that $3\sqrt{3}/2$ is the least value of $|\rho|$ for $y=\ln x$.
10. Find the radius of curvature at any point for the following curves:
 (i) $x=a\cos^3\theta,\ \ y=a\sin^3\theta$ (ii) $x=3a\cos t-a\cos 3t$, $y=3a\sin t-a\sin 3t$
 (iii) $x=a\sin 2\theta(1+\cos 2\theta)$, $y=a\cos^2\theta(1-\cos 2\theta)$
11. Find the radius of curvature at the origin of the two branches of the curve given by $x=1-t^2, y=t-t^3$.
12. Show that the radius of curvature at an end of the major axis of the ellipse $x^2/a^2+y^2/b^2=1$ is equal to the semi-latus rectum of the ellipse.
13. Prove that the radius of curvature at the point $(-2a, 2a)$ on the curve $x^2y=a(x^2+y^2)$ is, $-2a$.
14. For the curve $y=ax/(a+x)$, if ρ is the radius of curvature at any point (x, y), then show that $(2\rho/a)^{2/3}=(x/y)^2+(y/x)^2$.
15. Show that the radius of curvature at any point of the curve

 $x=a\left[\ln\tan\frac{\theta}{2}+\cos\theta\right]$, $y=a\sin\theta$,

 where θ is the parameter and a is a constant, varies inversely as the length of the normal intercepted between the point on the curve and the x-axis.
16. Show that in the curve, $x=\frac{3}{2}a(\sinh t\cosh t+t)$, $y=a\cosh^2 t$, if the normal at $P(x, y)$ meets the axis of x in Q, the radius of curvature at P is equal to $3PQ$.

4.9.4 Radius of Curvature at (0, 0). Newtonian Method

If a curve passes through the origin and the axis of x is tangent there, then

$$\rho \text{ at } (0,0) = \lim_{x \to 0} \left(\frac{x^2}{2y} \right). \qquad ...(4.99)$$

Since x-axis is tangent at (0, 0) so $\frac{dy}{dx}$ at (0, 0) is zero, thus from (4.91), $\rho = 1/y_2$ at $(0, 0)$.

Consider $\lim_{x \to 0} \left(\frac{x^2}{2y} \right) = \lim_{x \to 0} \left(\frac{2x}{2y_1} \right) = \lim_{x \to 0} \left(\frac{1}{y_2} \right) = 1/y_2$ at $(0, 0)$. This proves (4.99).

Similarly, *if a curve passes through the origin and the axis of y is tangent there, then*

$$\rho \text{ at } (0,0) = \lim_{y \to 0} \left(\frac{y^2}{2x} \right) \qquad ...(4.100)$$

When the curve passes through the origin and neither of the axes is tangent there, then the radius of curvature at $(0, 0)$ can be found by the method of expansion as follows.

Let the curve $y = f(x)$ pass through $(0, 0)$. Then $f(0) = 0$, and by Maclaurin's expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + ... = px + \frac{1}{2} qx^2 + ...$$

where $p = f'(0)$ and $q = f''(0)$.

Thus, $$\rho \text{ at } (0,0) = \frac{(1+p^2)^{3/2}}{q}. \qquad ...(4.101)$$

Remarks:

1. To find p and q in numerical problems, we substitute $y = px + \frac{1}{2} qx^2 + ...$ in the given equation and equate the coefficients of like powers of x on the both sides.
2. The equation (*s*) of the tangent at the origin is found by equating to zero the terms of the lowest degree in the equation of the curve.

Example 4.58: Find the radius of curvature at the origin for the following curves

(a) $x^3 + y^3 - 2x^2 + 6y = 0$

(b) $2x^4 + 4x^3y + xy^2 + 6y^3 - 3x^2 - 2xy + y^2 - 4x = 0$.

Solution: (a) The curve $x^3 + y^3 - 2x^2 + 6y = 0$ passes through (0, 0) and the equation of the tangent there is $y = 0$, the x-axis. Hence the radius of curvature ρ at (0, 0) is $\lim_{x \to 0} (x^2/2y)$.

Dividing each term in the equation by $2y$, we obtain $x\left(\frac{x^2}{2y}\right)+\frac{1}{2}y^2-2\left(\frac{x^2}{2y}\right)+3=0.$

Taking limit as $x\to 0$, (also $y\to 0$) and using $\rho=\lim_{x\to 0}(x^2/2y)$, we obtain $\rho = 3/2$.

(b) The curve $2x^4+4x^3y+xy^2+6y^3-3x^2-2xy+y^2-4x=0$ passes through (0, 0), and the equation of the tangent at origin is $x=0$, the y-axis. Hence, the radius of curvature ρ at (0, 0) is $\lim_{y\to 0}(y^2/2x)$.

Dividing each term in the equation by $2x$, we obtain

$$x^3+2x^2y+\frac{y^2}{2}+6y\left(\frac{y^2}{2x}\right)-\frac{3}{2}x-y+\frac{y^2}{2x}-2=0.$$

Taking limit as $x\to 0$, (also $y\to 0$) and using $\rho=\lim(y^2/2x)$, we obtain $\rho=2$.

Example 4.59: Show that the radii of curvature at the origin for the curve $x^3+y^3=3axy$ are each equal to $3a/2$.

Solution: The curve $x^3+y^3=3axy$ passes through (0, 0) and the tangents at the origin are obtained by equating to zero the lowest degree term, $xy=0$. It gives $x=0$, $y=0$. Thus, x-axis and y-axis both are tangents at (0, 0).

Therefore, ρ at (0, 0) is $\lim_{x\to 0}\frac{x^2}{2y}=\rho_1$ (say), or $\lim_{y\to 0}\frac{y^2}{2x}=\rho_2$ (say).

Dividing each term in the equation by $2xy$, we have

$$\frac{x^2}{2y}+\frac{y^2}{2x}=\frac{3a}{2}\ \text{or,}\ \frac{x^2}{2y}+\frac{1}{4}xy\frac{2y}{x^2}=\frac{3a}{2}.$$

Taking limit as $x\to 0$ and $y\to 0$, we get $\rho_1+0\frac{1}{\rho_1}=\frac{3a}{2}$, or $\rho_1=\frac{3a}{2}$.

Similarly, we can show that $\rho_2=\frac{3a}{2}$.

Example 4.60: Find the radii of curvature at the origin for the curve

$$y^2-3xy-4x^2+x^3+x^4y+y^3=0.$$

Solution: The curve passes through the origin but clearly neither of the axes is tangent at (0, 0).

To find the curvature at (0, 0) put $y=px+\frac{1}{2!}qx^2+\ldots$ in the given equation, we obtain

$$\left(px+\frac{1}{2!}qx^2+\ldots\right)^2-3x\left(px+\frac{1}{2!}qx^2+\ldots\right)-4x^2+x^3+x^4\left(px+\frac{1}{2!}qx^2+\ldots\right)+\left(px+\frac{1}{2!}qx^2+\ldots\right)^3=0$$

Simplifying, we obtain

$$(p^2-3p-4)x^2+\left(pq-\frac{3}{2}q+1\right)x^3+\ldots=0. \quad \ldots(4.102)$$

Equating to zero the coefficients of x^2 and x^3 on both sides of (4.102), we obtain respectively

$$p^2-3p-4=0, \text{ and } pq-\frac{3}{2}q+1=0$$

Solving these equations for p and q, we get $p=-1$, $q=2/5$, and $p=4$, $q=-2/5$ as the two solutions.

Also the radius of curvature ρ at (0, 0) is $\rho=\dfrac{(1+p^2)^{3/2}}{q}$, refer to (4.101).

For, $p=1$, $q=2/5$, $\rho=5/\sqrt{2}$, and for, $p=4$, $q=-2/5$, $\rho=\dfrac{-85\sqrt{17}}{2}$.

Thus, radii of curvature at origin are $5\sqrt{2}$ and $\dfrac{-85\sqrt{17}}{2}$.

Example 4.61: Obtain the radii of curvature for the curve $a(y^2-x^2)=x^3$ at the origin.

Solution: The given curve is $a(y^2-x^2)=x^3$. It gives

$$y=\pm x\left(1+\frac{x}{a}\right)^{\frac{1}{2}}=\pm x\left[1+\frac{1}{2}\left(\frac{x}{a}\right)+\ldots\right]=\pm\left[x+\frac{1}{2a}x^2+\ldots\right],$$

which is of the form $y=px+\dfrac{1}{2!}qx^2+\ldots$. Thus, we get $p=\pm 1$, $q=\pm\dfrac{1}{a}$

Hence the radius of curvature ρ at (0, 0) for $p=1$, $q=\dfrac{1}{a}$ is $\rho_1=\dfrac{(1+p^2)^{3/2}}{q}=2\sqrt{2}a$;

and for $p=-1$, $q=-\dfrac{1}{a}$ it is $\rho_2=\dfrac{(1+p^2)^{3/2}}{q}=-2\sqrt{2}a$.

Thus the radii of curvature at (0, 0) are $\pm 2\sqrt{2}a$.

EXERCISE 4.9

1. Find the radius of curvature at the origin for the curves

   (i) $y^4+x^3+a(x^2+y^2)-a^2y=0$ (ii) $2x^4+3y^4+4x^2y+xy-y^2+2x=0$

2. Find the radius of curvature at the origin for the curves

   (i) $y-x=x^2+2xy+y^2$ (ii) $y=6x+5x^2+x^3$

3. Find the radius of curvature at the origin for the curve $x = t - \frac{1}{3}t^3, y = t^2$.

4. Show that the radii of curvature of the curve $y^2 = x^2(a+x)/(a-x)$ at the origin are $\pm a\sqrt{2}$.

5. Find the radius of curvature at the origin for the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ using Newtonian method.

4.9.5 Radius of Curvature for Polar Curves: $r = f(\theta)$

To find the radius of curvature for polar curves we need to express $ds/d\psi$ in terms of r and its derivatives with respect to θ.

Let the tangent at any point $P(r, \theta)$ to the curve $r = f(\theta)$ make an angle ψ with the initial line OX and angle ϕ with the radius vector OP as shown in Fig. 4.18.

We have, $\psi = \theta + \phi$ and, therefore,

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\theta}{ds} + \frac{d\phi}{d\theta}\frac{d\theta}{ds} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right). \quad \text{...(4.103)}$$

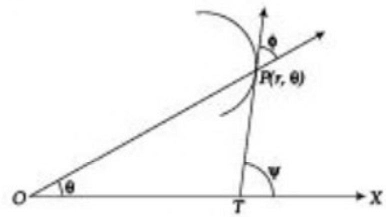


Fig. 4.18

Also, $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}}$, refer to (4.32a). Differentiating w.r.t. θ, we have

$$\sec^2\phi\frac{d\phi}{d\theta} = \frac{\frac{dr}{d\theta}\cdot\frac{dr}{d\theta} - r\cdot\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2} \text{ or, } \frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{(1+\tan^2\phi)\left(\frac{dr}{d\theta}\right)^2} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}. \quad \text{...(4.104)}$$

For the curve $r = f(\theta)$, refer to (4.29), we have

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}. \quad \text{...(4.105)}$$

From (4.103), (4.104) and (4.105), we obtain

$$\frac{d\psi}{ds} = \frac{1}{\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}}\left[1 + \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}\right] = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]^{3/2}} = \frac{r^2 + 2r_1^2 - rr_2}{(r^2 + r_1^2)^{3/2}}.$$

Therefore the radius of curvature ρ is given by

$$\rho = \frac{ds}{d\psi} = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}, \quad \text{...(4.106)}$$

where $r_1 = \dfrac{dr}{d\theta}$ and $r_2 = \dfrac{d^2r}{d\theta^2}$.

Remarks: **1.** In case the curve is of the form $u = f(\theta)$, where $u = 1/r$, then $r_1 = -u_1/u^2$, and $r_2 = (2u_1^2 - uu_2)/u^3$. Substituting in (4.106) and simplifying, we obtain

$$\rho = \frac{\left(u^2 + u_1^2\right)^{3/2}}{u^3\left(u + u_2\right)} \quad \text{...(4.107a)}$$

2. In case the initial line is tangent at the pole, then the radius of curvature ρ at (0, 0) is

$$\rho = \frac{1}{2} \lim_{\substack{r \to 0 \\ \theta \to 0}} \left(\frac{dr}{d\theta}\right).$$

For ρ at (0, 0) $= \lim\limits_{\substack{x \to 0 \\ y \to 0}} \left(\dfrac{x^2}{2y}\right) = \lim\limits_{\substack{r \to 0 \\ \theta \to 0}} \dfrac{r^2 \cos^2\theta}{2r \sin\theta} = \lim\limits_{\substack{r \to 0 \\ \theta \to 0}} \left[\dfrac{r}{2\theta} \cdot \dfrac{\theta}{\sin\theta} \cos^2\theta\right]$

$$= \frac{1}{2} \lim_{\substack{r \to 0 \\ \theta \to 0}} \left[\frac{r}{\theta}\right] = \frac{1}{2} \lim_{\substack{r \to 0 \\ \theta \to 0}} \left(\frac{dr}{d\theta}\right) \quad \text{...(4.107b)}$$

4.9.6 Radius of Curvature for Pedal Curves: $p = f(r)$

Let the tangent at any point P to the curve make an angle ψ with the initial line OX and let $OM = p$ be the length of the perpendicular from the origin to the tangent line to the curve at P, as shown in Fig. 4.19. Then (p, r) are the pedal co-ordinates of the point P.

From Fig. 4.19, $\psi = \theta + \phi$. It gives

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}. \quad \text{...(4.108)}$$

Also from Fig. 4.19, $p = r\sin\phi$. Differentiating this w.r.t. r, we have

$$\frac{dp}{dr} = \sin\phi + r\cos\phi \frac{d\phi}{dr}$$

$$= \sin\phi + r\cos\phi \frac{d\phi}{ds} \frac{ds}{dr} \quad \text{...(4.109)}$$

Further $\sin\phi = r\dfrac{d\theta}{ds}$, and $\cos\phi = \dfrac{dr}{ds}$, refer to (4.32b)

Using in (4.109), we have

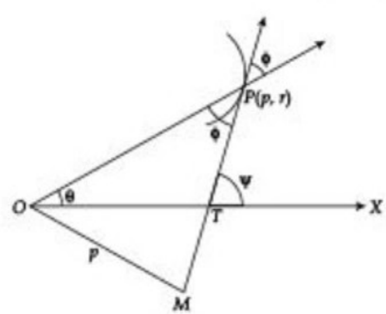


Fig. 4.19

$$\frac{dp}{dr} = r\frac{d\theta}{ds} + r\frac{dr}{ds}\frac{ds}{dr}\frac{d\phi}{ds}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right) = r\frac{d\psi}{ds}, \quad \text{using (4.108)}$$

which gives $$\frac{ds}{d\psi} = r\frac{dr}{dp},$$

or, $$\rho = r\frac{dr}{dp} \qquad ...(4.110)$$

as the desired result.

Remark: A relation between p and ψ, holding for every point of a curve, is called the *tangential polar equation*, and for the tangential polar curve $p = f(\psi)$ the radius of curvature is given by $\rho = p + \dfrac{d^2p}{d\psi^2}$.

Example 4.62: Find the radius of curvature at the point (r, θ) of the curve $r = a(1 - \cos\theta)$ and show that ρ^2 varies as r.

Solution: The equation is $r = a(1 - \cos\theta)$, hence $r_1 = a\sin\theta$ and $r_2 = a\cos\theta$. Therefore

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2} = \frac{(a^2(1-\cos\theta)^2 + a^2\sin^2\theta)^{3/2}}{2a^2\sin^2\theta + a^2(1-\cos\theta)^2 - a^2(1-\cos\theta)\cos\theta}$$

$$= \frac{a[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]^{3/2}}{2\sin^2\theta + 1 + 2\cos^2\theta - 3\cos\theta}$$

$$= \frac{a[2(1-\cos\theta)]^{3/2}}{3(1-\cos\theta)} = \frac{2a\sqrt{2}}{3}\sqrt{1-\cos\theta} = \frac{4a}{3}\sin\frac{\theta}{2}.$$

It gives, $$\rho^2 = \frac{16a^2}{9}\sin^2\theta/2 = \frac{8a}{9}[a(1-\cos\theta)] = \frac{8a}{9}r,$$

which implies that ρ^2 varies as r.

Example 4.63: For the curve $r^n = a^n\cos n\theta$, prove that radius of curvature is $\dfrac{a^n}{(n+1)r^{n-1}}$.

Solution: The curve is $r^n = a^n\cos n\theta$. Taking logarithm, we obtain

$$n\ln r = n\ln a + \ln\cos n\theta$$

Differentiating w.r.t. θ, $\dfrac{n}{r}\dfrac{dr}{d\theta} = -\dfrac{n\sin n\theta}{\cos n\theta}$. It gives $r_1 = -r\tan n\theta$.

Also, $r_2 = \dfrac{d^2 r}{d\theta^2} = -rn\sec^2 n\theta - \tan n\theta \dfrac{dr}{d\theta} = -rn\sec^2 n\theta + r\tan^2 n\theta.$

Thus, $$\rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2} = \frac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{2r^2\tan^2 n\theta + r^2 + nr^2\sec^2 n\theta - r^2\tan^2 n\theta}$$

$$= \frac{r\sec^3 n\theta}{\tan^2 n\theta + 1 + n\sec^2 n\theta} = \frac{r\sec^3 n\theta}{(n+1)\sec^2 n\theta}$$

$$= \frac{r\sec n\theta}{(n+1)} = \frac{r^n \sec n\theta}{(n+1)r^{n-1}} = \frac{a^n\cos n\theta \,.\sec n\theta}{(n+1)r^{n-1}} = \frac{a^n}{(n+1)r^{n-1}}.$$

Example 4.64: Show that at the point of intersection of the curves $r = a\theta$ and $r\theta = a$, the curvatures are in the ratio 3:1, $(0 < \theta < 2\pi)$.

Solution: The points of intersection of the curves $r = a\theta$ and $r\theta = a$ are given by $a\theta^2 = a$, or $\theta = \pm 1$.

For the curve $r = a\theta$, we have $r_1 = a, r_2 = 0$. If ρ_1 is the radius of curvature for this curve at $\theta = \pm 1$, then

$$\rho_1 = \left[\frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2}\right] = \frac{(a^2\theta^2 + a^2)^{3/2}}{2a^2 + a^2\theta^2} = \frac{a(1+\theta^2)^{3/2}}{2+\theta^2} = \frac{a\left(2\sqrt{2}\right)}{3}.$$

Next, for the curve $r\theta = a$, we have, $r_1 = -a/\theta^2$ and $r_2 = 2a/\theta^3$. If ρ_2 is the radius of curvature for this curve at $\theta = \pm 1$, then

$$\rho_2 = \left[\frac{\left(\dfrac{a^2}{\theta^2} + \dfrac{a^2}{\theta^4}\right)^{3/2}}{2\dfrac{a^2}{\theta^4} + \dfrac{a^2}{\theta^2} - \dfrac{2a^2}{\theta^4}}\right] = \frac{a(1+\theta^2)^{3/2}}{\theta^4} = 2a\sqrt{2}.$$

Thus, $$\frac{\rho_2}{\rho_1} = \frac{3}{1} \text{ or, } \rho_2 : \rho_1 = 3:1.$$

Example 4.65: Find the radius of curvature at the point (p, r) on the ellipse $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2b^2}$.

Solution: The given curve is $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2b^2}$. Differentiating w.r.t. r, we have

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2r}{a^2b^2}. \text{ It gives } r\frac{dr}{dp} = \frac{a^2b^2}{p^3}, \text{ or } \rho = \frac{a^2b^2}{p^3}$$

as the required radius of curvature.

Example 4.66: Find the radius of curvature of the curve $p = a \sin \psi \cos \psi$.

Solution: The curve is, $p = a \sin \psi \cos \psi = \frac{a}{2} \sin 2\psi$. Differentiating we obtain

$$\frac{dp}{d\psi} = a \cos 2\psi, \quad \frac{d^2p}{d\psi^2} = -2a \sin 2\psi.$$

Thus the radius of curvature is

$$\rho = p + \frac{d^2p}{d\psi^2} = \frac{a}{2} \sin 2\psi - 2a \sin 2\psi = -\frac{3a}{2} \sin 2\psi = 3p \text{ (numerically)}.$$

Example 4.67: Find the radius of curvature of the curve $r = a \sin n\theta$ at the pole.

Solution: Equation of the curve is $r = a \sin n\theta$. This gives $\frac{dr}{d\theta} = na \cos n\theta$.

Therefore, $$\tan \phi = r \frac{d\theta}{dr} = \frac{a \sin n\theta}{na \cos n\theta} = \frac{1}{n} \tan n\theta.$$

Further when $\theta = 0$, we have $r = 0$ and also $\tan \phi = 0$, which gives, $\phi = 0$.

Thus, the curve passes through the pole and also initial line is the tangent to the curve at pole.

By Newton's method, ρ at pole $= \lim\limits_{\substack{r \to 0 \\ \theta \to 0}} \left(\frac{1}{2} \frac{dr}{d\theta} \right)$, refer to (4.107)

$$= \lim_{\substack{r \to 0 \\ \theta \to 0}} \left(\frac{1}{2} . na \cos n\theta \right) = \frac{na}{2}.$$

Example 4.68: Find the radius of curvature at any point (r, θ) on the curve $\frac{l}{r} = 1 + e \cos \theta$.

Solution: The curve is $\frac{l}{r} = 1 + e \cos \theta$, or $u = \frac{1}{l}(1 + e \cos \theta)$, where $u = \frac{1}{r}$.

It gives $u_1 = \frac{du}{d\theta} = \frac{-e \sin \theta}{l}$, $\quad u_2 = \frac{d^2u}{d\theta^2} = \frac{-e \cos \theta}{l}$

Thus, the radius of curvature, refer to (4.107a), is

$$\rho = \frac{\left(u^2 + u_1^2\right)^{3/2}}{u^3(u + u_2)} = \frac{\left[\frac{(1 + e \cos \theta)^2}{l^2} + \frac{e^2 \sin^2 \theta}{l^2}\right]^{3/2}}{\left(\frac{1 + e \cos \theta}{l}\right)^3 \left[\frac{1 + e \cos \theta}{l} - \frac{e \cos \theta}{l}\right]}$$

$$= \frac{(1+e^2+2e\cos\theta)^{3/2}}{l^3} \frac{l^4}{(1+e\cos\theta)^3} = \frac{l(1+e^2+2e\cos\theta)^{3/2}}{(1+e\cos\theta)^3}.$$

EXERCISE 4.10

1. Find the radius of curvature at the point (r, θ) of the following curves:

   (i) $\sqrt{r}\cos\left(\frac{\theta}{2}\right) = \sqrt{a}$ (ii) $r^2 = a^2 \cos 2\theta$

2. Find the radius of curvature at any point (r, θ) of the following curves:

   (i) $\frac{2a}{r} = 1 + \cos\theta$ (ii) $r = ae^{\theta \cot\alpha}$

3. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a(1 + \cos\theta)$ which passes through the pole, then show that

   $$\rho_1^2 + \rho_2^2 = 16a^2/9.$$

4. Show that for the curve $r = a(1 + \cos\theta)$, the radius of curvature $\rho = \frac{4a}{3}\cos\frac{\theta}{2}$. Also find the radius of curvature where the tangent is parallel to the initial line.
5. Prove that in the curve $r^2 = a^2 \sin 2\theta$ the curvature varies as the radius vector.
6. A line is drawn through the origin meeting the cardioids $r = a(1 - \cos\theta)$ in the points P, Q and the normals at P, Q meet in C. Show that the radii of curvature at P and Q are proportional to PC and QC.
7. Find the radius of curvature at the pole to the curve $r = a(1 - \cos\theta)$ by Newtonian method.
8. Find the radius of curvature at any point (p, r) on the following curves:

   (i) $p^2 = ar$ (ii) $2ap^2 = r^3$

   (iii) $pa^2 = r^3$ (iv) $p^2 + a^2 = r^2$

   (v) $p^2(a^2 + b^2 - r^2) = a^2b^2$ (vi) $\frac{1}{p^2} = \frac{A}{r^2} + B.$

9. Find the radius of curvature for the ellipse $p^2 = a^2\cos^2\psi + b^2\sin^2\psi$.
10. For any curve $r = f(\theta)$, prove that $\frac{r}{\rho} = \sin\phi\left(1 + \frac{d\phi}{d\theta}\right)$.

4.9.7 Centre of Curvature. Circle of Curvature.

The centre of curvature for any point P of a curve is the point which lies on the positive direction of the normal at P and is at a distance ρ, the radius of curvature at P, from it.

The positive direction of the normal is obtained by rotating the positive direction of the tangent through $\pi/2$ in the anti-clockwise direction, where the positive direction of the tangent to $y = f(x)$ is the one in which x increases.

Co-ordinates of the centre of curvature: Let $C(X, Y)$ be the centre of curvature corresponding to any point $P(x, y)$ on the curve and ρ be the radius of curvature at the point P, then $PC = \rho$, as shown in Fig. 4.20.

Let the tangent PT make an angle ψ with the positive direction of the x-axis. Draw PL and CM perpendicular on x-axis and draw PN perpendicular on CM.

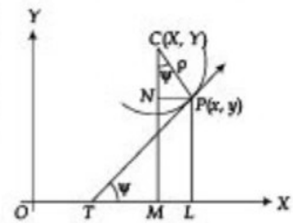


Fig. 4.20

Consider $\angle NCP = \dfrac{\pi}{2} - \angle NPC$

$$= \frac{\pi}{2} - \left(\frac{\pi}{2} - \angle XTP\right) = \angle XTP = \psi$$

Thus, $X = OM = OL - ML = OL - NP$

$$= OL - PC \sin\psi = x - \rho \sin\psi. \quad ...(4.111)$$

$$Y = MC = MN + NC = LP + NC$$

$$= LP + PC \cos\psi = y + \rho\cos\psi. \quad ...(4.112)$$

Further, $\tan\psi = \dfrac{dy}{dx} = y_1$ gives $\sin\psi = \dfrac{y_1}{\sqrt{1 + y_1^2}}$ and $\cos\psi = \dfrac{1}{\sqrt{1 + y_1^2}}$.

Also, $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}$.

Using these in (4.111) and (4.112), we obtain

$$X = x - \frac{y_1(1 + y_1^2)}{y_2}, \quad Y = y + \frac{(1 + y_1^2)}{y_2} \quad ...(4.113)$$

as the co-ordinates of the centre of curvature.

Circle of curvature: *The circle of curvature of any point P of a curve is the circle whose centre is at the centre of curvature and whose radius is ρ, the radius of curvature at P.*

If ρ is the radius of curvature and (X, Y) the centre of curvature at the given point P, then the equation of the circle of curvature of P is given by

$$(x - X)^2 + (y - Y)^2 = \rho^2. \quad ...(4.114)$$

Obviously the circle of curvature will touch the curve at P and its curvature will be the same as that of the curve at the point P.

4.9.8 Evolute

The locus of the centres of curvature of a curve is called its evolute and the curve is said to be involute of its evolute.

To find evolute of a curve we eliminate the parameters x and y between

$$X = x - \frac{y_1(1 + y_1^2)}{y_2} \quad \text{and} \quad Y = y + \frac{1 + y_1^2}{y_2},$$

the co-ordinates of the centre of curvature, and then the required equation of the evolute is obtained by changing $X \to x$ and $Y \to y$ in the relation obtained between X and Y.

Example 4.69: Find the centre of curvature of the parabola $x = at^2, y = 2at$ at the point t and hence find its evolute.

Solution: Curve is $x = at^2$, $y = 2at$. Therefore,

$$y_1 = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t} \text{ and, } y_2 = \frac{d}{dx}\left(\frac{1}{t}\right) = -\frac{1}{t^2}\frac{dt}{dx} = -\frac{1}{t^2}\frac{1}{2at} = -\frac{1}{2at^3}.$$

Thus the co-ordinates of the centre of curvature (X, Y) are

$$X = at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}} = at^2 + 2a(1+t^2) = a(2+3t^2)$$

$$Y = 2at + \frac{\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}} = 2at - 2at(1+t^2) = -2at^3$$

Eliminating t between X and Y, we obtain

$$(X-2a)^3 = (3a)^3\left(-\frac{Y}{2a}\right)^2 \quad \text{or,} \quad 4(X-2a)^3 = 27a\,Y^2.$$

Changing X to x and Y to y, the required equation of the evolute is $4(X-2a)^3 = 27a\,Y^2$.

Example 4.70: Show that the evolute of the tractrix $x = c\cos t + c\ln\tan\frac{t}{2}, y = c\sin t$ is the catenary $y = c\cosh\frac{x}{c}$.

Solution: Equation of the curve is $x = c\cos t + c\ln\tan\frac{t}{2}$, $y = c\sin t$.

It gives $\dfrac{dx}{dt} = -c\sin t + \dfrac{c}{\tan\frac{t}{2}}\dfrac{1}{2}\sec^2\dfrac{t}{2} = \dfrac{c\cos^2 t}{\sin t}$, $\dfrac{dy}{dt} = c\cos t$.

Therefore, $$y_1 = \frac{dy}{dt}\Big/\frac{dx}{dt} = c\cos t\,\frac{\sin t}{c\cos^2 t} = \tan t.$$

$$y_2 = \frac{d}{dx}(\tan t) = \sec^2 t\,\frac{dt}{dx} = \sec^2 t\,\frac{\sin t}{c\cos^2 t} = \frac{\sin t}{c\cos^4 t}.$$

If (X, Y) is the centre of curvature at any point on the curve, then

$$X = x - \frac{y_1(1+y_1^2)}{y_2} = c\cos t + c\ln\tan\frac{t}{2} - \frac{\tan t(1+\tan^2 t)}{\dfrac{\sin t}{c\cos^4 t}}$$

$$= c\cos t + c\ln\tan\frac{t}{2} - c\cos t = c\ln\tan\frac{t}{2} \qquad ...(4.115)$$

$$Y = y + \frac{1+y_1^2}{y_2} = c\sin t + \frac{1+\tan^2 t}{\dfrac{\sin t}{c\cos^4 t}} = c\sin t + \frac{c\cos^2 t}{\sin t} = \frac{c}{\sin t}. \qquad ...(4.116)$$

For evolute we eliminate t between X and Y.

From (4.115) $\quad \ln\tan\frac{t}{2} = \frac{X}{c}$, or $\quad \tan\frac{t}{2} = e^{\frac{X}{c}} \qquad ...(4.117)$

From (4.116) $\frac{Y}{c} = \frac{1}{\sin t} = \frac{1+\tan^2\frac{t}{2}}{2\tan\frac{t}{2}} = \frac{1}{2}\left[\frac{1}{\tan\frac{t}{2}} + \tan\frac{t}{2}\right] = \frac{1}{2}\left[e^{-\frac{X}{c}} + e^{\frac{X}{c}}\right]$, using (4.117)

$$= \cosh\frac{X}{c}$$

or,
$$Y = c\cosh\frac{X}{c}.$$

Changing X to x and Y to y, we obtain, $y = c\cosh\frac{x}{c}$, as the equation of the evolute.

Example 4.71: Find the circle of curvature at the point $t = \frac{\pi}{2}$ on the ellipse $x = a\cos t;\ y = b\sin t$.

Solution: The curve is $x = a\cos t$, $y = b\sin t$. It gives $\frac{dx}{dt} = -a\sin t$, $\frac{dy}{dt} = b\cos t$. Therefore,

$$y_1 = \frac{dy}{dt}\Big/\frac{dx}{dt} = -\frac{b\cos t}{a\sin t} = -\frac{b}{a}\cot t.$$

$$y_2 = \frac{d}{dx}\left(-\frac{b}{a}\cot t\right) = \frac{b}{a}\operatorname{cosec}^2 t\,\frac{dt}{dx} = \frac{b}{a}\operatorname{cosec}^2 t\left(-\frac{\operatorname{cosec} t}{a}\right) = -\frac{b}{a^2}\operatorname{cosec}^3 t.$$

At $\quad t = \pi/2;\ (x, y) = (0, b),\ y_1 = 0,\ y_2 = -b/a^2$, therefore

$$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = -\frac{a^2}{b}$$

and, the centre of curvature (X, Y) is $\left(x-\frac{y_1(1+y_1^2)}{y_2},\ y+\frac{1+y_1^2}{y_2}\right)=\left(0, \frac{b^2-a^2}{b}\right)$.

Hence the circle of curvature at $t=\pi/2$ is $x^2+\left(y-\frac{b^2-a^2}{b}\right)^2=\frac{a^4}{b^2}$.

Example 4.72: Find the equation of the evolute of the curve $x^{2/3}+y^{2/3}=a^{2/3}$.

Solution: The parametric equations of the curve are $x=a\cos^3\theta,\quad y=a\sin^3\theta$.

We obtain, $$\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta,\quad \frac{dy}{d\theta}=3a\sin^2\theta\cos\theta.$$

Therefore, $$\frac{dy}{dx}=\frac{dy}{d\theta}\bigg/\frac{dx}{d\theta}=-\frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta}=-\tan\theta.$$

$$\frac{d^2y}{dx^2}=-\frac{d}{dx}(\tan\theta)=-\sec^2\theta\frac{d\theta}{dx}=-\frac{1}{\cos^2\theta}\frac{1}{(-3a\cos^2\sin\theta)}=\frac{1}{3a\sin\theta\cos^4\theta}.$$

If (X, Y) is the centre of curvature at point θ, then

$$X=x-\frac{y_1(1+y_1^2)}{y_2}=a\cos^3\theta-\frac{-\tan\theta(1+\tan^2\theta)}{\dfrac{1}{3a\sin\theta\cos^4\theta}}=a\cos\theta(\cos^2\theta+3\sin^2\theta)$$

and $$Y=y+\frac{1+y_1^2}{y_2}=a\sin^3\theta+\frac{1+\tan^2\theta}{\dfrac{1}{3a\sin\theta\cos^4\theta}}=a\sin\theta(\sin^2\theta+3\cos^2\theta).$$

To find the equation of the evolute, we eliminate θ between X and Y.

We have, $X+Y=a(\cos^3\theta+\sin^3\theta)+3a\sin\theta\cos\theta(\sin\theta+\cos\theta)=a[\cos\theta+\sin\theta]^3$

or, $$(X+Y)^{2/3}=a^{2/3}[\cos\theta+\sin\theta]^2 \qquad ...(4.118)$$

Similarly, $$(X-Y)^{2/3}=a^{2/3}[\cos\theta-\sin\theta]^2 \qquad ...(4.119)$$

From (4.118) and (4.119), we have

$$(X+Y)^{2/3}+(X-Y)^{2/3}=a^{2/3}[(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2]=2a^{2/3}.$$

Changing X to x and Y to y the equation of evolute is $(x+y)^{2/3}+(x-y)^{2/3}=2a^{2/3}$.

EXERCISE 4.11

1. Find the co-ordinates of the centre of curvature at the point (x, y) on the parabola $y^2=4ax$. Also find the equation of its evolute.

2. Find the co-ordinates of the centre of curvature of the curve $a^2y = x^3$.
3. Show that the evolute of ellipse $x = a\cos\theta$, $y = b\sin\theta$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.
4. Prove that the centres of curvature at points of a cycloid lie on another cycloid.
5. Show that the parabolas $y = -x^2 + x + 1$, $x = -y^2 + y + 1$ have the same circle of curvature at the point $(1, 1)$.
6. Show that $\left(x - \frac{3}{4}a\right)^2 + \left(y - \frac{3}{4}a\right)^2 = \frac{1}{2}a^2$ is the circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$.
7. Find the radius of curvature and the centre of curvature for the curve $y = \tan x$ at the point when $x = \pi/4$.
8. Show that the evolute of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ is $x^2 + y^2 = a^2$.
9. Find the circle of curvature at the origin for the curve $x + y = ax^2 + by^2 + cx^3$.
10. Show that the circle of curvature at the origin of the parabola $y = mx + \frac{x^2}{a}$, is $(x^2 + y^2) = a(1 + m^2)(y - mx)$.

4.10 ENVELOPES

Consider the following one parameter family of straight lines

$$y = mx + a/m \quad \text{...(4.120)}$$

Here m is parameter and a is some constant.

For different values of m, (4.120) represents different straight lines. All these straight lines thus obtained constitute a family whose members touch the parabola $y^2 = 4ax$. In fact $y^2 = 4ax$ is the locus of the point of intersection two consecutive members of the family (4.120). Consider two members of the family

$$y = mx + \frac{a}{m} \quad \text{...(4.121)}$$

$$y = (m + \delta m)x + \frac{a}{m + \delta m}. \quad \text{...(4.122)}$$

The point of intersection of these lines is $P\left(\frac{a}{m(m + \delta m)}, \frac{a(2m + \delta m)}{m(m + \delta m)}\right)$.

Keeping m fixed and making $\delta m \to 0$, then the point of intersection P tends to the point

$$Q\left(\frac{a}{m^2}, \frac{2a}{m}\right). \quad \text{...(4.123)}$$

The point Q is the limiting position of the point of intersection of the two lines (4.121) and (4.122), when the latter tends to coincide with the former, and it lies on the line (4.121). A similar point will lie on every line of the family (4.120) and the locus of such points is obtained by eliminating m between x and y given by

$$x = \frac{a}{m^2}, \quad y = \frac{2a}{m} \qquad \text{...(4.124)}$$

which gives the parabola $y^2 = 4ax$.

The *locus* obtained such is called the *envelope* of the given family.

Thus $y^2 = 4ax$ is the envelope of the one-parameter family (4.20) of straight lines as shown in Fig. 4.21.

As another example, the circle $x^2 + y^2 = p^2$ is the envelope of the family of straight lines $x \cos \alpha + y \sin \alpha = p$. All members of this family touch the circle $x^2 + y^2 = p^2$.

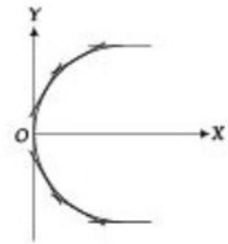


Fig. 4.21

A formal definition is given as follows.

The envelope of a one-parameter family of curves given by $f(x, y, \alpha) = 0$ is the locus of the limiting position of the points of intersection of any two consecutive members of the family when one tends to coincide with the other which is fixed.

The envelope of the one-parameter family of curves $f(x, y, \alpha) = 0$ is obtained by eliminating α between

$$f(x, y, \alpha) = 0, \quad \text{and} \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0. \qquad \text{...(4.125)}$$

In case of two-parameters family of curves $f(x, y, \alpha, \beta) = 0$; α, β being two independent parameters, envelope is obtained by eliminating α and β from

$$f(x, y, \alpha, \beta) = 0, \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha, \beta) = 0 \text{ and } \frac{\partial f}{\partial \beta}(x, y, \alpha, \beta) = 0. \qquad \text{...(4.126)}$$

Another result of interest is

Evolute of a curve is the envelope of the normals to that curve.

In the examples to follow, we will learn that a family of curves may have no envelope, or one envelope, or more than one envelope.

Example 4.73: Find the envelope of the family of curves given by

(a) $y = mx + \dfrac{a}{m}$ (b) $x \cos \alpha + y \sin \alpha = p$

α, m being paramers and a, p are constants.

Solution: (a) The family of curves is

$$f(x, y, m) = y - mx - \frac{a}{m} = 0. \qquad \text{...(4.127)}$$

Differentiating with respect to m, we have

$$\frac{\partial f}{\partial m} = -x + \frac{a}{m^2} = 0. \qquad \text{...(4.128)}$$

From (4.128) $\quad m^2 = a/x.$ ...(4.129)

Also from (4.127) $\quad y^2 = m^2x^2 + \dfrac{a^2}{m^2} + 2ax$ ...(4.130)

Substituting for m^2 from (4.129) in (4.130), we obtain $y^2 = 4ax$ as the envelope.

(b) The family of curves is

$$f(x, y, \alpha) = x \cos \alpha + y \sin \alpha - p = 0. \quad ...(4.131)$$

It gives $\quad \dfrac{\partial f}{\partial \alpha} = -x \sin \alpha + y \cos \alpha = 0.$ ...(4.132)

From (4.132), we have $\tan \alpha = y/x$, which gives

$$\sin \alpha = \frac{\pm y}{\sqrt{x^2 + y^2}}, \text{ and } \cos \alpha = \frac{\pm x}{\sqrt{x^2 + y^2}}. \quad ...(4.133)$$

Substituting for $\sin \alpha$ and $\cos \alpha$ from (4.133) in (4.131), we obtain

$$\pm \frac{(x^2 + y^2)}{\sqrt{x^2 + y^2}} = p, \text{ or} \quad ...(4.134)$$

Squaring and simplifying (4.134), we obtain $x^2 + y^2 = p^2$ as the envelope.

Example 4.74: Find the envelope of the family

$$x^2(x - a) + (x + a)(y - m)^2 = 0,$$

where m is a parameter and a is some constant.

Solution: The family of curves is

$$x^2(x - a) + (x + a)(y - m)^2 = 0. \quad ...(4.135)$$

Differentiating it partially w.r.t. m, we obtain

$$-2(x + a)(y - m) = 0 \quad ...(4.136)$$

Eliminating m between (4.135) and (4.136), we obtain

$$x^2(x - a) = 0 \quad ...(4.137)$$

as the envelope.

From (4.137) it is clear that the envelope consists of two lines as $x = 0$ and $x = a$, as shown in Fig. 4.22.

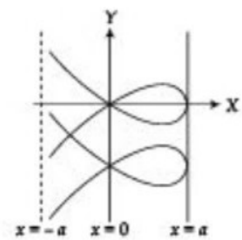


Fig. 4.22

Example 4.75: Check the family of circles $x^2 + (y - b)^2 = b^2$ for its envelope.

Solution: The family of circles is

$$x^2 + (y - b)^2 - b^2 = 0 \quad ...(4.138)$$

Differentiating partially w.r.t. b, we obtain

$$2(y - b)(-1) - 2b = 0, \quad \text{or } y = 0 \quad ....(4.139)$$

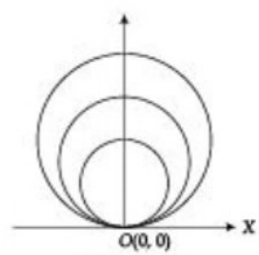


Fig. 4.23

Eliminating b between the two equations we obtain $x^2 = 0$, or $x = 0$. Taking it along with (4.139), it gives (0, 0) as the envelope or simply, *no envelope* as shown in Fig. 4.23.

Example 4.76: Find the envelope of the family of ellipses $x^2/a^2 + y^2/b^2 = 1$, where the two parameters a, b are connected by the relation $a + b = c$.

Solution: The family can be expressed as

$$\frac{x^2}{a^2} + \frac{y^2}{(c-a)^2} = 1. \quad \text{...(4.140)}$$

Differentiating partially w.r.t. a, we obtain

$$-x^2/a^3 + \frac{y^2}{(c-a)^3} = 0, \quad \text{or} \quad \frac{c-a}{a} = \frac{y^{2/3}}{x^{2/3}}. \quad \text{...(4.141)}$$

From (4.141), we have

$$a = cx^{2/3}/(x^{2/3} + y^{2/3}) \quad \text{and,} \quad c - a = cy^{2/3}/(x^{2/3} + y^{2/3})$$

Substituting these in (4.140), we obtain

$$x^{2/3}(x^{2/3} + y^{2/3})^2 + y^{2/3}(x^{2/3} + y^{2/3})^2 = c^2, \quad \text{or} \quad x^{2/3} + y^{2/3} = c^{2/3}$$

as the required envelope.

Example 4.77: Using the result that the evolute of a curve is the envelope of its normals, find the evolute to the parabola $y^2 = 4ax$.

Solution: Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$.

Equation of the normal at t is

$$y + xt = 2at + at^3. \quad \text{...(4.142)}$$

Differentiating it partially w.r.t. t, we obtain $x = 2a + 3at^2$, which gives $t = \sqrt{\frac{x-2a}{3a}}$

Substituting for t in (4.142), we obtain

$$y + x\sqrt{\frac{x-2a}{3a}} = 2a\sqrt{\frac{x-2a}{3a}} + a\left(\frac{x-2a}{3a}\right)^{3/2}$$

or,
$$y = \sqrt{\frac{x-2a}{3a}}\left[2a - x - a\left(\frac{x-2a}{3a}\right)\right] = \sqrt{\frac{x-2a}{3a}}\left(\frac{4a}{3} - \frac{2x}{3}\right).$$

Squaring both sides and simplifying, we get $27ay^2 = 4(x-2a)^3$ as the required envelope, also refer to Example 4.69.

Example 4.78: Find the envelope of straight lines drawn at right angles to the radii vectors of the cardiod $r = a(1 + \cos\theta)$ through their extremities.

Solution: Let P be any point on the cardiod $r = a(1 + \cos\theta)$. If α be its vectorial angle, then the radius vector $OP = a(1 + \cos\alpha)$.

The equation of a line through P at right angle to the radius vector OP is

$$r\cos(\theta - \alpha) = a(1 + \cos\alpha) \quad \text{...(4.143)}$$

here α is the parameter, since for different α the lines are different.

Differentiating w.r.t. α, we obtain,

$$r\sin(\theta - \alpha) = -a\sin\alpha$$

or, $$r\sin\theta\cos\alpha - (r\cos\theta - a)\sin\alpha = 0, \text{ which gives } \tan\alpha = r\sin\theta/(r\cos\theta - a).$$

Hence, $\sin\alpha = \dfrac{r\sin\theta}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}$, and $\cos\alpha = \dfrac{r\cos\theta - a}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}$

Also from (4.143), we have $(r\cos\theta - a)\cos\alpha + r\sin\theta\sin\alpha = a$

Substituting for $\sin\alpha$ and $\cos\alpha$, we obtain

$$\frac{(r\cos\theta - a)^2 + r^2\sin^2\theta}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} = a, \text{ or } \frac{r^2 + a^2 - 2a^2\cos\theta}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} = a$$

or, $$\sqrt{r^2 + a^2 - 2ar\cos\theta} = a, \text{ or } r^2 + a^2 - 2ar\cos\theta = a^2, \text{ or } r = 2a\cos\theta$$

as the required envelope.

EXERCISE 4.12

1. Find the envelope of the following family of lines
   (i) $y = mx \pm \sqrt{a^2m^2 + b^2}$, m being the parameter.
   (ii) $y = mx - 2am - am^3$, m being the parameter.
   (iii) $y = mx + am^3$, m being the parameter.
   (iv) $x\cos\alpha - y\sin\alpha = c$, α being the parameter.
   (v) $x\tan\alpha + y\sec\alpha = c$, α being the parameter
   (vi) $x\cos^n\alpha + y\sin^n\alpha = c$, α being the parameter
2. Find the envelope of the family of circles $(x - \alpha)^2 + y^2 = 2\alpha$, where α is the parameter.
3. Find the envelope of the family of semi-cubical parabolas $y^2 - (x + \alpha)^3 = 0$; α being the parameter.
4. Find the envelope of a system of concentric and co-axial ellipses of constant area A.
5. Find the envelope of a family of lines $x/a + y/b = 1$, where the parameters a and b are connected by the relation $a^n + b^n = c^n$.
6. Circles are described on the double ordinates of $y^2 = 4ax$ as diameter. Show that the envelope is $y^2 = 4a(x + a)$.

7. Show that the envelope of a circle whose centre lies on the parabola $y^2 = 4ax$ and which passes through its vertex is the cissoid $y^2(2a + x) + x^3 = 0$.
8. Considering that the evolute of a curve is the envelope of its normals, show that the evolute of the (i) ellipse $x^2/a^2 + y^2/b^2 = 1$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$ (ii) hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$

4.11 ASYMPTOTES

The concept of asymptotes is associated with those curves which extend to infinity like parabola and hyperbola. For example, the two infinite branches of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ are $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$, since here as $x \to \pm\infty$, y also tends to $\pm\infty$. On the other hand, in case of a circle, say $x^2 + y^2 = a^2$, x and y both can take only finite values.

A point $P(x, y)$ on an infinite branch of a curve is said to tend to ∞ along the curve when either x, or y, or x and y both tend to ∞ or $-\infty$, as P moves along the infinite branch of the curve.

Next we define asymptote:

A straight line at a finite distance from the origin is said to be an 'asymptote' to an infinite branch of a curve if the perpendicular distance of a point P on the curve from the straight line approaches zero as the point P moves to infinity along the curve.

In other words, *asymptote is the limiting position of the tangent line to the curve as the point of contact recedes to infinity along the infinite branch of the curve.*

Asymptotes parallel to x-axis are called *horizontal asymptotes* and those parallel to y-axis are called '*vertical asymptotes*'. Asymptotes which are not parallel to any of the axes are called *oblique asymptotes*. In some typical cases asymptotes are curves also.

4.11.1 Asymptotes Parallel to Axes

For the curve $y = f(x)$: First we find the asymptotes parallel to y-axis. If $x = k$ is an asymptote for the curve $y = f(x)$, then, by definition of an asymptote, $|x - k|$, the distance of the line $x = k$ from a point $P(x, y)$ on the curve, must tend to zero as $P \to \infty$ (in this case $y \to \infty$) along the curve. Thus, *to find asymptotes parallel to y-axis we find from the given equation of the curve the definite values k_1, k_2, ... to which x tends as y tends to ∞, or $-\infty$ and, then $x = k_1$, $x = k_2$, ... are equations of the asymptotes.*

Similarly, *to find asymptotes parallel to x-axis we find from the given equation of the curve the definite values l_1, l_2, ... to which y tends as x tends to ∞, or $-\infty$ and, then $y = l_1$, $y = l_2$ ... are equations of the asymptotes.*

For the curve $f(x, y) = 0$: Let the given equation $f(x, y) = 0$, after arranging in descending powers of y, be

$$\phi_0(x)\, y^n + \phi_1(x) y^{n-1} + \phi_2(x) y^{n-2} + \ldots + \phi_n(x) = 0 \qquad \ldots(4.144)$$

where $\phi_i(x)$, $i = 0, 1, 2, \ldots, n$ are polynomials in x.

Dividing (4.144) throughout by y^n, we obtain

$$\phi_0(x) + \phi_1(x)\frac{1}{y} + \phi_2(x)\frac{1}{y^2} + \ldots + \phi_n(x)\frac{1}{y^n} = 0 \qquad \ldots(4.145)$$

To find asymptotes $x = k$, parallel to y-axis, as above, we find from (4.145) the definite values k_1, k_2 ... to which x tends as $y \to \pm\infty$. These are given by $\phi_0(x) = 0$. Thus *to find asymptotes parallel to y-axis for the algebraic curve f(x, y) = 0 equate to zero the coefficient of the highest power of y, present in the equation of the curve. Resolve it into real linear factors. Then these factors equated to zero give the equation of the asymptotes. If the coefficient of the highest power of y is either a constant or not resolvable into real linear factors, then there are no asymptotes parallel to the y-axis.*

Similarly, *to find the asymptotes parallel to x-axis for the algebraic curve f(x, y) = 0, equate to zero the coefficient of the highest power of x, present in the equation of the curve. Resolve it into real linear factors. Then these factors equated to zero give the equation of the asymptotes. If the coefficient of the highest power of x is either a constant or not resolvable into real linear factors, then there are no asymptotes parallel to the x-axis.*

Example 4.79: Find the asymptotes parallel to the axes for the curves:

(i) $x^2y^2 = a^2(x^2 + y^2)$ (ii) $\dfrac{a^2}{x^2} - \dfrac{b^2}{y^2} = 1.$

Solution: (i) The equation of the curve is $x^2y^2 - a^2(x^2 + y^2) = 0$.

To find asymptotes parallel to x-axis equate to zero the coefficient of the highest power of x, we have

$$(y^2 - a^2) = 0, \text{ or } (y - a)(y + a) = 0, \text{ or } y = \pm a,$$

as the required asymptotes parallel to x-axis.

Next, to find asymptotes parallel to y-axis equate to zero the coefficient of the highest power of y, we have

$$(x^2 - a^2) = 0, \text{ or } (x - a)(x + a) = 0, \text{ or } x = \pm a,$$

as the required asymptotes parallel to y-axis.

(ii) The equation of the curve is $a^2/x^2 - b^2/y^2 = 1$ or, $x^2y^2 - a^2y^2 + b^2x^2 = 0$.

Equating to zero the coefficient of the highest power of x we have $(y^2 + b^2) = 0$. Since it can't be resolved into real linear factors. Hence there is no asymptote parallel to x-axis.

Proceeding on the same lines as in (i) above, the asymptotes parallel to y-axis are $x = \pm a$.

Example 4.80: Find the asymptotes parallel to axes to the curves

(i) $y = e^x$ (ii) $y = \sec x$.

Solution: (i) Equation of the curve is $y = e^x$.

Here, $\lim_{x\to\infty} y = \lim_{x\to\infty} e^x \to \infty$ and $\lim_{x\to-\infty} y = \lim_{x\to-\infty} e^x = 0$.

Therefore, $y = 0$ is an asymptote parallel to x-axis.

Further $y = e^x$ gives $x = \ln y$

Here, $\lim_{y\to\infty} x = \lim_{y\to\infty} \ln y \to \infty$, and $\lim_{y\to-\infty} x = \lim_{y\to-\infty} \ln y$, which does not exist.

Therefore, there is no asymptote parallel to y-axis.

(ii) Equation of the curve is $y = \sec x$.

Here $\lim\limits_{x\to\pm\infty} y = \lim\limits_{x\to\pm\infty} \sec x$ does not tend to a unique finite value. Therefore there is no asymptote parallel to x-axis.

However, $y \to \pm\infty$ gives $\cos x \to 0$, which implies $x = (2n+1)\pi/2$, for integral values of n. Thus, $x = (2n+1)\pi/2$ are asymptotes parallel to the y-axis.

EXERCISE 4.13

Find the asymptotes parallel to the axes for the following curves

1. $x(y^2 - x^2) = a(x^2 + y^2)$
2. $x^2y^2 - y^2 - 2 = 0$
3. $x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$
4. $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1$
5. $y = \sin x$
6. $y = \operatorname{cosec} x$.

4.11.2 Oblique Asymptotes

For the curve $y = f(x)$. Let $y = mx + c$ be the equation of the asymptote. If d is the distance of $y = mx + c$ from any point $P(x, y)$ on an infinite branch of the curve, then

$$d = \frac{|y - mx - c|}{\sqrt{1+m^2}}. \text{ This gives}$$

$$y - mx = c \pm d\sqrt{1+m^2} \qquad \text{...(4.146)}$$

or,

$$\frac{y}{x} = m + \frac{c}{x} \pm \frac{d}{x}\sqrt{1+m^2}. \qquad \text{...(4.147)}$$

Making the point $P(x, y) \to \infty$ along the curve, (4.147) gives $\lim\limits_{x\to\infty}\left(\dfrac{y}{x}\right) = m$, since d, by definition of the asymptote, tends to zero, as $P \to \infty$.

Also as $P \to \infty$, (4.146) gives $\lim\limits_{x\to\infty}(y - mx) = c$.

Hence, *if $y = mx + c$ is an oblique asymptote to the curve $y = f(x)$, then*

$$m = \lim_{x\to\infty}(y/x) \text{ and } \quad c = \lim_{x\to\infty}(y - mx).$$

For the curve $f(x, y) = 0$. Write the equation of the curve in the form

$$x^n\phi_n(y/x) + x^{n-1}\phi_{n-1}(y/x) + x^{n-2}\phi_{n-2}(y/x) + \ldots = 0, \qquad \text{...(4.148)}$$

where $\phi_r(y/x)$ is a polynomial of degree r in (y/x).

To find the point of intersection of the line $y = mx + c$ with (4.148), put $y/x = m + c/x$, and expand each of $\phi_r\left(m + \dfrac{c}{x}\right)$ by Taylor's series, we obtain

$$x^n\left[\phi_n(m)+\frac{c}{x}\phi_n'(m)+\frac{c^2}{2!\,x^2}\phi_n''(m)+\ldots\right]+x^{n-1}\left[\phi_{n-1}(m)+\frac{c}{x}\phi_{n-1}'(m)+\frac{c^2}{2!\,x^2}\phi_{n-1}''(m)\right]+\ldots=0$$

$$\phi_n(m)x^n+[c\phi_n'(m)+\phi_{n-1}(m)]x^{n-1}+\left[\frac{c^2}{2!}\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)\right]x^{n-2}+\ldots=0.$$

Dividing throughout by x^n, we obtain

$$\phi_n(m)+[c\phi_n'(m)+\phi_{n-1}(m)].\frac{1}{x}+\left[\frac{c^2}{2!}\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)\right]\frac{1}{x^2}+\ldots=0 \qquad \text{...(4.149)}$$

Also from (4.148)

$$\phi_n(y/x)+\frac{1}{x}\phi_{n-1}(y/x)+\frac{1}{x^2}\phi_{n-2}(y/x)+\ldots=0 \qquad \text{...(4.150)}$$

If $y = mx + c$ is the asymptote to the given curve, then $\lim\limits_{x\to\infty}(y/x) = m$ and hence from (4.150), we obtain

$$\phi_n(m) = 0. \qquad \text{...(4.151)}$$

The real roots of the Eq. (4.151) give the slopes of the asymptotes.

Substituting $\phi_n(m) = 0$ in (4.149) and multiplying throughout by x and taking $x \to \infty$ we get

$$c\phi_n'(m)+\phi_{n-1}(m)=0, \text{ or } \quad c=\frac{-\phi_{n-1}(m)}{\phi_n'(m)}, \text{provided} \quad \phi_n'(m)\neq 0. \qquad \text{...(4.152)}$$

Thus, if $m_1, m_2, \ldots$ are the distinct values of m as obtained from (4.151) and $c_1, c_2, \ldots$ are corresponding values of c given by (4.152), then the equations of the asymptotes are: $y_1 = m_1x + c_1$, $y = m_2x + c_2 \ldots$, provided $\phi_n'(m) \neq 0$.

If $\phi_n'(m) = 0$ but $\phi_{n-1}(m) \neq 0$, then from (4.152) values of c are infinite and thus there is no asymptote in this case.

But if $\phi_n'(m) = 0$ and $\phi_{n-1}(m) = 0$, then $c\phi_n'(m) + \phi_{n-1}(m) = 0$ becomes an identity. Now since $\phi_n'(m) = 0$. Thus $\phi_n(m)$ has repeated values of m. Proceeding on the similar lines, corresponding to the repeated values of m we get two values of c say c_1 and c_2 from

$$\frac{c^2}{2!}\phi_n''(m)+c\phi_{n-1}''(m)+\phi_{n-2}(m)=0 \qquad \text{...(4.153)}$$

provided $\phi_n''(m) \neq 0$; and so on.

Working rules to find the oblique asymptotes.

1. *To obtain $\phi_n(m)$ put $x = 1$, $y = m$ in the highest degree terms. Equate it to zero and solve for m, and let $m_1, m_2, \ldots$ be its real roots.*
2. *To obtain $\phi_{n-1}(m)$, put $x = 1$, $y = m$ in the next lower degree terms. Similarly for $\phi_{n-2}(m)$, etc.*

3. *Find the values $c_1, c_2, \ldots$ corresponding to values $m_1, m_2 \ldots$ by using the relation* $c = \dfrac{\phi_{n-1}(m)}{\phi'_n(m)}$, *provided* $\phi'_n(m) \neq 0$, *the corresponding asymptotes are:* $y = m_1x + c_1, \quad y = m_2x + c_2, \ldots$
4. *If* $\phi'_n(m) = 0$ *for some value of m, but* $\phi_{n-1}(m) \neq 0$, *then corresponding to that value of m there is no asymptote.*
5. *If* $\phi'_n(m) = 0 = \phi_{n-1}(m)$ *for some value of m, that is, two roots of* $\phi_n(m) = 0$ *are equal, then the values of c are obtained from the equation,*

$$\frac{c^2}{2!}\phi''_n(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$$

This gives two values of c, provided $\phi''_n(m) \neq 0$ *and the two parallel asymptotes, corresponding to this value of m, are obtained. However, if* $\phi''_n(m) = 0$, *we proceed further on similar lines as above.*

Remark: Before finding the oblique asymptotes, find the asymptotes parallel to the axes, if any. If the number of asymptotes parallel to the axes is less than the degree of the equation of the curve only then proceed to find for the oblique asymptotes since *the total number of asymptotes of a curve cannot exceed the degree of the equation of the curve.*

Example 4.81: Find the oblique asymptotes, if any, to the curves

(i) $y = e^x$ (ii) $y = \sec x$.

Solution: (i) Let $y = mx + c$ be an oblique asymptote, then $m = \lim\limits_{x\to\infty} \dfrac{y}{x} = \lim\limits_{x\to\infty} \dfrac{e^x}{x} \to \infty$.

Therefore there exists no finite value of m hence the curve has no oblique asymptote.

(ii) In this case, $$m = \lim_{x\to\infty} \frac{y}{x} = \lim_{x\to\infty} \frac{\sec x}{x} = \lim_{x\to\infty} \frac{1}{x\cos x} = 0.$$

The asymptote corresponding to $m = 0$, if any, will be parallel to x-axis, which does not exist refer to Example 4.80(ii). Hence, the curve does not have any oblique asymptote.

Example 4.82: Find all the asymptotes to the parabola $y^2 - 4ax = 0$.

Solution: Since the coefficient of the highest degree terms both in x and y are constants, so the curve has no asymptotes parallel to any of the axes.

To find the oblique asymptotes put $x = 1$, $y = m$ in the highest degree term, and the next lower degree term, we obtain

$$\phi_2(m) = m^2, \quad \phi_1(m) = -4a$$

Now, $\phi_2(m) = 0$ gives $m^2 = 0$, or $m = 0, 0$

Also, $\phi'_2(m) = 2m$, thus $\phi'_2(m) = 0$, at $m = 0$

but $\phi_1(m) \neq 0$. Hence no oblique asymptote. Otherwise also there can't be any oblique asymptote corresponding to $m = 0$. Thus, the curve $y^2 - 4ax = 0$ has no asymptote.

Example 4.83: Find the asymptotes of the curve $y^3-6xy^2+11x^2y-6x^3+x+y=0$.

Solution: Since the coefficients of y^3 and that of x^3, the highest degree terms in x and y are constants, therefore, the curve has no asymptote parallel to any of the axes.

To find the oblique asymptotes, put $x=1$, $y=m$ in the highest degree terms and next lower degree terms, we get

$$\phi_3(m)=m^3-6m^2+11m-6,\ \phi_2(m)=0 \text{ and } \phi_1(m)=1+m.$$

Now, $\phi_3(m)=0$, gives $m^3-6m^2+11m-6=0$, or $(m-1)(m-2)(m-3)=0$, or $m=1,2,3$.

Next, c is given by $c=-\dfrac{\phi_2(m)}{\phi_3'(m)}=-\dfrac{0}{3m^2-12m+11}=0,$

as $\phi_2(m)=0$ for $m=1,2,3$ and $\phi_3'(m)\neq 0$.

Therefore, the three asymptotes of the curve are $y=x$, $y=2x$, and $y=3x$.

Example 4.84: Find all the asymptotes of the curve

$$x^3+4x^2y+4xy^2+5x^2+15xy+10y^2-2y+1=0.$$

Solution: Since, the coefficient of x^3, the highest degree term in x is constant, thus, no asymptote parallel to x-axis.

Next equating to zero the coefficient of y^2, the highest degree term in y, gives $4x+10=0$, or $2x+5=0$, gives the equation of the asymptote parallel to y-axis.

To find oblique asymptotes, put $x=1$, $y=m$ in the third degree terms and the next lower degree terms, we have

$$\phi_3(m)=1+4m+4m^2,\quad \phi_2(m)=5+15m+10m^2,\ \phi_1(m)=-2m.$$

Now, $\phi_3(m)=0$, gives $4m^2+4m+1=0$, or $(2m+1)^2=0$, which gives, $m=-\dfrac{1}{2},-\dfrac{1}{2}$.

Also, $\phi_3'(m)=4+8m=4(1+2m)$.

Next, $c=-\dfrac{\phi_2(m)}{\phi_3'(m)}=-\dfrac{5+15m+10m^2}{4(1+2m)}=\dfrac{0}{0}$ at $m=-\dfrac{1}{2}$.

Therefore, c is given by

$$\frac{c^2}{2!}\phi_3''(m)+c\,\phi_2'(m)+\phi_1(m)=0. \qquad \text{...(4.154)}$$

We have, $\phi_3''(m)=8$, $\phi_2'(m)=15+20m$ and $\phi_1(m)=-2m$

At $m=-\dfrac{1}{2}$, $\phi_3''(m)=8$, $\phi_2'(m)=5$ and $\phi_1(m)=1$

Substituting in (4.154), gives

$$\frac{c^2}{2!}(8)+c(5)+1=0, \text{ or } 4c^2+5c+1=0, \text{ or } (4c+1)(c+1)=0 \text{ or, } c=-1,\ -\frac{1}{4}$$

Thus the oblique asymptotes are: $y = -\frac{1}{2}x - 1$, $y = -\frac{1}{2}x - \frac{1}{4}$.

or, $$x + 2y + 2 = 0, \quad 2x + 4y + 1 = 0$$

Hence, the asymptotes are: $2x + 5 = 0$, $x + 2y + 2 = 0$, $2x + 4y + 1 = 0$.

Example 4.85: Find the asymptotes to the curve

$$y^4 - 2xy^3 + 2x^3y - x^4 - 3x^3 + 3x^2y + 3xy^2 - 3y^3 - 2x^2 + 2y^2 - 1 = 0.$$

Solution: Since, the coefficients of y^4 and x^4, the highest degree terms in x and y, are constants, so there are no asymptotes parallel to any of the axes.

To find oblique asymptotes, put $x = 1$ and $y = m$ in the highest degree terms and the subsequent lower degree terms, we have

$$\phi_4(m) = m^4 - 2m^3 + 2m - 1,\ \phi_3(m) = -3m^3 + 3m^2 + 3m - 3,\ \phi_2(m) = 2(m^2 - 1), \text{ and } \phi_1(m) = 0.$$

$\phi_4(m) = 0$ gives $m^4 - 2m^3 + 2m - 1 = 0$. By inspection $m = 1$ is a root of this equation. Using synthetic division

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & 0 & 2 & -1 \\ & & 1 & -1 & -1 & 1 \\ \hline & 1 & -1 & -1 & 1 & \boxed{0} \end{array}$$

The depressed equation is $m^3 - m^2 - m + 1 = 0$

Again $m = 1$ is its root. Using synthetic division,

$$\begin{array}{r|rrrr} 1 & 1 & -1 & -1 & 1 \\ & & 1 & 0 & -1 \\ \hline & 1 & 0 & -1 & \boxed{0} \end{array}$$

The depressed equation is $m^2 - 1 = 0$ which gives $m = \pm 1$

Hence roots are: $m = 1, 1, 1$ and -1.

Also, $\phi_4'(m) = 4m^3 - 6m^2 + 2$, $\phi_4''(m) = 12m^2 - 12m$ $\phi_4'''(m) = 24m - 12$;

$\phi_3'(m) = -9m^2 + 6m + 3$, $\phi_3''(m) = -18m + 6$, and $\phi_2'(m) = 4m$.

Value of c of the asymptote with slope $m = -1$ is given by

$$c = \frac{-\phi_3(m)}{\phi_4'(m)} = -\frac{-3m^3 + 3m^2 + 3m - 3}{4m^3 - 6m^2 + 2} = -\frac{3 + 3 - 3 - 3}{-4 - 6 + 2} = 0.$$

Hence, equation of the asymptote is $y = -x$.

The values of c for the three parallel asymptotes corresponding to $m = 1$ are given by

$$\frac{c^3}{3!}\phi_4'''(m) + \frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

or, $$\frac{c^3}{6}(24m-12)+\frac{c^2}{2}(-18m+6)+c(4m)+0=0$$

or, $$c[(4m-2)c^2+(-9m+3)c+4m]=0$$

Put $m=1$, we get $2c(c^2-3c+2)=0$, or $c(c-1)(c-2)=0$, or $c=0,1,2$.

Thus the three asymptotes for $m=1$ are $y=x$, $y=x+1$, and $y=x+2$.

Therefore, the four asymptotes are $y=-x$, $y=x$, $y=x+1$ and $y=x+2$.

EXERCISE 4.14

Find the asymptotes of the following curves:

1. $x^3-2y^3+xy(2x-y)+y(x-1)+1=0$
2. $x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0$
3. $y^3-xy^2-x^2y+x^3+x^2-y^2-1=0$
4. $y^3-2xy^2-x^2y+2x^3+3y^2-7xy+2x^2+2y+2x+1=0$
5. $x^2(x-y)^2-a^2(x^2+y^2)=0$
6. $x^2(x-y)^2+a(x^2-y^2)-a^2xy=0$
7. $xy(x^2-y^2)(x^2-4y^2)+xy(x^2-y^2)+x^2+y^2-7=0$
8. $(2x-3y+1)^2(x+y)=8x-2y+9$
9. $(x+y)(x^4+y^4)-a(x^4+a^4)=0$
10. $(a_1x+b_1y+c_1)(a_2x+b_2x+c_2)+c_3=0$
11. Show that the asymptotes of the curve $x^2y^2=a^2(x^2+y^2)$ form a square of side $2a$.
12. Show that the asymptotes of the curve $x^2y^2-a^2(x^2+y^2)-a^3(x+y)+a^4=0$ form a square, through two of whose angular points the curve passes.
13. Find the asymptotes of the curve $x^2y^2-4(x-y)^2+2y-3=0$ and show that they form a square.

4.11.3 Some Further Results on Asymptotes

We state a few more results on asymptotes which are of vital importance.

1. *If the equation of a curve of degree n can be put in the form $F_n+F_{n-2}=0$, where F_n consists of n non-repeated linear factors and F_{n-2} is of degree $(n-2)$ at the most, then every linear factor of F_n equated to zero will give an asymptote, provided no two linear factors differ by a constant.*

For example, the equation of the curve $x^2y-xy^2+xy-y^2+x+y=0$ ...(4.155)
is of the form $F_n+F_{n-2}=0$, with $F_n=x^2y-xy^2+xy-y^2$, and $F_{n-2}=x+y$.

$F_n=0$ gives $x^2y-xy^2+xy-y^2=0$, or $y(x+1)(x-y)=0$.

Therefore, the asymptotes of the curve (4.155) are $y=0$, $x+1=0$, $x-y=0$.

This method of finding the asymptotes is called the *method of inspection*.

2. Any asymptote of an algebraic curve of the nth degree cuts the curve in $(n-2)$ points. Thus, if a curve of degree n has n asymptotes then they all intersect the curve in $n(n-2)$ points.

Further, *if the curve can be put in the form $F_n + F_{n-2} = 0$, then these $n(n-2)$ points of intersection, of the curve $F_n + F_{n-2} = 0$ and its asymptotes $F_n = 0$, lie on the curve $F_{n-2} = 0$.*

For example, for the curve (4.155) of degree 3, the three asymptotes cut it in $3(3-2) = 3$ points which lie on a curve of degree $(3-2) = 1$, that is, on a straight line. In this case it is $x + y = 0$; and it can be verified easily.

Example 4.86: Show that the points of intersection of the curve $xy(x^2 - y^2) + a^2y^2 + b^2x^2 - a^2b^2 = 0$ with its asymptotes lie on an ellipse.

Solution: The equation of the curve is

$$xy(x^2 - y^2) + a^2y^2 + b^2x^2 - a^2b^2 = 0 \qquad \text{...(4.156)}$$

Equating the coefficient of the highest power of x to zero, we get $y = 0$ as the asymptote parallel to x-axis.

Equating the coefficient of the highest power of y to zero, we get $x = 0$ as the asymptote parallel to y-axis.

To find the oblique asymptotes, put $x = 1$ and $y = m$ in the terms of degree 4 and the next lower degree terms, we have

$$\phi_4(m) = m(1 - m^2), \quad \phi_3(m) = 0, \quad \phi_2(m) = a^2m^2 + b^2.$$

Now, $\quad \phi_4(m) = 0$, or $\quad m(1 - m^2) = 0$, or $\quad m = 0, \pm 1$. Also, $\phi_4'(m) = 1 - 3m^2$.

The value of c is given by $c = \dfrac{\phi_3(m)}{\phi_4'(m)} = 0$, since $\phi_3(m) = 0$ and $\phi_4'(m) \neq 0$, for $m = 0, \pm 1$. Therefore, the asymptotes are: $y = 0,\ y = x,\ y = -x$.

Thus, the four asymptotes of the curve (4.156) are

$$x = 0, \quad y = 0, \quad x + y = 0, \text{ and } \quad x - y = 0.$$

These asymptotes cut the curve in $4(4-2) = 8$ points.

To find the curve on which these eight points will lie, first we find the joint equation of the asymptotes, which is

$$xy(x + y)(x - y) = 0, \text{ or } \quad xy(x^2 - y^2) = 0. \qquad \text{...(4.157)}$$

Subtracting (4.157) from (4.156), we obtain

$$a^2y^2 + b^2x^2 - a^2b^2 = 0, \text{ or } \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

an ellipse, as the curve on which points of intersection lie.

Alternatively, the equation of the given curve is

$$xy(x^2 - y^2) + a^2y^2 + b^2x^2 - a^2b^2 = 0$$

It is of the form $F_n + F_{n-2} = 0$, where $\quad F_n = xy(x^2 - y^2)$, and $\quad F_{n-2} = a^2y^2 + b^2x^2 - a^2b^2$.

$F_n = 0$ gives $xy(x^2 - y^2) = 0$, or $x = 0$, $y = 0$, $x - y = 0$, and $x + y = 0$
as the asymptotes to the given curve.

The eight points of intersection of these four asymptotes with the given curve lie on the conic

$F_{n-2} = 0$, that is, on $a^2y^2 + b^2x^2 - a^2b^2 = 0$, or $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

which is an ellipse.

Example 4.87: Find the equation of a quartic which has $x = 0$, $y = 0$, $y = x$, and $y = -x$ as its asymptotes and passes through a point (a, b) and it cuts its asymptotes again in eight points that lie on a circle $x^2 + y^2 = a^2$.

Solution: The joint equation of the asymptotes is

$$F_4 = xy(y - x)(y + x) = 0 \quad \text{...(4.158)}$$

Therefore the equation of the quartic is of the form

$$F_4 + F_2 = 0, \quad \text{...(4.159)}$$

where F_2 is of at the most of degree 2.

Also the curve of points of intersection of (4.158) and (4.159) is $x^2 + y^2 = a^2$, therefore (4.159) will be of the form

$$xy(y - x)(y + x) + k(x^2 + y^2 - a^2) = 0, \quad \text{...(4.160)}$$

where k is a constant.

Since it passes through the point (a, b), thus

$$ab(b - a)(b + a) + k(a^2 + b^2 - a^2) = 0, \text{ or } k = \frac{a(a^2 - b^2)}{b}.$$

Substituting the value of k in (4.160), the equation of the required quartic is

$$xy(y - x)(y + x) + \frac{a(a^2 - b^2)}{b}(x^2 + y^2 - a^2) = 0$$

or,
$$bxy(y^2 - x^2) + a(a^2 - b^2)(x^2 + y^2 - a^2) = 0$$

Example 4.88: Find the equation of a cubic which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + 4x + 5y + 7 = 0$ and which passes through the points $(0, 0)$, $(0, 2)$ and $(2, 0)$.

Solution: The equation of the given curve is

$$x^3 - 6x^2y + 11xy^2 - 6y^3 + 4x + 5y + 7 = 0$$

or,
$$(x - y)(x - 2y)(x - 3y) + (4x + 5y + 7) = 0$$

which is of the form $F_3 + F_1 = 0$.

Thus the asymptotes are given by $F_3 = 0$.

or,
$$(x - y)(x - 2y)(x - 3y) = 0$$

or, $\quad x-y=0, \quad x-2y=0,$ and $\quad x-3y=0.$

Now the equation of cubic which has three asymptotes given by $F_3=0$ is $F_3+F_1=0$, where F_1 is at most of degree 1. So in this case cubic will be of the form

$$(x-y)(x-2y)(x-3y)+ax+by+c=0 \quad \text{...(4.161)}$$

where a, b, c are arbitrary constants.

Since, this cubic passes through (0, 0), (0, 2) and (2, 0), therefore, these points must satisfy (4.161), which give $a=-4$, $b=24$ and $c=0$. Substituting for a, b and c in (4.161), we obtain

$$(x-y)(x-2y)(x-3y)-4x+24y=0$$

or,

$$x^3-6x^2y+11xy^2-6y^3-4x+24y=0$$

as the required cubic.

EXERCISE 4.15

1. Show that the points of intersection of the curve
$$2y^3-2x^2y-4xy^2+4x^3-14xy+6y^2+4x^2+6y+1=0$$
and its asymptotes lie on the straight line $8x+2y+1=0$.
2. Find the asymptotes of the curve $4(x^4+y^4)-17x^2y^2-4x(4y^2-x^2)+2(x^2-2)=0$ and show that they pass through the points of intersection of the curve with the ellipse $x^2+4y^2=4$.
3. Show that asymptotes of the curve $xy(x^2-y^2)+9x^2+25y^2=114$ cut the curve in eight points which lie on an ellipse of eccentricity 4/5.
4. Show that the asymptotes of the curve
$$(x^2-y^2)(y^2-4x^2)+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy-1=0$$
cut the curve again in eight points which lie on a circle of radius unity.
5. Find the equation of the cubic which has the same asymptotes as the curve $x^3-6x^2y+11xy^2-6y^3+x+y+1=0$ and which touches the axis of y at the origin and passes through the point (3, 2).
6. Find the equation of the cubic which has the same asymptotes as the curve $x^3-6x^2y+11xy^2-6y^3+x+y+1=0$ and which passes through the points (0, 0), (1, 0) and (0, 1).
7. Find the equation of the conic which has the same asymptotes as the conic $3x^2-2xy-5y^2+7x-9y=0$ and passes through the point (2, 2).

4.11.4 Asymptotes of Polar Curves

To find the asymptotes of polar curves we have the following result:

If θ_1 is the root of the equation $f(\theta)=0$, then equation of the asymptote of the curve $1/r=f(\theta)$ is $r\sin(\theta-\theta_1)=1/f'(\theta_1)$, provided $f'(\theta_1)\neq 0$.

Working rules to find the asymptotes of polar curves:

1. *Write the equation of the curve in the form $(1/r)=f(\theta)$ and change all the T-ratios, if any, into $\sin\theta$ and $\cos\theta$.*
2. *Solve the equation $f(\theta)=0$. Let the roots be $\theta=\theta_1, \theta_2$ etc.*

3. *Find $f'(\theta)$ and calculate $f'(\theta)$ for $\theta = \theta_1, \theta_2$ etc.*
4. *The asymptote corresponding to $\theta = \theta_i$ is $r \sin(\theta - \theta_i) = 1/f'(\theta_i)$ provided $f'(\theta_i) \neq 0$, $i = 1, 2$, etc.*

Example 4.89: Find the asymptotes of the curve $r\theta = a$.

Solution: The equation of the curve is $\frac{1}{r} = \frac{\theta}{a} = f(\theta)$, say.

Here, $$f(\theta) = 0 \text{ gives, } \frac{\theta}{a} = 0, \text{ or } \theta = 0.$$

Also, $$f'(\theta) = \frac{1}{a}, \text{ therefore } f'(0) = \frac{1}{a}.$$

Thus equation of the asymptote is

$$r \sin(\theta - 0) = a, \text{ that is } a = r\sin\theta.$$

Example 4.90: Prove that the curve $r = \frac{a}{1-\cos\theta}$ has no asymptotes.

Solution: The equation of the curve is

$$\frac{1}{r} = \frac{1-\cos\theta}{a} = f(\theta), \text{ say.}$$

Here, $$f(\theta) = 0, \text{ gives } \frac{1-\cos\theta}{a} = 0,$$

or, $$\cos\theta = 1, \text{ or } \theta = 2n\pi,$$

where n is an integer

Further $$f'(\theta) = \frac{\sin\theta}{a}, \text{ gives } f'(2n\pi) = \frac{\sin(2n\pi)}{a} = 0.$$

The equation of the asymptote corresponding to $\theta = \theta_i$ is $r\sin(\theta - \theta_i) = 1/f'(\theta_i)$ and since for the given curve $f'(\theta_i) = 0$ at $\theta_1 = 2n\pi$, so the curve has no asymptote.

Example 4.91: Find the asymptotes of the curve $r = a\sec\theta + b\tan\theta$.

Solution: The equation of the curves is $\frac{1}{r} = \frac{1}{a\sec\theta + b\tan\theta} = \frac{\cos\theta}{a + b\sin\theta} = f(\theta)$, say.

Here, $f(\theta) = 0$ gives $\cos\theta = 0$, that is, $\theta = (2n+1)\frac{\pi}{2}$, where n is an integer.

Also, $f'(\theta) = \frac{-\sin\theta(a + b\sin\theta) - b\cos\theta\cos\theta}{(a + b\sin\theta)^2}$. Futher

$$\sin(2n+1)\frac{\pi}{2} = \sin\left(\frac{\pi}{2} + n\pi\right) = \cos n\pi = (-1)^n, \text{ and } \cos(2n+1)\frac{\pi}{2} = \cos\left(\frac{\pi}{2} + n\pi\right) = -\sin n\pi = 0.$$

Therefore, $f'\left((2n+1)\frac{\pi}{2}\right) = \frac{(-1)^{n+1}}{[a+b(-1)^n]}$.

The equation of the asymptote corresponding to root $\theta = \theta_1$ is, $r\sin(\theta - \theta_1) = \frac{1}{f'(\theta)}$; it becomes

$$r\sin\left(\theta - (2n+1)\frac{\pi}{2}\right) = \frac{a+(-1)^n b}{(-1)^{n+1}}, \text{ or } -r\sin\left(n\pi + \frac{\pi}{2} - \theta\right) = \frac{a+(-1)^n b}{(-1)^{n+1}}, \text{ or } (-1)^{n+1} r\cos\theta = \frac{a+(-1)^n b}{(-1)^{n+1}}$$

or, $r\cos\theta = a + (-1)^n b$.

If n is odd, asymptote is $r\cos\theta = a - b$. If n is even, asymptote is $r\cos\theta = a + b$.

Hence, the curve has two asymptotes, $r\cos\theta = a \pm b$.

Circular asymptotes: *Let the equation of the curve be $r = f(\theta)$. If $\lim_{\theta\to\infty} f(\theta) = a$, then the circle $r = a$ is called the circular asymptote of the curve $r = r(\theta)$.*

Example 4.92: Find the circular asymptote of the curves

(i) $r = \dfrac{a\theta}{\theta - 1}$ (ii) $r(\theta + \sin\theta) = 2\theta + \cos\theta$.

Solution: (i) The equation of the given curve is $r = \dfrac{a\theta}{\theta-1} = f(\theta)$, say.

Consider $$\lim_{\theta\to\infty} f(\theta) = \lim_{\theta\to\infty}\frac{a\theta}{\theta-1} = \lim_{\theta\to\infty}\frac{a}{1-\frac{1}{\theta}} = a.$$

Hence the required circular asymptote is $r = a$

(ii) The equation of the given curve is $r = \dfrac{2\theta + \cos\theta}{\theta + \sin\theta} = f(\theta)$, say.

Here, $$\lim_{\theta\to\infty} f(\theta) = \lim_{\theta\to\infty}\frac{2\theta+\cos\theta}{\theta+\sin\theta} = \lim_{\theta\to\infty}\frac{2+\frac{\cos\theta}{\theta}}{1+\frac{\sin\theta}{\theta}} = \frac{2+0}{1+0} = 2.$$

Hence, the required circular asymptote is $r = 2$.

EXERCISE 4.16

Find the asymptotes of the polar curves

1. $\dfrac{2}{r} = 1 + 2\sin\theta$
2. $r = 4(\sec\theta + \tan\theta)$
3. $r\theta\cos\theta = a\cos 2\theta$
4. $r = a\,\mathrm{cosec}\,\theta + b$

5. $r \ln \theta = a$

6. $r = \dfrac{3a \sin\theta \cos\theta}{\sin^3\theta + \cos^3\theta}$

7. Show that the curve $r^n \sinh\theta = a^n$ has no asymptotes.

Find the circular asymptotes of the curves (8-10)

8. $r(e^\theta - 1) = a(e^\theta + 1)$

9. $r(2\theta^2 + \theta + 1) = 3\theta^2 + 2\theta + 1$

10. $r\cos\theta = a\sin\theta$.

4.12 CURVE TRACING

In applications such as computing the areas, length, volumes of solids of revolution, and surfaces of solids of revolution, it is very useful to know the shape of the curve represented by the given equation. The knowledge of curve tracing helps us to obtain the rough shape of the curve without actual plotting a large number of points on it. We shall study tracing of curves in cartesian, polar and parametric form.

4.12.1 Tracing of Cartesian Curves

While tracing curve of cartesian form we need to observe the following characteristics of the curve from the equation given.

1. *Symmetry:* (a) If only even powers of y occur in the equation, then the curve is *symmetrical about x-axis*. In this case the equation of the curve remains unchanged if y is changed to $-y$. For example, the curve $y^2 = 4ax$ is symmetrical about x-axis, refer to Fig. 4.24a.

(b) If only even powers of x occur in the equation, then the curve is *symmetrical about y-axis*. In this case the equation of the curve remains unchanged if x is changed to $-x$. For example, the curve $x^2 = 4ay$ is symmetrical about y-axis, refer to Fig. 4.24b.

(c) If the equation of the curve contains even powers in both of x and y, then the curve is *symmetrical about both the axes*. For example, the circle $x^2 + y^2 = a^2$, refer to Fig. 4.24c.

(d) If the equation of the curve remains unchanged when both x and y are changed to $-x$ and $-y$ respectively, then the curve is *symmetrical in opposite quadrants*. This is also called the *symmetry about the origin*. For example, the curve $xy = c^2$ is symmetrical in opposite quadrants, refer to Fig. 4.24d. We note that symmetry about both the axes implies symmetry about origin also but the converse is not true.

(e) If the equation of the curve remains unchanged when x and y are interchanged, then the curve is *symmetrical about the line* $y = x$. For example, the curve $x^3 + y^3 = 3axy$ is symmetric about the line $y = x$, refer to Fig. 4.24e.

2. *Passing through the origin; tangents at the origin:* (a) If there is no constant term in the equation of the curve, then the curve passes through the origin. For example, the curve $y^2 = 4ax$ passes through (0, 0).

(b) If the curve passes through the origin, to find the equation (s) of the tangent (s) to the curve at the origin, equate to zero the lowest degree terms in the equation of the curve. For example, $x = 0$, that is, y-axis is tangent to the curve $y^2 = 4ax$ at (0, 0), refer to Fig. 4.24a.

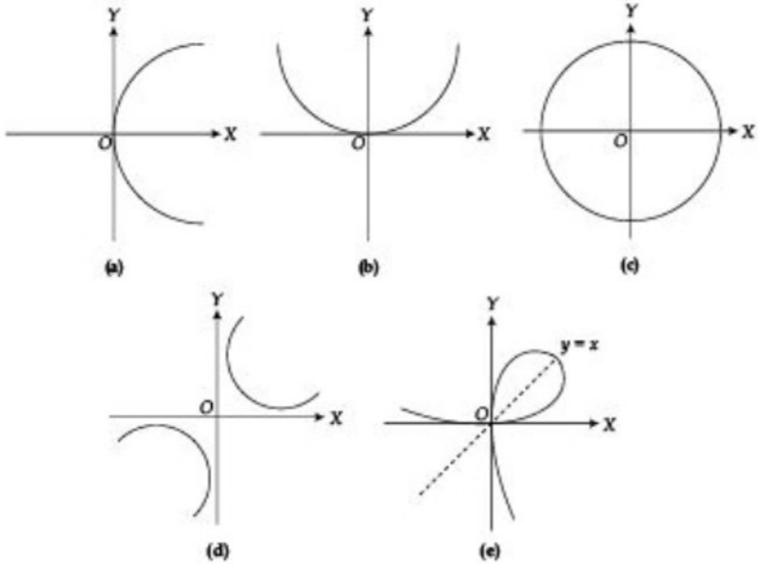


Fig. 4.24

(c) If there is only one tangent, find whether the curve lies below or above the tangent in the neighbourhood of the origin.

(d) If there are two or more tangents to the curve at the origin then the origin is a *multiple point*. In particular, if there are two tangents, the origin is called a *double point*. Further, the origin is called *a node, a cusp* or an *isolated point* according as the two tangents there are real and distinct, real and coincident, or imaginary. For example, in the curve $y^2(a-x) = x^2(a+x)$ as shown in Fig. 4.25a, the origin is a node; in the curve $y^2 - x^3 = 0$ as shown in Fig. 4.25b the origin is a cusp; and in the curve $x^2y^2 = a^2(x^2 + y^2)$ the origin is an isolated point as shown in Fig. 4.34, (refer to p. 277).

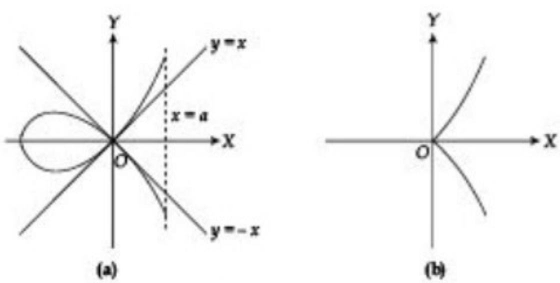


Fig. 4.25

3. ***Intersection with the co-ordinate axes:*** (a) To find the points where the curve cuts the x-axis, put $y = 0$ in equation of the curve and solve the resulting equation for x. Similarly, to find the points where the curve cuts the y-axis, put $x = 0$ in the equation of the curve and solve the resulting equation for y.

(b) To find the tangents to the curve at its point of intersection with the co-ordinate axes, we first shift the origin to this point and then by equating to zero the lowest degree term, we find the equation (s) of the tangent (s).

(c) If $y = x$, or $y = -x$ is a line of symmetry, then find the points of intersection of the curve and the line and also the tangents at that point.

4. ***Asymptotes:*** Find the equations of all the asymptotes parallel to the axes and also of oblique asymptotes as discussed in Section 4.11.

5. ***Regions in which the curve does not exist:*** To obtain such a region, solve the given equation for one variable in terms of the other and find out the set of values of one variable which make the other imaginary or undefined. For example in the curve $y^2(1-x) = x^3$, y is imaginary when $x < 0$ and when $x > 1$ hence the curve does not exist on the right of the line $x = 1$ and on the left of y-axis as shown in Fig. 4.26.

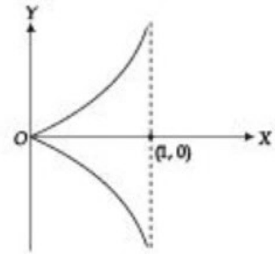


Fig. 4.26

6. ***The sign of*** $\dfrac{dy}{dx}$: Find the points on the curve where $\dfrac{dy}{dx} = 0$, or ∞, that is, the points where the tangents are parallel to x-axis or perpendicular to the x-axis. Also find the intervals where $\dfrac{dy}{dx}$ remains positive throughout, or negative throughout.

Find the points of maxima and minima of the curve and the points of inflexion, if any, by finding the value of $\dfrac{d^2y}{dx^2}$ at the points where $\dfrac{dy}{dx} = 0$. Also find the interval where $\dfrac{d^2y}{dx^2} > 0$, so the curve is concave upward there and interval where $\dfrac{d^2y}{dx^2} < 0$, so the curve is concave downward there, refer to Section 4.8.5.

7. Find the co-ordinates of some special points if necessary, which help in tracing out the curve. Sometimes it helps to convert the equation from the cartesian to polar form.

Taking all these observations in consideration we can draw the approximate shape of the curve as worked out in examples to follow.

Example 4.93: Trace the cissoid $y^2(2a - x) = x^3$.

Solution: (i) The curve is symmetrical about the x-axis, since only even powers of y occur.

(ii) The curve passes through the origin, since there is no constant term in its equation. Equating to zero the lowest degree terms, the tangents at the origin are $y = 0$, $y = 0$. Tangent being coincident, therefore origin is cusp.

(iii) The curve has an asymptote $x = 2a$, obtained by equating to zero the coefficient of y^2.

(iv) Also the curve meets the axes at (0, 0) only.

Further from the equation of the curve $y^2 = x^3/(2a - x)$.

When x is $-ve$, y^2 is $-ve$, so that no portion of the curve lies to the left of the y-axis. Also when $x > 2a$, y^2 is again $-ve$, so that no portion of the curve lies to the right of the line $3a = 2a$. The approximate shape of the curve is shown in Fig. 4.27.

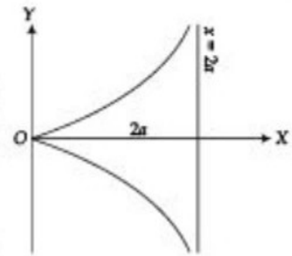


Fig. 4.27

Example 4.94: Trace the semi-cubical parabola $y^2 = x^3$.

Solution: (i) The curve is symmetrical about x-axis, since only even power of y occurs.

(ii) Curve passes through the origin, since there is no constant term. By equating to zero the lowest degree terms the tangent at the origin are $y^2 = 0$, which gives, $y = 0, y = 0$.

The tangents being coincident, there is a cusp at the origin.

(iii) There are no asymptotes parallel to either axis.

(iv) Solving for y, we get $y = \pm(x)^{3/2}$. When x is negative, the value of y becomes imaginary. Hence, the curve does not lie on the left of y-axis.

As x increases from 0 to ∞, y also increases from 0 to ∞. The approximate shape of the curve is shown in Fig. 4.28.

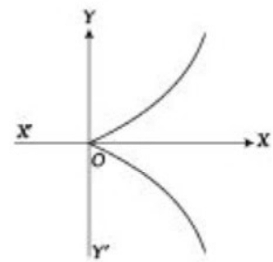


Fig. 4.28

Example 4.95: Trace the folium of Descartes $x^3 + y^3 = 3axy$.

Solution: (i) The curve is symmetrical about the line $y = x$, since the equation remains unchanged when x and y are interchanged.

(ii) It passes through the origin and tangents at the origin are given by $xy = 0$, that is $x = 0, y = 0$, thus origin is a node.

(iii) It has no asymptote parallel to the axes, since the coefficient of the highest powers of x and y in the equation are constant.

To find oblique asymptotes put $y = m$, $x = 1$ in third degree terms, we obtain

$$\phi_3(m) = 1 + m^3, \quad \phi_3(m) = 0 \text{ gives } \quad m = -1. \text{ Also } \phi_2(m) = -3\,am.$$

Therefore, $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\left(\frac{-3am}{3m^2}\right) = \frac{a}{m} = -a, \text{ when } m = -1.$$

Hence $y = -x - a$ is an asymptote.

(iv) It meets the axes at the origin only.

Also when $y = x$, $2x^3 = 3ax^2$ which gives $x = 0$ or $3a/2$.

Thus the curve crosses the line $y = x$ at $(3a/2,\ 3a/2)$.

The approximate shape of the curve is shown in Fig. 4.29.

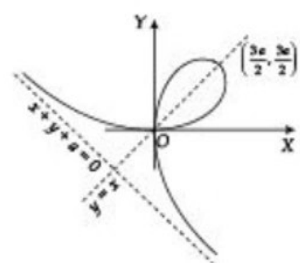


Fig. 4.29

Example 4.96: Trace the curve $y(x^2+a^2)=a^3$.

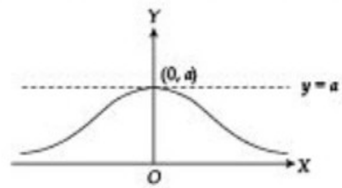


Fig. 4.30

Solution: (i) The curve is symmetrical about y-axis, since it involves only even power in x.

(ii) The curve does not meet x-axis but meets y-axis at $(0, a)$. Shifting the origin to $(0, a)$ the equation becomes $(y+a)(x^2+a^2)=a^3$, or $y(x^2+a^2)+ax^2=0$ and equating to zero the lowest degree term we get $y=0$, as the equation of the tangent at the new origin $(0, a)$ and w.r.t. $O(0,0)$ it is $y=a$.

(iii) Solving the given equation for x, we get $x=\pm a\sqrt{\dfrac{a-y}{y}}$ which shows that x is imaginary when $y<0$ and also when $y>a$. Thus there is no curve below x-axis and above the line $y=a$.

(iv) Equating to zero the coefficient of the highest degree term in x we get $y=0$, thus x-axis is asymptote to the curve.

(v) The approximate shape of the curve is shown in Fig. 4.30.

Example 4.97: Trace the curve $y(x^2-1)=x^2+1$.

Solution: (i) Since there are only even powers in x, so the curve is symmetrical about y-axis.

(ii) The curve does not pass through the origin.

(iii) The y-axis cuts the curve at $(0, -1)$. Shifting the origin to $(0, -1)$, the equation of the curve becomes

$$(y-1)(x^2-1)=x^2+1 \text{ or, } yx^2-2-2x^2=0.$$

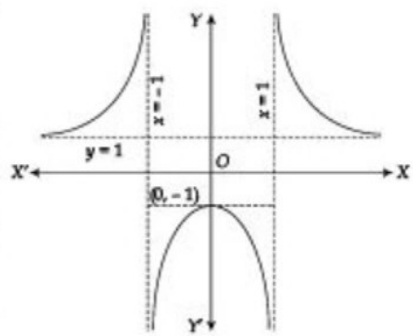


Fig. 4.31

Now equating to zero, the lowest degree terms, we get the equation of the tangent at the new origin $(0, -1)$ as $y=0$ and w.r.t. old origin as $y=-1$.

(iv) Equating to zero, the coefficients of the highest degree terms in x and y respectively, we get the asymptotes parallel to the axes as $y-1=0$ and $x^2-1=0$ i.e., $x=1$, $x=-1$.

(v) Rewriting as $y=[(x^2+1)/(x^2-1)]$. As x varies from 0 to 1, y varies from -1 to ∞ and as x varies from 1 to ∞, y varies from ∞ to 1. The approximate shape of the curve is shown in Fig. 4.31.

Example 4.98: Trace the asteroid $x^{2/3}+y^{2/3}=a^{2/3}$.

Solution: (i) Writing the equation of the curve as $(x^{1/3})^2+(y^{1/3})^2=a^{2/3}$. It is clear that the curve is symmetrical about x-axis and y-axis both.

(ii) The curve does not pass through the origin but meets the x-axis at $(\pm a, 0)$ and the y-axis at $(0, \pm a)$.

(iii) Differentiating the given equation, we get

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

At $(\pm a, 0)$ the slope of the tangent is 0 hence x-axis is the tangent at these points. At $(0, \pm a)$ the slope of the tangent is infinity, hence y-axis is tangent at these points.

(iv) Writing the equation in parametric form as $x = a\cos^3 t$, $y = a\sin^3 t$ which shows that $-a \le x \le a$ and $-a \le y \le a$. Therefore no point of the curve lies outside the square $x = \pm a, y \pm a$.

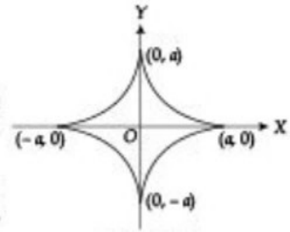


Fig. 4.32

(v) The approximate shape of the curve is shown in Fig. 4.32.

Example 4.99: Trace the Lemniscate of Bernoulli $y^2(a^2 + x^2) = x^2(a^2 - x^2)$.

Solution: (i) Since the equation contains even powers both in x and y, the curve is symmetrical about both the axes.

(ii) The equation does not contain any constant term, therefore the curve passes through the origin. Equating to zero the lowest degree term; the tangents at the origin are given by $y^2 = x^2 \Rightarrow y = \pm x$, which being real and distinct, therefore origin is a node.

(iii) The curve meets the x-axis at $O(0, 0)$, $A(a, 0)$ and $A'(-a, 0)$.

Shifting the origin to the point $A(a, 0)$, the equation becomes

$$y^2[a^2 + (x + a)^2] = (x + a)^2[a^2 - (x + a)^2]$$

or, $$y^2(x^2 + 2ax + 2a^2) = -(x + a)^2(x^2 + 2ax)$$

Equating to zero the lowest degree terms we get the tangent at the new origin $(a, 0)$ as $x = 0$, w.r.t. $O(0, 0)$ it is $x = a$, a line parallel to the y-axis.

Similarly, the tangent at the point $A'(-a, 0)$ is $x = -a$, again a line parallel to y-axis.

(iv) Clearly the curve has no asymptotes parallel to the axes. Also we can show very easily that the curve has no oblique asymptotes, (since $\phi_4(m) = 1 + m^2$ and $\phi_4(m) = 0$ gives imaginary values of m).

(vi) Solving the equations for y, we get $y = \pm x\sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}$, y is real only for $-a \le x \le a$.

Therefore the curve lies between the lines $x = \pm a$.

In the first quadrant $y = x\sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}$, therefore

$$\frac{dy}{dx} = \left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{\frac{1}{2}}\left[\frac{a^4 - x^4 - 2a^2x^2}{(a^2 - x^2)(a^2 + x^2)}\right] = \frac{a^4 - x^4 - 2a^2x^2}{(a^2 - x^2)^{1/2}(a^2 + x^2)^{3/2}}.$$

$\frac{dy}{dx}=0$ gives $a^4-x^4-2a^2x^2=0$, or $x=a\sqrt{\sqrt{2}-1}$. Thus, in the first quadrant when $x=0, y=0$; as x increases, y also increases and maxima reaches at $x=a\sqrt{\sqrt{2}-1}$; as x increases further from $a\sqrt{\sqrt{2}-1}$ to a, y decreases and ultimately becomes zero when $x=a$. The maximum value of y at $x=a\sqrt{\sqrt{2}-1}$ is $a(\sqrt{2}-1)$.

The approximate shape of the curve is shown in Fig. 4.33.

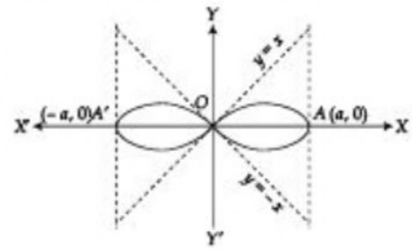


Fig. 4.33

Example 4.100: Trace the curve $x^2y^2=a^2(x^2+y^2)$.

Solution: (i) The curve is symmetrical about x-axis and y-axis both. Also it is symmetrical about the line $y=x$. Curve is symmetrical in opposite quadrants also.

(ii) The point (0, 0) lies on the curve and the tangents at the origin are given by $x^2+y^2=0$. But these are the imaginary, so the origin is an isolated point.

(iii) Equating to zero the coefficients of highest powers of x and y respectively, we get the asymptotes as $x=\pm a$ and $y=\pm a$.

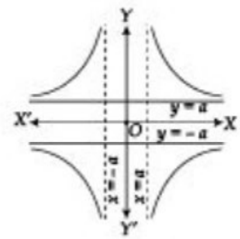


Fig. 4.34

(iv) Solving for y, we have $y^2=\frac{a^2x^2}{(x^2-a^2)}$ which gives $y=\pm\frac{ax}{\sqrt{(x^2-a^2)}}$.

When $0<x<a$, y is imaginary. It shows that the curve does not exist in the region bounded by the lines $x=0$ and $x=a$. As x varies from a to ∞, y varies from ∞ to a. The approximate shape of the curve is shown in Fig. 4.34.

Example 4.101: Trace strophoid $y^2(x+a)=x^2(3a-x)$.

Solution: (i) The curve is symmetrical about x-axis.

(ii) The curve passes through the origin. Equating to zero the lowest degree term, we get $ay^2=3ax^2$ or $y=\pm x\sqrt{3}$ as two tangents to the curve at the origin.

(iii) The curve meets y-axis only at the origin while it meets x-axis at $(3a, 0)$ also. Shifting the origin to this point the equation of the curve becomes

$$y^2(x+4a)=-(x+3a)^2x.$$

Equating to zero the lowest degree term we get $x=0$ as the tangent at the new origin $(3a, 0)$ and w.r.t. $O(0, 0)$, it is $x=3a$. Thus, at $(3a, 0)$ tangent is $x=3a$.

(iv) y is imaginary for $x<-a$ and for $x>3a$, therefore, there is no portion of the curve on the left of the line $x=-a$ and on the right of the line $x=3a$.

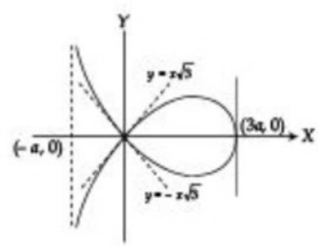


Fig. 4.35

(v) There is no asymptote parallel to x-axis, and for asymptote parallel to y-axis, we equate to zero the coefficient of the highest degree term in y so as to get $x + a = 0$ as the asymptote parallel to y-axis.

(vi) Also we can check that y is maximum at $x = a\sqrt{3}$. The approximate shape of the curve is shown in the Fig. 4.35.

EXERCISE 4.17

Trace the following curves

1. $y^2a = x^3 + y^2x, \quad a > 0$ (cissoid)
2. $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ (astroid)
3. $y = c\cosh\frac{x}{c}$ (catenary)
4. $x^2y^2 = a^2(y^2 - x^2)$
5. $x^3 + y^3 = a^2x$
6. $4ay^2 = x(x - 2a)^2, \quad a > 0$
7. $x(x^2 + y^2) = a(x^2 - y^2)$ (strophoid)
8. $y^2x = a^2(x - a), \quad a > 0$
9. $(x^2 - a^2)(y^2 - b^2) = a^2b^2, \quad a, b > 0$
10. $xy^2 + (x + a)(x + 2a) = 0, \quad a > 0$
11. $y = x^2/1 - x^2$
12. $y^3 = x(a^2 - x^2)$
13. $a^2/x^2 - b^2/y^2 = 1$
14. $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$
15. $y^2(a + x) = (a - x)^3$.

4.12.2 Tracing of Polar Curves

Similar to the curves in cartesian co-ordinates, we need to observe the following characteristics of the polar curves from the equation given.

1. *Symmetry:* (a) If θ is changed to $-\theta$ and the equation of the curve remains unchanged, then the curve is symmetrical about the initial line $\theta = 0$, (that is x-axis). For example, cardiod $r = a(1 + \cos\theta)$ is symmetrical about the initial line, refer to Fig. 4.36a.

(b) If θ is changed to $(\pi - \theta)$ and the equation of the curve remains unchanged, then the curve is symmetrical about the line $\theta = \pi/2$ (that is, y-axis). For example, the cardiod $r = a(1 + \sin\theta)$ is symmetrical about the line $\theta = \pi/2$, refer to Fig. 4.36b.

(c) If θ is changed to $\pi + \theta$ and the equation of the curve remains unchanged, then the curve is symmetrical about the pole (symmetry in opposite quadrant). For example, the curve $r = a\sin 2\theta$, (four leaved rose) has symmetry about the pole, refer to Fig. 4.36c.

(d) If θ is changed to $(\pi/2 - \theta)$ and the equation of the curve remains unchanged, then curve is symmetrical about the line $\theta = \pi/4$, (that is, the line $y = x$). For example, the curve $r = a\sin 2\theta$, refer to Fig. 4.36c.

2. *Passing through pole:* If $r = 0$ for some $\theta = \alpha$, then the curve passes through the pole and the line $\theta = \alpha$ is the tangent to the curve at the pole. For example, at $\theta = \pi$, $r = a(1 + \cos\theta) = 0$. Thus the curve passes through the pole and $\theta = \pi$ is the tangent there as shown in Fig. 4.36a.

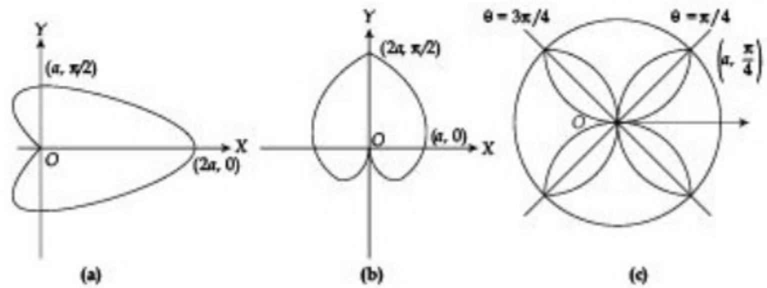


Fig. 4.36

3. ***Direction of the tangent:*** Find $\tan\phi = r\dfrac{d\theta}{dr}$, where ϕ is the angle between the tangent to the curve at a point $P(r, \theta)$ and its radius vector. It gives direction of the tangent at the point $P(r, \theta)$.

The points of specific interest are the points where the tangent concides with the radius vector or is perpendicular to it, that is, where $\tan\phi$ is zero or infinity.

For example, for the Lemniscates $r^2 = a^2 \sin 2\theta$, we have $2r\dfrac{dr}{d\theta} = 2a^2 \cos 2\theta$, which gives $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{r^2}{a^2 \cos 2\theta} = \tan 2\theta$, thus $\phi = 2\theta$. So when $\theta = \pi/4$, $\phi = \dfrac{\pi}{2}$. Thus, tangent to the curve at $\left(a, \dfrac{\pi}{4}\right)$ is perpendicular to the line $\theta = \pi/4$, refer to Fig. 4.37.

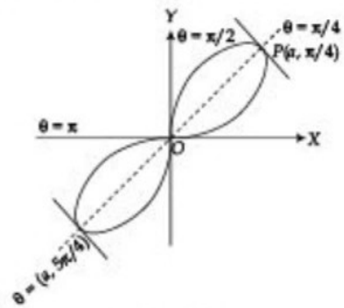


Fig. 4.37

4. ***Asymptotes:*** If $\dfrac{1}{r}$ becomes zero for $\theta = \theta_1$, then equation of the asymptote to the curve $\dfrac{1}{r} = f(\theta)$ at $\theta = \theta_1$ is $r \sin(\theta - \theta_1) = 1/f'(\theta_1)$. provided $f'(\theta_1) \neq 0$. For example, the spiral $r\theta = a$ has $r \sin\theta = a$ (that is, $y = a$) as asymptote, refer to Example 4.89.

5. ***Interaction with axes:*** Determine the points where the curve meets the lines $\theta = 0$, $\theta = \pi/4$, $\theta = \pi/2$, $\theta = \pi$ and $\theta = 3\pi/2$, etc.

6. ***Region:*** Determine the limits, if any, to which r and θ are confined. Find the greatest value of r (numerically), so as to find whether the curve lies within a circle or not. For example, $r = a \sin 2\theta$ lies within the circle $r = a$, refer to Fig. 4.36c.

Also find those values of θ for which r is imaginary. For example, $r^2 = a^2 \sin 2\theta$ does not lie between the lines $\theta = \pi/2$ and $\theta = \pi$, refer to Fig. 4.37.

7. *Loop*: If a curve meets a line at points A and B and the curve is symmetrical about that line, then a loop of the curve exists between A and B. For example, the curve $r = a\sin 2\theta$ is symmetrical about the line $\theta = \pi/4$ and meets it at the pole and at the point $(a, \pi/4)$. Hence it forms a loop about the line $\theta = \pi/4$ between the pole and the point $(a, \pi/4)$ as shown in Fig. 4.36c.

Example 4.102: Trace the cardiod $r = a(1 - \cos\theta)$.

Solution: (i) By changing θ to $-\theta$, the equation of the curve remains unchanged hence the curve is symmetrical about the initial line.

(ii) $r = 0$ gives $1 - \cos\theta = 0$, which implies that $\theta = 0$, thus the curve passes through the pole and initial line is tangent to it.

(iii) As θ increase from 0 to π, r increases from 0 to $2a$.

(iv) Also $\tan\phi = r\dfrac{d\theta}{dr} = \tan(\theta/2)$, which gives $\phi = \theta/2$. At $\theta = \pi$, $\phi = \pi/2$. So the tangent at $(2a, \pi)$ is perpendicular to the initial line.

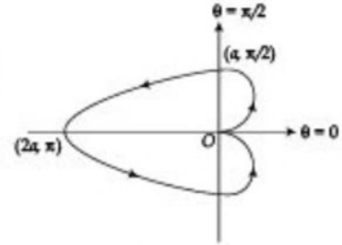


Fig. 4.38

The approximate shape of the curve is shown in Fig. 4.38.

Example 4.103: Trace $r = a + b\cos\theta$; $a > b$ (Lamicon).

Solution: (i) The curve is symmetrical about the initial line.

(ii) At $\theta = 0$, $r = a + b$; $\theta = \pi$, $r = a - b$

Also the curve does not pass through pole as $r = 0$ gives $\theta = \cos^{-1}(-a/b)$, $a > b$; thus there is no real value of θ for $r = 0$.

(iii) Also, $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a + b\cos\theta}{-b\sin\theta}$.

At $\theta = 0$, $\phi = \pi/2$; $\theta = \pi$, $\phi = \pi/2$

thus the tangents to the curve at $(a + b, 0)$ and $(a - b, \pi)$ are perpendicular to the initial line. The shape is shown in Fig. 4.39.

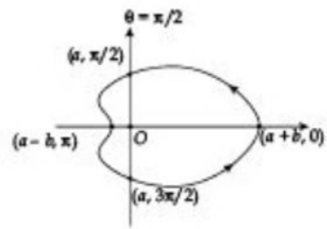


Fig. 4.39

Example 4.104: Trace the curve $r = a\sin 3\theta$, (three leaved rose)

Solution: (i) The curve is symmetrical about the line $\theta = \pi/2$, since changing $\theta \to \pi - \theta$ does not change the equation.

(ii) The curve wholly lies within the circle $r = a$, since $r = a\sin 3\theta$ gives $|r| < a$; and obviously it has no asymptotes.

(iii) Also $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a\sin 3\theta}{3a\cos 3\theta} = \dfrac{1}{3}\tan 3\theta$

Thus, $\phi = 0$, when $\theta = 0, \ \pi/3, \ldots$

$\phi = \pi/2$, when $\theta = \pi/6, \pi/2, \ldots$

Consider the following table for $0 \le \theta \le \pi/2$.

θ	r	Portion
0 to $\pi/6$	0 to a	O to A
$\pi/6$ to $\pi/3$	a to 0	A to O
$\pi/3$ to $\pi/2$	0 to $-a$	O to B

For, from $\pi/2$ to π, portions of the curve from B to O, O to C and C to O the curve is traced by symmetry about the line $\theta = \pi/2$.

Hence, the curve consists of three loops as shown in Fig. 4.40.

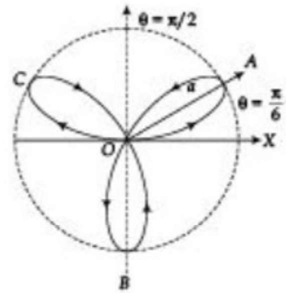


Fig. 4.40

Example 4.105: Trace the Leminiscate of Bernoulli $r^2 = a^2 \cos 2\theta$.

Solution: (i) The curve is symmetric about initial line, the line $\theta = \pi/2$ and about the pole.

(ii) The curve lies wholly within the circle $r = a$ since $r^2 \le a^2$. Also no portion of the curve lies between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$, since r^2 is negative for $\pi/4 < \theta < \frac{3\pi}{4}$.

(iii) Further $\tan\phi = r\frac{d\theta}{dr} = -\cot 2\theta = \tan\left(\frac{\pi}{2} + 2\theta\right)$, that is, $\phi = \frac{\pi}{2} + 2\theta$. Thus $\phi = 0$, when $\theta = -\pi/4$ and the tangent at A is perpendicular to the initial line.

(b) The variations of r and θ and the portion traced is given as:

θ	r	Portion
0 to $\pi/4$	a to 0	ABO
$3\pi/4$ to π	0 to a	OCD

As θ increases from π to 2π, the portion can be traced by symmetry about the initial line. The approximate shape of the curve is shown in Fig. 4.41.

Example 4.106: Trace the curve $r = a\,(\cos\theta + \sec\theta)$.

Solution: (i) Changing θ to $-\theta$ does not change the equation, so the curve is symmetric about the initial line.

(ii) $r = 0$, gives $\cos^2\theta = -1$, hence the curve does not pass through pole for any real θ.

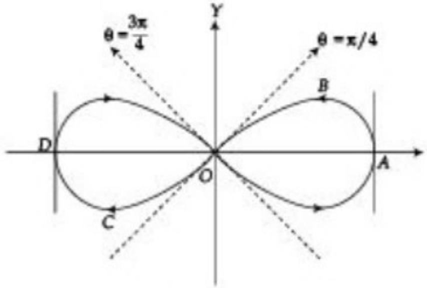


Fig. 4.41

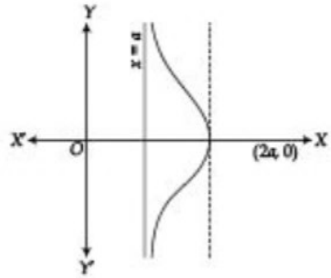


Fig. 4.42

(iii) $r = a(\cos\theta + \sec\theta)$ gives $r^2 = a\left[r\cos\theta + \dfrac{r^2}{r\cos\theta}\right]$; changing to cartesian co-ordinates it gives $x(x^2 + y^2) = a(2x^2 + y^2)$.

Equating the coefficient of highest power of y to zero gives $x = a$ as asymptote to the curve.

(iv) Also the curve meets the x-axis at $(2a, 0)$, the line $x = 2a$ is tangent at $(2a, 0)$.

(v) Also $y = \dfrac{x\sqrt{2a - x}}{\sqrt{x - a}}$ gives that y is defined only for $a < x \le 2a$ so the curve lies between the lines $x = a$ and $x = 2a$.

The approximate shape of the curve is shown in Fig. 4.42.

Example 4.107: Trace the equiangular spiral $r = ae^{b\theta}$ where $a, b > 0$ are constants.

Solution: (i) There is no symmetry in this curve.

(ii) At $\theta = 0$, we have $r = a$ when θ increases indefinitely r also increases indefinitely.

(iii) Consider $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{ae^{b\theta}}{abe^{b\theta}} = \dfrac{1}{b}$, a constant.

Thus, $\phi = \tan^{-1}\dfrac{1}{b} = \alpha$, say.

Hence the radius vector at any point of the curve always makes a constant angle with the tangent at that point.

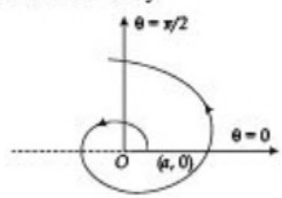


Fig. 4.43

The approximate shape of the curve is shown in Fig. 4.43.

Example 4.108: Trace the curve $r^2\cos 2\theta = a^2$.

Solution: (i) Since changing θ to $-\theta$ and θ to $\pi - \theta$, the equation of the curve remains unchanged, so the curve is symmetric about the initial line and about the line $\theta = \pi/2$.

(ii) Changing r to $-r$, the equation remains unchanged so the curve is symmetric about the pole.

(iii) Since $r^2 \cos 2\theta = a^2$ gives, $r^2 \geq a^2$, thus the curve is only in the region defined by $r \geq a$.

(iv) We have $r^2 \cos 2\theta = r^2(\cos^2\theta - \sin^2\theta) = x^2 - y^2$; thus the curve is $x^2 - y^2 = a^2$, a hyperbola.

Thus, $y = \pm x$, which give $\theta = \pm\pi/4$, are its asymptotes.

The approximate shape of the curve is shown in Fig. 4.44.

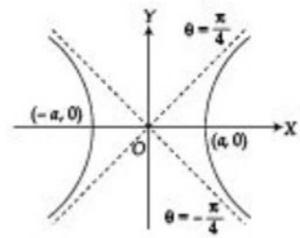


Fig. 4.44

EXERCISE 4.18

Trace the following curves

1. $r = a(1 + \sin\theta)$ (cardiod)
2. $r = 1 + \sqrt{2}\cos\theta$ (cardiod)
3. $r = ae^{\theta\cot\alpha}$, $a > 0$ (equiangular spiral)
4. $r\cos\theta = a\sin^2\theta$, $a > 0$
5. $r = a\cos 2\theta$ (four-leaved rose)
6. $r = a\cos 3\theta$ (three-leaved rose)
7. $r = a\sin 4\theta$ (eight-leaved rose)
8. $r = a\sin 5\theta$ (five-leaved rose)

4.12.3 Tracing of Parametric Curves

In case of the curve $x = f(t), y = g(t)$, we try to eliminate the parameter t to get the curve in cartesian form since it is comparatively easy to plot the curve in cartesian form. However, to plot the curve in parametric form itself, we account for the following characteristics.

1. ***Origin:*** If for any value of t, $x = 0$ and $y = 0$, then the curve passes through the origin.

2. ***Intercept with the axes:*** Find the values of t for which $f(t) = 0$, then find $y = g(t)$ for those values of t, the curve meets the y-axis at these points.

Similarly, find the point of intersection of the curve with the x-axis.

3. ***Regions:*** Find the greatest and the least values of x and y which give the region in which the curve lies, that is find the limits of the parameter 't' beyond which x or y cannot lie.

4. ***Asymptotes:*** In case there is some $t = t_1$ such that $\lim\limits_{t\to t_1} x = \infty$ and $\lim\limits_{t\to t_1} y = \infty$, then $t = t_1$ is an asymptote.

5. ***Derivatives:*** In general, it is difficult to check the various symmetries for the curve in parametric form so finding derivatives is very essential. Find dx/dt and dy/dt and note the values of t for which x, y are increasing or decreasing functions of t. Find $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$. Determine the points where tangent is parallel or perpendicular to the x-axis, that is the points where $\frac{dy}{dx} = 0$ or ∞.

Also find $\frac{d^2y}{dx^2}$ and check for the values of parameter t for which $\frac{d^2y}{dx^2} = 0$, $\frac{d^2y}{dx^2} > 0$, or $\frac{d^2y}{dx^2} < 0$ to know about the points of inflexion, concavity or convexity of the curve.

To plot the curve, assign the parameter t different values; find the corresponding values of x, y and also behaviour of dy/dx. We get different points on the curve and slope of the tangents at these points. Taking into consideration the characteristics mentioned above join these points to draw the curve.

Example 4.109: Trace the asteroid $x = a\cos^3 t, y = a\sin^3 t$.

Solution: The curve is symmetric about x-axis, since $t \to -t$ does not change x.

Further $|x|, |y| \le a$, hence the curve lies within the square bounded by the lines $x = \pm a$ and $y = \pm a$.

Also,
$$\frac{dx}{dt} = -3a\cos^2 t \sin t,$$
$$\frac{dy}{dt} = 3a\sin^2 t\cos t.$$

Hence, $\frac{dy}{dx} = -\tan t$. Therefore, $\frac{dy}{dx} = 0$ at $t = 0$ and $\frac{dy}{dx} \to \infty$ at $t = \pi/2, 3\pi/2$.

Following table gives the value of $t, x, y, dy/dx$ and the corresponding portion traced.

t	x	y	dy/dx	Portion
0 to $\pi/2$	+ve, decreases a to 0	+ve, increases 0 to a	0 to $-\infty$ through $-$ve	A to B
$\pi/2$ to π	$-$ve, decreases 0 to $-a$	+ve, decreases a to 0	∞ to 0 through +ve	B to A

Since, curve is symmetric about initial line, portion from π to 2π is the image of 0 to π in the initial line. For $t > 2\pi$, the curve repeats. An approximate shape is shown in Fig. 4.45.

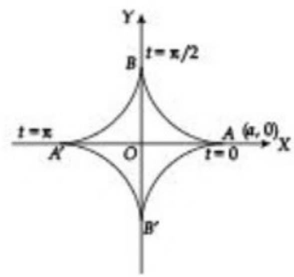


Fig. 4.45

Example 4.110: Trace the cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$.

Solution: (i) Since changing θ to $-\theta$ curve remains unchanged, thus the curve is symmetrical about y-axis, so it is sufficient to consider the curve only for $\theta > 0$.

(ii) The greatest and the least value for y are $2a$ and zero. Hence, the curve lies between the lines $y = 2a$ and $y = 0$.

(iii) The curve passes through the origin at $\theta = 0$. It meets the y-axis only at $(0, 0)$ while it meets x-axis where $1 - \cos\theta = 0$, that is, at $(0, 0), (2a\pi, 0), (4a\pi, 0)$ etc.

(iv) $\frac{dx}{d\theta} = a(1 + \cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$, thus $dy/dx = \tan\theta/2$.

Hence, $dy/dx = 0$, at $\theta = 0. 2\pi, 4\pi$, etc.
$= \infty$ at $\theta = \pi, 3\pi, 5\pi$, etc.

Tangent is parallel to y-axis at $x = a\pi, 3a\pi, 5a\pi$ etc. Following table gives the values of $\theta, x, y, dy/dx$ and the corresponding portion traced.

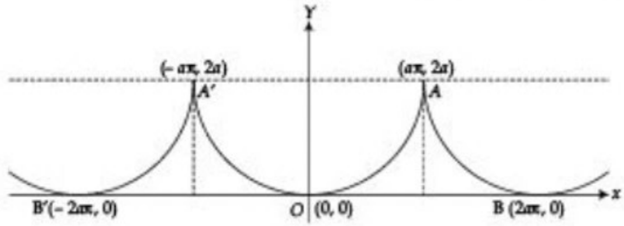


Fig. 4.46

θ	x	y	dy/dx	Portion
0 to π	Inc. 0 to $a\pi$	Inc. 0 to $2a$	0 to ∞	O to A
π to 2π	Inc. $a\pi$ to $2a\pi$	dec. $2a$ to 0	∞ to 0	A to B

For $-2\pi \le \theta \le 0$, the curve is image of the portion $0 \le \theta \le 2\pi$ in the y-axis. $A'OA$ is one cycloid corresponding to $-\pi \le \theta \le \pi$. The curve extends to ∞ on both the sides as shown in Fig. 4.46.

Example 4.111: Trace the tractrix $x = a(\cos t + \ln|\tan(t/2)|)$, $y = a \sin t$.

Solution: Rewriting the equation as $x = a\left[\cos t + \frac{1}{2}\ln\tan^2\frac{t}{2}\right]$, $y = a \sin t$.

(i) Since t changes to $-t$, x remains unchanged so the curve is symmetrical about x-axis.

(ii) The origin does not lie on the curve.

For $t = \pm\pi/2$, we get $x = 0$ and $y = \pm a$. Hence, the curve meets the y-axis at points $(0, \pm a)$

(iii) $|y| < a$, hence curve lies within the lines $y = a$ and $y = -a$.

(iv) As $t \to 0$, we have $x \to \infty$ and $y \to 0$, therefore x-axis is an asymptote to the curve.

(v) Consider $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = a\cos t \Big/ \frac{a\cos^2 t}{\sin t} = \tan t$.

Hence, $\frac{dy}{dx} = 0$, when $t = 0$

$= \infty$, when $t = \pm\pi/2$,

that is, at $(0, \pm a)$, so y-axis is tangent there.

Following table gives the values of t, x, y and dy/dx and the corresponding portion of the curve

t	x	y	dy/dx	Portion of the curve
0 to $\pi/2$	$-\infty$ to 0	Inc. 0 to a	0 to ∞	$-\infty$ to A
$\pi/2$ to π	Inc. 0 to ∞	dec. a to 0	∞ to 0	A to $+\infty$

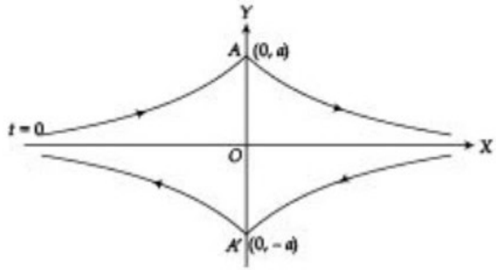


Fig. 4.47

Since it is symmetric about x-axis, curve for the portion π to 2π is the image in x-axis of the portion $0 \le x \le \pi$. The shape is shown in Fig. 4.47.

EXERCISE 4.19

Trace the following curve

1. $x = a\sec t,\quad y = b\tan t$
2. $x = a\cos^3 t,\quad y = b\sin^3 t$
3. $x = a(t - \sin t),\quad y = a(1 - \cos t),\quad 0 \le t \le 2\pi$
4. $x = a(t - \sin t),\quad y = a(1 + \cos t),\quad 0 \le t \le 2\pi$
5. $x = a(t + \sin t),\quad y = a(1 + \cos t)$
6. $x = a\sin 2t(1 + \cos t),\quad y = a\cos 2t(1 - \cos 2t)$.

ANSWERS

Exercise 4.1 (p. 188)

2. $f'(x) = 2|x|$

3. (a) $(-1)^{n-1}(n-1)!\left[\dfrac{1}{(x-1)^n} + \dfrac{1}{(x+2)^n}\right]$ (b) $(-1)^n \dfrac{n!c^{n-1}}{(cx+d)^{n+1}}(bc - ad)$

7. $y_n(0) = \begin{cases} \{m^2-(n-2)^2\}\ \{m^2-(n-4)^2\}\ \ldots\ (m^2) \\ \{m^2-(n-2)^2\}\ \{m^2-(n-4)^2\}\ \ldots\ (m^2-1^2)(m) \end{cases}$

Exercise 4.2 (p. 190)

1. 36π 2. ± 0.3 3. $25/\pi$ 4. 36.35 5. 0.0016 cm.

Exercise 4.3 (p. 200)

6. $a\cos t$ 9. $\pi/3$ 11. (a) $r^3 = a^2 p$ (b) $p^2 + a^2 = r^2$ 12. $\dfrac{(3x+a)}{2\sqrt{3ax}}$.

Exercise 4.4 (p. 206)

1. $(1, 1)$ 4. 3.0049 5. 1.43
8. Increasing in $[-1, 1]$; decreasing in $(-\infty, -1)$, and $(1, \infty)$.

Exercise 4.5 (p. 216)

2. (i) $\cos x = \dfrac{1}{\sqrt{2}} - \left(x - \dfrac{\pi}{4}\right)\dfrac{1}{\sqrt{2}} - \dfrac{\left(x - \frac{\pi}{4}\right)^2 \frac{1}{\sqrt{2}}}{2!} + \dfrac{\left(x - \frac{\pi}{4}\right)^3 \frac{1}{\sqrt{2}}}{3!} + \ldots$

(ii) $e^2\left[1 + (x-2) + \dfrac{(x-2)^2}{2!} + \dfrac{(x-2)^3}{3!} + \ldots\right]$

(iii) $1 - \dfrac{1}{2!}\left(x - \dfrac{\pi}{2}\right)^2 + \dfrac{1}{4!}\left(x - \dfrac{\pi}{2}\right)^4 - \ldots$, 0.9998.

(iv) $\dfrac{\pi}{4} + \dfrac{1}{2}(x-1) - \dfrac{1}{4}(x-1)^2 + \dfrac{1}{12}(x-1)^3 + \ldots$

(v) $45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$

3. 0.8482 4. 2.6121 5. 8.849 6. 4.123

9. (i) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$ (ii) $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$

(iii) $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \ldots$ (iv) $x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \ldots$

(v) $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7}$ (vi) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$

(vii) $1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \ldots$ (viii) $\dfrac{\pi}{4} + \dfrac{x}{2} - \dfrac{x^2}{4} + \dfrac{x^3}{12} - \ldots$

(ix) $a^h\left[1 + x(\ln a) + \dfrac{x^2}{2!}(\ln a)^2 + \dfrac{x^3}{3!}(\ln a)^3 + \ldots\right]$ 16. 6

Exercise 4.6 (p. 225)

1. 0 2. 3/2 3. 1 4. 1/2 5. $a=2$; limit -1

6. 1 7. ln 8. 1 9. $-\dfrac{1}{3}$ 10. 1

11. $-\dfrac{e}{2}$ 12. $-\dfrac{1}{2}$ 13. 1 15. continuous at the origin.

Exercise 4.7 (p. 231)

1. Minima at $(3-\sqrt{17})/4$ and 3; maxima at $(3+\sqrt{17})/4$. Neither maxima nor minima at -1.
2. Minimum value 0 at $x=0$.
3. Maximum $4/e^2$ at $x=\pm\sqrt{2}$; minimum 0 at $x=0$.
4. Maximum 14 at $t=-1$, minimum -17.25 at $t=3/2$.
5. Minima at $\pi/2, 3\pi/2$; maxima at $\pi/6, 5\pi/6$.
6. Minima at 0; maxima at -3 and 1.
7. Minima at $5\pi/3$; maxima at $\pi/3$.
8. Height = radius = $3\sqrt{v_0/\pi}$. 10. $\sin^{-1} 1/3$.

13. Pts. are $\left(-2-\sqrt{3}, \dfrac{-\sqrt{3}-1}{4}\right)$, $\left(-2+\sqrt{3}, \dfrac{1+\sqrt{3}}{4}\right)$, (1,1).

Exercise 4.8 (p. 238)

1. (i) $a\cos\psi$ (ii) $\dfrac{4}{3}a\cos\dfrac{1}{3}\psi$ (iii) $2a\sec^3\psi$ 2. (i) $\dfrac{a}{\sqrt{2}}$ (ii) $\dfrac{5\sqrt{5}}{4}$

(iii) $\dfrac{a}{2}$ 4. $2\sqrt{2}$ 10. (i) $3a\sin\theta\cos\theta$ (ii) $3a\sin t$

(iii) $4a\cos^3\theta$ 11. $2\sqrt{2}$

Exercise 4.9 (p. 242)

1. (i) $a/2$, (ii) 1 2. (i) $\dfrac{1}{2\sqrt{2}}$, (ii) $\dfrac{37\sqrt{37}}{10}$

3. $\dfrac{1}{2}$ 5. $4a$.

Exercise 4.10 (p. 248)

1. (i) $2r^{3/2}/a^{1/2}$ (ii) $\dfrac{a^2}{3r}$ 2. (i) $\dfrac{2r^{3/2}}{\sqrt{a}}$ (ii) $r\operatorname{cosec}\alpha$ 4. $\dfrac{2\sqrt{3}}{3}a$

7. 0 8. (i) $\dfrac{2r^{3/2}}{\sqrt{a}}$ (ii) $\dfrac{2}{3}\sqrt{2ar}$ (iii) $\dfrac{a^2}{3r}$ (iv) $\sqrt{r^2-a^2}$

(v) a^2b^2/p^3 (vi) r^4/Ap^3 9. a^2b^2/p^3.

Exercise 4.11 (p. 252)

1. $\left(3x+2a_1-2a^{-\frac{1}{2}}x^{3/2}\right)$, $4(x-2a)^3=27\,ay^2$

2. $\left[\dfrac{x}{2}\left(1-\dfrac{9x^4}{a^4}\right),\left(\dfrac{5x^3}{2a^2}+\dfrac{a^2}{6x}\right)\right]$ 7. $\left(\dfrac{\pi}{3}-\dfrac{3}{2},\dfrac{7}{4}\right)$ 9. $(a+b)(x^2+y^2)=2x+2y$

Exercise 4.12 (p. 257)

1. (i) $x^2/a^2+y^2/b^2=1$ (ii) $27ay^2=4(x-2a)^3$ (iii) $8x^3+27ay^2=0$
   (iv) $x^2-y^2=c^2$ (v) $y^2=a^2+x^2$ (vi) $x^{2/2-n}+y^{2/2-n}=c^{2/2-n}$

2. $y^2=2x+1$ 3. $y=0$ 4. $2xy=A$.

Exercise 4.13 (p. 260)

1. $x=0$ 2. $y=0,\ x=\pm1$ 3. $x=\pm a,\ y=\pm a$ 4. $x=\pm a,\ y=\pm b,$

5. No asymptote 6. $x=n\pi$.

Exercise 4.14 (p. 265)

1. $y=x+1/6,\ y=-x-1/2,\ y=-x/2+1/3$

2. $y=x+1,\ y=-x+1,\ x+2y=0$ 3. $y+x=0,\ y-x=0,\ y=x+1$

4. $y-x=0,\ y=-\dfrac{1}{2}x+\dfrac{1}{2},\ y=-\dfrac{1}{2}x-\dfrac{1}{2}$

5. $x+a=0,\ x-a=0,\ x-y+\sqrt{2}a=0,\ x-y-\sqrt{2}a=0$

6. $x=\pm a,\ y=x+a,\ y=x-a$

7. $x=0,\ y=0,\ x-y=0,\ x+y=0,\ x-2y=0,\ x+2y=0$

8. $y+x=0,\ 3y-2x-3=0,\ 3y-2x+1=0$

9. $y=-x+\dfrac{1}{2}a$ 10. $a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0.$

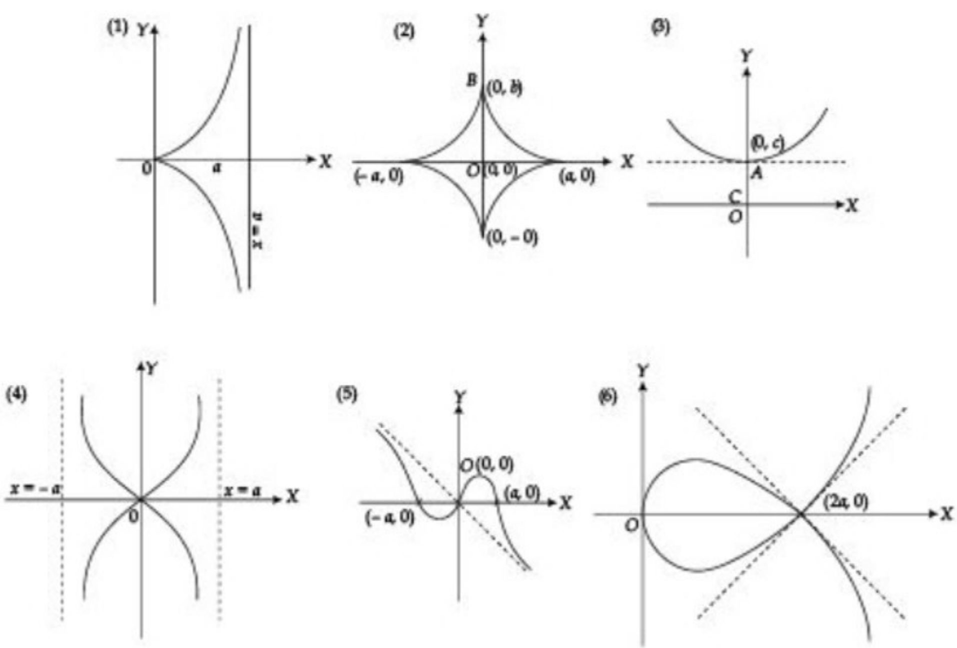
Exercise 4.15 (p. 268)

2. $x-2y=0, x+2y=0, \quad 2x-y+1=0, \quad 2x+y+1=0$
5. $x^3-6x^2y+11xy^2-6y^3-x=0$
6. $x^3-6x^2y+11xy^2-6y^3-x+6y=0$
7. $3x^2-2xy-5y^2+7x-9y-16=0.$

Exercise 4.16 (p. 270)

1. $r\sin\left(\theta\pm\frac{\pi}{6}\right)=2\sqrt{3}$
2. $r\cos\theta=8$
3. $r\sin\theta=a; \quad r\cos\theta=\frac{2a}{(2k+1)\pi}$
4. $r\sin\theta=a$
5. $r\sin(\theta-1)=a$
6. $r(\sin\theta+\cos\theta)+a=0$
8. $r=a$
9. $r=3/2$
10. $r=n\pi$, n integer.

Exercise 4.17 (p. 278)



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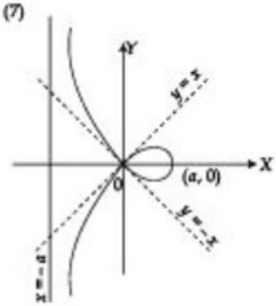

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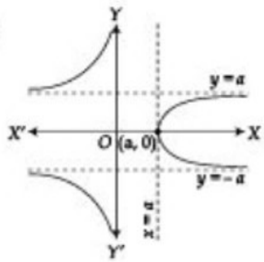

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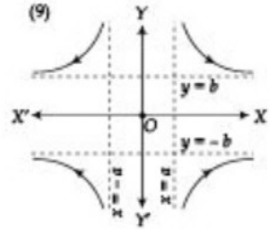

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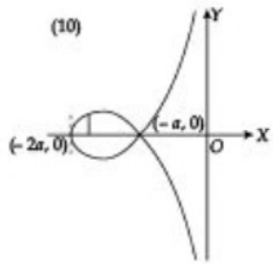

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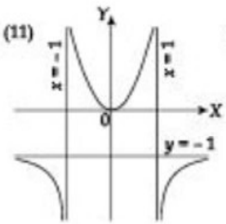

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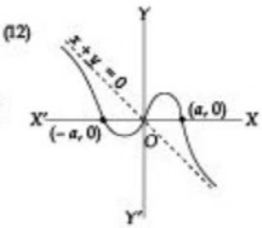

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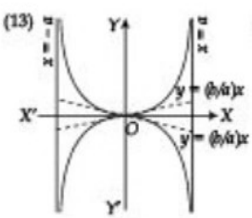

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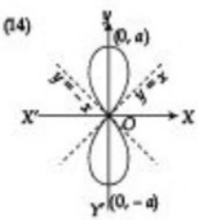

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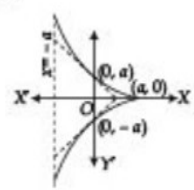
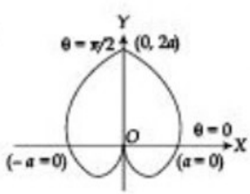


Exercise 4.18 (p. 283)

(1)



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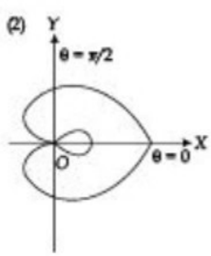

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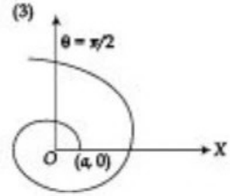

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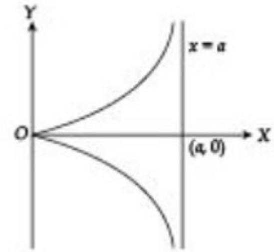

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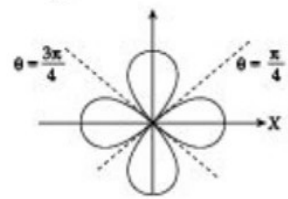

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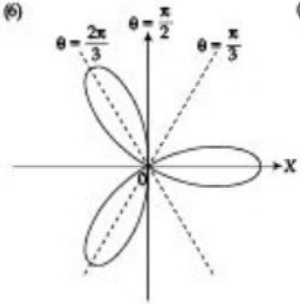

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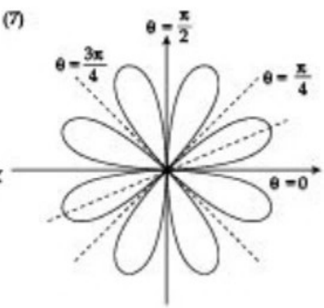

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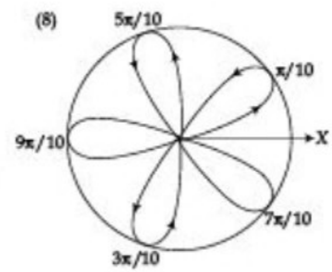
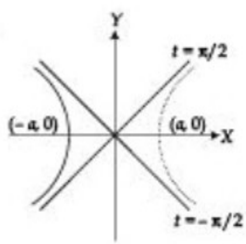


Exercise 4.19 (p. 286)

(1)



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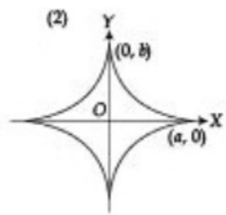

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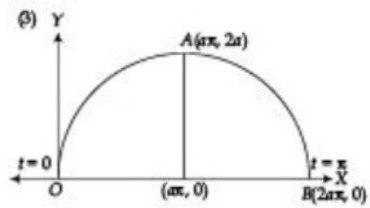

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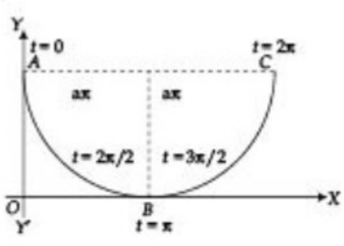

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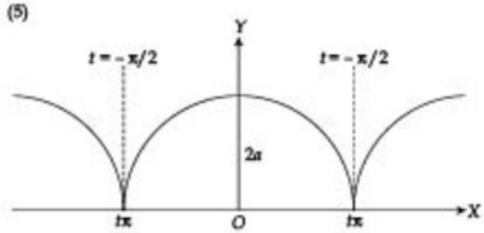

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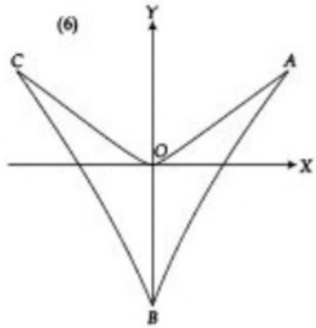


5 CHAPTER Partial Differentiation and Its Applications

Multivariable calculus is the branch of calculus that studies functions of two or more independent variables. Such functions appear more often in practical problems than functions of a single variable. Partial derivatives are the generalizations of ordinary derivatives. These are more varied and interesting because of the different ways in which variables can interact, and find applications in various fields like fluid dynamics, electricity, physics and chemistry, economics, probability and statistics to mention only a few.

5.1 FUNCTION, LIMITS AND CONTINUITY

We extend the concept of function of a single real variable defined by $y = f(x)$ to functions of two or more variables and study the notions of limit and continuity in relation to that.

5.1.1 Functions of Two Variables

The real-valued functions of two or more independent variables are defined in the same way as function of a single variable. Thus, $z = f(x, y)$ is said to be a *real-valued function of two independent real variables x and y, if corresponding to every pair of values of x and y there exists a real number z defined by* $f(x, y)$.

In finding the domain of definition we follow the usual practice of excluding inputs that lead to complex numbers or division by zero.

For example, if $f(x, y) = \sqrt{1 - x^2 - y^2}$, then domain is $x^2 + y^2 \le 1$ and the range is $[0, 1]$. Next if, $f(x, y) = \sin xy$, then the domain is the entire plane and the range is $[-1, 1]$. In case of $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$, domain is the entire space except $(0, 0, 0)$ and the range is $(0, \infty)$.

A *geometrical representation* of the function of two independent variables can be obtained by plotting the surface $z = f(x, y)$.

The set of all points $(x, y, f(x, y))$ in space for x, y belonging to the domain of f, is called the *graph* of f. Also the graph of $z = f(x, y) = c$, where c is a constant, is called a *level curve* of f.

For example for the function $f(x, y) = \sqrt{x^2 + y^2}$, the level curves are the circles $x^2 + y^2 = c^2$.

The concept of level surfaces of functions of three or more variables can be extended on similar lines. However for more than two independent variables, no simple representation for the graph of f is possible.

5.1.2 Limit and Continuity

Prior to considering the concept of limit of a function of two or more independent variables, we examine the notion of neighbourhood of a point in context to function of two independent variables (x, y).

Neighbourhood of a point: Let $P(x_0, y_0)$ be a point in $R \times R$. Then the δ–*neighbourhood* of the point $P(x_0, y_0)$, denoted by $N_\delta(P)$, is the set of all points (x, y) which lie inside a circle of radius δ with centre at the point $P(x_0, y_0)$, therefore

$$N_\delta(P) = \left\{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}.$$

It is called a *circular neighbourhood* of the point $P(x_0, y_0)$, as in Fig. 5.1a.

Another neighbourhood, a *square neighbourhood* of the point $P(x_0, y_0)$ is defined to be as

$$N_\delta(P) = \{(x, y) : |x - x_0| < \delta \text{ and } |y - y_0| < \delta\},$$

that is, the set of all points which lie inside a square of side 2δ with centre at (x_0, y_0) and sides parallel to the coordinate axes, as in Fig. 5.1b.

If the point $P(x_0, y_0)$ is not included in the set $N_\delta(P)$, then it is called the *deleted neighbourhood* of the point, that is, the set of points

$$\left\{(x, y) : 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}$$

or, $$\{(x, y) : 0 < |x - x_0| < \delta, \quad 0 < |y - y_0| < \delta\}$$

are the deleted neighbourhoods of the point $P(x_0, y_0)$.

Next we define the limit of a function $f(x, y)$ in terms of a circular neighbourhood.

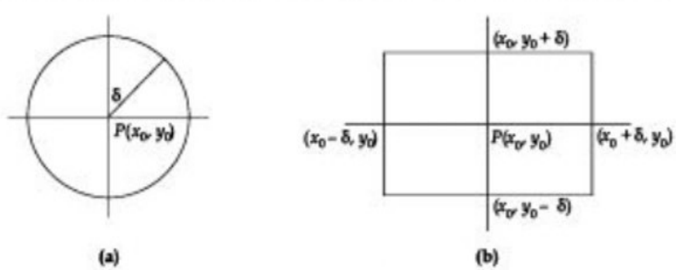


Fig. 5.1

Limit of a function $f(x, y)$: *A function $f(x, y)$ is said to approach the limit l as (x, y) approaches (x_0, y_0), and write $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l$, if for every number $\epsilon > 0$, there exists a number $\delta > 0$ such that for all (x, y) in the δ-neighbourhood of $P(x_0, y_0)$*

$$|f(x, y) - l| < \epsilon, \text{ whenever, } \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

For the limit to exist, the function $f(x, y)$ may or may not be defined at (x_0, y_0). If the function $f(x, y)$ is not defined at (x_0, y_0), then we write

$$|f(x, y) - l| < \epsilon, \text{ whenever, } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

In the two-dimensional path, (x, y) may approach (x_0, y_0) through infinite number of paths joining (x, y) to (x_0, y_0). The limit exists if it is same along all these paths, that is, if it is independent of the path. If the limit is dependent on a path, then the limit does not exist.

The following results on limits are easy to establish:

If $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l$, and $\lim_{(x,y)\to(x_0,y_0)} g(x, y) = m$, then

1. $\lim_{(x,y)\to(x_0,y_0)} [f(x, y) \pm g(x, y)] = l \pm m$
2. $\lim_{(x,y)\to(x_0,y_0)} k f(x, y) = kl$, for any number k.
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) g(x, y) = lm$
4. $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{l}{m}, \quad m \neq 0$
5. $\lim_{(x,y)\to(x_0,y_0)} [f(x, y)]^{a/b} = l^{a/b}$, provided a, b are reals and $l^{a/b}$ is also real.

Another result of interest in connection with the limit of a function $f(x, y)$ is the *Sandwich Theorem for function of two variables* stated as below

Sandwich Theorem: *If $g(x, y) \le f(x, y) \le h(x, y)$, for all $(x, y) \neq (x_0, y_0)$ in a circular nbd. of (x_0, y_0) and if g and h have the same finite limit l as $(x, y) \to (x_0, y_0)$, then $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l$.*

This result is sometimes useful to find the limits of certain functions, whose lower and upper bounds are specified.

Continuity at a point: *A function $f(x, y)$ is continuous at a point $P(x_0, y_0)$, if*

1. *$f(x, y)$ is defined at (x_0, y_0)*
2. *$\operatorname{limit}_{(x,y)\to(x_0,y_0)} f(x, y)$ exists, and*
3. *$\operatorname{limit}_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.*

Thus function $f(x, y)$ is continuous at (x_0, y_0), if for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - f(x_0, y_0)| < \epsilon, \textit{ whenever, } \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

If any one of the conditions given at 1, 2 or 3 above is not satisfied, then the function is said to be *discontinuous* at the point $P(x_0, y_0)$.

However, if $f(x_0, y_0)$ is defined and $\lim_{(x, y) \to (x_0, y_0)} f(x) = l$ exists but is not equal to $f(x_0, y_0)$, then the point of discontinuity (x_0, y_0) is called a *removable discontinuity*. The function can be made continuous by redefining it as $f(x_0, y_0) = l$.

A function is said to be continuous if it is continuous at every point of its domain of definition.

Also if $z = f(x, y)$ is continuous at some point $P(x_0, y_0)$, and $w = g(z)$ is continuous at $z_0 = f(x_0, y_0)$, then the composite function $w = g(f(x, y))$ is also continuous at (x_0, y_0). Thus, e^{x-y}, $\ln(1 + x^2 y^2)$, $\cos(x + y)$ are continuous functions of x and y.

The definition of limit and continuity for functions of two variables and the results about the limits and continuity for sums, products, quotients, powers and composite extend to functions of three or more variables also.

Example 5.1: Evaluate the following limits, if these exist

(i) $\lim_{(x,y) \to (0,0)} (1 + x^2) \dfrac{\sin y}{y}$ (ii) $\lim_{(x,y) \to (0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$

(iii) $\lim_{(x,y) \to (0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ (iv) $\lim_{(x,y) \to (0,1)} \tan^{-1}\left(\dfrac{y}{x}\right)$

Solution: (i) $\lim_{(x,y) \to (0,0)} (1 + x^2) \dfrac{\sin y}{y} = \lim_{(x,y) \to (0,0)} (1 + x^2) \lim_{(x,y) \to (0,0)} \dfrac{\sin y}{y} = (1)(1) = 1.$

(ii) To evaluate $\lim_{(x,y) \to (0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ we choose the path $y = mx$. As $(x, y) \to (0, 0)$, we get $x \to 0$.

Therefore, $\lim_{(x,y) \to (0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \dfrac{x^2 - m^2 x^2}{x^2 + m^2 x^2} = \lim_{x \to 0} \dfrac{1 - m^2}{1 + m^2} = \dfrac{1 - m^2}{1 + m^2}$,

which depends on m. For different values of m, we obtain different limits. Hence the limit does not exist.

(iii) $$\lim_{(x,y) \to (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y) \to (0,0)} \frac{x(x - y)\left(\sqrt{x} + \sqrt{y}\right)}{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)}$$

$$= \lim_{(x,y) \to (0,0)} x\left(\sqrt{x} + \sqrt{y}\right) = 0\left(\sqrt{0} + \sqrt{0}\right) = 0.$$

(iv) We have, $\lim_{(x,y) \to (0,1)} \tan^{-1} \dfrac{y}{x} = \tan^{-1}(\pm \infty) = \pm \pi/2$

depending on whether the point $(0,1)$ is approached from left or right along the line $y=1$. Thus, the limit does not exist.

Example 5.2: Show that the following functions are discontinuous at the given points

(i) $f(x, y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$

(ii) $f(x, y) = \begin{cases} \dfrac{x^2-xy+x-y}{x-y}, & (x, y) \neq (2,2) \\ 1, & (x, y) = (2,2) \end{cases}$

Solution: (i) Choose the path $y = mx^2$. As $(x, y) \to (0, 0)$, we have,

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2mx^4}{x^4+m^2x^4} = \lim_{x\to 0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

which depends on m.

Since, the limit does not exist, hence the function is not continuous at $(0, 0)$.

(ii) $$\lim_{(x,y)\to(2,2)} \frac{x^2-xy+x-y}{x-y} = \lim_{(x,y)\to(2,2)} \frac{(x+1)(x-y)}{(x-y)} = \lim_{(x,y)\to(2,2)} (x+1) = 3.$$

Since $\lim_{(x,y)\to(2,2)} f(x, y) \neq f(2, 2)$, the function is discontinuous at the point $(2, 2)$.

Example 5.3: Using the definition of continuity at a point show that the following functions are continuous at the point $(0, 0)$.

(i) $$f(x, y) = \frac{2x^4+3y^4}{x^2+y^2}, \quad (x, y) \neq (0,0)$$
$$= 0, \quad (x, y) = (0, 0)$$

(ii) $$f(x, y, z) = \frac{xy+yz+zx}{\sqrt{x^2+y^2+z^2}}, \quad (x, y, z) \neq (0, 0, 0)$$
$$= 0, \quad (x, y, z) = (0, 0, 0).$$

Solution: (i) Let $x = r\cos\theta$, $y = r\sin\theta$. Then $r = \sqrt{x^2+y^2} \neq 0$. Consider

$$|f(x, y) - f(0, 0)| = \left|\frac{2x^4+3y^4}{x^2+y^2}\right| = \left|\frac{r^4(2\cos^4\theta+3\sin^4\theta)}{r^2}\right| < r^2[2|\cos^4\theta|+3|\sin^4\theta|] < 5r^2 < \epsilon.$$

This gives $$r = \sqrt{x^2+y^2} < \sqrt{\epsilon/5}.$$

Choose $\delta < \sqrt{\epsilon/5}$, thus given $\epsilon > 0$, we can find $\delta\left(<\sqrt{\epsilon/5}\right)$ such that

$$|f(x, y) - f(0, 0)| < \epsilon, \text{ whenever, } 0 < \sqrt{x^2+y^2} < \delta.$$

Therefore, $\lim_{(x,y)\to(0,0)} f(x, y) = 0$. Hence $f(x, y)$ is continuous at $(0, 0)$.

(ii) We have,

$$|xy| \le (x^2+y^2)/2, \quad |yz| \le (y^2+z^2)/2, \text{ and } \quad |zx| \le (z^2+x^2)/2 \qquad \text{...(5.1)}$$

Consider, $|f(x, y, z) - f(0, 0, 0)| = \left|\dfrac{xy+yz+zx}{\sqrt{x^2+y^2+z^2}} - 0\right| = \dfrac{|xy+yz+zx|}{\sqrt{x^2+y^2+z^2}}$

$$\le \frac{|xy|+|yz|+|zx|}{\sqrt{x^2+y^2+z^2}} \le \frac{(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} = \sqrt{x^2+y^2+z^2} < \epsilon, \text{ using (5.1).}$$

Choose $\delta < \epsilon$, given $\epsilon > 0$ we can find $\delta(<\epsilon)$, such that

$$\left|\frac{xy+yz+zx}{\sqrt{x^2+y^2+z^2}} - 0\right| < \epsilon, \text{ whenever } \sqrt{x^2+y^2+z^2} < \delta.$$

Therefore, $\lim_{(x,y)\to(0,0)} f(x, y) = 0$. Hence, $f(x, y)$ is continuous at $(0, 0)$.

EXERCISE 5.1

Evaluate the following limits (if exist)

1. $\lim_{(x,y)\to(0,0)} e^{y}\dfrac{\sin x}{x}$

2. $\lim_{(x,y)\to(0,0)} \cos\dfrac{x^2+y^2}{x+y+1}$

3. $\lim_{(x,y)\to(1,2)} \dfrac{(x-1)^3}{(x-1)^2+(y-2)}$

4. $\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \dfrac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$

5. $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

6. $\lim_{(x,y)\to(0,0)} \dfrac{y}{\sqrt{x^2+y^2}}$

7. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy+yz+xz}{\sqrt{x^2+y^2+z^2}}$

8. $\lim_{(x,y)\to(0,0)} \dfrac{x+\sqrt{y}}{x^2+y^2}$

Discuss the continuity of the following functions at the point (0, 0)

9. $f(x, y) = \begin{cases} x^2y/(x^4+y^2), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

10. $f(x, y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y) \neq (0,0) \\ 1/2, & (x,y) = (0,0) \end{cases}$

11. $f(x, y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

12. $f(x, y) = \begin{cases} \tan^{-1}\left(\dfrac{|x|+|y|}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ \pi/2, & (x,y) = (0,0) \end{cases}$

Assign a suitable value for $f(0, 0)$ such that $f(x, y)$ is continuous at the point $(0, 0)$

13. $f(x, y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$ 14. $f(x, y) = \dfrac{2xy^2}{x^2+y^2}$

15. Discuss the continuity of the function

$$f(x, y, z) = \begin{cases} \dfrac{xyz}{x^2+y^2+z^2}, & (x, y z) \neq (0,0,0) \\ 0, & (x, y z) = (0,0,0) \end{cases}$$

at $(0, 0, 0)$.

Using the $\delta - \epsilon$ definition show that

16. $f(x, y, z) = x + y - z$ is continuous at every point.
17. $f(x, y, z) = x^2 + y^2 + z^2$ is continuous at the origin.
18. $f(x, y) = (x + y)/(2 + \cos x)$ is continuous everywhere by taking $\epsilon = 0.02$.
19. $f(x, y) = \tan^2 x + \tan^2 y + \tan^2 z$ is continuous everywhere by taking $\epsilon = 0.03$.
20. Does knowing that $|\cos(1/y)| \leq 1$ tell anything about $\lim\limits_{(x,y)\to(0,0)} x\cos\dfrac{1}{y}$? Give reasons for your answer.

5.2 PARTIAL DERIVATIVES

When we keep all but one of the independent variables of a function constant and differentiate the function with respect to that one variable, we get a partial derivative.

Consider a function $z = f(x, y)$ of two independent variables x and y. Let (x_0, y_0) be a point in the domain of $f(x, y)$. The partial derivative of f with respect to x at the point (x_0, y_0) is the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x},$$

provided the limit exists. In case it exists, it is denoted by $\left(\dfrac{\partial z}{\partial x}\right)_{(x_0, y_0)}$, *or* $\left(\dfrac{\partial f}{\partial x}\right)_{(x_0, y_0)}$, *or* $f_x(x_0, y_0)$.

In fact the partial derivative of f with respect to x at the point (x_0, y_0) is the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$.

Similarly, *the partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is the limit* $\lim\limits_{\Delta y \to 0} \dfrac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$, *provided the limit exists, and is denoted by* $\left(\dfrac{\partial z}{\partial y}\right)_{(x_0, y_0)}$, *or* $\left(\dfrac{\partial f}{\partial y}\right)_{(x_0, y_0)}$ or $f_y(x_0, y_0)$. Also the partial derivative of f with respect to y at the point (x_0, y_0) is the ordinary derivative of $f(x_0, y)$ with respect to y at the point $y = y_0$.

For example, if $z = e^{ax} \sin by$, then $\dfrac{\partial z}{\partial x} = ae^{ax} \sin by$, and $\dfrac{\partial z}{\partial y} = be^{ax} \cos by$.

In case we consider a function $u = f(x, y, z)$ of three independent variables, then we have three partial derivatives of first order denoted by $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial z}$. Here $\dfrac{\partial u}{\partial x}$ is obtained by differentiating u with respect to x, treating both y and z as constants; and

$$\left.\frac{\partial u}{\partial x}\right|_{(x_0, y_0, z_0)} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x} \text{ etc.}$$

For example, if $f(x, y, z) = x^2 + y^2 + z^2 + xye^z$, then

$$f_x = 2x + y(1 + x)e^x, f_y = 2y + xe^x \text{ and } f_z = 2z.$$

Remark: A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point without being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity. For example, consider the function

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

Here the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along the line $y = x$ is 0, but $f(0, 0) = 1$ and hence the function $f(x, y)$ is not continuous at $(0, 0)$ but

$$\left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1-1}{\Delta x} = \lim_{\Delta x \to 0} (0) = 0, \text{ and}$$

similarly, $\left(\dfrac{\partial f}{\partial y}\right)_{(0,0)} = 0$. Thus both partial derivatives exist at $(0, 0)$.

Geometrical Interpretation of $\partial z/\partial x$ and $\partial z/\partial y$: The partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ of a function $z = f(x, y)$ have a very simple geometric interpretation. The function $z = f(x, y)$ represents a surface in space. The equation $y = y_0$ then represents a vertical plane intersecting this surface in a curve $z = f(x, y_0)$. The partial derivative $\dfrac{\partial z}{\partial x}$ at the point (x_0, y_0) is the slope of the tangent to the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ as shown in Fig. 5.2.

Similarly, the partial derivative $\frac{\partial z}{\partial y}$ at (x_0, y_0) is the slope of the tangent to the curve $z = f(x_0, y)$ at the point $(x_0, y_0, f(x_0, y_0))$.

Partial derivatives of second and higher orders: The derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called partial derivatives of first order. By differentiating these derivatives once again we obtain the four partial derivatives of second order. These are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

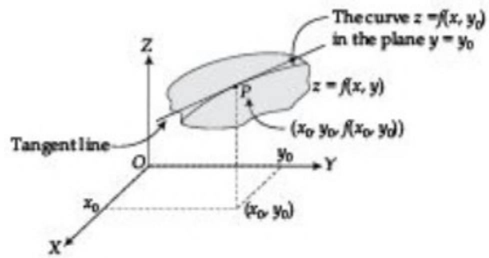


Fig. 5.2

It can be shown that if f_x, f_y and f_{xy} exist and are continuous, then the two mixed partial derivatives are equal and so the order of differentiation does not matter. In practical applications these conditions are satisfied and therefore we shall assume that the order of differentiation is immaterial.

By differentiating the partial derivatives of second order again with respect to x and y we obtain partial derivatives of the third and higher orders.

Example 5.4: If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

Solution: We have

$$\frac{\partial u}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}} \frac{1}{x} - 2y \tan^{-1}\frac{x}{y} + y^2 \frac{1}{1+\frac{x^2}{y^2}} \frac{x}{y^2}$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y}$$

$$= x - 2y \tan^{-1}\frac{x}{y}.$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 1 - 2y\frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y}$$

$$= 1 - \frac{2y^2}{y^2+x^2} = \frac{x^2-y^2}{x^2+y^2}.$$

Example 5.5: If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

Solution: Consider

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1. \quad ...(5.2)$$

Differentiating partially w.r.t. x, we have

$$\frac{2x}{a^2+u} - \frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x} - \frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x} - \frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x} = 0$$

or,

$$\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x} = \frac{2x}{a^2+u}$$

or,

$$\frac{\partial u}{\partial x} = \frac{2x}{a^2+u} \bigg/ \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] \quad ...(5.3)$$

Similarly differentiating (5.2) partially w.r.t. y and z, we get respectively

$$\frac{\partial u}{\partial y} = \frac{2y}{a^2+u} \bigg/ \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] \quad ...(5.4)$$

and,

$$\frac{\partial u}{\partial z} = \frac{2z}{a^2+u} \bigg/ \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] \quad ...(5.5)$$

Squaring and adding (5.3), (5.4) and (5.5), we obtain

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]}{\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]^2}$$

$$= \frac{4}{\dfrac{x^2}{(a^2+u)^2} + \dfrac{y^2}{(b^2+u)^2} + \dfrac{z^2}{(c^2+u)^2}} \quad \text{...(5.6)}$$

Also, from (5.3), (5.4) and (5.5), we have

$$2\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right\} = \frac{4\left[\dfrac{x^2}{a^2+u} + \dfrac{y^2}{b^2+u} + \dfrac{z^2}{c^2+u}\right]}{\dfrac{x^2}{(a^2+u)^2} + \dfrac{y^2}{(b^2+u)^2} + \dfrac{z^2}{(c^2+u)^2}}$$

$$= 4\bigg/\left(\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right), \quad \text{...(5.7)}$$

using (5.2). From (5.6) and (5.7), we obtain the desired result.

Example 5.6: Find the value of n, so that the equation $v = r^n(3\cos^2\theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = 0.$$

Solution: Consider $\quad v = r^n(3\cos^2\theta - 1). \quad$ ...(5.8)

Differentiating it partially w.r.t. r, we obtain $\dfrac{\partial v}{\partial r} = nr^{n-1}(3\cos^2\theta - 1)$

or, $$r^2\frac{\partial v}{\partial r} = nr^{n+1}(3\cos^2\theta - 1).$$

Differentiating again partially w.r.t. r, we get

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = n(n+1)r^n(3\cos^2\theta - 1). \quad \text{...(5.9)}$$

Next, differentiating (5.8) w.r.t. θ, we obtain $\dfrac{\partial v}{\partial\theta} = -6r^n\cos\theta\sin\theta$

or, $$\sin\theta\frac{\partial v}{\partial\theta} = -6r^n\cos\theta\sin^2\theta. \quad \text{...(5.10)}$$

Differentiating (5.10) w.r.t. θ, we get

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = -6r^n[-\sin^3\theta + 2\sin\theta\cos^2\theta] = 6r^n\sin\theta[\sin^2\theta - 2\cos^2\theta]$$

or, $$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial v}{\partial\theta}\right\} = 6r^n(\sin^2\theta - 2\cos^2\theta)$$

$$= -6r^n(3\cos^2\theta - 1). \quad \text{...(5.11)}$$

Adding (5.9) and (5.11), we get

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=\{n(n+1)-6\}r^n\{3\cos^2\theta-1\} \qquad ...(5.12)$$

The expression on the left side of (5.12) will be zero for all θ and r, provided

$$n(n+1)-6=0, \text{ or } n^2+n-6=0 \text{ or } \quad n=2,-3.$$

Example 5.7: If $x^x y^y z^z = c$, show that at $x = y = z$, $\dfrac{\partial^2 z}{\partial x\,\partial y} = -(x\ln ex)^{-1}$.

Solution: We have, $x^x y^y z^z = c$. Taking logarithm, we get

$$x\ln x + y\ln y + z\ln z = \ln c. \qquad ...(5.13)$$

Differentiating (5.13) w.r.t. x, we obtain $(\ln x+1)+\dfrac{\partial z}{\partial x}\ln z+\dfrac{\partial z}{\partial x}=0$,

which gives

$$\frac{\partial z}{\partial x}=-\frac{1+\ln x}{1+\ln z}=-\frac{\ln ex}{\ln ez}. \qquad ...(5.14)$$

Similarly, $$\frac{\partial z}{\partial y}=-\frac{\ln ey}{\ln ez}. \qquad ...(5.15)$$

Differentiating (5.15) w.r.t. x, we obtain

$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\ln ey}{(\ln ez)^2}\frac{1}{ez}e\frac{\partial z}{\partial x}=\frac{\ln ey}{z(\ln ez)^2}\frac{\partial z}{\partial x} \qquad ...(5.16)$$

Using (5.14) in (5.15), we get

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{(\ln ex)(\ln ey)}{z(\ln ez)^3}. \qquad ...(5.17)$$

At $x= y= z$, (5.17) becomes $\dfrac{\partial^2 z}{\partial x\partial y}=-(x\ln ex)^{-1}$.

Example 5.8: If $u = \ln(x^3+y^3+z^3-3xyz)$, prove that $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u=-9(x+y+z)^{-2}$.

Solution: Consider, $u = \ln(x^3+y^3+z^3-3xyz)$. Differentiating w.r.t. x, we have

$$\frac{\partial u}{\partial x}=\frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}.$$

Finding corresponding expressions for $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial z}$ and then adding, we obtain

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$
$$=3/(x+y+z). \qquad ...(5.18)$$

Also, $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \qquad \text{using (5.18)}$$

$$=-3\left[\frac{1}{(x+y+z)^2}+\frac{1}{(x+y+z)^2}+\frac{1}{(x+y+z)^2}\right]=-9/(x+y+z)^2.$$

Example 5.9: If (x, y) and (r, θ) are respectively cartesian and polar co-ordinates of a point P, find

$$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial y}{\partial \theta}, \frac{\partial y}{\partial r}, \frac{\partial x}{\partial \theta} \text{ and } \frac{\partial x}{\partial r}.$$

Solution: The relation between the cartesian and polar co-ordinates is

$$x=r\cos\theta, \quad y=r\sin\theta. \qquad ...(5.19)$$

We have, $\frac{\partial x}{\partial r}=\cos\theta, \quad \frac{\partial x}{\partial \theta}=-r\sin\theta$ and, $\frac{\partial y}{\partial r}=\sin\theta, \quad \frac{\partial y}{\partial \theta}=r\cos\theta.$

Also from (5.19), we have

$$r^2=x^2+y^2 \qquad ...(5.20)$$

and, $$\tan\theta=y/x. \qquad ...(5.21)$$

From (5.20), we have $2r\frac{\partial r}{\partial x}=2x$, or $\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta$. Similarly, $\frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta.$

Next differentiating (5.21) w.r.t. x, we obtain $\sec^2\theta\frac{\partial \theta}{\partial x}=-\frac{y}{x^2}$,

or, $$\frac{\partial \theta}{\partial x}=-\frac{y}{x^2}\cos^2\theta=-\frac{y}{x^2}\frac{x^2}{r^2}=-\frac{y}{r^2}=-\frac{\sin\theta}{r}.$$

Similarly, $$\frac{\partial \theta}{\partial y}=\frac{1}{x}\cos^2\theta=\frac{1}{x}\frac{x^2}{r^2}=\frac{x}{r^2}=\frac{\cos\theta}{r}.$$

Remark: In the above example, we have four variables x, y, r, θ connected by the two relations $x=r\cos\theta$ and $y=r\sin\theta$. To find $\frac{\partial r}{\partial x}$ we need a relation between r and x and such a relation will contain one more variable θ or y as given by $r=x\sec\theta$, $r^2=x^2+y^2$. We can find $\frac{\partial r}{\partial x}$ from any of these relations but there is no reason to suppose that the two values of $\frac{\partial r}{\partial x}$ as determined by these, where we regard θ and y respectively as constants, are equal. To avoid confusion we can denote first by $\left(\frac{\partial r}{\partial x}\right)_\theta$ to mean the partial derivative of r with respect to x, keeping θ constant and, second by $\left(\frac{\partial r}{\partial x}\right)_y$ to mean the partial derivative of r with respect to x, keeping y constant.

In general, when no indication is given regarding the variable to be kept constant, then $(\partial/\partial x)$ means $\left(\frac{\partial}{\partial x}\right)_y$ and $\left(\frac{\partial}{\partial y}\right)$ means $\left(\frac{\partial}{\partial y}\right)_x$; similarly $\frac{\partial}{\partial r}$ mean $\left(\frac{\partial}{\partial r}\right)_\theta$ and $\left(\frac{\partial}{\partial \theta}\right)$ means $\left(\frac{\partial}{\partial \theta}\right)_r$.

Example 5.10: If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$.

Solution: We have $u = f(r)$, and thus

$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x}, \quad \frac{\partial^2 u}{\partial x^2} = f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2};$$

and

$$\frac{\partial u}{\partial y} = f'(r)\frac{\partial r}{\partial y}, \quad \frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2}.$$

Here $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$, etc. are to be evaluated from $x = r\cos\theta$, $y = r\sin\theta$.

Also, $r^2 = x^2 + y^2$, and thus, $\frac{\partial r}{\partial x} = \frac{x}{r}$, and $\frac{\partial^2 r}{\partial x^2} = \frac{r - x^2/r}{r^2} = \frac{y^2}{r^3}$.

Similarly,
$$\frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}.$$

Thus,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\} + f'(r)\left\{\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right\}$$

$$= f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{x^2}{r^3} + \frac{y^2}{r^3}\right]$$

$$= f''(r) + \frac{1}{r} f'(r).$$

EXERCISE 5.2

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if

   (i) $z = y \sin xy$ (ii) $z = x^2 + 3xy + y + 1$

   (iii) $x + y + z = \ln z$ (iv) $z = \int_x^y g(t)dt.$

2. Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial y}$, if

   (i) $u = \sin^{-1} xyz$ (ii) $\tanh(x + 2y + 3z).$

3. If $z = \sin^{-1}(x/y)$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$

4. If $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

5. If $u = \ln \frac{x^2 + y^2}{xy}$, verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$

6. Show that $z = f(x - cy) + g(x + cy)$ satisfies the equation $\frac{\partial^2 z}{\partial y^2} = c^2\frac{\partial^2 z}{\partial x^2}.$

7. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, if $u = \tan^{-1}\left[\frac{2xy}{x^2 - y^2}\right].$

8. If $f(x, y) = (1 - 2xy + y^2)^{-1/2}$, show that $\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial f}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2\frac{\partial f}{\partial y}\right] = 0.$

9. Let $r^2 = x^2 + y^2 + z^2$ and $u = r^m$, prove that $u_{xx} + u_{yy} + u_{zz} = m(m + 1)r^{m-2}.$

10. If $u = \ln(x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1.$

11. If $u = e^{x-at}\cos(x - at)$, show that $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}.$

12. Given that $u = x(x^2 + y^2 + z^2)^{-3/2}$, show that $xu_x + yu_y + zu_z = -2u.$

13. Show that $v = (r^n + r^{-n})\sin n\theta$ satisfies Laplace equation in polar co-ordinates, that is,

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$$

14. If $w = \sin^{-1} u$, $u = (x^2 + y^2 + z^2)/(x + y + z)$, then $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = \tan w$.

15. If $z = \ln(u^2 + v)$, $u = e^{x+y^2}$, $v = x + y^2$, then $2y\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$.

16. If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$.

17. If $u = f(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r}f'(r)$.

18. If $z = \ln(e^x + e^y)$, show that $rt - s^2 = 0$, where $r = \dfrac{\partial^2 z}{\partial x^2}$, $t = \dfrac{\partial^2 z}{\partial y^2}$ and $s = \dfrac{\partial^2 z}{\partial x\,\partial y}$.

19. If $z = y + f(u)$, $u = \dfrac{x}{y}$, then show that $u\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$.

20. The equation $\dfrac{\partial u}{\partial t} = \mu\dfrac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation.

    Show that if $u = Ae^{-gx}\sin(nt - gx)$, where A, g, n are positive constants, then $g = \sqrt{(n/2\mu)}$.

5.3 TOTAL DIFFERENTIAL AND APPROXIMATION

We define the concept of total differential of a function of two or more variables and then apply it to estimate errors in calculations.

5.3.1 Total Differential of a Function

Consider a function $z = f(x, y)$ defined throughout in some domain D. Let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be two neighbouring points in D, so that Δx, Δy are the changes in the independent variables and let Δz be the corresponding change in z. Then

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \qquad ...(5.22)$$

is called the *total increment* in z corresponding to the increments Δx in x and Δy in y.

Subtracting and adding $f(x + \Delta x, y)$ in (5.22), we have

$$\Delta z = \{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)\} + \{f(x + \Delta x, y) - f(x, y)\} \qquad ...(5.23)$$

Applying Lagrange's mean value theorem, refer to Eq. (3.50), under the assumption that $f(x, y)$ satisfies the desired conditions in D, we have

$$\Delta z = \Delta y f_y(x + \Delta x, y + \theta_1 \Delta y) + \Delta x\, f_x(x + \theta_2 \Delta x, y),\ 0 < \theta_1, \theta_2 < 1 \qquad ...(5.24)$$

Writing $f_x(x + \theta_2 \Delta x, y) - f_x(x, y) = \epsilon_1$, and $f_y(x + \Delta x, y + \theta_1 \Delta y) - f_y(x, y) = \epsilon_2$, and substituting in (5.24) we obtain

$$\Delta z = \{f_x(x, y)\Delta x + f_y(x, y)\Delta y\} + (\epsilon_1 \Delta x + \epsilon_2 \Delta y), \quad ...(5.25)$$

where $\epsilon_1 = \epsilon_1(\Delta x, \Delta y)$ and $\epsilon_2 = \epsilon_2(\Delta x, \Delta y)$ are infinitesimal small such that $\epsilon_1 \to 0, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Thus the change Δz in z given by (5.25) consists of two parts.

The part $f_x \Delta x + f_y \Delta y$ in (5.25) which is linear in Δx and Δy is called the *total differential* or simply the *differential* of z and is denoted by dz or df, thus

$$dz = f_x\Delta x + f_y\Delta y \quad ...(5.26)$$

In the limiting case (5.25) becomes

$$dz = f_x dx + f_y dy \quad ...(5.27)$$

We note that the differential dz of the dependent variable z is not the same as the change Δz; it is the *principal part* of the increment Δz.

Similarly, for a function of more than two independent variables say $u = f(x, y, z)$, the *total differential* is

$$du = f_x dx + f_y dy + f_z dz \quad ...(5.28)$$

and so on.

Example 5.11: Find the total differential of the following functions.

(i) $z = \sin^{-1}(x/y)$ (ii) $\ln(x^2 + y^2 + z^2)$

Solution: (i) Here $f(x, y) = \sin^{-1}(x/y)$, which gives

$$f_x = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) = \frac{1}{\sqrt{y^2-x^2}}; f_y = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) = -\frac{x}{y\sqrt{y^2-x^2}}.$$

Therefore, the total differential is

$$dz = f_x dx + f_y dy = \frac{dx}{\sqrt{y^2-x^2}} - \frac{x dy}{y\sqrt{y^2-x^2}} = \frac{y dx - x dy}{y\sqrt{y^2-x^2}}.$$

(ii) Here $f(x, y, z) = \ln(x^2 + y^2 + z^2)$, which gives

$$f_x = \frac{2x}{x^2+y^2+z^2}, \quad f_y = \frac{2y}{x^2+y^2+z^2}, \quad f_z = \frac{2z}{x^2+y^2+z^2}.$$

Therefore, the total differential is $dz = f_x dx + f_y dy + f_z dz = \dfrac{2(x dx + y dy + z dz)}{x^2+y^2+z^2}$.

5.3.2 Approximation by Total Differentials

From above we observe that the approximate change dz in z corresponding to the small changes dx in x and dy in y is $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$, where the partial derivatives are evaluated at the given point (x, y). This result has applications in estimating errors in calculations.

Since, the partial derivatives and errors in arguments can be both positive and negative. Thus, the *maximum absolute error* is given by $|df| = |f_x||\Delta x| + |f_y|\Delta y$. The ratio $|df|/|f|$ is defined as the *maximum relative error* and $(|df|/|f|) \times 100$ as the *maximum percentage error*. These concepts can be extended to functions of more than two variables also.

An important result useful in calculated approximate value of a function is obtained from (5.25) by replacing Δz with $f(x+\Delta x, y+\Delta y) - f(x, y)$ and rewriting it as

$$f(x+\Delta x, y+\Delta y) = f(x, y) + f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

and it gives

$$f(x+\Delta x, y+\Delta y) \approx f(x, y) + f_x(x, y)\Delta x + f_y(x, y)\Delta y. \qquad ...(5.29)$$

Example 5.12: Find the percentage error in the computed area of an ellipse when an error of 2% is made in measuring its major and minor axes.

Solution: If a, b, A are the semi-major axis, semi-minor axis and area of an ellipse respectively, then $A = \pi ab$. It gives $dA = \pi b da + \pi a db$.

It is given that $da = 0.02a$, $db = 0.02b$. Therefore, $|dA| = 0.04\pi ab = 0.04A$, and hence,

$$\text{percentage error} = \left|\frac{dA}{A}\right| \times 100 = 4\%.$$

Example 5.13: Suppose that the variables r and h change from the initial values of $r_0 = 1.0$, $h_0 = 5$ by the amount $dr = 0.03$ and $dh = -0.1$. Estimate the resulting absolute, relative and percentage changes in the volume of the right circular cone of initial radius r_0 and height h_0.

Solution: The volume of a right circular cone with radius r and height h is $v = \frac{1}{3}\pi r^2 h$.

Hence, the absolute change, $dv = \frac{1}{3}\pi[2rh\,dr + r^2 dh]$

$$= \frac{1}{3}\pi[2(1)(5)(0.03) + (1)^2(-0.1)] = \frac{1}{3}\pi[0.3 - 0.1] = \frac{0.2\pi}{3}.$$

$$\text{Relative change} = \frac{dv}{v(r_0, h_0)} = \frac{0.2\pi}{3} \Big/ \frac{1}{3}\pi r_0^2 h_0 = \frac{0.2\pi}{\pi(1)^2 5} = 0.04.$$

$$\text{Percentage change} = \frac{dv}{v(r_0, h_0)} \times 100 = 0.04 \times 100 = 4\%$$

Example 5.14: The volume $v = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of r and h. Suppose that r is measured with an error of no more than 2% and h with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of v.

Solution: Here $\left|\frac{dr}{r} \times 100\right| \le 2$ and $\left|\frac{dh}{h} \times 100\right| \le 0.5$. Also $\frac{dv}{v} = \frac{2\pi rh\,dr + \pi r^2 dh}{\pi r^2 h} = \frac{2dr}{r} + \frac{dh}{h}$.

Thus, $$\left|\frac{dv}{v} \times 100\right| \le 2\left|\frac{dr}{r} \times 100\right| + \left|\frac{dh}{h} \times 100\right| = 2 \times 2 + 0.5 = 4.5.$$

Thus, the resulting possible error in the calculation of v is 4.5%

Example 5.15: Find a reasonable square about the point $(r, h) = (5, 12)$ in which the value of $v = \pi r^2 h$ will not vary by more than ± 0.1.

Solution: We have $\quad dv = 2\pi rh\,dr + \pi r^2 dh.$

About the point (5, 12), $dv = 2\pi(5)(12)dr + \pi(5)^2 dh = 120\pi dr + 25\pi dh$

Since the region to be restricted is a square, thus $dr = dh$ and, therefore,

$$dv = 120\pi\, dr + 25\pi\, dr = 145\pi\, dr.$$

Now $|dv| \le 0.1$ implies $|145\pi dr| \le 0.1$, or $|dr| \le \dfrac{0.1}{145\pi} \approx 2.1 \times 10^{-4}$.

Thus, the required square is given by

$$|r - 5| \le 2.1 \times 10^{-4}, \quad |h - 12| \le 2.1 \times 10^{-4}.$$

Example 5.16: Using differentials, find an approximate value of $f(x, y) = x^y$ at (2.1, 3.2).

Solution: Take $\quad (x, y) = (2, 3), \quad \Delta x = 0.1, \quad \Delta y = 0.2.$

Here $\quad f(x, y) = x^y$, thus $f_x(x, y) = yx^{y-1}$ and $f_y(x, y) = x^y \ln x$.

We have, $f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x, y)\,\Delta x + f_y(x, y)\Delta y$

Thus,
$$\begin{aligned} f(2.1, 3.2) &\approx f(2, 3) + f_x(2, 3)(0.1) + f_y(2, 3)(0.2) \\ &= 8 + 12\,(0.1) + 8\ln 2\,(0.2) = 10.309. \end{aligned}$$

EXERCISE 5.3

1. Suppose T is to be measured from the formula $T = x(e^y + e^{-y})$ where x and y are found to be 2 and ln 2 with maximum possible error of $|dx| = 0.1$ and $|dy| = 0.02$. Estimate the maximum possible error in the computed value of T.
2. Give a reasonable square about (1, 1) over which the value of $f(x, y) = x^3 y^4$ will not vary by more than ± 0.1.
3. When an x ohm and y ohm resistors are in parallel, by about what percentage their resultant resistance R will change if x increases from 20 to 20.1 ohms and y decreases from 25 to 24.9 ohms?
4. The power consumed in an electric resistor is given by $P = E^2/R$ watts. If $E = 80$ volts and $R = 5$ ohms, by how much the power consumption will change if E is increased by 3 volts and R is decreased by 0.1 ohms.
5. At a distance of 50 meters from the foot of a tower the elevation of its top is 30°. If the possible errors in measuring the distance and elevation are 2 cm and 0.05 degrees, find the approximate error in calculating the height.
6. If the sides of a plane triangle ABC vary in such a way that its circum-radius remains constant, prove that

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

7. If the radius r and the altitude h of a cone are measured with an absolute error of 1% in each measurement, then find the approximate percentage change in the lateral area of the cone if the measured values are $r = 3ft$ and $h = 4ft$.
8. A balloon in the form of right circular cylinder of radius 1.5 m and length 4.0 m is surmounted by hemispherical ends. If the radius is increased by 0.01 m, and the length by 0.05 m, find the percentage change in the volume of the balloon.
9. Using differentials, obtain the approximate values of the following:

   (a) $(4.05)^{1/2}(7.97)^{1/3}$ (b) $\dfrac{1}{\sqrt{1.05}} + \dfrac{1}{\sqrt{3.97}} + \dfrac{1}{\sqrt{9.01}}$

10. Using differentials, obtain the approximate values of the following.

    (a) $\cos 44° \sin 32°$ (b) $\sin 26° \cos 57° \tan 48°$.

5.4 THE CHAIN RULE: DIFFERENTIATION OF COMPOSITE AND IMPLICIT FUNCTIONS

In this section we study the differentiation of composite and implicit functions by the application of chain rule to functions of two or more variables.

5.4.1 Derivative of a Composite Function

Let $z = f(x, y)$ be a function of two independent variables x and y where x and y themselves are functions of an independent variable t, say $x = \phi(t)$ and $y = \psi(t)$. Then $z = f[\phi(t), \psi(t)]$ is said to define a *composite function of the independent variable t*.

Again if $x = \phi(u, v)$ and $y = \psi(u, v)$, so that x, y are functions of the variables u and v, then $z = f[\phi(u, v), \psi(u, v)]$ is said to define a *composite function of u and v*.

Now, if we wish to know the rate at which f changes with respect to t, we need to differentiate this composite function with respect to t, provided the derivative exists. Sometime we can do this by substituting the formula for $\phi(t)$ and $\psi(t)$ into the formula for f and then differentiating directly with respect to t, which may be a little bit cumbersome and so we prefer to apply *chain rule* given as follows.

Let $\Delta x, \Delta y$ and Δz be the increments respectively in x, y and z corresponding to the increment Δt in t. Then from (5.25)

$$\Delta z = \left(\frac{\partial z}{\partial x}\right)\Delta x + \left(\frac{\partial z}{\partial y}\right)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \quad ...(5.30)$$

where $\epsilon_1 = \epsilon_1(\Delta x, \Delta y)$ and $\epsilon_2 = \epsilon_2(\Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Dividing both sides of (5.30) by Δt and letting $\Delta t \to 0$, and hence, $\Delta x, \Delta y, \Delta z$ and ϵ_1, ϵ_2 approach zero, we get the chain rule as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad ...(5.31)$$

In case z is a composite function of two independent variables u and v, then the chain rule gives

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad ...(5.32)$$

and,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \qquad ...(5.32a)$$

The rule can easily be extended to functions of more than two independent variables.

Example 5.17: Find dz/dt when $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$ by chain rule and verify by direct substitution.

Solution: We have, $\frac{\partial z}{\partial x} = y^2 + 2xy$, $\frac{\partial z}{\partial y} = 2xy + x^2$, $\frac{dx}{dt} = 2at$, and $\frac{dy}{dt} = 2a$.

By chain rule, $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$. Substituting for $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{dx}{dt}, \frac{dy}{dt}$, we obtain

$$\begin{aligned}\frac{dz}{dt} &= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)\\ &= (4a^2t^2 + 4a^2t^3)(2at) + (4a^2t^3 + a^2t^4)2a = 2a^3t^3(8 + 5t).\end{aligned}$$

Also, $z = x^2y + xy^2 = 2a^3t^5 + 4a^3t^4$

Therefore, $\frac{dz}{dt} = 10a^3t^4 + 16a^3t^3 = 2a^3t^3(8 + 5t)$, hence the verification.

Example 5.18: If $z = f(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial^2 z}{\partial x^2}$.

Solution: We have $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$, and hence

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + y^2/x^2}(-y/x^2) = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$$

and,

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + y^2/x^2}\cdot\frac{1}{x} = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

We have,

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}\\ &= \frac{\partial z}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta} \qquad ...(5.33)\\ &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)z. \qquad ...(5.34)\end{aligned}$$

From (5.34),

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \qquad ...(5.35)$$

Thus $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$

$$= \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\right) \text{ using (5.33) and (5.35).}$$

$$= \cos\theta \frac{\partial}{\partial r}\left(\cos\theta \frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\right)$$

$$= \cos\theta\left[\cos\theta \frac{\partial^2 z}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial z}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 z}{\partial r \partial \theta}\right]$$

$$- \frac{\sin\theta}{r}\left[-\sin\theta \frac{\partial z}{\partial r} + \cos\theta \frac{\partial^2 z}{\partial \theta\, \partial r} - \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 z}{\partial \theta^2}\right]$$

$$= \cos^2\theta \frac{\partial^2 z}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial z}{\partial \theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 z}{\partial r\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial z}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 z}{\partial \theta^2}$$

Example 5.19: If $f = f(y - z, z - x, x - y)$, prove that $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z} = 0.$

Solution: Let $u = y - z$, $v = z - x$, $w = x - y$, so that $f = f(u, v, w)$ is a composite function of x, y, z.

Therefore, $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial x} = \dfrac{\partial f}{\partial u}(0) + \dfrac{\partial f}{\partial v}(-1) + \dfrac{\partial f}{\partial w}(1)$

$$= -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}. \quad ...(5.36)$$

Similarly, $$\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial w} + \frac{\partial f}{\partial u} \quad ...(5.37)$$

and, $$\frac{\partial f}{\partial z} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \quad ...(5.38)$$

Adding (5.36), (5.37) and (5.38) we get the desired result.

5.4.2 Derivative of an Implicit Function

The equation $f(x, y) = 0$ defines y implicitly as a function of x, say $y = h(x)$. Suppose the function $f(x, y)$ is differentiable, then applying chain rule to $w = f(x, y) = 0$, we obtain

$$0 = \frac{dw}{dx} = f_x\frac{dx}{dx} + f_y\frac{dy}{dx} = f_x + f_y\frac{dy}{dx},$$

or, $$\frac{dy}{dx} = -f_x/f_y, \text{ provided } f_y \neq 0. \quad ...(5.39)$$

Differentiating again with respect to x, regarding f_x and f_y as composite functions of x, we get

$$\frac{d^2y}{dx^2} = -\frac{\left(f_{xx} + f_{xy}\dfrac{dy}{dx}\right)f_y - f_x\left(f_{xy} + f_{yy}\dfrac{dy}{dx}\right)}{(f_y)^2}$$

$$= -\frac{(f_{xx}f_y - f_{xy}f_x)f_y - f_x(f_{xy}f_y - f_{yy}f_x)}{(f_y)^3}, \qquad \text{using (5.39)}$$

$$= -\frac{f_{xx}(f_y)^2 - 2f_{xy}f_x f_y + f_{yy}(f_x)^2}{(f_y)^3}. \qquad \text{...(5.40)}$$

Example 5.20: If $y^3 - 3ax^2 + x^3 = 0$, then using partial differentiation prove that

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

Solution: The implicit function is $f(x, y) = y^3 - 3ax^2 + x^3$. This gives

$$f_x = -6ax + 3x^2, \quad f_y = 3y^2, f_{xy} = 0, \quad f_{xx} = -6a + 6x, \text{ and } f_{yy} = 6y.$$

Substituting these values in (5.40), we obtain

$$\frac{d^2y}{dx^2} = -\frac{6(x-a)(9y^4) + 6y\,[9x^2(x-2a)^2]}{27y^6} = -\frac{2[(x-a)y^3 - x^2(x-2a)^2]}{y^5} \qquad \text{...(5.41)}$$

Now $y^3 - 3ax^2 + x^3 = 0 \Rightarrow y^3 = x^2(3a - x)$. Using this in (5.41) and simplifying, we obtain

$$\frac{d^2y}{dx^2} = \frac{-2a^2x^2}{y^5}, \text{ or } \frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

Example 5.21: Given that $F(x, y, z) = 0$, prove that $\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial z}\right) = -1.$

Solution: Since, $F(x, y, z) = 0$, we have

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0. \qquad \text{...(5.42)}$$

If z is kept constant, then $dz = 0$, from (5.42) we obtain

$$\left(\frac{dy}{dx}\right)_z = \frac{\partial y}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial y} \qquad \text{...(5.43)}$$

Similarly, if x is kept constant, then $dx = 0$, we get

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} \qquad \text{...(5.44)}$$

and, if y is kept constant, then $dy = 0$, we get

$$\frac{\partial x}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial x} \qquad \text{...(5.45)}$$

Multiplying (5.43), (5.44) and (5.45), we get

$$\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial z}\right) = -\frac{\partial F/\partial x}{\partial F/\partial y} \times \frac{\partial F/\partial y}{\partial F/\partial z} \times \frac{\partial F/\partial z}{\partial F/\partial x} = -1.$$

Example 5.22: Find $\partial w/\partial x$, if $w = x^2 + y^2 + z^2$, and $z^3 - xy + yz + y^3 = 1$, when x and y are independent variables.

Solution: It is not convenient to eliminate z from the two equations. We differentiate both w and z implicitly w.r.t. x treating x and y as independent variables. Thus,

$$\frac{\partial w}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} \qquad \text{...(5.46)}$$

and,

$$3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} = 0. \qquad \text{...(5.47)}$$

Solving (5.47) for $\frac{\partial z}{\partial x}$, we have $\frac{\partial z}{\partial x} = \frac{y}{y+3z^2}$.

Substituting this in (5.46), we obtain $\frac{\partial w}{\partial x} = 2x + \frac{2yz}{y+3z^2}$.

EXERCISE 5.4

1. Find $\frac{df}{dt}$ for the following functions:

   (a) $f = x^2 + y^2, \quad x = (t^2-1)/t, \quad y = t/(t^2+1)$ at $t = 1$

   (b) $f = e^{2x+3y}\cos 4z, \quad x = \ln t, \quad y = \ln(t^2+1), \quad z = t.$

   (c) $f = z\ln y + y\ln z + xyz, \quad x = \sin t, \quad y = t^2+1, \quad z = \cos^{-1} t$ at $t = 1$

2. If $z = f(x, y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$, then show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 2\left[x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right]$.

3. If $u = f(r, s, t)$, $r = x/y$, $s = y/z$, $t = z/x$, then show using chain rule that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

4. If $u = f(x^2 + 2yz, y^2 + 2zx)$ then prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$

5. Given that $z^3 + xy - y^2 z = 6$, obtain the expressions for $\partial y/\partial x$, $\partial z/\partial x$ in terms of x, y, z and find their values at the point (0, 1, 2).

6. Given that $u = f(x^2 + y^2 + z^2)$, where $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\phi$, find $\dfrac{\partial u}{\partial \theta}$, and $\dfrac{\partial u}{\partial \phi}$.

7. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, if $\cos xy + \cos yz + \cos zx = 1$.

8. Find $\dfrac{dy}{dx}$, if $x^y + y^x = a$, a is any constant and $x, y > 0$.

9. If $x + y = 2e^\theta \cos\phi$ and $x - y = 2ie^\theta \sin\phi$; prove that $\dfrac{\partial^2 v}{\partial \theta^2} + \dfrac{\partial^2 v}{\partial \phi^2} = 4xy \dfrac{\partial^2 v}{\partial x\, \partial y}$.

10. If v is a function of r only where $r^2 = \sum_{i=1}^{n} x_i^2$, then show that

$$\sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} = \frac{d^2 v}{dr^2} + \frac{n-1}{r}\frac{dv}{dr}.$$

11. If $x = r(\sec\theta + \tan\theta)$, and $y = r(\sec\theta - \tan\theta)$, then show that

$$4\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial r}\right)^2 - \frac{\cos^2\theta}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

12. If $x = r^2 \cosh\theta$, $y = r^2 \sinh\theta$, then prove that

$$\frac{1}{2}r^2\left\{\frac{\partial^2 u}{\partial r^2}\right\}\left\{\frac{\partial^2 u}{\partial \theta^2}\right\} = (2x^2 - y^2)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + (2y^2 - x^2)\frac{\partial^2 u}{\partial y^2}.$$

13. If $u = x^2 + y^2$, $v = 2xy$ and $f(x, y) = \phi(u, v)$, then show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left[\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right].$$

14. If $u = f(x, y, z)$ and $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \phi}\right)^2 + \frac{1}{r^2\sin^2\phi}\left(\frac{\partial f}{\partial \theta}\right)^2.$$

15. Find $\left(\dfrac{\partial w}{\partial y}\right)_x$ and $\left(\dfrac{\partial w}{\partial y}\right)_z$ at the point $(w, x, y, z) = (4, 2, 1, -1)$, if

$$w = x^2 y^2 + yz - z^3 \text{ and } x^2 + y^2 + z^2 = 6$$

5.5 JACOBIANS

The study of Jacobians is useful in connection with the transformations of variables applied while studying problems in partial differentiation and multiple integrals.

5.5.1 Definition and Properties

If $u = u(x, y)$ and $v = v(x, y)$ are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$ is defined as the Jacobian of u, v with respect to x and y and is denoted by $J\left(\frac{u, v}{x, y}\right)$ or $\frac{\partial(u, v)}{\partial(x, y)}$.

In the case of three functions $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$ of independent variables x, y and z, the jacobian of u, v, w with respect to x, y, z is defined as

$$J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Similarly, we can define Jacobians for four or more variables.

Properties of Jacobians: Following are some of the properties satisfied by Jacobians.

1. $$\frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u, v)} = 1$$

Consider $u = u(x, y)$ and $v = v(x, y)$, then

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v}.$$

Therefore, $$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

2. *If u, v are functions of x, y and x, y are themselves functions of s, t, then*

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial(u,v)}{\partial(s,t)} \text{ (The chain rule).}$$

Consider $$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{vmatrix} = \frac{\partial(u,v)}{\partial(s,t)}.$$

3. *The necessary and sufficient condition for the functions of two independent variables x, y, say $u(x, y)$ and $v(x, y)$ to be functionally dependent is* $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

We prove only the necessary part, let us assume that u, v are functionally dependent then there exists a functional relation of the type $F(u, v) = 0$.

Differentiating this with respect to x and y, we obtain respectively

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = 0 \quad ...(5.48)$$

and, $$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0 \quad ...(5.49)$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (5.48) and (5.49), we obtain

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0, \text{ that is, } \frac{\partial(u,v)}{\partial(x,y)} = 0.$$

4. *If the functions u, v, w of the variables x, y, z are defined by the relations $u = u(x)$, $v = v(x, y)$ and $w = w(x, y, z)$, then*

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}.$$

We observe that $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0, \text{ and also, } \frac{\partial v}{\partial z} = 0.$$

Thus, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & 0 & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}.$$

This result can be easily extended to more than three functions and the corresponding number of independent variables.

5. *(Jacobian of implicit functions). If* $f(u, v, w; x, y, z) = 0$, $g(u, v, w; x, y, z) = 0$, and $h(u, v, w; x, y, z) = 0$, then

$$\frac{\partial(f,g,h)}{\partial(u,v,w)}\,\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3\,\frac{\partial(f,g,h)}{\partial(x,y,z)}. \qquad ...(5.50)$$

From $f(u, v, w; x, y, z) = 0$, we have

$$\left.\begin{aligned} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} &= 0 \\ \frac{\partial f}{\partial z} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} &= 0 \end{aligned}\right\}, \text{ or } \left.\begin{aligned} \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} &= \frac{-\partial f}{\partial x} \\ \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} &= \frac{-\partial f}{fy} \\ \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} &= \frac{-\partial f}{\partial z} \end{aligned}\right\}, \qquad ...(5.51)$$

where the summation is over u, v and w only.

We can obtain similar sets of equations corresponding to the functions.

$g(u, v, w, x, y, z) = 0$, and $h(u, v, w, x, y, z) = 0$

Hence, $$\frac{\partial(f,g,h)}{\partial(u,v,w)}\,\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} & \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} & \sum \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} \\ \sum \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} & \sum \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} & \sum \frac{\partial g}{\partial u}\frac{\partial u}{\partial z} \\ \sum \frac{\partial h}{\partial u}\frac{\partial u}{\partial x} & \sum \frac{\partial h}{\partial u}\frac{\partial u}{\partial y} & \sum \frac{\partial h}{\partial u}\frac{\partial u}{\partial z} \end{vmatrix},$$

where summation is over u, v and w only.

Using (5.51) and similar equations, we have

$$\frac{\partial(f,g,h)}{\partial(u,v,w)}\,\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{-\partial f}{\partial x} & \frac{-\partial f}{\partial y} & \frac{-\partial f}{\partial z} \\ \frac{-\partial g}{\partial x} & \frac{-\partial g}{\partial y} & \frac{-\partial g}{\partial z} \\ \frac{-\partial h}{\partial x} & \frac{-\partial h}{\partial y} & \frac{-\partial h}{\partial z} \end{vmatrix} = (-1)^3 \frac{\partial(f,g,h)}{\partial(x,y,z)}.$$

Example 5.23: (i) If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$.

(ii) If $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, prove that $\dfrac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = r^2\sin\phi$.

Solution: (i) We have

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

(ii) $$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}$$

$$\begin{aligned} &= \cos\phi(r^2\cos\phi\sin\phi\cos^2\theta + r^2\cos\phi\sin\phi\sin^2\theta) \\ &\quad + r\sin\phi[r\sin^2\phi\cos^2\theta + r\sin^2\phi\sin^2\theta] \\ &= r^2\cos^2\phi\sin\phi + r^2\sin^3\phi = r^2\sin\phi(\cos^2\phi + \sin^2\phi) = r^2\sin\phi. \end{aligned}$$

Example 5.24: For $x = u$, $y = u\tan v$, $z = w$, verify $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 1$.

Solution: $$J_1 = \frac{\partial(x,y\,z)}{\partial(u,v\,w)} = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\sec^2 v.$$

Solving for u, v, w in terms of x, y and z, we have $u = x$, $v = \tan^{-1}(y/x)$, $w = z$. Thus,

$$J_2 = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{x}{x^2+y^2} = \frac{1}{x[1+(y/x)^2]} = \frac{1}{u[1+\tan^2 v]} = \frac{1}{u \sec^2 v}.$$

Hence, $J_1 J_2 = 1$.

Example 5.25: If $u = \sqrt{yz}, v = \sqrt{zx}, w = \sqrt{xy}$ and $x = r\sin\phi\cos\theta, y = r\sin\phi\sin\theta, z = r\cos\phi$, find

$$\frac{\partial(u, v, w)}{\partial(r, \phi, \theta)}.$$

Solution: By definition,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{z}{y}} & \frac{1}{2}\sqrt{\frac{y}{z}} \\ \frac{1}{2}\sqrt{\frac{z}{x}} & 0 & \frac{1}{2}\sqrt{\frac{x}{z}} \\ \frac{1}{2}\sqrt{\frac{y}{x}} & \frac{1}{2}\sqrt{\frac{x}{y}} & 0 \end{vmatrix} = \frac{1}{8}\left[\frac{\sqrt{xyz}}{\sqrt{xyz}} + \frac{\sqrt{xyz}}{\sqrt{xyz}}\right] = \frac{1}{4}$$

Also, $\qquad \dfrac{\partial(x, y, z)}{\partial(r, \phi, \theta)} = r^2 \sin\phi$, refer to Example 5.23 (ii).

By chain rule, $\dfrac{\partial(u, v, w)}{\partial(r, \phi, \theta)} = \dfrac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \dfrac{\partial(x, y, z)}{\partial(r, \phi, \theta)} = \dfrac{1}{4} r^2 \sin\phi.$

Example 5.26: Show that $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, $xy \neq 1$ are functionally dependent. Also find the relationship between u and v.

Solution: To show that u and v are functionally dependent, we prove that $\dfrac{\partial(u,v)}{\partial(x, y)} = 0$.

We have, $\dfrac{\partial(u,v)}{\partial(x, y)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \dfrac{1}{(1-xy)^2} - \dfrac{1}{(1-xy^2)} = 0.$

To find the relation between u and v, consider

$v = \tan^{-1} x + \tan^{-1} y = \tan^{-1} \dfrac{x+y}{1-xy}$, which gives $\tan v = \dfrac{x+y}{1-xy}$, or $u = \tan v$.

Example 5.27: If the three roots of the equation in λ given by $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ are u, v and w, then prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

Solution: The given equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ can be rewritten as

$$3\lambda^3 - 3(x+y+z)\lambda^2 + 3(x^2+y^2+z^2)\lambda - (x^3+y^3+z^3) = 0.$$

Since u, v, w are the three roots of this equation, therefore,

$$u+v+w = x+y+z,\ uv+vw+wu = x^2+y^2+z^2, \text{ and } uvw = \frac{1}{3}(x^3+y^3+z^3).$$

We define, $f(u, v, w; x, y, z) = \Sigma u - \Sigma x = 0$, $g(u, v, w; x, y, z) = \Sigma uv - \Sigma x^2 = 0$

and, $\quad h(u, v, w; x, y, z) = uvw - \dfrac{1}{3}\Sigma x^3 = 0.$

Here, u, v, w are the implicit functions of the three independent variables x, y and z, and thus from (5.50)

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\partial(f, g, h)}{\partial(x, y, z)} \Big/ \frac{\partial(f, g, h)}{\partial(u, v, w)}. \qquad ...(5.52)$$

Now, $$\frac{\partial(f,g,h)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

$$= -2[(y-x)(z^2-x^2) - (z-x)(y^2-x^2)]$$

$$= -2(y-x)(z-x)[z+x-y-x] = -2(x-y)(y-z)(z-x) \qquad ...(5.53)$$

Also, $$\frac{\partial(f,g,h)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ (v+w) & (w+u) & (u+v) \\ vw & wu & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} = (u-v)v(u-w) - w(u-w)(u-v)$$

$$= (u-v)(u-w)(v-w) = -(u-v)(v-w)(w-u) \qquad ...(5.54)$$

Substituting from (5.53) and (5.54) in (5.52), we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\,\frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

5.5.2 Applications in Change of Variables

Jacobians are applied in connection with the change of variables in multiple integrals to be discussed in Chapter 7. Here we discuss applications in connection with the change of variables in partial differentiation.

Suppose that $f(x, y)$ is a function of two independent variables x, y and x, y are functions of two new independent variables u, v given by $x = \phi(u, v)$, $y = \psi(u, v)$. By chain rule

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \qquad ...(5.55)$$

and,
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \qquad ...(5.56)$$

Suppose we want to determine $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ in terms of $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$. Solving (5.55) and (5.56) by Cramer's rule, we get

$$\frac{\partial f/\partial x}{\frac{\partial f}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial y}{\partial u}} = \frac{\partial f/\partial y}{\frac{\partial f}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial f}{\partial u}\frac{\partial x}{\partial v}} = \frac{1}{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}$$

Hence,
$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f,y)}{\partial(u,v)}\right] \text{ and } \frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f,x)}{\partial(u,v)}\right], \qquad ...(5.57)$$

where $J = \frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian of the variables of transformation.

Similarly, if $f(x, y, z)$ is a function of three independent variables x, y, z and x, y, z are functions of three new independent variables say u, v, w, then it can be shown that

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y, z)}{\partial(u, v, w)}\right], \frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x, z)}{\partial(u, v, w)}\right] \text{ and } \frac{\partial f}{\partial z} = \frac{1}{J}\left[\frac{\partial(f, x, y)}{\partial(u, v, w)}\right], \quad ...(5.58)$$

where $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ is the Jacobian of the variables of transformation.

Example 5.28: If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, then prove that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2.$$

Solution: We have $x = r\cos\theta$, $y = r\sin\theta$, thus

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r. \quad ...(5.59)$$

Also, $$\frac{\partial(f, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos\theta\frac{\partial f}{\partial r} - \sin\theta\frac{\partial f}{\partial \theta} \quad ...(5.60)$$

and, $$\frac{\partial(f, x)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \cos\theta & -r\sin\theta \end{vmatrix} = -r\sin\theta\frac{\partial f}{\partial r} - \cos\theta\frac{\partial f}{\partial \theta}. \quad ...(5.61)$$

Hence using (5.57), $$\frac{\partial f}{\partial x} = \frac{1}{J}\frac{\partial(f, y)}{\partial(r, \theta)} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta} \quad ...(5.62)$$

and, $$\frac{\partial f}{\partial y} = -\frac{1}{J}\frac{\partial(f, x)}{\partial(r, \theta)} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta} \quad ...(5.63)$$

Squaring and adding (5.62) and (5.63) and rearranging, we obtain

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = (\cos^2\theta + \sin^2\theta)\left(\frac{\partial f}{\partial r}\right)^2 + \frac{(\cos^2\theta + \sin^2\theta)}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2.$$

Example 5.29: If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z, then show that

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}.$$

Solution: We have $x = u + v + w,\ y = vw + wu + uv, \quad z = uvw.$ ...(5.64)

Thus,
$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} = -(u-v)(v-w)(w-u).$$

Also,
$$\frac{\partial(F, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix} = u^2(v-w)\frac{\partial F}{\partial u} + v^2(w-u)\frac{\partial F}{\partial v} + w^2(u-v)\frac{\partial F}{\partial w},$$

$$\frac{\partial(F, x, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ 1 & 1 & 1 \\ vw & wu & uv \end{vmatrix} = u(v-w)\frac{\partial F}{\partial u} + v(w-u)\frac{\partial F}{\partial v} + w(u-v)\frac{\partial F}{\partial w}$$

and,
$$\frac{\partial(F, x, y)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ 1 & 1 & 1 \\ v+w & w+u & u+v \end{vmatrix} = (v-w)\frac{\partial F}{\partial u} + (w-u)\frac{\partial F}{\partial v} + (u-v)\frac{\partial F}{\partial w}.$$

Using (5.58), we obtain

$$\frac{\partial F}{\partial x} = -\frac{u^2(v-w)\frac{\partial F}{\partial u} + v^2(w-u)\frac{\partial F}{\partial v} + w^2(u-v)\frac{\partial F}{\partial w}}{(u-v)(v-w)(w-u)} \quad \text{...(5.65)}$$

$$\frac{\partial F}{\partial y} = \frac{u(v-w)\frac{\partial F}{\partial u} + v(w-u)\frac{\partial F}{\partial v} + w(u-v)\frac{\partial F}{\partial w}}{(u-v)(v-w)(w-u)} \quad \text{...(5.66)}$$

$$\frac{\partial F}{\partial z} = -\frac{(v-w)\frac{\partial F}{\partial u} + (w-u)\frac{\partial F}{\partial v} + (u-v)\frac{\partial F}{\partial w}}{(u-v)(v-w)(w-u)} \quad \text{...(5.67)}$$

Multiplying (5.65) by x, (5.66) by $2y$ and (5.67) by $3z$ and adding; also using values for x, y and z from (5.64), we obtain

$$x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z} = (u+v+w)\times[\text{right side of }(5.65)]$$

$$+\,2(vw+wu+uv)\times[\text{right side of }(5.66)] + 3uvw\times[\text{right side of }(5.67)] \qquad \text{...(5.68)}$$

The coefficient of $\frac{\partial F}{\partial u}$ on the right side of (5.68) is

$$= \frac{-(u+v+w)(u^2)(v-w) + 2(vw+wu+uv)\,u\,(v-w) - 3uvw\,(v-w)}{(u-v)(v-w)(w-u)}$$

$$= \frac{[-(u+v+w)u + 2(vw+wu+uv) - 3vw]u}{(u-v)(w-u)} = \frac{[-u^2+wu+uv-vw]u}{(u-v)(w-u)} = u.$$

Similarly, we can show that coefficients of $\frac{\partial F}{\partial v}$ and $\frac{\partial F}{\partial w}$ on the right side of (5.68) are v and w respectively. Hence (5.68) simplifies to

$$x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial z}{\partial z} = u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial z}{\partial w}, \text{ the desired result.}$$

EXERCISE 5.5

1. If $x = r\cos\theta$, $y = r\sin\theta$, show that $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$.

2. If $x = e^u\cos v$, $y = e^u\sin v$, show that $\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)} = 1$.

3. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\frac{\partial(u,v)}{\partial(r,\theta)} = 6r^3\sin 2\theta.$$

4. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, show that $\frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \frac{1}{r}$.

5. If $F(x, y, u, v) = 0$ and $G(x, y, u, v) = 0$, prove that $\frac{\partial u}{\partial x}\frac{\partial y}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial y}{\partial v} = 0$.

6. If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\partial(x, y, z)/\partial(u, v, w) = u^2v$.

7. If $u^3 = xyz$, $\frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, $w^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{-v(y-z)(z-x)(x-y)(x+y+z)}{3u^2w(yz+zx+xy)}$$

8. If $u^3+v^3+w^3=x+y+z$, $u^2+v^2+w^2=x^3+y^3+z^3$, $u+v+w=x^2+y^2+z^2$, then prove that
$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}.$$

9. If $u=x^2+y^2+z^2$, $v=x+y+z$, $w=xy+yz+zx$, show that the Jacobian $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ vanishes identically. Also find the relation between u, v and w.

10. Show that the functions $u=x+y-z$, $v=x-y+z$, $w=x^2+y^2+z^2-2yz$ are not independent of one another. Also find the relation between them.

11. Show that the functions $u=\sin^{-1}x+\sin^{-1}y$, and $v=x\sqrt{1-y^2}+y\sqrt{1-x^2}$ are functionally dependent. Also find the relationship.

5.6 HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ is said to be a homogeneous function of order n, if the degree of each of its terms in x and y is equal to n. For example,

$$f(x,y)=a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\ldots+\ldots a_{n-1}xy^{n-1}+a_ny^n$$

is a homogeneous function of order n.

This definition of homogeneity is applicable only to polynomial functions. To enlarge the concept of homogeneity we say that *z is a homogeneous function of order n in x and y, if it can be expressed in the form $x^n f(y/x)$.* In addition to the polynomial functions, this brings transcendental functions also within its scope. For example, $x^n\sin(y/x)$ is a homogeneous function of degree n, while

$$\frac{\sqrt{y}+\sqrt{x}}{y+x}=x^{-1/2}\frac{(1+\sqrt{y/x})}{(1+y/x)}=x^{-1/2}f(y/x)$$

is a homogeneous function of degree $-1/2$.

To cover some other functions of the form $f(x,y)=x^nf(y/x)+y^ng(x/y)$, we say that *$F(x, y)$ is a homogeneous function of order n in x and y if it is expressible in the form $F(\lambda x,\lambda y)=\lambda^nF(x,y)$.*

Similarly, *a function $F(x, y, z)$ of three variables is said to be homogeneous function of order n in x, y and z if it can be expressible in the form*

$F(\lambda x,\lambda y,\lambda z)=\lambda^nF(x,y,z)$, or $F(x,y,z)=x^nf(y/x,z/x)$ or $y^ng(x/y,z/y)$, or $z^nh(x/z,y/z)$.

We have the following important result concerning homogeneous functions.

If u is a homogeneous function of order n in x and y, then $\partial u/\partial x$ and $\partial u/\partial y$ both are homogeneous functions of order $(n-1)$ in x and y.

To prove it, consider $u=x^nf(y/x)$. This gives

$$\frac{\partial u}{\partial x}=nx^{n-1}f(y/x)+x^nf'(y/x)(-y/x^2)=x^{n-1}\left[nf(y/x)-\frac{y}{x}f'(y/x)\right]=x^{n-1}g(y/x),$$

a homogeneous function of order n, where $g(y/x) = \left[nf(y/x) - \frac{y}{x}f'(y/x)\right]$.

Similarly, we can prove for $\partial u/\partial y$.

Next we consider an important theorem on homogeneous functions.

Theorem 5.1: **(Euler's Theorem on Homogeneous Functions)** *If $f(x, y)$ is a homogeneous function of order n in x and y, then*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf. \qquad ...(5.69)$$

Proof. Since $f(x, y)$ is a homogeneous function of order n in x and y, we can write

$$f(x, y) = x^n g(y/x). \qquad ...(5.70)$$

Differentiating this partially with respect to x and y separately, we obtain respectively

$$\frac{\partial f}{\partial x} = nx^{n-1}g(y/x) + x^n g'(y/x)(-y/x^2)$$
$$= nx^{n-1}g(y/x) - yx^{n-2}g'(y/x). \qquad ...(5.71)$$

and,
$$\frac{\partial f}{\partial y} = x^n g'(y/x)\,\frac{1}{x} = x^{n-1}g'(y/x). \qquad ...(5.72)$$

Multiplying (5.71) by x and (5.72) by y, we obtain

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n g(y/x) - yx^{n-1}g'(y/x) + yx^{n-1}g'(y/x) = nx^n g(y/x) = nf, \quad \text{using (5.70).}$$

Another result on homogeneous functions which follows from Euler's theorem is given as follows.

Theorem 5.2: *If f is a homogeneous function of x, y of order n, then*

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f \qquad ...(5.73)$$

Proof. Differentiating (5.69) partially with respect to x and y separately, we get

$$\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x\partial y} = n\frac{\partial f}{\partial x} \qquad ...(5.74)$$

and,
$$x\frac{\partial^2 f}{\partial y\partial x} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} = n\frac{\partial f}{\partial y} \qquad ...(5.75)$$

Multiplying (5.74) and (5.75) by x and y respectively, adding and rearranging the terms and assuming $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$, we obtain

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n\left[x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right]-\left[x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right]=n(nf)-nf=n(n-1)f, \text{ using (5.69)}$$

Euler's theorem on homogeneous functions can be extended to functions of three or more independent variables. For example, if f is a homogeneous function of three independent variables x, y, and z, then

$$f=x^n\phi\left(\frac{y}{x},\frac{z}{x}\right)=x^n\phi(u,v), \tag{5.76}$$

where $u=y/x$ and $v=z/x$.

We have,

$$f_x=nx^{n-1}\phi+x^n\left[\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}\right]$$

$$=n\,x^{n-1}\phi-y\,x^{n-2}\frac{\partial\phi}{\partial u}-zx^{n-2}\frac{\partial\phi}{\partial v}, \tag{5.77}$$

$$f_y=x^n\left[\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}\right]=x^{n-1}\frac{\partial\phi}{\partial u}, \tag{5.78}$$

and,

$$f_z=x^n\left[\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial z}\right]=x^{n-1}\frac{\partial\phi}{\partial v}. \tag{5.79}$$

Multiplying (5.77), (5.78), and (5.79) by x, y and z respectively and adding, then using (5.76), we obtain

$$xf_x+yf_y+zf_z=nf. \tag{5.80}$$

This is Euler's theorem on homogeneous functions for three variables.

Example 5.30: If $u=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\sin 2u$.

Solution: Here $u(x,y)=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ is not a homogeneous function, however,

$$z=\tan u=\frac{x^3+y^3}{x-y}=x^2\frac{1+(y/x)^3}{1-(y/x)},$$

is a homogeneous function of order two in x and y. Using Euler's theorem, we obtain

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z. \tag{5.81}$$

Here $\dfrac{\partial z}{\partial x}=\sec^2 u\dfrac{\partial u}{\partial x}$; $\dfrac{\partial z}{\partial y}=\sec^2 u\dfrac{\partial u}{\partial y}$.

Substituting in (5.81), we get $\sec^2 u \left\{ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right\} = 2\tan u$

or,
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sin u}{\cos u}\cos^2 u = \sin 2u$$

Example 5.31: If $u(x, y) = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

Solution: We have $u(\lambda x, \lambda y) = u(x, y)$, thus $u(x, y)$ is a homogeneous function of degree zero. Therefore by Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.u = 0.$

Example 5.32: If $u = \sin^{-1}\frac{x+2y+3z}{x^8+y^8+z^8}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Solution: Here $u(x, y, z) = \sin^{-1}\frac{x+2y+3z}{x^8+y^8+z^8}$ is not a homogeneous function. However,

$$f = \sin u = \frac{x+2y+3z}{x^8+y^8+z^8} = x^{-7}\frac{1+2(y/x)+3(z/x)}{1+(y/x)^8+(z/x)^8}$$

is a homogeneous function of order -7 in x, y, z. Hence, by Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = (-7)f \qquad ...(5.82)$$

But $\frac{\partial f}{\partial x} = \cos u\frac{\partial u}{\partial x}$, $\frac{\partial f}{\partial y} = \cos u\frac{\partial u}{\partial y}$, and $\frac{\partial f}{\partial z} = \cos u\frac{\partial u}{\partial z}$.

Substituting in (5.82), we get

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -7\sin u$$

or,
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u.$$

Example 5.33: If $z = x^m f(y/x) + x^n g(y/x)$, then show that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right). \qquad ...(5.83)$$

Solution: Let $z = x^m f(y/x) + x^n g(y/x) = z_1 + z_2$, say.

The function $z_1 = x^m f(y/x)$, is a homogeneous function of order m in x and y, thus

$$x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y} = mz_1 \qquad ...(5.84)$$

and, $$x^2 \frac{\partial^2 z_1}{\partial x^2} + 2xy \frac{\partial^2 z_1}{\partial x \partial y} + y^2 \frac{\partial^2 z_1}{\partial y^2} = m(m-1)z_1. \quad ...(5.85)$$

Similarly, for $z_2 = x^n g(y/x)$, we have

$$x \frac{\partial z_2}{\partial x} + y \frac{\partial z_2}{\partial y} = nz_2 \quad ...(5.86)$$

and, $$x^2 \frac{\partial^2 z_2}{\partial x^2} + 2xy \frac{\partial^2 z_2}{\partial x \partial y} + y^2 \frac{\partial^2 z_2}{\partial y^2} = n(n-1)z_2. \quad ...(5.87)$$

Adding (5.84) and (5.86) and using $z = z_1, + z_2$, we obtain

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = mz_1 + nz_2. \quad ...(5.88)$$

Similarly from (5.85) and (5.87), we obtain

$$x^2 \frac{\partial z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = m(m-1)z_1 + n(n-1)z_2 \quad ...(5.89)$$

Thus left side of (5.83), using (5.89) becomes

$$\begin{aligned} &= m(m-1)z_1 + n(n-1)z_2 + mnz \\ &= m(m-1)z_1 + mnz_1 + n(n-1)z_2 + mnz_2 \\ &= (m+n-1)mz_1 + (m+n-1)nz_2 \\ &= (m+n-1)(mz_1 + nz_2), \end{aligned}$$

using (5.88) it is the same as the right side of (5.83).

EXERCISE 5.6

1. Verify Euler's theorem for
   (a) $z = ax^2 + 2hxy + by^2$ (b) $z = (x^2 + xy + y^2)^{-1}$
   (c) $z = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$ (d) $z = x^n \ln \frac{y}{x}$
   (e) $z = x^2(x^2 - y^2)^3/(x^2 + y^2)^3$.
2. If u is a homogeneous function of order n in x, y, then prove that
   (a) $x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y \partial y} = (n-1) \frac{\partial u}{\partial x}$ (b) $x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = (n-1) \frac{\partial u}{\partial y}$
3. If $z = xy f(x/y)$, then show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$.

4. If $u=\cos^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=-\dfrac{1}{2}\cot u$.

5. If $u=\sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$.

6. If $u=\sin^{-1}(x/y)+\tan^{-1}(y/x)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=0$.

7. If $u=\sin(y/x)+x\sin^{-1}(y/x)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial z^2}=0$.

8. If $u=f(r,s,t), r=x/y, s=y/z, t=z/x$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=0$.

9. If $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=2\cos 3u\sin u$.

10. If $u=\tan^{-1}\dfrac{y^2}{x}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=-\sin 2u\sin^2 u$.

5.7 TAYLOR'S EXPANSION. APPROXIMATION AND ERROR ESTIMATION

We have already studied the Taylor's expansion for functions in one variable in Chapter 4. Here we extend this to the functions of more than one variable.

5.7.1 Taylor's Expansion for Two Variables

Theorem 5.3: **(Taylor's Expansion)** If $f(x, y)$ and its partial derivatives up to order $(n+1)$ are continuous throughout the domain D centered at a point (x_0, y_0), then throughout D

$$f(x_0+h, y_0+k)=f(x_0, y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0, y_0)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0)+\ldots+$$

$$+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0, y_0)+R_n \qquad \text{...(5.90)}$$

where R_n is the remainder term given by

$$R_n=\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k),\quad 0<\theta<1. \qquad \text{...(5.91)}$$

Proof. To prove this, let $x=x_0+\lambda h$ and $y=y_0+\lambda k$, where $0<\lambda<1$, and let

$$F(\lambda)=f(x_0+\lambda h, y_0+\lambda k).$$

By chain rule, $F'(\lambda) = \dfrac{dF}{d\lambda} = \dfrac{\partial f}{\partial x}\dfrac{dx}{d\lambda} + \dfrac{\partial f}{\partial y}\dfrac{dy}{d\lambda} = h\dfrac{\partial f}{\partial x} + k\dfrac{\partial f}{\partial y} = \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right) f(x, y).$

Similarly, $F''(\lambda) = \dfrac{d^2F}{d\lambda^2} = \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^2 f(x, y)$

......

$$F^{(n)}(\lambda) = \frac{d^nF}{d\lambda^{n-1}} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x, y), \text{ and } F^{(n+1)}(\lambda) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x, y).$$

Set $\lambda = 0$ in $F(\lambda), F'(\lambda), F''(\lambda), \ldots, F^n(\lambda)$ and $\lambda = \theta$ in $F^{(n+1)}(\lambda)$, we obtain

$$\left.\begin{aligned}
F(0) &= f(x_0, y_0)\\
F'(0) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0)\\
F''(0) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0)\\
&\vdots\\
F^{(n)}(0) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0),\\
\text{and,}\quad F^{(n+1)}(\theta) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k).
\end{aligned}\right\} \qquad \ldots(5.92)$$

By Taylor's theorem for a function of one variable, refer to (4.58), we have expansion of $F(1)$ as

$$F(1) = F(0) + F'(0) + \frac{1}{2!}F''(0) + \ldots + \frac{1}{n!}F^{(n)}(0) + \frac{1}{(n+1)!}F^{(n+1)}(\theta), \; 0 < \theta < 1 \qquad \ldots(5.93)$$

Hence, using (5.92) in (5.93), we obtain

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \ldots$$

$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h,\; y_0 + \theta k), \; 0 < \theta < 1.$$

which is the Taylor's expansion for a function of two variables. The result can be extended to functions of three or more variables on the similar lines.

5.7.2 Taylor's Expansion for $f(x, y)$ at the Origin: Maclaurin's Expansion

If $(x_0, y_0) = (0, 0)$, then we treat h and k as independent variables, replacing h by x and k by y in (5.90), we obtain Taylor's expansion for $f(x, y)$ at $(0, 0)$, also called the Maclaurin's expansion, given by

$$f(x, y) = f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f(0, 0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0) + \ldots$$

$$+ \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(0, 0) + \frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f(\theta x, \theta y), 0 < \theta < 1. \qquad \ldots(5.94)$$

When $R_n \to 0$ as $n \to \infty$, from (5.90) and (5.94), we get respectively, the Taylor's series and Maclaurin's series, respectively as

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0)$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0, y_0) + \ldots \qquad \ldots(5.95)$$

$$f(x, y) = f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0, 0) + \ldots \qquad \ldots(5.96)$$

The results for the Taylor's expansion can be extended to more than two variables on the similar line.

5.7.3 Approximation and Error Estimation

Taylor's formula (5.90) provides polynomial approximation to two variables functions. The first $n + 1$ terms give the polynomial of degree n and the last term gives approximation error.

For example for $n = 1$, the linear approximation of $f(x, y)$ about the point (x_0, y_0) is

$$f(x, y) = f(x_0 + \overline{x - x_0}, y_0 + \overline{y - y_0}) \approx f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) \qquad \ldots(5.97)$$

with error term, $R_1 = \frac{1}{2!}[(x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0) f_{xy} + (y - y_0)^2 f_{yy}]$ $\qquad \ldots(5.98)$

Here, f_{xx}, f_{xy} and f_{yy} are to be evaluated at the point $[x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)], 0 < \theta < 1$.

Since θ is unknown we cannot evaluate R_1 exactly, however, it is possible to find an upper bound to R_1 in a given rectangular region $R = \{(x, y) : |x - x_0| < \delta_1, |y - y_0| < \delta_2\}$. From (5.98)

$$|R_1| \le \frac{1}{2}[|x - x_0|^2 |f_{xx}| + 2|x - x_0||y - y_0||f_{xy}| + |y - y_0|^2 |f_{yy}|]$$

If $M = max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}$, for all $(x, y) \in R$, then

$$|R_1| \le \frac{M}{2}[|x - x_0|^2 + 2|x - x_0||y - y_0| + |y - y_0|^2]$$

$$= \frac{M}{2}[|x - x_0| + |y - y_0|]^2 \le \frac{M}{2}[\delta_1 + \delta_2]^2, \text{ for all } (x, y) \in R.$$

Thus we have the following result:

If $M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}$, then the maximum absolute error $|R_1|$ in the linear approximation of $f(x, y)$ in the rectangular region $R = \{(x, y) : |x - x_0| < \delta_1, |y - y_0| < \delta_2\}$ about the point (x_0, y_0), is

$$\frac{M}{2}[\delta_1 + \delta_2]^2. \qquad ...(5.99)$$

Similarly, it can be shown that *the maximum absolute error $|R_2|$ in the quadratic approximation of $f(x, y)$ in the rectangular region $R = \{(x, y) : |x - x_0| < \delta_1, |y - y_0| < \delta_2\}$ about the point (x_0, y_0), is*

$$\frac{M}{6}(\delta_1 + \delta_2)^3, \qquad ...(5.100)$$

where $M = \max.\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\}$.

We note that the *quadratic approximation of $f(x, y)$* about (x_0, y_0) is given by

$$f(x, y) \approx f(x_0, y_0) + (x - x_0)f_x + (y - y_0)f_y + \frac{1}{2!}[(x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0)f_{xy} + (y - y_0)^2 f_{yy}].$$

Example 5.34: Find the quadratic Taylor series polynomial approximation to the function $f(x, y) = \sin x \sin y$ about the origin. Obtain the maximum absolute error if $|x| \le 0.1$ and $|y| \le 0.1$.

Solution: We have

$$\begin{aligned}
f(x, y) &= \sin x \sin y, & f(0, 0) &= 0 \\
f_x(x, y) &= \cos x \sin y, & f_x(0, 0) &= 0 \\
f_y(x, y) &= \sin x \cos y, & f_y(0, 0) &= 0 \\
f_{xx}(x, y) &= -\sin x \sin y, & f_{xx}(0, 0) &= 0 \\
f_{xy}(x, y) &= \cos x \cos y, & f_{xy}(0, 0) &= 1 \\
f_{yy}(x, y) &= -\sin x \sin y, & f_{yy}(0, 0) &= 0 \\
f_{xxx}(x, y) &= -\cos x \sin y, & f_{xxy}(x, y) &= -\sin x \cos y, \\
f_{xyy}(x, y) &= -\cos x \sin y, & f_{yyy}(x, y) &= -\sin x \cos y.
\end{aligned}$$

The quadratic approximation about $(0, 0)$ is given by

$$f(x, y) \approx f(0, 0) + \{xf_x(0, 0) + yf_y(0, 0)\} + \frac{1}{2!}\{x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)\}$$

Substituting for $f(0, 0), f_x(0, 0), f_y(0, 0)$, etc., we obtain $\sin x \sin y \approx xy$.

The maximum absolute error in the quadratic approximation, refer to (5.100), is given by

$$|R_2| \le \frac{M}{6}[|x| + |y|]^3 \le \frac{M}{6}[0.1 + 0.1]^3 = 0.00133\,M,$$

where $M = \max.\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\}$ for all $(x, y) \in R : |x| \le 0.1$ and $|y| \le 0.1$.

Here the derivatives are to be evaluated at point $(\theta x, \theta y), 0 < \theta < 1$.

Since the third order derivatives in this case, as derived above, are products of sines and cosines thus they can never exceed unity and hence M can at the most be equal to 1; and thus $|R_2| \le 0.00133$.

Example 5.35: Find the linear and quadratic Taylor series polynomial approximation to the function $f(x, y) = x^2y + 3y - 2$ about the point $(-1, 2)$ and obtain the maximum absolute error in the region $|x + 1| < 0.1$ and $|y - 2| < 0.1$.

Solution: We have,

$$
\begin{aligned}
f(x, y) &= x^2y + 3y - 2, & f(-1, 2) &= 6 \\
f_x &= 2xy & f_x(-1, 2) &= -4 \\
f_y &= x^2 + 3 & f_y(-1, 2) &= 4 \\
f_{xx} &= 2y & f_{xx}(-1, 2) &= 4 \\
f_{xy} &= 2x & f_{xy}(-1, 2) &= -2 \\
f_{yy} &= 0 & f_{yy}(-1, 2) &= 0 \\
f_{xxx} &= 0 & f_{xxx}(-1, 2) &= 0 \\
f_{xxy} &= 2 & f_{xxy}(-1, 2) &= 2 \\
f_{xyy} &= 0 & f_{xyy}(-1, 2) &= 0 \\
f_{yyx} &= 0 & f_{yyx}(-1, 2) &= 0 \\
f_{yyy} &= 0 & f_{yyy}(-1, 2) &= 0.
\end{aligned}
$$

The linear approximation of $f(x, y)$ about $(-1, 2)$ is

$$
\begin{aligned}
f(x, y) &\approx f(-1, 2) + [(x+1)f_x(-1, 2) + (y-2)f_y(-1, 2)] \\
&= 6 + (x+1)(-4) + (y-2)(4) = 6 - 4(x+1) + 4(y-2).
\end{aligned}
$$

The maximum absolute error in the linear approximation, refer to (5.99), is

$$|R_1| \le \frac{M}{2}[|x+1| + |y-2|]^2 \le \frac{M}{2}[0.1 + 0.1]^2 = 0.02M$$

where $M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}$, for all $(x, y) \in R: |x+1| \le 0.1$, and $|y-2| \le 0.1$.

Now,

$$
\begin{aligned}
\max|f_{xx}| &= \max|2y| = 2\max|y| = 2\max|(y-2)+2| \\
&\le 2\max[|y-2| + 2] \le 2(0.1 + 2) = 4.2 \\
\max|f_{xy}| &= \max|2x| = 2\max|x| = 2\max|(x+1)-1| \\
&\le 2\max[|x+1| + 1] \le 2[0.1 + 1] = 2.2
\end{aligned}
$$

and,

$$\max|f_{yy}| = 0.$$

Hence, $M = 4.2$, and thus, $|R_1| \le 0.02(4.2) = .084$

Next, the quadratic approximation about $(-1, 2)$ is

$$f(x, y) \approx f(-1, 2) + [(x+1)f_x(-1, 2) + (y-2)f_y(-1, 2)] + \frac{1}{2!}[(x+1)^2 f_{xx}(-1, 2)$$

$$+2(x+1)(y-2)f_{xy}(-1,2)+(y-2)^2 f_{yy}(-1,2)]$$

$$=6+(x+1)(-4)+(y-2)4+\frac{1}{2}[4(x+1)^2+2(x+1)(y-2)(-2)]$$

$$=6-4(x+1)+4(y-2)+2(x+1)^2-2(x+1)(y-2).$$

The maximum absolute error in the quadratic approximation, refer (5.100), is

$$|R_2| \le \frac{M}{6}[|x+1|+|y-2|]^3 \le \frac{M}{6}[0.1+0.1]^3 = \frac{.008M}{6} \approx .00133\,M,$$

where $M = \max\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\}$, for all $(x, y) \in R: |x+1| \le 0.1$, and $|y-2| \le 0.1$.

Thus $M = \max\{0, 2, 0, 0\} = 2$, and hence, $|R_2| \le (0.00133)(2) = 0.00266$.

Example 5.36: Expand $f(x, y) = e^x \ln(1+y)$ in Taylor series about the origin up to the terms of degree three.

Solution: We have

$$f(x,y) = e^x \ln(1+y), \qquad f(0,0) = 0$$

$$f_x = e^x \ln(1+y) \qquad f_x(0,0) = 0$$

$$f_y = \frac{e^x}{1+y} \qquad f_y(0,0) = 1$$

$$f_{xx} = e^x \ln(1+y) \qquad f_{xx}(0,0) = 0$$

$$f_{xy} = \frac{e^x}{1+y} \qquad f_{xy}(0,0) = 1$$

$$f_{yy} = -\frac{e^x}{(1+y)^2} \qquad f_{yy}(0,0) = -1$$

$$f_{xxx} = e^x \ln(1+y) \qquad f_{xxx}(0,0) = 0$$

$$f_{xxy} = \frac{e^x}{1+y} \qquad f_{xxy}(0,0) = 1$$

$$f_{xyy} = -\frac{e^x}{(1+y)^2} \qquad f_{xyy}(0) = -1$$

$$f_{yyy} = \frac{2e^x}{(1+y)^3} \qquad f_{yyy}(0) = 2.$$

Taylor's series expansion of $f(x, y)$ about $(0,0)$ up to the terms of degree 3 is

$$f(x,y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)]$$
$$+ \frac{1}{6}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)].$$

Substituting the values for $f(0, 0)$, $f_x(0, 0)$, $f_y(0, 0)$ etc. we get

$$e^x \ln(1+y) = 0 + [x(0) + y(1)] + \frac{1}{2}[x^2(0) + 2xy(1) + y^2(-1)] + \frac{1}{6}[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)]$$

or
$$e^x \ln(1+y) = y + xy + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3,$$

as the required expansion.

Example 5.37: Expand $f(x, y) = \tan^{-1} xy$ in powers of $(x-1)$ and $(y-1)$ up to second degree terms. Hence compute $f(1.1, 0.8)$.

Solution: We have

$$f(x,y) = \tan^{-1} xy \qquad f(1,1) = \pi/4 = 0.7854$$

$$f_x = \frac{y}{1+x^2y^2} \qquad f_x(1,1) = \frac{1}{2};$$

$$f_y = \frac{x}{1+x^2y^2} \qquad f_y(1,1) = \frac{1}{2};$$

$$f_{xx} = \frac{-2xy^3}{(1+x^2y^2)^2} \qquad f_{xx}(1,1) = -\frac{1}{2};$$

$$f_{xy} = \frac{1-x^2y^2}{(1+x^2y^2)^2} \qquad f_{xy}(1,1) = 0;$$

$$f_{yy} = \frac{-2x^3y}{(1+x^2y^2)^2} \qquad f_{yy}(1,1) = -\frac{1}{2}.$$

Taylor's series expansion of $f(x, y)$ about $(1, 1)$ up to terms of degree 2 is

$$f(x,y) = f(1,1) + [(x-1)f_x(1,1) + (y-1)f_y(1,1)]$$
$$+ \frac{1}{2}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)].$$

Substituting the values for $f(1, 1)$, $f_x(1, 1)$, $f_y(1, 1)$ etc. we get

$$\tan^{-1} xy = 0.7854 + \left[(x-1)\left(\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right]$$
$$+ \frac{1}{2}\left[(x-1)^2\left(-\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right]$$

or, $$\tan^{-1} xy = 0.7854 + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) - \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2.$$

To compute the value of $\tan^{-1} xy$ at (1.1, 0.8), put $x = 1.1$ and $y = 0.8$, we obtain

$$f(1.1, 0.8) = 0.7854 + \frac{1}{2}(0.1) + \frac{1}{2}(-0.2) - \frac{1}{4}(0.1)^2 - \frac{1}{4}(-0.2)^2 = 0.7229.$$

Example 5.38: Evaluate $\ln[(1.03)^{1/3} + (0.98)^{1/4} - 1]$ approximately using linear Taylor's series approximation.

Solution: Let $f(x, y) = \ln[x^{1/3} + y^{1/4} - 1]$. Take $x_0 = 1, y_0 = 1, h = 0.03$ and $k = -0.02$.

We have, $$f_x = \frac{\frac{1}{3}x^{-2/3}}{x^{1/3} + y^{1/4} - 1}, \text{ and } f_y = \frac{\frac{1}{4}y^{-3/4}}{x^{1/3} + y^{1/4} - 1}$$

Thus, $f(1, 1) = 0$, $f_x(1, 1) = 1/3$, and $f_y(1, 1) = 1/4$.

Linear Taylor's series approximation is $f(x_0 + h, y_0 + k) \approx f(x_0, y_0) + [f_x(x_0, y_0) + f_y(x_0, y_0)]$.

Substituting the values, we get

$$\ln[(1.03)^{1/3} + (0.98)^{1/4} - 1] \approx 0 + (.03)(1/3) - (.02)(1/4) = .01 - .005 = .005 \text{ (approx.)}$$

EXERCISE 5.7

1. Express $x^2 + 3y^2 - 9x - 9y + 26$ in powers of $(x - 2)$ and $(y - 2)$ using the Taylor's series expansion.
2. Obtain the quadratic Taylor's series polynomial approximation to the function $f(x, y) = 2x^3 + 3y^3 - 4x^2y$ about the point (1, 2) and obtain the maximum absolute error in the region $|x - 1| < 0.01$, $|y - 2| < 0.1$.
3. Using Taylor's series find a quadratic approximation of $\cos x \cos y$ at the origin and also estimate the error in the approximation, if $|x| \le 0.1$ and $|y| \le 0.1$.
4. Find the quadratic Taylor's series approximation of $e^x \cos y$ about the point $(1, \pi/4)$.
5. Find the cubic Maclaurin's approximation of $e^{ax} \sin by$.
6. Expand $\dfrac{(x+h)(y+k)}{(x+h)+(y+k)}$ in powers of h and k up to the second degree terms.
7. If $x^2 - xy + y^2$ is to be approximated by a linear Taylor's series polynomial about the point (2, 3), then find a square with centre at (2, 3) such that the error of approximation is less than or equal to 0.1 in magnitude for all points within this square.

8. Find the cubic Taylor's series polynomial approximation of $f(x, y) = \tan^{-1}(y/x)$ and hence compute $f(1.1, 0.9)$ approximately.
9. Expand $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ using Taylor's series up to first degree terms about the point (2, 2, 1). Obtain the maximum error in the region $|x - 2| < 0.1$, $|y - 2| < 0.1$, $|z - 1| < 0.1$.
10. Expand $f(x, y, z) = e^z \sin(x + y)$ in Taylor's series up to second order terms about the point (0, 0, 0). Obtain the maximum error in the region $|x| \le 0.1$, $|y| \le 0.1$, $|z| \le 0.1$.

5.8 EXTREME VALUES OF FUNCTIONS OF TWO VARIABLES

Similar to the case of function of a single variable as discussed in Section 4.8, here we discuss the extreme values of a function $z = f(x, y)$ of two independent variables x and y.

5.8.1 Local Maxima and Local Minima. Saddle Point

A function $z = f(x, y)$ is said to have a *local maxima* value at a point $P(a, b)$, if for all positive or negative small values of h and k

$$f(a + h, b + k) - f(a, b) < 0.$$

Similarly, the function $z = f(x, y)$ is said to have a *local minima* at a point $P(a, b)$, if for all positive or negative small values of h and k

$$f(a + h, b + k) - f(a, b) > 0.$$

Thus, if $\Delta f = f(a + h, b + k) - f(a, b)$ is of the same sign for all positive or negative small values of h and k, then point (a, b) is the point of local maxima, if $\Delta f < 0$, or of local minima, if $\Delta f > 0$, and $f(a, b)$ is the corresponding extreme, maximum or minimum value. In Fig. 5.3, $f(a, b)$ is the maximum value of $f(x, y)$ in the neighbourhood (*nbd.*) of (a, b), and P is the point $P(a, b, f(a, b))$.

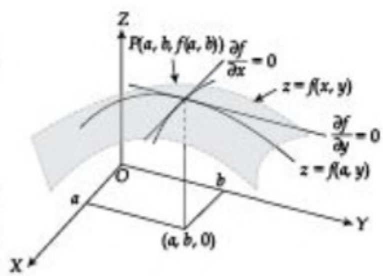


Fig. 5.3

In case the sign of Δf does not remain constant in the nbd. of (a, b), there will be neither a maxima nor a minima at (a, b), and then the point is said to be a *saddle point*.

5.8.2 Necessary Conditions For $f(x, y)$ to be Maximum or Minimum

We have seen that for $f(x, y)$ to be a maximum or minimum value at an arbitrary point (x, y), $\Delta f = f(x + h, y + k) - f(x, y)$ must keep the same sign for arbitrary small values of h and k. Using Taylor's expansion,

$$\Delta f = f(x + h, y + k) - f(x, y) = (hf_x + kf_y) + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}) + \ldots \qquad \ldots(5.101)$$

For small values of h and k, the second and higher order terms in h and k are still smaller and hence may be neglected, and thus sign of Δf depends on the sign of $(hf_x + kf_y)$, which changes with

h and k. Hence the necessary condition for $f(x, y)$ to be a maximum or minimum at an arbitrary point (x, y) is $h f_x + k f_y = 0$, for arbitrary small values of h and k which is feasible, if $f_x = 0, f_y = 0$.

Hence, the necessary conditions for $f(x, y)$ to have a maximum or a minimum at a point (x, y) are

$$f_x = 0, \quad f_y = 0. \qquad \text{...(5.102)}$$

A point satisfying the conditions in (5.102), is called a *stationary point* or a *critical point*.

The conditions in (5.102) are only the necessary and not sufficient one for a point (x, y) to be a point of maxima or minima. For example, consider the function $f(x, y) = y^2 - x^2$.

Here, $\quad f_x = -2x = 0, \Rightarrow x = 0$, and $f_y = 2y = 0 \Rightarrow y = 0$.

Therefore local maximum or minimum can occur only at the origin (0, 0). However, along the positive x-axis f has the value $f(x, 0) = -x^2 < 0$ and along the positive y-axis f has the value $f(0, y) = y^2 > 0$. Therefore, every neighbourhood centred at (0, 0) in the xy-plane contains points where the function is positive and points where it is negative, Thus (0, 0) is neither a point of maxima nor of minima for $f(x, y) = y^2 - x^2$.

5.8.3 Sufficient Conditions for $f(x, y)$ to be Maximum or Minimum

Suppose that a function $f(x, y)$ is continuous and possess first and second derivatives at a critical point (x, y), then $f_x = 0, f_y = 0$ at (x, y), and thus

$$\Delta f = f(x+h, y+k) - f(x, y) \approx \frac{1}{2}[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] + \dots$$

$$\approx \frac{1}{2f_{xx}}(h^2 f_{xx}^2 + 2hk f_{xy} f_{xx} + k^2 f_{xx} f_{yy})$$

$$\approx \frac{1}{2f_{xx}}[(h f_{xx} + k f_{xy})^2 + k^2(f_{xx} f_{yy} - f_{xy}^2)]. \qquad \text{...(5.103)}$$

Since $(hf_{xx} + kf_{xy})^2 > 0$, the sufficient condition for expression $[(hf_{xx} + kf_{xy})^2 + k^2(f_{xx}f_{yy} - f_{xy}^2)]$ to be positive is that $f_{xx}f_{yy} - f_{xy}^2 > 0$. Hence, if $f_{xx}f_{yy} - f_{xy}^2 > 0$, then $\Delta f < 0$, if $f_{xx} < 0$, and $\Delta f > 0$, if $f_{xx} > 0$.

Therefore, a sufficient condition for the critical point $P(x, y)$ to be a point of local maxima is

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \text{ and } f_{xx} < 0 \qquad \text{...(5.104)}$$

and, to be a point of local minima is

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \text{ and } f_{xx} > 0. \qquad \text{...(5.105)}$$

If $f_{xx}f_{yy} - f_{xy}^2 < 0$, then there will be neither a maxima nor a minima and point is called *saddle point*; and, if $f_{xx}f_{yy} - f_{xy}^2 = 0$, then the test is inconclusive.

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the *discriminant* of f. It is easy to remember the discriminant in the determinant form as

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

Example 5.39: Find the maximum and minimum values of the function $x^3 + y^3 - 3axy, a > 0$.

Solution: We have, $f(x, y) = x^3 + y^3 - 3axy$. Here $f_x = 3x^2 - 3ay$ and $f_y = 3y^2 - 3ax$.

For critical points $f_x = 0$ and $f_y = 0$. This gives $x^2 = ay$ and $y^2 = ax$.

Therefore, critical points are $(0, 0)$ and (a, a). Also, $f_{xx} = 6x$, $f_{xy} = -3a$, and $f_{yy} = 6y$.

At $(0, 0)$, $f_{xx}f_{yy} - f_{xy}^2 = 0 - 9a^2 = -9a^2 < 0$, therefore at $(0, 0)$, $f(x, y)$ is neither maximum nor minimum.

At (a, a), $f_{xx}f_{yy} - f_{xy}^2 = (6a)(6a) - 9a^2 = 27a^2 > 0$ and also $f_{xx} = 6a > 0$ and hence (a, a) is a point of minima and the minimum value of the function $f(x, y)$ is $a^3 + a^3 - 3a^3 = -a^3$.

Example 5.40: Find the absolute maximum and minimum values of the function

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular plate in the first quadrant bounded by $x = 0, y = 0, y = 9 - x$.

Solution: The function $f(x, y)$ can have maximum or minimum values at the critical points inside the triangle or on its boundary $OABO$, refer to Fig. 5.4.

We have $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ which gives

$$f_x = 2 - 2x, \quad \text{and} \quad f_y = 2 - 2y.$$

Now $f_x = 0, f_y = 0$ gives $(1, 1)$ as the critical point.

Further, $f_{xx} = -2, \; f_{xy} = 0, \; f_{yy} = -2$.

At $(1, 1)$, $f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 0 = 4 > 0$ and also $f_{xx} = -2 < 0$, and hence $(1, 1)$ is a point of maxima and the maximum value of $f(x, y)$ at $(1, 1)$ is $f(1, 1) = 4$.

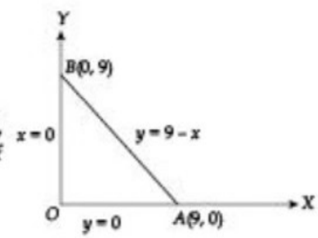


Fig. 5.4

On the boundary line OA, $y = 0$, and so the function

$$f(x, y) = f(x, 0) = g(x) = 2 + 2x - x^2,$$

which is a function of single variable on $0 \le x \le 9$.

Now, $dg/dx = 0$ gives $2 - 2x = 0$, or $x = 1$; also $\dfrac{d^2g}{dx^2} = -2 < 0$.

Therefore at $x = 1$, the function has a maxima and the maximum value is $g(1) = 3$.

Also at the end points $O(0, 0)$ and $A(9, 0)$ we have

$$f(0, 0) = g(0) = 2, \quad f(9, 0) = g(9) = 2 + 18 - 81 = -61$$

Next, on the boundary line OB, $x = 0$, and so the function

$$f(x, y) = f(0, y) = h(y) = 2 + 2y - y^2, \quad 0 \le y \le 9$$

As obtained above we have $f(0, 1) = h(1) = 3$, $f(0, 0) = h(0) = 2$, and $f(0, 9) = h(9) = -61$.

Next along the line AB, $y = 9 - x$ and thus

$$f(x, y) = f(x, 9 - x) = \phi(x) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2$$
$$= -61 + 18x - 2x^2, \quad 0 < x < 9.$$

Here, $\dfrac{d\phi}{dx} = 0$ gives $18 - 4x = 0$, or $x = 9/2$, also, $\dfrac{d^2\phi}{dx^2} = -4 < 0$.

Therefore at $x = 9/2$, the function has a maxima and the maximum value is

$$f\left(\frac{9}{2},\frac{9}{2}\right) = 2 + 2\left(\frac{9}{2}\right) + 2\left(\frac{9}{2}\right) - \frac{81}{4} - \frac{81}{4} = -\frac{41}{2}.$$

We have already evaluated $f(x, y)$ at the boundary points $A(9, 0)$ and $B(0, 9)$.

Therefore, the absolute maximum value is 4, which $f(x, y)$ assumes at $(1, 1)$ and the absolute minimum value is -61 which $f(x, y)$ assumes at $(0, 9)$ and $(9, 0)$.

Example 5.41: Find the points of maxima and minima of $x^3y^2(1 - x - y)$.

Solution: We have $f(x, y) = x^3y^2(1 - x - y)$, hence

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3, \text{ and } f_y = 2x^3y - 2x^4y - 3x^3y^2.$$

For critical points $f_x = 0$, and $f_y = 0$, which give respectively

$$x^2y^2(3 - 4x - 3y) = 0, \text{ and } x^3y(2 - 2x - 3y) = 0.$$

Solving for x and y, the critical points are $\left(\frac{1}{2},\frac{1}{3}\right)$ and $(0, 0)$.

Also, $f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3, f_{xy} = 6x^2y - 8x^3y - 9x^2y^2, f_{yy} = 2x^3 - 2x^4 - 6x^3y.$

At $\left(\frac{1}{2},\frac{1}{3}\right)$, $f_{xx}f_{yy} - f_{xy}^2 = \left(\frac{1}{3} - \frac{1}{3} - \frac{1}{9}\right)\left(\frac{1}{4} - \frac{1}{8} - \frac{1}{4}\right) - \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)^2 = \frac{1}{144} > 0.$

Also, at $\left(\frac{1}{2},\frac{1}{3}\right)$, $f_{xx} = \left(\frac{1}{3} - \frac{1}{3} - \frac{1}{9}\right) = -\frac{1}{9} < 0$, therefore $\left(\frac{1}{2},\frac{1}{3}\right)$ is a point of maxima.

Next, at $(0, 0)$, $f_{xx}f_{yy} - f_{xy}^2 = 0$ and thus further investigation is needed.

For points along the line $y = x$, $f(x, y) = x^5(1 - 2x)$ which is positive for x slightly greater than zero (say, $x = .01$) and negative for x slightly less than zero (say, $x = -.01$) and thus $f(x, y)$ changes sign around the point $(0, 0)$ and hence $(0, 0)$ is neither a point of maxima nor of minima.

Example 5.42: Find the points on the surface $z^2 = xy + 1$ nearest to the origin.

Solution: Let (x, y, z) be an arbitrary point on the surface

$$z^2 = xy + 1 \quad \text{...(5.106)}$$

and let d be the distance of this point from the origin, then

$$d = \sqrt{x^2 + y^2 + z^2}$$

or, $$d^2 = x^2 + y^2 + z^2 = x^2 + y^2 + xy + 1, \quad \text{using (5.106).}$$

Consider $f(x, y) = x^2 + y^2 + xy + 1$, then $f_x = 2x + y$, $f_y = 2y + x$.

For critical points $f_x = 0$, and $f_y = 0$, which give respectively $2x + y = 0$, and $2y + x = 0$.

Therefore critical point is $(0, 0)$.

Further $f_{xx} = 2, f_{xy} = 1, f_{yy} = 2$, and thus $f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$, and also $f_{xx} = 2 > 0$, therefore, $f(x, y)$ has minima at $(0, 0)$. Substituting, $x = 0, y = 0$ in (5.106), we get $z = \pm 1$.

Hence the points on $z^2 = xy + 1$ nearest to the origin are $(0, 0, \pm 1)$.

Example 5.43: Find the dimensions of a rectangular parallelopiped of maximum volume with edges parallel to co-ordinate axes which can be inscribed in the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Solution: If $P(x, y, z)$ be the co-ordinates of one of the vertex of the required rectangular parallelopiped in the positive octant inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad \text{...(5.107)}$$

then edges being parallel to the axes are of the lengths $2x$, $2y$ and $2z$ respectively.

Thus the volume of the rectangular parallelopiped is

$$V = (2x)(2y)(2z) = 8xyz. \qquad \text{...(5.108)}$$

We need to find the maxima of (5.108) subject to (5.107), we have

$$V^2 = 64x^2y^2z^2 = 64x^2y^2c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \text{ using (5.107)}$$

$$= 64c^2x^2y^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = f(x, y), \text{ say.}$$

Consider $$f(x, y) = 64c^2\left(x^2y^2 - \frac{x^4y^2}{a^2} - \frac{x^2y^4}{b^2}\right), \text{ then}$$

$$f_x = 64c^2\left(2xy^2 - \frac{4x^3y^2}{a^2} - \frac{2xy^4}{b^2}\right), f_y = 64c^2\left(2x^2y - \frac{2x^4y}{a^2} - \frac{4x^2y^3}{b^2}\right).$$

For critical points $f_x = 0$ and $f_y = 0$, which give respectively

$$xy^2\left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right) = 0 \qquad \text{...(5.109)}$$

and, $$x^2y\left(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2}\right) = 0. \qquad \text{...(5.110)}$$

Solving (5.109) and (5.110) and considering points with only positive values of abscissa and ordinate. The critical point obtained is $x = \frac{a}{\sqrt{3}}$, $y = \frac{b}{\sqrt{3}}$. Also,

$$f_{xx} = 64c^2\left(2y^2 - \frac{12x^2y^2}{a^2} - \frac{2y^4}{b^2}\right), f_{xy} = 64c^2\left(4xy - \frac{8x^3y}{a^2} - \frac{8xy^3}{b^2}\right), f_{yy} = 64c^2\left(2x^2 - \frac{2x^4}{a^2} - \frac{12x^2y^2}{b^2}\right).$$

At $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$,

$$f_{xx}f_{yy} - f_{xy}^2 = \left[64c^2\left(\frac{2b^2}{3} - \frac{12a^2b^2}{9a^2} - \frac{2b^4}{9b^2}\right)\right]\left[64c^2\left(\frac{2a^2}{3} - \frac{2a^4}{9a^2} - \frac{12a^2b^2}{9b^2}\right)\right] - \left[64c^2\left(\frac{4ab}{3} - \frac{8a^3b}{9a^2} - \frac{8ab^3}{9b^2}\right)\right]^2$$

$$= (64)^2\left(\frac{4}{3}\right)^2 c^4a^2b^2\left[\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right] > 0,$$

and,
$$f_{xx} = 64c^2\left(\frac{2}{3}b^2 - \frac{4}{3}b^2 - \frac{2}{9}b^2\right) = -\frac{512}{9}b^2c^2 < 0.$$

Hence $f(x, y)$ is maximum at $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}$.

Also at $x = \frac{a}{\sqrt{3}}$ and $y = \frac{b}{\sqrt{3}}$, $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} = \frac{c}{\sqrt{3}}$.

Thus dimensions of the desired rectangular parallelopiped are $\frac{2a}{\sqrt{3}}$, $\frac{2b}{\sqrt{3}}$ and $\frac{2c}{\sqrt{3}}$, and the corresponding volume is $V = 8xyz = \frac{8abc}{3\sqrt{3}}$.

5.9 CONSTRAINED EXTREME VALUES: LAGRANGE'S METHOD

Sometimes we need to find extreme values of a function which may be subject to some constraints. In such cases all variables in the function are not independent but are connected by some given relations. In general, we try to convert the given function to the one having the least number of independent variables using the given relation and then find the extreme values.

However, Lagrange's multiplier method is a powerful tool of finding the extreme values in case of constrained functions. The method was developed by Lagrange in 1755 to solve max-min problems in geometry.

5.9.1 Lagrange's Method

Suppose we need to find the extremum of the function $f(x, y, z)$ under the condition

$$\phi(x, y, z) = 0. \quad \text{...(5.111)}$$

We construct an auxiliary function of the form

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda\phi(x, y, z), \qquad \text{...(5.112)}$$

where λ is an undetermined parameter called the *Lagrange multiplier*.

To determine the stationary points of F, the necessary conditions obtained from (5.112) are

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0 \quad \text{and} \quad \frac{\partial F}{\partial z} = 0,$$

which give respectively the equations

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0, \text{ and } \quad \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0. \qquad \text{...(5.113)}$$

The Eqs. (5.111) and (5.113) give the values of x, y, z and λ for a maximum or minimum. The fact that whether the point is of maxima or minima is further established from the physical considerations of the problem.

In case the problem is subject to more than one constraint, say two, then two undetermined parameters are introduced and their values are determined.

Example 5.44: Find the maximum and minimum distances of the point $A(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$.

Solution: Let $P(x, y, z)$ be any point on the sphere $x^2 + y^2 + z^2 = 1$.

The distance of the given point $A(3, 4, 12)$ from (x, y, z) is

$$AP = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}.$$

We find the extreme values of the square of this distance, that is, of

$$(AP)^2 = (x-3)^2 + (y-4)^2 + (z-12)^2 = f(x, y, z), \quad \text{(say)} \qquad \text{...(5.114)}$$

subject to the condition

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0. \qquad \text{...(5.115)}$$

Consider the auxiliary function

$$F(x, y, z, \lambda) = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1),$$

where λ is Lagrange's constant.

For extreme values

$$\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x = 0, \qquad \text{...(5.116)}$$

$$\frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y = 0, \qquad \text{...(5.117)}$$

and, $$\frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z = 0. \qquad \text{...(5.118)}$$

Multiplying (5.116) by x, (5.117) by y, (5.118) by z and adding, we get

$$2(x^2 + y^2 + z^2) - 2(3x + 4y + 12z) + 2\lambda(x^2 + y^2 + z^2) = 0. \quad ...(5.119)$$

Using (5.115), and simplifying (5.119) becomes

$$3x + 4y + 12z = 1 + \lambda. \quad ...(5.120)$$

Next, from (5.116), (5.117) and (5.118), we have respectively

$$x = \frac{3}{1+\lambda}, \quad y = \frac{4}{1+\lambda}, \text{and} \quad z = \frac{12}{1+\lambda}. \quad ...(5.121)$$

Substituting these values of x, y, z in (5.120), we have

$$\frac{9}{1+\lambda} + \frac{16}{1+\lambda} + \frac{144}{1+\lambda} = 1 + \lambda,$$

which gives $(1 + \lambda)^2 = 169$, and hence $\lambda = 12$, or -14.

From (5.121) for $\lambda = 12$, the point is $P\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$, and for $\lambda = -14$, the point is $Q\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$. Thus, $AP = \sqrt{\left(3-\frac{3}{13}\right)^2 + \left(4-\frac{4}{13}\right)^2 + \left(12-\frac{12}{13}\right)^2} = 12$,

and,
$$AQ = \sqrt{\left(3+\frac{3}{13}\right)^2 + \left(4+\frac{4}{13}\right)^2 + \left(12+\frac{12}{13}\right)^2} = 14.$$

Therefore, the minimum and maximum distances of the point $A(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$ are 12 and 14, respectively.

Example 5.45: Find the shortest distance between the line $y = 10 - 2x$ and the ellipse $x^2/4 + y^2/9 = 1$.

Solution: Let (x, y) be a point on the ellipse $(x^2/4) + (y^2/9) - 1 = 0$ and, (u, v) be a point on the line $2x + y - 10 = 0$. The shortest distance between the line and the ellipse is the square root of the minimum value of

$$f(x, y, u, v) = (x - u)^2 + (y - v)^2 \quad ...(5.122)$$

subject to the conditions

$$\phi_1(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0, \quad ...(5.123)$$

and,
$$\phi_2(u, v) = 2u + v - 10 = 0. \quad ...(5.124)$$

Consider the auxiliary function

$$F(x, y, u, v, \lambda_1, \lambda_2) = (x - u)^2 + (y - v)^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + \lambda_2(2u + v - 10) \quad ...(5.125)$$

where λ_1 and λ_2 are the Lagrange constants.

For extreme values, from (5.125)

$$\frac{\partial F}{\partial x} = 2(x-u) + \frac{2\lambda_1 x}{4} = 0, \text{ or } \lambda_1 x = 4(u-x), \quad ...(5.126)$$

$$\frac{\partial F}{\partial y} = 2(y-u) + \frac{2\lambda_1 y}{9} = 0, \text{ or } \lambda_1 y = 9(v-y), \quad ...(5.127)$$

$$\frac{\partial F}{\partial u} = -2(x-u) + 2\lambda_2 = 0, \text{ or } \quad \lambda_2 = x-u, \quad ...(5.128)$$

and,
$$\frac{\partial F}{\partial v} = -2(y-v) + \lambda_2 = 0, \text{ or } \quad \lambda_2 = 2(y-v). \quad ...(5.129)$$

From (5.126) and (5.127)

$$4(u-x)y = 9(v-y)x, \quad ...(5.130)$$

and from (5.128) and (5.129)

$$(x-u) = 2(y-v). \quad ...(5.131)$$

Next from (5.130) and (5.131), we obtain $8y = 9x$, or $y = \frac{9}{8}x$. Substituting in (5.123), we get $\frac{x^2}{4} + \frac{9x^2}{64} = 1$, which gives $x^2 = \frac{64}{25}$, or $x = \pm\frac{8}{5}$ and thus $y = \pm 9/5$.

Substituting $x = 8/5$ and $y = 9/5$ in (5.131) we obtain $u = 2v - 2$. Substituting this in (5.124), we get $u = 18/5$ and $v = 14/5$.

Hence an extremum corresponds to $(x, y) = (8/5, 9/5)$ and $(u, v) = (18/5, 14/5)$.

The distance between these two points is

$$\sqrt{\left(\frac{8}{5} - \frac{18}{5}\right)^2 + \left(\frac{9}{5} - \frac{14}{5}\right)^2} = \sqrt{4+1} = \sqrt{5}.$$

Similarly corresponding to $(x, y) = (-8/5, -9/5)$ we obtain $(u, v) = (22/5, 6/5)$ and the corresponding distance between these two points is $3\sqrt{5}$.

Thus the shortest distance between the line and the ellipse is $\sqrt{5}$.

Example 5.46: The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the point on the ellipse that lie closest to and farthest from the origin.

Solution: Let (x, y, z) be a point on the desired ellipse in which the plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$. We find the extreme values of

$$f(x, y, z) = x^2 + y^2 + z^2, \quad ...(5.132)$$

the square of the distance of (x, y, z) from the origin subject to the constraints

$$\phi_1(x, y, z) = x^2 + y^2 - 1 = 0, \quad ...(5.133)$$

and, $\phi_2(x, y, z) = x + y + z - 1 = 0.$ ...(5.134)

Consider the auxiliary function

$$F(x, y, z, \lambda_1, \lambda_2) = x^2 + y^2 + z^2 + \lambda_1(x^2 + y^2 - 1) + \lambda_2(x + y + z - 1) \quad ...(5.135)$$

where λ_1, λ_2, are Lagrange constants.

For extreme values, from (5.135), we have

$$\frac{\partial F}{\partial x} = 2x + 2\lambda_1 x + \lambda_2 = 0, \quad ...(5.136)$$

$$\frac{\partial F}{\partial y} = 2y + 2\lambda_1 y + \lambda_2 = 0 \quad ...(5.137)$$

$$\frac{\partial F}{\partial z} = 2z + \lambda_2 = 0 \quad ...(5.138)$$

From (5.136) and (5.138), we obtain

$$(1 + \lambda_1)x = z. \quad ...(5.139)$$

From (5.137) and (5.138), we obtain

$$(1 + \lambda_1)y = z \quad ...(5.140)$$

The equations (5.139) and (5.140) are satisfied simultaneously if either $\lambda_1 = -1$ and $z = 0$, or, $\lambda_1 \neq -1$ and $x = y = \frac{z}{1+\lambda_1}$.

For $z = 0$ solving (5.133) and (5.134) simultaneously, the points are $P_1(1, 0, 0)$ and $P_2(0, 1, 0)$.

Next, for $x = y$, from (5.133) and (5.134), we get $2x^2 = 1$ and $z = 1 - 2x$. Solving these, we get $x = \pm\frac{\sqrt{2}}{2}, z = 1 \mp \sqrt{2}$. The corresponding points on the ellipse are

$$Q_1\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \text{ and } Q_2\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right).$$

The distance of $P_1(1, 0, 0)$ from origin is 1 and of $P_2(0, 1, 0)$ from the origin is 1.

Also the distance of Q_1, from the origin is $\sqrt{4 - 2\sqrt{2}}$, and of Q_2 from the origin is $\sqrt{4 + 2\sqrt{2}}$.

Hence $P_1(1, 0, 0)$ and $P_2(0, 1, 0)$ are the closest points to the origin and $Q_2\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right)$ is the farthest point to the origin lying on the desired ellipse.

EXERCISE 5.8

1. Find the maxima and minima of the functions

   (a) $x^3 + y^3 - 12x - 3y + 16$ (b) $x^3y + xy^3 + 2a^2xy - 3axy^2 - ax^3, \quad a > 0$

(c) $xy - x^2 - y^2 - 2x - 2y + 4$ (d) $x^4 + y^4 - 2x^2 + 4xy - 2y^2$

2. In a triangle ABC find the maximum value of $\cos A \cos B \cos C$.
3. Divide 36 into three parts such that the continued product of the first with the square of the second and the cube of the third is maximum.
4. If $xyz = 8$, find the values of x, y, z for which $5xyz/(x + 2y + 4z)$ is maximum.
5. Given the perimeter 20, determine the triangle of maximum area.
6. Given $x + y + z = a$, find the maximum value of $x^m y^n z^p$.
7. Find the point on the surface $y = x^2 + z^2$ nearest to the point $(3, 4, -6)$.
8. Find the maximum value of xyz subject to the condition $9x^2 + 36y^2 + 4z^2 = 36$. What is the geometrical interpretation of this problem?
9. Prove that of all rectangular parallelopiped of the same volume, the cube has the least surface.
10. Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.
11. A space probe in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x, y, z) on the probe's surface is $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. Find the hottest point on the probe's surface.
12. Find the point closest to the origin on the line of intersection of the planes $y + 2z = 12$ and $x + y = 6$.
13. Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.
14. Find the smallest and the largest distance between the points P and Q such that P lies on the plane $x + y + z = 2a$ and Q lies on the sphere $x^2 + y^2 + z^2 = a^2$, where a is any constant.
15. Using Lagrange's multipliers, show that the maximum and minimum values of $u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$, where $lx + my + nz = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ are given by
$$\frac{l^2a^4}{a^2u-1} + \frac{m^2b^4}{b^2u-1} + \frac{n^2c^4}{c^2u-1} = 0.$$

5.10 DIFFERENTIATION UNDER THE INTEGRAL SIGN: THE LEIBNITZ'S RULE

In applications sometimes we come across integrals of the form

$$\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)\,dx \qquad ...(5.141)$$

where α is a parameter and the integrand $f(x, \alpha)$ is such that it is not easily integrable. Leibnitz's rule gives us a procedure to evaluate these integrals by differentiating $\phi(\alpha)$ with respect to the parameter α to obtain $\phi'(\alpha)$ and then obtain the desired integral by integrating $\phi'(\alpha)$ with respect to α. The rule is stated as follows.

Theorem 5.4: ***(Leibnitz's rule)*** *If* $\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)\, dx$, *and* $a(\alpha)$, $b(\alpha)$, $f(x, \alpha)$ *are differentiable functions of* α *and* $\frac{\partial f}{\partial \alpha}$ *is continuous, then*

$$\frac{d\phi}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha} dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha}. \quad \text{...(5.142)}$$

Proof. Let $\Delta\alpha$ be an increment in α and Δa, Δb, be the corresponding increments in $a(\alpha)$ and $b(\alpha)$. If $\Delta\phi$ is the resultant increment in $\phi(\alpha)$, then

$$\Delta\phi = \phi(\alpha + \Delta\alpha) - \phi(\alpha) = \int_{a+\Delta a}^{b+\Delta b} f(x, \alpha + \Delta\alpha)dx - \int_a^b f(x, \alpha)dx$$

$$= \int_{a+\Delta a}^{a} f(x, \alpha + \Delta\alpha)dx + \int_a^b f(x, \alpha + \Delta\alpha)dx + \int_b^{b+\Delta b} f(x, \alpha + \Delta\alpha)dx - \int_a^b f(x, \alpha)dx,$$

or, $$\Delta\phi = \int_{a+\Delta a}^{a} f(x, \alpha + \Delta\alpha)dx + \int_a^b [f(x, \alpha + \Delta\alpha) - f(x, \alpha)]dx + \int_b^{b+\Delta b} f(x, \alpha + \Delta\alpha)dx.$$

Dividing by $\Delta\alpha$, we obtain

$$\frac{\Delta\phi}{\Delta\alpha} = \int_{a+\Delta a}^{a} \frac{f(x, \alpha + \Delta\alpha)}{\Delta\alpha}dx + \int_a^b \frac{f(x, \alpha + \Delta\alpha) - f(x, \alpha)}{\Delta\alpha}dx + \int_b^{b+\Delta b} \frac{f(x, \alpha + \Delta\alpha)}{\Delta\alpha}dx. \quad \text{...(5.143)}$$

Using the mean value theorem of integrals, refer to (6.8),

$$\int_{a+\Delta a}^{a} f(x, \alpha + \Delta\alpha)dx = -\Delta a f(\xi, \alpha + \Delta\alpha), \text{ for some } \xi, \alpha < \xi < a + \Delta a, \quad \text{...(5.144)}$$

and $$\int_b^{b+\Delta b} f(x, \alpha + \Delta\alpha)dx = \Delta b f(\eta, \alpha + \Delta\alpha), \text{ for some } \eta,\ b < \eta < b + \Delta b. \quad \text{...(5.145)}$$

Next using the Lagrange's mean value theorem,

$$f(x, \alpha + \Delta\alpha) - f(x, \alpha) = \Delta\alpha \frac{\partial f(x, \theta)}{\partial \alpha}, \text{ for some } \theta, \alpha < \theta < \alpha + \Delta\alpha. \quad \text{...(5.146)}$$

Using (5.144), (5.145) and (5. 146) in (5.143), we obtain

$$\frac{\Delta\phi}{\Delta\alpha} = -f(\xi, \alpha + \Delta\alpha)\frac{\Delta a}{\Delta\alpha} + \int_a^b \frac{\partial f(x, \theta)}{\partial \alpha} dx + f(\eta, \alpha + \Delta\alpha)\frac{\Delta b}{\Delta\alpha}. \quad \text{...(5.147)}$$

Taking limit as $\Delta\alpha \to 0$, and further we note that as $\Delta\alpha \to 0$, $\xi \to a$, $\eta \to b$ and $\theta \to \alpha$. Thus, (5.147) becomes

$$\frac{d\phi}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha}\,dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}.$$

In case $a(\alpha)$ and $b(\alpha)$ are independent of α, then Leibnitz's rule (5.142) becomes

$$\frac{d\phi}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha}\,dx. \quad \text{...(5.148)}$$

In case the integrand f is independent of α, then (5.142) gives

$$\frac{d\phi}{d\alpha} = f(b)\frac{db}{d\alpha} - f(a)\frac{da}{d\alpha}. \quad \text{...(5.149)}$$

Example 5.47: Evaluate $\int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx$, $\alpha \geq 0$ by applying differentiation under the integral sign.

Solution: Let $\phi(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx.$ ...(5.150)

Applying Leibnitz's rule, we obtain

$$\frac{d\phi}{d\alpha} = \int_0^1 \frac{\partial}{\partial\alpha}\left(\frac{x^\alpha - 1}{\ln x}\right)dx = \int_0^1 \frac{x^\alpha \ln x}{\ln x}\,dx = \int_0^1 x^\alpha dx = \frac{1}{\alpha+1}.$$

Integrating with respect to α, we get

$$\phi(\alpha) = \ln(\alpha+1) + c. \quad \text{...(5.151)}$$

From (5.150), $\phi(0) = 0$. Using in (5.151), we get $c = 0$. Thus, $\int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx = \ln(\alpha+1)$.

Example 5.48: Evaluate using the differentiation under the integral sign

$$\phi(\alpha) = \int_0^{\alpha^2} \tan^{-1}\left(\frac{x}{\alpha}\right)dx. \quad \text{...(5.152)}$$

Solution: Applying Leibnitz's rule, we obtain

$$\frac{d\phi}{d\alpha} = \int_0^{\alpha^2} \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{x}{\alpha}\right)\right)dx + \tan^{-1}\left(\frac{\alpha^2}{\alpha}\right)\frac{d}{d\alpha}(\alpha^2) - \tan^{-1}\left(\frac{0}{\alpha}\right)\frac{d}{d\alpha}(0)$$

$$= \int_0^{\alpha^2} \frac{1}{1+\frac{x^2}{\alpha^2}}\left(\frac{-x}{\alpha^2}\right)dx + 2\alpha\tan^{-1}\alpha = -\int_0^{\alpha^2} \frac{x}{\alpha^2+x^2}\,dx + 2\alpha\tan^{-1}\alpha$$

$$= -\frac{1}{2}[\ln(\alpha^2+x^2)]_0^{\alpha^2} + 2\alpha\tan^{-1}\alpha = 2\alpha\tan^{-1}\alpha - \frac{1}{2}\ln(1+\alpha^2).$$

Integrating with respect to α, we get

$$\phi(\alpha) = \int 2\alpha\tan^{-1}\alpha\, d\alpha - \frac{1}{2}\int \ln(1+\alpha^2)d\alpha + c$$

$$= \left[\tan^{-1}\alpha.\alpha^2 - \int \frac{\alpha^2}{1+\alpha^2}d\alpha\right] - \frac{1}{2}\left[\ln(1+\alpha^2).\alpha - \int \frac{2\alpha^2}{1+\alpha^2}d\alpha\right] + c$$

$$= \alpha^2\tan^{-1}\alpha - \int \frac{\alpha^2}{1+\alpha^2}d\alpha - \frac{1}{2}\alpha\ln(1+\alpha^2) + \int \frac{\alpha^2}{1+\alpha^2}d\alpha + c$$

$$= \alpha^2\tan^{-1}\alpha - \frac{1}{2}\alpha\ln(1+\alpha^2) + c. \qquad ...(5.153)$$

From (5.152), $\phi(0) = 0$. Using this in (5.153), we get $c = 0$. Thus,

$$\int_0^{\alpha^2} \tan^{-1}\left(\frac{x}{\alpha}\right)dx = \alpha^2\tan^{-1}\alpha - \frac{1}{2}\alpha\ln(1+\alpha^2).$$

Example 5.49: By successive differentiation of $\int_0^{\infty} e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{2\alpha}$, $(\alpha > 0)$, w.r.t. α, prove that

$$\int_0^{\infty} x^{2n}e^{-\alpha^2x^2}dx = \sqrt{\pi}\frac{(2n)!}{n!(2\alpha)^{2n+1}}, \text{ where } n \text{ is a positive integer.}$$

Solution: We have, $\int_0^{\infty} e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{2\alpha}$. ...(5.154)

Differentiating both sides of (5.154) w. r. t. α and using Leibnitz's rule, we obtain

$$\int_0^{\infty} -2\alpha x^2e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{2}\left(-\frac{1}{\alpha^2}\right)$$

or,

$$\int_0^{\infty} x^2e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{2}\frac{1}{2\alpha^3}. \qquad ...(5.155)$$

Again differentiating (5.155) w. r. t. α using Leibnitz's rule, and after simplification, we get

$$\int_0^\infty x^4 e^{-\alpha^2 x^2} dx = \frac{\sqrt{\pi}}{2} \frac{1.3}{2^2 \alpha^5}.$$

In general, we can write

$$\int_0^\infty x^{2n} e^{-\alpha^2 x^2} dx = \frac{\sqrt{\pi}}{2} \frac{1.3.5 \ldots (2n-1)}{2^n . \alpha^{2n+1}} = \frac{\sqrt{\pi}\,(2n)!}{n!\,(2\alpha)^{2n+1}}.$$

This proves the result.

EXERCISE 5.9

1. By differentiating under the integral sign, evaluate the integral $\int_0^\infty \frac{e^{-\alpha x}\sin x}{x} dx, \alpha > 0$, and hence show that $\int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \frac{\sin \alpha x}{x} dx = \frac{\pi}{2}, \alpha > 0.$

2. Evaluate the integral $\int_0^\infty \frac{e^{-x} - e^{-\alpha x}}{x \sec x} dx, \alpha > 0.$

3. Evaluate the integral $\int_0^\infty \frac{\tan^{-1}(\alpha x)}{x(1+x^2)} dx, \alpha \geq 0$ and $\alpha \neq 1$.

4. Using the result $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, evaluate $\int_0^\infty e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx.$

5. Show that $\int_0^\infty e^{-x}\left(\frac{1-\cos \alpha x}{x}\right) dx = \frac{1}{2} \ln(1+\alpha^2).$

Using the concept of differentiation under the integral sign, show that

6. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \ln \frac{b}{a}, a, b, > 0.$

7. $\int_0^1 \frac{x^a - x^b}{\ln x} dx = \ln[(a+1)/(b+1)], a, b, > -1.$

8. $\int_0^1 x^m (\ln x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$, $m > 1$ and $n > 0$ are integers.

9. $\int_0^{\pi/2} \ln\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\frac{d\theta}{\sin\theta} = \pi\sin^{-1}\frac{a}{b}, a > b.$

10. $\int_0^{\pi/2} \ln(\alpha\cos^2\theta + \beta\sin^2\theta)\, d\theta = \pi\ln\left[\frac{1}{2}\left(\sqrt{\alpha}+\sqrt{\beta}\right)\right], \alpha, \beta > 0.$

11. By successive differentiation of $\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$ w. r. t. a, show that

$$\int_0^\infty \frac{dx}{(x^2+a^2)^{n+1}} = \frac{(2n)!\pi}{(n!)^2(2a)^{2n+1}}$$

12. Evaluate $\int_0^\alpha \frac{\ln(1+\alpha x)}{1+x^2}dx$ by differentiating under the integral sign and hence show that

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2.$$

ANSWERS

Exercise 5.1 (p. 299)

1. 1	2. 1	3. 0	4. 4
5. 5	6. does not exist	7. 0	8. does not exist
9. discontinuous	8. continuous	11. discontinuous	12. continuous
15. continuous.			

Exercise 5.2 (p. 308)

1. (i) $y^2\cos xy$; $xy\cos xy + \sin xy$ (ii) $2x+3y$; $3x+1$

(iii) $\frac{z}{1-z}$, $\frac{z}{1-z}$ (iv) $-g(x)$; $g(y)$

2. (i) $\dfrac{yz}{\sqrt{1-x^2y^2z^2}}, \quad \dfrac{xz}{\sqrt{1-x^2y^2z^2}}, \quad \dfrac{xy}{\sqrt{1-x^2y^2z^2}}$

   (ii) $\operatorname{sech}^2(x+2y+3z), \quad 2\operatorname{sech}^2(x+2y+3z), \quad 3\operatorname{sech}^2(x+2y+3z).$

Exercise 5.3 (p. 312)

1. 0.31 2. $|x-1| \le 1/70, \ |y-1| \le 1/70$ 3. 0.1
4. 121.6 watts 5. 7 cm. 7. 2% 8. 2.39
9. (a) 4.02 (b) 1.81 10. (a) $\dfrac{1}{2\sqrt{2}}\left[1+\dfrac{\pi}{180}(2\sqrt{3}+1)\right]$

   (b) $\dfrac{1}{720}[180+\pi(6-\sqrt{3})]$.

Exercise 5.4 (p. 317)

1. (a) 0, (b) $2t(t^2-1)^2\{(4t^2+1)\cos 4t - 2t(t^2+1)\sin 4t\}$

   (c) $(\pi/2 - 2/\pi)$

5. $y/(2yz-x), \ y/(y^2-3z^2); \ \dfrac{1}{4}, \ -1/11$ 6. 0, 0

7. $-\dfrac{y(\sin xy)+z(\sin xz)}{y\sin(yz)+x(\sin xz)}, \quad \dfrac{(x\sin xy)+z(\sin yz)}{y(\sin yz)+x(\sin xz)}$

8. $-(yx^{y-1}+y^x\ln y)/(xy^{x-1}+x^y\ln x)$ 15. 5, 5.

Exercise 5.5 (p. 328)

9. $v^2 = u + 2w$ 10. $uv = w$ 11. $v = \sin u$.

Exercise 5.7 (p. 341)

1. $6-5(x-2)+3(y-2)+(x-2)^2+3(y-2)^2$
2. $18-10(x-1)+32(y-2)-2[(x-1)^2+4(x-1)(y-2)-9(y-2)^2]$; 0.004
3. $1-\dfrac{1}{2}x^2-\dfrac{1}{2}y^2$; 0.00134
4. $$\frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} + \left(y-\frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right)\right]$$
$$+\frac{1}{2!}\left[(x-1)^2\frac{e}{\sqrt{2}} + 2(x-1)\left(y-\frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right) + \left(y-\frac{\pi}{4}\right)^2\left(-\frac{e}{\sqrt{2}}\right)\right]$$

5. $(by + abxy) + \frac{1}{6}(3a^2bx^2y - b^3y^3)$

6. $\frac{xy}{x+y} + \frac{hy^2 + kx^2}{(x+y)^2} - \frac{h^2y^2 - 2hxy + k^2x^2}{(x+y)^3}$

7. $|x-2| \le 0.1581,\ |y-3| \le 0.1581$

8. $\frac{\pi}{4} - \frac{1}{2}[(x-1)-(y-1)] + \frac{1}{4}[(x-1)^2 - (y-1)^2]$

$- \frac{1}{12}[(x-1)^3 + 3(x-1)^2(y-1) - 3(x-1)(y-1)^2 - (y-1)^3];$ 0.6887

9. $3 + \frac{2}{3}[(x-2) + (y-2) + (z-1)];$ 0.017

10. $x + y + xz + yz;$ 0.005.

Exercise 5.8 (p. 351)

1. (a) Max at $(-2, -1)$, min at $(2, 1)$

(b) Max at $\left(\frac{a}{2}, \frac{a}{2}\right)$, min at $\left(-\frac{a}{2}, \frac{a}{2}\right)$

(c) Max at $(-2, -2)$

(d) Min at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

2. $\frac{1}{8}$

3. 6, 12, 18

4. 4, 2, 1

5. Equilateral triangle

6. $m^m n^n p^p a^{m+n+p}/(m+n+p)^{m+n+p}$

7. $(1, 5, -2)$

8. $\frac{2}{\sqrt{3}}$

10. $\frac{8}{3\sqrt{3}}$ cubic units

11. $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$

12. $(2, 4, 4)$

13. Max is 4 at $(0, 0, \pm 2)$ and min is 2 at $(\pm\sqrt{2}, \pm\sqrt{2}, 0)$

14. $P\left(\frac{2a}{3}, \frac{2a}{3}, \frac{2a}{3}\right)$, $Q\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}\right)$; $\frac{a}{\sqrt{3}}\sqrt{(7-4\sqrt{3})}$; $\frac{a}{\sqrt{3}}\sqrt{(7+4\sqrt{3})}$.

Exercise 5.9 (p. 356)

2. $\frac{1}{2}\ln\left(\frac{1+\alpha^2}{2}\right)$

3. $\frac{\pi}{2}\ln(1+\alpha)$

4. $\frac{\sqrt{\pi}}{2}e^{-2a}$.

6 CHAPTER Definite Integrals and Their Applications

Integral calculus is related to differential calculus by the fundamental theorem of integral calculus which says roughly that the derivative and integral are inverse operators. Integration is vital to many scientific areas since numerous powerful mathematical tools are based on integration. Applications of definite integrals include computations involving area, arc length, volume and surface of the solid generated.

6.1 ANTIDERIVATIVES: INDEFINITE INTEGRALS

A function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$ for all x in the domain of definition of $f(x)$. We observe that if $F(x)$ is an antiderivative of $f(x)$, then $F(x) + c$, where c is an arbitrary constant, is also antiderivative of $f(x)$. $F(x) + c$ is called the *indefinite integral of $f(x)$ with respect to x*, and is denoted by

$$\int f(x)\,dx = F(x) + c. \qquad \text{...(6.1)}$$

The function f is called the *integrand* and the constant c is called the *constant of integration*.

A number of indefinite integrals are found by reversing derivative formulae. We can evaluate an indefinite integral directly, or by using the method of substitution, or integration by parts.

Next, we list indefinite integrations of some commonly used functions. These results can be derived using the standard methods of integration.

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c, \ (n \neq -1)$
2. $\int \dfrac{1}{x} dx = \ln |x| + c$
3. $\int e^{ax} dx = \dfrac{1}{a} e^{ax} + c$
4. $\int \sin x dx = -\cos x + c$
5. $\int \cos x dx = \sin x + c$
6. $\int \tan x dx = -\ln |\cos x| + c$

7. $\int \cot x\,dx = \ln|\sin x| + c$

8. $\int \sec x\,dx = \ln|\sec x + \tan x| + c$

9. $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + c$

10. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

11. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

12. $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + c$

13. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + c$

14. $\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

15. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

16. $\int \tan^2 x\,dx = \tan x - x + c$

17. $\int \cot^2 x\,dx = -\cot x - x + c$

18. $\int \ln x\,dx = x\ln x - x + c$

19. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

20. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

21. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

22. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + c$

23. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right) + c$

24. $\int \sinh x\,dx = \cosh x + c$

25. $\int \cosh x\,dx = \sinh x + c$

26. $\int \operatorname{sech}^2 x\,dx = \tanh x + c$

27. $\int \operatorname{cosech}^2 x\,dx = \coth x + c$

28. $\int \tanh x\,dx = \ln|\cosh x| + c$

29. $\int \coth x\,dx = \ln|\sinh x| + c$

6.2 DEFINITE INTEGRALS AND THEIR PROPERTIES

Let $f(x)$ be a function defined and continuous on a closed interval $[a, b]$. Divide the interval $[a, b]$ into n subintervals by choosing $n-1$ points, say $x_1, x_2, \ldots, x_{n-1}$ between a and b such that

$$a < x_1 < x_2 < \ldots < x_{n-1} < b.$$

To make the notation consistent denote, $a = x_0$ and $b = x_n$, and let $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots n$.

Let $m_i = \min\limits_{x_{i-1} \le x \le x_i} f(x)$, and $M_i = \max\limits_{x_{i-1} \le x \le x_i} f(x)$ and c_i be any point in the interval $[x_{i-1}, x_i]$, $i = 1, 2 \ldots n$.

Corresponding to this partition $P = \{x_0, x_1, \ldots, x_n\}$, the sum $S_n(f) = \sum\limits_{i=1}^{n} f(c_i)\Delta x_i$, which depends on P and the choice of the numbers $c_i's$, is called a *Reimann sum for f on the interval [a, b]*. Next we define, lower sum $= L_n(f) = \sum\limits_{i=1}^{n} m_i \Delta x_i$; upper sum $= U_n(f) = \sum\limits_{i=1}^{n} M_i \Delta x_i$;

and hence,

$$L_n(f) \le S_n(f) \le U_n(f). \qquad \ldots(6.2)$$

Let $n \to \infty$ such that $\max(\Delta x_i) \to 0$, and if $\lim\limits_{n\to\infty} L_n(f) = \lim\limits_{n\to\infty} S_n(f) = \lim\limits_{n\to\infty} U_n(f) = I$, say

for any choice of the partition P and c_i in the interval $[x_{i-1}, x_i]$, then this limit I denoted by $\int\limits_a^b f(x)dx$ is called *the definite integral of f(x) over [a, b] and we say that f is integrable over [a, b].*

Geometrically, $\int\limits_a^b f(x)dx$ gives the area bounded by the curve $y = f(x)$, the axis of x and the two ordinates $x = a$ and $x = b$.

6.2.1 The Existence of Definite Integrals

Every function f(x) which is continuous and bounded on a closed interval [a, b] has a definite integral over it.

Further many discontinuous functions are also integrable. For example,

$$f(x) = \begin{cases} -1, & -1 < x < 0 \\ 0, & x = 0 \\ 1, & 0 < x < 1 \end{cases}$$

The function $f(x)$ is discontinuous at $x = 0$ but $\int\limits_{-1}^{1} f(x)dx$ exists.

In fact, the function f(x) may only be piecewise continuous over [a, b] for the definite integral to exist.

There are some discontinuous functions which are not integrable. For example, consider the function $f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ 0, & \text{where } x \text{ is irrational} \end{cases}$. This function is not integrable over $[0, 1]$, since in this case for any choice of the partition P, the lower and upper sums are

$L_n(f) = \Sigma \min x_i . \Delta x_i = \Sigma 0.\Delta x_i = 0$ and, $U_n(f) = \Sigma \max x_i \ \Delta x_i = \Sigma 1 \ \Delta x_i = \Sigma \Delta x_i = 1.$

and thus, $$\lim_{\substack{n \to \infty \\ \Delta x_i \to 0}} L_n(f) \neq \lim_{\substack{n \to \infty \\ \Delta x_i \to 0}} U_n(f).$$

6.2.2 Properties of Definite Integrals

We state below some important properties of the definite integrals.

1. Invariance Property. If $f(x)$ is integrable over $[0, a]$, then

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx. \quad ...(6.3)$$

2. $$\left.\begin{aligned} \int_{-a}^{a} f(x)dx &= 2\int_0^a f(x)dx, && \text{if } f(x) \text{ is even} \\ &= 0, && \text{if } f(x) \text{ is odd} \end{aligned}\right\} \quad ...(6.4)$$

3. $$\left.\begin{aligned} \int_0^{2a} f(x)dx &= 2\int_0^a f(x)dx, && \text{if } f(2a-x) = f(x) \\ &= 0, && \text{if } f(2a-x) = -f(x) \end{aligned}\right\} \quad ...(6.5)$$

4. Shift Property: If $f(x)$ is integrable and defined for the necessary values of x, then

$$\int_a^b f(x)dx = \int_{a-c}^{b-c} f(x+c)dx. \quad ...(6.6)$$

5. Max-Min Inequality: If M and m are the maximum and minimum values of $f(x)$ on $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a). \quad ...(6.7)$$

6. Mean-Value Theorem: If $f(x)$ is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{(b-a)} \int_a^b f(x)dx. \quad ...(6.8)$$

The expression, $\frac{1}{b-a}\int_a^b f(x)dx$ is called the *average, or mean value of $f(x)$ on $[a, b]$*.

7. First Fundamental Theorem of Integral Calculus: If $f(x)$ is continuous on the closed interval $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$, differentiable in (a, b), and $\frac{dF}{dx} = f(x),\ a \le x \le b.$

That is, for every continuous function the process of integration and differentiation are inverse of one another.

8. Second Fundamental Theorem of Integral Calculus: If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a). \quad \text{...(6.9)}$$

This theorem gives us a method to evaluate the definite integral of a continuous function $f(x)$ from a to b. The existence part is taken care of by the First Fundamental Theorem of Integral Calculus.

Example 6.1: Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}dx.$

Solution: Let $$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}dx = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}dx, \text{ using (6.3)}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}dx.$$

Thus, $$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}dx = \int_0^{\pi/2} dx = \pi/2, \text{ and hence } I = \pi/4.$$

Example 6.2: Evaluate $\displaystyle\int_0^{\pi/2} \ln \sin x\, dx.$

Solution: Let $$I = \int_0^{\pi/2} \ln \sin x\, dx = \int_0^{\pi/2} \ln \sin(\pi/2 - x)dx, \quad \text{using (6.3)}$$

$$= \int_0^{\pi/2} \ln \cos x\, dx.$$

Thus, $$2I = \int_0^{\pi/2} (\ln \sin x + \ln \cos x)dx = \int_0^{\pi/2} \ln \sin x \cos x\, dx = \int_0^{\pi/2} \ln \frac{\sin 2x}{2}dx$$

$$= \int_0^{\pi/2} \ln \sin 2x\, dx - \frac{\pi}{2} \ln 2 = \frac{1}{2} \int_0^{\pi} \ln \sin x\, dx - \frac{\pi}{2} \ln 2$$

$$= \int_0^{\pi/2} \ln \sin x\, dx - \frac{\pi}{2} \ln 2, \quad \text{using (6.5)}$$

$$= I - \frac{\pi}{2} \ln 2, \text{ and hence, } \quad I = -\frac{\pi}{2} \ln 2.$$

Example 6.3: Without actual integration, show that $\int_0^1 \sqrt{1+\cos x}\, dx \le \sqrt{2}$.

Solution: The maximum value of $\sqrt{1+\cos x}$ over the interval [0, 1] is $\sqrt{1+1} = \sqrt{2}$ and so

$$\int_0^1 \sqrt{1+\cos x}\, dx \le \max. \sqrt{1+\cos x}\,.(1-0) = \sqrt{2}, \text{ using (6.7).}$$

Example 6.4: Using the inequality $\cos x \ge 1 - \frac{x^2}{2}$, find a lower bound for the value of $\int_0^1 \cos x\, dx$.

Solution: We have $\cos x \ge 1 - \frac{x^2}{2}$ $\forall x$ in [0, 1], thus

$$\int_0^1 \cos x\, dx \ge \int_0^1 \left(1 - \frac{x^2}{2}\right) dx = \int_0^1 dx - \frac{1}{2}\int_0^1 x^2\, dx = 1 - \frac{1}{6} = \frac{5}{6}. \text{ Hence, } 5/6 \le \int_0^1 \cos x\, dx.$$

Example 6.5: Find the mean value of $f(x) = (1+x^2)$ over [2, 3]. Does $f(x)$ actually assume this value at some point in the interval [2, 3]? If so, find that point.

Solution: The mean value of $f(x) = 1 + x^2$ over [2, 3] is given by

$$\frac{1}{3-2}\int_2^3 (1+x^2)\, dx = \left[x + \frac{x^3}{3}\right]_2^3 = 12 - \frac{14}{3} = \frac{22}{3}.$$

The function f assumes this value at the point, where

$$1 + x^2 = \frac{22}{3} \text{ or } \quad x^2 = \frac{19}{3}, \text{ or } \quad x = \sqrt{\frac{19}{3}} \approx 2.52,$$

which lies in [2, 3], the negative value is rejected since it does not lie in [2, 3].

EXERCISE 6.1

1. Without actual integration, show that the value of $\int_0^1 \sin(x^2)dx$ cannot possibly be greater than 1.
2. Without actual integration, show that the value of $\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2}$ and 3.
3. Show that if f is continuous on $[a, b]$, $a \neq b$, and if $\int_a^b f(x)dx = 0$, then $f(c) = 0$ for at least one c in $[a, b]$.
4. Find the mean value of pressure p varying from 2 to 10 atm, if the pressure p and the volume v are related as $pv^{3/2} = 160$.
5. A body falling to the ground from a state of rest acquires a velocity $v = \sqrt{2gs}$ on covering a vertical path s. Find the average velocity over this path.
6. Show that the mean value of $f(x) = \dfrac{\cos^2 x}{\sin^2 x + 4\cos^2 x}$ over the interval $\left[0, \dfrac{\pi}{2}\right]$ is 1/6, also find the point where this mean value is attained.
7. Show that $\int_0^1 \sec x\,dx \geq 7/6$.
8. Evaluate

   (a) $\displaystyle\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$ (b) $\displaystyle\int_0^1 \frac{\sin^{-1} x}{x}dx$

9. Prove that

   (a) $\displaystyle\int_0^\pi \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi^2}{2ab}$ (b) $\displaystyle\int_0^\pi \frac{x\,dx}{1+\sin^2 x} = \frac{\pi^2}{2\sqrt{2}}$

10. Prove that

    (a) $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$ (b) $\displaystyle\int_0^1 \cot^{-1}(1 - x + x^2)dx = \frac{\pi}{2} - \ln 2.$

6.3 REDUCTION FORMULAE

Some integrals involving certain parameters are solved by the successive applications of integration by parts which results in *reduction formulae*. For example, consider the integral

$$I_n = \int x^n e^{ax} dx = x^n \frac{e^{ax}}{a} - \int n x^{n-1} \frac{e^{ax}}{a} dx$$

or,
$$I_n = \frac{x^n}{a} e^{ax} - \frac{n}{a} I_{n-1} \qquad \text{...(6.10)}$$

The index n has been reduced to $n - 1$ and thus the integral I_n can be solved by using the recurrence relation (6.10), the reduction formula for I_n. Next we derive a few reduction formulae generally used in solving various integrals.

6.3.1 Reduction Formula for

(a) $\int \sin^n x\, dx$ **(b)** $\int \cos^n x\, dx$ **(c)** $\int \sin^m x \cos^n dx$

where m, n are natural numbers.

(a) Let $I_n = \int \sin^n x\, dx = \int \sin^{n-1} x \sin x\, dx$.

Integrating it by parts, we obtain

$$I_n = \sin^{n-1} x(-\cos x) - \int (n-1) \sin^{n-2} x \cos x(-\cos x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x(1 - \sin^2 x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

or,
$$n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

or,
$$I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2} \qquad \text{...(6.11)}$$

as the reduction formula for $\int \sin^n x\, dx$.

Specifically, consider the definite integral $I_n = \int_0^{\pi/2} \sin^n x\, dx$.

Using (6.11), it gives

$$I_n = \int_0^{\pi/2} \sin^n x\, dx = -\frac{1}{n} [\sin^{n-1} x \cos x]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx$$

or,
$$I_n = 0 + \frac{n-1}{n} I_{n-2}$$

or, $$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx \qquad ...(6.12)$$

(b) Let $\quad I_n = \int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx.$

Integrating by parts, we obtain

$$I_n = \cos^{n-1} x \sin x - \int (n-1)\cos^{n-2} x(-\sin x)\sin x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n$$

or, $$nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2}$$

or, $$I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} \qquad ...(6.13)$$

as the reduction formula for $\int \cos^n x\,dx$.

Specifically, consider the definite integral, $I_n = \int_0^{\pi/2} \cos^n x\,dx,$

Using (6.13), it becomes

$$I_n = \frac{1}{n}[\cos^{n-1} x \sin x]_0^{\pi/2} + \frac{n-1}{n}I_{n-2} = 0 + \frac{n-1}{n}I_{n-2}$$

or, $$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx. \qquad ...(6.14)$$

From (6.12) and (6.14) we observe that if

$I_n = \int_0^{\pi/2} \sin^n x\,dx$, or $= \int_0^{\pi/2} \cos^n x\,dx$, both can be expressed as

$$I_n = \frac{n-1}{n}I_{n-2} = \frac{n-1}{n}\frac{n-3}{n-2}I_{n-4} = \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}I_{n-6}, \text{ and so on.}$$

Thus, $$I_n = \begin{cases} \dfrac{n-1}{n}\dfrac{n-3}{n-2}\dfrac{n-5}{n-4}\cdots\dfrac{3}{4}\dfrac{1}{2}I_0, \text{ if } n \text{ is even.} \\ \dfrac{n-1}{n}\dfrac{n-3}{n-2}\dfrac{n-5}{n-4}\cdots\dfrac{4}{5}\dfrac{2}{3}I_1, \text{ if } n \text{ is odd.} \end{cases}$$

Further, $I_0 = \int_0^{\pi/2} dx = \pi/2$ $\quad I_1 = \int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = 1$, if $I_n = \int_0^{\pi/2} \sin^n x\,dx$

and, $I_1 = \int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1$, if $I_n = \int_0^{\pi/2} \cos^n x\,dx$. Hence, in both cases $I_1 = 1$. Thus,

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\ldots 3.1}{n(n-2)(n-4)\ldots 4.2}\dfrac{\pi}{2}, & \text{if n is even.} \\ \dfrac{(n-1)(n-3)(n-5)\ldots 4.2}{n(n-2)(n-4)\ldots 5.3}1, & \text{if n is odd.} \end{cases} \quad \ldots(6.15)$$

(c) Let $I_{m,n} = \int \sin^m x \cos^n x\,dx$

Rewrite it as

$I_{m,n} = \int \sin^m x \cos^{n-1} x \cos x\,dx = \int \cos^{n-1} x\,(\sin^m x \cos x)dx$

Integrating it by parts taking $\cos^{n-1}x$ as first function, we obtain

$$I_{m,n} = \cos^{n-1} x\frac{\sin^{m+1} x}{m+1} - \int (n-1)\cos^{n-2} x(-\sin x)\frac{\sin^{m+1} x}{m+1}dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x(1-\cos^2 x)dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\,dx - \frac{n-1}{m+1}\int \sin^m x \cos^n x\,dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}I_{m,n-2} - \frac{n-1}{m+1}I_{m,n}$$

or, $$\left(\frac{m+n}{m+1}\right)I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}I_{m,n-2}$$

or, $$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n}I_{m,n-2} \quad \ldots(6.16)$$

as the reduction formula.

In (6.16) the power of cosine gets reduced by two. In case the power of sine is to be reduced by two, the corresponding reduction formula is

$$I_{m,n} = \int \sin^m x \cos^n x\,dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \quad \text{...(6.17)}$$

In case of definite integral with limits 0 to $\frac{\pi}{2}$, we have from (6.16)

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx = \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x dx$$

or, $I_{m,n} = 0 + \frac{n-1}{m+n} I_{m,n-2} = \frac{n-1}{m+n} I_{m,n-2}$ ...(6.18)

Similarly from (6.17), we have

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} \quad \text{...(6.19)}$$

Replacing n by $n-2$ in (6.18), we obtain $I_{m,n-2} = \frac{n-3}{m+n-2} I_{m,n-4}$, and hence (6.18) becomes

$$I_{m,n} = \frac{(n-1)(n-3)}{(m+n)(m+n-2)} I_{m,n-4} = \frac{(n-1)(n-3)(n-5)}{(m+n)(m+n-2)(m+n-4)} I_{m,n-6}, \text{ and so on.}$$

Case I: When n is even

$$I_{m,n} = \frac{(n-1)(n-3)(n-5)\ldots 3.1}{(m+n)(m+n-2)(m+n-4)\ldots(m+2)} I_{m,0} \text{ But } I_{m,0} = \int_0^{\pi/2} \sin^m x dx = I_m$$

Therefore, using (6.15)

$$I_{m,n} = \begin{cases} \frac{(n-1)(n-3)(n-5)\ldots 1}{(m+n)(m+n-2)\ldots(m+2)} \frac{(m-1)(m-3)\ldots 1}{m(m-2)\ldots 2} \frac{\pi}{2} & \text{if } m \text{ is even} \\ \frac{(n-1)(n-3)(n-5)\ldots 1}{(m+n)(m+n-2)\ldots(m+2)} \frac{(m-1)(m-3)\ldots 2}{m(m-2)\ldots 3} 1, & \text{if } m \text{ is odd} \end{cases} \quad \text{...(6.20)}$$

Case II: When n is odd

$$I_{m,n} = \frac{(n-1)(n-3)(n-5)\ldots 2}{(m+n)(m+n-2)\ldots(m+3)} I_{m,1}$$

But $$I_{m,1} = \int_0^{\pi/2} \sin^m x \cos x\,dx = \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2} = \frac{1}{m+1}.$$

Therefore, $I_{m,n} = \dfrac{(n-1)(n-3)\ldots 4.2}{(m+n)(m+n-2)\ldots(m+3)}\dfrac{1}{(m+1)}$ ...(6.21)

Example 6.6: Evaluate

(a) $\int_0^1 x^5 e^{2x}\,dx$ (b) $\int_0^{\pi/2} \cos^6 x\,dx$ (c) $\int_{-\pi}^{\pi} \sin^4 x \cos^2 x\,dx$

Solution: (a) Using (6.10) for $n = 5$ and $a = 2$, we obtain

$$I_5 = \int x^5 e^{2x}\,dx = x^5\frac{e^{2x}}{2} - \frac{5}{2}I_4 = \frac{x^5}{2}e^{2x} - \frac{5}{2}\left[x^4\frac{e^{2x}}{2} - \frac{4}{2}I_3\right]$$

$$= \frac{x^5}{2}e^{2x} - \frac{5x^4}{4}e^{2x} + 5\left[x^3\frac{e^{2x}}{2} - \frac{3}{2}I_2\right] = \frac{x^5}{2}e^{2x} - \frac{5}{4}x^4e^{2x} + \frac{5}{2}x^3e^{2x} - \frac{15}{2}\left[x^2\frac{e^{2x}}{2} - I_1\right]$$

$$= \frac{x^5}{2}e^{2x} - \frac{5}{4}x^4e^{2x} + \frac{5}{2}x^3e^{2x} - \frac{15x^2}{4}e^{2x} + \frac{15}{2}\left[x\frac{e^{2x}}{2} - \frac{e^{2x}}{4}\right]$$

$$= \left[\frac{x^5}{2} - \frac{5x^4}{4} + \frac{5x^3}{2} - \frac{15x^2}{4} + \frac{15x}{4} - \frac{15}{8}\right]e^{2x}.$$

Thus,

$$\int_0^1 x^5 e^{2x}\,dx = \left(\frac{1}{2} - \frac{5}{4} + \frac{5}{2} - \frac{15}{4} + \frac{15}{4} - \frac{15}{8}\right)e^2 + \frac{15}{8} = -\frac{1}{8}e^2 + \frac{15}{8} = \frac{1}{8}(15 - e^2).$$

(b) $\displaystyle\int_0^{\pi/2} \cos^6 x\,dx = \frac{5.3.1}{6.4.2}.\frac{\pi}{2} = \frac{5\pi}{32}$, using (6.15)

(c) Since the integrand $\sin^4 x \cos^2 x$ is an even function of x, thus we have

$$\int_{-\pi}^{\pi} \sin^4 x \cos^2 x\,dx = 2\int_0^{\pi} \sin^4 x \cos^2 x\,dx = 4\int_0^{\pi/2} \sin^4 x \cos^2 x\,dx, \text{ using (6.5)}$$

$$= 4\frac{(2-1)}{4+2}\int_0^{\pi/2} \sin^4 x\,dx, \text{ using (6.20)}$$

$$= \frac{4}{6}\frac{(4-1)(4-3)}{4(4-2)}\frac{\pi}{2} = \frac{\pi}{8}.$$

Example 6.7: Evaluate

(a) $\int_0^1 x^4(1-x^2)^{3/2}dx$ (b) $\int_0^{2a} x^3\sqrt{2ax - x^2}\,dx$

Solution: (a) Let $$I = \int_0^1 x^4(1-x^2)^{3/2}\,dx$$

Put $x = \sin\theta$, so that $dx = \cos\theta\,d\theta$ and when x goes from 0 to 1, θ goes from 0 to $\pi/2$. Thus, I becomes

$$I = \int_0^{\pi/2} \sin^4\theta(\cos^2\theta)^{3/2}\cos\theta\,d\theta = \int_0^{\pi/2} \sin^4\theta\cos^4 d\theta = \frac{3.1\times 3.1}{8.6.4.2}\frac{\pi}{2} = \frac{3\pi}{256}.$$

(b) Let $$I = \int_0^{2a} x^3\sqrt{2ax - x^2}\,dx.$$

Put $x = 2a\sin^2\theta$, so that $dx = 4a\sin\theta\cos\theta d\theta$. Thus I becomes

$$I = \int_0^{\pi/2} (2a\sin^2\theta)^3\sqrt{4a^2\sin^2\theta - 4a^2\sin^4\theta}\,(4a\sin\theta\cos\theta d\theta)$$

$$= 64a^5\int_0^{\pi/2} \sin^8\theta\cos^2\theta d\theta = 64a^5\,\frac{7.5.3.1\times 1}{10.8.6.4.2}\frac{\pi}{2} = \frac{7\pi a^5}{8}.$$

6.3.2 Reduction Formulae for

(a) $\int \tan^n x\,dx$ (b) $\int \sec^n x\,dx$

(a) Let $I_n = \int \tan^n x\,dx.$

$$= \int \tan^{n-2}x\tan^2 x\,dx = \int \tan^{n-2}x(\sec^2 x - 1)dx$$

$$= \int \tan^{n-2}x\sec^2 x\,dx - \int \tan^{n-2}x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$$

or, $$I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2} \quad \text{...(6.22)}$$

as the required reduction formula.

(b) Let $I_n = \int \sec^n x\,dx = \int \sec^{n-2}x\sec^2 x\,dx.$

Integrating by parts, we obtain

$$I_n = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \tan x dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x(\sec^2 x - 1) dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

or, $(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

or, $$I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2} \qquad ...(6.23)$$

as the required reduction formula.

Similarly reduction formulae for $\int \cot^n x dx$ and $\int \operatorname{cosec}^n x dx$ can be obtained.

6.3.3 Reduction Formulae for

(a) $\int x^n \sin mx dx$ (b) $\int x^n \cos mx dx$

(c) $\int \cos^m x \sin nx dx$ (d) $\int \cos^m x \cos nx dx$

(a) Let $I_n = \int x^n \sin mx dx.$

Integrating it by parts taking x^n as first function, we obtain

$$I_n = -x^n \frac{\cos mx}{m} - \int n x^{n-1} \frac{(-\cos mx)}{m} dx$$

$$= \frac{-1}{m} x^n \cos mx + \frac{n}{m} \int x^{n-1} \cos mx dx$$

Again integrating by parts taking x^{n-1} as first function, we obtain

$$I_n = -\frac{1}{m} x^n \cos mx + \frac{n}{m}\left[x^{n-1} \frac{\sin mx}{m} - \int (n-1) x^{n-2} \frac{\sin mx}{m} dx\right]$$

or, $$I_n = \frac{-x^n}{m} \cos mx + \frac{n}{m^2} x^{n-1} \sin mx - \frac{n(n-1)}{m^2} I_{n-2} \qquad ...(6.24)$$

as the required reduction formula.

(b) Let $I_n = \int x^n \cos mx\,dx.$

Proceeding as above we obtain the reduction formula

$$I_n = \frac{x^n \sin mx}{m} + \frac{n}{m^2} x^{n-1} \cos mx - \frac{n(n-1)}{m^2} I_{n-2} \quad ...(6.25)$$

(c) Let $I_{m,n} = \int \cos^m x \sin nx\,dx.$

Integrating by parts taking $\cos^m x$ as first function, we obtain

$$I_{m,n} = \cos^m x \frac{(-\cos nx)}{n} - \int m \cos^{m-1} x(-\sin x) \frac{(-\cos nx)}{n} dx$$

$$= -\frac{1}{n} \cos^m x \cos nx - \frac{m}{n} \int \cos^{m-1} x \cos nx \sin x\,dx. \quad ...(6.26)$$

Also, $\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$

or, $\cos nx \sin x = \sin nx \cos x - \sin(n-1)x.$

Using this in (6.26), we obtain

$$I_{m,n} = -\frac{1}{n} \cos^m x \cos nx - \frac{m}{n} \int \cos^{m-1} x\{\sin nx \cos x - \sin(n-1)x\} dx$$

$$= -\frac{1}{n} \cos^m x \cos nx - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

or, $$\left(\frac{m+n}{n}\right) I_{m,n} = -\frac{1}{n} \cos^m x \cos nx + \frac{m}{n} I_{m-1,n-1}$$

or, $$I_{m,n} = -\frac{1}{m+n} \cos^m x \cos nx + \frac{m}{m+n} I_{m-1,n-1} \quad ...(6.27)$$

as the required reduction formula.

(d) Let $I_{m,n} = \int \cos^m x \cos nx\,dx.$

Integrating by parts taking $\cos^m x$ as first function, we obtain

$$I_{m,n} = \cos^m x \frac{\sin nx}{n} - \int m \cos^{m-1} x(-\sin x) \frac{\sin nx}{n} dx$$

$$= \cos^m x \frac{\sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \sin nx \sin x\,dx. \quad ...(6.28)$$

Also, $\cos(n-1)x = \cos nx \cos x + \sin nx \sin x$

or, $\sin nx \sin x = \cos(n-1)x - \cos nx \cos x. \quad ...(6.29)$

Using (6.29) in (6.28), we obtain

$$I_{m,n} = \cos^m x \frac{\sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x\{\cos (n-1)x - \cos nx \cos x\} dx$$

$$= \cos^m x \frac{\sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \cos (n-1)x dx - \frac{m}{n} \int \cos^m x \cos nx \, dx$$

or, $\left(\frac{m+n}{n}\right) I_{m,n} = \cos^m x \frac{\sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$

or, $$I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}. \quad ...(6.30)$$

as the required reduction formula.

6.3.4 Reduction Formulae for

(a) $\int e^{ax} \sin^n x \, dx$ (b) $\int e^{ax} \cos^n x dx$ (c) $\int x^m (\ln x)^n dx$

(a) Let $I_n = \int e^{ax} \sin^n x dx$.

Integrating it by parts, taking $\sin^n x$ as first function, we obtain

$$I_n = \sin^n x \frac{e^{ax}}{a} - \int n \sin^{n-1} x \cos x \frac{e^{ax}}{a} dx$$

$$= \frac{1}{a} \sin^n x e^{ax} - \frac{n}{a} \int e^{ax} \sin^{n-1} x \cos x \, dx.$$

Again integrating by parts taking $\sin^{n-1} x \cos x$ as first function, gives

$$I_n = \frac{1}{a} \sin^n x e^{ax} - \frac{n}{a}\left[\sin^{n-1} x \cos x \frac{e^{ax}}{a} - \int \{(n-1) \sin^{n-2} x \cos^2 x - \sin^{n-1} x \sin x\} \frac{e^{ax}}{a} dx\right]$$

$$= \frac{1}{a} \sin^n x e^{ax} - \frac{n}{a^2} e^{ax} \sin^{n-1} \cos x + \frac{n}{a^2} \int e^{ax}\{(n-1) \sin^{n-2} x(1 - \sin^2 x) - \sin^n x\} dx$$

$$= \frac{e^{ax}}{a^2} \sin^{n-1} x\{a \sin x - n \cos x\} + \frac{n(n-1)}{a^2} \int e^{ax} \sin^{n-2} x dx - \frac{n^2}{a^2} \int e^{ax} \sin^n x dx$$

or,

$$\left(\frac{n^2+a^2}{a^2}\right) I_n = \frac{e^{ax}}{a^2} \sin^{n-1} x(a \sin x - n \cos x) + \frac{n(n-1)}{a^2} I_{n-2}$$

or, $$I_n = \int e^{ax} \sin^n x dx = \frac{e^{ax} \sin^{n-1} x}{n^2+a^2}(a \sin x - n \cos x) + \frac{n(n-1)}{n^2+a^2} I_{n-2},$$

as the required reduction formula.

(b) Let $I_n = \int e^{ax} \cos^n x\, dx.$

Proceeding as in (a) above, we obtain

$$I_n = \int e^{ax} \cos^n x\, dx = \frac{e^{ax} \cos^{n-1} x}{n^2 + a^2}(a \cos x + n \sin x) + \frac{n(n-1)}{n^2 + a^2} I_{n-2}$$

as the required reduction formula.

(c) Let $$I_{m,n} = \int x^m (\ln x)^n dx.$$

Integrating it by parts, taking $(\ln x)^n$ as first function, we obtain

$$I_{m,n} = (\ln x)^n \frac{x^{m+1}}{m+1} - \int n(\ln x)^{n-1} \frac{1}{x} \frac{x^{m+1}}{m+1} dx$$

$$= \frac{x^{m+1}}{m+1} (\ln x)^n - \frac{n}{m+1} \int (\ln x)^{n-1} x^m dx$$

or, $$I_{m,n} = \frac{x^{m+1}}{m+1} (\ln x)^n - \frac{n}{m+1} I_{m,n-1}$$

as the required reduction formula.

Example 6.8: Show that $\int_0^{\pi/2} \cos^m x \cos nx\, dx = \frac{m}{m+n} \int_0^{\pi/2} \cos^{m-1} x \cos (n-1)x\, dx.$

Hence, deduce that $\int_0^{\pi/2} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}.$

Solution: Let $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx\, dx$. Then first part follows immediately from (6.30), since

$\left[\frac{\cos^m x \sin nx}{m+n}\right]_0^{\pi/2} = 0$. Thus, $I_{m,n} = \frac{m}{m+n} I_{m-1,n-1}$. Set $m = n$, we obtain

$$I_n = \frac{1}{2} I_{n-1}, \text{ where } I_n = \int_0^{\pi/2} \cos^n x \cos nx\, dx.$$

$$= \frac{1}{2} \cdot \frac{1}{2} I_{n-2} = \frac{1}{2^n} \int_0^{\pi/2} (\cos x)^0 dx = \frac{\pi}{2^{n+1}}.$$

Example 6.9: If $I_n = \int_0^{\pi/4} \tan^n \theta\, d\theta$, show that $I_{n-1} + I_{n+1} = \frac{1}{n}$ and hence evaluate $\int_0^a x^5(2a^2 - x^2)^{-3} dx$.

Solution: Consider

$$I_{n+1} = \int_0^{\pi/4} \tan^{n+1} \theta\, d\theta = \int_0^{\pi/4} \tan^{n-1} \theta (\sec^2 \theta - 1) d\theta$$

$$= \int_0^{\pi/4} \tan^{n-1} \theta \sec^2 \theta\, d\theta - \int_0^{\pi/4} \tan^{n-1} \theta\, d\theta = \left[\frac{\tan^n \theta}{n}\right]_0^{\pi/4} - I_{n-1} = \frac{1}{n} - I_{n-1}$$

or,
$$I_{n+1} + I_{n-1} = \frac{1}{n}. \qquad \text{...(6.31)}$$

Next, let $I = \int_0^a x^5(2a^2 - x^2)^{-3} dx$. Put $x = \sqrt{2} a \sin \theta$, we have

$$I = \int_0^{\pi/4} (\sqrt{2}\, a \sin \theta)^5 (2a^2 - 2a^2 \sin^2 \theta)^{-3} \sqrt{2}\, a \cos \theta\, d\theta = \int_0^{\pi/4} \tan^5 \theta\, d\theta = I_5.$$

For $n = 4$, (6.31) gives $I_5 + I_3 = 1/4$. For $n = 2$, (6.31) gives $I_3 + I_1 = 1/2$.

Subtracting, we obtain $I_5 = I_1 - \frac{1}{4}$.

Now, $I_1 = \int_0^{\pi/4} \tan \theta\, d\theta = [\ln \sec \theta]_0^{\pi/4} = \frac{1}{2} \ln 2$. Thus, $I_5 = \frac{1}{2} \ln 2 - \frac{1}{4}$.

Example 6.10: Evaluate $\int_0^{\pi} x \sin^8 x \cos^4 x\, dx$.

Solution: Let $I = \int_0^{\pi} x \sin^8 x \cos^4 x\, dx$. Using (6.3), we have

$$I = \int_0^{\pi} (\pi - x) \sin^8 (\pi - x) \cos^4 (\pi - x) dx = \pi \int_0^{\pi} \sin^8 x \cos^4 x\, dx - I.$$

This gives $2I = \pi \int_0^{\pi} \sin^8 x \cos^4 x\, dx = 2\pi \int_0^{\pi/2} \sin^8 x \cos^4 x\, dx$

$$= 2\pi . \frac{(7.5.3.1.)\,(3.1)}{12.10.8.6.4.2}\,\frac{\pi}{2} = \frac{7\pi^2}{1024}.\ \text{Hence, } I = \frac{7\pi^2}{2048}.$$

EXERCISE 6.2

1. Evaluate

(a) $\int_0^{\pi/3} \cos^6 x\,dx$ (b) $\int_0^{\pi/2} \sin^3\theta \cos^4\theta \cos 2\theta\,d\theta$

(c) $\int_0^{\infty} \frac{dx}{(1+x^2)^5}$ (d) $\int_0^{\pi} \theta \sin^7\theta \cos^4\theta\,d\theta.$

2. Evaluate

(b) $\int_0^{a} \frac{x^7}{\sqrt{a^2-x^2}}dx$ (b) $\int_0^{\infty} \frac{dx}{(a^2+x^2)^n}$ (c) $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta\,d\theta.$

3. If $I_n = \int_0^{\pi/2} x \cos^n x\,dx$, then prove that $I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n^2}$.

4. Evaluate $\int_0^{\pi} \frac{\sin^4\theta \sqrt{1-\cos\theta}}{(1+\cos\theta)^2} d\theta$.

5. Show that $\int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta\,d\theta = \frac{3\pi}{16\sqrt{2}}$.

6. If $I_{m,n} = \int x^m (ax^p + b)^n dx$, then show that $I_{m,n} = \frac{x^{m+1}(ax^p+b)^n}{np+m+1} + \frac{bnp}{np+m+1} I_{m,n-1}$.

7. If $I_n = \int x^n (a-x)^{1/2} dx$, then show that $(2n+3)I_n = 2an\,I_{n-1} - 2x^n(a-x)^{3/2}$.

8. If $I_n = \int_{\pi/4}^{\pi/2} \cot^n x\,dx$, show that $I_n = \frac{1}{n-1} - I_{n-2}$, $(n > 2)$ and hence evaluate $\int_{\pi/4}^{\pi/2} \cot^4 x\,dx$.

9. If $I_n = \int_0^{\infty} e^{-x} \sin^n x\,dx$, show that $(1+n^2)I_n = n(n-1)I_{n-2}$, $n \geq 2$. Hence evaluate I_4.

10. If $I_n = \int_0^{\pi/2} x^n \cos ax\,dx$, show that $I_n = \frac{1}{a}\left(\frac{\pi}{2}\right)^n \left[\sin\frac{a\pi}{2} + \frac{2n}{a\pi}\cos\frac{a\pi}{2}\right] - \frac{n(n-1)}{a^2} I_{n-2}$.

6.4 AREAS OF BOUNDED REGIONS

Definite integrals are applied to find the areas of the bounded regions, the volumes and surface areas of solids of revolution, the length of curves, centroid of area and volume of revolution, etc. In this section, we find the areas of bounded regions.

6.4.1 Area Under a Curve in Cartesian Form

The area A of a region, bounded above by a curve $y = f(x)$, below by the axis of x and on left by ordinate $x = a$ and on right by $x = b$, as shown in Fig. 6.1, is given by

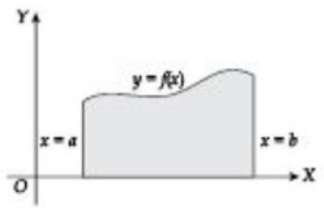


Fig. 6.1

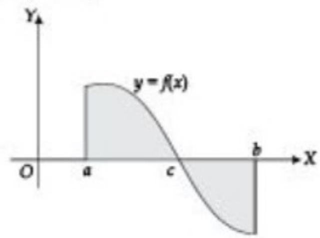


Fig. 6.2

$$A = \int_a^b f(x)dx. \quad ...(6.32)$$

If the curve $y = f(x)$ is below the x-axis, then the value of the integral $\int_a^b f(x)$ is negative; we take the area as the magnitude of this value.

In case the curve $y = f(x)$ crosses the x-axis at a point c, $a < c < b$, as shown Fig. 6.2, then area is given by

$$A = \int_a^c f(x)dx + \left|\int_c^b f(x)dx\right| \quad ...(6.33)$$

The area bounded by the curve $x = f(y)$, the axis of y and the two abscissas $y = c$ and $y = d$ as shown in Fig. 6.3, is given by

$$A = \int_c^d f(y)dy. \quad ...(6.34)$$

The area of the region bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$ and the lines $x = a$, $x = b$, as shown in Fig. 6.4, is given by

$$A = \int_a^b [f(x) - g(x)]dx. \quad ...(6.35)$$

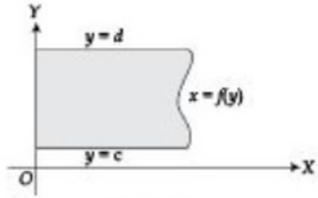


Fig. 6.3

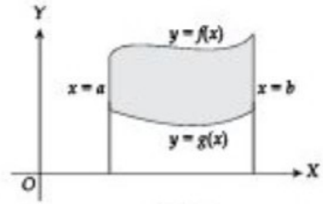


Fig. 6.4

6.4.2 Area Under a Curve in Polar Form

If $r = f(\theta)$ is the equation of a curve in polar form, then the area $OABO$ of the sector bounded by the curve $r = f(\theta)$ and the radial lines $\theta = \alpha$ and $\theta = \beta$, when $f(\theta)$ is continuous on $[\alpha, \beta]$ as shown in Fig. 6.5 can be found as follows.

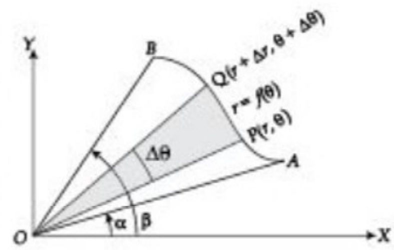


Fig. 6.5

Let $P(r, \theta)$ and $Q(r + \Delta r, \theta + \Delta\theta)$ be two points on the curve $r = f(\theta)$, Q being close to P. If ΔA is the area of the elementary strip OPQ, then for small value of $\Delta\theta$, PQ can be approximated with a circular arc of radius r, and thus

$$\Delta A = \frac{1}{2} r^2 \Delta\theta.$$

Hence, the area A of the sector $OABO$, when θ goes from α to β, is given by

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta. \qquad ...(6.36)$$

6.4.3 Area of a Curve in Parametric Form

If a curve is in the parametric form as $x = \phi(t)$, $y = \psi(t)$, $a \le t \le b$, where $\phi(t)$, $\psi(t)$ are continuous on $[a, b]$, then the area bounded above by the curve $y = f(x)$, below by x-axis and the ordinates $x = \phi(a)$ and $x = \phi(b)$, is given by

$$A = \int_{\phi(a)}^{\phi(b)} y\,dx = \int_{a}^{b} \psi(t)\phi'(t)\,dt. \qquad ...(6.37)$$

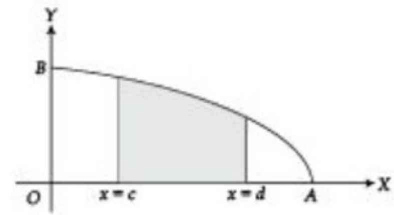
Example 6.11: Find the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, the ordinates $x = c$, $x = d$ and the x-axis, in the first quadrant.

Solution: Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. In the first quadrant, it is $y = \frac{b}{a}\sqrt{1 - \frac{x^2}{a^2}}$.

The required area A, as shown in Fig. 6.6, is

$$A = \int_c^d y\,dx = \frac{b}{a}\int_c^d (a^2 - x^2)^{1/2} dx$$

$$= \frac{b}{a}\left[\frac{1}{2}x(a^2 - x^2)^{1/2} + \frac{1}{2}a^2 \sin^{-1}\frac{x}{a}\right]_c^d$$



$$= \frac{b}{2a}\left[d(a^2 - d^2)^{1/2} - c(a^2 - c^2)^{1/2} + a^2\left(\sin^{-1}\frac{d}{a} - \sin^{-1}\frac{c}{a}\right)\right] \text{ sq. unit.}$$

Fig. 6.6

Example 6.12: Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

Solution: The points of intersection P and Q of the parabola $x^2 = 4y$ and the straight line $x = 4y - 2$ are given by

$$x = x^2 - 2, \text{ or } \quad x^2 - x - 2 = 0$$

or, $(x + 1)(x - 2) = 0$, or $\quad x = -1, 2.$

The required area A, as shown in Fig. 6.7, is

$$A = \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx = \frac{1}{4}\int_{-1}^{2}(x + 2 - x^2)dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{1}{4}\left(\frac{10}{3} + \frac{7}{6}\right) = \frac{9}{8} \text{ sq. unit.}$$

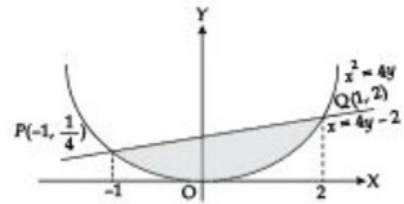


Fig. 6.7

Example 6.13: Find the area between the curve $y^2(2a - x) = x^3$ and its asymptotes.

Solution: The curve $y^2(2a - x) = x^3$ is symmetric about x-axis. It passes through the origin (0, 0) and for real and finite y, $x \in [0, 2a)$.

In fact, $y = 0$ is the cuspidal tangent at (0, 0) and $x = 2a$ is asymptote to this curve parallel to the y-axis. The required area as shown in Fig. 6.8, is

$$A = 2\int_0^{2a} \frac{x^{3/2}}{\sqrt{2a - x}}\,dx.$$

Put $x = 2a \sin^2 \theta$, it gives $dx = 4a \sin\theta \cos\theta\, d\theta$.

Thus, $$A = 2\int_0^{\pi/2} \frac{(2a)^{3/2} \sin^3\theta . 4a \sin\theta \cos\theta}{(2a)^{1/2} \cos\theta} d\theta$$

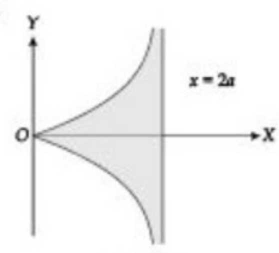


Fig. 6.8

$$= 16a^2 \int_0^{\pi/2} \sin^4 \theta \, d\theta = 16a^2 \, \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 3\pi a^2 \text{ sq. unit.}$$

Example 6.14: Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by x-axis and the line $y = x - 2$.

Solution: The parabola $y = \sqrt{x}$ and the line $y = x - 2$, meet where

$\sqrt{x} = x - 2$ or, $x^2 - 5x + 4 = 0$ or $x = 1, 4$.

Only the value $x = 4$, satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous roots. Corresponding to $x = 4$, the point is $Q(4, 2)$.

The required area A, as shown in Fig. 6.9, is given by

$$A = \int_0^2 \{(2 + y) - y^2\} dy, \quad \text{refer to (6.34)}$$

$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 = 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3} \text{ sq. unit.}$$

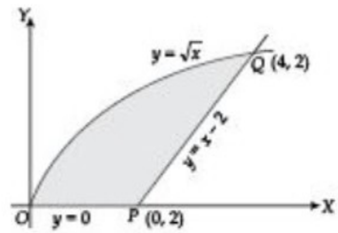


Fig. 6.9

Example 6.15: For the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ obtain the area between its base and the portion of the curve from cusp to cusp.

Solution: The cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is symmetrical about y-axis. The required area A, as shown in Fig. 6.10, is

$$A = 2\int_0^{2a} x \, dy = 2\int_0^{\pi} a(t + \sin t) a \sin t \, dt$$

$$= 2a^2 \int_0^{\pi} [t \sin t + \sin^2 t] dt$$

$$= 2a^2 \left[(-t \cos t)_0^{\pi} - \int_0^{\pi} 1.(-\cos t) \, dt + \frac{1}{2}\int_0^{\pi} (1 - \cos 2t) \, dt\right]$$

$$= 2a^2(0 + \pi) + 2a^2 [\sin t]_0^{\pi} + a^2\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi}$$

$$= 2\pi a^2 + 0 + \pi a^2 - 0 = 3\pi a^2 \text{ sq. unit.}$$

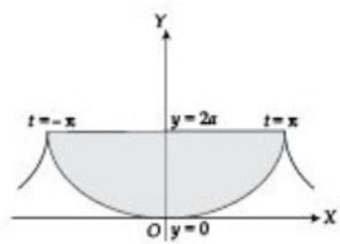


Fig. 6.10

Example 6.16: Find the area of the loop of the curve $x^3 + y^3 = 3axy$.

Solution: The required area is as shown in Fig. 6.11.

To calculate the area we transform the curve $x^3+y^3=3xy$ into polar form.

Put $x=r\cos\theta$, $y=r\sin\theta$, the equation becomes

$$r^3(\cos^3\theta+\sin^3\theta)=3ar^2\cos\theta\sin\theta$$

or,
$$r=\frac{3a\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}.$$

The loop is completed as θ goes from 0 to $\frac{\pi}{2}$, thus the required area is given as

$$A=\int_0^{\pi/2}\frac{1}{2}r^2d\theta=\frac{9a^2}{2}\int_0^{\pi/2}\frac{\sin^2\theta\cos^2\theta}{(\cos^3\theta+\sin^3\theta)^2}d\theta.$$

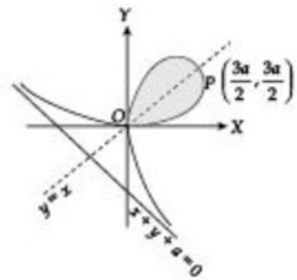


Fig. 6.11

Dividing the numerator and denominator of the integrand by $\cos^6\theta$, we obtain

$$A=\frac{9a^2}{2}\int_0^{\pi/2}\frac{\tan^2\theta\sec^2\theta d\theta}{(1+\tan^3\theta)^2}.$$

Substituting $1+\tan^3\theta=t$, which gives $3\tan^2\theta\sec^2\theta d\theta=dt$, we obtain

$$A=\frac{3a^2}{2}\int_1^{\infty}\frac{dt}{t^2}=\frac{3a^2}{2}\left[-\frac{1}{t}\right]_1^{\infty}=\frac{3a^2}{2}\text{ sq. unit.}$$

Example 6.17: Find the area of the region that lies inside the circle $r=a\cos\theta$ and outside the cardiod $r=a(1-\cos\theta)$.

Solution: The required area is as shown in Fig. 6.12.

The points of intersection of the circle $r=a\cos\theta$ and the cardiod $r=a(1-\cos\theta)$ are given by $a\cos\theta=a(1-\cos\theta)$, which gives $\cos\theta=\frac{1}{2}$ or $\theta=\pm\frac{\pi}{3}$.

Hence the area $A=\frac{1}{2}.2\int_0^{\pi/3}(r_1^2-r_2^2)d\theta$

$$=a^2\int_0^{\pi/3}[\cos^2\theta-(1-\cos\theta)^2]d\theta$$

$$=a^2\int_0^{\pi/3}(2\cos\theta-1)d\theta=a^2[2\sin\theta-\theta]_0^{\pi/3}$$

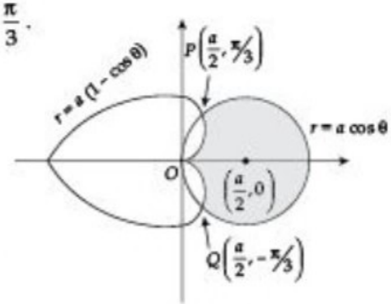


Fig. 6.12

$$= a^2\left[2\,\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right] = \frac{a^2}{3}(3\sqrt{3} - \pi) \text{ sq. unit.}$$

Example 6.18: Find the area between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.

Solution: The equation of the curve is $r = a(\sec\theta + \cos\theta)$.

Transforming it into cartesian form by substituting $x = r\cos\theta$ and $y = r\sin\theta$, we obtain

$$r = a\left(\frac{r}{x} + \frac{x}{r}\right), \text{ or } xr^2 = a(x^2 + r^2).$$

Substituting $r^2 = x^2 + y^2$ and simplifying, we get $y^2 = \dfrac{x^2(2a - x)}{x - a}$.

The curve is symmetrical about x-axis, $x = a$ is asymptote to the curve and $x = 2a$ is the tangent to the curve at the point $P(2a, 0)$. For real and finite y, $x \in (a, 2a]$. The required area, as shown in Fig. 6.13, is

$$A = 2\int_a^{2a} y\,dx = 2\int_a^{2a} x\left(\frac{2a - x}{x - a}\right)^{1/2} dx = 2\int_a^{2a} x\left(\frac{a - (x - a)}{x - a}\right)^{1/2} dx.$$

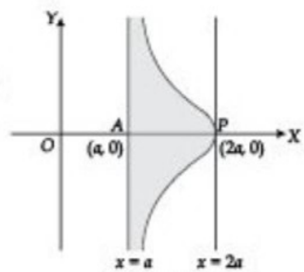


Fig. 6.13

Substituting $(x - a) = a\sin^2 t$, which gives $dx = 2a\sin t\cos t\,dt$, and thus

$$A = 2\int_0^{\pi/2} a(1 + \sin^2 t)\left\{\frac{a - a\sin^2 t}{a\sin^2 t}\right\}^{1/2} 2a\sin t\cos t\,dt$$

$$= 4a^2\int_0^{\pi/2}(1 + \sin^2 t)\cos^2 t\,dt$$

$$= 4a^2\left[\int_0^{\pi/2}\cos^2 t\,dt + \int_0^{\pi/2}\sin^2 t\cos^2 t\,dt\right]$$

$$= 4a^2\left[\frac{1}{2}\cdot\frac{\pi}{2} + \frac{2-1}{2+2}\cdot\frac{2-1}{2}\cdot\frac{\pi}{2}\right] = 4a^2\left[\frac{\pi}{4} + \frac{\pi}{16}\right] = \frac{5}{4}\pi a^2 \text{ sq. units.}$$

EXERCISE 6.3

1. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
2. Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

3. Find the area bounded by the parabola $y^2 = x$ and the straight lines $x = 0$, $x = 1$ and $y - x = 0$.
4. Find the area enclosed by the curve $a^2x^2 = y^3(a - y)$.
5. Find the area enclosed by the curves $x^2 = 4ay$ and $y = \dfrac{8a^3}{x^2 + 4a^2}$.
6. Show that the area cut off a parabola by any double ordinate is two-third of the corresponding rectangle formed by that double ordinate and its distance from the vertex.
7. Find the area enclosed by the curve $x^2(x^2 + y^2) = a^2(x^2 - y^2)$.
8. Find the area bounded by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote.
9. Find the area between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.
10. Obtain the area common to two circles $r = a\sqrt{2}$ and $r = 2a \cos \theta$.
11. Find the area common to the two cardiodes $r = a(1 + \cos \theta)$ and $r = a(1 - \cos \theta)$.
12. Find the area common to the two ellipses $a^2x^2 + b^2y^2 = 1$ and $b^2x^2 + a^2y^2 = 1$, $(0 < a < b)$.
13. Show that the area enclosed by the curve $|x| + |y| = 2a$, $(a > 0)$ is $8a^2$.
14. Find the area of one loop of the curve $r = a \sin n\theta$, $n \in N$ and is odd. What is the total area of all the loops?
15. Show that the larger of the two areas into which the circle $x^2 + y^2 = 64a^2$ is divided by the parabola $y^2 = 12ax$ is $\dfrac{16}{3}a^2(8\pi - \sqrt{3})$.
16. Show that the area bounded by cissoid $x = a \sin^2 t$, $y = a \sin^3 t/\cos t$ and its asymptote is $3\pi a^2/4$.
17. Find the area enclosed by the curve $x = a \cos^3 t$, $y = b \sin^3 t$.
18. Find the area included between the portion of the cycloid from cusp to cusp and its base $x = a(t - \sin t)$, $y = a(1 - \cos t)$.
19. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.
20. The Fig. 6.14 here shows the triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

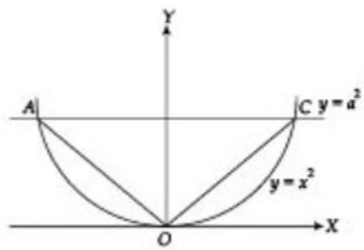


Fig. 6.14

6.5 ARC LENGTHS OF PLANE CURVES

In this section we apply definite integral to find arc lengths of curves in a plane. We consider it for curves in cartesian, parametric and polar forms.

Cartesian Form: Consider a portion AB of the curve $y = f(x)$, $a \le x \le b$ and let dy/dx be continuous on the interval $[a, b]$. Then the arc length between A and B as shown in Fig. 6.15, is given by

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx, \quad \text{refer to (4.22)} \quad ...(6.38)$$

If the curve is defined as $x = g(y)$, $c \le y \le d$, then

$$s = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy, \quad \text{refer to (4.23)} \quad ...(6.39)$$

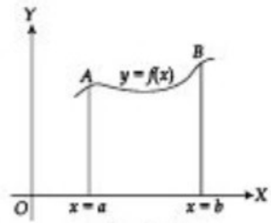


Fig. 6.15

Parametric Form: If x and y are given as functions of parameter t, say $x = \phi(t)$ and $y = \psi(t)$, $t_0 \le t \le t_1$, where $\phi(t)$, $\psi(t)$ have continuous first order derivatives on $[t_0, t_1]$, then the arc length is given by

$$s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt, \quad \text{refer to (4.24)} \quad ...(6.40)$$

Polar Form: In case the curve is given in polar form $r = f(\theta)$, $\alpha \le \theta \le \beta$, where $f(\theta)$ has continuous first order derivative on $[\alpha, \beta]$, then the arc length is given by

$$s = \int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta, \quad \text{refer to (4.29)} \quad ...(6.41)$$

$$s = \int_\alpha^\beta \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\, dr, \quad \text{refer to (4.30)} \quad ...(6.42)$$

Example 6.19: Find the total length of the curve given by $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Solution: The curve as shown in Fig. 6.16 is symmetrical about both the axes. Thus the total length s of the curve, refer to (6.40), is

$$s = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 4\int_0^{\pi/2} \sqrt{(-3a\cos^2\theta\sin\theta)^2+(3a\sin^2\theta\cos\theta)^2}\, d\theta$$

$$= 4\int_0^{\pi/2} 3a\cos\theta\sin\theta\, d\theta = 6a\int_0^{\pi/2} \sin 2\theta\, d\theta$$

$$= 6a\left(\frac{-\cos 2\theta}{2}\right)_0^{\pi/2} = 6a\left(\frac{1+1}{2}\right) = 6a.$$

Example 6.20: Find the length of the curve $y = \left(\frac{x}{2}\right)^{2/3}$ from $x = 0$ to $x = 2$.

Solution: The curve is $y = \left(\frac{x}{2}\right)^{2/3}, \quad 0 \le x \le 2$ ...(6.43)

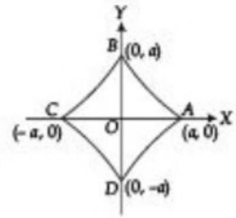


Fig. 6.16

The derivative $\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-\frac{1}{3}}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{x}{2}\right)^{-\frac{1}{3}}$ is discontinuous at $x = 0$, so we can't find the arc length using (6.38). Rewriting (6.43) as

$$x = 2y^{3/2}, \quad 0 \le y \le 1.$$

The derivative $\frac{dx}{dy} = 3y^{1/2}$ is continuous in the interval [0, 1]. Thus the arc length, refer to (6.39) is

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy = \left[\frac{2}{3}\frac{(1+9y)^{3/2}}{9}\right]_0^1 = \frac{2}{27}(10\sqrt{10} - 1).$$

Example 6.21: Find the arc length of the loop of the curve

$$9ay^2 = (x - 2a)\,(x - 5a)^2.$$

Solution: The curve

$$9ay^2 = (x - 2a)\,(x - 5a)^2 \quad \text{...(6.44)}$$

is symmetrical about x-axis, and the loop of the curve lies between $x = 2a$ and $x = 5a$ as shown in Fig. 6.17.

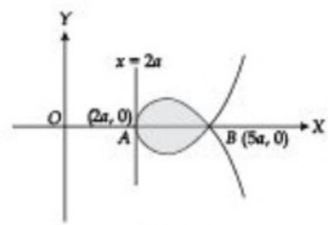


Fig. 6.17

Differentiating w.r.t x, we get

$$18ay\frac{dy}{dx} = (x - 5a)^2 + 2(x - 2a)(x - 5a) \text{ or, } \frac{dy}{dx} = \frac{(x - 5a)\,(x - 3a)}{6ay}.$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x - 5a)^2(x - 3a)^2}{36a^2y^2} = \frac{(x - a)^2}{4a(x - 2a)}, \text{ using (6.44) and simplifying.}$$

Thus the arc length of the loop of the curve, refer to (6.38), is

$$s = 2\int_{2a}^{5a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 2\int_{2a}^{5a} \frac{(x - a)}{2\sqrt{a}\,\sqrt{x - 2a}}\, dx.$$

Put $x - 2a = t$, which gives $dx = dt$, thus

$$s = \frac{1}{\sqrt{a}}\int_0^{3a} \frac{a + t}{\sqrt{t}}\, dt = \frac{1}{\sqrt{a}}\int_0^{3a}\left(at^{-\frac{1}{2}} + t^{\frac{1}{2}}\right)dt = \frac{1}{\sqrt{a}}\left[2at^{\frac{1}{2}} + \frac{2}{3}t^{3/2}\right]_0^{3a} = \frac{1}{\sqrt{a}}\left[2a\sqrt{3a} + \frac{2}{3}(3a)^{3/2}\right] = 4\sqrt{3}a.$$

Example 6.22: Find the perimeter of the cardiod $r = a(1 + \cos\theta)$ and show that the upper half of the curve is bisected at $\theta = \pi/3$.

Solution: The curve $r = a(1 + \cos\theta)$ is symmetrical about the line $\theta = 0$, as shown in Fig. 6.18.

The perimeter, refer to (6.14), is

$$s = 2\int_0^{\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= 2\int_0^{\pi}[a^2(1+\cos\theta)^2 + (-a\sin\theta)^2]^{\frac{1}{2}}\,d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi}(1+\cos\theta)^{\frac{1}{2}}\,d\theta$$

$$= 4a\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta = 8a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 8a.$$

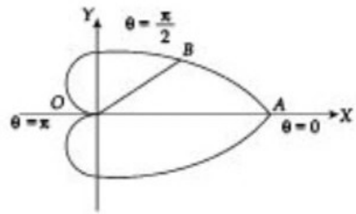


Fig. 6.18

The length of the upper half is $4a$, and let the upper half is bisected at $\theta = \theta_1$, then

$$\int_0^{\theta_1}\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta = 2a, \text{ which gives}$$

$$2a\int_0^{\theta_1}\cos\frac{\theta}{2}\,d\theta = 2a, \text{ or } \int_0^{\theta_1}\cos\frac{\theta}{2}\,d\theta = 1, \text{ or, } \sin\frac{\theta_1}{2} = \frac{1}{2}, \quad \text{or} \quad \theta_1 = \pi/3.$$

EXERCISE 6.4

1. Find the length of the portion of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ from $x = 0$ to $x = a$ in the first quadrant.
2. Find the parameter of the curve $x^2 + y^2 = a^2$.
3. Prove that the length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum is $a\left[\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right]$.
4. Find the perimeter of the one loop of the curve $x^2(a^2 - x^2) = 8a^2y^2$.
5. Find the entire length of the curve given by $r = a\sin^3(\theta/3)$.
6. Prove that the whole length of an arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ is $8a$.
7. Prove that the cardiod $r = a(1 + \cos\theta)$ is divided by the line $4r\cos\theta = 3a$ into two parts such that the length of the arcs on either side of this line are equal.
8. Show that on the curve $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$, $s = 16a\sin^2(\psi/6)$ where s is the length of the arc of the curve from the point $t = 0$ to the point where the tangent makes an angle ψ with the tangent at the point $t = 0$.
9. Find the length of the arc of equiangular spiral $r = ae^{\theta\cot\alpha}$ between the points for which radii vectors are r_1 and r_2.

10. Find the length of the arc of the parabola $(l/r) = 1 + \cos\theta$ cut off by its latus rectum.

6.6 VOLUMES OF SOLIDS OF REVOLUTION

The solids of revolution are solids which can be generated by revolving plane regions about axes, e.g., a sphere. We can find their volume by the application of definite integral.

6.6.1 Cartesian Co-ordinates

(a) Revolution about x-axis

Let the area bounded by the curve $y = f(x)$ from $x = a$ to $x = b$ and the x-axis, as shown in Fig. 6.19, be rotated about the x-axis through four right angles, which generates a solid of revolution. Divide the arc AB into n parts by considering subintervals $[x_0, x_1]$, $[x_1, x_2]$..., $[x_{n-1}, x_n]$; $a = x_0 < x_1 < x_2 ... < x_n = b$ and let $P'Q' = \Delta x_i = x_{i+1} - x_i$, $i = 0, 1, 2, ... n-1$.

Then the volume ΔV_i of the solid formed by rotating the strip $P'Q'QP$ about x-axis through four right angles can be approximated as $\Delta V_i = \pi y_i^2 \Delta x_i$, where $P'P = y_i$.

Thus an estimate of the total volume for n strips is given by

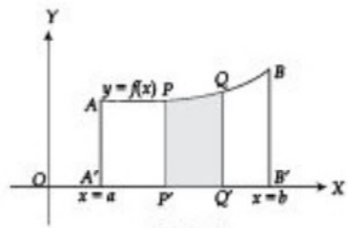


Fig. 6.19

$$\sum_i \pi y_i^2 \Delta x_i \qquad ...(6.45)$$

Let $n \to \infty$ such that max $\Delta x_i \to 0$, the summation (6.45) tends to coincide with the actual volume of the solid of revolution and is given by

$$V = \int_a^b \pi y^2 dx \qquad ...(6.46)$$

(b) Revolution about the y-axis

Similarly if the area bounded by curve $x = g(y)$, the y-axis, and the lines $y = c$ and $y = d$, as shown in Fig. 6.20, is revolved about the y-axis, then the volume of the solid of revolution generated is given by

$$V = \int_c^d \pi x^2 \, dy \qquad ...(6.47)$$

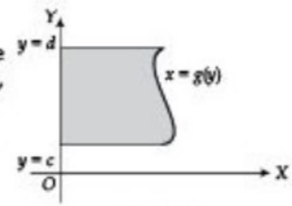


Fig. 6.20

(c) Revolution about the line y = c

In case the area bounded by the curve $y = f(x)$, the line $y = c$ and the lines $x = a$, $x = b$, is revolved about the line $y = c$, then the volume of the solid of revolution is given by

$$V = \int_a^b \pi (y - c)^2 dx. \qquad ...(6.48)$$

(d) Revolution about the line $x = a$

Similarly, if the area bounded by the curve $x = g(y)$, the line $x = a$ and the lines $y = c, y = d$ is revolved about the line $x = a$, then the volume of solid of revolution formed is given by

$$V = \int_c^d \pi(x-a)^2 dy. \qquad ...(6.49)$$

(e) Revolution about any line

To obtain the volume of the solid generated by rotation about any axis CD the area bounded by the curve AB, the axis CD and the perpendicular AC and BD on the axis CD, take O, any fix point on the line CD, as the origin and the line OCD as x-axis. A line through O and perpendicular to OD is taken as the y-axis.

If $P(x, y)$ be any point on the curve AB, and $PN \perp OX$, as shown in Fig. 6.21, then the required volume of revolution

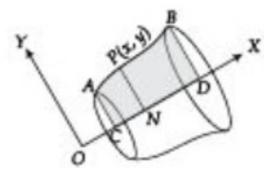


Fig. 6.21

$$V = \pi \int_a^b y^2 dx = \pi \int_{OC}^{OD} (PN)^2 \, d(ON), \qquad ...(6.50)$$

where $OC = a$ and $OD = b$

(f) Revolving area bounded by curves $y = f_1(x)$ and $y = f_2(x)$

The volume of the solid generated by rotating about x-axis the region bounded by the curves $y = f_1(x)$, $y = f_2(x)$ and the ordinates $x = a$ and $x = b$ is

$$V = \pi \int_a^b [f_1(x)^2 - f_2(x)^2] dx, \qquad ...(6.51)$$

when $f_1(x) \geq f_2(x)$, for $x \in [a, b]$.

6.6.2 Polar Co-ordinates

(a) Revolution about the initial line $\theta = 0$

The elementary area for the sector OPQ, as shown in Fig. 6.22, is $r^2 \Delta\theta/2$, and its centroid can be considered to lie on OP at $\left(\frac{2}{3} r\cos\theta, \frac{2}{3} r\sin\theta\right)$. The elementary volume generated when this area is evolved about initial line $\theta = 0$ is $\Delta V = \frac{2\pi}{3} r^3 \sin\theta\Delta\theta$. Hence the volume V of the solid generated by rotating the sector OAB about the initial line, when the area OAB is bounded by the lines $\theta = \alpha$, $\theta = \beta$ and the curve $r = f(\theta)$ is given by

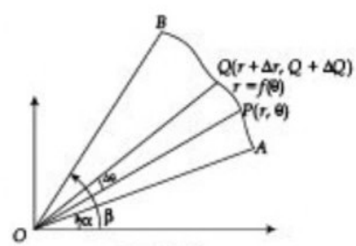


Fig. 6.22

$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^3 \sin\theta \, d\theta. \quad ...(6.52)$$

In case the area OAB is rotated about the line $\theta = \pi/2$, then volume V generated is given by

$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^3 \cos\theta \, d\theta. \quad ...(6.53)$$

6.6.3 Parametric Co-ordinates

When the curve is in parametric form $x = \phi(t)$, $y = \psi(t)$, $t_0 \le t \le t_1$, then the volume of the solid generated by rotating the area bounded by the curve about x-axis is

$$V = \pi\int_{t_0}^{t_1} [\psi(t)]^2 \phi'(t) dt \quad ...(6.54)$$

and, about y-axis is

$$V = \pi\int_{t_0}^{t_1} [\phi(t)]^2 \psi'(t) dt \quad ...(6.55)$$

Example 6.23: A segment is cut off a sphere of radius a by a plane at a distance $a/2$ from the centre. Show that the volume of the segment is (5/32) of the volume of the sphere.

Solution: Consider a circle $x^2 + y^2 = a^2$ of radius a. The sphere of radius a is generated by revolving the semi-circular area $AOBDA$ about x-axis, while the segment is generated by revolving the area $ALMA$ about the x-axis, where $OL = \frac{a}{2}$ as shown in Fig. 6.23.

The volume of the sphere is

$$V = \pi\int_{-a}^{a} y^2 \, dx = \pi\int_{-a}^{a} (a^2 - x^2) dx = 2\pi\left[a^2 x - \frac{x^3}{3}\right]_0^a = \frac{4}{3}\pi a^3.$$

Next, the volume of the segment is

$$V' = \pi\int_{\frac{a}{2}}^{a} y^2 dx = \pi\int_{\frac{a}{2}}^{a} (a^2 - x^2) dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_{\frac{a}{2}}^{a}$$

$$= \pi\left[a^3 - \frac{a^3}{3} - \frac{a^3}{2} + \frac{a^3}{24}\right] = \frac{5}{24}\pi a^3 = \frac{5}{32} V.$$

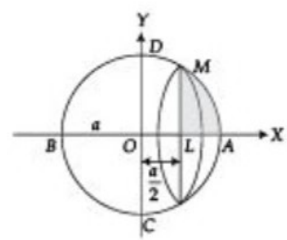


Fig. 6.23

This proves the result.

Example 6.24: Find the volume of the solid generated by revolving the region bounded by the curves $y = 3 - x^2$ and $y = -1$ about the line $y = -1$.

Solution: The region $BCAB$ bounded by the parabola $y = 3 - x^2$ and the line $y = -1$, as shown in Fig. 6.24, is revolved about the line $y = -1$.

The volume generated is

$$V = \pi \int_{-2}^{2} (1+y)^2 dx = \pi \int_{-2}^{2} (1+3-x^2)^2 dx$$

$$= 2\pi \int_{0}^{2} (4-x^2)^2 dx$$

$$= 2\pi \int_{0}^{2} (16 - 8x^2 + x^4) dx = 2\pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2$$

$$= 2\pi \left[32 - \frac{64}{3} + \frac{32}{5}\right] = \frac{512}{15}\pi$$

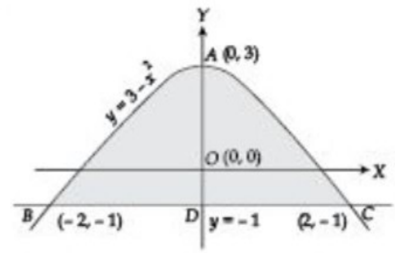


Fig. 6.24

Example 6.25: Find the volume of the solid of revolution obtained by revolving the region bounded between the curves $y^2 = x^3$ and $x^2 = y^3$ about the x-axis.

Solution: The two curves are $y^2 = x^3$ and $x^2 = y^3$.

Their points of intersection are given by

$$x^2 = y^3 = (x^{3/2})^3 = x^{9/2}$$

or, $$x^2(1 - x^{5/2}) = 0, \text{ or } x = 0, 1.$$

Thus points of intersection are $O(0, 0)$ and $A(1, 1)$.

The region $OPAR$ bounded by the curves is shown in Fig. 6.25.

The required volume of revolution is given by

$$V = \pi \int_{0}^{1} (y_1^2 - y_2^2)\, dy,$$

where y_1 and y_2 are the upper and lower curves of the bounded region.

Thus, $$V = \pi \int_{0}^{1} \left[(x^{2/3})^2 - (x^{3/2})^2\right] dx$$

$$= \pi \int_{0}^{1} (x^{4/3} - x^3) dx$$

$$= \pi \left[\frac{3}{7}x^{7/3} - \frac{x^4}{4}\right]_0^1 = \pi \left[\frac{3}{7} - \frac{1}{4}\right]$$

$$= \frac{5\pi}{28} \text{ cubic unit.}$$

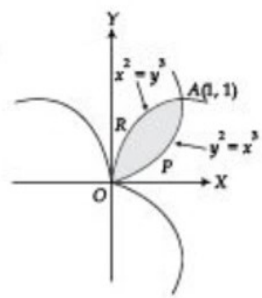


Fig. 6.25

Example 6.26: The area cut off the parabola $y^2 = 4ax$ by the chord joining the vertex to one end of the latus rectum is revolved about the chord. Find the volume of the solid generated.

Solution: The equation of the chord passing through the points $O(0, 0)$ and $L(a, 2a)$ is $y = 2x$. The region bounded by the parabola $y^2 = 4ax$ and the chord $y = 2x$ is as shown in Fig. 6.26.

Let $P(x, y)$ be any point on the parabola and PN be the perpendicular from P on OL, then

$$PN = \frac{|y-2x|}{\sqrt{1^2+(-2)^2}} = \frac{1}{\sqrt{5}}|y-2x|.$$

Next, $ON = \sqrt{OP^2 - PN^2}$

$$= \sqrt{x^2+y^2-\frac{1}{5}(y-2x)^2}$$

$$= \sqrt{\frac{5x^2+5y^2-y^2-4x^2+4xy}{5}}$$

$$= \frac{x+2y}{\sqrt{5}}.$$

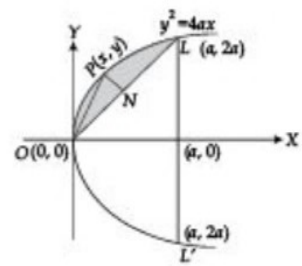


Fig. 6.26

Therefore, $d(ON) = \frac{1}{\sqrt{5}}(dx + 2dy)$.

From the equation $y^2 = 4ax$, we have, $2y\,dy = 4a\,dx$ or $dy = \frac{2a}{y}dx$. Thus,

$$d(ON) = \frac{1}{\sqrt{5}}\left(dx + \frac{4a}{y}dx\right) = \frac{1}{\sqrt{5}}\left(1+\frac{4a}{y}\right)dx.$$

The required volume generated, refer to (6.50), is

$$V = \int_0^a \pi(PN)^2\,d(ON) = \pi\int_0^a \frac{(y-2x)^2}{5}\ \frac{1}{\sqrt{5}}\left(1+\frac{4a}{y}\right)dx$$

$$= \frac{\pi}{5\sqrt{5}}\int_0^a\left[(y-2x)^2 + \frac{4a}{y}(y-2x)^2\right]dx$$

$$= \frac{\pi}{5\sqrt{5}}\int_0^a\left[y^2+4x^2-4xy+4ay+\frac{16ax^2}{y}-16ax\right]dx$$

$$= \frac{\pi}{5\sqrt{5}}\int_0^a\left[4ax+4x^2-8x\sqrt{ax}+8a\sqrt{ax}+8x\sqrt{ax}-16ax\right]dx$$

$$= \frac{\pi}{5\sqrt{5}} \int_0^a \left[4ax + 4x^2 + 8a\sqrt{ax} - 16ax\right] dx$$

$$= \frac{\pi}{5\sqrt{5}} \left[2ax^2 + \frac{4}{3}x^3 + \frac{16}{3}a^{3/2}x^{3/2} - 8ax^2\right]_0^a$$

$$= \frac{\pi a^3}{5\sqrt{5}} \left[2 + \frac{4}{3} + \frac{16}{3} - 8\right] = \frac{2\pi a^3}{15\sqrt{5}} \text{ cubic unit.}$$

Example 6.27: The area lying inside the cardioid $r = 2a(1 + \cos\theta)$ and outside the parabola $r(1 + \cos\theta) = 2a$ is revolved about the initial line $\theta = 0$. Find the volume of the solid generated.

Solution: The points of intersection of the two curves, the cardiod $r = 2a(1 + \cos\theta)$ and the parabola $r = 2a/(1+ \cos\theta)$, are given by

$$2a(1 + \cos\theta) = \frac{2a}{(1 + \cos\theta)}$$

or, $$(1 + \cos\theta)^2 = 1, \text{ or } \cos\theta = 0, \text{ or } \theta = \frac{\pi}{2}, \frac{-\pi}{2}.$$

The bounded region is shown in Fig. 6.27.

The required volume is generated by rotating just the upper half (or, the lower half) portion for which θ varies from 0 to $\frac{\pi}{2}$. Therefore, $V = \frac{2\pi}{3}\int_0^{\pi/2} (r_1^3 - r_2^3)\sin\theta\, d\theta$, where r_1 is the value of r for the cardiod and r_2 is the value of r for the parabola, thus

$$V = \frac{2\pi}{3}\int_0^{\pi/2}\left[8a^3(1 + \cos\theta)^3 - \frac{8a^3}{(1 + \cos\theta)^3}\right]\sin\theta\, d\theta$$

$$= \frac{16\pi a^3}{3}\int_0^{\pi/2}\left[(1 + \cos\theta)^3 - (1 + \cos\theta)^{-3}\right]\sin\theta\, d\theta$$

$$= \frac{-16\pi a^3}{3}\left[\frac{(1 + \cos\theta)^4}{4} - \frac{(1 + \cos\theta)^{-2}}{-2}\right]_0^{\pi/2}$$

$$= \frac{-16\pi a^3}{3}\left[\frac{1}{4} + \frac{1}{2} - 4 - \frac{1}{8}\right] = 18\pi a^3.$$

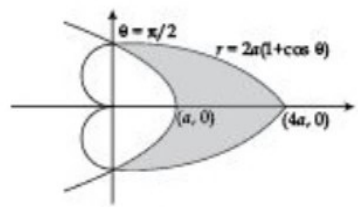


Fig. 6.27

Example 6.28: The area bounded by an arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$ and the x-axis is revolved around x-axis. Find the volume of the solid generated.

Solution: The area bounded by an arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is shown in Fig. 6.28.

The volume generated when this area is revolved around x-axis is given by

$$V = \pi\int_0^{2\pi} y^2\,dx$$

$$= \pi\int_0^{2\pi} a^2(1-\cos\theta)^2\, a(1-\cos\theta)d\theta = \pi a^3\int_0^{2\pi}(1-\cos\theta)^3 d\theta$$

$$= \pi a^3\int_0^{2\pi}(1-3\cos\theta+3\cos^2\theta-\cos^3\theta)\,d\theta.$$

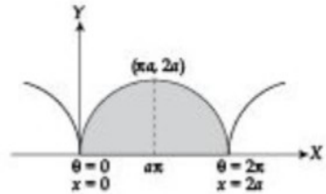


Fig. 6.28

$$= \pi a^3\int_0^{2\pi}\left[1-3\cos\theta+\frac{3}{2}(1+\cos 2\theta)-\frac{1}{4}(\cos 3\theta+3\cos\theta)\right]d\theta$$

$$= \pi a^3\int_0^{2\pi}\left[\frac{5}{2}-\frac{15}{4}\cos\theta+\frac{3}{2}\cos 2\theta-\frac{1}{4}\cos 3\theta\right]d\theta$$

$$= \pi a^3\left[\frac{5}{2}\theta-\frac{15}{4}\sin\theta+\frac{3}{4}\sin 2\theta-\frac{1}{12}\sin 3\theta\right]_0^{2\pi} = 5\pi^2 a^3 \text{ cubic units.}$$

EXERCISE 6.5

1. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.
2. Find the volume of the solid generated by revolving about x-axis the area bounded by the parabola $y = (x^2/4) + 2$ and the straight line $8y = 5x + 14$.
3. The area bounded by the parabola $y^2 = 4x$ and the straight line $4x - 3y + 2 = 0$ is rotated about the y-axis. Find the volume of the solid generated.
4. The ellipse $(x^2/a^2) + (y^2/b^2) = 1$ is revolved about y-axis. Find the volume of the solid generated.
5. A cone is formed by revolving about y-axis the line joining the origin to the point (a, b). Find the volume of the cone generated.
6. Find the volume of the solid generated by the revolution of the circle $x^2 + (y-b)^2 = a^2$, $b > a$ about the axis of x.
7. Find the volume of the solid generated by revolving the area included between the curve $(y+8)/x = x - 2$ and the x-axis about the line $x + 5 = 0$.
8. The loop of the curve $r = a\cos\theta$ lying between $\theta = -\pi/6$ and $\theta = \pi/6$ revolves about the initial line. Find the volume of the solid generated.
9. The area of the cardioid $r = a(1+\cos\theta)$ included between $\theta = -\pi/2$ and $\theta = \pi/2$ is rotated about the line $\theta = \pi/2$. Show that the volume generated is $2[2 + (5\pi/8)]\pi a^3$.

10. Find the volume of the solid obtained by revolving the cissoid $y^2(2a - x) = x^3$ about its asymptotes.
11. Find the volume of the solid generated by revolving the astroid $x = \cos^3\theta$, $y = a\sin^3\theta$ about x-axis.
12. Show that the volume of the solid generated by revolving the leminiscate $r^2 = a^2\cos 2\theta$ about the line $\theta = \pi/2$, is $\pi a^3/4\sqrt{2}$.

6.7 SURFACE AREAS OF SOLIDS OF REVOLUTION

In this section, we find the surface areas of the solids of revolution by the application of definite integral. We discuss the various cases as given below.

6.7.1 Cartesian Co-ordinates

(a) *Revolution about x-axis*

An estimate of the curved surface area of the slice generated may be found by assuming that it is formed by rotating the chord PQ instead of the arc PQ, refer to Fig. 6.19, about x-axis. Under this assumption the solid formed is a slice of a cone with surface area approximately given by

$$\Delta S_i = 2\pi y_i \Delta s_i$$

where $\Delta s_i = \widehat{PQ}$. Hence the curved surface area S of the solid of revolution is given by

$$S = \int_{x=a}^{x=b} 2\pi y\, ds = \int_a^b 2\pi y \frac{ds}{dx} dx = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \quad \text{refer to (4.22)} \qquad ...(6.56)$$

(b) *Revolution about the y-axis*

Similarly if the area bounded by the curve $x = g(y)$, the y-axis and the lines $y = c$ and $y = d$ is revolved about the y-axis, refer to Fig. (6.20), then surface area of the solid of revolution generated is given by

$$S = \int_c^d 2\pi x ds = 2\pi \int_c^d x \frac{ds}{dy} dy = 2\pi \int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy, \quad \text{refer to (4.23)} \qquad ...(6.57)$$

6.7.2 Polar Co-ordinates

(a) *Revolution about the initial line θ = 0*

The formula for the surface area of revolution in terms of polar co-ordinates, when the area is revolved around the initial line $\theta = 0$, is given by

$$S = \int 2\pi y\, ds = \int_{\theta=\alpha}^{\beta} 2\pi y \frac{ds}{d\theta} d\theta = \int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta, \quad \text{refer to (4.29).} \qquad ...(6.58)$$

(b) *Revolution about the line $\theta = \pi/2$*

Similarly when the area is revolved around the line $\theta = \frac{\pi}{2}$, the surface area S is given by

$$S = \int 2\pi x\, ds = \int_{\theta=\alpha}^{\beta} 2\pi x \frac{ds}{d\theta}\, d\theta = \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta, \qquad \text{...(6.59)}$$

refer to (4.29).

6.7.3 Parametric Co-ordinates

In terms of parametric co-ordinates $x = \phi(t)$, $y = \psi(t)$, $t_0 \le t \le t_1$, the surface of the solid of revolution obtained by rotating the area about the x-axis is,

$$S = \int 2\pi y\, ds = \int 2\pi y \frac{ds}{dt}\, dt = \int_{t_0}^{t_1} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{...(6.60)}$$

and about y-axis is

$$S = \int_{t_0}^{t_1} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \qquad \text{...(6.61)}$$

refer to (4.24).

Example 6.29: Find the surface area of the solid generated by revolving the loop of the curve $3ay^2 = x(x - a)^2$ about x-axis.

Solution: The loop of the given curve extends from $x = 0$ to $x = a$ as shown in Fig. 6.29.

Therefore, the required surface area S obtained by revolving the loop about x-axis is

$$S = \int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad \text{...(6.62)}$$

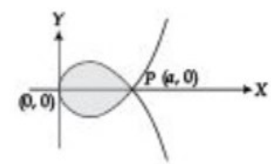


Fig. 6.29

For the given equation $3ay^2 = x(x - a)^2$, we have

$$6ay\frac{dy}{dx} = (x - a)^2 + 2x(x - a) \text{ or, } \frac{dy}{dx} = \frac{(x - a)(3x - a)}{6ay}.$$

Thus,

$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = y\sqrt{1 + \frac{(x - a)^2(3x - a)^2}{36a^2y^2}}$$

$$= \sqrt{y^2 + \frac{(x - a)^2(3x - a)^2}{36a^2}} = \sqrt{\frac{x(x - a)^2}{3a} + \frac{(x - a)^2(3x - a)^2}{36a^2}}$$

$$= \frac{(x - a)}{6a}\sqrt{12ax + (3x - a)^2} = \frac{(x - a)(3x + a)}{6a} = \frac{3x^2 - 2ax - a^2}{6a}.$$

Hence, (6.62) becomes

$$S = \frac{\pi}{3a}\int_0^a (3x^2 - 2ax - a^2)\,dx = \frac{\pi}{3a}[x^3 - ax^2 - a^2x]_0^a = \frac{-\pi a^2}{3}.$$

Thus $S = \frac{\pi a^2}{3}$, taking the numerical value.

Example 6.30: Find the surface area of the solid generated by revolving an arc of the cycloid, $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$ about the x-axis.

Solution: The area bounded by the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$ and the x-axis is shown in Fig. 6.28. The surface area of the solid generated by revolving the area about x-axis is given by

$$S = 2\pi\int_0^{2\pi} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

We have, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \frac{2a\sin\theta/2\cos\theta/2}{2a\sin^2\theta/2} = \cot\theta/2$. Thus

$$\therefore \quad S = 2\pi\int_0^{2\pi} a(1 - \cos\theta)\sqrt{1 + \cot^2\frac{\theta}{2}}\,a(1 - \cos\theta)d\theta.$$

$$= 8\pi a^2\int_0^{2\pi} \sin^4\frac{\theta}{2}\operatorname{cosec}\frac{\theta}{2}d\theta = 8\pi a^2\int_0^{2\pi}\sin^3\frac{\theta}{2}d\theta = 2\pi a^2\int_0^{2\pi}\left(3\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right)d\theta$$

$$= 2\pi a^2\left[-6\cos\frac{\theta}{2} + \frac{2}{3}\cos\frac{3\theta}{2}\right]_0^{2\pi} = 2\pi a^2\left[6 - \frac{2}{3} + 6 - \frac{2}{3}\right] = \frac{64}{3}\pi a^2.$$

Example 6.31: The lemniscate $r^2 = a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the surface generated is $4\pi a^2$.

Solution: The curve $r^2 = a^2\cos 2\theta$ has two loops so the required surface area is twice the surface area generated by one loop when it revolves around the tangent, say MOM' as shown in Fig. 6.30.

Let $P(r, \theta)$ be any point on the loop of the lemniscate and $PN \perp MOM'$, then $PN = r\sin\left(\theta + \frac{\pi}{4}\right)$.

Also from $r^2 = a^2\cos 2\theta$, we obtain

$$2r\frac{dr}{d\theta} = -2a^2\sin 2\theta, \text{ or } \frac{dr}{d\theta} = \frac{-a^2}{r}\sin 2\theta.$$

Then surface area S generated is given by

$$S = 2\int_{-\pi/4}^{\pi/4} 2\pi\, PN\,ds = 4\pi\int_{-\pi/4}^{\pi/4} r\sin\left(\theta + \frac{\pi}{4}\right)\frac{ds}{d\theta}\,d\theta$$

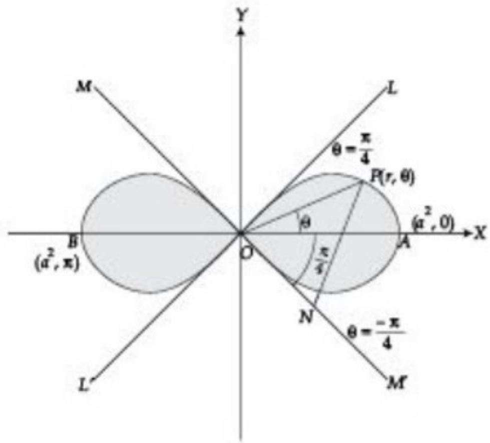


Fig. 6.30

$$= 4\pi \int_{-\pi/4}^{\pi/4} r \sin\left(\theta + \frac{\pi}{4}\right)\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta, \quad \text{refer to (4.29)}$$

$$= 4\pi \int_{-\pi/4}^{\pi/4} r \sin\left(\theta + \frac{\pi}{4}\right)\left[r^2 + \frac{a^4}{r^2}\sin^2 2\theta\right]^{\frac{1}{2}} d\theta$$

$$= 4\pi \int_{-\pi/4}^{\pi/4} \sin\left(\theta + \frac{\pi}{4}\right)\sqrt{r^4 + a^4 \sin^2 2\theta}\, d\theta$$

$$= 4\pi \int_{-\pi/4}^{\pi/4} \sin\left(\theta + \frac{\pi}{4}\right)\sqrt{a^4 \cos^2 2\theta + a^4 \sin^2 2\theta}\, d\theta = 4\pi a^2 \int_{-\pi/4}^{\pi/4} \sin\left(\theta + \frac{\pi}{4}\right) d\theta$$

$$= 4\pi a^2 \left[-\cos\left(\theta + \frac{\pi}{4}\right)\right]_{-\pi/4}^{\pi/4} = -4\pi a^2\left[\cos\frac{\pi}{2} - \cos 0\right] = 4\pi a^2.$$

Example 6.32: Find the surface of the solid generated by the revolution of the asteroid $x = a\cos^3 t$, $y = a\sin^3 t$ about the y-axis.

Solution: The asteroid is symmetrical about the axes, as shown in Fig. 6.31.

For its portion in the first quadrant, $0 \le t \le \pi/2$, we have

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{9a^2\cos^4 t\sin^2 t+9a^2\sin^4 t\cos^2 t}$$

$$= 3a\sin t\cos t.$$

Hence, the required surface area is

$$S = 2\times 2\pi\int_0^{\pi/2} x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = 4\pi\int_0^{\pi/2} a\cos^3 t.3a$$

$$\sin t\cos t dt$$

$$= 12\pi a^2\int_0^{\pi/2}\sin t\cos^4 t\,dt = 12\pi a^2\frac{3.1}{5.3.1} = \frac{12\pi a^2}{5}$$

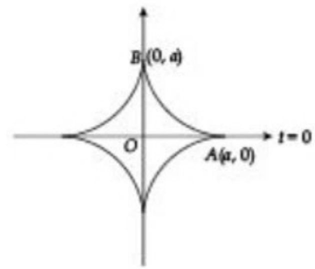


Fig. 6.31

EXERCISE 6.6

1. The portion between the consecutive cusps of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ is revolved about the x-axis. Prove that the ratio of the area of the surface generated to the area of the cycloid is 64:9.
2. A quadrant of a circle of radius a revolves around its chord. Show that the surface of the spindle generated is $2\pi a^2\sqrt{2}\,[1 - \pi/4]$.
3. Find the area of the surface generated by revolving the loop of the curve $3ay^2 = x(x - a)^2$ about x-axis.
4. Find the volume and surface area of the solid generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis.
5. A circular arc revolves about the chord. Show that the area of the surface generated is $4\pi a^2(\sin\alpha - \alpha\cos\alpha)$ where 2α is the angle subtended by the arc at the centre.
6. Prove that surface area of the solid generated by revolving tractrix $x = a\cos t + \frac{a}{2}\ln\tan^2(t/2)$, $y = a\sin t$ about x-axis is $4\pi a^2$.
7. Prove that the surface and volume of the solid generated by revolving the loop of the curve $x = t^2$, $y = t - t^3/3$ about x-axis are respectively 3π and $3\pi/4$.
8. Find the volume and surface area of the right circular cone obtained by the revolution of a right angled triangle about a side which contains the right angle.
9. Find the area of the surface generated by revolving about the x-axis a closed contour formed by the curves $y = x^2$ and $x = y^2$.
10. An arc of the catenary $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$ whose end points have abscissas 0 and x respectively, revolves about the x-axis. If S and V are respectively the surface and the volume generated show that $S = 2V/a$.

6.8 CENTROIDS OF ARC, LAMINA, VOLUME AND SURFACE OF REVOLUTION

The centroids of a body is a point where the total mass of the body may be supposed to lie.

6.8.1 Centroid of an Arc of a Curve

Let ρ be the density per unit length and let $P(s)$ and $Q(s+\delta s)$ be two neighbouring points, as shown in Fig. 6.32, such that arc $PQ = \delta s$.

Then the centroid $(\bar{x}, \bar{y})$ of the arc AB is given by

$$\bar{x} = \frac{\Sigma x\rho\delta s}{\Sigma \rho\delta s}, \quad \bar{y} = \frac{\Sigma y\rho\delta s}{\Sigma \rho\delta s}.$$

In the limiting case it tends to

$$\bar{x} = \frac{\int x\rho ds}{\int \rho ds}, \quad \bar{y} = \frac{\int y\rho ds}{\int \rho ds}, \qquad ...(6.63)$$

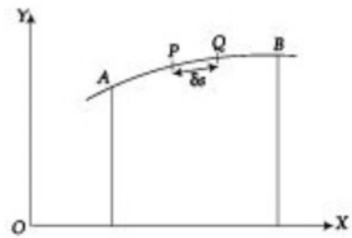


Fig. 6.32

where the integration is to be taken over the whole length of the arc AB.

6.8.2 Centroid of a Plane Lamina

Let ρ be the density per unit area of the plane lamina and δA be the elementary area about P. Then the centroid $(\bar{x}, \bar{y})$ of the plane lamina as shown in Fig. 6.33 is given by

$$\bar{x} = \frac{\Sigma x\rho\delta A}{\Sigma \rho\delta A}, \quad \bar{y} = \frac{\Sigma y\rho\delta A}{\Sigma \rho\delta A}.$$

In the limiting case it tends to

$$\bar{x} = \frac{\int x\rho dA}{\int \rho dA}, \quad \bar{y} = \frac{\int y\rho dA}{\int \rho dA} \qquad ...(6.64)$$

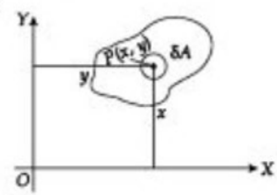


Fig. 6.33

where the integration is to be taken over the whole area.

(a) If the area is bounded above by $y = f(x)$, below by x-axis and on sides by the ordinates $x = a$ and $x = b$, then its centriod $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{\int_a^b x\rho y dx}{\int_a^b \rho y dx}, \quad \bar{y} = \frac{\int_a^b \frac{y}{2}\rho y dx}{\int_a^b \rho y dx}, \qquad ...(6.65)$$

since the C.G. of elementary strip can be taken at $(x, y/2)$, as shown in Fig. 6.34.

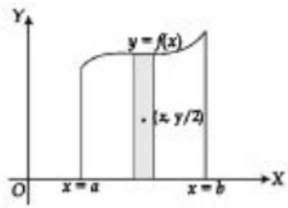


Fig. 6.34

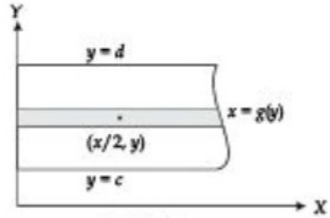


Fig. 6.35

Similarly, if the area is bounded by the curve $x = g(y)$, y-axis and the abscissas $y = c$ and $y = d$ then its centroid $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{\int \frac{x}{2}\,\rho y dx}{\int \rho y dx}, \quad \bar{y} = \frac{\int y\,.\,\rho y dx}{\int \rho y dx}, \qquad ...(6.66)$$

since in this case C.G. of the elementary strip can be taken at $(x/2, y)$ as shown in Fig. 6.35.

(c) In case of polar co-ordinates when the area is bounded by the curve $r = f(\theta)$, and the radii vectors $\theta = \alpha$ and $\theta = \beta$, then the C.G. of the elementary area, $\delta A = \frac{1}{2} r^2 \delta\theta$ of the elementary triangular strip, as shown in Fig. 6.36, is $\left(\frac{2}{3} r\cos\theta, \frac{2}{3} r\sin\theta\right)$. Thus, the centroid $(\bar{x}, \bar{y})$ of the sector OAB is

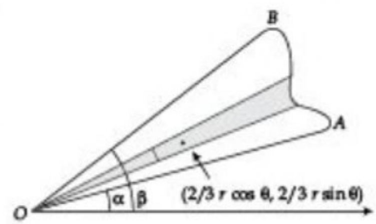


Fig. 6.36

$$\left.\begin{aligned} \bar{x} &= \frac{\int_\alpha^\beta \rho . \frac{2}{3} r\cos\theta . \frac{1}{2} r^2 d\theta}{\int_\alpha^\beta \rho . \frac{1}{2} r^2 d\theta} \\ \bar{y} &= \frac{\int_\alpha^\beta \rho . \frac{2}{3} r\sin\theta . \frac{1}{2} r^2 d\theta}{\int_\alpha^\beta \rho . \frac{1}{2} r^2 d\theta} \end{aligned}\right\} \quad \text{or,} \quad \left.\begin{aligned} \bar{x} &= \frac{\frac{1}{3}\int_\alpha^\beta \rho r^3 \cos\theta\, d\theta}{\frac{1}{2}\int_\alpha^\beta \rho r^2 d\theta} \\ \bar{y} &= \frac{\frac{1}{3}\int_\alpha^\beta \rho r^3 \sin\theta\, d\theta}{\frac{1}{2}\int_\alpha^\beta \rho r^2 d\theta} \end{aligned}\right\} \qquad ...(6.67)$$

6.8.3 Centroid of the Volume of Solid of Revolution

In case the solid is formed by *rotating the area about x-axis*, the centroid is $(\bar{x}, 0)$, where

$$\bar{x} = \frac{\int \rho x dv}{\int \rho dv} = \frac{\int \rho x \pi y^2 dx}{\int \rho \pi y^2 dx} = \frac{\int \rho x y^2 dx}{\int \rho y^2 dx}. \quad ...(6.68)$$

In case the solid is formed by *rotating the area about y-axis*, the centroid is $(0, \bar{y})$, where

$$\bar{y} = \frac{\int \rho y dv}{\int \rho dv} = \frac{\int \rho y \pi x^2 dy}{\int \rho \pi x^2 dy} = \frac{\int \rho y x^2 dx}{\int \rho x^2 dy}. \quad ...(6.69)$$

6.8.4 Centroid of the Surface of Solid of Revolution

In case the surface is formed by rotating the curve about x-axis, the centroid is $(\bar{x}, 0)$, where

$$\bar{x} = \frac{\int \rho x . 2\pi y ds}{\int \rho 2\pi y ds}. \quad ...(6.70)$$

In case the surface is formed by rotating the curve about y-axis, then the centroid is $(0, \bar{y})$, where

$$\bar{y} = \frac{\int \rho y . 2\pi x ds}{\int \rho 2\pi x ds}. \quad ...(6.71)$$

Remark: In case the density ρ is constant throughout the region, then it cancels out from the numerator and the denominator and the corresponding formulae becomes independent of ρ.

Example 6.33: Find the centroid of the area bounded by the curve $y = 3x^2$, the x-axis and the ordinates $x = 0$ and $x = 2$.

Solution: If $\bar{x}, \bar{y}$ are co-ordinates of the centroid of the given area then

$$\bar{x} = \frac{\int_0^2 xy dx}{\int_0^2 y dx} = \frac{\int_0^2 x(3x^2) dx}{\int_0^2 3x^2 dx} = \frac{\left[\frac{3x^4}{4}\right]_0^2}{[x^3]_0^2} = \frac{12}{8} = 1.5$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^2 y^2 dx}{\int_0^2 y dx} = \frac{\frac{1}{2}\int_0^2 (3x^2)^2 dx}{8} = \frac{\frac{9}{2}\left[\frac{x^5}{5}\right]_0^2}{8} = \frac{18}{5} = 3.6$$

Example 6.34: Find the centroid of the quadrant of a uniform elliptic lamina $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the first quadrant.

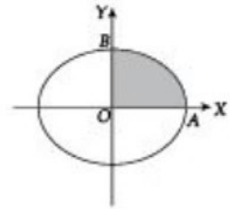


Fig. 6.37

Solution: The required portion of the elliptic lamina is shown in Fig. 6.37. If $\bar{x}$, $\bar{y}$ are the co-ordinates of the centroid of this portion, then

$$\bar{x}=\frac{\int_0^a xy\,dx}{\int_0^a y\,dx},\quad \bar{y}=\frac{\int_0^a \frac{y}{2}y\,dx}{\int_0^a y\,dx}$$

Now, $$\int_0^a y\,dx=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_0^a=\frac{\pi ab}{4},$$

$$\int_0^a xy\,dx=\int_0^a \frac{b}{a}x\sqrt{a^2-x^2}\,dx=\frac{-b}{3a}\left[(a^2-x^2)^{3/2}\right]_0^a=\frac{ba^2}{3},$$

and, $$\frac{1}{2}\int_0^a y^2dx=\frac{1}{2}\int_0^a \frac{b^2}{a^2}(a^2-x^2)dx=\frac{b^2}{2a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a=\frac{b^2a}{3}.$$

Hence, $\bar{x}=\frac{ba^2}{3}\Big/\frac{\pi ab}{4}=\frac{4a}{3\pi}$ and $\bar{y}=\frac{ab^2}{3}\Big/\frac{\pi ab}{4}=\frac{4b}{3\pi}$.

Example 6.35: Find the centroid of the area bounded by the ellipse $4x^2+9y^2=36$ and the circle $x^2+y^2=9$ and situated in the first quadrant.

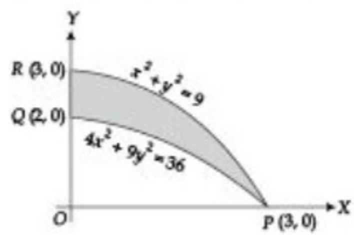


Fig. 6.38

Solution: The required area is shown in Fig. 6.38.

If $(\bar{x},\bar{y})$ be the centroid of the area, then

$$\bar{x}=\frac{\int_0^3 x(y_2-y_1)\,dx}{A},\quad \bar{y}=\frac{\frac{1}{2}\int_0^3 (y_2^2-y_1^2)\,dx}{A}$$

where $y_1=\frac{2}{3}\sqrt{9-x^2}$ and $y_2=\sqrt{9-x^2}$

and A is the area of the shaded region. Also we have

$$\int_0^3 x(y_2-y_1)\,dx=\frac{1}{3}\int_0^3 x\sqrt{9-x^2}\,dx=\frac{-1}{9}\left[(9-x^2)^{3/2}\right]_0^3=3,$$

$$\frac{1}{2}\int_0^3 (y_2^2 - y_1^2)\,dx = \frac{1}{2}\int_0^3 \frac{5}{9}(9 - x^2)\,dx = \frac{5}{18}\left[9x - \frac{x^3}{3}\right]_0^3 = 5,$$

and, A = (area of quadrant of a circle) – (area of quadrant of an ellipse)

$$= \frac{1}{4}(9\pi) - \frac{1}{4}(6\pi) = \frac{9\pi}{4} - \frac{3\pi}{2} = \frac{3\pi}{4}.$$

Thus, $$\bar{x} = 3\Big/\frac{3\pi}{4} = 4/\pi,\ \bar{y} = 5\Big/\frac{3\pi}{4} = 20/3\pi.$$

Example 6.36: Find the centroid of the hemispherical shell of radius a.

Solution: The hemispherical shell is generated when a quadrant of a circle is rotated about one of its binding radius. Take the centre of the circle as origin and binding radii as axes as shown in Fig. 6.39.

The parametric equations of the circle are

$$x = a\cos\theta,\quad y = a\sin\theta,\quad 0 \le \theta \le \pi/2$$

and the hemispherical shell is generated by rotating the arc AB, say about x-axis.

If $(\bar{x}, \bar{y})$ is the centroid of the surface generated then by symmetry $\bar{y} = 0$, and

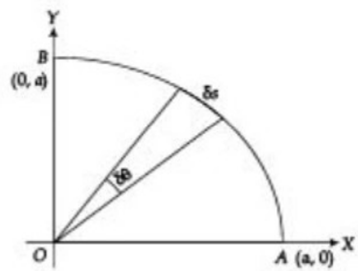


Fig. 6.39

$$\bar{x} = \frac{\int xy\,ds}{\int y\,ds} = \frac{\int_0^{\pi/2} xya\,d\theta}{\int_0^{\pi/2} ya\,d\theta},$$ since $\delta s = a\delta\theta$ in case of circle.

$$= \frac{\int_0^{\pi/2} a\cos\theta . a\sin\theta\,d\theta}{\int_0^{\pi/2} a\sin\theta\,d\theta} = \frac{\frac{a}{2}\int_0^{\pi/2}\sin 2\theta\,d\theta}{\int_0^{\pi/2}\sin\theta\,d\theta} = \frac{\frac{a}{4}[\cos 2\theta]_0^{\pi/2}}{[\cos\theta]_0^{\pi/2}} = a/2.$$

Example 6.37: Find the centroid of a semicircle of radius a.

Solution: Taking the centre of the circle O as the pole and the central radius as the initial line as shown in Fig. 6.40.

By symmetry the centroid $(\bar{x}, \bar{y})$ will lie on the initial line and thus $\bar{y} = 0$. Also,

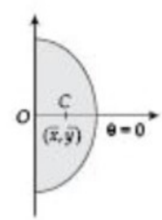


Fig. 6.40

$$\bar{x} = \frac{\int_{-\pi/2}^{\pi/2} \frac{2}{3} a \cos\theta . \frac{1}{2} a^2 \, d\theta}{A} = \frac{\frac{1}{3} a^3 \int_{-\pi/2}^{\pi/2} \cos d\theta}{\frac{1}{2} \pi a^2} = \frac{2a}{3\pi} (\sin\theta)_{-\pi/2}^{\pi/2} = \frac{4a}{3\pi}.$$

6.9 THEOREMS OF PAPPUS

Two theorems of Pappus relate centroids to surfaces and volumes of revolution. These two results provide shortcuts to a number of otherwise lengthy problems.

Theorem 6.1: **(Pappus's Theorem for Volumes)** *If a plane area is rotated about an axis in its own plane but not intersecting it, the volume of the solid formed is given by the product of the area and the distance moved by the centroid of the area.*

Proof. Let the area A revolves about OX, which does not intersect with A, through four right angles resulting in a ring-shaped solid of volume V. Let ΔA be the elementary area about a point $P(x, y)$ of the area A. The volume ΔV generated by the elementary area ΔA is given by

$$\Delta V = 2\pi y \Delta A.$$

Hence, the volume V of the solid is given by

$$V = \Sigma\, 2\pi y\, \Delta A \qquad ...(6.72)$$

Also,

$$\bar{y} = \frac{\Sigma\, y\Delta A}{\Sigma\, \Delta A}, \qquad ...(6.73)$$

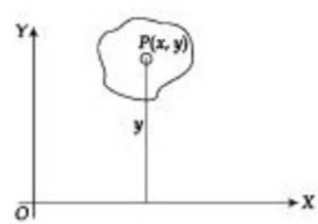


Fig. 6.41

where $\bar{y}$ is the distance of the centroid of the area A from the x-axis, and hence from (6.72) and (6.73)

$$V = 2\pi \bar{y} A. \qquad ...(6.74)$$

This proves the theorem.

Example 6.38: Find the centroid of the semicircular region of radius a by Pappus Theorem.

Solution: Let the semicircular region be the region between $y = \sqrt{a^2 - x^2}$, $-a \le x \le a$ and the x-axis as shown in Fig. 6.42. Imagine revolving the region about the x-axis to generate a solid sphere. If $G(\bar{x}, \bar{y})$ is the centroid of the region then by symmetry $\bar{x} = 0$ and $\bar{y}$, by Pappus theorem on volumes is

$$\bar{y} = \frac{V}{2\pi A} = \frac{4/3\, \pi a^3}{2\pi \frac{1}{2} \pi a^2} = \frac{4a}{3\pi}.$$

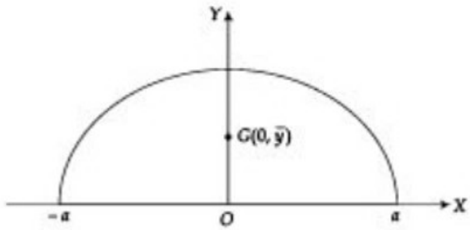


Fig. 6.42

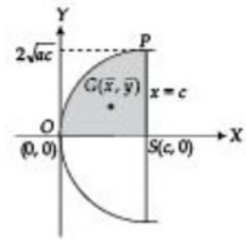


Fig. 6.43

Example 6.39: Find centroid of the area bounded by $y^2 = 4ax$, the x-axis and the ordinate $x = c$.

Solution: Let $G(\bar{x}, \bar{y})$ be the centroid of the area A as shown in Fig. 6.43.

Here, $$A = \int_0^c y\,dx = \int_0^c 2\sqrt{ax}\,dx = 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^c = \frac{4}{3}\sqrt{a}\,c^{3/2} = \frac{4}{3}c\sqrt{ac}.$$

Imagine the area is revolved about x-axis, then V_x, the volume of the solid generated is given by

$$V_x = \int_0^c \pi y^2 dx = \pi\int_0^c 4ax\,dx = 4a\pi\frac{c^2}{2} = 2a\pi c^2.$$

Similarly, if the area is revolved about y-axis, then V_y the volume of solid generated is given by

$$V_y = \int_0^{2\sqrt{ac}} \pi(c^2 - x^2)dy = \pi\int_0^{2\sqrt{ac}}\left(c^2 - \frac{y^4}{16a^2}\right)dy = \pi\left[c^2 y - \frac{y^5}{80a^2}\right]_0^{2\sqrt{ac}}$$

$$= \pi\left[2c^2\sqrt{ac} - \frac{32a^2c^2\sqrt{ac}}{80a^2}\right] = \pi\left[2c^2\sqrt{ac} - \frac{2}{5}c^2\sqrt{ac}\right] = \frac{8}{5}\pi c^2\sqrt{ac}.$$

By Pappus theorem on volumes

$$\bar{x} = \frac{V_y}{2\pi A} = \frac{8\pi c^2\sqrt{ac}}{5}\,\frac{3}{2\pi.4c\sqrt{ac}} = \frac{3c}{5},\ \bar{y} = \frac{V_x}{2\pi A} = 2\pi ac^2\,\frac{3}{2\pi.4c\sqrt{ac}} = \frac{3}{4}\sqrt{ac}.$$

Theorem 6.2: **(Pappus Theorem for Surfaces)** *If an arc of a smooth plane curve is rotated about an axis in its own plane but not intersecting it, the area of the surface generated is given by the product of the perimeter of the arc and the distance moved by the centroid of the arc length.*

The proof follows on the similar lines as in theorem on volumes.

Example 6.40: Using Pappus theorem for surfaces find the area of the curved surface of a right circular cone.

Solution: A right circular cone may be imagined to be formed by the revolution of a straight line PA about a fix line PO, as shown in Fig. 6.44.

Let $OA = r$ and $PA = l$. The centroid of the line AP may be taken to lie at its midpoint G. From G, draw a line $GM \perp OP$. Obviously MG is parallel to OA and

$$MG = \frac{1}{2}OA = \frac{1}{2}r.$$

Hence, by Pappus theorem on surfaces, the desired surface area is

$$S = PA\left(2\pi\frac{r}{2}\right) = \pi rl.$$

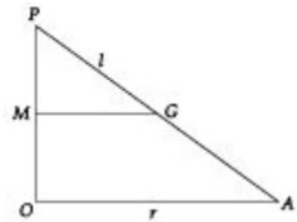


Fig. 6.44

Example 6.41: Find the co-ordinates of centroid of

(i) the quadrant of a uniform circular lamina,

(ii) the quadrant of a circular arc,

(iii) also find the volume and the surface of the solid generated when the quadrant is rotated about the tangent at either of its extremities.

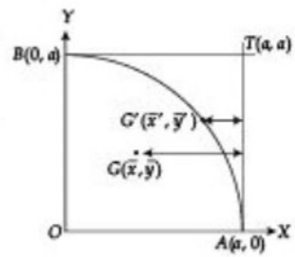


Fig. 6.45

Solution: (i): Let $G(\bar{x}, \bar{y})$ be the centroid of the quadrant as shown in Fig. 6.45.

Then by symmetry $\bar{x} = \bar{y}$.

By Pappus theorem on volumes

$$\bar{y} = \frac{V}{2\pi A} = \frac{2}{3}\pi a^3 \Big/ 2\pi\frac{\pi a^2}{4} = \frac{4a}{3\pi}.$$

Thus, centroid G is $\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$.

(ii) Let $G'(\bar{x}', \bar{y}')$ be the centroid of the arc AB, then by symmetry $\bar{x}' = \bar{y}'$.

By Pappus theorem on surfaces

$$\bar{y}' = \frac{S}{2\pi L} = \frac{2\pi a^2}{2\pi\left(\frac{\pi a}{2}\right)} = \frac{2a}{\pi}.$$

Thus, centroid G' is $\left(\frac{2a}{\pi}, \frac{2a}{\pi}\right)$.

(iii) Let the region be rotated about the tangent AT at $A(a, 0)$, then the volume generated is

$$V = \frac{\pi a^2}{4} 2\pi(a - \bar{x}) = \frac{\pi^2 a^2}{2}\left(a - \frac{4a}{3\pi}\right) = \frac{\pi a^3}{6}(3\pi - 4).$$

When the arc is rotated about the tangent AT at $(0, 0)$, then the surface generated is

$$S = \frac{\pi a}{2} 2\pi(a - \bar{x}') = \pi^2 a\left(a - \frac{2a}{\pi}\right) = \pi a^2(\pi - 2).$$

EXERCISE 6.7

1. If s is the length of an arc of the catenary $y = a\cosh(x/a)$ measured from the vertex to any point $P(x, y)$, show that centroid of the portion considered is $\left(x + \frac{a(y-a)}{s}, \frac{ax}{2s} + \frac{y}{2}\right)$.
2. If s is the length of the arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ from the origin to the point (x, y), show that $s^2 = 8ay$. Also find the centroid of the arc lying in the first quadrant.
3. Find the centroid of the thin flat plate covering the triangular region in the first quadrant bounded by the y-axis, the parabola $y = x^2/4$ and the line $y = 4$.
4. Find the centre of mass of a thin flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$ if the density function is $\rho(y) = 1 + y$.
5. Obtain the centroid of the solid formed by revolving the area bounded by the parabola $y^2 = 4ax$, the x-axis and the latus rectum, about the latus ractum.
6. Find the co-ordinates of the centroid of region bounded by the straight line $y = \frac{2}{\pi}x$ and the sinusoid $y = \sin x$, $(x \geq 0)$.
7. Find the centroid of the volume of the solid generated when the area bounded by $a^2y = x^3$, the x-axis and the line $x = 2a$ is revolved about the line $x = 2a$.
8. The loop of the curve $2ay^2 = x(a - x)^2$ revolves about the line $y = a$. Find the volume of the solid generated using the Pappus theorem on volumes.
9. The curve $y = f(x)$, $0 \leq x \leq b$ is rotated about the x-axis to form a solid. Given that the volume of the solid is $b^2 + b$ for all b, find the value of the function $f(x)$.
10. A cone is formed by revolving about the y-axis, the line joining the origin to the point (a, b). Find the volume of this cone.
11. An equilateral triangle with side a revolves about an axis parallel to the base and situated at a distance $b > a$ from the base. Find the volume of the solid of resolution.
12. Using Pappus theorem prove that centroid of a triangle is one-third of the altitude distant from its base.
13. A sector of a circle whose central angle is 2α is rotated about the line through the centre parallel to the chord of the arc. Find the surface and the volume generated by the complete revolution.
14. Find the centroid of the region in the first quadrant bounded by two concentric circles and the co-ordinate axes, if the circles have radii a and b, $0 < a < b$ and their centre are at the origin. Find the limits of the co-ordinates of the centroid as a approaches b and discuss the obtained result.

ANSWERS

Exercise 6.1 (p. 366)

4. 32 atm.
5. $2v/3$
8. (a) $\pi^2/4$ (b) $(\pi/2)\ln 2$.

Exercise 6.2 (p. 378)

1. (a) $\frac{3\sqrt{3}}{32}+\frac{5\pi}{48}$ (b) $\frac{2}{315}$ (c) $\frac{35\pi}{256}$ (d) $\frac{16\pi}{1155}$
2. (a) $\frac{16}{35}a^7$ (b) $\frac{1}{a^{2n-1}}\frac{(2n-3)(2n-5)\ldots 3.1}{(2n-2)(2n-4)\ldots 4.2}\cdot\frac{\pi}{2}$ (c) $\frac{1}{15}$
4. $\frac{64\sqrt{2}}{15}$
8. $(3\pi-8)/12$
9. $24/85$

Exercise 6.3 (p. 384)

1. πab
2. $\frac{8}{3}a^2$
3. $\frac{1}{6}$
4. $\frac{\pi a^2}{8}$
5. $2a^2\left(\pi-\frac{2}{3}\right)$
7. $a^2(\pi-2)$
8. $4\pi a^2$
9. $4a^2$
10. $(\pi-1)a^2$
11. $\frac{a^2}{2}(3\pi-8)$
12. $\frac{4}{ab}\tan^{-1}\left(\frac{a}{b}\right)$
14. $\frac{\pi a^2}{4n}, \frac{\pi a^2}{4}$
17. $\frac{3}{8}\pi ab$
18. $3\pi a^2$
19. $\frac{11}{3}$

Exercise 6.4 (p. 388)

1. $\frac{3a}{2}$
2. $2\pi a$
4. $\frac{\pi a}{\sqrt{2}}$
5. $\frac{3a\pi}{2}$
9. $(r_2-r_1)\sec\alpha$
10. $l\left[\sqrt{2}+\ln\left(\sqrt{2}+1\right)\right]$.

Exercise 6.5 (p. 395)

1. $\frac{7\pi}{6}$
2. $\frac{891}{1280}\pi$
3. $\pi/20$
4. $\frac{4}{3}\pi ab^2$
5. $\frac{1}{3}\pi a^2 b$
6. $2\pi^2 a^2 b$
7. 432π
8. $\frac{7\pi a^3}{96}$

10. $2\pi^2 a^3$ 11. $\dfrac{32\pi a^3}{105}$.

Exercise 6.6 (p. 400)

3. $\dfrac{\pi a^2}{3}$ 4. $\dfrac{12\pi a^2}{5}$

8. $\dfrac{1}{2}\pi r^2 h;\ \pi r\sqrt{r^2+h^2}$, where r is the base and h is the height and it is rotated about h.

9. $\dfrac{67\sqrt{5}\pi}{48} - \dfrac{\pi}{32}\ln\left(2+\sqrt{5}\right) - \dfrac{\pi}{6}$.

Exercise 6.7 (p. 409)

2. $a\left(\pi - \dfrac{4}{3}\right), \dfrac{2a}{3}$ 3. $(3/2, 12/5)$ 4. $(9/5, 11/10)$ 5. $\left(a, \dfrac{5a}{8}\right)$

6. $\left(\dfrac{\pi}{6(4-\pi)}, \dfrac{12-\pi^2}{12-3\pi}\right)$ 7. $\left(2a, \dfrac{10a}{7}\right)$ 8. $\dfrac{8\sqrt{2}\pi a^3}{15}$ 10. $\dfrac{1}{3}\pi a^2 b$

11. $\pi\left(\dfrac{a^2 b\sqrt{3}}{2} \mp \dfrac{a^3}{3}\right)$ 13. $\dfrac{4}{3}\pi a^3 \sin\alpha,\ 4\pi a^2 \sin\alpha$.

14. $\bar{x} = \bar{y} = 4(a^2+ab+b^2)/3\pi(a+b),\ (2a/\pi, 2a/\pi)$.

7 Multiple Integrals and Their Applications

CHAPTER

Multiple integrals are definite integrals of functions of several variables. Double and triple integrals arise in evaluating quantities such as area, volume, mass, moments, centroid and moments of inertia and are used in many applications in science and engineering. If the number of variables is higher, then one will arrive at hypervolumes which cannot be graphed.

7.1 DOUBLE INTEGRALS

Let $f(x, y)$ be a continuous and single valued function of x and y defined over a simple region R bounded by a closed curve C as shown in Fig. 7.1.

Subdivide the region R by drawing lines parallel to co-ordinate axes. Number the rectangles which are inside R, in some order, say from 1 to n. Choose an arbitrary point (x_k, y_k) in each $\Delta A_k = \Delta x_k \, \Delta y_k$, the area of the kth rectangle and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k \qquad \text{...(7.1)}$$

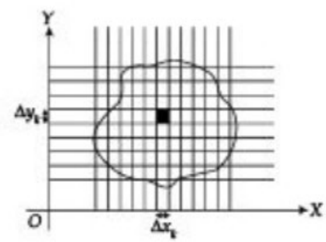


Fig. 7.1

The limit of this sum as $n \to \infty$ and $\Delta A_k \to 0$ is defined as the double integral of $f(x, y)$ over the region R and is denoted by

$$I = \iint_R f(x, y)\,dA, \text{ or } \quad \iint_R f(x, y)\,dx\,dy. \qquad \text{...(7.2)}$$

The continuity of $f(x, y)$ is a sufficient condition for the existence of the double integral, but not a necessary one. The limit under consideration exists for many discontinuous functions as well.

Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are defined to be bounded and continuous functions over the region R, then

1. $\iint_R k f(x, y)\,dx\,dy = k \iint_R f(x, y)\,dx\,dy$, for any number k.

2. $\iint_R [f(x, y) \pm g(x, y)]\,dx\,dy = \iint_R f(x, y)\,dx\,dy \pm \iint_R g(x, y)\,dx\,dy$.

3. $\iint_R f(x, y)\,dx\,dy \geq 0$, if $f(x, y) \geq 0$ on R.

4. $\iint_R f(x, y)\,dx\,dy \geq \iint_R g(x, y)\,dx\,dy$, if $f(x, y) \geq g(x, y)$ on R.

5. $\iint_R f(x, y)\,dx\,dy = \iint_{R_1} f(x, y)\,dx\,dy + \iint_{R_2} f(x, y)\,dx\,dy$,

where R is the union of two non-overlapping regions R_1 and R_2.

Evaluation of double Integrals

The double integral in terms of the limit of sums is only applicable to some specific computational problems. In fact, the double integral over a region R is evaluated by two successive single integrations as explained below.

Case I: Let the region R be the rectangular region expressed in the form

$$R = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\},$$

as shown in Fig. 7.2.

In this case the limits, both for x and y, are constants, so it is immaterial whether we first integrate w.r.t. x or w.r.t. y.

Thus, in this case

$$\iint_R f(x, y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy, \text{ or}$$

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$

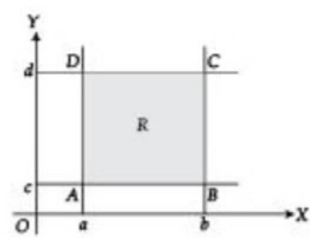


Fig. 7.2

Case II: Let the region R be expressed in the form

$$R = \{(x, y) : a \leq x \leq b,\ g(x) \leq y \leq h(x)\}$$

as shown in Fig. 7.3.

In this case limits for x are constants, but for y are functions of x. We assume $g(x)$ and $h(x)$ to be both integrable functions and $g(x) \leq h(x)$, for $x \in [a, b]$.

Here we first integrate w.r.t. y and then w.r.t. x. Thus

$$\iint_R f(x, y)dxdy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x, y)dy \right) dx.$$

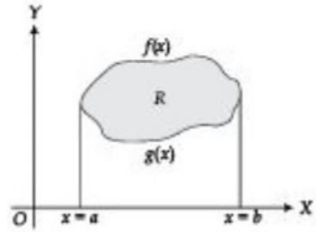


Fig. 7.3

Case III: Let the region R be expressed in the form

$$R = \{(x, y): g(y) \le x \le h(y),\ c \le y \le d\}$$

as shown in Fig. 7.4.

In this case limits for x are functions of y, but limits for y are constants, we assume $g(y)$ and $h(y)$ to be both integrable functions and $g(y) \le h(y)$, for $y \in [c, d]$.

Here we first integrate w.r.t. x and then w.r.t. y. Thus

$$\iint_R f(x, y)dxdy = \int_c^d \left(\int_{g(y)}^{h(y)} f(x, y)dx \right) dy.$$

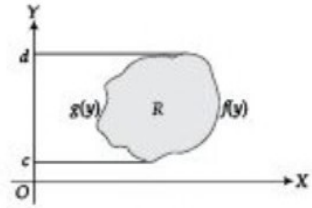


Fig. 7.4

Example 7.1: Evaluate $\iint_R f(x, y)dA$ for $f(x, y) = 1 - 6x^2y$ and $R: \{0 \le x \le 2, -1 \le y \le 1\}$.

Solution: Let $\quad I = \iint_R f(x, y)dA = \int_{-1}^{1} \left[\int_0^2 (1 - 6x^2y)dx \right] dy = \int_{-1}^{1} \left[x - 2x^3y \right]_0^2 dy$

$$= \int_{-1}^{1} (2 - 16y)dy = [2y - 8y^2]_{-1}^{1} = 4.$$

We may verify that the double integral I, evaluated as $I = \int_0^2 \left(\int_{-1}^{1} (1 - 6x^2y)dy \right) dx$ also yields the same value as above, since the limits of integration are constants in this case and, therefore, the order of integration is immaterial.

Example 7.2: Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1 + x^2 + y^2} dy\, dx.$

Solution: Let

$$I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} dy\, dx = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} dy \right] dx$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^{\sqrt{1+x^2}} \right) dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} (\tan^{-1} 1 - \tan^{-1} 0) \right] dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}} = \frac{\pi}{4} \left[\ln\left(x + \sqrt{1+x^2} \right) \right]_0^1$$

$$= \frac{\pi}{4} \left[\ln\left(1+\sqrt{2}\right) - \ln 1 \right] = \frac{\pi}{4} \ln\left(1+\sqrt{2}\right)$$

Example 7.3: Calculate $\iint_R \frac{\sin x}{x} dA$, where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.

Solution: The region of integration

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le x\}$ is shown in Fig. 7.5.

Integrating first w.r.t. y and, then w.r.t. x, we get

$$I = \int_0^1 \frac{\sin x}{x} \left(\int_0^x dy \right) dx = \int_0^1 \frac{\sin x}{x} x\, dx$$

$$= \int_0^1 \sin x\, dx = [-\cos x]_0^1 = 1 - \cos 1.$$

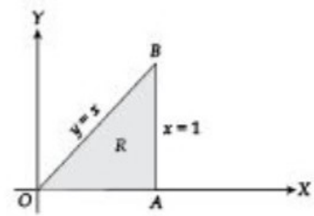


Fig. 7.5

Change of order of Integration

In the preceding example if we represent the region as $R = \{(x, y): y \le x \le 1, 0 \le y \le 1\}$, then

$$I = \int_0^1 \left(\int_y^1 \frac{\sin x}{x} dx \right) dy$$

that is the order of integration is reversed. In this case we observe that $\int \frac{\sin x}{x} dx$ can't be expressed in terms of the elementary function, and thus, it is not easy to calculate the integration in this form.

There is no definite rule to foresee that which order of integration will work well. Sometimes it is convenient to evaluate the integral by changing the order and modify the limits suitably.

In the next example, we illustrate the procedure for changing of order of integration and finding the modified limits.

Example 7.4: Evaluate $\iint_R xy\,dA$, where R is the positive quadrant of the circle $x^2 + y^2 = a^2$ integrating, (a) first w.r.t. to x and then w.r.t. y, (b) first w.r.t. to y and then w.r.t. x.

Solution: The region of integration R is shown in Fig. 7.6. To evaluate the integration over R by integrating first w.r.t. x, imagine an elementary strip PQ through R in the direction of increasing x.

Mark the x values where the strip enters and leaves the region R as shown in Fig. 7.7.

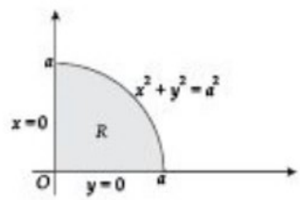


Fig. 7.6

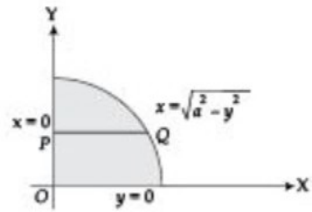


Fig. 7.7

Thus, for x the limits of integration are: $x = 0$ to $x = \sqrt{a^2 - y^2}$.

To cover the entire region R such strips should start with the minimum value of y, that is, $y = 0$ to the maximum value of y, that is, $y = a$. Thus the limits of y are: $y = 0$ to $y = a$, and hence the integral I is therefore expressed as

$$I = \int_0^a \left(\int_0^{\sqrt{a^2-y^2}} xy\,dx \right) dy = \int_0^a \left(\frac{x^2}{2} y \right)_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2} \int_0^a (a^2 - y^2) y\,dy = \frac{1}{2} \left[\frac{a^2 y^2}{2} - \frac{y^4}{4} \right]_0^a = \frac{a^4}{8}.$$

Next, to evaluate the integration over R by integrating first w.r.t. y, imagine an elementary strip through R in the direction of increasing y. Mark the y values where the strip enters and leaves the region R as shown in Fig. 7.8.

Thus for y, the limits of integration are: $y = 0$ to $y = \sqrt{a^2 - x^2}$.

To cover the entire region R such strip should start with the minimum value of x, that is, $x = 0$, to the maximum value of x, that is, $x = a$. Thus the limits of x are: $x = 0$ to $x = a$, and hence the integral I is therefore expressed as

$$I = \int_0^a \left(\int_0^{\sqrt{a^2-x^2}} xy\,dy \right) dx = \int_0^a \left(\frac{y^2}{2} x \right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_0^a (a^2 - x^2) x\,dx = \frac{1}{2} \left[\frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a = \frac{a^4}{8}.$$

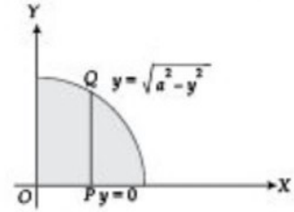


Fig. 7.8

Example 7.5: Change the order of integration and hence evaluate

$$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx.$$

Solution: The region of integration is $R = \{(x, y) : 0 \le x \le 1,\ x^2 \le y \le 2 - x\}$, as shown in Fig. 7.9.

In the given integral $I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$, the integration is first w.r.t. y and then w.r.t. x.

To change the order of integration the elementary strip PQ is to be taken parallel to x-axis. This requires the splitting of the region R into two subregions R_1 and R_2 by the line AB, $y = 1$, refer to Fig. 7.9.

For R_1, the elementary strip PQ goes from $x = 0$ to $x = \sqrt{y}$ and to cover the region min.$y = 0$ and max.$y = 1$. For R_2, the elementary strip $P'Q'$ goes from $x = 0$ to $x = 2 - y$ and to cover the region min.$y = 1$ and max.$y = 2$. Thus the regions R_1 and R_2 are given by

$$R_1 = \{(x, y) : 0 \le x \le \sqrt{y},\ 0 \le y \le 1\}, \text{ and } R_2 = \{(x, y) : 0 \le x \le 2 - y,\ 1 \le y \le 2\}.$$

For R_1, $$I_1 = \int_0^1 \left(\int_0^{\sqrt{y}} xy\,dx \right) dy = \int_0^1 \left(\frac{x^2 y}{2} \right)_0^{\sqrt{y}} dy = \frac{1}{2} \int_0^1 y^2\,dy = \left(\frac{y^3}{6} \right)_0^1 = \frac{1}{6};$$

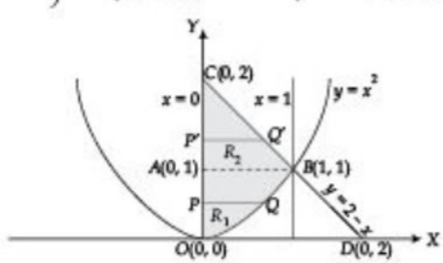


Fig. 7.9

and, for R_2, $I_2 = \int_1^2 \left(\int_0^{2-y} xy\,dx \right) dy = \int_1^2 \left(\frac{x^2 y}{2} \right)_0^{2-y} dy$

$$= \frac{1}{2} \int_1^2 y(2-y)^2 dy = \frac{1}{2} \left(2y^2 - \frac{4y^3}{3} + \frac{y^4}{4} \right)_1^2 = \frac{5}{24}.$$

Hence, $I = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$.

Example 7.6: Express $\int_0^{\frac{a}{\sqrt{2}}} \int_0^x x\,dy\,dx + \int_{\frac{a}{\sqrt{2}}}^a \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$ as a single integral and evaluate it.

Solution: Let $I_1 = \int_0^{\frac{a}{\sqrt{2}}} \int_0^x x\,dy\,dx$ and $I_2 = \int_{\frac{a}{\sqrt{2}}}^a \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$.

If R_1 and R_2 are the regions of integration in case of I_1 and I_2 respectively, then

$$R_1 = \left\{ (x, y) : 0 \le x \le \frac{a}{\sqrt{2}}, 0 \le y \le x \right\} \quad \text{and}$$

$$R_2 = \left\{ (x, y) : \frac{a}{\sqrt{2}} \le x \le a, 0 \le y \le \sqrt{a^2 - x^2} \right\}.$$

These are shown in Fig. 7.10.

It is obvious that R_1 and R_2 are non-overlapping regions and let $R = R_1 \cup R_2$. Then,

$$I = I_1 + I_2 = \iint_R x\,dx\,dy$$

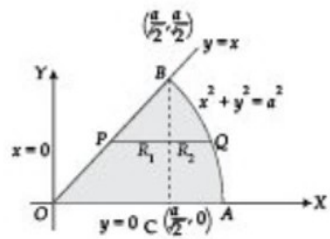


Fig. 7.10

where $R = \left\{ (x, y) : y \le x \le \sqrt{a^2 - y^2}, 0 \le y \le \frac{a}{\sqrt{2}} \right\}$.

To evaluate I, we consider an elementary strip PQ parallel to x-axis. The limits for x are: y to $\sqrt{a^2 - y^2}$. To cover the entire region R, the min. y is zero and max. y is $a/\sqrt{2}$ and, therefore,

$$I = \int_0^{\frac{a}{\sqrt{2}}} \left(\int_y^{\sqrt{a^2-y^2}} x\,dx \right) dy = \int_0^{\frac{a}{\sqrt{2}}} \left(\frac{x^2}{2} \right)_y^{\sqrt{a^2-y^2}} dy = \frac{1}{2} \int_0^{\frac{a}{\sqrt{2}}} (a^2 - 2y^2) dy$$

$$= \frac{1}{2}\left(a^2y - \frac{2y^3}{3}\right)_0^{\frac{a}{\sqrt{2}}} = \frac{1}{2}\left[\frac{a^3}{\sqrt{2}} - \frac{a^3}{3\sqrt{2}}\right] = \frac{a^3}{3\sqrt{2}}.$$

Example 7.7: Evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$ by changing the order of integration.

Solution: The region of integration $R = \left\{(x, y): 0 \le x \le 4a, \frac{x^2}{4a} \le y \le 2\sqrt{ax}\right\}$ is shown in Fig. 7.11.

The points of intersection of the two parabolas $y^2 = 4ax$ and $x^2 = 4ay$ are $O(0, 0)$ and $A(4a, 4a)$.

In the given integral, the integration is first w.r.t. y and then w.r.t. x. To change the order of integration consider an elementary strip PQ parallel to x–axis. The limits for x are: $y^2/4a$ to $2\sqrt{ay}$. To cover the entire region R the minimum of y is zero and maximum is $4a$ and, therefore,

$$I = \int_0^{4a}\left(\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right)dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$= \left(2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right)_0^{4a} = \frac{32}{3}\sqrt{a}\,a^{3/2} - \frac{64a^3}{12a} = \frac{16}{3}a^2.$$

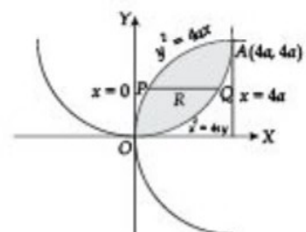


Fig. 7.11

Example 7.8: Evaluate $\int_0^{\infty}\int_0^{x} xe^{-x^2/y}\,dy\,dx$ by change of order of integration.

Solution: The region of integration $R = \{(x, y): 0 \le x < \infty,\ 0 \le y \le x\}$ is shown in Fig. 7.12.

In the given integral, the integration is first w.r.t. y and then w.r.t. x.

To change the order of integration, take an elementary strip PQ starting from P parallel to x-axis. Thus, the limits for x are: $x = y$ to ∞. To cover the entire region R the minimum y is zero and maximum is infinity, and, therefore

$$I = \int_0^{\infty}\left(\int_y^{\infty} xe^{-x^2/y}dx\right)dy.$$

To evaluate I, put $x^2 = t$, this implies $2xdx = dt$, we get

$$I = \int_0^{\infty}\left(\frac{1}{2}\int_{y^2}^{\infty} e^{-t/y}dt\right)dy = \frac{1}{2}\int_0^{\infty}\left(-ye^{-t/y}\right)_{y^2}^{\infty} dy$$

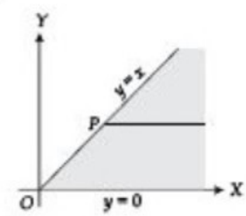


Fig. 7.12

$$= \frac{1}{2}\int_0^\infty ye^{-y}dy = \frac{1}{2}\left[(-ye^{-y})_0^\infty - \int_0^\infty 1(-e^{-y})dy\right]$$

$$= -\frac{1}{2}\left[e^{-y}\right]_0^\infty = \frac{1}{2}.$$

Example 7.9: Evaluate $\int_0^{a/\sqrt{2}}\int_0^{y} \ln(x^2+y^2)\,dx\,dy + \int_{a/\sqrt{2}}^{a}\int_0^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy$ by change of order of integration.

Solution: The regions of integration are:

$$R_1 = \left\{(x, y):\ 0 \le x \le y,\ 0 \le y \le \frac{a}{\sqrt{2}}\right\},\ R_2 = \left\{(x, y):\ 0 \le x \le \sqrt{a^2 - y^2},\ \frac{a}{\sqrt{2}} \le y \le a\right\}$$

These are non-overlapping; the joint region, $R = R_1 \cup R_2$, given by

$$R = \left\{(x, y): y \le x \le \sqrt{a^2 - y^2}, 0 \le y \le a\right\}$$

is shown in Fig. 7.13. In the given integral the integration is first w.r.t. x and then w.r.t. y. To change the order of integration, take an elementary strip PQ parallel to y-axis. The limits for y are: $y = x$ to $y = \sqrt{a^2 - x^2}$. To cover the entire region R, minimum x is zero and maximum x is $\frac{a}{\sqrt{2}}$, and, therefore

$$I = \int_0^{\frac{a}{\sqrt{2}}}\left(\int_x^{\sqrt{a^2-x^2}} \ln(x^2 + y^2)dy\right)dx.$$

To evaluate I, we change to polar co-ordinates. Put $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^a (\ln r^2)r\,dr\,d\theta = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\int_0^a \ln r\,(r dr)\right)d\theta$$

$$= \frac{\pi}{2}\left[\ln r.\frac{r^2}{2} - \int \frac{1}{r}.\frac{r^2}{2}dr\right]_0^a = \frac{\pi}{2}\left[\frac{r^2}{2}\ln r - \frac{r^2}{4}\right]_0^a$$

$$= \frac{\pi a^2}{4}\left[\ln a - \frac{1}{2}\right],\quad \text{since } \lim_{r\to 0} r^2 \ln r = 0.$$

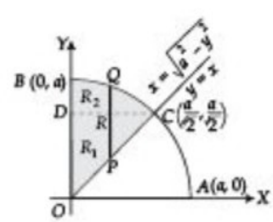


Fig. 7.13

7.2 DOUBLE INTEGRALS IN POLAR CO-ORDINATES

Let $f(r, \theta)$ be defined over a region R bounded by the radii vectors $\theta = \alpha$, $\theta = \beta$ and the continuous curves $r = g_1(\theta)$, $r = g_2(\theta)$ as shown in Fig. 7.14.

Suppose that $a \le g_1(\theta) \le g_2(\theta) \le b$ for every value of $\theta \in [\alpha, \beta]$. Then the region R lies in the region $ABCD$ defined by $\{(r, \theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$.

Divide the radial interval $[a, b]$ into h parts by concentric circular arc at interval $\Delta r = \dfrac{b-a}{h}$ and divide the angular interval $[\alpha, \beta]$ into k parts by radial lines at interval $\Delta\theta = \dfrac{\beta - \alpha}{k}$.

Thus, the whole region $ABCD$ has been divided into polar rectangles. Number these polar rectangles that lie inside R from 1 to n and let their areas be $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, where

$$\Delta A_i = \frac{1}{2}\left(r_i + \frac{1}{2}\Delta r\right)^2 \Delta\theta - \frac{1}{2}\left(r_i - \frac{1}{2}\Delta r\right)^2 \Delta\theta$$

$$= \frac{1}{2}(2r_i \Delta r \Delta\theta) = r_i \Delta r \Delta\theta;$$

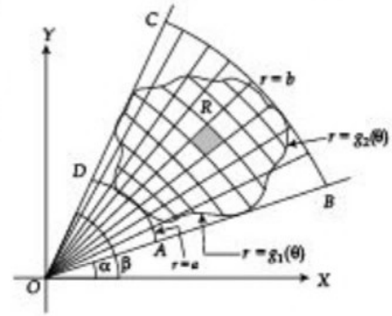


Fig. 7.14

(r_i, θ_i) being the co-ordinates of the centre of the polar rectangle of area ΔA_i. Form the sum

$$S_n = \sum_{i=1}^{n} f(r_i, \theta_i)\Delta A_i = \sum_{i=1}^{n} f(r_i, \theta_i) r_i \Delta r \Delta\theta. \qquad \ldots(7.3)$$

If f is continuous throughout R, then S_n approaches a limit S as $\Delta r \to 0$ and $\Delta\theta \to 0$, that is, as $n \to \infty$. This limit is defined as the double integral of f over R and we write

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} f(r_i, \theta_i) r_i \Delta r \Delta\theta = \iint_R f(r, \theta)\, r\, dr\, d\theta = \int_{\alpha}^{\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta. \qquad \ldots(7.4)$$

In general, to evaluate the double integral in polar coordinates, we first integrate w.r.t. r and then w.r.t. θ. However, the order of integration may be changed with suitable changes in the limits.

Example 7.10: Evaluate $\iint r^3\, dr\, d\theta$ over the region bounded by the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

Solution: The region of integration

$$R = \left\{(r, \theta) : 2\cos\theta \le r \le 4\cos\theta, \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$$

is shown in Fig. 7.15.

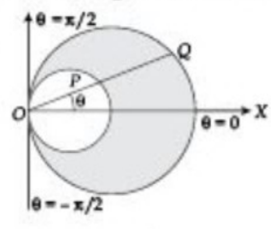


Fig. 7.15

Therefore $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_{2\cos\theta}^{4\cos\theta} r^3 dr \right) d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \right]_{2\cos\theta}^{4\cos\theta} d\theta$

$$= \frac{1}{4} \int_{-\pi/2}^{\pi/2} (256 \cos^4 \theta - 16 \cos^4 \theta) d\theta = 120 \int_0^{\pi/2} \cos^4 \theta \, d\theta = 120 \times \frac{3}{4} \times \frac{1}{2} \frac{\pi}{2} = \frac{45\pi}{2}.$$

Example 7.11: Evaluate $\iint_R \frac{r\,dr\,d\theta}{\sqrt{r^2 + a^2}}$ where R is the area of one loop of the lemniscate $r^2 = a^2 \cos 2\theta$.

Solution: The region of integration

$$R = \left\{ (r, \theta), 0 \le r \le a\sqrt{\cos 2\theta}, \frac{-\pi}{4} \le \theta \le \frac{\pi}{4} \right\}$$

is shown in Fig. 7.16. Therefore,

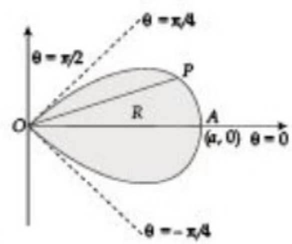


Fig. 7.16

$$I = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left(\int_0^{a\sqrt{\cos 2\theta}} \frac{r\,dr}{\sqrt{r^2 + a^2}} \right) d\theta = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left[(r^2 + a^2)^{\frac{1}{2}} \right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2a \int_0^{\frac{\pi}{4}} \left(\sqrt{1 + \cos 2\theta} - 1 \right) d\theta = 2a \int_0^{\frac{\pi}{4}} \left(\sqrt{2} \cos \theta - 1 \right) d\theta = 2a \left[\sqrt{2} \sin \theta - \theta \right]_0^{\pi/4}$$

$$= 2a \left[\sqrt{2} \sin \frac{\pi}{4} - \frac{\pi}{4} \right] = 2a \left(1 - \frac{\pi}{4} \right).$$

Example 7.12: Sketch the region of integration of $\int_a^{ae^{\pi/4}} \int_{2\ln\left(\frac{r}{a}\right)}^{\pi/2} f(r, \theta).r\,d\theta\,dr$ and change the order of integration.

Solution: The region of integration is

$$R = \left\{ (r, \theta) : a \le r \le ae^{\frac{\pi}{4}}, 2\ln \frac{r}{a} \le \theta \le \frac{\pi}{2} \right\}$$

Here $r = a$ and $r = ae^{\pi/4}$ are circles with pole as the centre and radii a and $ae^{\pi/4}$ respectively. The curve $\theta = 2\ln \frac{r}{a}$ is $r = ae^{\theta/2}$, an equiangular spiral. At $\theta = 0$, $r = a$ and at $\theta = \pi/2$, $r = ae^{\pi/4}$.

The region R is shown in Fig. 7.17.

To change the order of integration take a strip PQ passing through the pole O. Thus the limits for r are: a to $r = ae^{\theta/2}$. To cover the entire region R the minimum of θ is zero and maximum is $\pi/2$. Thus the region R can be rewritten as

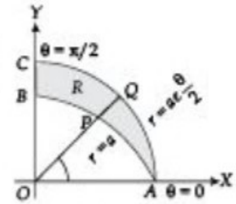


Fig. 7.17

$$R = \begin{cases} a \le r \le ae^{\theta/2} \\ 0 \le \theta \le \pi/2 \end{cases}$$

and, therefore, the given integral becomes

$$I = \int_0^{\pi/2} \left(\int_a^{ae^{\theta/2}} f(r, \theta) r\, dr \right) d\theta.$$

EXERCISE 7.1

1. Evaluate the following integrals

(a) $\int_0^1 \int_{x^2}^{x} (x^2 + 3y + 2)\, dy\, dx$ (b) $\int_0^1 \int_x^{x^2} e^{y/x}\, dy\, dx$

(c) $\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} r \sin\theta\, dr\, d\theta$ (d) $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \cos\theta\, dr\, d\theta$

2. (a) Evaluate $\iint \dfrac{xy\, dx\, dy}{\sqrt{1-x^2}}$ over the positive quadrant of the disc $x^2 + y^2 \le 1$.

(b) Evaluate $\iint y\, dx\, dy$ over the positive quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

3. Evaluate $\iint_R xy(x + y)\, dx\, dy$, where R is the region bounded by $y = x^2$ and $y = x$.

4. Evaluate $\iint \sin(ax + by)\, dx\, dy$ over the triangular area bounded by $x = 0$, $y = 0$ and $ax + by = 1$.

5. Evaluate, $\iint_R xy\, dx\, dy$, where R is the region bounded by $y = 0$, $x = 4a$ and the curve $x^2 = 4ay$.

6. Evaluate $\iint_R \sqrt{(y^2 - xy)}\, dy\, dx$, where R is the triangular region with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$.

Evaluate the following integrals (7-12) by changing the order of integration.

7. $\int_0^1 \int_x^{2-x} dy\,dx$

8. $\int_0^\infty \int_x^\infty (e^{-y}/y)\,dy\,dx$

9. $\int_0^1 \int_{4y}^4 e^{x^2}\,dx\,dy$

10. $\int_0^\infty \int_0^x x e^{-x^2/y}\,dy\,dx$

11. $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4 - a^2x^2}}$

12. $\int_0^a \int_0^x \frac{f'(y)}{\sqrt{(a-x)(x-y)}}\,dy\,dx$

13. Calculate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

14. Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the cardioid $r = a(1-\cos\theta)$ above the initial line.

15. Show that $\iint r\,dr\,d\theta = a^2$ over the area of the lemniscate $r^2 = a^2\cos 2\theta$.

7.3 TRANSFORMATION OF VARIABLES IN DOUBLE INTEGRAL

Similar to the case of definite integrals in case of a single variable, the evaluation of a double integral sometimes is simplified by the transformation of variables of integration, e.g., from cartesian to polar or to some general system, say (u, v).

Let the variables x, y be defined in a region R_{xy} in the xy-plane be transformed to the new variables u, v as $x = x(u, v)$, $y = y(u, v)$, where $x(u, v)$ and $y(u, v)$ are continuous and have continuous first order derivatives in the region R'_{uv} in the $uv-$plane corresponding to the region R_{xy} in the $xy-$plane. Let the inverse transformations be $u = u(x, y)$, $v = v(x, y)$ where $u(x, y)$ and $v(x, y)$ are defined and have continuous first order derivatives in the region R_{xy} in the $xy-$plane. Then,

$$\iint_{R_{xy}} f(x, y)\,dx\,dy = \iint_{R'_{uv}} f[x(u, v), y(u, v)]\,|J|\,du\,dv = \iint_{R'_{uv}} g(u, v)\,du\,dv.$$

Here $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$ is the Jacobian of the variable of transformation from (x, y) to (u, v). For example, in case of polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$ we have $J = r$ and thus

$$\iint_{R_{xy}} f(x, y)\,dx\,dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \iint_{R'_{r\theta}} g(r, \theta)\,dr\,d\theta$$

where $R'_{r\theta}$ is the region in the $r\theta-$plane corresponding to the region R_{xy} in the $xy-$plane.

Example 7.13: Evaluate the integral $\iint (a^2 - x^2 - y^2)dx\,dy$ over the area of the circle $x^2 + y^2 = a^2$.

Solution: The region of integration R is, $R = \left\{(x, y): -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}, -a \le x \le a\right\}$

and, thus $I = \int_{-a}^{a} \left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2)dy \right) dx.$

To simplify the computation we transform to polar co-ordinates system, by substituting $x = r\cos\theta$, $y = r\sin\theta$, $J = r$, and, therefore, I becomes

$$I = \int_0^{2\pi}\int_0^a (a^2 - r^2) r\,dr\,d\theta = 2\pi \int_0^a (a^2 r - r^3)dr = 2\pi \left(a^2 \frac{r^2}{2} - \frac{r^4}{4} \right)_0^a = \frac{\pi a^4}{2}.$$

Example 7.14: Evaluate $\int\int \left[\dfrac{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}{1 + \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}} \right]^{1/2} dx\,dy$ over the positive quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Solution: Applying the transformation $x = au$, $y = bv$, we obtain

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab.$$

The region of integration, the positive quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$, transforms to the positive quadrant of the circle $u^2 + v^2 = 1$, and, therefore the integral becomes

$$I = ab \int_0^1 \int_0^{\sqrt{1-u^2}} \left(\frac{1 - u^2 - v^2}{1 + u^2 + v^2} \right)^{1/2} du\,dv.$$

To simplify the computation further we transform to polar co-ordinates system by substituting $u = r\cos\theta$, $v = r\sin\theta$, which gives $J = \dfrac{\partial(u, v)}{\partial(r, \theta)} = r$ and, therefore, I becomes

$$I = ab \int_0^{\pi/2} \left(\int_0^1 \left(\frac{1 - r^2}{1 + r^2} \right)^{\frac{1}{2}} r\,dr \right) d\theta = \frac{\pi ab}{2} \int_0^1 \left(\frac{1 - r^2}{1 + r^2} \right)^{\frac{1}{2}} r\,dr$$

$$= \frac{\pi ab}{4} \int_0^1 \left(\frac{1 - t}{1 + t} \right)^{\frac{1}{2}} dt = \frac{\pi ab}{4} \int_0^1 \frac{1 - t}{\sqrt{1 - t^2}} dt, \quad \text{using } r^2 = t,$$

$$= \frac{\pi ab}{4}\left[\int_0^1 \frac{1}{\sqrt{1-t^2}}dt - \int_0^1 \frac{t}{\sqrt{1-t^2}}dt\right] = \frac{\pi ab}{4}\left[\sin^{-1} t + \sqrt{1-t^2}\right]_0^1 = \frac{\pi ab}{4}\left[\frac{\pi}{2} - 1\right].$$

Example 7.15: Evaluate $\iint_R \cos\left(\frac{x-y}{x+y}\right)dx\,dy$, where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$.

Solution: We apply the transformation $u = x - y$, $v = x + y$, which gives

$$x = \frac{u+v}{2}, \text{ and } y = \frac{v-u}{2}.$$

Thus the line $x = 0$ transforms to the line $u + v = 0$ that is, $v = -u$, the line $y = 0$ transforms to the line $v = u$ and the line $x + y = 1$ transforms to $v = 1$. Thus the region R as shown in Fig. 7.18a in the xy plane transforms to the region R' in the uv plane as shown in Fig. 7.18b.

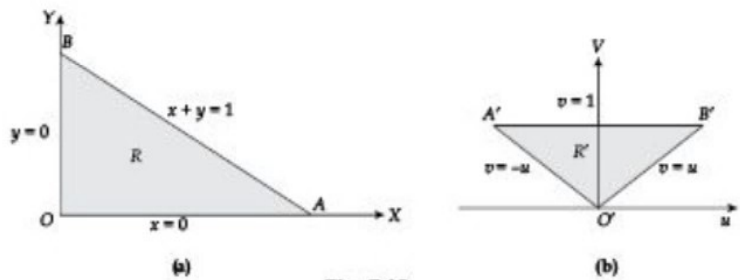


Fig. 7.18

Also, $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$. Thus,

$$I = \iint_R \cos\left(\frac{x-y}{x+y}\right)dx\,dy = \frac{1}{2}\iint_{R'} \cos\frac{u}{v}\,du\,dv = \frac{1}{2}\int_0^1\left(\int_{-v}^{v} \cos\frac{u}{v}\,du\right)dv = \frac{1}{2}\int_0^1\left[v\sin\frac{u}{v}\right]_{-v}^{v} dv$$

$$= \frac{1}{2}\int_0^1 v[\sin(1) - \sin(-1)]dv = \sin(1)\int_0^1 v\,dv = \frac{1}{2}\sin(1).$$

EXERCISE 7.2

Evaluate the following integrals by changing the variables to polars coordinates

1. $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$

2. $\int_0^{a}\int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,dx\,dy$

3. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{a\,dx\,dy}{(x^2+y^2+a^2)^{3/2}(x^2+y^2+b^2)^{1/2}}$

4. $\int_0^{2a}\int_0^{\sqrt{(2ax-x^2)}} (x^2+y^2)\,dy\,dx$

5. Change the order of integration in the double integral $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y)\,dy\,dx$.

6. Evaluate $\iint y^2 dx\,dy$ over the area outside the circle $x^2+y^2-ax=0$ and inside the circle $x^2+y^2-2ax=0$.

7. Evaluate $\iint \frac{dx dy}{(1+x^2+y^2)^{3/2}}$ over the unit square $0 \le x \le 1, 0 \le y \le 1$.

8. Evaluate $\iint \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy$ over the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

7.4 APPLICATIONS OF DOUBLE INTEGRALS

Double integrals are used to calculate the areas of bounded regions in the plane and also the mass, moments, centres of mass, etc. of thin plates covering these regions. In addition to this, double integrals are used to calculate the volume of the region below the surface $z = f(x, y)$ and above the xy-plane and volumes of the solids generated by revolution.

In this section we consider these applications of double integrals.

7.4.1 Area, Mass and Centre of Mass of the Bounded Regions in Plane

If we take $f(x, y) = 1$ in the definition of the double integral over a region R, then (7.2) gives *the area A of the bounded plane region R in cartesian co-ordinates, that is,*

$$A = \iint_R dx\,dy. \qquad ...(7.5)$$

In *polar co-ordinates system, the area A of the bounded plane region R* is given by

$$A = \iint_R r\,dr\,d\theta. \qquad ...(7.6)$$

The expression

$$\frac{1}{A}\iint_R f(x, y)\,dx\,dy \qquad \text{...(7.7)}$$

is defined as the *average value of $f(x, y)$ over the region R*, where A is the area of the region R.

For example, if $f(x, y)$ denotes the distance of an arbitrary point $P(x, y)$ in R from a fixed point T then (7.7) gives the average distance of R from T.

Next, if $\rho(x, y)$ is the density function, then

$$M = \iint_R \rho(x, y)\,dx\,dy \qquad \text{...(7.8)}$$

gives *the mass M of the thin plate covering region R* in the xy–plane.

Further, $M_x = \iint_R y\rho(x, y)\,dx\,dy$ and $M_y = \iint_R x\rho(x, y)\,dx\,dy$ ...(7.9)

give *the first moments about x–axis and y–axis*, respectively, and

$$\bar{x} = M_y/M, \quad \bar{y} = M_x/M \qquad \text{...(7.10)}$$

give *the co-ordinates of the centre of mass* of the mass M in R.

7.4.2 Moments of Inertia and Radii of Gyration

If $\rho(x, y)$ is the density function, then

$$I_x = \iint_R y^2\rho(x, y)\,dx\,dy, \quad I_y = \iint_R x^2\rho(x, y)\,dx\,dy \qquad \text{...(7.11)}$$

are the *moments of inertia, or second moments of the mass M in R about x–axis and y–axis respectively*. Also

$$I_0 = \iint_R (x^2 + y^2)\rho\,dx\,dy = I_x + I_y \qquad \text{...(7.12)}$$

is the *moment of inertia about the origin, or the polar moment of the mass M in R*, and

$$I = \iint_R d^2(x, y)\,\rho(x, y)\,dx\,dy \qquad \text{...(7.13)}$$

is the *moment about a line l of the mass M in R*, where $d(x, y)$ is the distance of an arbitrary point $P(x, y)$ from the line l.

The *radii of gyration* R_x, R_y and R_0 respectively about x–axis, y–axis and the origin are defined as

$$I_x = MR_x^2, \quad I_y = MR_y^2 \text{ and } \quad I_0 = MR_0^2. \qquad \text{...(7.14)}$$

Example 7.16: Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.

Solution: The average height $\bar{h}$ of the paraboloid $z = x^2 + y^2$ over the region $R = \{(x, y),\ 0 \le x, y \le 2\}$, a square of area **4** sq. unit, is given by

$$\bar{h} = \frac{1}{4}\int_0^2\int_0^2 (x^2+y^2)dx\,dy = \frac{1}{4}\int_0^2 \left(x^2y+\frac{y^3}{3}\right)_0^2 dx$$

$$= \frac{1}{4}\int_0^2 \left(2x^2+\frac{8}{3}\right)dx = \frac{1}{4}\left(\frac{2x^3}{3}+\frac{8}{3}x\right)_0^2 = \frac{1}{4}\left(\frac{16}{3}+\frac{16}{3}\right) = 8/3.$$

Example 7.17: Find the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$ using the double integration.

Solution: Using the symmetry we consider the area only in the first quadrant as shown in Fig. 7.19 and thus

$$A = 4\int_0^b \int_0^{a\sqrt{1-\frac{y^2}{b^2}}} dx\,dy = 4a\int_0^b \sqrt{1-\frac{y^2}{b^2}}\,dy$$

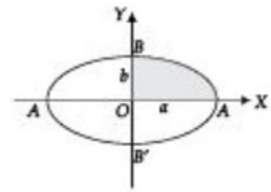


Fig. 7.19

Substituting $y = b\sin\theta$, we get

$$A = 4a\int_0^{\pi/2} \cos\theta\,(b\cos\theta)d\theta$$

$$= 4ab\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}d\theta = 2ab\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \pi ab.$$

Example 7.18: Find the area lying inside the circle $x^2 + y^2 - 2ax = 0$ and outside the circle $x^2+y^2=a^2$ using double integration.

Solution: The centres of the two given circles are $(a, 0)$ and $(0, 0)$ and both are of radius a. The required area is shown in Fig. 7.20.

Changing to polar co-ordinates, the equations of the circle are $r = a$ and $r = 2a\cos\theta$. Their points of intersection A, A' are given by

$$2a\cos\theta = a, \text{ or } \cos\theta = \frac{1}{2}, \text{ or } \theta = \pm\frac{\pi}{3}.$$

Thus the required area is equal to

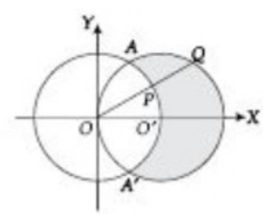


Fig. 7.20

$$A = 2\int_0^{\frac{\pi}{3}}\int_a^{2a\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_a^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/3}(4a^2\cos^2\theta - a^2)d\theta = a^2\int_0^{\pi/3}[2(1+\cos 2\theta)-1]d\theta$$

$$= a^2 \int_0^{\pi/2} (1 + 2\cos 2\theta)\, d\theta = a^2 [\theta + \sin 2\theta]_0^{\pi/3} = a^2\left[\frac{\pi}{3} + \sin\frac{2\pi}{3}\right] = a^2\left[\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right].$$

Example 7.19: Find the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptotes.

Solution: The curve $r = a(\sec\theta + \cos\theta)$ is symmetrical about the initial line $\theta = 0$, and has an asymptote $r = a\sec\theta$. The required area is shown in Fig. 7.21. It is given by

$$A = 2\int_0^{\pi/2} \int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\, dr\, d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2 \int_0^{\pi/2} [(\sec\theta + \cos\theta)^2 - \sec^2\theta]\, d\theta = a^2 \int_0^{\pi/2} (\cos^2\theta + 2)\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (\cos 2\theta + 5)\, d\theta = \frac{a^2}{2}\left[\frac{\sin 2\theta}{2} + 5\theta\right]_0^{\pi/2} = \frac{5\pi a^2}{4}.$$

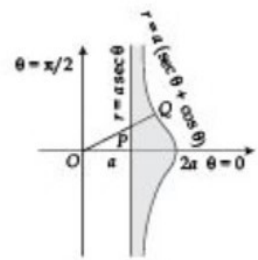


Fig. 7.21

Example 7.20: A thin plate covers the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $\rho(x, y) = 6(x + y + 1)$. Find the plate's mass, first moments, centre of mass, moments of inertia and radii of gyration about the co-ordinate axes.

Solution: The triangular region R as shown in Fig. 7.22, is given by

$R = \{(x, y) : 0 \le y \le 2x,\ 0 \le x \le 1\}$.

The mass M of the plate is

$$M = \iint_R \rho(x, y)\, dy\, dx = \int_0^1 \int_0^{2x} 6(x + y + 1)\, dy\, dx$$

$$= 6\int_0^1 \left[xy + \frac{y^2}{2} + y\right]_0^{2x} dx = 6\int_0^1 [2x^2 + 2x^2 + 2x]\, dx$$

$$= 12\int_0^1 (2x^2 + x)\, dx = 12\left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1 = 14.$$

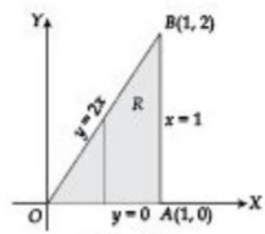


Fig. 7.22

The first moment about x-axis, refer to (7.9), is

$$M_x = \iint_R y\rho(x, y)\, dy\, dx = 6\int_0^1 \int_0^{2x} y(x + y + 1)\, dy\, dx = 6\int_0^1 \left(\frac{xy^2}{2} + \frac{y^3}{3} + \frac{y^2}{2}\right)_0^{2x} dx$$

$$= 6\int_0^1 \left(2x^3 + \frac{8}{3}x^3 + 2x^2\right)dx = 6\int_0^1 \left(\frac{14}{3}x^3 + 2x^2\right)dx = 6\left[\frac{7}{6}x^4 + \frac{2x^3}{3}\right]_0^1 = 11.$$

The first moment about the y-axis, refer to (7.9), is

$$M_y = \iint_R x\rho(x, y)dy\,dx = 6\int_0^1\int_0^{2x} x(x + y + 1)dy\,dx = 6\int_0^1 \left(x^2y + \frac{xy^2}{2} + xy\right)_0^{2x} dx$$

$$= 6\int_0^1 (2x^3 + 2x^3 + 2x^2)dx = 12\int_0^1 (2x^3 + x^2)dx = 12\left(\frac{x^4}{2} + \frac{x^3}{3}\right)_0^1 = 10.$$

Thus the co-ordinates of the centre of mass G $(\bar{x}, \bar{y})$, refer to (7.10), are

$$\bar{x} = \frac{M_y}{M} = \frac{10}{14} = \frac{5}{7}, \quad \bar{y} = \frac{M_x}{M} = \frac{11}{14}.$$

The moment of inertia about x-axis, refer to (7.11), is

$$I_x = \iint_R y^2\rho(x, y)dy\,dx = 6\int_0^1\int_0^{2x} y^2(x + y + 1)dy\,dx = 6\int_0^1 \left[\frac{xy^3}{3} + \frac{y^4}{4} + \frac{y^3}{3}\right]_0^{2x} dx$$

$$= 6\int_0^1 \left(\frac{8x^4}{3} + 4x^4 + \frac{8x^3}{3}\right)dx = 6\int_0^1 \left(\frac{20x^4}{3} + \frac{8x^3}{3}\right)dx = 6\left(\frac{4x^5}{3} + \frac{2x^4}{3}\right)_0^1 = 12.$$

The moment of inertia about y-axis is

$$I_y = \iint_R x^2\rho(x, y)dy\,dx = 6\int_0^1\int_0^{2x} x^2(x + y + 1)dy\,dx = 6\int_0^1 \left(x^3y + \frac{x^2y^2}{2} + x^2y\right)_0^{2x} dx$$

$$= 6\int_0^1 (2x^4 + 2x^4 + 2x^3)dx = 12\int_0^1 (2x^4 + x^3)dx = 12\left(\frac{2x^5}{5} + \frac{x^4}{4}\right)_0^1 = \frac{39}{5}.$$

The moment of inertia about the z-axis is

$$I_0 = \iint_R (x^2 + y^2)\rho(x, y)dy\,dx = I_x + I_y = 12 + \frac{39}{5} = \frac{99}{5}.$$

The radii of gyration about the x-axis, y-axis and z-axis are

$$R_x = \sqrt{I_x/M} = \sqrt{12/14} = \sqrt{6/7}, \quad R_y = \sqrt{I_y/M} = \sqrt{\left(\frac{39}{5}\right)\Big/14} = \sqrt{39/70}$$

and, $R_0 = \sqrt{I_0/M} = \sqrt{\left(\frac{99}{5}\right)\Big/14} = \sqrt{99/70}$ respectively.

EXERCISE 7.3

1. Show that the area enclosed between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $16a^2/3$.
2. Show that the area enclosed by the line $y = x$ and the parabola $y = x^2$ in the first quadrant is $9/2$.
3. Find the area enclosed by the curve $r = a(1 + \cos\theta)$.
4. Find the area enclosed by one loop of the curve $r^2 = a^2 \cos 2\theta$.
5. Find by double integration, the area lying inside the circle $r = a \sin\theta$ and outside the cardiod $r = a(1 - \cos\theta)$.
6. Find the area enclosed by the curve $y = 3x/(x^2 + 2)$ and $y^2 = 4(1 - x)$.
7. Find the centroid of the area of the circle $x^2 + y^2 = a^2$ in the first quadrant.
8. Find the average height of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ above the disc $x^2 + y^2 \le a^2$ in the xy-plane.
9. Find the centroid of the region of constant density in the first quadrant bounded by the x-axis, the parabola $y^2 = 2x$ and the line $x + y = 4$.
10. Find by double integration the centre of gravity of the area of the cardioid $r = a(1 + \cos\theta)$, the density being constant.
11. Find the centroid of a loop of the lemniscate $r^2 = a^2 \cos 2\theta$.
12. A plane in the form of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ is of small but varying thickness, the thickness at any point being proportional to the product of the distances of that point from the axes, show that the co-ordinates of the centroid are $(8a/15, 8b/15)$.
13. Find the centre of mass, moment of inertia and radius of gyration about y-axis of a thin rectangular plate cut from the first quadrant by the lines $x = 6$ and $y = 1$ if the density $\rho(x, y) = x + y + 1$.
14. Find the centre of mass, moments of inertia, radii of gyration about the co-ordinate axes of a thin triangular plate bounded by the lines $y = x$, $y = -x$ and $y = 1$ if density $\rho(x, y) = y + 1$.
15. Find moment of inertia of uniform area bounded by the curve $r^2 = a^2 \cos 2\theta$ about its axis.
16. Find the moment of inertia of a solid right circular cone of uniform density having base radius r and height h about (i) its axis, (ii) an axis through the vertex and perpendicular to its axis. (iii) a diameter of its base.

7.4.3 Area of a Curved Surface: $z = f(x, y)$

Let S' be the projection of the surface $S: z = f(x, y)$, on the xy-plane. Divide the region S' into area elements by drawing lines parallel to the axis of x and y as shown in Fig. 7.23. If the element $PQRS$ of area $\delta x \delta y$ is the projection of the surface element δS, then $\delta x \delta y = \cos\gamma\, \delta S$, where γ is the angle between $\hat{k}$, the normal to the xy-plane, and $\hat{N}$, the outward normal to δS, the elementary surface.

Since the d.c.'s of the normal to S, that is, $f(x, y) - z = 0$ are proportional to $-\dfrac{\partial z}{\partial x}, -\dfrac{\partial z}{\partial y}, 1$ and those to the z-axis are 0, 0, 1,

therefore
$$\cos\gamma = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}},$$

and thus
$$\delta S = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;\delta x\,\delta y.$$

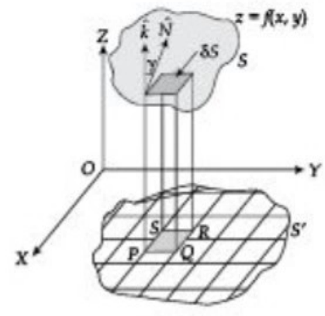


Fig. 7.23

Hence,
$$S = \lim_{\delta S \to 0} \sum \delta S = \iint_{S'} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;dx\,dy, \qquad \text{...(7.15)}$$

where integration is over S', the orthogonal projection of S on the xy–plane.

7.4.4 Volume of the Region Below the Surface $z = f(x, y)$ and Above the xy-plane

Consider a surface $S, z = f(x, y)$ and let S' be the orthogonal projection of S on the xy-plane as shown in Fig. 7.23.

Divide S' into elementary rectangular areas $\delta x\,\delta y$ by drawing lines parallel to x-axis and y-axis. With each of these rectangles as base, erect a prism with lengths parallel to z-axis. Then the volume of this typical prism between S' and the surface $z = f(x, y)$ is $\delta V = z\delta x\,\delta y = f(x, y)\delta x\,\delta y$.

Therefore, the volume of the solid cylinder bounded above by the surface $z = f(x, y)$ and below by S' orthogonal projection of S in the xy-plane, with generators parallel to the z-axis is given by

$$V = \iint_{S'} f(x, y)\,dx\,dy. \qquad \text{...(7.16)}$$

In terms of polar co-ordinates, it is given by

$$V = \iint_{S'} zr\,dr\,d\theta. \qquad \text{...(7.17)}$$

7.4.5 Volume of Solid of Revolution

Let the region R bounded above by $y = f(x)$, below by x-axis and the ordinates $x = a$ and $x = b$, as shown in Fig. 7.24, is revolved about x-axis. We need to determine the volume of the solid generated as such.

Consider an elementary area $PQRS$, where $P = (x, y)$ and $R = (x + \delta x, y + \delta y)$.

The volume of the solid generated by this elementary rectangle $PQRS$ of area $\delta x\,\delta y$ when revolved about x-axis is $\delta V = \pi[(y + \delta y)^2 - y^2]\delta x \approx 2\pi y\,\delta y\,\delta x$, neglecting the second order differentials.

Therefore, the total volume of the solid generated is given by

$$V = 2\pi \iint_R y\,dy\,dx, \quad \text{...(7.18)}$$

where $R = \{(x, y): a \le x \le b,\ 0 \le y \le f(x)\}$.

In case the area is revolved about y–axis, then the volume of the solid generated is

$$V = 2\pi \iint_R x\,dx\,dy. \quad \text{...(7.19)}$$

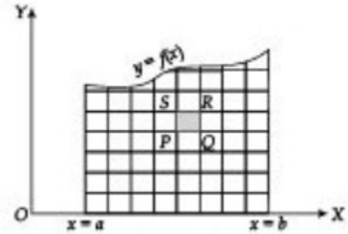


Fig. 7.24

In terms of *polar coordinates*, the corresponding formulae for the volumes of solid generated, when R is revolved about x-axis and y-axis are respectively

$$V = 2\pi \iint_R r\sin\theta . r\,dr\,d\theta = 2\pi \iint_R r^2 \sin\theta\,dr\,d\theta \quad \text{...(7.20)}$$

and, $$V = 2\pi \iint_R r\cos\theta . r\,dr\,d\theta = 2\pi \iint_R r^2 \cos\theta\,dr\,d\theta. \quad \text{...(7.21)}$$

In case the area *is revolved about any line L*, then the volume of the solid generated is

$$V = 2\pi \iint_R d(x, y)\,dx\,dy, \quad \text{...(7.22)}$$

where $d(x, y)$ is the perpendicular distance of an arbitrary point $P(x, y)$ in R from the line L.

Example 7.21: A circular hole of radius b is made centrally through a sphere of radius a, find the volume of the remaining portion of the sphere.

Solution: Let the centre of the sphere be taken as the origin and axis of the hole be taken as the z-axis as shown in Fig. 7.25.

The volume of the upper half of the hole is $\iint_R z\,dx\,dy$,

where $z = \sqrt{a^2 - x^2 - y^2}$ and R is the orthogonal projection of the surface for the hollow portion $z = \sqrt{a^2 - x^2 - y^2}$ in the xy-plane, that is, $R: x^2 + y^2 = b^2$. Hence the volume V_1 of the circular hole is

$$V_1 = 2 \iint_{x^2 + y^2 = b^2} \sqrt{a^2 - x^2 - y^2}\,dx\,dy.$$

Using the polar co-ordinates, we obtain

$$V_1 = 2\int_0^{2\pi}\int_0^b \sqrt{a^2 - r^2}\,r\,dr\,d\theta = 4\pi\int_0^b \sqrt{a^2 - r^2}\,r\,dr$$

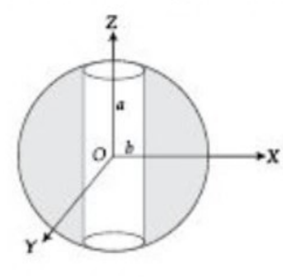


Fig. 7.25

$$= 4\pi\left[-\frac{1}{3}(a^2 - r^2)^{3/2}\right]_0^b = \frac{4\pi}{3}[a^3 - (a^2 - b^2)^{3/2}].$$

Since the volume of the sphere is $V_2 = \frac{4}{3}\pi a^3$, hence the volume V of the remaining portion is

$$V = V_2 - V_1 = \frac{4}{3}\pi a^3 - \frac{4}{3}\pi[a^3 - (a^2 - b^2)^{3/2}] = \frac{4}{3}\pi(a^2 - b^2)^{3/2}.$$

Example 7.22: Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the double integration.

Solution: The required volume V is 8 times the volume in the first octant, thus $V = 8\iint\limits_R z\,dx\,dy$,

where $z = c\sqrt{1 - x^2/a^2 - y^2/b^2}$ and R is the projection of this surface in the xy-plane which is the region in the first quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$. Therefore,

$$V = 8\int_0^a\left(\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dy\right)dx.$$ Substituting $y = b\sqrt{1 - \frac{x^2}{a^2}}\sin\theta$, we get

$$V = 8c\int_0^a\left(\int_0^{\pi/2}\sqrt{1 - \frac{x^2}{a^2}}\cos\theta\left(b\sqrt{1 - \frac{x^2}{a^2}}\cos\theta\right)d\theta\right)dx = 8bc\int_0^a\left(\int_0^{\pi/2}\left(1 - \frac{x^2}{a^2}\right)\cos^2\theta\,d\theta\right)dx$$

$$= 4bc\int_0^a\left\{\left(1 - \frac{x^2}{a^2}\right)\int_0^{\pi/2}(\cos 2\theta + 1)d\theta\right\}dx = 4bc\int_0^a\left(1 - \frac{x^2}{a^2}\right)\left(\frac{\sin 2\theta}{2} + \theta\right)_0^{\pi/2}dx$$

$$= 2\pi bc\int_0^a\left(1 - \frac{x^2}{a^2}\right)dx = 2\pi\,bc\left[x - \frac{x^3}{3a^2}\right]_0^a = \frac{4\pi bc}{3}\text{ cubic unit.}$$

Example 7.23: Find by double integration, the volume generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about its axis $\theta = 0$.

Solution: The volume generated by the revolution of the cardioid about x-axis, that is, about $\theta = 0$ is obtained by considering the area above (or below) the initial line and is given by

$$V = 2\pi\iint\limits_R r^2\sin\theta\,dr\,d\theta,$$

where $R = \{(r, \theta): 0 \le r \le a(1 - \cos\theta),\ 0 \le \theta \le \pi\}$, as shown in Fig. 7.26.

Thus, $V = 2\pi \int_0^{\pi} \left(\int_0^{a(1-\cos\theta)} r^2 dr \right) \sin\theta \, d\theta$

$$= \frac{2\pi}{3} \int_0^{\pi} \left[r^3 \right]_0^{a(1-\cos\theta)} \sin\theta \, d\theta = \frac{2\pi a^3}{3} \int_0^{\pi} (1-\cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \left[\frac{(1-\cos\pi)^4}{4} \right] = \frac{8\pi a^3}{3}.$$

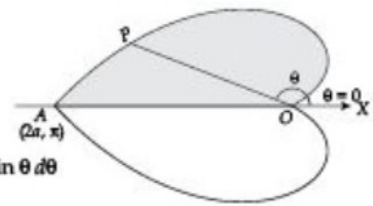


Fig. 7.26

Example 7.24: Find the volume of the solid generated obtained by revolving the circle $x^2 + y^2 = 4$ about the line $x = 3$.

Solution: The required volume is

$$V = \iint_R 2\pi d(x, y) dx\, dy,$$

where $R = \{(x, y): -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, -2 \le x \le 2\}$ is the region enclosed by the circle $x^2 + y^2 = 4$, as shown in Fig. 7.27, and $d(x, y)$ is the perpendicular distance of an arbitrary point $P(x, y)$ in R from the line $x - 3 = 0$, that is, $d(x, y) = |3 - x| = 3 - x$, for $-2 \le x \le 2$. Thus,

$$V = 2\pi \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-x) dx\, dy = 2\pi \int_{-2}^{2} \left\{ (3-x) \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \right\} dx$$

$$= 4\pi \int_{-2}^{2} (3-x)\sqrt{4-x^2}\, dx = 4\pi \left[\int_{-2}^{2} 3\sqrt{4-x^2} - \int_{-2}^{2} \sqrt{4-x^2}\, x\, dx \right]$$

$$= 4\pi \left[3\left\{ \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} \right\} + \frac{1}{3}(4-x^2)^{3/2} \right]_{-2}^{2}$$

$$= 4\pi[3\pi + 3\pi] = 24\pi^2.$$

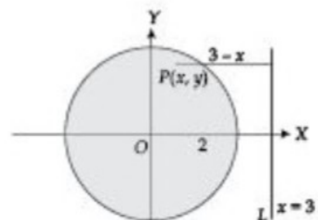


Fig. 7.27

Example 7.25: Show that the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ is $4\pi a^2$.

Solution: The orthogonal projection of the sphere $x^2 + y^2 + z^2 = a^2$ in the xy-plane is the circle $x^2 + y^2 = a^2$.

For the surface $z^2 = a^2 - x^2 - y^2$, we have $\frac{\partial z}{\partial x} = -\frac{x}{z}$, $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

Thus, $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2}{z^2} + \frac{y^2}{z^2} + 1 = \frac{a^2}{a^2 - x^2 - y^2}$. Therefore,

$S = 2\iint\limits_R \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx\,dy$, where R is the circle $x^2 + y^2 = a^2$ in the xy-plane.

Changing to polar co-ordinates $x = r\cos\theta,\ y = r\sin\theta$

$$S = 2\int_0^{2\pi}\int_0^a \frac{a}{\sqrt{a^2 - r^2}} r\,dr\,d\theta = -2\pi a\int_0^a (a^2 - r^2)^{-\frac{1}{2}}(-2r)dr = -4\pi a\left[(a^2 - r^2)^{1/2}\right]_0^a = 4\pi a^2.$$

Example 7.26: Find the area of the portion of the surface of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.

Solution: The projection of the one-fourth of the desired surface area in the xy-plane is the semicircle $x^2 + y^2 = 3y$ in the first quadrant as shown in the Fig. 7.28. For the surface of the sphere $x^2 + y^2 + z^2 = 9$, we have

$$\frac{\partial z}{\partial x} = -\frac{x}{z},\quad \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

Thus $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2 + y^2 + z^2}{z^2} = \frac{9}{9 - x^2 - y^2}$.

Thus the required surface area S is

$$S = 4\iint\limits_R \frac{3}{\sqrt{9 - x^2 - y^2}}\,dx\,dy,$$

where R is the semicircle $x^2 + y^2 = 3y$ in the first quadrant.

Changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the region is

$$R = \left\{(r, \theta) : 0 \le r \le 3\sin\theta,\ 0 \le \theta \le \frac{\pi}{2}\right\}, \text{ and hence}$$

$$S = 4\int_0^{\frac{\pi}{2}}\int_0^{3\sin\theta} \frac{3}{\sqrt{9 - r^2}} r\,dr\,d\theta = -12\int_0^{\pi/2}\left[(9 - r^2)^{\frac{1}{2}}\right]_0^{3\sin\theta} d\theta$$

$$= 36\int_0^{\pi/2}(1 - \cos\theta)d\theta = 18(\pi - 2).$$

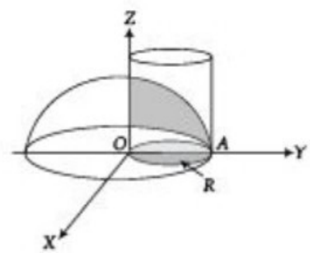


Fig. 7.28

EXERCISE 7.4

1. Find the volume of the sphere $x^2+y^2+z^2=a^2$, using double integration.
2. Find the volume of the region bounded by the surfaces $y=x^2$ and $x=y^2$ and the planes $z=0$, $z=3$.
3. Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.
4. Using double integration find the volume of the tetrahedron bounded by the coordinate planes and the plane $x/a+y/b+z/c=1$ in the first octant.
5. Using double integration show that the volume generated by the revolution of the cardioid $r=a(1+\cos\theta)$ about its axis is $8\pi a^3/3$.
6. The area bounded by the parabola $y^2=4x$ and the straight lines $x=1$ and $y=0$ in the first quadrant is revolved about the line $y=2$. Find by double integral the volume of the solid generated.
7. Find the volume generated by the revolution of the curve $y^2(2a-x)=x^3$ about its asymptote through four right angles.
8. Find the area of the portion of the cylinder $x^2+z^2=4$ lying inside the cylinder $x^2+y^2=4$.
9. Compute the area of that part of the plane $x+y+z=2a$ which lies in the first octant and is bounded by the cylinder $x^2+y^2=a^2$.
10. Find area of the surface of the cylinder $x^2+z^2=a^2$ that lies inside the cylinder $x^2+y^2=a^2$.

7.5 TRIPLE INTEGRALS

Let $f(x, y, z)$ be a continuous and single valued function of x, y and z defined over a closed and bounded region D in space. Subdivide the region D into a number of parallelopipeds by drawing planes parallel to the coordinate planes. Number the parallelopipeds which are inside D in some order say from 1 to n. Choose an arbitrary point (x_k, y_k, z_k) in each ΔV_k, where $\Delta V_k=\Delta x_k\Delta y_k\Delta z_k$ is the volume of the kth parallelopiped, and form the sum

$$S_n=\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta x_k\,\Delta y_k\,\Delta z_k. \qquad ...(7.23)$$

The limit of this sum as $n\to\infty$ and $\Delta V_k\to 0$ is defined as the triple integral of $f(x, y, z)$ over the region D and is denoted by

$$I=\iiint_D f(x, y, z)\,dV, \text{ or } \iiint_D f(x, y, z)\,dx\,dy\,dz. \qquad ...(7.24)$$

Similar to double integrals, here also the continuity of $f(x, y, z)$ is a sufficient condition for the existence of the triple integrals, but not a necessary one. Also triple integrals satisfy the properties similar to that of double integrals.

Further, as in case of double integrals, the triple integrals are also hardly evaluated as the limits of the sums. These are evaluated by three successive integrations. If the region D is given by

$$D=\{(x, y, z): x_1\le x\le x_2,\ y_1(x)\le y\le y_2(x),\ z_1(x, y)\le z\le z_2(x, y)\},$$

then the triple integral is evaluated as

$$\iiint_D f(x, y, z)\,dx\,dy\,dz = \int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,dz\right]dy\right]dx.$$

However, the order of integration depends on the form of the problem given.

Example 7.27: Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2 + y^2 + z^2)\,dx\,dy\,dz.$

Solution: Let

$$I = \int_{-c}^{c}\left(\int_{-b}^{b}\left(\int_{-a}^{a}(x^2 + y^2 + z^2)\,dx\right)dy\right)dz = \int_{-c}^{c}\left(\int_{-b}^{b}\left[\frac{x^3}{3} + y^2x + z^2x\right]_{-a}^{a} dy\right)dz$$

$$= 2\int_{-c}^{c}\left(\int_{-b}^{b}\left[\frac{a^3}{3} + a(y^2 + z^2)\right]dy\right)dz = 2\int_{-c}^{c}\left[\frac{a^3}{3}y + a\left(\frac{y^3}{3} + z^2y\right)\right]_{-b}^{b} dz$$

$$= 4\int_{-c}^{c}\left(\frac{ba^3}{3} + \frac{ab^3}{3} + z^2ba\right)dz = 4\left[\frac{ba^3}{3}z + \frac{ab^3z}{3} + \frac{z^3ba}{3}\right]_{-c}^{c}$$

$$= \frac{8}{3}\left[a^3bc + ab^3c + abc^3\right] = \frac{8abc}{3}(a^2 + b^2 + c^2).$$

Example 7.28: Evaluate the triple integral $\iiint xyz\,dx\,dy\,dz$ over the volume enclosed by three co-ordinate planes and the plane $x + y + z = 1$.

Solution: Let D be the volume enclosed by three co-ordinate planes and the plane $x + y + z = 1$. The plane $x + y + z = 1$ meets the co-ordinate axis in $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$. The projection of the region D on the xy-plane is the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$, as shown in Fig. 7.29.

Hence, $D = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}$ and, therefore, the triple integral is

$$I = \iiint_D xyz\,dx\,dy\,dz = \int_0^1\left(\int_0^{1-x}\left(\int_0^{1-x-y} xyz\,dz\right)dy\right)dx$$

$$= \int_0^1\left(\int_0^{1-x}\left[\frac{xyz^2}{2}\right]_0^{1-x-y} dy\right)dx = \frac{1}{2}\int_0^1\left(\int_0^{1-x} xy(1 - x - y)^2\,dy\right)dx$$

$$= \frac{1}{2}\int_0^1\left(\int_0^{1-x}(xy + x^3y + xy^3 - 2x^2y - 2xy^2 + 2x^2y^2)\,dy\right)dx$$

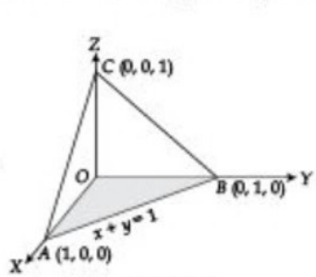


Fig. 7.29

$$= \frac{1}{2}\int_0^1 \left[\frac{xy^2}{2} + \frac{x^3y^2}{2} + \frac{xy^4}{4} - x^2y^2 - \frac{2xy^3}{3} + \frac{2x^2y^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x(1-x)^2}{2} + \frac{x^3(1-x)^2}{2} + \frac{x(1-x)^4}{4} - x^2(1-x)^2 - \frac{2x(1-x)^3}{3} + \frac{2x^2(1-x)^3}{3}\right] dx$$

$$= \frac{1}{24}\int_0^1 (x + 8x^2 - 30x^3 + 32x^4 - 11x^5) dx$$

$$= \frac{1}{24}\left[\frac{x^2}{2} + \frac{8x^3}{3} - \frac{30x^4}{4} + \frac{32x^5}{5} - \frac{11x^6}{6}\right]_0^1 = \frac{1}{24}\left[\frac{1}{2} + \frac{8}{3} - \frac{15}{2} + \frac{32}{5} - \frac{11}{6}\right] = \frac{7}{720}.$$

Example 7.29: Evaluate the triple integral $\iiint_D y\,dx\,dy\,dz$, where D is the region bounded by the surfaces $x = y^2$, $x = y + 2$, $4z = x^2 + y^2$ and $z = y + 3$.

Solution: The variable z varies from $(x^2 + y^2)/4$ to $y + 3$. The projection of D on the xy-plane is the region bounded by the curves $x = y^2$ and $x = y + 2$, which intersect when $y^2 = y + 2$, that is, when $y = -1, 2$. For $-1 \le y \le 2$, we have, $y^2 \le x \le y + 2$, hence, the required region is

$$D = \left\{(x, y, z) : -1 \le y \le 2, y^2 \le x \le y + 2, \frac{x^2 + y^2}{4} \le z \le y + 3\right\}. \text{ Thus,}$$

$$I = \iiint_D y\,dx\,dy\,dz = \int_{-1}^{2}\left(\int_{y^2}^{y+2}\left(\int_{\frac{x^2+y^2}{4}}^{y+3} y\,dz\right)dx\right)dy$$

$$= \int_{-1}^{2}\left(\int_{y^2}^{y+2} y\left(y + 3 - \frac{x^2 + y^2}{4}\right)dx\right)dy = \int_{-1}^{2}\left[\left(y^2 + 3y - \frac{y^3}{4}\right)x - \frac{x^3y}{12}\right]_{y^2}^{y+2} dy$$

$$= \int_{-1}^{2}\left[\left(y^2 + 3y - \frac{y^3}{4}\right)(y + 2 - y^2) - \frac{y}{12}\{(y+2)^3 - y^6\}\right]dy$$

$$= \int_{-1}^{2}\left[\frac{16y}{3} + 4y^2 - 3y^3 - \frac{4y^4}{3} + \frac{y^5}{4} + \frac{y^7}{12}\right]dy$$

$$= \left[\frac{8y^2}{3} + \frac{4y^3}{3} - \frac{3y^4}{4} - \frac{4y^5}{15} + \frac{y^6}{24} + \frac{y^8}{96}\right]_{-1}^{2} = \frac{92}{15} - \frac{433}{480} = \frac{837}{160}.$$

7.6 TRANSFORMATION OF VARIABLES IN TRIPLE INTEGRALS

The method is analogous to double integrals except that here we work in three dimensions instead of two. We define x, y, z as functions of the three new variables u, v, w as $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ having continuous first order partial derivatives.

Suppose that the region D in the xyz-space is tranformed to the region G in the uvw-space, and the function $f(x, y, z)$ becomes $g(u, v, w)$, then

$$\iiint_D f(x, y, z)\,dx\,dy\,dz = \iiint_G g(u, v, w)\,|J|\,du\,dv\,dw, \quad \text{...(7.25)}$$

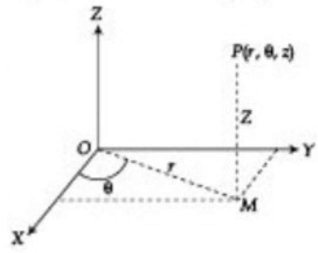


Fig. 7.30

where $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad \text{...(7.26)}$$

is the Jacobian of the variable of transformation.

For example, in case of change to cylindrical co-ordinates r, θ and z from the cartesian co-ordinates x, y and z, refer to Fig. 7.30, we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad \text{...(7.27)}$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r. \quad \text{...(7.28)}$$

Thus, $$\iiint_D f(x, y, z)\,dx\,dy\,dz = \iiint_G g(r, \theta, z)\,r\,dr\,d\theta\,dz. \quad \text{...(7.29)}$$

In case of change to *spherical co-ordinates* r, θ *and* ϕ from the cartesian co-ordinates x, y and z, refer to Fig. 7.31, we have

$$\left.\begin{aligned} x &= r\sin\phi\cos\theta, \\ y &= r\sin\phi\sin\theta, \\ z &= r\cos\phi, \end{aligned}\right\} \quad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi. \quad \text{...(7.30)}$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} = \begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}$$

$$= r^2\sin\phi, \quad \text{...(7.31)}$$

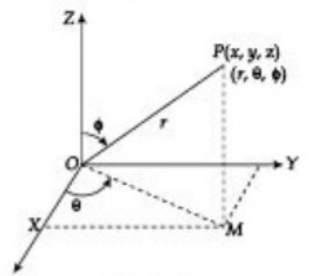


Fig. 7.31

refer to Example 5.23 (ii).

Example 7.30: Evaluate $\int_0^3 \int_0^4 \int_{x=y/2}^{y/2+1} \left(\frac{2x-y}{2}+\frac{z}{3}\right) dx\,dy\,dz$ by applying the transformation $x = u + v$, $y = 2v$, $z = 3w$.

Solution: The region of integration D in the xyz-space is given by

$$D = \{(x, y, z): y/2 \le x \le y/2 + 1,\ 0 \le y \le 4,\ 0 \le z \le 3\}.$$

Under the transformation $x = u + v$, $y = 2v$, $z = 3w$ the region D transforms to the region G in the uvw-space given by $G = \{(u, v, w): 0 \le u \le 1, 0 \le v \le 2, 0 \le w \le 1\}$.

Also,
$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6,$$

and
$$\frac{2x-y}{2} + \frac{z}{3} = \frac{2(u+v)-2v}{2} + \frac{3w}{3} = u + w.$$

Thus the given integral becomes

$$I = \int_0^1 \left(\int_0^2 \left(\int_0^1 (u+w)|J|\,du \right) dv \right) dw = 6\int_0^1 \left(\int_0^2 \left[\frac{u^2}{2} + uw \right]_0^1 dv \right) dw$$

$$= 6\int_0^1 \left(\int_0^2 \left(w + \frac{1}{2} \right) dv \right) dw = 6\int_0^1 \left[wv + \frac{1}{2}v \right]_0^2 dw = 6\int_0^1 (2w+1)\,dw = 6\left[w^2 + w\right]_0^1 = 12.$$

Example 7.31: Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$ using

(i) cartesian co-ordinates,
(ii) cylindrical co-ordinates,
(iii) spherical polar co-ordinates.

Solution: (i) The region of integration is clearly the volume of the sphere $x^2 + y^2 + z^2 = a^2$ in the positive octant. We have

$$I = \int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\int_0^{\sqrt{a^2-x^2-y^2}} dz \right) dy \right) dx = \int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2-y^2} \right) dy \right) dx$$

$$= \int_0^a \left(\int_0^t \left(\sqrt{t^2 - y^2} \right) dy \right) dx, \text{ where } \quad t = \sqrt{a^2 - x^2}$$

$$= \int_0^a \left[\frac{y\sqrt{t^2 - y^2}}{2} + \frac{t^2}{2} \sin^{-1} \frac{y}{t} \right]_0^t dx = \int_0^a \frac{t^2}{2} \sin^{-1}(1) dx$$

$$= \frac{\pi}{4} \int_0^a (a^2 - x^2) dx = \frac{\pi}{4} \left[a^2 x - \frac{x^3}{3} \right]_0^a = \frac{\pi a^3}{6}.$$

(ii) Changing to cylindrical co-ordinates r, θ and z by substituting $x = r\cos\theta, y = r\sin\theta, z = z$, the equation of the sphere $x^2 + y^2 + z^2 = a^2$ becomes $r^2 + z^2 = a^2$. The region of integration, the volume of the sphere $x^2 + y^2 + z^2 = a^2$ in the positive octant transforms to

$$\{(r, \theta, z) : 0 \le r \le a, \; 0 \le \theta \le \pi/2, \; 0 \le z \le \sqrt{a^2 - r^2}\}.$$

The volume element $dx\,dy\,dz$ becomes $|J|\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz$. Thus,

$$I = \int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} dz\,dy\,dx = \int_0^a \int_0^{\pi/2} \int_0^{\sqrt{a^2 - r^2}} r\,dr\,d\theta\,dz = \int_0^a \int_0^{\pi/2} \sqrt{a^2 - r^2}\, r\,dr\,d\theta$$

$$= -\frac{\pi}{4} \int_0^a \sqrt{a^2 - r^2}\,(-2r)dr = -\frac{\pi}{6} \left[(a^2 - r^2)^{3/2} \right]_0^a = -\frac{\pi}{6}[-a^3] = \frac{\pi a^3}{6}.$$

(iii) Changing to spherical polar co-ordinates r, θ and ϕ by substituting

$$x = r \sin\phi \cos\theta, \quad y = r \sin\phi \sin\theta, \quad z = r\cos\phi.$$

the equation of the sphere $x^2 + y^2 + z^2 = a^2$ becomes $r = a$ and region of integration transforms to

$$\{(r, \theta, \phi) : 0 \le r \le a, \; 0 \le \theta \le \frac{\pi}{2}, \; 0 \le \phi \le \frac{\pi}{2}\}.$$

The volume element $dxdydz$ becomes $|J|\,dr\,d\theta\,d\phi = r^2 \sin\phi\,dr\,d\theta\,d\phi$. Thus,

$$I = \int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} dx\,dy\,dz = \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\phi\,dr\,d\theta\,d\phi$$

$$= \left(\int_0^a r^2\,dr \right) \left(\int_0^{\pi/2} d\theta \right) \left(\int_0^{\pi/2} \sin\phi\,d\phi \right) = \left[\frac{r^3}{3} \right]_0^a [\theta]_0^{\pi/2} [-\cos\phi]_0^{\pi/2} = \frac{\pi a^3}{6}.$$

Example 7.32: Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_{\sqrt{x^2+y^2}}^1 \frac{dz\,dy\,dx}{\sqrt{x^2 + y^2 + z^2}}$.

Solution: Changing to spherical polar co-ordinates (r, θ, ϕ), we have $x = r \sin \phi \cos \theta$, $y = r \sin \phi \sin \theta$, $z = r \cos \phi$,

we have $\sqrt{x^2 + y^2 + z^2} = r$, and $dx\,dy\,dz = r^2 \sin \phi \, dr\, d\theta\, d\phi$.

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant. Thus θ varies from 0 to $\pi/2$, ϕ from 0 to $\pi/4$ and r from 0 to $\sec \phi$, as shown in Fig. 7.32.

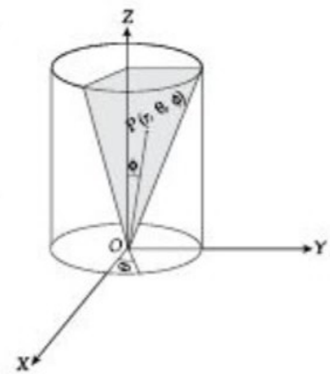


Fig. 7.32

Hence the given integral becomes

$$I = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sec\phi} r \sin \phi \, dr\, d\theta\, d\phi = \left(\int_0^{\pi/2} d\theta\right) \int_0^{\pi/4} \left(\frac{r^2}{2}\right)_0^{\sec\phi} \sin \phi\, d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/4} \sec \phi \tan \phi\, d\phi = \frac{\pi}{4}[\sec \phi]_0^{\pi/4} = \frac{(\sqrt{2}-1)\pi}{4}.$$

EXERCISE 7.5

1. Evaluate the following triple integrals

(a) $\displaystyle\int_0^{\ln 2}\int_0^{x}\int_0^{x+\ln y} e^{x+y+z} dz\, dy\, dx$ (b) $\displaystyle\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx.$

2. Evaluate $\displaystyle\iiint_V \sqrt{x^2+y^2}\, dx\, dy\, dz$, where V is the volume bounded by surfaces $z = 0, z = 1$, $x^2 + y^2 = 1$.

3. Evaluate $\iiint z(x^2 + y^2 + z^2) dx\, dy\, dz$ over the volume of the cylinder $x^2 + y^2 = a^2, z = 0, z = h$ by changing to cylindrical co-ordinates.

4. Evaluate $\displaystyle\int_0^{a}\int_0^{\sqrt{a^2-z^2}}\int_0^{\sqrt{a^2-y^2-z^2}} (x^2 + y^2 + z^2) dx\, dy\, dz.$

5. Evaluate $\displaystyle\int_0^{a}\int_0^{x}\int_0^{x+y} e^{x+y+z} dz\, dy\, dx$ stating precisely the region of integration.

6. Evaluate $\displaystyle\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dx\, dy\, dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar co-ordinates.

7. Evaluate $\iiint x^2 dx\,dy\,dz$ over the volume bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

8. Evaluate $\iiint (x+y+z)$ over the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.

9. Evaluate $\iiint x^2 y^2 z^2 dx\,dy\,dz$ over the volume bounded by $xy=4$, $xy=9$, $yz=1$, $yz=4$, $zx=25$, $zx=49$.

10. Evaluate $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$ over the region bounded by $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, $a>b$.

7.7 APPLICATIONS OF TRIPLE INTEGRALS

Triple integrals are used to calculate the volume, mass, moment, centroid, moment of inertia, etc. in case of objects in three dimensions.

7.7.1 Volume, Mass, the Centre of Mass of the Bounded Regions in Space

In case we take $f(x, y, z) = 1$ in (7.24), then I gives the volume V of the bounded region D in the cartesian co-ordinates. Thus,

$$V = \iiint_D dx\,dy\,dz \qquad \text{...(7.32)}$$

in cylindrical co-ordinates (7.32) becomes

$$V = \iiint_D r\,dr\,d\theta\,dz \qquad \text{...(7.33)}$$

and, in spherical co-ordinates we have

$$V = \iiint_D r^2 \sin\phi\,dr\,d\theta\,d\phi. \qquad \text{...(7.34)}$$

If $f(x, y, z)$ is an integrable function defined over a region D of measurable volume V, then the expression,

$$\frac{1}{V}\iiint_D f(x, y, z)\,dx\,dy\,dz \qquad \text{...(7.35)}$$

is defined as *the average value of* $f(x, y, z)$ over D.

For example, if $f(x, y, z) = \sqrt{x^2+y^2+z^2}$, then (7.35) gives the average distance of D from the orgin (0, 0, 0).

If $\rho(x, y, z)$ is the density function, then

$$M = \iiint_D \rho(x, y, z)\,dx\,dy\,dz \qquad ...(7.36)$$

gives the mass M of the solid bounded by the region D in space.

The expressions

$$M_{yz} = \iiint_D \rho f(x, y, z)\,dx\,dy\,dz, \quad M_{zx} = \iiint_D y\rho(x, y, z)\,dx\,dy\,dz, \quad M_{xy} = \iiint_D z\rho(x, y, z)\,dx\,dy\,dz \quad ...(7.37)$$

are called the *first moments about the co-ordinate planes*; and

$$\bar{x} = M_{yz}/M, \quad \bar{y} = M_{zx}/M, \quad \bar{z} = M_{xy}/M \qquad ...(7.38)$$

give the co-ordinates of the *centre of mass* or *the centroid* of the mass M in D.

7.7.2 Moments of Inertia of the Solid Covering Region *D* in Space

If $\rho(x, y, z)$ is the density function, then

$$I_x = \iiint_D (y^2 + z^2)\rho\,dx\,dy\,dz, \; I_y = \iiint_D (x^2 + z^2)\rho\,dx\,dy\,dz, \quad I_z = \iiint_D (x^2 + y^2)\rho\,dx\,dy\,dz. \qquad ...(7.39)$$

are the *moments of inertia, or the second moments of the mass M in D about x-axis, y-axis and z-axis* respectively.

In general, if $r = (x, y, z)$ is the distance of an arbitrary point (x, y, z) in D from a line L, then

$$I_L = \iiint_D r^2\rho(x, y, z)\,dx\,dy\,dz \qquad ...(7.40)$$

is the *moment of the mass M in D about the line L*.

Example 7.33: Find the volume bounded above by the surface $z = 1 - (x^2 + y^2)$, on the sides by the planes $x = 0$, $y = 0$, $x + y = 1$ and below by the plane $z = 0$.

Solution: The region of integration is $D = \{(x, y, z) : 0 \le z \le 1 - (x^2 + y^2),\ 0 \le y \le 1 - x,\ 0 \le x \le 1\}$.

Hence, the volume bounded by the region D is

$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} dz\,dy\,dx = \int_0^1 \int_0^{1-x} (1 - x^2 - y^2)\,dy\,dx = \int_0^1 \left[y - x^2 y - \frac{y^3}{3} \right]_0^{1-x} dx$$

$$= \int_0^1 \left[(1 - x) - (1 - x)\,x^2 - \frac{(1-x)^3}{3} \right] dx = \left[x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \frac{(1-x)^4}{12} \right]_0^1 = \frac{1}{3}.$$

Example 7.34: Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ay$.

Solution: The volume inside the cylinder bounded by the sphere is twice the volume of the shaded region as shown in Fig. 7.33. Its projection on the xy-plane is the circle $x^2 + y^2 = ay$.

Changing to cylindrical co-ordinates (r, θ, z), we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

The equation of the sphere $x^2 + y^2 + z^2 = a^2$ becomes $r^2 + z^2 = a^2$; and that of the circle $x^2 + y^2 = ay$ in the xy-plane becomes $r = a\sin\theta$. The volume element $dx\,dy\,dz = r\,dr\,d\theta\,dz$ and the region of integration is $D = \{(r, \theta, z): -\sqrt{a^2 - r^2} \le z \le \sqrt{a^2 - r^2},\ 0 \le r \le a\sin\theta,\ 0 \le \theta \le \pi\}$.

Thus volume

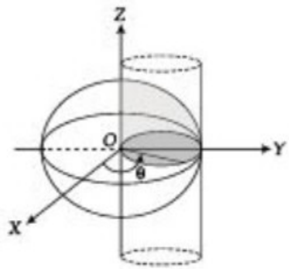


Fig. 7.33

$$V = \int_0^{\pi}\int_0^{a\sin\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = 2\int_0^{\pi}\int_0^{a\sin\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$

$$= -\int_0^{\pi}\int_0^{a\sin\theta}\sqrt{a^2-r^2}\,(-2r)\,dr\,d\theta = -\frac{2}{3}\int_0^{\pi}\left[(a^2-r^2)^{3/2}\right]_0^{a\sin\theta} d\theta$$

$$= \frac{2a^3}{3}\int_0^{\pi}(1-\cos^3\theta)\,d\theta = \frac{2a^3}{3}\int_0^{\pi}\left[1 - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta\right]d\theta$$

$$= \frac{2a^3}{3}\left[\theta - \frac{3}{4}\sin\theta - \frac{1}{12}\sin 3\theta\right]_0^{\pi} = \frac{2\pi a^3}{3} \text{ cubic unit.}$$

Example 7.35: Find the volume of the upper region D cut from the solid sphere $x^2 + y^2 + z^2 \le 1$ by the cone, $\phi = \pi/3$.

Solution: Using the spherical polar coordinates $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, the volume

$$V = \iiint_D r^2\sin\phi\,dr\,d\theta\,d\phi,$$

where D is the region shown in Fig. 7.34 and is given by

$$D = \{(r, \theta, \phi): 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/3\}.$$

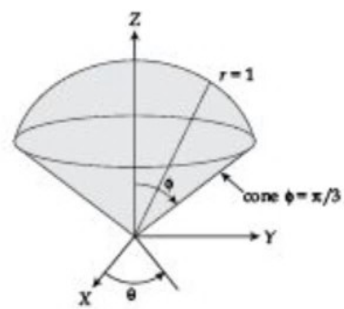


Fig. 7.34

Thus,

$$V = \int_0^{\pi/3}\int_0^{2\pi}\int_0^1 r^2\sin\phi\,dr\,d\theta\,d\phi = \left(\int_0^1 r^2 dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/3}\sin\phi\,d\phi\right)$$

$$= \left(\frac{r^3}{3}\right)_0^1 [\theta]_0^{2\pi} [-\cos\phi]_0^{\pi/3} = \frac{1}{3} \times 2\pi \times \frac{1}{2} = \frac{\pi}{3} \text{ cubic unit.}$$

Example 7.36: A solid fills the region between two concentric spheres of radii a and b, $0 < a < b$. The density at each point is inversely proportional to its square of distance from the origin. Find the total mass.

Solution: The region of integration D is shown as the shaded area in Fig. 7.35. The density $\rho(x, y, z)$ at a point (x, y, z) is given by

$$\rho(x, y, z) = \frac{k}{x^2 + y^2 + z^2},$$ where k is a constant. Thus,

$$M = \iiint_D \frac{k}{x^2 + y^2 + z^2} \, dx\, dy\, dz,$$

where D is given by $a^2 \le x^2 + y^2 + z^2 \le b^2$.

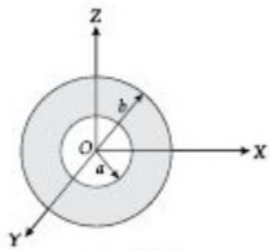


Fig. 7.35

Changing to spherical polar co-ordinates

$x = r \sin\phi \cos\theta$, $y = r \sin\phi \sin\theta$, $z = r\cos\phi$, $a < r < b$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

The volume element $dxdydz = r^2 \sin\phi \, drd\theta d\phi$, and the region of integration is $\{(r, \theta, \phi) : a \le r \le b, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$. Thus,

$$M = k \int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{r^2 \sin\phi}{r^2} \, dr d\phi d\theta = k(b-a)\int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi d\theta = 2k\pi(b-a)\left[-\cos\phi\right]_0^{\pi} = 4k\pi(b-a) \text{ units.}$$

Example 7.37: If the density at any point of the positive octant of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ varies as xyz, find the co-ordinates of the centroid of the solid.

Solution: The density $\rho(x, y, z)$ at a point (x, y, z) is given by $\rho(x, y, z) = kxyz$, where k is a constant. Thus if M is the mass of the solid octant, then

$$M = \iiint_D kxyz\,dx\,dy\,dz,$$

where integration is to be taken over the region D, the positive octant of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Applying the transformation $x = aX$, $y = bY$, $z = cZ$, the given ellipsoid transforms to the sphere $X^2 + Y^2 + Z^2 = 1$ and the volume element $dx\,dy\,dz = abc\,dx\,dy\,dz$, therefore, M becomes

$$M = ka^2b^2c^2 \iiint_{D'} XYZ\,dX\,dY\,dZ,$$

where region D' is the positive octant of the sphere $X^2 + Y^2 + Z^2 = 1$.

On the same lines M_{yz}, the moment about the yz-plane is given by

$$M_{yz} = ka^3b^2c^2 \iiint_{D'} X^2YZ\,dX\,dY\,dZ,$$

and if, $G(\bar{x}, \bar{y}, \bar{z})$ is the centroid of the solid octant, then

$$\bar{x} = \frac{M_{yz}}{M} = \frac{a\iiint_D X^2YZ\,dX\,dY\,dZ}{\iiint_D XYZ\,dX\,dY\,dZ}.$$

Changing to spherical polar co-ordinates $X = r\sin\phi\cos\theta$, $Y = r\sin\phi\sin\theta$, $Z = r\cos\phi$, we have volume element $dX\,dY\,dZ = r^2\sin\phi\,dr\,d\theta\,d\phi$, and thus

$$\bar{x} = \frac{a\int_0^1\int_0^{\pi/2}\int_0^{\pi/2} r^6\sin\theta\cos^2\theta\sin^4\phi\cos\phi\,dr\,d\theta\,d\phi}{\int_0^1\int_0^{\pi/2}\int_0^{\pi/2} r^5\sin\theta\cos\theta\sin^3\phi\cos\phi\,dr\,d\theta\,d\phi}$$

$$= \frac{a\left(\int_0^1 r^6\,dr\right)\left(\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta\right)\left(\int_0^{\pi/2}\sin^4\phi\cos\phi\,d\phi\right)}{\left(\int_0^1 r^5\,dr\right)\left(\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right)\left(\int_0^{\pi/2}\sin^3\phi\cos\phi\,d\phi\right)} = \frac{6a}{7}\,\frac{\left(\frac{1}{3.1}\right)\left(\frac{3.1}{5.3.1}\right)}{\left(\frac{1}{2}\right)\frac{2}{4.2}} = \frac{16a}{35}.$$

Similarly, $\bar{y} = \dfrac{16b}{35}$, $\bar{z} = \dfrac{16c}{35}$, and thus the centroid is $\left(\dfrac{16a}{35}, \dfrac{16b}{35}, \dfrac{16c}{35}\right)$.

Example 7.38: Find the moment of inertia of the mass of the tetrahedron bounded by the co-ordinate planes and the plane $x + y + z = 1$ about the z-axis, assuming the density to be uniform throughout.

Solution: Let ρ be the uniform density then the moment of inertia of the mass of the tetrahedron about the z-axis is $I_z = \rho\iiint_D (x^2 + y^2)\,dx\,dy\,dz$, where integration is over the region D given as

$$D = \{(x, y, z): 0 < x < 1,\ 0 < y < 1 - x,\ 0 < z < 1 - x - y\}.$$

Thus, $I_z = \rho\int_0^1\int_0^{1-x}\int_0^{1-x-y}(x^2 + y^2)\,dz\,dy\,dx = \rho\int_0^1\int_0^{1-x}(x^2 + y^2)(1 - x - y)\,dy\,dx$

$$= \rho\left[\int_0^1\int_0^{1-x}(1-x-y)x^2\,dy\,dx + \int_0^1\int_0^{1-x}(1-x-y)y^2\,dy\,dx\right]$$

$$= \rho\left[\int_0^1\left\{(1-x)y - \frac{y^2}{2}\right\}_0^{1-x} x^2\,dx + \int_0^1\left\{\frac{(1-x)y^3}{3} - \frac{y^4}{4}\right\}_0^{1-x} dx\right]$$

$$= \rho\left[\frac{1}{2}\int_0^1 x^2(1-x)^2\,dx + \frac{1}{12}\int_0^1 (1-x)^4\,dx\right]$$

$$= \rho\left[\frac{1}{2}\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1 + \frac{1}{12}\left[-\frac{(1-x)^5}{5}\right]_0^1\right] = \rho\left(\frac{1}{60}+\frac{1}{60}\right) = \frac{\rho}{30}.$$

Example 7.39: Find the centre of mass of a solid of constant density ρ bounded below by the disk $x^2 + y^2 \le 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$.

Solution: If $G(\bar{x}, \bar{y}, \bar{z})$ is the centre of mass of the solid region shown in Fig. 7.36, then by symmetry

$$\bar{x} = \bar{y} = 0, \text{ and } \bar{z} = \frac{M_{xy}}{M},$$

where $M_{xy} = \iiint_D \rho z\,dx\,dy\,dz$ is the moment about the xy-plane and $M = \iiint_D \rho\,dx\,dy\,dz$ is the mass of the solid region. The region of integration D is

$$\{(x, y, z): 0 \le z \le 4 - x^2 - y^2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\ \ -2 \le x \le 2\}.$$

Thus, $M_{xy} = \rho\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-x^2-y^2} z\,dz\,dy\,dx = \dfrac{\rho}{2}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-x^2-y^2)^2\,dy\,dx.$

Changing to polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$, so that, $dx\,dy = r\,dr\,d\theta$, we have

$$M_{xy} = \frac{\rho}{2}\int_0^{2\pi}\left(\int_0^2 (4-r^2)^2 r\,dr\right)d\theta = -\frac{\rho}{4}\left\{\int_0^2 (4-r^2)^2(-2r)\,dr\right\}\left\{\int_0^{2\pi} d\theta\right\}$$

$$= -\frac{\pi\rho}{6}\left[(4-r^2)^3\right]_0^2 = \frac{32\pi\rho}{3}.$$

Also, $M = \rho \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-x^2-y^2} dz\,dy\,dx$

$$= \rho \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2)dy\,dx = \rho \int_0^{2\pi}\int_0^2 (4-r^2)r\,dr\,d\theta$$

$$= -\frac{\rho}{2}\int_0^{2\pi}\int_0^2 (4-r^2)(-2r)dr\,d\theta = -\frac{\pi\rho}{2}\left[(4-r^2)^2\right]_0^2 = 8\pi\rho.$$

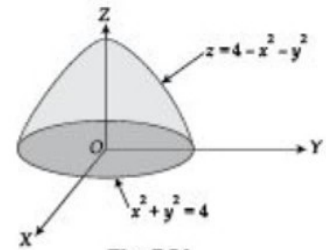


Fig. 7.36

Therefore, $\bar{z} = (M_{xy}/M) = \frac{4}{3}$; thus the centre of mass is $(0, 0, 4/3)$.

Example 7.40: If $f(x, y, z) = xyz$ is the density of a solid cube bounded by the co-ordinate planes $x = 2$, $y = 2$, and $z = 2$ in the first octant, then find the average density of the solid.

Solution: The average density $\bar{\rho}$ of the solid cube is given by

$$\bar{\rho} = \frac{1}{V}\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz,$$

where V is the volume of the cube; which is $2\times2\times2 = 8$ cubic units. Thus,

$$\bar{\rho} = \frac{1}{8}\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = \frac{1}{8}\left(\int_0^2 x\,dx\right)\left(\int_0^2 y\,dy\right)\left(\int_0^2 z\,dz\right) = \frac{1}{8}\left(\frac{x^2}{2}\right)_0^2\left(\frac{y^2}{2}\right)_0^2\left(\frac{z^2}{2}\right)_0^2 = 1 \text{ mass/volume}.$$

EXERCISE 7.6

1. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
2. Find the volume of the region enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.
3. Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$.
4. Find the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = mx$ and $z = nx$, $n > m$.
5. Show that the volume of the solid surrounded by the surface $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1$ is $4\pi abc/35$.
6. Find the volume of the region bounded by the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = b^2$.
7. Find the region between the planes $x + y + 2z = 2$ and $2x + 2y + z = 4$ in the first octant.
8. Find the average distance from a point $P(x, y, z)$ in the cube in the first octant bounded by the co-ordinate planes and the planes $x = 1$, $y = 1$ and $z = 1$.

9. Find the x co-ordinate of the centre of gravity of the solid lying inside the cylinder $x^2 + y^2 = 2ax$ between the plane $z = 0$ and the paraboloid $x^2 + y^2 = az$.
10. Find the centre of mass of the solid hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, if the density at any point is proportional to the distance from the origin.
11. Find the mass of the solid bounded by the planes $x + z = 1$, $x - z = -1$, $y = 0$ and the surface $y = \sqrt{z}$. The density of the solid is $\rho(x, y, z) = 2y + 5$.
12. Find the moment of inertia of a solid right circular cylinder about its axis and about a diameter of the base.
13. Find the centre of gravity of the volume common to the cylinder $x^2 + y^2 = ax$ and the sphere $x^2 + y^2 + z^2 = a^2$ above the plane $z = 0$.
14. Obtain the moment of inertia of the sphere of radius about a diameter in terms of mass M of the sphere.
15. A hemisphere of radius r has a cylindrical hole of radius a drilled through it, the axis of the hole being along the radius normal to the plane face of the hemisphere. Find its radius of gyration about a diameter of this face.

7.8 IMPROPER INTEGRALS AND THEIR CONVERGENCE

In the definite integral $\int_a^b f(x)dx$, in general, we assume two conditions. First, the interval of integration from a to b is finite and second, the integrand $f(x)$ is bounded for all x in $[a, b]$. In practice, we frequently come across problems that fail to meet one or both of these conditions. For example, we might be interested to find the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$, which is an example of infinite domain, such integrals are called *improper integrals*.

7.8.1 Kinds of Improper Integrals

If in the definite integral $\int_a^b f(x)dx$, a or b, or both a and b are infinite, then the integral is called *improper integral of the first kind, or improper integral with infinite limits*. But if $f(x)$ becomes infinite at $x = a$ or $x = b$ or at one or more points within the interval (a, b), then the integral is called *improper integral of the second kind, or improper integral with unbounded integrand*. For example, the integral $\int_1^\infty \frac{dx}{x^p}$, is an improper integral of the first kind, while the integral $\int_0^3 \frac{dx}{(x-1)^{2/3}}$ is an improper integral of the second kind, since the integrand $f(x) = 1/(x-1)^{2/3}$ is unbounded at $x = 1$.

Initially, we shall assume that the integrand $f(x)$ is of the same sign within the range of integration, generally it is assumed that $f(x) \geq 0$; also we will consider $f(x)$ to be continuous over each finite subinterval contained in the range of integration.

7.8.2 Convergence of Improper Integrals of the First Kind

When the limits involved exist, we evaluate such integrals by the following procedure.

(a) If f is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx. \qquad ...(7.41)$$

(b) If f is continuous on $(-\infty, b)$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx. \qquad ...(7.42)$$

(c) If f is continuous on $[a, b]$ and c is any finite constant including zero in (a, b), then

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx + \lim_{b\to\infty}\int_c^b f(x)dx. \qquad ...(7.43)$$

In each case if the limits on the right exist and are finite, we say that the corresponding improper integral *converges* and the limit is the *value* of the improper integral. In case the limit fails to exist we say that the improper integral *diverges*.

Example 7.41: Evaluate the following improper integrals, if they exist.

(a) $\int_{-\infty}^{0} x \sin x\, dx$ (b) $\int_0^{\infty} \frac{dx}{a^2+x^2},\ (a>0)$

(c) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx$ (d) $\int_a^{\infty} \frac{dx}{x^p},\ (a>0),\ p\neq 1.$

Solution: (a) $\int_{-\infty}^{0} x \sin x dx = \lim_{a\to-\infty}\int_a^0 x\sin x dx = \lim_{a\to-\infty}[-x\cos x + \sin x]_a^0$

$$= \lim_{a\to-\infty}(a\cos a - \sin a).$$

Since $\cos a$ and $\sin a$ oscillate between ± 1, thus improper integral diverges to $-\infty$.

(b) $\int_0^{\infty}\frac{dx}{a^2+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{a^2+x^2} = \lim_{b\to\infty}\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^b = \frac{1}{a}\lim_{b\to\infty}\tan^{-1}\frac{b}{a} = \frac{\pi}{2a}.$

Thus, the improper integral converges to $\frac{\pi}{2a}$.

(c) $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} [\tan^{-1} x]_a^0 + \lim_{b\to\infty} [\tan^{-1} x]_0^b = -\lim_{a\to-\infty} \tan^{-1} a + \lim_{b\to\infty} \tan^{-1} b = -(-\pi/2) + \pi/2 = \pi.$$

Thus, the improper integral converges to π.

(d) $$\int_{a}^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty} \int_{a}^{b} \frac{dx}{x^p} = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_a^b = -\lim_{b\to\infty} \left[\frac{b^{-(p-1)}}{p-1}\right] + \frac{a^{1-p}}{p-1}.$$

But, $-\lim_{b\to\infty} \left[\frac{b^{-(p-1)}}{p-1}\right] \to 0$, if $p > 1$ and diverges to $+\infty$, if $p < 1$. Hence,

$$\int_{a}^{\infty} \frac{dx}{x^p} = \frac{a^{1-p}}{p-1}, \text{ if } \quad p > 1.$$

For $p=1$, $\int_{a}^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_{a}^{b} \frac{dx}{x} = \lim_{b\to\infty} [\ln x]_a^b = \lim_{b\to\infty} \ln b - \ln a$, which diverges to ∞ as $b \to \infty$.

Thus, the improper integral converges to $\frac{a^{1-p}}{p-1}$, if $p > 1$.

Example 7.42: Evaluate the following improper integrals with infinite limits.

(a) $\int_{e^2}^{\infty} \frac{dx}{x(\ln x)^3}$ (b) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5}$ (c) $\int_{0}^{\infty} e^{-x} \sin x\, dx.$

Solution: (a) We have,

$$\int_{e^2}^{\infty} \frac{dx}{x(\ln x)^3} = \lim_{b\to\infty} \int_{e^2}^{b} \frac{dx}{x(\ln x)^3} = \lim_{b\to\infty} \left[\frac{1}{-2(\ln x)^2}\right]_{e^2}^{b} = -\left(\frac{1}{2}\right) \lim_{b\to\infty} \left[\frac{1}{(\ln b)^2} - \frac{1}{4}\right] = \frac{1}{8}.$$

(b) We have,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5} = \lim_{a\to-\infty} \int_{a}^{c} \frac{dx}{x^2+2x+5} + \lim_{b\to\infty} \int_{c}^{b} \frac{dx}{x^2+2x+5}, \quad a < c < b$$

$$= \lim_{a\to-\infty} \left[\frac{1}{2} \tan^{-1} \frac{x+1}{2}\right]_a^c + \lim_{b\to\infty} \left[\frac{1}{2} \tan^{-1} \frac{x+1}{2}\right]_c^b$$

$$= \frac{1}{2}\tan\frac{c+1}{2} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{c+1}{2} = \pi/2.$$

(c) We have,

$$\int_0^{\infty} e^{-x}\sin x\,dx = \lim_{b\to\infty}\int_0^{b} e^{-x}\sin x\,dx = \lim_{b\to\infty}\left[-\frac{e^{-x}}{2}(\sin x + \cos x)\right]_0^b$$

$$= -\frac{1}{2}\lim_{b\to\infty}[e^{-b}(\sin b + \cos b) - 1] = \frac{1}{2}.$$

Example 7.43: Evaluate the integral $I = \int_0^{\infty}\frac{x^2}{1+x^4}dx$.

Solution: Apply $x = 1/t$, then I becomes

$$I = \int_0^{\infty}\frac{x^2}{1+x^4}dx = \int_{\infty}^{0}\frac{1/t^2}{1+1/t^4}(-1/t^2 dt) = \int_0^{\infty}\frac{1}{1+t^4}dt.$$

Adding an other integral I to the both sides, we obtain

$$2I = \int_0^{\infty}\frac{1}{1+t^4}dt + \int_0^{\infty}\frac{t^2}{1+t^4}dt = \int_0^{\infty}\frac{1+t^2}{1+t^4}dt = \int_0^{\infty}\frac{1/t^2+1}{t^2+1/t^2}dt.$$

Apply $z = t - \frac{1}{t}$, we obtain

$$I = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dz}{z^2+2} = \frac{1}{2}\left[\lim_{a\to-\infty}\int_a^0\frac{dz}{z^2+2} + \lim_{b\to\infty}\int_0^b\frac{dz}{z^2+2}\right]$$

$$= -\frac{1}{2\sqrt{2}}\lim_{a\to-\infty}\tan^{-1}\frac{a}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\lim_{b\to\infty}\tan^{-1}\frac{b}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

7.8.3 Comparison Tests

It may not always be possible to discuss the convergence or divergence of certain improper integrals directly. For example, the integral $\int_0^{\infty} e^{-x^2}dx$ can't be integrated directly. We introduce some comparison tests which are used to discuss the convergence or divergence of such improper integrals. Though by the applications of test we can't find the value to which the improper integral converges, yet we may be able to find a bound to the integral by applying the comparison tests. In such cases we approximate the integral numerically.

Direct comparison test: *If $f(x)$ and $g(x)$ are continuous on $[a, \infty)$ and $0 \le f(x) \le g(x)$ for all $x \ge a$, then*

1. $\int_a^\infty f(x)dx$ *converges, if* $\int_a^\infty g(x)dx$ *converges,*

2. $\int_a^\infty g(x)dx$ *diverges, if* $\int_a^\infty f(x)dx$ *diverges.*

For example, $\int_1^\infty e^{-x^2}\,dx$ converges, since $0 \le e^{-x^2} \le e^{-x}$ for all $x \ge 1$, and

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}dx = \lim_{b\to\infty}[-e^{-x}]_1^b = -\lim_{b\to\infty}[e^{-b} - e^{-1}] = \frac{1}{e}.$$

Thus, we can say that the value of $\int_1^\infty e^{-x^2}dx$ is less than $\frac{1}{e}$.

Similarly, $\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}$ diverges because $\frac{1}{\sqrt{x^2-0.1}} \ge \frac{1}{x}$ for all $x \ge 1$, and

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}dx = \lim_{b\to\infty}\left[\ln x\right]_1^b = \lim_{b\to\infty}[\ln b - 0] \to \infty.$$

Example 7.44: Test the convergence of the integral $\int_0^\infty x^{100}e^{-0.01x}\,dx$.

Solution: Here $f(x) = x^{100}e^{-0.01x} = \frac{x^{100}}{e^{0.01x}}$

$$= \frac{x^{100}}{1 + (0.01)x + \frac{(0.01x)^2}{2!} + \ldots} < \frac{x^{100}}{\frac{(0.01x)^{102}}{102!}} = \frac{(102)!(10)^{102}}{x^2}.$$

The improper integral $\int_0^\infty \frac{(102)!(10)^{102}}{x^2}dx$ converges (since, $p = 2 > 1$), and hence by the direct comparison test the integral $\int_0^\infty x^{100}e^{-0.01x}dx$ also converges.

Limit comparison test: *If $f(x)$ and $g(x)$ are two positive functions continuous on $[a, \infty)$ and $\lim_{x\to\infty}\frac{f(x)}{g(x)} = l$, $0 < l < \infty$, then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or diverge simultaneously. However, it should be clear that the two integrals do not have the same value in case of convergence. Further, in case $l = 0$, we can conclude only that convergence of $\int_a^\infty g(x)dx$ implies the convergence of $\int_a^\infty f(x)dx$.*

The integral $\int_1^\infty \frac{3}{e^x+5}dx$ converges by the limit comparison test, since $\int_1^\infty \frac{dx}{e^x}$ converges and $\lim_{x\to\infty}\frac{1/e^x}{3/(e^x+5)} = \lim_{x\to\infty}\frac{e^x+5}{3e^x} = \lim_{x\to\infty}\frac{1+5e^{-x}}{3} = \frac{1}{3}$ is a positive finite limit.

Example 7.45: Check for the convergence of the integrals

(a) $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ (b) $\int_1^\infty e^{-x}x^p\,dx$, p is real.

Solution: (a) We have, $0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$ on $[1, \infty)$. But $\int_1^\infty \frac{1}{x^2}dx$ is convergent, therefore, by the direct comparison test $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ is also convergent.

(b) Consider $g(x) = \frac{1}{x^2}$. Then $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{e^{-x}x^p}{x^{-2}} = \lim_{x\to\infty}\frac{x^{(p+2)}}{e^x} = 0.$

But $\int_1^\infty \frac{1}{x^2}dx$ is convergent, therefore, by the limit comparison test $\int_1^\infty e^{-x}x^p dx$ is also convergent.

7.8.4 Convergence Tests for Improper Integrals of the Second Kind

Here we introduce tests for the convergence of the improper integrals of the form $\int_a^b f(x)dx$, where a and b are finite constants but $f(x)$ has infinite discontinuity at $x = a$, or $x = b$, or $x = a$ and $x = b$ both, or $f(x)$ has infinite discontinuities at one or more finite number of points $c_1, c_2, \ldots, c_k$ in (a, b). When the limit(s) involved exist, we evaluate such integrals as follows.

(a) If $f(x)$ has infinite discontinuity at $x = a$, then

$$\int_a^b f(x)dx = \lim_{h\to 0}\int_{a+h}^b f(x)dx. \qquad ...(7.44)$$

(b) If $f(x)$ has infinite discontinuity at $x = b$, then

$$\int_a^b f(x)dx = \lim_{k\to 0} \int_a^{b-k} f(x)dx. \qquad ...(7.45)$$

(c) If $f(x)$ has infinite discontinuity at $x = a$ and $x = b$, both, then for $a < c < b$,

$$\int_a^b f(x)dx = \lim_{h\to 0} \int_{a+h}^{c} f(x)dx + \lim_{k\to 0} \int_c^{b-k} f(x). \qquad ...(7.46)$$

(d) If $f(x)$ has infinite discontinuity at c, $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{k\to 0} \int_a^{c-k} f(x)\,dx + \lim_{h\to 0} \int_{c+h}^{b} f(x)dx \qquad ...(7.47)$$

(e) If $f(x)$ has infinite discontinuities at $c_1, c_2, a < c_1 < c_2 < b$, then

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{b} f(x)dx. \qquad ...(7.48)$$

The improper integrals on the right of (7.48) are evaluated separately on the lines as discussed above. In case the each integral converges separately, we say that the given improper integral on the left of (7.48) converges and its limit is the sum of the limits of the improper integrals on the left. If any of the integral on the left of (7.48) fails to converge, then the given improper integral does not converge.

Example 7.46: Discuss the convergence of the following integrals

(a) $\int_a^b \frac{dx}{(a-x)^2}$ (b) $\int_0^3 \frac{dx}{(x-1)^{2/3}}$ (c) $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$ (d) $\int_0^3 \frac{dx}{3x-x^2}$

Solution: (a) The integrand $f(x) = \frac{1}{(a-x)^2}$ has infinite discontinuity at $x = a$. We have

$$\int_a^b \frac{dx}{(a-x)^2} = \lim_{h\to 0} \int_{a+h}^{b} \frac{dx}{(a-x)^2} = \lim_{h\to 0} \left[\frac{1}{(a-x)}\right]_{a+h}^{b} = \frac{1}{a-b} - \lim_{h\to 0} \frac{1}{(-h)} \to \infty.$$

Hence the improper integral diverges.

(b) The integrand $f(x) = \frac{1}{(x-1)^{2/3}}$ has infinite discontinuity at $x = 1$. We have

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{k\to 0} \int_0^{1-k} \frac{dx}{(x-1)^{2/3}} + \lim_{h\to 0} \int_{1+h}^{3} \frac{dx}{(x-1)^{2/3}}$$

$$= 3\lim_{k\to 0}\left[(x-1)^{1/3}\right]_0^{1-k} + 3\lim_{h\to 0}\left[(x-1)^{1/3}\right]_{1+h}^{3}$$

$$= 3\left[\lim_{k\to 0}\left\{(-k)^{\frac{1}{3}} - (-1)^{\frac{1}{3}}\right\}\right] + 3\left[\lim_{h\to 0}\left\{(2)^{\frac{1}{3}} - (h)^{\frac{1}{3}}\right\}\right] = 3 + 3\,(2)^{1/3}.$$

Thus the improper integral converges to $3\left(1+\sqrt[3]{2}\right)$.

(c) The integrand $f(x) = \dfrac{1}{\sqrt{4-x^2}}$ has infinite discontinuity at $x = 2$. We have

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{k\to 0}\int_0^{2-k}\frac{dx}{\sqrt{4-x^2}} = \lim_{k\to 0}\left[\sin^{-1}\frac{x}{2}\right]_0^{2-k} = \lim_{k\to 0}\sin^{-1}\left(1-\frac{k}{2}\right) = \sin^{-1}1 = \pi/2.$$

Thus the improper integral converges to $\pi/2$.

(d) The integrand $f(x) = \dfrac{1}{3x-x^2}$ has infinite discontinuities both at $x = 0$ and $x = 3$. Take any point, say $x = 2$, inside $(0, 3)$ at which $f(x)$ is defined. We have

$$\int_0^3 \frac{dx}{3x-x^2} = \int_0^2 \frac{dx}{3x-x^2} + \int_2^3 \frac{dx}{3x-x^2} = \lim_{h\to 0}\int_h^2 \frac{dx}{3x-x^2} + \lim_{k\to 0}\int_2^{3-k}\frac{dx}{3x-x^2}$$

$$= \frac{1}{3}\lim_{h\to 0}\left[\ln\left(\frac{x}{3-x}\right)\right]_h^2 + \frac{1}{3}\lim_{k\to 0}\left[\ln\left(\frac{x}{3-x}\right)\right]_2^{3-k}$$

$$= \frac{1}{3}\lim_{h\to 0}\left[\ln 2 - \ln\left(\frac{h}{3-h}\right)\right] + \frac{1}{3}\lim_{k\to 0}\left[\ln\frac{3-k}{k} - \ln 2\right].$$

Since the limits on the right side do not exist therefore, the given improper integral diverges.

Remark. Direct comparison tests and limit comparison tests discussed in case of improper integrals of first kind can also be applied to ascertain the convergence or divergence of the improper integrals of the second kind also.

Example 7.47: Discuss the convergence of the following improper integrals

(a) $\displaystyle\int_1^2 \frac{\sqrt{x}}{\ln x}\,dx$ (b) $\displaystyle\int_0^{\pi/2}\frac{\cos^m x}{x^n}\,dx.$

Solution: The integrand $f(x) = \dfrac{\sqrt{x}}{\ln x}$ has $x = 1$ as its point of infinite discontinuity in the interval

[1, 2]. Also $f(x) = \dfrac{\sqrt{x}}{\ln x} > 0, \quad 1 < x \le 2$. Consider $g(x) = \dfrac{1}{x \ln x}$, then we have

$$\lim_{x \to 1+} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(1+h)}{g(1+h)} = \lim_{h \to 0} (1+h)^{3/2} = 1.$$

Thus improper integrals $\int_1^2 f(x)dx$ and $\int_1^2 g(x)dx$ converge or diverge together. We have

$$\int_1^2 g(x)dx = \int_1^2 \frac{dx}{x \ln x} = \lim_{h \to 0} \int_{1+h}^2 \frac{dx}{x \ln x} = \lim_{h \to 0} [\ln(\ln x)]_{1+h}^2 = \lim_{h \to 0} [\ln(\ln 2) - \ln(\ln(1+h))] = \infty.$$

Therefore, $\int_1^2 \dfrac{dx}{x \ln x}$ is divergent and hence by the limit comparison test the improper integral $\int_1^2 \dfrac{\sqrt{x}}{\ln x} dx$ is also divergent.

(b) The integrand $f(x) = \dfrac{\cos^m x}{x^n}$ has $x = 0$ as its point of infinite discontinuity in the domain $\left[0, \dfrac{\pi}{2}\right]$.

Also, $f(x) = \dfrac{\cos^m x}{x^n} < \dfrac{1}{x^n}$ for $x \in (0, \pi/2]$. Take $g(x) = \dfrac{1}{x^n}$. Consider

$$\int_0^{\pi/2} g(x)dx = \int_0^{\pi/2} \frac{1}{x^n} dx = \lim_{h \to 0} \int_h^{\pi/2} \frac{1}{x^n} dx = \lim_{h \to 0} \left[\frac{x^{-n+1}}{-n+1}\right]_h^{\pi/2}$$

$$= \lim_{h \to 0} \frac{1}{1-n}\left[(\pi/2)^{-n+1} - \frac{1}{h^{n-1}}\right] = \frac{1}{1-n}\left(\frac{\pi}{2}\right)^{1-n}, \text{ for } n < 1.$$

Therefore $\int_0^{\pi/2} g(x)dx$ is convergent for $n < 1$, and hence, by the direct comparison test, $\int_0^{\pi/2} \dfrac{\cos^m x}{x^n} dx$ is also convergent for $n < 1$.

7.8.5 Cauchy Principal Value

We have seen that if in the case of the definite integral $\int_a^b f(x)dx$, the integrand $f(x)$ has infinite discontinuity at an interior point c such that $a < c < b$, then we write

$$\int_a^b f(x)dx = \lim_{k\to 0}\int_a^{c-k} f(x)dx + \lim_{h\to 0}\int_{c+h}^{b} f(x)dx \qquad ...(7.49)$$

where k and h tend to zero independently. It may sometimes happen that the two limits on the right of (7.49) do not converge separately but if we set $h = k$, then we may find a finite answer since the unbounded parts of the two limits cancel out. The value so obtained is called the *Cauchy Principal value* of the integral, written as $pr.v.\int_a^b f(x)dx$.

Example 7.48: Evaluate the principal value of the integral $\int_1^4 \frac{dx}{(x-2)^3}$.

Solution: The integrand $f(x) = \frac{1}{(x-2)^3}$ has a point of infinite discontinuity at $x = 2$ in the domain [1, 4]. Write,

$$\int_1^4 \frac{dx}{(x-2)^3} = \int_1^2 \frac{dx}{(x-2)^3} + \int_2^4 \frac{dx}{(x-2)^3} = \lim_{k\to 0}\int_1^{2-k} \frac{dx}{(x-2)^3} + \lim_{h\to 0}\int_{2+h}^{4} \frac{dx}{(x-2)^3}$$

$$= \lim_{k\to 0}\frac{1}{2}\left(-\frac{1}{(x-2)^2}\right)_1^{2-k} + \lim_{h\to 0}\frac{1}{2}\left(-\frac{1}{(x-2)^2}\right)_{2+h}^{4} = \lim_{k\to 0}\left(\frac{1}{2}-\frac{1}{2k^2}\right) + \lim_{h\to 0}\left(-\frac{1}{8}+\frac{1}{2h^2}\right).$$

Since, both the limits on the right diverge to infinity and hence the given improper integral diverges. But in case we set $k = h$, then we get

$$\int_1^4 \frac{dx}{(x-2)^3} = \lim_{h\to 0}\left\{\frac{1}{2}-\frac{1}{2k^2}-\frac{1}{8}+\frac{1}{2h^2}\right\} = \frac{3}{8}.\ \text{Thus,}\quad pr.v\int_1^4 \frac{dx}{(x-2)^3} = \frac{3}{8}.$$

7.8.6 Absolute Convergence of Improper Integrals

So far we have assumed that $f(x)$ is of the same sign throughout the range of integration. In case $f(x)$ changes sign within the interval of integration, we consider the absolute convergence of the improper integral.

The improper integral $\int_a^b f(x)dx$ is said to be absolutely convergent, if $\int_a^b |f(x)|\,dx$ is convergent. Also, an absolutely convergent improper integral is convergent but the converse may not be true. Further, since $|f|$ is always non-negative within the range of integration thus all the comparison tests discussed earlier to check the convergence of improper integrals may be applied to check the absolute convergence also.

For example, consider the improper integral $\int_2^{\infty} \frac{\sin x}{3x^2+1} dx$. Here the integrand $f(x) = \frac{\sin x}{3x^2+1}$ is not everywhere positive in the interval of integration $[2, \infty]$. But $\left|\frac{\sin x}{3x^2+1}\right| \le \frac{1}{3x^2+1} < \frac{1}{3x^2}$ and both $\frac{1}{3x^2+1}$, and $\frac{1}{3x^2} > 0$ for $x \in [2, \infty]$.

Since, the integral $\int_2^{\infty} \frac{dx}{3x^2}$ is convergent, $(p = 2 > 1)$; thus by direct comparison test $\int_2^{\infty} \frac{dx}{3x^2+1}$, and hence, $\int_2^{\infty} \left|\frac{\sin x}{3x^2+1}\right| dx$ is also convergent. Thus the integral $\int_2^{\infty} \frac{\sin x}{3x^2+1}$ converges absolutely.

Example 7.49: Prove that $I = \int_0^{\infty} \frac{\sin x}{x} dx$ converges conditionally.

Solution: Rewrite I as the sum of the two integrals as

$$I = \int_0^{\infty} \frac{\sin x}{x} dx = \int_0^{\pi/2} \frac{\sin x}{x} dx + \int_{\pi/2}^{\infty} \frac{\sin x}{x} dx. \qquad ...(7.50)$$

The first integral on the right of (7.50) is a proper integral, since $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Next, consider the second integral

$$\int_{\pi/2}^{\infty} \frac{\sin x}{x} dx = \lim_{b \to \infty} \int_{\pi/2}^{b} \frac{\sin x}{x} dx = \lim_{b \to \infty} \left[\left[-\frac{\cos x}{x} \right]_{\pi/2}^{b} - \int_{\pi/2}^{b} \frac{\cos x}{x^2} dx \right] = -\int_{\pi/2}^{\infty} \frac{\cos x}{x^2} dx,$$

$$\text{since } \lim_{b \to \infty} \left[-\frac{\cos x}{x} \right]_{\pi/2}^{b} = 0.$$

The improper integral $\int_{\pi/2}^{\infty} \frac{\cos x}{x^2} dx$ converges absolutely, since $\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$ and the integral $\int_{\pi/2}^{\infty} \frac{dx}{x^2}$ converges. Therefore the integral $\int_{\pi/2}^{\infty} \frac{\sin x}{x} dx$ converges and thus from (7.50), the integral

$\int_0^{\infty} \frac{\sin x}{x} dx$ converges.

Next, $\left|\frac{\sin x}{x}\right| \geq \left|\frac{\sin^2 x}{x}\right| = \frac{1-\cos 2x}{2x}$, but the integral

$$\int_{\pi/2}^{\infty} \frac{1-\cos 2x}{2x} dx = \lim_{b\to\infty}\left[\frac{1}{2}\int_{\pi/2}^{b} \frac{dx}{x} - \frac{1}{2}\int_{\pi/2}^{\infty} \frac{\cos 2x}{x} dx\right] = \frac{1}{2}\left[\lim_{b\to\infty} \ln b - \frac{1}{2}\ln \pi/2 - \frac{1}{2}\int_{\pi/2}^{\infty} \frac{\cos 2x}{x} dx\right]$$

tends to ∞, since the integral $\int_{\pi/2}^{\infty} \frac{\cos 2x}{x} dx$ converges. Thus, the integral $\int_{\pi/2}^{\infty} \left|\frac{1-\cos 2x}{2x}\right| dx$, and, therefore, the integral $\int_{\pi/2}^{\infty} \left|\frac{\sin x}{x}\right| dx$ diverges, and hence, the integral $\int_0^{\infty} \frac{\sin x}{x} dx$ converges conditionally only.

Example 7.50: Prove that the integral $I = \int_0^{\infty} \sin(x^2) dx$ converges.

Solution: Put $x = \sqrt{t}$, we have $I = \int_0^{\infty} \sin(x^2) dx = \frac{1}{2}\int_0^{\infty} \frac{\sin t}{\sqrt{t}} dt$.

Writing the integral on the right as the sum of two integrals as

$$\int_0^{\infty} \frac{\sin t}{\sqrt{t}} dt = \int_0^{\pi/2} \frac{\sin t}{\sqrt{t}} dt + \int_{\pi/2}^{\infty} \frac{\sin t}{\sqrt{t}} dt. \quad ...(7.51)$$

The first integral on the right of (7.51) is convergent, since $\lim_{t\to 0^+} \frac{\sin t}{\sqrt{t}} = 0$. Consider the second integral

$$\int_{\pi/2}^{\infty} \frac{\sin t}{\sqrt{t}} dt = \left[-\frac{\cos t}{\sqrt{t}}\right]_{\pi/2}^{\infty} - \frac{1}{2}\int_{\pi/2}^{\infty} \frac{\cos t\,dt}{t^{3/2}} = -\frac{1}{2}\int_{\pi/2}^{\infty} \frac{\cos t}{t^{3/2}} dt. \quad ...(7.52)$$

The last integral in (7.52) converges absolutely, since $\frac{|\cos t|}{t^{3/2}} \leq \frac{1}{t^{3/2}}$ and the integral $\int_{\pi/2}^{\infty} \frac{1}{t^{3/2}} dt$ converges, hence the integral $\int_{\pi/2}^{\infty} \frac{\sin t}{\sqrt{t}} dt$, and thus I converges.

EXERCISE 7.7

1. Show whether the following integrals converge or diverge.

(a) $\int_0^\infty \frac{dx}{x^4+2}$ (b) $\int_0^\infty \frac{x^{32}\,dx}{x^4+100}$ (c) $\int_4^\infty \frac{\sin^2 x\,dx}{\sqrt{x}(x-1)}$ (d) $\int_0^1 \frac{dx}{x^2 \cos x}$

2. Evaluate the following improper integrals, if they exist

(a) $\int_0^\infty x \sin x\,dx$ (b) $\int_0^\infty e^{-ax} \cos px\,dx,\ a > 0$

(c) $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$ (d) $\int_0^1 \frac{x^p - x^{-p}}{x-1}\,dx.$

3. Enter the change $x = 1/\xi$ in the improper integral $\int_2^\infty \frac{dx}{x^4-2}$; check whether the resultant integral is improper or not.

4. For what range of α does the given integrals converge ?

(a) $\int_0^\infty \frac{dx}{x^\alpha+3}$ (b) $\int_0^\infty \frac{x^\alpha dx}{x+1}$ (c) $\int_1^2 (x^2-1)^\alpha dx$

(d) $\int_1^\infty \frac{dx}{x^\alpha}$ (e) $\int_0^1 \frac{dx}{x^\alpha}.$

5. Discuss the convergence of the integrals

(a) $\int_{-\infty}^\infty xe^{-x^2 dx}$ (b) $\int_{-\pi/2}^{\pi/2} \tan x\,dx$ (c) $\int_{-1}^{+1} \sqrt{\frac{1+x}{1-x}}\,dx$

6. Check for the absolute convergence of the following improper integrals

(a) $\int_0^1 \frac{\sin(1/x)}{x^p}\,dx$ (b) $\int_2^\infty \frac{\sin x}{x(\ln x)^2}\,dx$ (c) $\int_{-\infty}^\infty \frac{\sin 3x}{1+x^4}\,dx$

7. Find the Cauchy principal value of $\int_{-\infty}^\infty \frac{x}{8-x^3}\,dx.$

8. Prove that the following integrals converge

(a) $\int_0^\infty \cos(x^2)dx$ (b) $\int_0^\infty 2x\cos(x^4)dx$.

9. Prove the convergence of the integral $I = \int_0^{\pi/2} \ln(\sin x)dx$ and evaluate it.

10. Prove that the integral $\int_0^\pi \frac{dx}{(\sin x)^k}$ converges if $k < 1$, and diverges if $k \geq 1$.

7.9 THE GAMMA FUNCTION

Euler's gamma function with parameter α, denoted by $\Gamma(\alpha)$, is defined as the integral

$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx, \quad \alpha > 0 \qquad ...(7.53)$$

This integral arises frequently in many science and engineering applications and has been studied extensively.

Convergence of the gamma integral We observe that gamma integral is improper for two reasons, first, the upper limit is ∞ and second, the integrand has infinite discontinuity at $x = 0$, for $0 < \alpha < 1$. To check its convergence or divergence, we rewrite the gamma integral as

$$\int_0^\infty e^{-x}x^{\alpha-1}dx = \int_0^\tau e^{-x}x^{\alpha-1}dx + \int_\tau^\infty e^{-x}x^{\alpha-1}dx, \quad 0 < \tau < \infty$$

$$= I_1 + I_2, \text{ say.}$$

First consider the convergence of the integral I_1, at $x = 0$ and $0 < \alpha < 1$.

Take $f(x) = e^{-x}x^{\alpha-1}$ and $g(x) = x^{\alpha-1}$; we have

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{h\to 0} \frac{f(h)}{g(h)} = \lim_{h\to 0} e^{-h} = 1.$$

Since, $\int_0^\tau g(x)dx = \int_0^\tau \frac{dx}{x^{1-\alpha}}$ converges for $1 - \alpha < 1$, or $\alpha > 0$. Thus I_1 is also convergent for all $\alpha > 0$.

Next, consider the convergence of the integral I_2 at ∞.

Take $f(x) = e^{-x}x^{\alpha-1}$ and $g(x) = 1/x^2$, we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{\alpha+1}}{e^x} = 0, \text{ for all } \quad x \geq \tau.$$

Since $\int_1^\infty \frac{dx}{x^2}$ is convergent, therefore, by the limit comparison test the integral $\int_1^\infty e^{-x} x^{\alpha-1} dx$ also converges for all α.

Hence the gamma integral $\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$ *is convergent for* $\alpha > 0$.

Replacing x by x^2 in (7.53), we obtain

$$\Gamma(\alpha) = \int_0^\infty e^{-x^2} x^{2\alpha-2}(2x)dx = 2\int_0^\infty e^{-x^2} x^{2\alpha-1} dx, \qquad ...(7.54)$$

another form of the gamma function with parameter α.

Integrating (7.53) by parts, we obtain

$$\Gamma(\alpha+1) = \int_0^\infty e^{-x} x^{\alpha} dx = -\left[x^{\alpha} e^{-x}\right]_0^\infty + \alpha \int_0^\infty e^{-x} x^{\alpha-1} dx.$$

Thus, $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$. ...(7.55)

Also $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$. Hence, if α is a positive integer m, then by the repeated applications of (7.55), we obtain

$$\Gamma(m+1) = m!, \quad m = 0, 1, 2, ... \qquad ...(7.56)$$

Thus the gamma function can be regarded as a *generalized factorial function*.

Also from (7.55), we obtain

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha} = \frac{\Gamma(\alpha+2)}{\alpha(\alpha+1)} = ... = \frac{\Gamma(\alpha+k+1)}{\alpha(\alpha+1)...(\alpha+k)}.$$

Thus, $\Gamma(\alpha) = \dfrac{\Gamma(\alpha+k+1)}{\alpha(\alpha+1)...(\alpha+k)}$, $(\alpha \neq 0, -1, -2, ..., -k)$ ...(7.57)

We can use (7.57) to define the gamma function for negative α, $\alpha \neq -1, -2, ...$, choosing k to be the smallest integer such that $\alpha + k + 1 > 0$. In fact, expressions (7.55) and (7.57) may be considered together to give a definition of $\Gamma(\alpha)$ for all α not equal to zero or a negative integer. The graph of $y = \Gamma(\alpha)$ is shown in Fig. 7.37. Clearly it is a continuous function of α for $\alpha > 0$.

Consider, $\Gamma(1/2) = \int_0^\infty e^{-x} x^{\frac{-1}{2}} dx = 2\int_0^\infty e^{-t^2} dt$, where $x = t^2$. Write,

$$[\Gamma(1/2)]^2 = \left[2\int_0^\infty e^{-t^2} dt\right]\left[2\int_0^\infty e^{-s^2} ds\right] = 4\int_0^\infty \int_0^\infty e^{-(s^2+t^2)} ds\, dt$$

$$= 4 \int_{\theta=0}^{\pi/2} \int_{0}^{\infty} r e^{-r^2} dr\, d\theta, \text{ where } s = r \cos\theta, t = r \sin\theta$$

$$= 2\pi \int_0^{\infty} r e^{-r^2} dr = -\pi \int_0^{\infty} e^{-r^2}(-2r) dr = -\pi \left[e^{-r^2} \right]_0^{\infty} = \pi.$$

It gives $\Gamma(1/2) = \sqrt{\pi}$, a value of practical importance.

Also from (7.57) for $k = 0$, $\Gamma(\alpha) = \dfrac{\Gamma(\alpha+1)}{\alpha}$, $\alpha < 0$.

Substituting $\alpha = -\dfrac{1}{2}$, we obtain $\Gamma(-1/2) = -2\sqrt{\pi}$, another value used frequently.

An approximation of the gamma function for large positive α is given by the *stirling formula*

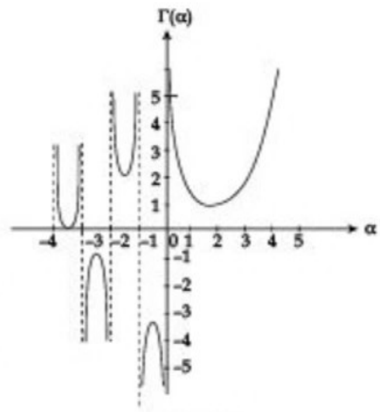


Fig. 7.37

$$\Gamma(\alpha+1) \approx \sqrt{2\pi\alpha}\left(\frac{\alpha}{e}\right)^{\alpha},$$

where e is the base of natural logarithm.

We will find that many integrals which occur in practical applications are not themselves gamma function integrals but can be evaluated by reducing to the gamma functions by making suitable change of variables.

7.10 THE BETA FUNCTION

The beta function with parameters l and m, denoted by $B(l, m)$ is defined as the integral

$$B(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1} dx, \; l > 0, m > 0. \quad ...(7.58)$$

Convergence of the beta function The integral $B(l, m)$ is an improper integral for $0 < l < 1$, $0 < m < 1$. It has points of infinite discontinuity at $x = 0$, when $l < 1$, and at $x = 1$ when $m < 1$.

When $l < 1$ and $m < 1$, take a number $c \in (0, 1)$ and write the improper integral as

$$I = \int_0^1 x^{l-1}(1-x)^{m-1} dx = \int_0^c x^{l-1}(1-x)^{m-1} dx + \int_c^1 x^{l-1}(1-x)^{m-1} dx = I_1 + I_2, \text{ say.}$$

Integral $I_1 = \int_0^c x^{l-1}(1-x)^{m-1} dx$ is improper because $x = 0$ is a point of infinite discontinuity, and integral $I_2 = \int_c^1 x^{l-1}(1-x)^{m-1} dx$ is improper because $x = 1$ is a point of infinite discontinuity.

First, consider the convergence of I_1 at $x = 0$, $0 < l < 1$. The integrand is $f(x) = x^{l-1}(1-x)^{m-1}$ Consider another function $g(x) = x^{l-1}$, we have

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} (1-x)^{m-1} = 1.$$

The integral $\int_0^c g(x)dx = \int_0^c \frac{dx}{x^{1-l}}$ is convergent when $1 - l < 1$, or $l > 0$.

Therefore by limit comparison test $\int_0^c f(x)dx$, that is, I_1 is also convergent for $l > 0$.

Next consider the convergence of I_2 at $x = 1$, for $0 < m < 1$. The integrand is $f(x) = x^{l-1}(1-x)^{m-1}$.

Consider another function $g(x) = (1-x)^{m-1}$. We have, $\lim_{x \to 1^-} \frac{f(x)}{g(x)} = \lim_{x \to 1^-} x^{l-1} = 1$.

The integral $\int_c^1 g(x)dx = \int_c^1 \frac{dx}{(1-x)^{1-m}} = \int_0^{1-c} \frac{dx}{x^{1-m}}$ is convergent, when $1 - m < 1$, or $m > 0$.

Therefore, by limit comparison test, $I_2 = \int_c^1 f(x)dx$ is also convergent for $m > 0$.

Hence the beta integral $B(l, m)$ converges for $l > 0$, $m > 0$.

By substituting $x = (1 - t)$ in (7.58), it is very easy to see that the *beta function is symmetric with respect to its parameters l, m*, that is,

$$B(l, m) = B(m, l). \qquad ...(7.59)$$

Next, substituting $x = \sin^2\theta$ so that, $dx = 2\sin\theta\cos\theta d\theta$, in (7.58), we obtain

$$B(l, m) = 2\int_0^{\pi/2} \sin^{2l-1}\theta \cos^{2m-1}\theta \, d\theta, \qquad ...(7.60)$$

another form of the beta function.

Relation between the beta and the gamma function The relation is

$$B(l, m) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}, \; l > 0, \; m > 0. \qquad ...(7.61)$$

To prove (7.61), consider

$\Gamma(l) = \int_0^\infty e^{-x} x^{l-1} dx = 2\int_0^\infty e^{-t^2} t^{2l-1} dt$, where $x = t^2$. Similarly, $\Gamma(m) = 2\int_0^\infty e^{-s^2} s^{2m-1} ds$. Therefore,

$$\Gamma(l)\Gamma(m) = \left(2\int_0^\infty e^{-t^2} t^{2l-1} dt\right)\left(2\int_0^\infty e^{-s^2} s^{2m-1} ds\right) = 4\int_0^\infty\int_0^\infty e^{-(s^2+t^2)} s^{2m-1} t^{2l-1} ds\,dt. \qquad ...(7.62)$$

Changing to polar co-ordinates $s = r\cos\theta$, $t = r\sin\theta$ so that $ds\,dt = r\,dr\,d\theta$, (7.62) becomes

$$\Gamma(l)\Gamma(m) = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}.r^{2(l+m)-1}\cos^{2m-1}\theta\,\sin^{2l-1}\theta\,d\theta\,dr$$

$$= \left[2\int_0^{\pi/2}\sin^{2l-1}\theta\cos^{2m-1}\theta\,d\theta\right]\left[2\int_0^{\infty}e^{-r^2}r^{2(l+m)-1}dr\right]$$

$= B(l, m)\,\Gamma(l+m)$, using (7.60) and (7.54). This gives (7.61).

Next, substituting $l = m = \frac{1}{2}$ in (7.61), we obtain

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)} = \pi, \qquad \text{...(7.63)}$$

and, for $l = m = 1$, we have

$$B(1, 1) = \frac{\Gamma(1)\,\Gamma(1)}{\Gamma(2)} = 1. \qquad \text{...(7.64)}$$

The results (7.63) and (7.64) can be obtained from (7.60) also.

Another form of the beta function $B(l, m)$ is

$$B(l, m) = \int_0^{\infty}\frac{x^{l-1}}{(1+x)^{l+m}}\,dx, \qquad \text{...(7.65)}$$

obtained from (7.58) by substituting $x = t/(1+t)$.

The beta function has its applications in statistics and also in science and engineering because of its close relation to the gamma function. Also a few definite integrals of some trigonometric function, which arise in some practical problems, can be evaluated in terms of beta function. For example, set $p = 2l - 1$ and $q = 2m - 1$ in (7.60), we obtain

$$\int_0^{\pi/2}\sin^p x\cos^q x\,dx = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right), \qquad \text{...(7.66)}$$

an important result to remember.

In particular, set $p = n$ and $q = 0$, we obtain

$$\int_0^{\pi/2}\sin^n x\,dx = \frac{1}{2}B\left(\frac{n+1}{2},\frac{1}{2}\right). \qquad \text{...(7.67)}$$

Similarly, $$\int_0^{\pi/2}\cos^n dx = \frac{1}{2}B\left(\frac{1}{2},\frac{n+1}{2}\right). \qquad \text{...(7.68)}$$

Example 7.51: Express the following integrals in terms of gamma functions

(a) $\int_0^\infty x^{2/3}e^{-\sqrt{x}}dx$ (b) $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ (c) $\int_0^\infty a^{-bx^2}dx$ (d) $\int_0^1 x^{3/2}(1-\sqrt{x})^{1/2}dx.$

Solution: (a) Consider $I = \int_0^\infty x^{2/3}e^{-\sqrt{x}}dx$. Set $\sqrt{x} = t$, I becomes

$$I = \int_0^\infty (t^2)^{2/3}e^{-t}\cdot 2tdt = 2\int_0^\infty e^{-t}t^{7/3}dt = 2\Gamma(10/3) = 2\,\frac{7}{3}\,\frac{4}{3}\,\frac{1}{3}\,\Gamma(1/3) = \frac{56}{27}\Gamma(1/3).$$

(b) Consider $I = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$. Set $x^2 = \sin\theta$, it becomes

$$I = \int_0^{\pi/2} \frac{\frac{1}{2}(\sin\theta)^{-\frac{1}{2}}\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} = \frac{1}{2}\int_0^{\pi/2}(\sin\theta)^{-\frac{1}{2}}d\theta = \frac{1}{2}B\left(\frac{1}{4},\frac{1}{2}\right), \text{ using (7.67)}$$

$$= \frac{1}{2}\,\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)} = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(1/4)}{\Gamma(3/4)}.$$

(c) Consider $I = \int_0^\infty a^{-bx^2}dx = \int_0^\infty e^{-(b\ln a)x^2}dx.$

Set $(b\ln a)x^2 = t$, so that, $2(b\ln a)\,xdx = dt$, thus I becomes

$$I = \frac{1}{2\sqrt{b\ln a}}\int_0^\infty e^{-t}t^{-1/2}dt = \frac{\Gamma(1/2)}{2\sqrt{b\ln a}} = \frac{\sqrt{\pi}}{2\sqrt{b\ln a}}.$$

(d) Consider $I = \int_0^1 x^{3/2}(1-\sqrt{x})^{1/2}dx$. Set $\sqrt{x} = t$, it becomes

$$I = 2\int_0^1 t^3(1-t)^{1/2}t\,dt = 2\int_0^1 t^4(1-t)^{1/2}dt = 2B(5, 3/2)$$

$$= 2\frac{\Gamma(5)\,\Gamma(3/2)}{\Gamma(13/2)} = 2\frac{4!(1/2)\sqrt{\pi}}{\frac{11}{2}\,\frac{9}{2}\,\frac{7}{2}\,\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}} = \frac{512}{3465}.$$

Example 7.52: Given $\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}, 0 < n < 1$, show that $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$.

Solution: Consider $\Gamma(n)\,\Gamma(1-n) = B(n, 1-n) = B(1-n, n)$

$$= \int_0^1 x^{-n}(1-x)^{n-1}dx. \qquad ...(7.69)$$

Set $x = \frac{1}{1+y}$. It gives $dx = -\frac{dy}{(1+y)^2}$. Thus (7.69) becomes

$$\Gamma(n)\,\Gamma(1-n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{-n}\left(\frac{y}{1+y}\right)^{n-1}\left(-\frac{dy}{(1+y)^2}\right) = \int_0^\infty \frac{y^{n-1}}{1+y}dy = \frac{\pi}{\sin n\pi}.$$

Example 7.53: Show that

(a) $\int_0^1 x^m(\ln x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$, where $n > 0$ is an integer and $m > -1$.

(b) $\int_0^1 y^{n-1}\left(\ln\frac{1}{y}\right)^{m-1} dy = \frac{\Gamma(m)}{n^m}$, where $m, n > 0$.

Solution: (a) Consider $I = \int_0^1 x^m(\ln x)^n dx$. Set $\ln x = t$, so that, $x = e^t$ and $dx = e^t dt$, thus I becomes

$$I = \int_{-\infty}^0 e^{mt} t^n e^t dt = \int_{-\infty}^0 e^{(m+1)t} t^n dt.$$

Further setting $(m+1)\,t = -\tau$, we obtain

$$I = \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-\tau}\tau^n d\tau = \frac{(-1)^n\,\Gamma(n+1)}{(m+1)^{n+1}} = \frac{(-1)^n\, n!}{(m+1)^{n+1}}.$$

(b) Consider $I = \int_0^1 y^{n-1}\left(\ln\frac{1}{y}\right)^{m-1} dy$. Set $\ln\frac{1}{y} = t$, so that, $y = e^{-t}$, thus I becomes

$$I = -\int_\infty^0 e^{-(n-1)t} t^{m-1} e^{-t} dt = \int_0^\infty e^{-nt} t^{m-1} dt.$$

Further setting $nt = \tau$, we obtain

$$I = \int_0^\infty \frac{e^{-\tau}\tau^{m-1}}{n^{m-1}}\frac{d\tau}{n} = \frac{1}{n^m}\int_0^\infty e^{-\tau}\tau^{m-1}d\tau = \frac{\Gamma(m)}{n^m}.$$

Example 7.54: Show that $\int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$.

Solution: Consider, $I_1 = \int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}}dx$. Set $x^2 = \sin\theta$, I_1 becomes

$$I_1 = \frac{1}{2}\int_0^{\pi/2} \frac{\sin\theta\cos\theta}{\cos\theta\sqrt{\sin\theta}}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta = \frac{1}{4}B\left(\frac{3}{4},\frac{1}{2}\right).$$

Next consider, $I_2 = \int_0^1 \frac{dx}{\sqrt{(1+x^4)}}$. Set $x^2 = \tan\theta$, I_2 becomes

$$I_2 = \frac{1}{2}\int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{\sqrt{\tan\theta}\sec\theta} = \frac{1}{2}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\sin\theta\cos\theta}}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\sin 2\theta}} = \frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}} = \frac{1}{4\sqrt{2}}B\left(\frac{1}{4},\frac{1}{2}\right).$$

Thus $\int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{16\sqrt{2}}B\left(\frac{3}{4},\frac{1}{2}\right)B\left(\frac{1}{4},\frac{1}{2}\right)$

$$= \frac{1}{16\sqrt{2}}\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(5/4)}\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)} = \frac{\pi}{4\sqrt{2}}.$$

Example 7.55: Show that

(a) $\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2-b^2}{(a^2+b^2)^2}$ (b) $\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}$

Solution: Let $I = \int_0^\infty xe^{-ax}\cos bx\,dx + i\int_0^\infty xe^{-ax}\sin bx\,dx = \int_0^\infty xe^{-ax}(\cos bx + i\sin bx)dx$

$$= \int_0^\infty xe^{-ax}e^{ibx}dx = \int_0^\infty xe^{-(a-ib)x}dx = \frac{\Gamma(2)}{(a-ib)^2}, \text{ since } \int_0^\infty xe^{-mx}dx = \frac{\Gamma(2)}{m^2}$$

$$= \frac{1}{(a-ib)^2} = \frac{(a+ib)^2}{(a-ib)^2(a+ib)^2} = \frac{a^2-b^2+2iab}{(a^2+b^2)^2} = \frac{a^2-b^2}{(a^2+b^2)^2} + i\frac{2ab}{(a^2+b^2)^2}$$

Equating the real and imaginary parts on both sides, we obtain

$$\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2+b^2}{(a^2+b^2)^2} \text{ and } \int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}.$$

Example 7.56: If l, m are positive real numbers then

$$\iint_D x^{l-1}y^{m-1}dx\,dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}, \quad ...(7.70)$$

where D is the domain $x \geq 0$, $y \geq 0$ and $x + y \leq 1$.

Solution: The region of integration is $D = \{(x, y): 0 \leq y \leq 1-x,\ 0 \leq x \leq 1\}$.

$$\text{Let } I = \iint_D x^{l-1}y^{m-1}dx\,dy = \int_0^1\left(\int_0^{1-x} x^{l-1}\,y^{m-1}\,dy\right)dx = \frac{1}{m}\int_0^1 x^{l-1}(1-x)^m dx = \frac{1}{m}B(l, m+1)$$

$$= \frac{1}{m}\frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)} = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}.$$

Example 7.57: If l, m, n are positive real numbers, then

$$\iiint_D x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}, \quad ...(7.71)$$

where D is the region $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y + z \leq 1$.

Solution: The region of integration is $D = \{(x, y, z): 0 \leq z \leq 1-x-y,\ 0 \leq y \leq 1-x,\ 0 \leq x \leq 1\}$.

$$\text{Let, } I = \iiint_D x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^{l-1}y^{m-1}z^{n-1}dz\,dy\,dx.$$

$$\text{Consider } I' = \int_0^{1-x}\int_0^{1-x-y} y^{m-1}z^{n-1}dz\,dy. \text{ Set } 1-x = h,\ I' \text{ becomes}$$

$$I' = \int_0^h\int_0^{h-y} y^{m-1}z^{n-1}dz\,dy = \frac{1}{n}\int_0^h y^{m-1}(h-y)^n dy.$$

Next, set $y = hY$, I' becomes

$$I' = \frac{1}{n}\int_0^1 (hY)^{m-1}h^n(1-Y)^n h\,dY = \frac{h^{m+n}}{n}\int_0^1 Y^{m-1}(1-Y)^n dY$$

$$= \frac{h^{m+n}}{n} B(m, n+1) = \frac{(1-x)^{m+n}}{n} B(m, n+1).$$

Thus $$I = \frac{B(m, n+1)}{n} \int_0^1 x^{l-1}(1-x)^{m+n} dx$$

$$= \frac{B(m, n+1)}{n} B(l, m+n+1) = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}.$$

Remark: The integral (7.70) is known as *Dirichlet's integral for two variables*, and the integral (7.71) is called *Dirichlet's integral for three variables*. Dirichlet's integrals are used to evaluate some specific area and volume integrals, as illustrated in the example considered next.

Example 7.58: Evaluate $\iiint_D xyz\,dx\,dy\,dz$, where D is the region enclosed by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Solution: Since the region of integration $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$ is symmetrical in all the eight octants, therefore the given integral is

$$I = 8 \iiint_{D'} xyz\,dx\,dy\,dz,$$

where D' is given as $x \ge 0$, $y \ge 0$, $z \ge 0$ and $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$.

Set $x^2/a^2 = u$, $y^2/b^2 = v$, and $z^2/c^2 = w$, so that, $2xdx = a^2 du$, $2ydy = b^2 dv$, and $2zdz = c^2\,dw$. The integral I becomes

$$I = a^2 b^2 c^2 \iiint_{D''} du\,dv\,dw,$$

where D'' is given as $u \ge 0$, $v \ge 0$, $w \ge 0$ and $u + v + w \le 1$. Rewriting I as,

$$I = a^2 b^2 c^2 \iiint_{D''} u^{1-1} v^{1-1} w^{1-1} du\,dv\,dw = a^2 b^2 c^2 \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(4)}, \text{ using (7.71)}$$

$$= \frac{1}{6} a^2 b^2 c^2.$$

EXERCISE 7.8

1. Evaluate the following improper integrals in terms of gamma function.

(a) $\int_0^\infty \sqrt{x}\, e^{-x^3} dx$ (b) $\int_0^\infty e^{-x^3} dx$ (c) $\int_{-\infty}^\infty e^{-x^2} dx$ (d) $\int_0^\infty \frac{x^a}{a^x} dx.$

2. Evaluate the following integrals using the gamma and beta functions.

(a) $\int_0^1 x^m(1-x^n)^p dx$ (b) $\int_0^{\pi/2} \sin^7\theta\cos^5\theta\, d\theta$ (c) $\int_0^{\pi/2} \sin^{10}\theta\, d\theta$

(d) $\int_0^a \frac{x^{3/2}}{\sqrt{a^2-x^2}} dx$ (e) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$ (f) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

(g) $\int_0^\infty e^{-ax}x^{m-1}\sin bx\, dx.$

3. Prove that

(a) $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

(b) $\int_{-1}^1 (1-x^2)^n dx = \frac{2^{2n+1} + (n!)^2}{(2n+1)!}$, n is a positive integer

(c) $\int_0^\infty x^m e^{-\alpha x^n} dx = \frac{1}{n\alpha^{(m-1)/n}} \Gamma\left(\frac{m+1}{n}\right) \quad m, n, \alpha > 0$

(d) $\int_0^m x^n\left(1-\frac{x}{m}\right)^{m-1} dx = m^{n+1}B(m, n+1),\ m, n > 0.$

4. Prove that

(a) $B(m, n) = B(m+1, n) + B(m, n+1)$

(b) $\frac{B(m+1, n)}{m} = \frac{B(m, n+1)}{m} = \frac{B(m, n)}{m+n}$

(c) $B(l, m) = \int_0^1 \frac{x^{l-1} + x^{m-1}}{(1+x)^{l+m}} dx.$

5. Show that

(a) $\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma\left(n+\frac{1}{2}\right)\Gamma(n)$, and hence, $\Gamma(1/4)\,\Gamma(3/4) = \pi\sqrt{2}$

(b) $B(n, n) = \dfrac{\Gamma(n)\sqrt{\pi}}{2^{2n-1}\,\Gamma\left(n+\frac{1}{2}\right)}$.

6. Show that $\iint x^{m-1}y^{n-1}\,dx\,dy$ over the positive quadrant of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\dfrac{a^m b^n}{2n}\,B\left(\dfrac{m}{2}, \dfrac{n}{2}+1\right)$.

7. Evaluate the integral $\iiint\limits_D x^{p-1}y^{q-1}z^{r-1}\,dx\,dy\,dz$, $p, q, r > 0$, where D is the region of the tetrahedron bounded by $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z \leq 1$.

8. The plane $x/a + y/b + z/c = 1$ meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron $OABC$. Also find its mass if the density at any point is $kxyz$, where k is a constant.

9. Evaluate $\iiint\limits_D x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz$, using Dirichlet's integrals where the region of integration D is given by $x, y, z \geq 0$ and $(x/a)^p + (y/b)^q + (z/c)^r \leq 1$.

7.11 THE ERROR FUNCTION

The *error function of x*, denoted by $erf(x)$, is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt. \qquad ...(7.72)$$

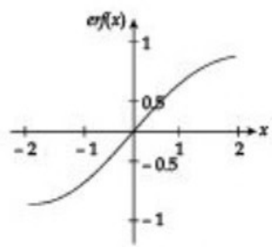


Fig. 7.38

The graph of $erf(x)$ is shown in Fig. 7.38.
It shows that it is odd in x.
In the series form, it can be expressed as

$$erf(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{1!3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} + \ldots\right]. \qquad ...(7.73)$$

The *complementary error function*, denoted by $erf_c(x)$, is defined as

$$erf_c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt. \qquad ...(7.74)$$

Set $t^2 = u$ in (7.72), we obtain

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^{x^2} e^{-u} \frac{1}{2} u^{-1/2} du = \frac{1}{\sqrt{\pi}} \int_0^{x^2} u^{-1/2} e^{-u} du. \quad ...(7.75)$$

This is *another form of the error function*.

The error functions arise in the theory of probability and solution of some partial differential equations and find applications in physics and various engineering disciplines. Next, we study some properties of the error function.

Properties of Error Function

1. $erf(0) = 0$ and $erf(\infty) = 1$. The result $erf(0) = 0$ is obvious. For $erf(\infty) = 1$, from (7.75)

$$erf(\infty) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} u^{-1/2} e^{-u} du = \frac{1}{\sqrt{\pi}} \Gamma(1/2) = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1.$$

2. $erf(x) + erf_c(x) = 1$. Consider

$$erf(x) + erf_c(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt + \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt = 1, \quad \text{refer to (7.54)}$$

3. $erf(-x) = -erf(x)$. By definition, $erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2} dt$. Set $t = -y$, we have

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-y^2} (-dy) = -\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-y^2} dy = -erf(x).$$

4. Derivative of error function: $\frac{d}{dx}[erf(ax)] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$. By definition

$erf(ax) = \frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-t^2} dt$. Differentiating under the integral sign, using Leibnitz's rule,

$$\frac{d}{dx} erf(ax) = \frac{2}{\sqrt{\pi}} \left[\int_0^{ax} \frac{\partial}{\partial x}(e^{-t^2}) dt + \frac{d}{dx}(ax) e^{-a^2x^2} - \frac{d}{dx}(0)\, 1 \right] = \frac{2}{\sqrt{\pi}} [a e^{-a^2x^2}] = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}.$$

5. Integral of error function: $\int_0^t erf(ax) dx = t\, erf(at) + \frac{1}{a\sqrt{\pi}} \left[e^{-a^2t^2} - 1 \right]$.

$$\int_0^t erf(ax) dx = [x\, erf(ax)]_0^t - \int_0^t x \left(\frac{d}{dx} erf(ax) \right) dx = t\, erf(at) - \int_0^t x \frac{2a}{\sqrt{\pi}} e^{-a^2x^2} dx$$

$$= t\, erf(at) - \frac{a}{\sqrt{\pi}} \int_0^{t^2} e^{-a^2 x}\, dx, \quad \text{(replacing } x^2 \text{ by } x\text{)}$$

$$= t\, erf(at) - \frac{a}{\sqrt{\pi}} \left[\frac{e^{-a^2 x}}{-a^2} \right]_0^{t^2} = t\, erf(at) + \frac{1}{a\sqrt{\pi}} \left(e^{-a^2 t^2} - 1 \right).$$

EXERCISE 7.9

1. Show that

(a) $erf_c(-x) + erf_c(x) = 2$ (b) $\int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2} [erf(b) - erf(a)]$

(c) $\int_0^\infty e^{-x^2 - 2ax} dx = \frac{\sqrt{\pi}}{2} e^{a^2} [1 - erf(a)]$.

2. Show that

(a) $\frac{d}{dx} [erf_c(ax)] = -\frac{2a}{\sqrt{\pi}} e^{-a^2 x^2}$ (b) $\int_0^t erf_c(ax) dx = t\, erf_c(at) \frac{1}{a\sqrt{\pi}} \left[e^{-a^2 t^2} - 1 \right]$.

3. Prove that $erf_c(x/2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma\left(1 - \frac{k}{2}\right)} x^k$.

4. Expanding $erf(x)$ in series, show that $\int_0^\infty e^{-pt} erf(\sqrt{t})\, dt = \frac{1}{p\sqrt{p+1}}$.

ANSWERS

Exercise 7.1 (p. 423)

1. (a) $7/12$ (b) $1/2$ (c) $a^2/6$ (d) $\frac{5}{8}\pi a^3$

2. (a) $1/6$ (b) $\frac{ab^2}{3}$ 3. $\frac{3}{56}$

4. $\dfrac{(\sin 1 - \cos 1)}{ab}$ 5. $\dfrac{64\,a^4}{3}$ 6. 6 7. 1

8. 1 9. $(1/8)(e^{16} - 1)$ 10. 1/2 11. $\pi a^2/6$

12. $-\pi[f(a) - f(0)]$ 13. 22.5π 14. $4a^2/3$.

Exercise 7.2 (p. 427)

1. $8\left(\dfrac{\pi}{2} - \dfrac{5}{3}\right)$ 2. $\dfrac{\pi a^5}{20}$ 3. $\dfrac{2\pi}{a+b}$ 4. $\dfrac{3\pi a^4}{4}$

6. $\dfrac{15\pi a^4}{64}$ 7. $\dfrac{\pi}{6}$ 8. $2\pi ab/3$.

Exercise 7.3 (p. 432)

3. $(3/2)a^2\pi$ 4. $a^2/2$ 5. $a^2\left(1 - \dfrac{\pi}{4}\right)$ 6. $\dfrac{3}{2}\ln 3 - 2/3$

7. $(4a/3\pi, 4a/3\pi)$ 8. $2a/3$ 9. $\bar{x} = 64/35,\ \bar{y} = 5/7$

10. $(5a/6, 0)$ 11. $\left(\dfrac{\pi a\sqrt{2}}{8}, 0\right)$ 13. $\bar{x} = 11/3,\ \bar{y} = 14/27,\ I_y = 432,\ R_y = 4$

14. $\bar{x} = 0,\ \bar{y} = 7/10,\ I_x = 9/10,\ I_y = 3/10,\ I_0 = 6/5$

15. $\dfrac{\rho a^4}{48}(3\pi - 8)$ 16. $\dfrac{\pi\rho r^4 h}{10},\ \dfrac{\pi\rho r^2 h}{20}(r^2 + 4h^2),\ \dfrac{\pi\rho r^2 h}{60}(3r^2 + 2h^2)$.

Exercise 7.4 (p. 438)

1. $\dfrac{4}{3}\pi a^3$ 2. 1 3. 16π 4. $\dfrac{1}{6}abc$

6. $\dfrac{10\pi}{3}$ 7. $2\pi^2 a^3$ 8. 32 9. $\dfrac{3\pi a^2}{4}$

10. $8a^2$.

Exercise 7.5 (p. 444)

1. (a) $\dfrac{8}{3}\ln 2 - \dfrac{19}{9}$ (b) 1/48 2. $\pi/6$ 3. $\dfrac{\pi}{4}a^2h^2(a^2 + h^2)$

4. $\dfrac{\pi a^5}{10}$ 5. $\dfrac{1}{8}(e^{4a} - 6e^{2a} + 8e^a - 3)$ 6. $\dfrac{\pi^2}{8}$ 7. $\dfrac{4}{15}\pi a^3 bc$

8. 1/4 9. 2325.04 10. $4\pi\ln(a/b)$.

Exercise 7.6 (p. 451)

1. $\dfrac{4\pi abc}{3}$ 2. $8\pi\sqrt{2}$ 3. $\left(\dfrac{4}{3}\sqrt{2}-2\right)\pi a^3$ 4. $(n-m)\pi a^3/8$

6. $\dfrac{\pi b^4}{2a}$ 7. 2 8. 1 9. $4a/3$

10. $(0, 0, 2a/5)$ 11. 3 12. $\dfrac{1}{2}\rho\pi r^6 h$, $\dfrac{1}{12}\rho\pi r^2 h\,(3r^2+4h^2)$

13. $\left(0, 0, \dfrac{45a\pi}{64(3\pi-4)}\right)$ 14. $\dfrac{2}{5}Ma^2$ 15. $[(4r^2+a^2)/10]^{1/2}$

Exercise 7.7 (p. 464)

1. (a) converges (b) diverges (c) converges (d) diverges.
2. (a) diverges (b) $a/(a^2+p^2)$ (c) $\pi/2$ (d) $\dfrac{1}{p}-\pi\cot p\pi$, $(-1<p<1)$.
3. The resultant integral is not improper.
4. (a) $1<\alpha<\infty$ (b) $-1<\alpha<0$ (c) $\alpha>-1$ (d) $\alpha>1$ (e) $\alpha<1$.
5. (a) converges to zero (b) divergent (c) converges to π.
6. (a) converges absolutely for $p<1$ (b) converges absolutely
   (c) converges absolutely.
7. $-\sqrt{3}\,\pi/6$ 9. $-(\pi/2)\ln 2$

Exercise 7.8 (p. 474)

1. (a) $(1/2)\,\Gamma(3/4)$ (b) $1/3\,\Gamma(1/3)$ (c) $\sqrt{\pi}$ (d) $\dfrac{\Gamma(a+1)}{(\ln a)^{a+1}}$

2. (a) $\dfrac{1}{n}B\left(\dfrac{m+1}{n}, p+1\right)$ (b) $1/120$ (c) $\dfrac{63\pi}{512}$

   (d) $\dfrac{1}{2}a^{3/2}B\left(\dfrac{5}{4}, \dfrac{1}{2}\right)$ (e) $\sqrt{\pi}$ (f) $\dfrac{\sqrt{\pi}\,\Gamma(1/3)}{\Gamma(5/4)}$

   (g) $\dfrac{\Gamma(m)}{r^m}\sin m\theta$; $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\dfrac{b}{a}$.

7. $\dfrac{\Gamma(p)\,\Gamma(q)\,\Gamma(r)}{\Gamma(p+q+r+1)}$ 8. $abc/6$; $ka^2b^2c^2/720$ 9. $\dfrac{a^l\, b^m\, c^n}{pqr}\cdot\dfrac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p+m/q+n/r+1)}$.

PART C

Vector Calculus

8 Vector Differential Calculus

CHAPTER

Most of the problems in science and engineering deal with the analysis of forces, velocities and various other quantities which are vectors. These are not generally static but vary with position and time and are thus functions of one or more variables. Vector differential calculus extends the concept of differential calculus over a real line to these vector functions, enabling to analyze the problems over curves and surfaces in three dimensions and finds applications in fluid-flow, heat flow, solid mechanics, electrostatic and many other branches.

8.1 DIFFERENTIATION OF A VECTOR FUNCTION

Let $\bar{V}(t)$ be a vector, continuous and single valued function of a scalar variable t. The basic concepts of calculus such as convergence, continuity and differentiability can be defined for vector functions in a simple and natural way. Among these the concept of derivative is the most important one from applications point of view.

8.1.1 Convergence. Limit and Continuity

An infinite sequence of vectors $\bar{v}_n(t)$, $n = 1, 2, \ldots$ is said to converge if there exists a vector $\bar{v}(t)$ such that, $\lim_{n\to\infty} |\bar{v}_n(t) - \bar{v}(t)| = 0$. The vector $\bar{v}(t)$ is called the limit vector of that sequence, and we write $\lim_{n\to\infty} \bar{v}_n(t) = \bar{v}(t)$.

Assuming the co-ordinate system to be cartesian, this sequence of vectors converges to $\bar{v}(t)$ if, and only if the three sequences of components of the vectors converge to the corresponding components of the vector $\bar{v}(t)$.

Similarly, a vector function $\bar{v}(t)$ of a real variable t is said to have the limit $\bar{l}$ as t approaches t_0, if $\bar{v}(t)$ is defined in some neighbourhood of t_0, possibly except at t_0, and $\lim_{t\to t_0} |\bar{v}(t) - \bar{l}| = \bar{0}$, and then we write $\lim_{t\to t_0} |\bar{v}(t) - \bar{l}| = \bar{l}$.

A vector function $v(t)$ is said to be continuous at $t = t_0$, if it is defined in some neighbourhood of t_0 and $\lim_{t \to t_0} \bar{v}(t) = \bar{v}(t_0)$.

In case of cartesian co-ordinate system, we may write $\bar{v}(t) = v_1(t)\hat{i} + v_2(t)\hat{j} + v_3(t)\hat{k}$,

where $\hat{i}, \hat{j}, \hat{k}$ have their usual meanings. Then continuity of $\bar{v}(t)$ at t_0 implies and is implied by the continuity of its three components at t_0.

Next, we define the derivative of a vector function.

8.1.2 Derivatives of a Vector Function

A vector function $\bar{v}(t)$ is said to be differentiable at a point t, if the limit $\lim_{\delta t \to t_0} \frac{\bar{v}(t+\delta t) - \bar{v}(\delta t)}{\delta t}$ *exists. If so, then this limit, denoted by $\frac{d\bar{v}}{dt}$, or $\bar{v}'(t)$, is called the derivative of the vector function $\bar{v}(t)$.*

If $\bar{v}(t) = v_1(t)\hat{i} + v_2(t)\hat{j} + v_3(t)\hat{k}$, then $\bar{v}(t)$ is differentiable at a point t if, and only if its three components are differentiable at t, and the derivative of $\bar{v}(t)$ is obtained by differentiating each component separately.

Since $\frac{d\bar{v}}{dt}$ is itself a vector function of t, its derivative, denoted by $\frac{d^2\bar{v}}{dt^2}$, is called the *second order derivative of $\bar{v}$ with respect to t*. Similarly, we can define higher order derivatives of $\bar{v}(t)$.

General rules of differentiation. If ϕ is a scalar function and $\bar{u}, \bar{v}, \bar{w}$ are vector functions of a scalar variable t, then we have the following general rules of differentiation of vector functions similar to those of ordinary differential calculus, provided the order of factors in vector product is maintained.

1. $\frac{d}{dt}(\bar{u} \pm \bar{v}) = \frac{d\bar{u}}{dt} \pm \frac{d\bar{v}}{dt}$
2. $\frac{d}{dt}(\bar{u} \cdot \bar{v}) = \bar{u} \cdot \frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt} \cdot \bar{v}$
3. $\frac{d}{dt}(\bar{u} \times \bar{v}) = \bar{u} \times \frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt} \times \bar{v}$
4. $\frac{d}{dt}(\phi\bar{u}) = \phi\frac{d\bar{u}}{dt} + \frac{d\phi}{dt}\bar{u}$
5. $\frac{d}{dt}[\bar{u}\ \bar{v}\ \bar{w}] = \left[\frac{d\bar{u}}{dt}\ \bar{v}\ \bar{w}\right] + \left[\bar{u}\ \frac{d\bar{v}}{dt}\ \bar{w}\right] + \left[\bar{u}\ \bar{v}\ \frac{d\bar{w}}{dt}\right]$
6. $\frac{d}{dt}[\bar{u} \times (\bar{v} \times \bar{w})] = \frac{d\bar{u}}{dt} \times (\bar{v} \times \bar{w}) + \bar{u} \times \left(\frac{d\bar{v}}{dt} \times \bar{w}\right) + \bar{u} \times \left(\bar{v} \times \frac{d\bar{w}}{dt}\right)$

As an illustration, we prove 3. By definition

$$\frac{d}{dt}(\bar{u} \times \bar{v}) = \lim_{\delta t \to 0} \frac{(\bar{u} + \delta\bar{u}) \times (\bar{v} + \delta\bar{v}) - (\bar{u} \times \bar{v})}{\delta t} = \lim_{\delta t \to 0} \frac{(\bar{u} \times \bar{v} + \bar{u} \times \delta\bar{v} + \delta\bar{u} \times \bar{v} + \delta\bar{u} \times \delta\bar{v}) - (\bar{u} - \bar{v})}{\delta t}$$