Here $m = 4$, $n = 3$, $N = 12$, and $G = 25$. Further

$\sum\sum x_{ij}^2 = 0 + 4 + 36 + 4 + 1 + 49 + 9 + 25 + 100 + 9 + 64 + 16 = 317.$

Thus,
$$TSS = 317 - \frac{(25)^2}{12} = 317 - 52.08 = 264.92,$$

$$SSR = \frac{16 + 100 + 324 + 49}{3} - \frac{(25)^2}{12} = 163 - 52.08 = 110.92,$$

$$SSC = \frac{4 + 16 + 729}{4} - \frac{(25)^2}{12} = 187.25 - 52.08 = 135.17,$$

and, $SSE = 264.92 - 110.92 - 135.17 = 18.83.$

The ANOVA table is

Source of variation	*df*	*SS*	*MS*	*F*
Row (Detergent)	3	110.92	$MSR = 36.91$	$F_R = 11.75$
Col. (Engine)	2	135.17	$MSC = 67.58$	$F_C = 25.52$
Error	6	18.83	$MSE = 3.14$	
Total	11	264.92		

From Table IVA, $F_{(3,6)[.05]} = 4.76$. Since F_R calculated exceeds the F tabulated so we conclude that there are differences in the effectiveness of the 4 detergents.

Similarly, from Table IVA, $F_{(2,6)[.05]} = 5.14$, and thus F_C calculated exceeds the F tabulated, therefore, there are differences due to engines also.

Example 25.45: Following data gives the monthly phone costs of four different companies at three different usage levels. Test the hypothesis that there is no difference among the companies by taking $\alpha = .05$.

Usage Level	*Company*			
	A	*B*	*C*	*D*
Low	27	24	31	23
Middle	68	76	65	67
High	308	326	312	300

Solution: The table is

Usage Level	*Company*				*Row total* (R_i)
	A	*B*	*C*	*D*	
Low	27	24	31	23	105
Middle	68	76	65	67	276
High	308	326	312	300	1246
Col. total (C_j)	403	426	408	390	1627

Here $m = 3$, $n = 4$, $N = 12$ and $G = 1627$. Further

$$\sum\sum x_{ij}^2 = (27)^2 + (24)^2 + \dots\dots + (300)^2 = 410393.$$

Thus,

$$TSS = 410393 - \frac{(1627)^2}{12} = 410393 - 220594.08 = 189798.92$$

$$SSR = \frac{(105)^2 + (276)^2 + (1246)^2}{4} - \frac{(1627)^2}{12} = 409929.25 - 220594.08 = 189335.17$$

$$SSC = \frac{(403)^2 + (426)^2 + (408)^2 + (390)^2}{3} - \frac{(1627)^2}{12} = 220816.33 - 220594.08 = 222.25$$

and, $SSE = TSS - SSR - SSC = 189798.92 - 189335.17 - 222.25 = 241.5.$

Therefore, $$MSC = \frac{222.25}{3} = 74.08,\ MSE = \frac{241.5}{6} = 40.25$$

and thus, $$F_C = \frac{MSC}{MSE} = \frac{74.08}{40.25} = 1.84.$$

From Table IVA $F_{(3,\ 6)[.05]} = 4.76$. Since tabulated value of F is greater than the value of F calculated, thus there is no significance difference among the companies. Thus, the hypothesis may be accepted.

EXERCISE 25.6

1. Suppose we wish to compare the means of five populations based on independent random samples each of which contains 8 observations. Insert in an ANOVA table the sources of variation and their respective degrees of freedom. Also mention the assumptions made.
2. A college administrator claims that there is no difference in first-year grade point averages for students entering the college from any of the three different city high schools. The following data give the first-year grade point averages of 12 randomly chosen students, 4 from each of the three high schools. Test the administrator's claim at 5% level of significance.

School 1	School 2	School 3
3.2	3.4	2.8
3.4	3.0	2.6
3.3	3.7	3.0
3.5	3.3	2.7

3. The calcium content of a powdered mineral substance was analyzed five times by each of the three methods with similar deviations:

Method	*Percent calcium*				
1	.0279	.0276	.0270	.0275	.0281
2	.0268	.0274	.0267	.0263	.0267
3	.0280	.0279	.0282	.0278	.0283

Use an appropriate test to compare the three methods of measurements at $\alpha = 0.05$.

4. Given the following observations collected according to the one-way analysis variance design

Treatment 1: 6 4 5 5
Treatment 2: 11 10 13 12 14
Treatment 3: 7 9 11
Treatment 4: 3 5 1 4 2

Construct the ANOVA table and test the equality of treatments at $\alpha = 0.05$.

5. To compare the prices of nuts in four different states, five suppliers have been randomly selected in each of the four states. The prices per kilogram in rupees are given in the table

States

A	*B*	*C*	*D*
241	216	230	245
235	220	225	250
238	205	235	238
247	213	228	255
250	220	240	255

Test using $\alpha = 0.05$ whether the data provide sufficient evidence to indicate that the average price per kilogram of nuts differ among the four states.

6. It is suspected that the environmental temperature in which batteries are activated affects their activated life. Thirty homogeneous batteries were tested, six at each of five temperatures, and the data shown below were obtained. Carry out the analysis of variance.

Temp.(°C)	*Activated life in sec.*					
0	55	55	57	54	54	56
25	60	61	60	60	60	60
50	70	72	73	68	77	77
75	72	72	72	70	68	69
100	65	66	60	64	65	65

7. Following data gives the times in seconds to exhaustion of 6 participants when put on a treadmill, who where put on three different diets for a period of six days.

Diet	Participant					
	1	2	3	4	5	6
A	84	35	91	57	56	45
B	91	48	71	45	61	61
C	122	53	110	71	91	122

Perform the ANOVA. Use 0.01 level of significance to determine if there are significant differences among the diets.

8. Following table gives the observations on temperature of a computer chip when four different types of cooling fans were tried on each of the five different computers. Construct the analysis of variance table and test for difference among the cooling fans using $\alpha = 0.05$.

Cooling Fan	Computer				
	A	B	C	D	E
I	26	18	23	12	21
II	26	22	28	21	28
III	24	19	22	21	24
IV	24	21	23	18	19

9. The following data represents the number of different macroinvertebrate species collected at 6 stations, located in the vicinity of a thermal discharge from 1970 to 1977. Test the hypotheses using $\alpha = 0.01$ that the data are unchanging (a) from year to year, and (b) from station to station.

Year	Station					
	1	2	3	4	5	6
1970	53	35	31	37	40	43
1971	36	34	17	21	30	18
1972	47	37	17	31	45	26
1973	55	31	17	23	43	37
1974	40	32	19	26	45	37
1975	52	42	20	27	26	32
1976	39	28	21	21	36	28
1977	40	32	21	21	36	35

ANSWERS

Exercise 25.1 (p. 1436)

1. Yes
3. No
4. (a) 0.023 (b) 0.0038
5. 0.0179 (b) 0.740 (c) 0.242.
6. (a) Approximately normal with mean 0.75 and S. E. 0.0306 (b) 0.0516 (c) .69 to .81.

Exercise 25.2 (p. 1452)

1. $z = -1.25$, rejected
2. $z = -1.06$, rejected
3. (8.61, 15.38)
4. 20% to 29%
5. $z = 0.37$, accepted
6. $z = 2.5$, unlikely to be hidden
7. $z = 2.38$, second production line does superior work
8. $z = 1.489$, accepted
9. 0.0013
10. $z = -2.681$, claim may be considered to be valid.
11. $n = 35$
12. (–1022, 4622) interval contains zero, cannot conclude that type B is superior to type A
13. (2.80, 3.40), yes
14. $z = 3.535$, yes
15. (a) .9772 (b) 0.0062.

Exercise 25.3 (p. 1462)

1. $t = 1.32$, not viable
2. 41.5 ± 1.6
3. (a) 7.496 (b) $t = -2.849$, reject H_0 (c) yes
4. $t = 2.89$, yes!
5. $t = -0.609$; don't differ significantly
6. $t = 4.03$; program is effective, [4.0, 6.4]
7. $t = 0.95$; accepted
8. (i) $t = 2.17$; B is superior (ii) $t = 5.09$; significant (iii) 0.2 ± 0.075
10. 0.47.

Exercise 25.4 (p. 1468)

1. $\chi^2 = 5.29$, claim accepted, (3.53, 206.07)
2. $\chi^2 = 22.45$, reject H_0
3. $\chi^2 = 10.89$, H_0 accepted
4. $F = 1.15$, H_0 accepted
5. $F = 2.4$, No
6. $H_0 : \mu_1 = \mu_2$; $t = 1.9$ not significant
   $H_0 : \sigma_1^2 = \sigma_1^2$; $F = 4.08$ not significant, yes!
7. $F = 1.26$ accepted.

Exercise 25.5 (p. 1476)

1. $\chi^2 = 58.542$, H_0 rejected
2. $\chi^2 = 19.63$, H_0 rejected
3. $\chi^2 = 3.05$, H_0 accepted
4. $\chi^2 = 40.937$, H_0 rejected
5. $\chi^2 = 20.179$, H_0 rejected
6. $\chi^2 = 31.17$, Non-homogeneous response
7. $\chi^2 = 79.83$, H_0 rejected
8. $\chi^2 = 19.172$, yes!
9. Vaccine is effective
10. Vaccine is efficient.

Exercise 25.6 (p. 1486)

2. Rejected
3. $F = 16.38$, methods are different.
4. $F = 33.55$, treatments are different.
5. $F = 26.44$, prices are different.
6. $F = 70.27$, temperature effects the activated life.
7. $F = 11.86$, significant difference between the diets.
8. $F = 4.24$, not significant.
9. (a) $F = 3.73$, slightly significant (b) $F = 22.48$, significant.

26 Linear Programming

CHAPTER

Operations research is scientific knowledge through interdisciplinary team effort for the purpose of best utilization of the limited resources. Linear programming applies to optimize when the objective function and constraints are strictly linear. The simplex method is a powerful computational technique to solve linear programming models.

26.1 INTRODUCTION

Linear programming (LP) is one of the most important and fully-developed optimization technique in operations research (OR), a field which was initiated in England during the World War II by a team of British scientists entrusted with the task to make decisions regarding the best utilization of the limited military resources. In any OR model, the first crucial step is the identification of the *decision variables* involved, and next is to construct the *objective function* and the *constraints* of the model using these decision variables.

A solution of the model is *feasible* if it satisfies all the constraints and further it is *optimal* also if it yields the optimum (maximum or minimum) value of the objective function. *Linear programming technique is designed for the OR models when the objective function and constraints both are strictly linear.* A few other important techniques for solving OR models include *integer programming* (variables assume integer values), *dynamic programming* (model is decomposed into smaller sub-problems), *network programming* (problem is modelled as a network), and *non-linear programming* (objective function and/or constraints are non-linear). LP models describe a variety of situations encountered in administration, economics, industry and engineering. In addition to this, LP models are quite useful in obtaining results in the field of graph theory, combinatorial analysis, approximation theory and numerical analysis.

In this chapter, we shall study the basic results of linear programming models, their mathematical formulations, graphical solution of two variables LP-models leading to the *simplex method* (an algebraic technique to solve a general LPP), the concept of duality in linear programming problems, primal-dual relationship, transportation and assignment problems.

26.2 MATHEMATICAL FORMULATION OF LP MODEL

To begin with, we consider a two-variable LP model and describe the various steps involved for the mathematical formulation of the problem, and also define the various terms which appear in this process.

Example 26.1 A company produces two types of paints P_1 and P_2 from two raw materials R_1 and R_2. Each unit of paint P_1 requires 6 units of raw material R_1 and 1 unit of raw material R_2, and each unit of paint P_2 requires 4 units of R_1 and 2 units of R_2. Profit per unit from the sale of paint P_1 is Rs. 5,000 and from P_2 is Rs. 4,000. The maximum availability of the raw material per day for R_1 is 24 units and for R_2 is 6 units. Formulate it as linear programming problem to maximize the daily profit, if daily demand for the paint P_2 can't exceed that of P_1 by more than one unit and also the maximum daily demand of P_2 is 2 units.

Solution: The data given is summarized as follows:

	Units of raw material required per unit of the paint		*Maximum daily availability (units)*
	P_1	P_2	
Raw material R_1	6	4	24
Raw material R_2	1	2	6
Profit per unit (in Rs 000')	5	4	

For the problem under consideration, we need to determine the amounts of paints P_1 and P_2 to be produced daily. Let x_1 and x_2 units be the daily production of the paints P_1 and P_2 respectively. Since the amounts cannot be negative, thus $x_1 \geq 0$ and $x_2 \geq 0$.

The objective is to maximize the daily profit from the sale. Assuming that total produced is sold in the market, the objective function (in '000 Rs) is given by $z = 5x_1 + 4x_2$.

The constraints are the limited availability of the row materials and the demand of the produce. From the data given

Use of raw material R_1 per day $= 6x_1 + 4x_2$ units.

Use of raw material R_2 per day $= x_1 + 2x_2$ units.

Thus, $6x_1 + 4x_2 \leq 24$ and $x_1 + 2x_2 \leq 6$.

Further, the daily demand restrictions imply that $x_2 - x_1 \leq 1$, and $x_2 \leq 2$.

Thus, the LP model can be put in the following mathematical form

Maximize $z = 5x_1 + 4x_2$, subject to

$$
\begin{aligned}
6x_1 + 4x_2 &\leq 24 \\
x_1 + 2x_2 &\leq 6 \\
-x_1 + x_2 &\leq 1 \\
x_2 &\leq 2 \\
x_1, \text{ and } x_2 &\geq 0
\end{aligned}
$$

The variables x_1 and x_2 that enter into the problem are called the *decision variables*. Any values of these variables that satisfy the constraints of the LP model form a *feasible solution*. For example, $x_1 = 1$ and $x_2 = 2$ is a feasible solution. The value of the objective function is then $z = 13$ (in '000 Rs). Another feasible solution is $x_1 = 3$, $x_2 = 1$ and the corresponding value of the objective function is $z = 19$ (in '000 Rs). However, our interest lies in finding the *optimum feasible solution* that yields the maximum total daily profit while satisfying all the constraints.

Example 26.2: A technician assembles two types of printed circuits. The requirement of transistors, resistors and capacitors for each type of printed circuit along with their respective availabilities is given in the table below. The profit from the sale of a circuit of Type I is Rs. 50 and from that of Type II is Rs. 80. Formulate it as a linear programming problem to maximize the profit.

Items	*Requirements for circuit of*		*Available stock*
	Type I	*Type II*	
Transistors	15	10	180
Resistors	10	20	200
Capacitors	15	20	210

Solution: Let x_1 and x_2 respectively be the Type I and Type II circuits manufactured, the objective function is then $z = 50x_1 + 80x_2$

The constraints due to the availability of the stock are

$$15x_1 + 10x_2 \le 180$$
$$10x_1 + 20x_2 \le 200$$
$$15x_1 + 20x_2 \le 210$$

or, equivalently

$$3x_1 + 2x_2 \le 36$$
$$x_1 + 2x_2 \le 20$$
$$3x_1 + 4x_2 \le 42.$$

In addition to this, there is an obvious restriction that x_1 and x_2 are non-negative, that is, $x_1 \ge 0$ and $x_2 \ge 0$.

Thus, LP model can be put in the following mathematical form

Maximize $z = 50x_1 + 80x_2$, subject to

$$3x_1 + 2x_2 \le 36$$
$$x_1 + 2x_2 \le 20$$
$$3x_1 + 4x_2 \le 42$$
$$x_1 \ge 0 \text{ and } x_2 \ge 0$$

Example 26.3: Following table gives the minimum requirements of support staff in a city hospital during the various time slots in the total span of 24 hrs. The staff reports to the hospital at the beginning of each slot and remains on duty for 8 consecutive hours. The hospital wants to determine the minimum requirement of the staff for each slot. Formulate it as a linear programming problem.

Slot	*Time period*	*Minimum supporting staff*
1	6:00 A.M. – 10:00 A.M.	3
2	10:00 A.M. – 2:00 P.M.	7
3	2:00 P.M. – 6:00 P.M.	10
4	6:00 P.M. – 10:00 P.M.	15
5	10:00 P.M. – 2:00 A.M.	9
6	2:00 A.M. – 6:00 A.M.	5

Solution: Let x_j be the number of supporting staff required to report for duty at the beginning of the jth slot, ($j = 1, 2, 3, 4, 5, 6$). Then the objective function is $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$.

The constraints are: $x_1 + x_2 \geq 7$, $x_2 + x_3 \geq 10$, $x_3 + x_4 \geq 15$, $x_4 + x_5 \geq 9$, $x_5 + x_6 \geq 5$, $x_6 + x_1 \geq 3$.

The natural constraints are x_1, x_2, x_3, x_4, x_5 and $x_6 \geq 0$.

Hence, the LP model can be put in the following form

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

subject to $x_1 + x_2 \geq 7$

$$
\begin{aligned}
x_2 + x_3 &\geq 10 \\
x_3 + x_4 &\geq 15 \\
x_4 + x_5 &\geq 9 \\
x_5 + x_6 &\geq 5 \\
x_6 + x_1 &\geq 3 \\
x_1, x_2, x_3, x_4, x_5 &\text{ and } x_6 \geq 0.
\end{aligned}
$$

EXERCISE 26.1

1. A company has three operational departments weaving, processing and packing, with capacity to produce three different types of clothes, namely suiting, shirtings and woollens yielding a profit of Rs. 2, Rs. 4 and Rs. 3 per metre respectively. One metre of suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing. Similarly, one metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing. One metre of woollen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours for weaving, processing and packing respectively. Formulate the linear programming problem to find the product mix to maximize the profit.
2. The manager of an oil refinery must decide on the optimum mix of possible blending processes of which the input and output production runs are as follows:

Process	Input		Output	
	Crude A	Crude B	Gasoline x	Gasoline y
I	6	4	6	9
II	5	6	5	5

The maximum amount of crudes A and B are 250 units and 200 units respectively. Market demand shows that at least 150 units of gasoline X and 130 units of gasoline Y must be produced. The profit per product run from process I and process II are Rs. 4 and Rs. 5 respectively. Formulate the LPP for maximizing the profit.

3. A private sector bank is in the process of devising a loan policy that involves a maximum of Rs. 12 million. The following table provides the data about available types of loans:

Types of loan	*Interest rate*	*Recovery ratio*
Personal	.140	.90
Car	.130	.93
Home	.120	.97
Farm	.125	.95
Commercial	.100	.98

Interest revenue is earned only on the recoverable loans.

Competition with other financial institutions requires the bank to allocate at least 40% of the funds to farm and commercial loans. For the growth of the housing industry, home loans must equal at least 50% of the personal, car and home loans. As a policy matter, the overall loan recovery rate must be greater than or equal to 96%. Formulate the problem mathematically to maximize the interest revenue.

4. Managing Director of a sports company wishes to determine how many advertisements to place in the selected three monthly magazines A, B and C. His objective is to advertise in such a way that total exposure to principal buyers of expensive sports good is maximized. Percentage of principal buyers among the readers for each magazine are known. Exposure in any particular magazine is the number of advertisements placed multiplied by the number of principal buyers. In this context following data has been obtained

	Magazine		
	A	*B*	*C*
Readers (in lacs)	10	6	4
Principal buyers (%)	20	15	8
Cost per advt. (in Rs)	80000	60000	50000

The budgeted amount for advt. is at most Rs. 10 lac. It has already been decided that magazine A should not have more than 10 advt. and that B and C each have at least 8 advt. Formulate an LP model for the problem.

5. An airline operates its fleet of m types of aircraft on n different routes. Suppose $a_i (i = 1, 2, ... m)$, is the number of aircrafts of type i in the fleet and b_j, $(j = 1, ..., n)$, is the number of passengers requiring passage on the jth route in a given period. Also, suppose that an aircraft of type i can accomodate p_{ij} passengers at an operating cost c_{ij} if it is assigned to the route j during the period. Airline wants to assign its various types of aircraft to different routes in order to satisfy the passengers demand at the least operating cost. Formulate this as LP model.

6. Three grades of coal A, B and C contain ash and phosphorus as impurities. In a particular industrial process a fuel obtained by blending the above grades containing not more than 25% ash and 0.03% phosphorus is required. The maximum demand of the fuel is 100 tons. Percentage impurities and costs of the various grades of coal is shown below. Assuming that there is an unlimited supply of each grade of coal and there is no loss in blending, formulate this as an LP model to minimize the cost.

Coal Grade	% Ash	% Phosphorus	Cost per ton (in Rs)
A	30	0.02	240
B	20	0.04	300
C	25	0.03	280

7. A firm making castings uses electric furnace to melt iron with the following specifications:

	Minimum	Maximum
Carbon	3.20%	3.40%
Silicon	2.25%	2.35%

Specifications and costs of various raw materials used for this purpose are given below:

Material	Carbon %	Silicon %	Cost (Rs)/ton
Steel scrap	0.4	0.15	8,500
Cast iron scrap	3.80	2.40	9,000
Remelt from foundry	3.50	2.30	5,000

If the total charge of iron metal required is 4 tons, write this as LP model to find the weight in kg. of each raw material that must be used in the optimal mix at minimum cost.

26.3 GRAPHICAL SOLUTION OF LP MODEL

The two-variable LP model can be effectively solved by graphical method. The graphical solution provides the necessary basis for the development of a general algebraic method (the simplex method) for solving a general linear programming problem (LPP). After writing the LP model in the mathematical form, the graphical procedure constitutes the following two steps:

1. Determining the solution space that defines all feasible solutions of the model.
2. Determining the optimum solution from among all the feasible points in the solution space.

We illustrate this by considering next a few examples.

Example 26.4: A unit makes two kinds of leather belts. Belt A is of a high quality belt and belt B is of a lower quality. The respective profits are Rs. 4.00 and Rs. 3.00 per belt. Each belt of type A requires twice as much time as a belt of type B, and if all belts were of type B, the unit could make 1000 belts per day. The supply of leather is sufficient for only 800 belts per day, both A and B combined. Belt A requires a fancy buckle, only 400 of which are available per day. For belt B at the most 700 buckles are available per day.

Formulate it as LPP and find the optimal solution graphically.

Solution: Let x and y be the number of belts of type A and B respectively. Then the mathematical formulation of the given LP model is

Maximize $z = 4x + 3y$, subject to

$$2x + y \le 1000 \quad ①$$
$$x + y \le 800 \quad ②$$
$$x \le 400 \quad ③$$

$$y \le 700 \qquad ④$$
$$x \ge 0 \qquad ⑤$$
$$\text{and, } y \ge 0 \qquad ⑥$$

We plot the graph by considering the cartesian rectangular axis in the (x, y)-plane. First, we account for the non-negative constraints $x \ge 0$ and $y \ge 0$. This restricts the solution space area to the first quadrant only.

To account for the remaining constraints, we replace each inequality with an equation and then plot the resulting straight lines as shown in Fig. 26.1. Next we consider the effect of the inequality. An inequality divides the (x, y)-plane into two half spaces and only one of these satisfy the inequality. To determine the correct side, normally we select (0, 0) as the reference point (another point is selected if origin lies on the line), if it satisfies the inequality, then the side in which it lies is the feasible half-space, otherwise, the alternate side is the feasible one.

Applying this criterion, the feasible space OABCDE is shown in Fig. 26.1.

Any point within and on the boundary of the feasible space OABCDE satisfies the constraints ① to ⑥ and is a feasible solution to the LPP. Since it consists of infinite number of points, so we need to identify the point(s) at which the optimum value of the objective function exists and that point(s) is(are) called *optimal solution(s).*

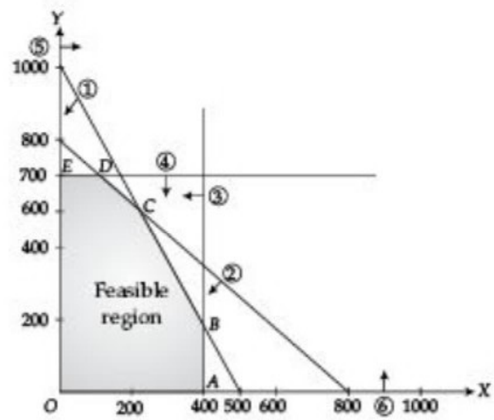


Fig. 26.1

To find the optimal solution(s), we use the following result which we state without proof.

If R is the convex feasible region for an LP model, then the objective function has the optimal value (maximum or minimum) at the vertices of the convex region.

So to determine the optimal value we find the corner points O, A, B, C, D and E and find the value of the objective function corresponding to these corner points.

Corner Point	Coordinate (x, y)	$z = 4x + 3y$
O	(0, 0)	0
A	(400, 0)	1600
B	(400, 200)	2200
C	(200, 600)	2600 ← maximum
D	(100, 700)	2500
E	(0, 700)	2100

Hence, the optimum solution for the LP model is (200, 600) and the optimum (maximum) profit is Rs. 2600 per day.

Alternatively, we assign an arbitrary value to the objective function say $z = 600$ and draw the dotted line $4x + 3y = 600$, as shown in Fig. 23.2. Next we move this profit line parallel to itself and farther from the orgin and we observe that this line touches only the point C (200, 600) of the feasible region before leaving it. Thus, the optimum solution occurs at C itself, a point beyond which any further increase in the objective function value will put us in the infeasible region. Hence, the optimal value is $z = 4(200) + 3(600) =$ Rs. 2600.

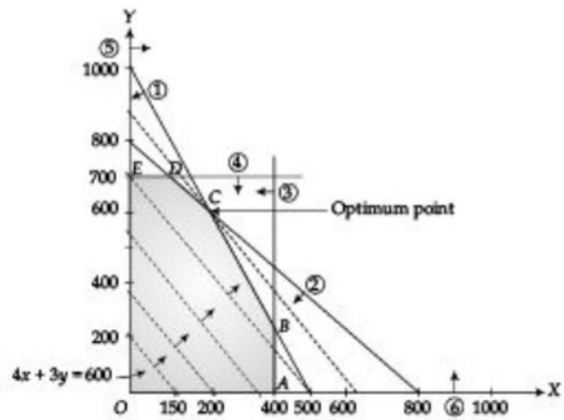


Fig. 26.2

Remarks:

1. In case the feasible region is empty, then LP model is not feasible and so question of optimal solution does not arise.
2. When feasible region is a convex polyhedron, we get a finite optimum solution as observed in the preceding example.
3. If the feasible region is unbounded convex set, we may either get a finite optimum solution or an unbounded solution M as the maximum value of the objective function, $z = ax + by$, if the open half-plane $ax + by > M$ has no point in common with the feasible region otherwise, z has no maximum value. Similarly, m is the minimum value of the objective function $z = ax + by$, if the open half-plane determined by $ax + by < m$ has no point in common with the feasible region, otherwise, z has no minimum value.

Example 26.5: Solve the following LPP graphically

Maximize $z = 3x + 9x$, subject to

$$x + 3y \le 60 \quad ①$$
$$x + y \ge 10 \quad ②$$
$$x \le y \quad ③$$
$$x \ge 0 \quad ④$$
$$\text{and, } y \ge 0 \quad ⑤$$

Solution: The non-negativity criteria of the decision variables x and y restrict the solution space area to the first quadrant of the XOY-plane. Ploting the constraints ① to ⑤ as equality in the plane and then accounting for each inequality, the feasible region ABCD is shown in Fig. 26.3.

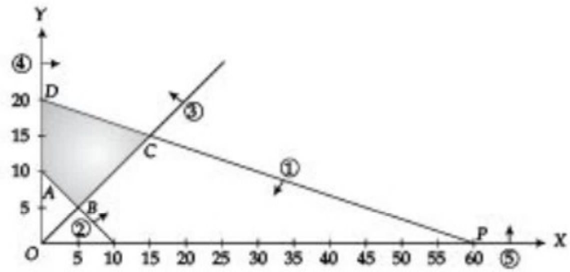


Fig. 26.3

The feasible region is convex polygon so the optimum solution(s) exists at the corner points A, B, C and D. We proceed as follows.

Corner point	Coordinate (x, y)	$z = 3x + 9y$	
A	(0, 10)	90	
B	(5, 5)	60	
C	(15, 15)	180	← Maximum
D	(0, 20)	180	

The maximum value 180 of the objective function exists at two points C (15, 15) and D (0, 20). In fact every point on the line segment CD is the optimum solution and gives the same maximum value 180. So in this case LP model is having no unique optimum solution, there are infinite number of solutions.

Example 26.6: Solve the following LPP graphically

Minimize $z = -50x + 20y$, subject to

$$2x - y \ge -5 \quad ①$$
$$3x + y \ge 3 \quad ②$$
$$2x - 3y \le 12 \quad ③$$

$$x \geq 0 \qquad ④$$

$$\text{and, } y \geq 0 \qquad ⑤$$

Solution: The feasible region of the given LP model is shown in Fig. 26.4 and we observe that the region is convex unbounded.

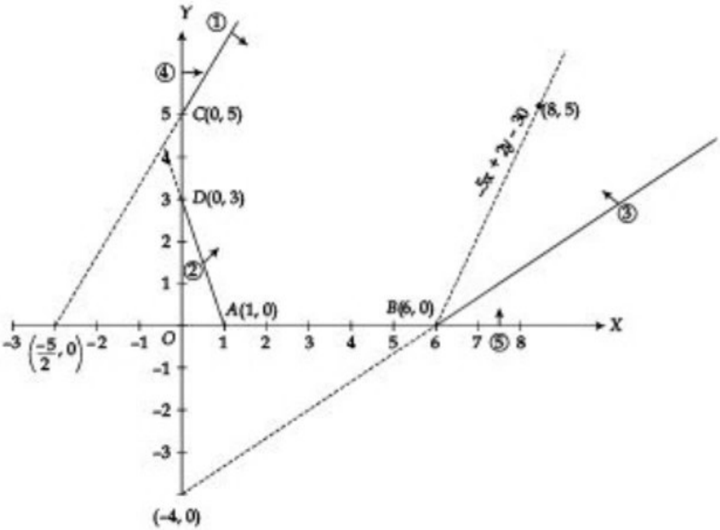


Fig. 26.4

Evaluating z at the corner points, we have

Corner point	Coordinate (x, y)	$z = -50x + 20y$
A	(1, 0)	−50
B	(6, 0)	−300
C	(0, 5)	100
D	(0, 3)	60

In the table obtained −300 is the smallest value of z at (6, 0). Since the regions is unbounded, it may or may not be the minimum value of z. To ascertain this, we graph the inequality

$$-50x + 20y < -300, \text{ or, } -5x + 2y < -30$$

and check whether the resulting open half-plane has points in common with the feasible region or not. If it has common points, then −300 will not be the minimum value of z, otherwise, it will be so.

Plotting the equality $-5x + 2y = -30$ as dotted line, we observe from the Fig. 26.4 that the region $-5x + 2y < -30$ has points common with the solution space, therefore, $z = -50x + 20y$ has no minima subject to the given constraints. In fact, the minimum value of z occurs at infinity and thus the LP model has unbounded solution.

Remark: Applying the parallel argument, we can verify that the objective function $z = -50x + 20y$ has the maximum value at the point C (0, 5).

Example 26.7: Using graphical method, solve the LPP

Maximize $z = 4x + 3y$, subject to

$$x - y \le -1 \quad ①$$
$$-x + y \le 0 \quad ②$$
$$x \ge 0 \quad ③$$
$$\text{and, } y \ge 0 \quad ④$$

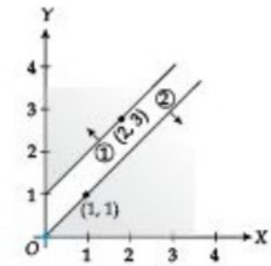


Fig. 26.5

Solution: The constraints plotted as equality are shown in Fig. 26.5. Since there is no common point (x, y) between the both shaded regions, thus the constraints are inconsistent and hence the problem has no solution.

Remark. From the examples discussed above, we observe that an L.P.P. may have, (i) a unique optimal solution, or (ii) an infinite number of optimal solutions, or (iii) an unbounded solution, or (iv) no solution at all.

EXERCISE 26.2

1. A fast food company can purchase potatoes from two sources which differ in their yields of various sizes and quality. From source I, there is a 20% yield of French fries, a 20% yield of hash browns and a 30% yield of flakes; the remaining 30% is waste. The corresponding figures for the source II are respectively 30%, 10%, 30% and 30%. The profit contribution per ton purchased is 5 for source I and 6 for source II. Solve this problem graphically to maximize the profit when the sales limitations for French fries, hash brown, and flakes are respectively 1.8, 1.2, and 2.4 tons.
2. Find graphically the maximum value of $z = 2x + 3y$ subject to the constraints:

$$x + y \le 30,\ y \ge 3,\ 0 \le y \le 12,\ x - y \ge 0, \text{and } 0 \le x \le 20.$$

3. An agro company uses at least 800 lb. of special feed daily which is a mixture of corn and soyabean meal with the following compositions:

	lb./gm of feedstuff		
Feedstuff	*Protein*	*Fiber*	*(Cost in $/lb)*
Corn	.09	.02	.30
Soyabean meal	.60	.06	.90

The dietary requirements of the special feed are at least 30% protein and at most 5% fibre. The company wishes to determine the daily minimum-cost feed mix. Write the problem as LP-model and solve it graphically.

4. In a pig farm, the pigs are fed on various products grown on the farm. In order to ensure certain nutrient constituents, say X, Y and Z, it is necessary to buy two additional products

A and *B*. One kg. of product *A* contains 36 gm. of *X*, 3 gms of *Y* and 20 gms of *Z*, and one kg. of product *B* contains 6 gms of *X*, 12 gms of *Y* and 10 gms of *Z*. The minimum requirement of nutrients *X*, *Y* and *Z* is 108 gm, 36 gms and 100 gms respectively, product *A* costs Rs. 20 per kg. and *B* costs Rs. 40 per kg.
Formulate the problems as LP-model to minimize the total cost and solve it graphically.

5. Using graphical method, solve the LPP:

   Maximize $z = 2x + 3y$, subject to $x - y \le 2$, $x + y \ge 4$, $x, y \ge 0$.

6. Using graphical method, solve the LPP:

   Maximize $z = 4x + 3y$, subject to $x + y \le 1$, $3x + 4y \ge 12$, $x, y \ge 0$.

7. Using graphical method, solve the LPP:

   Maximize $z = 2x + 4y$, subject to $x + 2y \le 5$, $x + y \le 4$, $x, y \ge 0$.

8. Solve graphically the LPP:

   Minimize $z = x - 2y$, subject to $-x + y \le 2$, $2x + y \le 2$, $x, y \ge 0$.

9. A production manager wants to determine the quantity to be produced per month of products *A* and *B*. The data on resources required and availability of resources are given below:

Resources	*Requirements*		*Available per month*
	Product A	*Product B*	
Raw material (kg)	60	120	12,000
Machine hrs/piece	8	5	600
Assembly manhrs/piece	3	4	500
Sale price/piece (in Rs.)	30	40	

   Formulate the problem as LP-model and solve it graphically to find the product mix to maximize the profit.

10. A brewery has two bottling plants one each at locations *A* and *B*. Each plant produces three drinks: whisky, beer and fruit juices, with the daily production data as given below.

Drink	*Plant location*		*Min. Demand during a particular month*
	A	*B*	
Whisky	1,500	1,500	20,000
Beer	3,000	1,000	40,000
Fruit juices	2,000	5,000	44,000

   The operating costs per day for plants at *A* and *B* are 600 and 400 monetary units. Solve graphically that for how many days each plant be run in a particular month so as to minimize the production cost, while still meeting the market demand.

26.4 GENERAL FORMULATION OF AN LPP

The general form of an LPP is given as follows.

Optimize (Maximize or Minimize) $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ ...(26.1)

subject to the constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n\ (\le, =, \ge)\, b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n\ (\le, =, \ge)\, b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n\ (\le, =, \ge)\, b_m \\ x_1, x_2, \dots, x_n \ge 0. \end{array}\right\} \quad \text{...(26.2)}$$

In concise form this can be expressed as

$$\text{Maximize (or minimize) } z = \sum_{j=1}^{n} c_jx_j \quad \text{...(26.3)}$$

$$\text{subject to} \sum_{j=1}^{n} a_{ij}x_j\ (\le, =, \ge)\, b_i,\ i = 1, 2, \dots, m \text{ and, } x_j \ge 0,\ j = 1, 2, \dots, n. \quad \text{...(26.4)}$$

An n-tuple $(x_1, x_2, \dots, x_n)$ that satisfies all the constraints, including the non-negative restrictions, is called a *feasible solution* of LPP.

A feasible solution which optimizes the objective function is called an *optimal solution* of LPP.

For solving an LPP algebraically, it is put in the *standard form*. The characteristics of the standard form are:

1. *All the constraints are expressed in the form of equation, except the non-negative restrictions which remain inequalities* $(\ge, 0)$.
2. *The right-hand side of each constraint equation is non-negative.*
3. *All the decision variables are non-negative.*
4. *The objective function is of the maximization form (this can always be made, since minimize (z) is equivalent to maximize (−z)).*

The inequality constraints are changed to equality constraints by adding or subtracting a non-negative variable from the left hand sides of such constraints. The new variable when added, which converts the inequality $(\le)$ to equality is called a *slack variable*, because it takes up the slack in the inequality; and the new variable when subtracted, which converts the inequality $(\ge)$ to equality is called a *surplus variable*.

Further, in the standard form the decision variables $x_1, x_2, \dots, x_n$ are all non-negative. In actual practice any of these can be zero or negative. In case a variable x_i is unrestricted, it is expressed as $x_i = x_i' - x_i''$, where x_i' and x_i'' are non-negative variables.

Example 26.8: Express the following LPP in the standard form

Maximize $z = 3x_1 + 2x_2 + 5x_3$ subject to

$$2x_1 - 3x_2 \le 3$$
$$x_1 + 2x_2 + 3x_3 \ge 5$$
$$3x_1 + 2x_3 \le 2$$
$$x_1 \ge 0, x_2 \ge 0.$$

Solution: The decision variables x_1 and x_2 are restricted to be non-negative, but x_3 is unrestricted. Express x_3 as $x_3 = x_3' - x_3''$, where $x_3' \ge 0$ and $x_3'' \ge 0$. Thus, the constrains can be expressed as

$$2x_1 - 3x_2 \le 3$$
$$x_1 + 2x_2 + 3x_3' - 3x_3'' \ge 5$$
$$3x_1 + 2x_3' - 2x_3'' \le 2$$
$$x_1, x_2, x_3', x_3'' \ge 0$$

Next, introducing the slack and surplus variables to convert inequalities into equalities, the standard form is

Maximize $z = 3x_1 + 2x_2 + 5x_3' - 5x_3''$, subject to

$$2x_1 - 3x_2 + s_1 = 3$$
$$x_1 + 2x_2 + 3x_3' - 3x_3'' - s_2 = 5$$
$$3x_1 + 2x_3' - 2x_3'' + s_3 = 2$$
$$x_1, x_2, x_3', x_3'', s_1, s_2 \text{ and } s_3 \ge 0.$$

Example 26.9: Express the following LPP in the standard form

Minimize $z = 3x_1 - 4x_2$, subject to

$$2x_1 - x_2 \ge -4$$
$$3x_1 + 5x_2 \le 10$$
$$x_1 - 4x_2 = 12$$
$$x_1 \ge 0.$$

Solution: The decision variable x_1 is restricted to be non-negative but x_2 is unrestricted. Writing $x_2 = x_2' - x_2''$, where $x_2' \ge 0$ and $x_2'' \ge 0$. Introducing the slack and surplus variables for changing inequality constraints into equality making the right-hand side of constraints non-negative and rewriting the objective function in the maximization form, the standard form is

Maximize $z'(=z) = -3x_1 - 4x_2' + 4x_2''$, subject to

$$-2x_1 + x_2' - x_2'' + s_1 = 4$$
$$3x_1 + 5x_2' - 5x_2'' + s_2 = 10$$
$$x_1 - 4x_2' + 4x_2'' = 12$$
$$x_1, x_2', x_2'', s_1, s_2 \ge 0$$

26.5 THE SIMPLEX METHOD

The graphical method of solving an LPP studied so far is confined only to two variables x_1, x_2. However, the most practical problems involve more than two variables. The simplex method developed by G.B. Dantzig, an American mathematician in 1948 is an algebraic iterative procedure which will solve exactly any LPP in a finite number of steps, or give an indication that there is an unbounded solution. Before applying this method, the problem is first put in the standard form.

We have seen that in case of graphical method, the optimum solution always lies on a corner point of the convex solution space. This characteristic of the graphical method has been exploited to develop the simplex method for solving a general LP model. The method entails a computational procedure that determines the corner points of the solution space algebraically. It is an iterative method and each iteration moves the solution to a new corner point, improving the value of the objective function. The process ends only when no further improvement is realized. Since the number of corner points is finite, the method leads to an optimum solution in a finite number of steps.

In simplex method following procedure is adopted to achieve a systematic reduction from an infinite number of solutions to a finite number of promising solutions.

If there are m equality constraints in $n + m$ variables, then the system has infinity of feasible solutions. The starting solution is obtained by setting n variables equal to zero, and then solving the m equations for the remaining m variables, provided the unique solution exists. The n zero variables are called the *non-basic variables* and the remaining m variables are called the *basic variables*, which form a *basic solution*. If all the basic variables are non-negative, then the solution is *basic feasible solution*, otherwise, it is *infeasible*. Obviously, the number of basic feasible solutions are less than or equal to C_n^{n+m}.

Further, if all the variables in the basic feasible solution are positive, then the solution is called *non-degenerate solution*, and if a few of the basic variables are zero, the solution is called *degenerate solution*.

A new basic feasible solution is obtained from the one under consideration by setting one of the m basic variables equal to zero and replacing it by a new non-basic variable. The variable set equal to zero is called a *leaving variable* and the new variable entered is called an *entering variable*. The entering variable is selected so, that it improves the value of the objective function over the previous one and this is ensured by the *optimality condition*, according to which the selected entering variable produces the largest per unit gain in the objective function.

The process continues until no further improvement in the value of the objective function is possible. The end solution is called an *optimal basic feasible solution* or simply an *optimal solution*. This is the solution which optimizes the objective function and satisfies all the constraints including the non-negativity conditions of the decision variables. However, this happens only when the optimum value for the objective function is finite.

In real-life situations m and n are considerable large, say in hundreds or thousands, and thus, the number of possible basic feasible solutions are very large. The search procedure in simplex method is structured so intelligently that it needs to investigate only a fraction of the all possible basic solutions to reach at the optimal solution. The method ensures that if the starting solution is basic feasible, the subsequent solution will also be basic feasible and will improve the value of the objective function.

Example 26.10: For the following LPP

$$\text{Maximize} \quad z = 40x_1 + 88x_2, \text{ subject to}$$
$$x_1 + 4x_2 \le 30$$
$$5x_1 + 2x_2 \le 60$$
$$x_1, x_2 \ge 0$$

find all the basic solutions and then find the optimal solution.

Solution: The solution space is represented by the standard form given as

$$\text{Maximize} \quad z = 40x_1 + 88x_2, \text{ subject to}$$
$$x_1 + 4x_2 + s_1 = 30$$
$$5x_1 + 2x_2 + s_2 = 60$$
$$x_1, x_2, s_1 \text{ and } s_2 \ge 0.$$

The system has two equations and four variables. Thus, the two basic variables can be determined by setting two non-basic variables equal to zero. The following table provides all the basic solutions of the given LPP.

Non-basic	Basic variables	Basic solutions	Feasible	Objective value
(x_1, x_2)	(s_1, s_2)	(30, 60)	Yes	Zero
(x_1, s_1)	(x_2, s_2)	(7.5, 45)	Yes	660
(x_1, s_2)	(x_2, s_1)	(30, –90)	No	–
(x_2, s_1)	(x_1, s_2)	(30, –45)	No	–
(x_2, s_2)	(x_1, s_1)	(12, 18)	Yes	480
(s_1, s_2)	(x_1, x_2)	(10, 5)	Yes	**840**

Thus, the optimum solution exists at (10, 5) and the maximum value is 840.

Example 26.11: For the following LPP

$$\text{Maximize} \quad z = 3x_1 + 9x_2, \text{ subject to}$$
$$x_1 + 3x_2 \le 60$$
$$x_1 + x_2 \ge 10$$
$$x_1 - x_2 \le 0$$
$$x_1 \ge 0, x_2 \ge 0,$$

find all the basic solutions and then find the optimal solution.

Solution: The solution space is represented by the standard form given as

$$\text{Maximize} \quad z = 3x_1 + 9x_2, \text{ subject to}$$
$$x_1 + 3x_2 + s_1 = 60$$
$$x_1 + x_2 - s_2 = 10$$
$$x_1 - x_2 + s_3 = 0$$
$$x_1, x_2, s_1, s_2, s_3 \ge 0.$$

The system has three equations and five variables. Thus, the three basic variables can be determined by setting two non-basic variables equal to zero. The following table provides all the basic solutions of the given LPP.

Non-basic variables	*Basic variables*	*Basic solution*	*Feasible*	*Objective value*
(x_1, x_2)	(s_1, s_2, s_3)	(60, –10, 0)	No	-
(x_1, s_1)	(x_2, s_2, s_3)	(20, 10, 20)	Yes	**180**
(x_1, s_2)	(x_2, s_1, s_3)	(10, 30, 10)	Yes	90
(x_1, s_3)	(x_2, s_1, s_2)	(0, 60, –10)	No	-
(x_2, s_1)	(x_1, s_2, s_3)	(60, –50, –60)	No	-
(x_2, s_2)	(x_1, s_1, s_3)	(10, 50, –10)	No	-
(x_2, s_3)	(x_1, s_1, s_2)	(0, 60, –10)	No	-
(s_1, s_2)	(x_1, s_2, s_3)	(–15, 25, 40)	No	-
(s_1, s_3)	(x_1, x_2, s_2)	(15, 15, 20)	Yes	**180**
(s_2, s_3)	(x_1, x_2, x_1)	(5, 5, 40)	Yes	60

Thus, the optimum solution exists at two points (0, 20) and (15, 15), as already found in Example 26.5.

EXERCISE 26.3

1. Convert the following LPP to the standard form

   Maximize $z = 3x_1 + 5x_2 + 7x_3$

   subject to
   $$6x_1 - 4x_2 \le 5$$
   $$3x_1 + 2x_2 + 5x_3 \ge 11$$
   $$4x_1 + 3x_3 \le 2,$$
   $$x_1, x_2 \ge 0 \text{ and } x_3 \text{ unrestricted.}$$

2. Express the following LPP

   Maximize $z = 2x_1 + 3x_2$

   subject to
   $$x_1 + 3x_2 \le 6$$
   $$3x_1 + 2x_2 \le 6$$
   $$x_1, x_2 \ge 0$$

   in the standard form. Determine all the basic solutions of the problem, classify them as feasible and infeasible and by direct substitution find the optimum solution.

3. Express the following LPP into the standard form

   Minimize $z = 2x_1 + 3x_2$

   subject to
   $$2x_1 - 3x_2 - x_3 = -4$$
   $$3x_1 + 4x_2 - x_4 = -6$$
   $$2x_1 + 5x_2 + x_5 = 10$$
   $$4x_1 - 3x_2 + x_6 = 18$$
   $$x_3, x_4, x_5, x_6 \ge 0.$$

4. Find all the basic solutions to following LPP

   Maximize $z = x_1 + 3x_2 + 3x_3$

   subject to $x_1 + 2x_2 + 3x_3 = 4$

   $2x_1 + 3x_2 + 5x_3 = 7$

   Also find which of the basic solutions are

   (a) basic feasible

   (b) non-degenerate basic feasible

   (c) optimal basic feasible.

5. Find an optimal solution to the following LPP by computing all basic solutions:

   Maximize $z = 2x_1 + 3x_2 + 4x_3 + 7x_4$

   subject to $2x_1 + 3x_2 - x_3 + 4x_4 = 8$

   $x_1 - 2x_2 + 6x_3 - 7x_4 = -3$

   $x_1, x_2, x_3, x_4 \geq 0.$

6. Show that the following system of linear equations has two degenerate feasible basic solutions and the non-degenerate basic solution is not feasible:

   $$2x_1 + x_2 - x_3 = 2, \quad 3x_1 + 2x_2 + x_3 = 3.$$

7. Show that all the basic solutions of the following LPP are infeasible

   Maximize $z = x_1 + x_2$

   subject to $x_1 + 2x_2 \leq 6$

   $2x_1 + x_2 \geq 16,$

   $x_1, x_2 \geq 0.$

26.6 COMPUTATIONAL DETAILS OF THE SIMPLEX METHOD

In this section, we illustrate the computational details of the simplex method for solving an LPP. This includes the rules for determining the entering and leaving variables as well as for terminating the computations when the optimum solution has been achieved. To illustrate all this, we reconsider below the problem described in Example 26.10.

Example 26.12: Solve the LPP as given in Example 26.10 by applying simplex method describing the computational details.

Solution: Writing the LPP in the standard form as

$$\begin{aligned} z - 40x_1 - 88x_2 \qquad &= 0 \\ x_1 + 4x_2 + s_1 \qquad &= 30 \\ 5x_1 + 2x_2 \qquad + s_2 &= 60 \\ x_1, x_2, s_1 \text{ and } s_2 &\geq 0. \end{aligned}$$

The variables s_1 and s_2 are the slacks associated with the respective constraints.

The *initial simplex table* T_0 is represented as follows:

Table T_0

	Basic	z	x_1	x_2 ↓	s_1	s_2	solution	
	z	1	–40	–88	0	0	0	z-row
←	s_1	0	1	$\boxed{4}$	1	0	30	s_1-row
	s_2	0	5	2	0	1	60	s_2-row

The simplex algorithm starts at the origin $(x_1, x_2) = (0, 0)$. Thus, non-basic (zero) variables are (x_1, x_2) and basic variables are (s_1, s_2).

Every simplex table gives a basic feasible solution which is obtained by setting the non-basic variables to zero. Thus, the initial Table T_0 gives the basic feasible solution as

$$s_1 = 30/1 = 30,\ s_2 = 60/1 = 60,\ z = 0,$$

with s_1 obtaining from the s_1-row and s_2 from the s_2-row.

Is the basic feasible solution achieved optimum? This is so when all the z-row coefficients of the non-basic variables are non-negative. By looking at the z-row, we note that this is not the case yet. Then to improve the objective function, we select x_2 as the *entering variable*, (marked '↓') since it has the *most negative coefficient* in the z-row.

To determine the *leaving variable* we compute the ratios by dividing the enteries under the solution column with the corresponding enteries in the column corresponding to the entering variable x_2. The ratios are $30/4 = 7.5$ and $60/2 = 30$.

The minimum non-negative ratio corresponding to the basic s_1, specifies that s_1 is the leaving variable, (marked '←') that its value will be zero in the next iteration. Thus, the new non-basic (zero) variables are (x_1, s_1) and basic variables are (s_2, x_2).

The x_2-column is called the *pivot column* and the s_1-row is called the *pivot row* and the intersection of these is called the *pivot element*. Here 4 is the pivot element.

To determine the new basic solution with (s_2, x_2) as the basic variables, we need to perform the appropriate row-operations called the *Gauss-Jorden row operations*. This involve the following computations.

I. *New pivot s_1-row = Current pivot row ÷ pivot element.*

II. *All others rows = Current row – (its pivot column coefficient) × new pivot row.*

These row operations will make the pivot element as unity and also give zero element above and below the pivot element.

In context with the Table T_0 given above these computations are

New pivot s_1-row = (Current s_1-row) ÷ 4
New z-row = Current z-row – (–88) (New pivot row)
New s_2-row = Current s_2-row – (2) (New pivot row)

This gives the new simplex Table T_1, corresponding to the new basic solution (x_2, s_2), as

Table T_1

	Basic	z	x_1 ↓	x_2	s_1	s_2	Solution
	z	1	–18	0	22	0	660
	x_2	0	1/4	1	1/4	0	15/2
←	s_2	0	$\boxed{9/2}$	0	–1/2	1	45

The basic feasible solution is

$$x_2 = 15/2, \quad s_2 = 45, \quad z = 660$$

Since one of the enteries in the z-row of the Table T_1 is still negative, thus the solution $z = 660$ is not optimal.

Proceeding as earlier, we select x_1 as the new entering variable, s_2 as the new leaving variable, on the basis of minimum ratio and, thus, 9/2 is the pivot element.

To determine the new basic solution with (x_1, x_2) as the basic variable, we perform the following row operations.

New pivot s_2-row = (Current s_2-row) + (9/2)

New z-row = (Current z-row) – (–18) New pivot row

New x_2-row = (Current x_2-row) – (1/4) New pivot row

This gives the new simplex Table T_2, corresponding to the new basic solution (x_1, x_2) as

Table T_2

Basic	z	x_1	x_2	s_1	s_2	Solution
z	1	0	0	20	4	840
x_2	0	0	1	5/12	1/18	5
x_1	0	1	0	–1/9	2/9	10

Since all the enteries in the z-row are non-negative, thus the optimum solution has been achieved and it is given by $x_1 = 10$, $x_2 = 5$, and $z = 840$.

Remarks:

1. The rules for selecting the entering and leaving variables are respectively referred to as the *optimality* and *feasibility* conditions.
2. In case we need to minimize z (instead to maximize), the entering variable is the non-basic variable having the most positive coefficient (instead of negative) in the z-row. Ties are broken arbitrarily. The optimum is reached at the iteration when all the entries in the z-rows are non-positive.

Example 26.13: For the following LPP

Maximize $z = 4x_1 + 8x_2 + 5x_3$

subject to $x_1 + 2x_2 + 3x_3 \le 18, 2x_1 + 6x_2 + 4x_3 \le 15, x_1 + 4x_2 + x_3 \le 6, x_1, x_2, x_3 \ge 0.$

Solution: Writing the LPP in the standard form as

$$\begin{aligned} z - 4x_1 - 8x_2 - 5x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 + s_1 &= 18 \\ 2x_1 + 6x_2 + 4x_3 + s_2 &= 15 \\ x_1 + 4x_2 + x_3 + s_3 &= 6 \end{aligned}$$

$$x_1, x_2, x_3, s_1, s_2, \text{ and } s_3 \ge 0.$$

The variables s_1, s_2, and s_3 are the slacks associated with the respective constraints. The system has three equations and seven variables. The starting basic feasible solution is

$$x_1 = x_2 = x_3 = 0 \text{ (non-basic)}$$

$$s_1 = 18, s_2 = 15, s_3 = 6 \text{ (basic)}, z = 0.$$

The initial simplex Table T_0 is represented as follows:

Table T_0

Basic	x_1	x_2 ↓	x_3	s_1	s_2	s_3	Solution
z	–4	–8	–5	0	0	0	0
s_1	1	2	3	1	0	0	18
s_2	2	$\boxed{6}$	4	0	1	0	15
← s_3	1	4	1	0	0	1	6

We observe that the z-row contains the negative coefficients so $z = 0$ is not the optimal value.

Here x_2 is the entering variable, s_3 is the leaving variable on the basis of the minimum ratio, and thus 4 is the pivot element. To determine the new basic solution with s_1, s_2, x_2 as the basic variables, we perform the following operations:

New pivot s_3-row = (Current s_3-row) ÷ 4

New z-row = (Current z-row) – (–8)(New pivot row)

New s_1-row = (Current s_1-row) – 2(New pivot row)

New s_2- row = (Current s_2-row) – 6(New pivot row)

and obtain the simplex Table T_1 corresponding to the new basic solution (s_1, s_2, x_2) as

The basic feasible solution is $s_1 = 15, s_2 = 6, x_2 = 3/2, z = 12.$

Since all the entries in the z-row of the Table T_1 are not non-negative, so this solution is not optimal. Now x_3 is the entering variable, s_2 is the leaving variable on the basis of minimum ratio, and thus, 5/2 is the pivot element. To determine the new basic solution with s_1, x_3, x_2 as the basic variables, we perform the following row operations:

Table T_1

Basic	x_1	x_2	x_3 ↓	s_1	s_2	s_3	Solution
z	–2	0	–3	0	0	2	12
s_1	1/2	0	5/2	1	0	–1/2	15
← s_2	1/2	0	$\boxed{5/2}$	0	1	–3/2	6
x_2	1/4	1	1/4	0	0	1/4	3/2

New pivot s_2-row = (Current pivot s_2-row) ÷ (5/2)
New z-row = (Current z-row) – (–3)(New pivot row)
New s_1-row = (Current s_1-row) – (5/2)(New pivot row)
New x_2-row = (Current x_2-row) – (1/4)(New pivot row)

and obtain the simplex Table T_2 corresponding to the new basic solution (s_1, x_3, x_2) as

Table T_2

Basic	x_1 ↓	x_2	x_3	s_1	s_2	s_3	Solution
z	–7/5	0	0	0	6/5	1/5	96/5
s_1	0	0	0	1	–1	1	9
x_3	1/5	0	1	0	2/5	–3/5	12/5
← x_2	$\boxed{1/5}$	1	0	0	–1/10	2/10	9/10

The basic feasible solution is

$$s_1 = 9, x_3 = 12/5, x_2 = 9/10 \text{ and } z = 96/5$$

Since all the entries in the z-row of the Table T_2 are not non-negative, thus this solution is not optimal. Now x_1 is the entering variable, x_2 is the leaving variable on the basis of minimum ratio, and thus, 1/5 is the pivot element to determine the new basic solution. We perform the following row operations

New pivot x_2-row = (Current pivot x_2-row) ÷ (1/5)
New z-row = (Current z-row) – (–7/5)(New pivot row)
New s_1-row = Current s_1-row
New x_3-row = (Current x_3-row) – (1/5)(New pivot row)

and obtain the simplex Table T_3 corresponding to the new basic solution (s_1, x_3, x_1) as

The basic feasible solution is

$$s_1 = 9, \quad x_3 = 3/2, \quad x_1 = 9/2, \quad z = 51/2$$

Table T_3

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	0	7	0	0	1/2	3	51/2
s_1	0	0	0	1	–1	1	9
x_3	0	–1	1	0	1/2	–1	3/2
x_1	1	5	0	0	–1/2	2	9/2

Since all the entries in the z-row of the Table T_3 are non-negative, thus the current solution is optimal. Hence, the solution to the given LPP is $x_1 = 9/2$, $x_2 = 0$, $x_3 = 3/2$, and $z = 51/2$.

Example 26.14: For the following product-mix selection LPP

Maximize $$z = 4x_1 + 5x_2 + 9x_3 + 11x_4$$

subject to $$x_1 + x_2 + x_3 + x_4 \le 15,\ 7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120,$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100,\ x_1, x_2, x_3, x_4 \ge 0$$

Solution: Writing the LPP in the standard form as

$$\begin{aligned} z - 4x_1 - 5x_2 - 9x_3 - 11x_4 \qquad\qquad &= 0 \\ x_1 + x_2 + x_3 + x_4 + s_1 \qquad &= 15 \\ 7x_1 + 5x_2 + 3x_3 + 2x_4 + s_2 \quad &= 120 \\ 3x_1 + 5x_2 + 10x_3 + 15x_4 + s_3 &= 100 \end{aligned}$$

$$x_1, x_2, x_3, x_4, s_1, s_2, \text{ and } s_3 \ge 0.$$

The initial simplex Table T_0 is represented as follows:

Table T_0

	Basic	x_1	x_2	x_3	↓ x_4	s_1	s_2	s_3	Solution
	z	–4	–5	–9	–11	0	0	0	0
	s_1	1	1	1	1	1	0	0	15
	s_2	7	5	3	2	0	1	0	120
←	s_3	3	5	10	$\boxed{15}$	0	0	1	100

Since the z-row contains negative entries, thus, the basic solution

$$s_1 = 15,\ s_2 = 120,\ s_3 = 100 \text{ and } z = 0$$

is not optimal.

Here x_4 is the entering variable, s_3 is the leaving variable on the basis of minimum ratio, and thus, 15 is the pivot element. T_0 obtain the basic solution, we perform the following row operations:

New pivot s_3-row = (Current pivot s_3-row) ÷ 15
New z-row = (Current z-row) − (−11)(New pivot row)
New s_1-row = (Current s_1-row) − (New pivot row)
New s_2-row = (Current s_2-row) − 2(New pivot row)

and obtain the new simplex Table T_1 corresponding to the new basic solution (s_1, s_2, x_4) as

Table T_1

	Basic	↓ x_1	x_2	x_3	x_4	s_1	s_2	s_3	Solution
	z	−9/5	−4/3	−5/3	0	0	0	11/15	220/3
←	s_1	$\boxed{4/5}$	2/3	1/3	0	1	0	−1/15	25/3
	s_2	33/5	13/3	5/3	0	0	1	−2/15	320/3
	x_4	1/5	1/3	2/3	1	0	0	1/15	20/3

Since the z-row contains the negative entries, thus the solution

$$x_1 = x_2 = x_3 = 0,\ x_4 = 20/3 \text{ and } z = 220/3$$

is not optimal.

Here x_1 is the entering variable, s_1 is the leaving variable on the basis of minimum ratio, and thus, 4/5 is the pivot element. To obtain the new basic solution, we perform the following row operations:

New pivot s_1-row = (Current pivot s_1-row) ÷ (4/5)
New z-row = (Current z-row) − (−9/5)(New pivot row)
New s_2-row = (Current s_2-row) − (33/5)(New pivot row)
New x_4-row = (Current x_4-row) − (1/5)(New pivot row)

and obtain the new simplex Table T_2 corresponding to the new basic solution (x_1, s_2, x_4) as

Table T_2

	Basic	x_1	x_2	↓ x_3	x_4	s_1	s_2	s_3	Solution
	z	0	1/6	−11/12	0	9/4	0	7/12	1105/12
	x_1	1	5/6	5/12	0	5/4	0	−1/12	125/12
	s_2	0	−7/6	−13/12	0	−33/4	1	5/12	455/12
←	x_4	0	1/6	$\boxed{7/12}$	1	−1/4	0	1/12	55/12

Since the z-row contains negative entries, thus the basic solution

$$x_1 = 125/12,\quad s_2 = 455/12,\quad x_4 = 55/12,\quad z = 1105/12$$

is not optimal.

Here x_3 is the entering variable, x_4 is the leaving variable on the basis of minimum ratio, and thus, 7/12 is the pivot element. To obtain the new basic solution, we perform the following row operations:

New pivot x_4-row = (Current pivot x_4-row) ÷ (7/12)

New z-row = (Current z-row) – (–11/12)(New pivot row)

New x_1-row = (Current x_1-row) – (5/12)(New pivot row)

New s_2-row = (Current s_2-row) – (–13/12)(New pivot row)

and obtain the next simplex Table T_3 corresponding to the new basic solution (x_1, s_2, x_4) as

Table T_3

Basic	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Solution
z	0	3/7	0	11/7	13/7	0	5/7	695/7
x_1	1	5/7	0	–5/7	10/7	0	–1/7	50/7
s_2	0	–6/7	0	13/7	–61/7	1	4/7	325/7
x_3	0	2/7	1	12/7	–3/7	0	1/7	55/7

Since the z-row contains only non-negative elements, the solution

$$x_1 = 50/7, \quad s_2 = 325/7, \quad x_3 = 55/7, \quad \text{and } z = 695/7$$

is optimal. Thus, the solution to the given LPP is

$$x_1 = 50/7, \quad x_2 = 0, \quad x_3 = 55/7, \quad x_4 = 0, \quad \text{and } z = 695/7$$

EXERCISE 26.4

Solve the following problems by simplex method assuming all the x_i to be non-negative

1. Maximize $z = 30x_1 + 20x_2$
   subject to
   $$-x_1 + x_2 \le 5$$
   $$2x_1 + x_2 \le 10$$
2. Maximize $z = 2x_1 + x_2 + 3x_3$
   subject to $4x_1 + 3x_2 + 6x_3 \le 12.$
3. Maximize $z = 5x_1 + 3x_2$
   subject to
   $$x_1 + x_2 \le 2$$
   $$5x_1 + 2x_2 \le 10$$
   $$3x_1 + 8x_2 \le 12$$
   $$x_1, x_2 \ge 0.$$

4. Minimize $z = 3x_1 + 5x_2 + 4x_3$
   subject to
$$2x_1 + 3x_2 \leq 8$$
$$2x_2 + 5x_3 \leq 10$$
$$3x_1 + 2x_2 + 4x_3 \leq 15$$
$$x_1, x_2, x_3 \geq 0.$$
5. Minimize $z = x_1 - 3x_2 + 2x_3$
   subject to
$$3x_1 - x_2 + 2x_3 \leq 7$$
$$-2x_1 + 4x_2 \leq 12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$
6. Suppose that x_1 and x_2 batteries of type A are being produced by process P_1 and P_2 respectively and, x_3 and x_4 batteries of type B are being produced by process P_3 and P_4 respectively. Let the profit per battery be Rs. 100 for battery of type A and Rs. 200 for battery of type B. Maximize the total profit sup;'bject to the constraints
$$12x_1 + 8x_2 + 6x_3 + 4x_4 \leq 120 \quad \text{(machine hrs.)}$$
$$3x_1 + 6x_2 + 12x_3 + 24x_4 \leq 180 \quad \text{(labour hrs.)}$$
7. Find the maximum as well as minimum value of the objective function $z = 4x + 5y$ subject to $2x + y \leq 6$, $x + 2y \leq 5$, $x - 2y \leq 2$, $-x + y \leq 2$, $x + y \geq 1$ and $x, y \geq 0$.
8. Two products A and B are to be manufactured. One single unit of A requires 2.5 minutes of punch press time and 5 minutes of assembly time, the profit for product A is Rs. 0.60 per unit. One single unit of product B requires 3 minutes of punch press time and 2.5 minutes of welding time. The profit for product B is Rs. 0.70 per unit. The capacity of the punch press deptt available for these products is 1,200 minutes/week. For the welding unit available capacity is 600 minutes per week and for assembly department it is 1500 minutes per week. Apply simplex method to find the quantities of products A and B so that the total weekly profit is maximum.
9. A firm produces three products which are processed on the machines. The relevant data is given below

Machine	*Time per unit (minutes) for product*			*Machine capacity (minutes/day)*
	A	B	C	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

The profit per unit for products A, B and C are respectively Rs. 4, 3 and 6. Determine the daily number of units to be manufactured for each product, assuming that units produced are consumed in the market.

10. A company manufactures purses, shaving bag and belts. The following table gives the availability of the resources, their usage by the three products, and the profit per unit. Find the optimum number of the three products to be produced to maximize the profit.

Resource	*Resource requirements per unit*			*Daily availability*
	Purse	*Bag*	*Belt*	
Leather (ft^2)	2	1	1.5	45 ft^2
Sewing (hr)	2	1	1	40 hrs
Finishing (hr)	1	.5	.75	45 hrs
Profit per unit (Rs)	25	20	22	

26.7 SOME EXCEPTIONAL CASES IN SIMPLEX METHOD

So far we have seen that in most of the LPP discussed in the preceding section a solution was obtained. However, as seen in Section 26.3, there are LPP for which no solution exists, or more than one solution exists, or the solution is unbounded one. Since such special cases have theoretical as well as practical importance, we study here these cases explicitly. We consider the following four special cases that arise in the application of the simplex method and illustrate them by considering an example in each case:

1. *Degeneracy and cyclic*
2. *Alternative optima*
3. *Unbounded solutions*
4. *Infeasible solutions*

26.7.1 Degeneracy and Cyclic

In the application of the simplex method, while selecting a leaving variable a tie (in cases it arises) for the minimum ratio is broken arbitrarily, and when it happens at least one basic variable becomes zero in the subsequent iteration. This gives rise to a degenerate basic feasible solution. The basis obtained in a specific iteration may get repeated at a later iteration resulting into never improving the objective function and never terminating the computation. This problem is said to be a *cyclic* problem. There are methods for eliminating cyclic but they lead to slowdown in computations considerably and are thus, generally not included in the LP codes. Further it is not advisable to stop the computations at the iteration when the degeneracy appears first and the solution is not optimum, since the solution may be temporarily degenerate. From practical point, the degeneracy means that model has at least one redundant constraint, indicating that some resources are superfluous. This information may lead to removing irregularities in the construction of the model. However, such situations occur rarely in practice.

Example 26.15: Maximize $z = 5x_1 + 3x_2$, subject to

$$x_1 + x_2 \le 2 \quad ①$$

$$5x_1 + 2x_2 \le 10 \quad ②$$

$$x_1, x_2 \ge 0.$$

Solution: Introducing the slacks s_1 and s_2 and writing the LPP in the standard form, we have

Maximize $z = 5x_1 + 3x_2$, subject to

$$x_1 + x_2 + s_1 \quad = 2$$
$$5x_1 + 2x_2 + s_2 = 10$$
$$x_1, x_2, s_1, s_2 \geq 0.$$

Using s_1 and s_2 as initial basic variables the starting simplex Table T_0 is

Table T_0

	Basic	x_1 ↓	x_2	s_1	s_2	Solution
	z	–5	–3	0	0	0
←	s_1	$\boxed{1}$	1	1	0	2
	s_2	5	2	0	1	10

Since all the entries in the z-row are not non-negative the solution $z = 0$ is not optimal.

Here x_1 is the entering variable, and the variables s_1 and s_2 tie for the leaving variable on the feasibility of minimum ratio. Breaking the tie arbitrarily and taking s_1 as the leaving variable, and hence 1 is the pivot element. Performing the necessary row operations, the resultant simplex Table T_1 is

Table T_1

Basic	x_1	x_2	s_1	s_2	Solution
z	0	2	5	0	10
x_1	1	1	1	0	2
s_2	0	–3	–5	1	0

Since all the entries in the z-row are non-negative the solution $x_1 = 2$, $x_2 = 0$, $z = 10$ is optimal. It is being a degenerate solution. The graphical solution of the given LP-model is represented in Fig. 26.6.

From Fig. 26.6, we observe that the point A (2, 0) is over determined and the constraint $5x_1 + 2x_2 \leq 10$ is redundant.

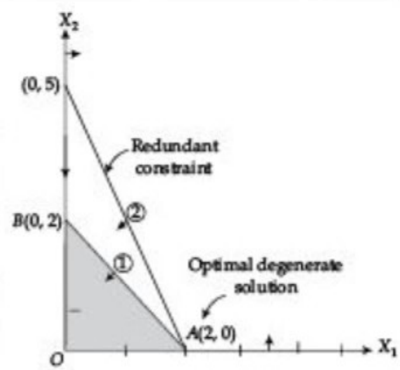


Fig. 26.6

26.7.2 Alternative Optima

When the objective function is parallel to a binding constraint, the objective function will assume the same optimum value at more than one solution point. In fact in such a situation there are infinity of such solutions. The practical implication of alternative optima is that, they give us flexibility to choose from many solutions without experiencing any deterioration in the objective function.

Example 26.16: Maximize $z = 2x_1 + 4x_2$, subject to

$$x_1 + 2x_2 \le 5 \quad ①$$
$$x_1 + x_2 \le 4 \quad ②$$
$$x_1\, x_2 \ge 0.$$

Solution: Introducing the slacks s_1 and s_2 and writing the given LPP in the standard form, we have

Maximize $z = 2x_1 + 4x_2$
subject to $x_1 + 2x_2 + s_1 \quad = 5$
$x_1 + x_2 \quad + s_2 = 4$
x_1, x_2, s_1, and $s_2 \ge 0$.

Using s_1 and s_2 as initial basic variables, the starting simplex Table T_0 is

Table T_0

	Basic	x_1	x_2 ↓	s_1	s_2	Solution
	z	–2	–4	0	0	0
←	s_1	1	$\boxed{2}$	1	0	5
	s_2	1	1	0	1	4

Since the entries in the z-row are not non-negative, the solution $z = 0$ is not optimal. Here x_2 is the entering variable, s_1 is the leaving variable on the basis of minimum ratio, and thus, 2 is the pivot element.

Performing the necessary row operations, the resultant simplex Table T_1 is

Table T_1

	Basic	x_1 ↓	x_2	s_1	s_2	Solution
	z	0	0	2	0	10
	x_2	1/2	1	1/2	0	5/2
←	s_2	$\boxed{1/2}$	0	–1/2	1	3/2

The solution $x_1 = 0, x_2 = 5/2, z = 10$ is optimum, since the entries in the z-row are non-negative.

An alternative optimum is obtained by considering further x_1 as the entering variable, s_2 as the leaving variable on the basis of minimum ratio, and 1/2 as the pivot element. Performing the necessary row operations, the resultant simplex Table T_2 is

Table T_2

Basic	x_1	x_2	s_1	s_2	Solution
z	0	0	2	0	10
x_2	0	1	10	−1	1
x_1	1	0	−1	2	3

The solution $x_1 = 3, x_2 = 1, z = 10$ is alternative optima.

Thus, simplex method gives, say $P(0, 5/2)$ and $Q(3, 1)$ as two solutions to the given LPP. Mathematically, given $0 \le \alpha \le 1$, then all the points on the line segment PQ given by

$$x_1 = \alpha(0) + (1 - \alpha)3 = 3 - 3\alpha$$

$$x_2 = \alpha(5/2) + (1 - \alpha).1 = 1 + \frac{3}{2}\alpha$$

is the solution to the given LPP.

The graphical solution of the given LP-model is represented in Fig. 26.7.

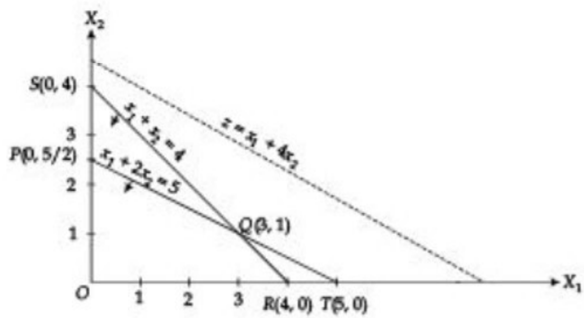


Fig. 26.7

We observe that the objective function $z = 2x_1 + 4x_2$ is parallel to the equality constraint $x_1 + 2x_2 = 5$.

26.7.3 Unbounded Solutions

If in case of some LP models, the values of the decision variables may be increased indefinitely without violating any of the constraints, then the solution space is unbounded. This results in the value of the objective function increasing (in case of maximization) or decreasing (in case of minimization) indefinitely. In case of unboundedness, at any iteration all the constraint coefficients of some non-basic variables are zero or negative. Practically, it indicates that the model has not been designed properly, that is, either one or more essential constraints have not been accounted for, or the constraints being used are not correct.

Example 26.17: Maximize $z = 3x_1 + 2x_2$, subject to

$$x_1 - x_2 \le 2 \quad ①$$
$$x_1 \le 3 \quad ②$$

Solution: Introducing the slacks s_1 and s_2 and writing the LPP in the standard form, we have

Maximize $z = 3x_1 + 2x_2$, subject to

$$x_1 - x_2 + s_1 = 2$$
$$x_1 + s_2 = 3$$

Using s_1 and s_2 as initial basic variables, the starting simplex Table T_0 is

Table T_0

Basic	x_1	x_2	s_1	s_2	Solution
z	–3	–2	0	0	0
s_1	1	–1	1	0	2
s_2	1	0	0	1	3

In the starting Table T_0, x_1 being with the most negative coefficient is the most suitable entering variable, however, we observe that all the constraint coefficients under x_2 are negative or zero, thus x_2 can be increased indefinitely without violating any of the constraints, resulting in an indefinite increase of z. Thus, solution space is unbounded in the direction of x_2 and so the given LPP has an unbounded solution, as shown in Fig. 26.8.

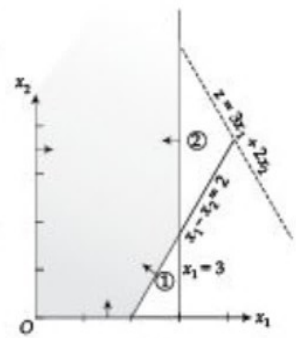


Fig. 26.8

26.7.4 Infeasible Solution

When the constraints are not satisfied simultaneously, the LP-model has no feasible solution. This cannot happen if all the constraints are of the ≤ type, since the slacks provide a feasible solution. For the constraints of the type =, and, ≥ we use artificial variables (refer to Section 26.9) which need to reduce to zero at the optimum in case of feasible solution. But if at least one artificial variable appears positive in the optimum iteration this makes the solution to be infeasible. Practically, the infeasible solution space indicates that the model has not been designed correctly.

Example 26.18: Maximize $z = 3x_1 + 2x_2$, subject to

$$2x_1 + x_2 \le 2 \quad ①$$
$$3x_1 + 4x_2 \ge 12 \quad ②$$
$$x_1, x_2 \ge 0.$$

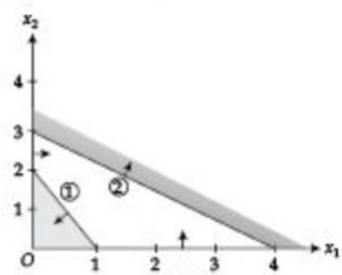


Fig. 26.9

Solution: The graphical solution of the LP-model is given in Fig. 26.9. There is no point (x_1, x_2) which can lie in the regions satisfied by both the constraints, hence, the model has infeasible solution or we may call it a *pseudo-optimal* solution.

For simplex treatment of the LPP, refer to Section 26.9, Example 26.21.

26.8 SENSITIVITY OR POST OPTIMALITY ANALYSIS

We have seen that if a linear optimization model has a finite optimal solution, then it finds an optimal basic solution also. Practically, while using an LP-model user is hardly interested only in the optimal values. His main interest lies to know how far the input parameters values can vary without much changes in the optimal solution, that is, the basic variables and their values remain unchanged. Such post-optimal studies are referred to as *sensitivity*, or *post optimality analysis*, since it studies the sensitivity of the optimal solution to the changes made in the model. A few of these studies include:

1. Changes in the objective function coefficients.
2. Changes in the availability of resources.
3. Changes due to the addition and deletion of some variables
4. Changes due to the addition and deletion of some linear constraints.

One or more of these changes, when effected in an LPP, may result in any of the following cases:

(a) The optimal solution remains unchanged both in terms of the basic variables and their values.

(b) The basic variables remain the same but their values are changed.

(c) The basic solution gets changed completely.

In general, in face of the changes effected in an LPP, the revised LP-model is not solved afresh but, a considerable use of the previous optimal solution is made. The sensitivity analysis in case of a few simple cases of two-variable LP-models may be investigated on the basis of the graphical solutions. However, to learn the concept of sensitivity analysis, in general, we study it in the context of the following LP-model.

$$\text{Maximize} \quad z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, \ldots, m$$

$$x_j \ge 0, \quad j = 1, 2, \ldots, n.$$

Specifically, we consider Example 26.13. We reproduce here *the initial system of equations* given by

$$z - 4x_1 - 8x_2 - 5x_3 + 0.s_1 + 0.s_2 + 0.s_3 = 0 \quad (z\text{-row}) \qquad \ldots(\mathrm{I}_1)$$

$$x_1 + 2x_2 + 3x_3 + s_1 + 0.s_2 + 0.s_3 = 18 \quad (s_1\text{-row}) \qquad \ldots(\mathrm{I}_2)$$

$$2x_1 + 6x_2 + 4x_3 + 0.s_1 + s_2 + 0.s_3 = 15 \quad (s_2\text{-row}) \qquad \ldots(\mathrm{I}_3)$$

$$x_1 + 4x_2 + x_3 + 0.s_1 + 0.s_2 + s_3 = 6 \quad (s_3\text{-row}) \qquad \ldots(\mathrm{I}_4)$$

and, *the final (optimal) system of equations* given by

$$z + 0.x_1 + 7x_2 + 0.x_3 + 0.s_1 + \frac{1}{2}s_2 + 3s_3 = \frac{51}{2} \qquad \ldots(\mathrm{F}_1)$$

$$0x_1 + 0x_2 + 0.x_3 + s_1 - s_2 + s_3 = 9 \quad \text{...}(F_2)$$

$$0x_1 - x_2 + x_3 + 0s_1 + \frac{1}{2}s_2 - s_3 = \frac{3}{2} \quad \text{...}(F_3)$$

$$x_1 + 5x_2 + 0x_3 + 0.s_1 - \frac{1}{2}s_2 + 2s_3 = \frac{9}{2} \quad \text{...}(F_4)$$

The optimal solution of the problem is

$$x_1 = \frac{9}{2}, \quad x_2 = 0, \quad x_3 = \frac{3}{2} \quad \text{and } z = \frac{51}{2}.$$

26.8.1 Changes in the Objective Function Coefficients

Here we investigate if there is a range of variation for the objective function coefficients that will keep the current optimum solution unaltered.

Consider the coefficient of the non-basic variable x_2 in z-row of the initial system of equations. It is obvious that if it is made less profitable, the current solution remains optimal. But in case its profitability is sufficiently increased, then there may be some scope of improvement on the current optimal solution. Our interest may be to know the revised level of unit profit for x_2 for which the current solution becomes non-optimal.

Suppose the revised unit-profit coefficient for x_2 is $(8 + p_2)$, $p_2 \geq 0$. The z-row in the initial system becomes

$$z - 4x_1 - (8 + p_2)\, x_2 - 5x_3 = 0 \quad \text{...}(I_1')$$

Since in performing each simplex iteration, we simply add a multiple of a row to z-row, thus in the final system of equations the z-row must take the form

$$z + (7 - p_2)\, x_2 + \frac{1}{2}s_2 + 3s_3 = \frac{51}{2} \quad \text{...}(F_1')$$

We note that in case of $p_2 > 7$, the coefficient of x_2 in row (F_1') is negative and then the current solution will not be optimal and x_2 will be the new entering variable.

Thus, the z-row coefficients of the non-basic variables in the final system of equations represent the largest positive increments to the original objective function coefficients that will keep the current optimum solution unaltered.

Next we investigate for the range of variation of the coefficient of the variables x_1 or x_3, which are in the basis. Suppose that the revised unit-profit coefficient for x_1 is $(4 + p_1)$, the z-row in the initial system becomes

$$z - (4 + p_1)x_1 - 8x_2 - 5x_3 = 0 \quad \text{...}(I_1'')$$

and in the final system it becomes

$$z - p_1x_1 + 7x_2 + \frac{1}{2}s_2 + 3s_3 = \frac{51}{2}. \quad \text{...}(F_1'')$$

To draw any conclusion, we must make the coefficient of x_1 equal to zero. For this, multiplying the row (F_4) with p_1 and adding it to the row(F_1''), we obtain the new z-row as

$$z + (7 + 5p_1)x_2 + \frac{1}{2}(1 - p_1)s_2 + (3 + 2p_1)s_3 = \frac{51}{2} + \frac{9}{2}p_1$$

From the above equation, we observe that for the range $-3/2 \le p_1 \le 1$, the coefficients in the new z-row remain non-negative and the current solution remains optimal. In case p_1 falls outside this range, the current solution does not remain optimal.

Similarly, we can investigate the range of variation for the increment p_3 in the coefficient of the variable x_3. In case the change is in more than one variable, basic or non-basic, sensitivity analysis can be explained on the same approach.

26.8.2 Changes in the Availability of the Resources

Changes in the availability of the resources lead to change in the right-hand side constants in the constraints. Here we want to investigate whether a previously optimal basis remains feasible in case of change or not. In case the basis remains feasible the new solution is optimal, because the coefficients in the z-row are unchanged.

Let us suppose that the constant on the right-hand side of s_1-row, that is, in row (I_2) is changed to $(18 + k)$. We note that the slack s_1 for this row is in the final basis and so s_1 will change by the amount k. Thus, from the row(F_2) in the final set of equations, the present solution remains feasible so long as $k \ge -18$, since the value of $s_1 = 0$, if $k_1 = -18$.

Next, consider the constant on the right-hand side of s_2-row, that is, the row (I_3), of the initial set. Suppose it is changed to $(15 + m)$. Our interest lies in the range of m that does not alter the current solution.

Performing the simplex algorithm while carring m along with 15, on the right side of the row (I_3), we note that contribution due to m is the same as due to s_2 on the left side of this row. Thus, after the final iteration we arrive at the following set of equations:

$$z + 0.x_1 + 7x_2 + 0.x_3 + 0.s_1 + \frac{1}{2}s_2 + 3s_3 = \frac{51}{2} + \frac{1}{2}m \qquad \text{...}(F_1^*)$$

$$0.x_1 + 0.x_2 + 0.x_3 + s_1 - s_2 + s_3 = 9 - m \qquad \text{...}(F_2^*)$$

$$0.x_1 - x_2 + x_3 + 0.s_1 + \frac{1}{2}s_2 - s_3 = \frac{3}{2} + \frac{1}{2}m \qquad \text{...}(F_3^*)$$

$$x_1 + 5x_2 + 0.x_3 + 0.s_1 - \frac{1}{2}s_2 + 2s_3 = \frac{9}{2} - \frac{1}{2}m \qquad \text{...}(F_4^*)$$

Setting the non-basic variables x_3, s_2 and s_3 to zero yields the values for the basic variables, which are now in terms of m. For feasibility of the current solution, thus all the constants on the right-hand side of the above set of equations must be non-negative, that is,

$$\frac{51}{2} + \frac{1}{2}m \ge 0, \quad 9 - m \ge 0, \quad \frac{3}{2} + \frac{1}{2}m \ge 0, \quad \text{and} \quad \frac{9}{2} - \frac{1}{2}m \ge 0$$

which implies that $-3 \le m \le 9$.

Therefore, the current solution remains feasible for $-3 \le m \le 9$. This approach can be generalized to simultaneous changes in several of the right-hand side coefficients and in that case we obtain set of inequalities needed to the satisfied by the amounts by which the coefficients have been increased. Similarly, we can continue to develop the methods to test the sensitivity to the coefficients a_{ij}. But all this is an adhoc approach, a comprehensive treatment of the sensitivity can be studied by the use of concept of duality and the primal-dual relationship.

EXERCISE 26.5

1. Solve the LPP

   Maximize $z = 2x_1 + x_2$

   subject to
   $$4x_1 + 3x_2 \le 12$$
   $$4x_1 + x_2 \le 8$$
   $$4x_1 - x_2 \le 8$$
   $$x_1, x_2 \ge 0.$$

2. For the following LPP

   Maximize $z = 3x_1 + 2x_2$

   subject to
   $$4x_1 - x_2 \le 8$$
   $$4x_1 + 3x_2 \le 12$$
   $$4x_1 + x_2 \le 8$$
   $$x_1, x_2 \ge 0.$$

   show that the associated simplex iterations are temporarily degenerate and also solve the problem.

3. Solve the LPP

   Maximize $z = 10x_1 + 6x_2$

   subject to $5x_1 + 3x_2 \le 30$, $x_1 + 2x_2 \le 18$, and $x_1, x_2 \ge 0$.

4. Show by simplex method that the LPP

   Maximize $z = 2x_1 + x_2$

   subject to $x_1 - x_2 \le 10$, $2x_1 - x_2 \le 40$ and $x_1, x_2 \ge 0$

   has an unbounded optimal solution.

5. Show by simplex method that the LPP

   Minimize $z = -40x_1 - 100x_2$

   subject to $10x_1 + 5x_2 \le 250$, $2x_1 + 5x_2 \le 100$, $2x_1 + 3x_2 \le 90$ and $x_1, x_2 \ge 0$

   has infinite solutions.

6. Show by simplex method that the LPP

   Maximize $z = 20x_1 + 10x_2 + x_3$

   subject to $3x_1 - 3x_2 + 5x_3 \le 50$, $x_1 + x_3 \le 10$, $x_1 - x_2 + 4x_3 \le 20$ and $x_1, x_2, x_3 \ge 0$

   has an unbounded solution.

7. A manufacturing firm produces two products X and Y which are equally profitable. It has to supply 40 units of X and 20 units of Y per week. The technology process implies that production of X must always be at least as large as of Y. There are two raw material constraints to be satisfied: $5X + 8Y \le 400$, and $55X + 50Y \le 2750$

   Solve the problem and comment on the solution so obtained.

8. A steel company produces two kinds of iron I_1, I_2 by using three kinds of raw material R_1, R_2 and R_3 as per the details given below. Maximize the daily profit.

Raw material	Raw material needed per ton		Availability per day (tons)
	Iron I_1	Iron I_2	
R_1	2	1	16
R_2	1	1	8
R_3	0	1	3.5
Net profit per ton (Rs.)	1500	3000	

26.9 DIFFICULTIES IN THE STARTING SOLUTION: THE ARTIFICIAL VARIABLES

In case of LPP in which all constraints are of the type ($\le$) with non-negative right-hand sides, all-slack basic feasible solution is a convenient choice as starting solution. But when the LPP involves (=) and/or ($\ge$) constraints, then it may sometimes be difficult to find a starting basic feasible solution. In such a situation, the concept of *artificial variable(s)* is helpful. These variables are introduced to play the role of slacks at the first iteration, and are then driven to zero level at some later iteration. We present here two methods:

1. *The M-method, or the penalty method*
2. *The two-phase method.*

26.9.1 The *M*-Method

The *M*-method starts with the LPP in the standard form. We add an artificial variable R_i to the left-hand side of the each constraint of the (=, or $\ge$) type with the purpose to obtain an initial basic feasible solution. Since the artificial variables are extraneous to the LP model, we use a feedback mechanism so that these variables do not appear in the final solution. This is achieved by penalizing (assigning coefficient $-M$ in maximization problems and $+M$ in minimization problems, where M is very large) the artificial variables in the objective function. Thus, by assigning a high penality cost it is ensured that they will be driven to zero, when the objective function is optimized. Once an artificial variable leaves the basis, it will never enter again in the subsequent iterations. Further if in the optimal table, the artificial variable remains with positive level, then the LP model is said to be *infeasible*. But if the artificial variable remains in the optimal table at zero level, then the solution is said to be *pseudo-optimal*. We explain all this in the examples considered next.

Example 26.19: Maximize $z = 3x_1 + 2x_2 + 3x_3$

subject to $2x_1 + x_2 + x_3 \le 2$, $3x_1 + 4x_2 + 2x_3 \ge 8$, $x_1, x_2, x_3 \ge 0$.

Solution: Introducing x_4 as a slack in the first constraint and x_5 as a surplus in the second constraint, the standard form of the given LPP is

$$\text{Maximize} \quad z = 3x_1 + 2x_2 + 3x_3$$

$$\text{subject to} \quad \begin{aligned} 2x_1 + x_2 + x_3 + x_4 \quad &= 2 \\ 3x_1 + 4x_2 + 2x_3 \quad - x_5 &= 8 \\ x_1, x_2, x_3, x_4 \text{ and } x_5 &\geq 0 \end{aligned}$$

Since the second equality constraint does not have a variable to play the role of slack, we add artificial variable R_1 in the second equation and penalize it in the objective function with the coefficient $-M$, M being large positive. The resultant LPP is

$$\text{Maximize} \quad z = 3x_1 + 2x_2 + 3x_3 - MR_1$$

$$\text{subject to} \quad \begin{aligned} 2x_1 + x_2 + x_3 + x_4 \quad\quad &= 2 \\ 3x_1 + 4x_2 + 2x_3 \quad - x_5 + R_1 &= 8 \\ x_1, x_2, x_3, x_4, x_5, \text{ and } R_1 &\geq 0. \end{aligned}$$

We can use now x_4 and R_1 as an initial basic solution. The starting simplex Table T_0 is given as

Table T_0

Basic	x_1	x_2	x_3	x_5	x_4	R_1	Solution
z	–3	–2	–3	0	0	M	0
x_4	2	1	1	0	1	0	2
R_1	3	4	2	–1	0	1	8

Before proceeding further, we need to modify the z-row in the Table T_0 consistent with the rest of the table.

In Table T_0, the solution is $x_1 = x_2 = x_3 = x_5 = 0$, $x_4 = 2$ and $R_1 = 8$, which gives $z = -8M$ instead of $z = 0$, as indicated in the z-row under the column 'solution'. This inconsistency is eliminated by multiplying the R_1-row by $(-M)$ and adding it to the z-row. Thus,

$$\text{New } z\text{-row} = \text{Current } z\text{-row} - M(\text{Current } R_1\text{-row});$$

and the modified table is given below as

Table T_0 (modified)

	Basic	x_1	x_2 ↓	x_3	x_5	x_4	R_1	Solution
	z	$-3-3M$	$-2-4M$	$-3-2M$	M	0	0	$-8M$
←	x_4	2	$\boxed{1}$	1	0	1	0	2
	R_1	3	4	2	–1	0	1	8

We observe that $z = -8M$ is consistent with the initial basic feasible solution $x_4 = 2, R_1 = 8$.

Here x_2 is the entering variable (being with the most negative coefficient since M is large) and x_4 is the leaving variable (we can consider R_1 instead of x_4 as entering variable, as both ratios are the same). Thus, 1 is the pivot element. To obtain the new basic solution, we perform the following operations.

New x_4-row = Current x_4-row

New z-row = (Current z-row) − (−2 − 4M)(New x_4-row)

New R_1-row = (Current R_1-row) − 4(New x_4-row)

and obtain the new simplex Table T_1 corresponding to the new basic solution (x_2, R_1) as

Table T_1

Basic	x_1	x_2	x_3	x_5	x_4	R_1	Solution
z	$1+5M$	0	$-1+2M$	M	$2+4M$	0	4
x_2	2	1	1	0	1	0	2
R_1	−5	0	−2	−1	−4	1	0

Since all the entries in the z-row are non-negative, the optimality criteria is satisfied. We observe that the artificial variable R_1 is still in the basic feasible solution but it is with value zero. Thus, the optimum solution exists and is given by

$$x_1 = 0, x_2 = 2, x_3 = 0, z = 4.$$

Example 26.20: Minimize $z = 4x_1 + x_2$,

subject to $3x_1 + x_2 = 3, 4x_1 + 3x_2 \geq 6, x_1 + 2x_2 \leq 4, x_1, x_2 \geq 0$.

Solution: Introducing x_3 as surplus and x_4 as slack in the second and third constraints respectively and taking $z' = -z$, standard form of the LPP is given as

$$\begin{aligned} \text{Maximize} \quad & (z') = -4x_1 - x_2 \\ \text{subject to} \quad & 3x_1 + x_2 = 3 \\ & 4x_1 + 3x_2 - x_3 = 6 \\ & x_1 + 2x_2 + x_4 = 4 \\ & x_1, x_2, x_3, \text{ and } x_4 \geq 0 \end{aligned}$$

Since the first and second equality constraints do not have slack variables, we add artificial variables R_1 and R_2 in first and second equation respectively and penalize these in the objective function with the penality $-M$, M being large positive. The resultant LPP is

$$\begin{aligned} \text{Maximize} \quad & (z') = -4x_1 - x_2 - MR_1 - MR_2 \\ \text{subject to} \quad & 3x_1 + x_2 + R_1 = 3 \\ & 4x_1 + 3x_2 - x_3 + R_2 = 6 \\ & x_1 + 2x_2 + x_4 = 4 \end{aligned}$$

We can now use R_1, R_2 and x_4 as an initial basic solution.

The starting simplex Table T_0 is given as

Table T_0

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z'	4	1	0	M	M	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

Eliminating the inconsistency in the Table T_0 by substituting out R_1 and R_2 in the z-row, that is

New z'-row = Current z'-row − M(Current R_1-row + Current R_2-row);

the modified table is

Table T_0 (modified)

	Basic	x_1 ↓	x_2	x_3	R_1	R_2	x_4	Solution
	z'	$4-7M$	$1-4M$	M	0	0	0	$-9M$
←	R_1	3	1	0	1	0	0	3
	R_2	4	3	−1	0	1	0	6
	x_4	1	2	0	0	0	1	4

The solution $R_1 = 3$, $R_2 = 4$, $x_4 = 4$ and $z' = -9M$

in the modified Table T_0 is not optimal, since the z'-row has negative entries. The variable x_1 has the most negative coefficient in the z'-row, thus is the entering variable and R_1 is the leaving variable on the basis of the minimum ratio condition. Hence, 3 is the pivot element. To obtain the new basic solution with variables x_1, R_2, x_4, we perform the following operations:

New R_1-row = (Current R_1-row) ÷ 3

New z'-row = Current z-row − $(4-7M)$(New R_1-row)

New R_2-row = Current R_2-row − 4(New R_1-row)

New x_4-row = Current x_4-row − (New R_1-row)

The resultant simplex Table T_1 is

The solution $x_1 = 1$, $R_2 = 2$, $x_4 = 3$ and $z' = -(4 + 2M)$

in the Table T_1 is not optimal, since z'-row has negative entries. The variable x_2 has the most negative coefficient in the z'-row, thus is the entering variable, and R_2 is the leaving variable on the basis of the minimum ratio condition. Hence, 5/3 is the pivot element. To obtain the new basic solution with variables x, x_2, x_4, we perform the following operations:

Table T_1

Basic	x_1	x_2 ↓	x_3	R_1	R_2	x_4	Solution
z'	0	$-\frac{1+5M}{3}$	M	$\frac{7M-4}{3}$	0	0	$-(4+2M)$
x_1	1	1/3	0	1/3	0	0	1
← R_2	0	$\boxed{5/3}$	−1	−4/3	1	0	2
x_4	0	5/3	0	−1/3	0	1	3

New R_2-row = (Current R_2-row) ÷ $\frac{5}{3}$

New z'-row = Current z'-row + $\frac{1+5M}{3}$(New R_2-row),

New x_1-row = Current x_1-row − $\frac{1}{3}$(New R_2-row),

New x_4-row = Currnet x_4-row − $\frac{5}{3}$(New R_2-row).

The resultant simplex Table T_2 is

Table T_2

Basic	x_1	x_2	x_3 ↓	R_1	R_2	x_4	Solution
z'	0	0	−1/5	$\frac{5M-8}{5}$	$\frac{5M+1}{5}$	0	−18/5
x_1	1	0	1/5	3/5	−1/5	0	3/5
x_2	0	1	−3/5	−4/5	3/5	0	6/5
← x_4	0	0	$\boxed{1}$	1	−1	1	1

The solution $x_1 = 3/5$, $x_2 = 6/5$, $x_4 = 1$ and $z' = -18/5$

in the Table T_2 is not optimal, since z'-row has negative entry. The variable x_3 has negative coefficient in the z'-row. Thus, it is the entering variable. Further, x_4 is the leaving variable on the basis of the minimum ratio condition. Hence, 1 is the pivot element. To obtain the new basic solution with variables x_1, x_2, x_3, we perform the following row operatons:

New x_4-row = Current x_4-row

New z'-row = (Current z'-row) + (1/5) (New x_4-row),

New x_1-row = (Current x_1-row) − (1/5)(New x_4-row),

New x_2-row = Current x_2-row + (3/5)(New x_4-row).

The resultant simplex Table T_3 is

Table T_3

Basic	x_1	x_2	x_3	R_1	R_2	x_4	solution
z'	0	0	0	$\frac{5M-3}{5}$	M	1/5	–17/5
x_1	1	0	0	2/5	0	–1/5	2/5
x_2	0	1	0	–1/5	0	3/5	9/5
x_3	0	0	1	1	–1	1	1

The solution $x_1 = 2/5$, $x_2 = 9/5$, $x_3 = 1$, and $z' = -17/5$

in the Table T_3 is optimal since all the entries in the z'-row are non-negative. Thus, the optimal solution to the given minimization problem is

$$x_1 = 2/5, x_2 = 9/5, \text{ and } z = 17/5.$$

Remark: The above problem can be solved in the minimization form also. The non-basic variable with the largest positive coefficient is selected as the entering variable. When all the enteries in the z-row are non-positive, the optimal solution is reached.

Example 26.21: Minimize $z = 3x_1 + 2x_2$

subject to

$$2x_1 + x_2 \leq 2$$
$$3x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

Solution: Introducing x_3 and x_4 as slack and surplus respectively in the first and second constraints and then introducing R_1 as artificial variable in the resulting second equality constraint; penalizing it with the penality $-M$ in the objective function, the LPP in the standard form is

Maximize $z = 3x_1 + 2x_2 - MR_1$

subject to

$$2x_1 + x_2 + x_3 = 2$$
$$3x_1 + 4x_2 - x_4 + R_1 = 12$$
$$x_1, x_2, x_3, x_4, \text{ and } R_1 \geq 0$$

using x_3, R_1 as initial basic variables, the starting simplex Table T_0 is

Table T_0

Basic	x_1	x_2	x_4	x_3	R_1	Solution
z	–3	–2	0	0	M	0
x_3	2	1	0	1	0	2
R_1	3	4	–1	0	1	12

Eliminating the inconsistency in the Table T_0 by substituting out R_1 in the z-row, that is,

New z-row = Current z-row − M (current R_1-row);

the modified table is

Table T_0 (modified)

Basic	x_1	x_2 ↓	x_4	x_3	R_1	Solution
z	$-3(1+M)$	$-2(1+2M)$	M	0	0	$-12M$
← x_3	2	$\boxed{1}$	0	1	0	2
R_1	3	4	−1	0	1	12

The solution in the modified Table T_0

$$x_3 = 2, R_1 = 12, z = -12M$$

is not optimal, since the z-row has negative entries. The variable x_2 has the most negative coefficient in the z-row, thus is the entering variable, and x_3 is the leaving variable on the basis of the minimum ratio condition. Hence, 1 is the pivot element. To obtain the new basic solution with variables x_2, and R_1 we perform the following row operations:

New x_3-row = Current x_3-row

New z-row = Current z-row + $2(1+2M)$(New x_3-row),

New R_1-row = Current R_1-row − 4(New x_3-row);

the resultant simplex Table T_1 is

Table T_1

Basic	x_1	x_2	x_4	x_3	R_1	Solution
z	$1+5M$	0	M	$2(1+2M)$	0	$4(1-M)$
x_2	2	1	0	1	0	2
R_1	−5	0	−1	−4	1	4

The solution $x_2 = 2, R_1 = 4, x_1 = x_3 = x_4 = 0, z = 4(1-M)$

in the Table T_3 is optimal since all the entries in the z-row are non-negative. But it contains the artificial variable R_1 at a positive level (=4), thus the given LPP does not have optimal basic feasible solution.

26.9.2 Two-Phase Method

The penality method sometimes leads into difficulty, especially when the problem is to be solved on a digital computer, where M is to be assigned some large numerical value. The simultaneous manipulation of large and small coefficients may result in round off error, affecting the accuracy of the end result. The two-phase method checks this problem by eliminating the use of penality M

altogether. Though the method was developed for use on digital computer, but is equally well suited for hand computations also. As the name suggests, in two-phase method we try to solve the LPP in two phases. In Phase I, we attempt to drive all artificial variables to zero and find a starting basic feasible solution, and if so, then in Phase II, we optimize the actual objective function, as illustrated below.

Phase I: Put the problem in the standard form and add the artificial variables to the constraints to find a starting solution. Formulate a new objective function r which consists of the sum of the artificial variables. Using simplex method, minimize (r) subject to the constraints, and obtain the optimum basic feasible solution. If the minimum (r) is positive, the LPP has no feasible solution; terminate the procedure since a positive value of an artificial variable signifies that original constraints are not compatible. In case minimum (r) = 0 and also no artificial variable remains basic at zero level at the end of Phase I, proceed to Phase II.

If minimum (r) = 0 and some of the artificial variables remain basic at zero level at the end of Phase I, continue with the Phase I and try to derive all artificial variables out of the basis and then proceed to Phase II.

Phase II: Use the optimum feasible solution obtained from Phase I as an initial solution for the original LPP and using simplex iterations, obtain an optimal basic feasible solution.

We illustrate all this in the examples considered next.

Example 26.22: Maximize $z = 2x_1 + x_2$

subject to $x_1 + x_2 \geq 2$, $x_1 + x_2 \leq 4$, $x_1, x_2 \geq 0$.

using the two-phase method.

Solution: Introducing x_3 as surplus and x_4 as slack in the first and second constraints respectively and then introducing R_1 as artificial variable in the resulting first constraint, the LPP model for the Phase I is

Minimize $r = R_1$

subject to

$$x_1 + x_2 - x_3 + R_1 = 2$$
$$x_1 + x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4, \text{ and } R_1 \geq 0$$

Using R_1, x_4 as initial basic variables, the starting simplex Table T_0 is

Table T_0

Basic	x_1	x_2	x_3	R_1	x_4	Solution
r	0	0	0	–1	0	0
R_1	1	1	–1	1	0	2
x_4	1	1	0	0	1	4

Substituting out R_1 in the r-row, that is,

New r-row = Current r-row + Current R_1-row;

the modified Table T_0 is

Table T_0 (modified)

	Basic	x_1 ↓	x_2	x_3	R_1	x_4	Solution
	r	1	1	–1	0	0	2
←	R_1	$\boxed{1}$	1	–1	1	0	2
	x_4	1	1	0	0	1	4

The solution $R_1 = 2, x_4 = 4, r = 2$

in the Table T_0 (modified) is not optimal, since the entries in the r-row are positive. Take x_1 as the entering variable (we can take x_2 also), R_1 as the leaving variable on the basis of minimum ratio condition, and 1 as the pivot element. We perform the following row operations:

New R_1-row = Current R_1 –row

New r-row = Current r-row – New R_1-row

New x_4-row = Current x_4-row – New R_1-row

the resultant simplex Table T_1 is

Table T_1

Basic	x_1	x_2	x_3	R_1	x_4	Solution
r	0	0	0	–1	0	0
x_1	1	1	–1	1	0	2
x_4	0	0	1	–1	1	2

The solution $x_1 = 2, x_2 = x_3 = R_1 = 0, x_4 = 2, r = 0$

is optimal, since all the entries in the r-row are non-positive. Because minimum $r = 0$, Phase I gives the basic feasible solution $x_1 = 2$ and $x_4 = 2$.

Eliminating the column corresponding to the artificial variable in the Table T_1 and proceeding to Phase II, we write the LPP as

$$\begin{aligned}
&\text{Maximize} \quad z = 2x_1 + x_2 \\
&\text{subject to} \quad x_1 + x_2 - x_3 = 2 \\
&\qquad\qquad\qquad x_3 + x_4 = 2 \\
&\qquad\qquad x_1, x_2, x_3, \text{ and } x_4 \geq 0
\end{aligned}$$

and take $x_1 = 2, x_4 = 2$ as the starting basic feasible solution.

The simplex Table T_0' associated with this LPP is

Again since the basic variable x_1 has non-zero coefficient in the z-row of Table T_0' substituting out this by considering

Table T_0'

Basic	x_1	x_2	x_3	x_4	Solution
z	–2	–1	0	0	0
x_1	1	1	–1	0	2
x_4	0	0	1	1	2

New z-row= Current z-row + 2(Current x_1-row);

the modified Table T_0' is

Table T_0' (modified)

	Basic	x_1	x_2	x_3 ↓	x_4	Solution
	z	0	1	–2	0	4
	x_1	1	1	–1	0	2
←	x_4	0	0	1	1	2

The solution $x_1 = 2$, $x_4 = 2$, $z = 4$

is not optimal, since the entries in the z-row are not non-negative. Here x_3 is the entering variable, x_4 is the leaving variable on the basis of minimum ratio conditions and, thus, 1 is the pivot element. We perform the following row operations:

New x_4-row = Current x_4-row

New z-row = Current z-row + 2(New x_4-row)

New x_1-row = Current z-row + New x_4-row,

the resultant simplex Table T_1' is

Table T_1'

Basic	x_1	x_2	x_3	x_4	Solution
z	0	1	0	2	8
x_1	1	1	0	1	4
x_3	0	0	1	1	2

The solution $x_1 = 4$, $x_3 = 2$, $x_2 = x_4 = 0$, $z = 8$

is optimal, since all the entries in the z-row are non-negative. Thus, the optimal solution is

$$x_1 = 4,\ x_2 = 0, \text{ and } z = 8.$$

Example 26.23: For the following LPP

Maximize $z = 2x_1 + x_2 - 3x_3$

subject to $x_1 + 2x_2 + 2x_3 \geq 12$, $3x_1 - 2x_2 + 4x_3 \leq 10$, $x_1 \geq 0$, $x_2 \leq 0$ and $x_3 \geq 0$

Solution: First setting $x_2' = -x_2$. Next introducing x_4 as slack and x_5 as surplus respectively in the first and second constraints and, then introducing R_1 as artificial variable in the resulting first equality constraint, the LP model for the Phase I is

Minimize $r = R_1$

subject to

$$\begin{aligned} x_1 - 2x_2 + 2x_3 - x_4 + R_1 &= 12 \\ 3x_1 + 2x_2' + 4x_3 + x_5 &= 10 \\ x_1, x_2, x_3, x_4, x_5 \text{ and } R_1 &\geq 0. \end{aligned}$$

Using R_1 and x_5 as initial basic variables, the starting simplex Table T_0 is

Table T_0

Basic	x_1	x_2'	x_3	x_4	R_1	x_5	Solution
r	0	0	0	0	–1	0	0
R_1	1	–2	2	–1	1	0	12
x_5	3	2	4	0	0	1	10

Substituting out R_1 in the r-row by considering

New r-row = Current r-row + Current R_1-row;

the modified Table T_0 is

Table T_0 (modified)

↓

	Basic	x_1	x_2'	x_3	x_4	R_1	x_5	Solution
	r	1	–2	2	–1	0	0	12
	R_1	1	–2	2	–1	1	0	12
←	x_5	3	2	$\boxed{4}$	0	0	1	10

The solution $R_1 = 12$, $x_5 = 10$, $x_1 = x_2' = x_3 = x_4 = 0$, $r = 12$ is not optimal since the r-row contain positive entries. The variable x_3, being with the most positive coefficient in the r-row, is the entering variable and x_5 is the leaving variable on the basis of the minimum ratio condition. Hence, 4 is the pivot element. We perform the following row operations:

New x_5-row = Current x_5-row ÷ 4

New r-row = Current r-row – 2(New x_5-row)

New R_1-row = Current R_1-row – 2(New x_5-row)

the resultant simplex Table T_1 is

Table T_1

Basic	x_1	x_2'	x_3	x_4	R_1	x_5	Solution
r	–1/2	–3	0	–1	0	–1/2	7
R_1	–1/2	–3	0	–1	1	–1/2	7
x_3	3/4	1/2	1	0	0	1/4	5/2

Since all the entries in the r-row are non-positive, the current solution $r = 7 > 0$ is optimal. This implies that there does not exist any feasible solution to the given LPP.

Example 26.24: For the following LPP

Minimize $z = 4x_1 + x_2$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

using two-phase method.

Solution: Introducing x_3 as surplus and x_4 as slack respectively in the second and third constraint and then introducing R_1 and r_2 as artificial variables respectively in the first and second equality constraints, the LP model for the Phase I is

Minimize $r = R_1 + R_2$

subject to

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 - x_3 + R_2 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4, R_1, \text{and } R_2 \geq 0,$$

Using R_1, R_2 and x_4 as initial basic variables, the starting simplex Table T_0 is

Table T_0

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	–1	–1	0	0
R_1	$\boxed{3}$	1	0	1	0	0	3
R_2	4	3	–1	0	1	0	6
x_4	1	2	0	0	0	1	4

Substituting out R_1 and R_2 in the r-row, by considering

New r-row = Current r-row + (R_1-row + R_2–row);

the modified Table T_0 is

Table T_0 (modified)

	Basic	↓ x_1	x_2	x_3	R_1	R_2	x_4	Solution
	r	7	4	−1	0	0	0	9
←	R_1	3	1	0	1	0	0	3
	R_2	4	3	−1	0	1	0	6
	x_4	1	2	0	0	0	1	4

The solution min $r = 9$ is not optimal since all the entries in the r-row are not non-positive. Here x_1 is the entering variable, R_1 is the leaving variable on the basis of the minimum ratio condition, and thus, 3 is the pivot element. We perform the following row operations:

New R_1-row = (Current R_1-row) ÷ 3,
New r-row = Current r-row − 7(New R_1-row),
New R_2-row = Current R_2-row − 4(New R_1-row),
New x_4-row = Current x_4-row − New R_1-row;
the resultant simplex Table T_1 is

Table T_1

	Basic	x_1	↓ x_2	x_3	R_1	R_2	x_4	Solution
	r	0	5/3	−1	−7/3	0	0	2
	x_1	1	1/3	0	1/3	0	0	1
←	R_2	0	$\boxed{5/3}$	−1	−4/3	1	0	2
	x_4	0	5/3	0	−1/3	0	1	3

The solution min $r = 2$ is not optimal, since all the entries in the r-row are not non-positive. Here x_2 is the entering variable, R_2 is the leaving variable on the basis of the minimum ratio condition, and thus, 5/3 is the pivot element. We perform the following row operations:

New R_2-row = (3/5) (Current R_2-row) ÷ (5/3),
New r-row = Current r-row − (5/3)(New R_2-row),
New x_1-row = Current x_1-row − (1/3)(New R_2-row),
New x_4-row = Current x_4-row − (5/3)(New R_2-row);
the resultant simplex Table T_2 is

Table T_2

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	−1	−1	0	0
x_1	1	0	1/5	3/5	−1/5	0	3/5
x_2	0	1	−3/5	−4/5	3/5	0	6/5
x_4	0	0	1	1	−1	1	1

The solution min. $r = 0$ is optimal since all the entries in the r-row are non-positive. The basic feasible solution produced by Phase I is $x_1 = 3/5$, $x_2 = 6/5$, and $x_4 = 1$.

Eliminating the columns corresponding to the artificial variables R_1 and R_2 in the Table T_2 and proceeding to Phase II, we write the LPP as

Minimize $z = 4x_1 + x_2$

subject to

$$\begin{aligned} x_1 \quad + (1/5)x_3 \quad &= 3/5 \\ x_2 - (3/5)x_3 \quad &= 6/5 \\ x_3 + x_4 &= 1 \end{aligned}$$

x_1, x_2, x_3, and $x_4 \geq 0$.

Taking x_1, x_2, x_4 as the basic variables, the simplex Table T_0' for this LPP is

Table T_0'

Basic	x_1	x_2	x_3	x_4	Solution
z	–4	–1	0	0	0
x_1	1	0	1/5	0	3/5
x_2	0	1	–3/5	0	6/5
x_4	0	0	1	1	1

Substituting out for the basic variables x_1 and x_2 in the z-row by considering

New z-row = Current z-row + 4(Current x_1-row) + (Current x_2- row);

the modified Table T_0' is

Table T_0 (modified)

Basic	x_1	x_2	x_3 ↓	x_4	Solution
z	0	0	1/5	0	18/5
x_1	1	0	1/5	0	3/5
x_2	0	1	–3/5	0	6/5
← x_4	0	0	$\boxed{1}$	1	1

The solution minimum $z = 18/5$ is not optimal, since the entries in the z-row are not non-positive. The variable x_3 is the entering variable and x_4 is the leaving variable on the basis of the minimum ratio, and thus, 1 is the pivot element.

Performing the row operations:

New x_4-row = Current x_4-row,

New z-row = Current z-row – (1/5)(New x_4-row),

New x_1-row = Current x_1-row − (1/5)(New x_4-row),
New x_2-row = Current x_2-row + (3/5)(New x_4-row);

the resultant simplex Table T_1'' is

Table T_1''

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	0	−1/5	17/5
x_1	1	0	0	−1/5	2/5
x_2	0	1	0	3/5	9/5
x_3	0	0	1	1	1

The solution minimum $z = 17/5$ is optimal, since the entries in the z-row are non-positive. Thus, the solution to the given LPP is

$$x_1 = 2/5, x_2 = 9/5, z = 17/5.$$

EXERCISE 26.6

Using the penalty method, solve the following LPP

1. Maximize $z = 2x_1 + x_2$

   subject to $x_1 - \frac{1}{2}x_2 \geq 1, x_1 - x_2 \leq 2, x_1 + x_2 \leq 4$, and $x_1, x_2 \geq 0$.

2. Minimize $z = 3x_1 + 2x_2$

   subject to $x_1 + x_2 \geq 2, x_1 + 3x_2 \leq 3, x_1 - x_2 = 1$, and $x_1, x_2 \geq 0$.

3. Maximize $z = 4x_1 + x_2 + 2x_3$

   subject to $x_1 + x_2 + x_3 \leq 1, x_1 + x_2 - x_3 \leq 0$, and $x_1, x_2, x_3 \geq 0$.

4. Maximize $z = 4x_1 + 8x_2 + 3x_3$

   subject to $x_1 + x_2 \geq 2, 2x_2 + x_3 \geq 5$, and $x_1, x_2, x_3 \geq 0$.

5. Minimize $z = -x_1 + 2x_2$

   subject to $x_1 + 2x_2 - x_3 = 1, -x_1 + x_2 + x_4 = 1$ and $x_1, x_2, x_3, x_4 \geq 0$.

6. Minimize $z = x_1 + x_2$

   subject to $x_1 + 2x_2 \leq 2, 3x_1 + 5x_2 \geq 15$, and $x_1, x_2 \geq 0$.

7. Using penalty method show that the LPP

   Maximize $z = 6x_1 + 4x_2$

   subject to $2x_1 + 3x_2 \leq 30, 3x_1 + 2x_2 \leq 24, x_1 + x_2 \geq 3$, and $x_1, x_2 \geq 0$

   has infinite solutions.

8. Using penalty method show that the LPP

   Maximize $z = x_1 + x_2 + x_4$ subject to

   $$x_1 + x_2 + x_3 + x_4 = 4, x_1 + 2x_2 + x_3 + x_5 = 4, x_1 + 2x_2 + x_3 = 4, \text{ and } x_1, x_2, x_3, x_4, x_5 \geq 0,$$

   has alternate solutions.

9. Using two-phase method show that the LPP

   Maximize $z = 2x_1 + 5x_2$

   subject to $3x_1 + 2x_2 \geq 6, 2x_1 + x_2 \leq 2$, and $x_1, x_2 \geq 0$

   has no feasible solution.

10. Use two-phase method to

    Maximize $z = 5x_1 + 8x_2$

    subject to $3x_1 + 2x_2 \geq 3, x_1 + 4x_2 \geq 4, x_1 + x_2 \leq 5$, and $x_1, x_2 \geq 0$.

11. Use two-phase method to

    Maximize $z = 5x_1 - 4x_2 + 3x_3$

    subject to $2x_1 + x_2 - 6x_3 = 20, 6x_1 + 5x_2 + 10x_3 \leq 76, 8x_1 - 3x_2 + 6x_3 \leq 50$, and $x_1, x_2, x_3 \geq 0$.

12. Use two-phase method to

    Maximize $z = 12x_1 + 15x_2 + 9x_3$

    subject to $8x_1 + 16x_2 + 12x_3 \leq 250, 4x_1 + 8x_2 + 10x_3 \geq 80, 7x_1 + 9x_2 + 8x_3 = 105, x_1, x_2, x_3 \geq 0$.

26.10 DUALITY IN LINEAR PROGRAMMING

For every linear programming problem, there is associated another linear programming problem involving the same data as of the original problem. The original problem is called the *primal* and the associated one its *dual*. The two are so closely related that the optimal solution of one problem automatically yields the optimal solution of the other.

Duality is an important and interesting concept in the theory of linear programming. Its importance is mainly due to the following two facts. Firstly, if the number of variables in the primal is considerably smaller than the number of constraints, then computations can be considerably reduced by converting it into the dual and then solving it, since the amount of simplex computations depends largely on the number of constraints. Secondly, duality gives additional information as to how the optimal solution changes when the parameters of the LP-model are changed. This forms the basis of post optimality or sensitivity analysis. In addition to these, the variables in the LP-models have interpretation from the cost or economic point of view which are helpful to the decision-makers in planning future/alternate course of action.

26.10.1 Formulation of the Dual Problem

Consider the primal in the equation form

Maximize (minimize) $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ ...(26.5)

subject to the constraints:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \\ x_1, x_2, \dots, x_n &\geq 0. \end{aligned}\right\} \quad ...(26.6)$$

If the above problem is referred to as *primal*, then its *associated dual* will be

Minimize (maximize) $w = b_1y_1 + b_2y_2 + \dots + b_my_m$ ...(26.7)

subject to the constraints:

$$\left.\begin{aligned} a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m &\geq (\leq)\, c_1 \\ a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m &\geq (\leq)\, c_2 \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m &\geq (\leq)\, c_n \end{aligned}\right\} \quad ...(26.8)$$

$y_1, y_2, \dots, y_m$ being unrestricted.

From the above primal-dual pair, we observe the following:

1. *If the primal contains n variables and m constraints, then the dual will contain m variables and n constraints.*
2. *If the objective function in the primal is of maximization (minimization) type, then the objective function in the dual will be of minimization (maximization) type and the constraints will be of the type $\geq (\leq)$.*
3. *The coefficients $c_1, c_2, \dots, c_n$ in the objective function of the primal define the right-hand side of the constraints in the dual.*
4. *The coefficients $b_1, b_2, \dots, b_m$ in the right-hand sides of the constraints of the primal appear as coefficients in the objective function in the dual.*
5. *The m variables $y_1, y_2, \dots, y_m$ in the dual will be unrestricted in nature.*

Example 26.25: Write the dual to the primal

Maximize $z = 3x_1 + 5x_2$

subject to $x_1 + 3x_2 \leq 25, 3x_1 + 2x_2 \leq 35, 5x_1 - 3x_2 \leq 10, x_2 \leq 20, x_1, x_2 \geq 0$.

Solution: Write the given primal in the equation form, we have

Maximize $z = 3x_1 + 5x_2 + 0.x_3 + 0.x_4 + 0.x_5 + 0.x_6$

subject to

$$x_1 + 3x_2 + x_3 + 0.x_4 + 0.x_5 + 0.x_6 = 25$$
$$3x_1 + 2x_2 + 0.x_3 + x_4 + 0.x_5 + 0.x_6 = 35$$

$$5x_1 - 3x_2 + 0.x_3 + 0.x_4 + x_5 + 0.x_6 = 10$$
$$0.x_1 + x_2 + 0.x_3 + 0.x_4 + 0.x_5 + x_6 = 20$$
$$x_1, x_2, x_3, x_4, x_5 \text{ and } x_6 \geq 0.$$

Let y_1, y_2, y_3 and y_4 be the dual variables, then the dual is

Minimize $w = 25y_1 + 35y_2 + 10y_3 + 20y_4$

subject to
$$y_1 + 3y_2 + 5y_3 + 0.y_4 \geq 3$$
$$3y_1 + 2y_2 - 3y_3 + y_4 \geq 5$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$y_3 \geq 0$$
$$y_4 \geq 0$$
y_1, y_2, y_3 and y_4 are unrestricted.

This can be rewritten as

Minimize $w = 25y_1 + 35y_2 + 10y_3 + 20y_4$

subject to
$$y_1 + 3y_2 + 5y_3 \geq 3$$
$$3y_1 + 2y_2 - 3y_3 + y_4 \geq 5$$
$$y_1, y_2, y_3, y_4 \geq 0.$$

Example 26.26: Write the dual to the primal

Minimize $z = 10x_1 + 8x_2$

subject to
$$x_1 + 2x_2 \geq 5$$
$$2x_1 - x_2 \geq 12$$
$$x_1 + 3x_2 \geq 4$$
$x_1 \geq 0$ and x_2 unrestricted.

Solution: Write $x_2 = x_2' - x_2''$ and express the given primal in the equation form, we have

Minimize $z = 10x_1 + 8x_2' - 8x_2'' + 0.x_3 + 0.x_4 + 0.x_5$

subject to
$$x_1 + 2x_2' - 2x_2'' - x_3 + 0.x_4 + 0.x_5 = 5$$
$$2x_1 - x_2' + x_2'' + 0.x_3 - x_4 + 0.x_5 = 12$$
$$x_1 + 3x_2' - 3x_2'' + 0.x_3 + 0.x_4 - x_5 = 4$$
$$x_1, x_2', x_2'', x_3, x_4 \text{ and } x_5 \geq 0.$$

Let y_1, y_2, y_3 be the dual variables, the dual is

Maximize $w = 5y_1 + 12y_2 + 4y_3$

subject to
$$y_1 + 2y_2 + y_3 \leq 10$$
$$2y_1 - y_2 + 3y_3 \leq 8$$
$$-2y_1 + y_2 - 3y_3 \leq -8$$

$$-y_1 \le 0$$
$$-y_2 \le 0$$
$$-y_3 \le 0$$

y_1, y_2, y_3, y_4 are unrestricted

This can be rewritten as

Maximize $w = 5y_1 + 12y_2 + 4y_3$

subject to
$$y_1 + 2y_2 + y_3 \le 10$$
$$2y_1 - y_2 + 3y_3 = 8$$
$$y_1, y_2, y_3 \ge 0.$$

Example 26.27: Write the dual to the primal

Maximize $z = 3x_1 + 10x_2 + 2x_3$

subject to $2x_1 + 3x_2 + 2x_3 \le 7, 3x_1 - 2x_2 + 4x_3 = 3, x_1, x_2, x_3 \ge 0.$

Solution: Write the given primal in the equation form, we have

Maximize $z = 3x_1 + 10x_2 + 2x_3 + 0.x_4$

subject to
$$2x_1 + 3x_2 + 2x_3 + x_4 = 7$$
$$3x_1 - 2x_2 + 4x_3 + 0.x_4 = 3$$

Let y_1 and y_2 be the dual variables, then the duals is

Minimize $w = 7y_1 + 3y_2$

subject to
$$2y_1 + 3y_2 \ge 3$$
$$3y_1 - 2y_2 \ge 10,$$
$$2y_1 + 4y_2 \ge 2,$$
$$y_1 \ge 0;$$

y_1, and y_2 unrestricted.

The dual can be rewritten as

Minimize $w = 7y_1 + 3y_2$

subject to
$$2y_1 + 3y_2 \ge 3$$
$$3y_1 - 2y_2 \ge 0$$
$$2y_1 + 4y_2 \ge 2$$

$y_1 \ge 0$, and y_2 unrestricted.

Remark: From Examples 26.26 and 26.27, we observe that an unrestricted primal variable corresponds to an equality constraint and a primal equality constraint results in an unrestricted dual variable.

EXERCISE 26.7

Write dual to the following LPP

1. Maximize $z = 40x_1 + 25x_2 + 50x_3$

   subject to $x_1 + 2x_2 + x_3 \le 36, 2x_1 + x_2 + 4x_3 \le 60, 2x_1 + 5x_2 + x_3 \le 45$ and $x_1, x_2, x_3 \ge 0.$

2. Minimize $z = 8x_1 + 3x_2 + 15x_3$
   subject to $2x_1 + 4x_2 + 3x_3 \geq 28, 3x_1 + 5x_2 + 6x_3 \geq 30$ and $x_1, x_2, x_3 \geq 0$.
3. Minimize $z = x_1 - 3x_2 - 2x_3$
   subject to $3x_1 - x_2 + 2x_3 \leq 7, 2x_1 - 4x_2 \geq 12, -4x_1 + 3x_2 + 8x_3 = 10$,
   $x_1, x_2 \geq 0$ and x_3 is unrestricted.
4. Minimize $z = x_1 - 2x_2 + 3x_3$
   subject to $-2x_1 + x_2 + 3x_3 = 2, 2x_1 + 3x_2 + 4x_3 = 1$ and $x_1, x_2, x_3 \geq 0$
5. Minimize $z = 2x_1 + 3x_2 + 4x_2$
   subject to $2x_1 + 3x_1 + 5x_3 \geq 12, 3x_1 + x_2 + 7x_3 = 3, x_1 + 4x_2 + 6x_3 \leq 5$,
   $x_1, x_2 \geq 0$ and x_3 unrestricted.

26.10.2 Primal-Dual Relationships

A few results concerning the primal and its dual are

1. *The dual of the dual is the primal.*
2. *If the primal has an unbounded solution, then the solution to its dual is infeasible.*
3. *(Duality Principle) If both the primal and its dual have feasible solutions, then both have optimal solutions and max. z = min. w.*

The Result 3, also called the *Fundamental Theorem of Duality*, suggests that an optimal solution to the primal can be directly obtained from that of the dual and vice versa.

Computing the optimal dual solution. The optimal dual solution can be computed directly from the optimum table of the primal. In the starting simplex Table T_0 of the primal, the constraint coefficients under the starting variables form an *identity matrix*. After subsequent iterations, in the optimum table this identity matrix is reduced to the matrix known as the *optimal primal inverse*, a matrix key to calculate the optimal values of the dual variables. We have the following result:

$$\begin{pmatrix} \textit{Optimal values} \\ \textit{of the dual variables} \end{pmatrix} = \begin{pmatrix} \textit{Row vector of the primal objective} \\ \textit{coefficients in the order in which the basic} \\ \textit{variables appear in the optimum simplex table} \end{pmatrix} \begin{pmatrix} \textit{Optimal primal} \\ \textit{inverse} \end{pmatrix}$$

and further, since the dual of the dual is the primal, this result can be used to determine the optimal solution of the primal from that of the dual.

We illustrate all this in the example considered below.

Example 26.28: Solve the following LP-model using its dual

Minimize $w = 15y_1 + 7y_2 + 100y_3$, subject to

$y_1 + 7y_2 + 3y_3 \geq 4, y_1 + 5y_2 + 5y_3 \geq 5, y_1 + 3y_2 + 10y_3 \geq 9, y_1 + 2y_2 + 15y_3 \geq 11, y_1, y_2, y_3 \geq 0$.

Solution: Write the given primal in the equation form, we have

Minimize $w = 15y_1 + 120y_2 + 100y_3 + 0.y_4 + 0.y_5 + 0.y_6 + 0.y_7$

$$y_1 + 7y_2 + 3y_3 - y_4 + 0.y_5 + 0.y_6 + 0.y_7 = 4$$

$$y_1 + 5y_2 + 5y_3 + 0.y_4 - y_5 + 0.y_6 + 0.y_7 = 5$$
$$y_1 + 3y_2 + 10y_3 + 0.y_4 + 0.y_5 - y_6 + 0.y_7 = 9$$
$$y_1 + 2y_2 + 15y_3 + 0.y_4 + 0.y_5 + 0.y_6 - 7 = 11$$

Let x_1, x_2, x_3, and x_4 be the dual variables, then the dual is

Minimize $z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

$$x_1 + x_2 + x_3 + x_4 \le 15$$
$$7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$$
$$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100$$
$$-x_1, -x_2, -x_3, \text{ and } -x_4 \le 0$$
$$\text{that is } x_1, x_2, x_3, x_4 \ge 0$$

The optimal solution to this LP model, refer to Example 26.14, is

Maximum $x_1 = 50/7, x_2 = 0, x_3 = 55/7, x_4 = 0, z = \dfrac{695}{7}$

The *optimal inverse matrix*, (refer, to Table T_3 on page 1515), is given by

$$\begin{bmatrix} 10/7 & 0 & -1/7 \\ -61/7 & 1 & 4/7 \\ -3/7 & 0 & 1/7 \end{bmatrix}$$

The row order of the variables listed in the Table T_3 is x_1, s_2, x_3. The elements of the original objective coefficients corresponding to these variables in this order are: 4, 0, 9.

Thus, the optimal primal variables y_1, y_2, y_3 are given by

$$[y_1 \quad y_2 \quad y_3] = [4 \quad 0 \quad 9]\begin{bmatrix} 10/7 & 0 & -1/7 \\ -61/7 & 1 & 4/7 \\ -3/7 & 0 & 1/7 \end{bmatrix}$$
$$= [13/7, \quad 0, \quad 5/7]$$

We can verify that the solution $y_1 = 13/7,\ y_2 = 0,$ and $y_3 = 5/7$ satisfies the constraints of the given LPP(primal), and also

$$\text{Min}\ (w) = 15(13/7) + 120(0) + 100(5/7) = \frac{695}{7} = \text{Max}\ (z).$$

This implies that the solution obtained is optimal one.

Example 26.29: Find the solution of the primal

Minimize $z = 8x_1 + 4x_2$

subject to $x_1 + x_2 \ge 3, 4x_1 + 2x_2 \ge 9, x_1, x_2 \ge 0.$
using that of its dual.

Solution: Writing the given primal in the equation form, we have

Minimize $z = 8x_1 + 4x_2 + 0.x_3 + 0.x_4$

subject to $x_1 + x_2 - x_3 + 0.x_4 = 3,$

$4x_1 + 2x_2 + 0x_3 - x_4 = 9$

$x_1, x_2, x_3, x_4 \geq 0.$

Let y_1, y_2 be the dual variables, then the dual is

Minimize $w = 3y_1 + 9y_2$

subject to $y_1 + 4y_2 \leq 8$

$y_1 + 2y_2 \leq 4$

$-y_1 \leq 0$

$-y_2 \leq 0$

y_1, y_2 being unrestricted.

This can be rewritten as

Maximize $w = 3y_1 + 9y_2$

subject to $y_1 + 4y_2 \leq 8,$

$y_1 + 2y_2 \leq 4,$

$y_1, y_2 \geq 0.$

Introducing y_3 and y_4 as the slack and writing the LPP in the standard form, we have

Maximize $w = 3y_1 + 9y_2$

$y_1 + 4y_2 + y_3 \quad = 8$

$y_1 + 2y_2 \quad + y_4 = 4$

$y_1, y_2, y_3,$ and $y_4 \geq 0.$

Using y_3 and y_4 as initial basic variables, the starting simple x Table T_0 is

Table T_0

	Basic	y_1	y_2 ↓	y_3	y_4	Solution
	w	-3	-9	0	0	0
←	y_3	1	$\boxed{4}$	1	0	8
	y_4	1	2	0	1	4

Since all the entries in the w-row are not non-negative, the solution $w = 0$ is not optimal. Here y_2 is the entering variable, and on the basis of the minimum ratio condition y_3 (we can select y_4 also) is the leaving variable, hence, 4 is the pivot element.

To find the new basic solution, we perform the following row operations:

New pivot y_3-row = (Current y_3-row) ÷ 4,

New w-row = (Current w-row) – (–9)(New y_3-row),
New y_4-row = (Current y_4-row) – 2(New y_3-row);

and obtain the resultant simplex Table T_1 as

Table T_1

	Basic	y_1 ↓	y_2	y_3	y_4	Solution
	w	–3/4	0	9/4	0	18
	y_2	1/4	1	1/4	0	2
←	y_4	$\boxed{1/2}$	0	–1/2	1	0

Since the entries in the w-row are still not non-negative, the solution $w = 18$ is not optimal. Here y_1 is the entering variable, and on the basis of minimum ratio condition y_4 is the leaving variable, hence, 1/2 is the pivot element. To obtain the new basic solution, we perform the following row operations:

New pivot y_4-row = (Current pivot y_4-row) ÷ (1/2),

New w-row = (Current w-row) – (–3/4)(New pivot y_4-row),

New y_2-row = (Current y_2-row) – (1/4)(New pivot y_4-row);

and obtain the resultant simplex Table T_2 as

Table T_2

Basic	y_1	y_2	y_3	y_4	Solution
w	0	0	3/2	3/2	18
y_2	0	1	1/2	–1/2	2
y_1	1	0	–1	2	0

Since the entries in the w-row are non-negative the solution $y_1 = 0$, $y_2 = 2$, and $w = 18$ is optimal.

The optimal inverse matrix is $\begin{bmatrix} 1/2 & -1/2 \\ -1 & 2 \end{bmatrix}$

The row-order of the variables in the optimal Table T_2 is y_2, y_1. The elements of the objective coefficients corresponding to these variables in the same order are: [9, 3]. Thus, the optimal primal variables x_1, x_2 are given by

$$[x_1 \quad x_2] = [9 \quad 3]\begin{bmatrix} 1/2 & -1/2 \\ -1 & 2 \end{bmatrix} = [3/2 \quad 3/2]$$

We can verify that the solution $x_1 = 3/2$ and $x_2 = 3/2$ satisfy the constraints and also

Minimum $z = 8(3/2) + 4(3/2) = 18 =$ Maximum w

EXERCISE 26.8

Write the associated dual problem in case of following LPP and find the solution of the primal

1. Maximize $z = 2x_1 + x_2$

   subject to $x_1 + 2x_2 \leq 10, x_1 + x_2 \leq 6, x_1 - x_2 \leq 2, x_1 - 2x_2 \leq 1$ and $x_1, x_2 \geq 0$.

2. Maximize $z = 3x_1 + 6x_2 + 5x_3$

   subject to $x_1 + x_3 \leq 1, 2x_1 + x_2 - 3x_3 \leq 1, 7x_2 + 5x_3 = 2$, and $x_1 \geq 0, x_2 \leq 0, x_3$ unrestricted.

3. Minimize $z = 10x_1 + 8x_2$

   subject to $x_1 + 2x_2 \geq 5, 2x_1 - x_2 \geq 12, x_1 + 3x_2 \geq 4$, and $x_1 \geq 0$, x_2 unrestricted.

4. Minimize $z = 0.7x_1 + 0.5x_2$

   subject to $x_1 \geq 4, x_2 \geq 6, x_1 + 2x_2 \geq 20, 2x_1 + x_2 \geq 18$, and $x_1, x_2 \geq 0$.

5. Find the optimal value of the objective function for the LPP

   Minimize $z = 10x_1 + 4x_2 + 5x_3$

   subject to $5x_1 - 7x_2 + 3x_3 \geq 50$ and $x_1, x_2, x_3 \geq 0$

   by only inspecting its dual.

26.11 TRANSPORTATION PROBLEM

The transportation problem is a subclass of linear programming problem that deals with shipping a commodity from different *sources* to *destinations*. Here objective is to determine the shipping schedule to minimize the total transportation cost subject to the availability of the commodity at sources and requirement at destinations. The solution is worked out under the assumption that shipping cost is proportional to the number of units shipped on a given route.

26.11.1 General Transportation Problem

Let there be m sources and n destinations and let the amount of supply at the sources i be a_i and the amount of requirement at the destination j be b_j. If c_{ij} is the cost of transporting one unit of commodity from the source i to the destination j, then objective is to determine the unknown quantities x_{ij} to be transported from the origin i to the destination j, for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$, such that the total transportation cost is minimum subject to the constraints of supply and requirements. Mathematically, the problem can be expressed as an *LPP* as follows:

$$\text{Minimize} \quad z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \qquad \text{...(26.9)}$$

$$\text{subject to the constraints: } \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, ..., m \qquad \text{...(26.10)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, ..., n \qquad \text{...(26.11)}$$

and $x_{ij} \geq 0$ for all i and j.

If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, that is, the total supply is the same as the total demand, the problem is said to be *balanced*, otherwise, *unbalanced*.

A necessary and sufficient condition for the existence of a feasible solution to the general transportation problem is that, it must be balanced one.

26.11.2 Representation of a Transportation Problem

The general problem can also be represented as network as shown in Fig. 26.10, with m sources (a_i, $i = 1, 2, ..., m$), and n destinations (b_j, $j = 1, 2, ..., m$), each representated by a *node*. An arc (i, j) represents the *route* joining a specific source i with a specific destination j; c_{ij} is the transportation cost per unit and x_{ij} the amount shipped along this route.

Since the transportation problem is a special class of general *LPP*, it can be solved by simplex method. However, by representing the transportation problem, whenever possible, in the tabular form it is simpler to solve it by *transportation algorithm* than by simplex method. The *general transportation tableau* is shown below in Fig. 26.11.

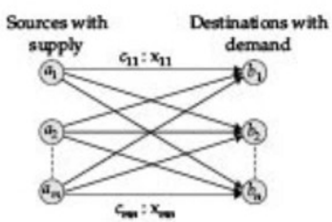


Fig. 26.10

		Destination 1	2	------	n	Supply
Source	1	x_{11} c_{11}	x_{12} c_{12}	------	x_{1n} c_{1n}	a_1
	2	x_{21} c_{21}	x_{22} c_{22}	------	x_{2n} c_{2n}	a_2
	⋮	⋮	⋮	⋮	⋮	⋮
	m	x_{m1} c_{m1}	x_{m2} c_{m2}	------	x_{mn} c_{mn}	a_m
Demand		b_1	b_2	------	b_n	

Fig. 26.11

The mn squares are called the *cells*. The per unit transportation cost c_{ij} is displayed in the lower right corner of the (i, j)th cell and feasible solution x_{ij}, if any, is displayed in the upper left corner of the (i, j)th cell.

The transportation algorithm follows the exact steps of the simplex method, only the special structure of the transportation problem allows us to organize the computation in a more convenient way. Further the algorithm is based on the assumption that the problem is balanced one. If it is unbalanced, we can always make it balanced by augmenting with a *dummy source*, or a *dummy destination*.

26.11.3 Solution of a Transportation Problem

A general transportation problem with m sources and n destinations has $m + n$ constraints one for each source and each destination. However, since the transportation model is always balanced (or, is made so, if not), one of these constraints is redundant. Thus, there are only $m + n - 1$ independent constraints, and the initial basic solution should consist of $(m + n - 1)$ basic variables. If any basic variable takes the value zero, then the basic feasible solution is said to *degenerate*. Like *LPP*, all non-basic variables take the value zero. The cells corresponding to basic variables are called *occupied cells* and that corresponding to non-basic variables are called *non-occupied cells*.

The solution of a transportation problem involves the following steps.

1. *Find an initial basic solution satisfying the supply and demand constraints.*
2. *Examine the solution obtained in Step 1 for optimality using the optimality condition (to be discussed in Section 26.11.5). If the optimality condition is satisfied, stop, otherwise, go to Step 3.*
3. *Modify the shipping schedule by including the unoccupied cell, whose inclusion may result in an improved solution and go to Step 2.*

Generally, the following three methods for finding an initial basic feasible solution are applied

1. *North west corner method* (NWCM)
2. *Least-cost method* (LCM)
3. *Vogel approximation method* (VAM)

In general, Vogel approximation method yields better initial basic feasible solution. We discuss here this method only.

26.11.4 Vogel Approximation Method

The method consists of the following steps:

1. *For each row (column) of the transportation table, find the difference between the smallest and the next-to-smallest unit costs. Display them alongside the transportation table as row (column) penalities.*
2. *Identify the row or column with the largest penalty, breaking ties (if, any) arbitrarily. Allocate the maximum feasible commodity amount to the least unit cost in the identified row or column. Adjust the supply and demand, and cross out the satisfied row or column. If a row and a column are satisfied simultaneously, then only one of the two is crossed out and the remaining row (column) is assigned zero supply (demand).*

3. *Recalculate the row (column) differences for the reduced transportation table and go to Step 2. Repeat the procedure until all the supply and demand entries are satisfied.*

We illustrate all these steps in the example considered next.

Example 26.30: Find an initial basic feasible solution of the transportation problem.

Destination

Source	D_1	D_2	D_3	D_4	Supply
S_1	3	2	4	1	20
S_2	2	4	5	3	15
S_3	3	5	2	6	25
S_4	4	3	1	4	40
Demand	30	20	25	25	

Solution: Since the total demand is equal to the total supply, the problem is balanced one. Calculating the row and column penalties, we obtain

					Supply	Row penalties
	3	2	4	20 1	20	1
	2	4	5	3	15	1
	3	5	2	6	25	1
	4	3	1	4	40	2
Demand	30	20	25	25		
Col. penalties	1	1	1	2		

There is tie in the largest penalty at row '4' and column '4'. Select the column '4' and allocate 20 in the (1, 4) cell. Cross-off the first row as supply is exhausted, adjust the corresponding demand to 5. The reduced table is

					Supply	Row penalties
	2	4	5	3	15	1
	3	5	2	6	25	1
	4	3	25 1	4	40	2
Demand	30	20	25	5		
Col. penalties	1	1	1	1		

The largest penalty corresponds to the row 3. Allocate 25 in the (3, 3) cell. Cross off the third column since the demand is exhausted, adjust the corresponding supply to 15. The reduced table is

				Supply	Row penalties
	2	4	3	15	1
	[25] 3	5	6	25	2
	4	3	4	15	1
Demand	30	20	5		
Col. penalties	1	1	1		

The largest penalty corresponds to the row 2. Allocate 25 in the (2, 1) cell. Cross off the second row since the supply is exhausted, adjust the corresponding demand to 5. The reduced table is

				Supply	Row penalties
	[5] 2	4	3	15	1
	4	3	4	15	1
Demand	5	20	5		
Col. penalties	2	1	1		

The largest penalty corresponds to the column 1. Allocate 5 in the (1, 1) cell. Cross off the first column since the demand is exhausted, adjust the corresponding supply to 10. The reduced table is

			Supply	Row penalties
	4	3	10	1
	[15] 3	4	15	1
Demand	20	5		
Col. penalties	1	1		

The largest penalty is 1 but there are ties. Select the second row and allocate 15 to (2, 1) cell. Cross off the second row since the supply is exhausted, adjust the corresponding demand to 5. The reduced table is

[5] 4	[5] 3	10
5	5	

The complete allocation is shown below.

3	2	4	(20) 1
(5) 2	(5) 4	5	(5) 3
(25) 3	5	2	6
4	(15) 3	(25) 1	4

The transportation cost according to this shipping route is given by

$$z = 20\times 1 + 5\times 2 + 5\times 4 + 5\times 3 + 25\times 3 + 15\times 3 + 25\times 1$$
$$= 20 + 10 + 20 + 15 + 75 + 45 + 25 = 210.$$

Example 26.31: Find an initial basic feasible solution by *VAM* for the following transportation problem.

	D_1	D_2	D_3	Supply
S_1	1	5	6	90
S_2	3	2	3	10
S_3	2	6	1	20
Demand	60	50	50	

Solution: Since demand exceeds the supply the problem is unbalanced, to make it balanced, we create a dummy source S_4 with supply of 40 and unit cost of transportation 0 to each destination. The problem can now be expressed as balanced one as

	D_1	D_2	D_3	
S_1	1	5	6	90
S_2	3	2	3	10
S_3	2	6	1	20
S_4	0	0	0	40
	60	50	50	

The solution for this transportation problem using VAM is obtained as follows.

At each iteration the differences between the two successive lowest costs for each row and each column are computed, and are written besides the corresponding rows or columns under the heading 'row penalties' and 'column penalties.' The largest penalty at each stage is encircled to make maximum possible assignment to the cell with lowest cost in that specific row or column, supply and demand are adjusted accordingly, and, then moving to the next iteration, and so on.

Following these, the condensed form of the solution table obtained is given below.

	D_1	D_2	D_3	Supply	Row penalties		
S_1	$\boxed{60}$ 1	$\boxed{30}$ 5	6	~~90~~ ~~30~~ 0	4	1	1
S_2	3	$\boxed{10}$ 2	3	~~10~~ 0	1	1	1
S_3	2	6	$\boxed{20}$ 1	~~20~~ 0	1	5	—
S_4	0	$\boxed{10}$ 0	$\boxed{30}$ 0	~~40~~ ~~10~~ 0	0	0	0
Demand	~~60~~ 0	~~50~~ ~~40~~ ~~20~~ 0	~~50~~ ~~30~~ 0				
Column penalities	1	2	1				
	—	2	1				
	—	2	3				
	—	2	—				

The solution is

$$x_{11} = 60, x_{12} = 30, x_{22} = 10, x_{33} = 20, x_{42} = 10, x_{43} = 30$$

Thus, the transportation cost is given by

$$z = 60 \times 1 + 30 \times 5 + 10 \times 2 + 20 \times 1 + 10 \times 0 + 30 \times 0$$

$$= 250, \text{ along with a short supply of 10 at } D_2 \text{ and 30 at } D_3.$$

Remarks:

1. In case for short supply there is a penalty say of 2, 3 and 4 by D_1, D_2 and D_3 respectively then the transportation problem is modified as shown below.

1	5	6	30
3	2	3	10
2	6	1	20
2	3	4	40
60	50	50	

and can be solved accordingly.

(2) In case supply exceeds the demand, then we add dummy destination with zero (or, non-zeros) penalties to make the problem balanced.

26.11.5 Method for Finding the Optimal Solution (UV-Method)

After finding the initial BFS, we take the following steps to test its optimality and if not found so, then to find the optimal solution:

1. *For each occupied cell (basic variable) in the current solution, solve the system of equations $u_i + v_j = c_{ij}$, starting initially with some $u_i = 0$ or $v_j = 0$.*
2. *For all unoccupied cells, (non-basic variables), calculate the net evaluation $\bar{c}_{ij} = u_i + v_j - c_{ij}$.*
   If all the net evaluation $\bar{c}_{ij}$ are less than or equal to zero, then the current solution is an optimal one and thus stop, otherwise, go to Step 3.
3. *Select the unoccupied cell, say (r, c) cell, with the largest positive net evaluation to enter the basis. Thus, x_{rc} is the entering variable. Construct a closed loop that starts and ends at the entering variable (r, s). The loop consists of connected horizontal and vertical segments only and each corner of the loop, except corresponding to the entering variable cell, must coincide with a basic variable. We must note that exactly one loop exists for a given entering variable.*
4. *Assign the maximum amount θ to the entering cell (r, c) and alternate between subtracting and adding the amount θ at the successive comers of the loop such that supply and demand constraints remain satisfied and the value at one of the corners (basic variable) is reduced to zero and the values at the remaining corners of the loop remain non-negative. The basic cell whose allocation is reduced to zero leaves the basis.*
5. *Go to Step 2, and repeat the process until an optimum solution is obtained.*

Example 26.32: Consider the initial basic feasible solution obtained in Example 26.30. Test for its optimality. If not, find the optimal one.

Solution: The values of u_i and v_j satisfying $c_{ij} = u_i + v_j$ for the occupied cells are shown as below.

3	2	4	[20] 1	$u_1 = 0$
[5] $+\theta$ 2	[5] $-\theta$ 4	5	[5] 3	$u_2 = 2$
[25] $-\theta$ 3	5	$+\theta$ 2	6	$u_3 = 3$
4	[15] $+\theta$ 3	[25] $-\theta$ 1	4	$u_4 = 1$
$v_1 = 1$	$v_2 = 2$	$v_3 = 0$	$v_4 = 1$	

The net evaluations for the unoccupied cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$ are given by

$\bar{c}_{11} = -3$, $\bar{c}_{12} = 0$, $\bar{c}_{13} = -4$, $\bar{c}_{23} = -3$, $\bar{c}_{32} = 0$, $\bar{c}_{33} = 1$, $\bar{c}_{34} = -2$, $\bar{c}_{41} = -3$, $\bar{c}_{44} = -2$.

Since all the net evaluations are not non-positive, therefore, the given solution is not optimal. Cell (3, 3) is with the largest positive value. Identify a loop starting and ending in the cell (3, 3) with comers at the occupied cells and assign an amount θ to the cell (3, 3) and subtract and add at the comer points (in occupied cells only), as shown in the table given above. Obviously max. θ = min. {5, 15, 25} = 5. The value at cell (2, 2) becomes zero and thus cell (2, 2) leaves the basis and cell (3, 3) enters the basis. The current solution is updated as below.

3	2	4	[20] 1	$u_1 = 0$
[10] 2	4	5	[5] 3	$u_2 = 2$
[20] 3	5	[5] 2	6	$u_3 = 3$
4	[20] 3	[20] 1	4	$u_4 = 2$
$v_1 = 0$	$v_2 = 1$	$v_3 = -1$	$v_4 = 1$	

Again specify the values of u_i and v_j satisfying $c_{ij} = u_i + v_j$ for the occupied cells.

The net evaluations for the unoccupied cells: $\bar{c}_{ij} = u_i + v_j - c_{ij}$ are given by

$\bar{c}_{11} = -3$, $\bar{c}_{12} = -1$, $\bar{c}_{13} = -5$, $\bar{c}_{22} = -1$, $\bar{c}_{23} = -4$, $\bar{c}_{32} = -1$, $\bar{c}_{34} = -2$, $\bar{c}_{41} = -2$, and $\bar{c}_{43} = -1$.

Since all the $\bar{c}_{ij}$ are non-positive thus the current solution

$$x_{14} = 20,\ x_{21} = 10,\ x_{24} = 5,\ x_{31} = 20,\ x_{33} = 5,\ x_{42} = 20,\ x_{43} = 20$$

is optimal and the minimum transportation cost is

$$\begin{aligned} z &= 20 \times 1 + 10 \times 2 + 5 \times 3 + 20 \times 3 + 5 \times 2 + 20 \times 3 + 20 \times 1 \\ &= 20 + 20 + 15 + 60 + 10 + 60 + 20 = 205. \end{aligned}$$

26.11.6 Degeneracy in Transportation Problems

A basic feasible solution for the general transportation problem is said to be *degenerate* when the number of occupied cells is less than $(m + n - 1)$. The degeneracy may occur in initial solution or it may arise in the subsequent iterations of uv-method. The degeneracy arises when the supply and the demand in the intermediate stages are equal corresponding to a selected cell for allocation.

To resolve degeneracy at any stage, a very small quantity ε (> 0) is allocated in an unoccupied cell so as to get $(m + n - 1)$ number of occupied cells, generally allocated to the cells with the lowest transportation costs. The quantity ε is used to evaluate unoccupied cells and once the purpose is over ε is made to tend to zero in the optimal solution.

EXERCISE 26.9

1. Compute the initial *BFS* for the transportation model given below using *VAM*

		To				
		1	2	3	4	Supply
	1	11	13	17	14	250
From	2	16	18	14	10	300
	3	21	24	13	10	400
Demand		200	225	275	250	

2. Use Vogel's approximation method to obtain an initial basic feasible solution of the transportation problem and obtain the minimum cost

		To 1	2	3	4	Supply
From	1	11	13	17	14	250
	2	16	18	14	10	300
	3	21	24	13	10	400
Demand		200	225	275	250	

3. For the following transportation problem find the initial *BFS* using *VAM*

		To 1	2	3	4	5	Supply
From	1	4	2	3	2	6	8
	2	5	4	5	2	1	12
	3	6	5	4	7	3	14
Demand		4	4	6	8	8	

4. Find an initial *BFS* for the following transportation problem, storage costs at each source of each unshipped item is also given.

		To 1	2	3	Supply	Storage cost
From	1	1	2	1	20	3
	2	0	4	5	40	4
	3	2	3	2	30	5
Demand		30	20	20		

5. Given $x_{13} = 50, x_{14} = 20, x_{21} = 55, x_{31} = 30, x_{32} = 35$ and $x_{34} = 15$. Is it an optimal solution to the transportation problem given below? If not, modify it to find the optimal one.

		To 1	2	3	4	Supply
From	1	6	1	9	3	70
	2	11	5	2	8	55
	3	10	12	4	7	90
Demand		85	35	50	45	

6. Find optimal solution of the following transportation problem.

	To 1	2	3	4	Supply
From 1	9	16	15	9	15
2	2	1	3	5	25
3	6	4	7	3	20
Demand	10	15	25	10	

7. Solve the following transportation problem subject to the constraint that the demand at D_1 must be met from S_4 only.

	To 1	2	3	Supply
From 1	5	1	0	20
2	3	2	4	10
3	7	5	2	15
4	9	6	0	15
Demand	5	10	15	

26.12 ASSIGNMENT PROBLEM

The assignment problem is a special case of the transportation problem in which the objective is to assign a number of sources (workers) to the equal number of destinations (jobs) at a minimum cost (or maximum profit). Since the units available at each source is one and also the units required at each detination is equal to one and thus, in this sense, an assignment problem is a completely degenerate form of the transpotation problem. This means exactly one occupied cell in each row and each column of the transportation table.

26.12.1 The Mathematical Formulation

Consider the general assignment model with n workers and n jobs represented as below

	Jobs 1	2	--------	n	Supply
Workers 1	c_{11}	c_{12}	--------	c_{1n}	1
2	c_{21}	c_{22}	--------	c_{2n}	1
⋮	⋮	⋮	--------	⋮	⋮
n	c_{n1}	c_{n2}	--------	c_{nn}	1
Demand	1	1	--------	1	

The element c_{ij} represents the cost of assigning the worker i to the job j ($i, j = 1, 2, ..., n$). There is no loss of generality in assuming that the number of workers are equal to the number of jobs since we can always add fictitious workers or fictitious jobs to balance the problem. Also the cost matrix $[c_{ij}]_{n\times n}$ in case of an assignment problem is same as that of the transportation problem except that the supply of workers and the demand at each of the destination is unity.

If x_{ij} denotes the assignment of the ith worker to the jth job, such that

$$x_{ij} = \begin{cases} 1, & \text{if } i\text{th worker is assigned the } j\text{th job} \\ 0, & \text{otherwise} \end{cases}$$

then the mathematical formulation of the assignment problem is

$$\text{Minimize} \quad z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \qquad ...(26.12)$$

$$\text{subject to} \quad \left.\begin{aligned} &\sum_{i=1}^{n} x_{ij} = 1, \text{ for all } j = 1, 2, ..., n \\ &\sum_{j=1}^{n} x_{ij} = 1, \text{ for all } i = 1, 2, ..., n \end{aligned}\right\} \qquad ...(26.13)$$

26.12.2 Solution of the Assignment Problem (Hungarian Method)

Since assignment problem is a special case of the transportation problem in which the workers represent the sources and jobs represent the destinations with supply and demand respectively at each source and destination being unity, thus, assignment problem can be solved as a regular transportation problem. However, the characteristics that all the supply and demand are unity has lead to an efficient method, called the *Hungarian method* for solving an assignment problem. Though the Hungarian method appears to be totally unrelated to transportation method, yet in actual practice, like transportation method it is completely based on the simplex method for solving an *LPP*. Before giving Hungarian method for solving an assignment problem, we state without proof the following result related to the assignment problems.

In an assignment problem if we add or subtract a constant to every element of any row (or, column) of the cost matrix $[c_{ij}]_{n\times n}$ then the optimal assignment is not affected.

Hungarian method. Assuming the cost matrix $[c_{ij}]$ of the given assignment problem to be square. Hungarian method envolves the following steps:

1. *For the cost matrix $[c_{ij}]_{n\times n}$ identify the smallest element in each row and then subtract the same from all the elements in that row.*
2. *For the resulting matrix from Step 1, identify the smallest element in each column and then subtract the same from all the elements in that column, resulting in each row and each column has at least one zero.*

3. *For the resulting matrix from Step 2, search for an optimal assignment as follows*

   (i) *Examine the rows successively until a row with a single zero is found. Enrectangle this zero* ($\boxed{0}$) *and cross off (X) all other zeros in its column. Continue this until all the rows are exhausted.*

   (ii) *Repeat the step 3(i) for columns of the reduced matrix.*

   (iii) *If a row and/or columns has two or more zeros and one cannot be chosen by inspection, then enrectangle arbitrary any one of these zeros and cross off all other zeros of that row/column.*

   (iv) *Repeat the Steps 3(i) to 3(iii) until all the zeros are enrectangled or crossed off.*

   If the number of assignment (enrectangled zeros) is equal to n, the order of the cost matrix, an optimal solution is obtained. If the number of assignments is less than n, then go to Step 4.

4. *Draw the minimum number of horizontal and/or vertical lines to cover all the zeros of the reduced matrix obtained from Step 2.*

5. *To obtain the new revised cost matrix, select the smallest uncovered element, subtract it from every uncovered element and then add it to every element at the intersection of any two lines.*

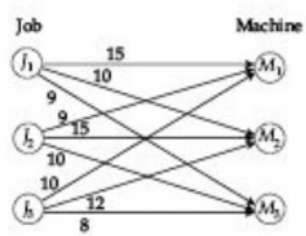
6. *Go to step 3 and repeat the procedure until an optimal assignment is obtained.*

Example 26.33: An engineer wants to assign 3 jobs J_1, J_2, J_3 to three machines M_1, M_2, M_3 in such a way that each job is assigned to some machine and no machine works on more than one job. The cost of assigning job J_i to machine M_j is given below.

	M_1	M_2	M_3
J_1	15	10	9
J_2	9	15	10
J_3	10	12	8

(a) Draw the associated network.
(b) Formulate it as LPP.
(c) Find the optimal assignment using Hungarian method.

Solution (a) The associated network is



(b) The associated *LPP* is

Minimize $z = 15x_{11} + 10x_{12} + 9x_{13} + 9x_{21} + 15x_{22} + 10x_{23} + 10x_{31} + 12x_{32} + 8x_{33}$

subject to

$$x_{11} + x_{12} + x_{13} = 1$$
$$x_{21} + x_{22} + x_{23} = 1$$
$$x_{31} + x_{32} + x_{33} = 1$$
$$x_{11} + x_{21} + x_{31} = 1$$
$$x_{12} + x_{22} + x_{32} = 1$$
$$x_{13} + x_{23} + x_{33} = 1$$

where $x_{ij} = 1$ or 0 for $i, j = 1, 2, 3$.

(c) Let p_i and q_j be row i and column j minima elements. Computing row minima from the original cost matrix as shown below.

$$\begin{array}{cl} \begin{bmatrix} 15 & 10 & 9 \\ 9 & 15 & 10 \\ 10 & 12 & 8 \end{bmatrix} & \begin{array}{l} p_1 = 9 \\ p_2 = 9 \\ p_3 = 8 \end{array} \end{array}$$

Subtracting row minima from the elements in the respective row the resultant matrix along with column minima, is

$$\begin{array}{c} \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 2 & 4 & 0 \end{bmatrix} \\ q_1 = 0 \quad q_2 = 1 \quad q_3 = 0 \end{array}$$

Subtracting column minima from the elements in the respective column, the reduced matrix is

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 1 \\ 2 & 3 & 0 \end{bmatrix}$$

Starting with row 1, enrectangling the single zeros successively first in rows and crossing all others in the column so marked, and then in column on the reduced matrix, we obtain

$$\begin{array}{|ccc|} 6 & \boxed{0} & \cancel{0} \\ \boxed{0} & 5 & 1 \\ 2 & 3 & \boxed{0} \end{array}$$

Now since each row and each column has one and only one assignment, an optimal assignment is achieved and is given by

$$J_1 \to M_2,\ J_2 \to M_1 \text{ and } J_3 \to M_3$$

The minimum total cost is

$$C = 10 + 9 + 8 = 27.$$

Remark: The total cost will always be equal to $\Sigma p_i + \Sigma q_j$, e.g., in the above problem it is given by

$$p_1 + p_2 + p_3 + q_1 + q_2 + q_3 = 9 + 9 + 8 + 0 + 1 + 0 = 27.$$

Example 26.34: A CEO has to assign four tasks to his four subordinates. His estimates of the time each of his subordinate would take to perform each task is given in the matrix below. How should the tasks be allocated, one to each one, so to minimize the total man-hours?

Tasks	Subordinates S_1	S_2	S_3	S_4
T_1	18	26	17	11
T_2	13	28	14	26
T_3	38	19	18	15
T_4	19	26	24	10

Solution: Using Hungarian method, we take the following steps:

1. Subtracting the row minima from the elements in the respective row, the resultant matrix is

7	15	6	0
0	15	1	13
23	4	3	0
9	16	14	0

2. Subtracting the column minima from the elements in the respective column, the reduced matrix is

7	11	5	0
0	11	0	13
23	0	2	0
9	12	13	0

3. Starting with row 1, rectangling the single zeros successively and crossing all other zeros in the column so marked and then repeating the same procedure for the columns, we obtain

7	11	5	$\boxed{0}$
$\boxed{0}$	11	$\times$	13
23	$\boxed{0}$	2	$\times$
9	12	13	$\times$

Since the column 3 and row 4 do not have any assignment so optimum assignment cannot be found at this stage, we go for the next step.

4. Drawing straight lines through row 2, row 3 and column 4, we obtain

7	11	5	0
0	11	0	13
23	0	2	0
9	12	13	0

5. To obtain the revised cost matrix, we subtract the smallest uncovered element, which is 5 in this case, from all the uncovered elements and adding it to all the elements at the intersection of the lines, the revised cost matrix is

2	6	0	0
0	11	0	18
23	0	2	5
4	7	8	0

6. Repeating steps on the revised cost matrix, we obtain

$$\begin{array}{|cccc|}
\hline
2 & 6 & \boxed{0} & \cancel{0} \\
\boxed{0} & 11 & \cancel{0} & 18 \\
23 & \boxed{0} & 2 & 5 \\
4 & 7 & 8 & \boxed{0} \\
\hline
\end{array}$$

Since each row and each column has one and only one assignment, thus an optimal solution is achieved and the optimal assignment is

$$T_1 \to S_3,\ T_2 \to S_1,\ T_3 \to S_2 \text{ and } T_4 \to S_4.$$

The minimum total time is

$$T = 17 + 13 + 19 + 10 = 59 \text{ man hrs.}$$

Remark: In some cases the pay off elements of the assignment problem may represent revenues or profit instead of costs so that the objective will be to maximize the total revenue or profit. The maximization problem of this type can be converted into the minimization one by selecting the largest element in the matrix and then subtracting from it all other elements in the matrix, and can then proceed to obtain the optimum solution for this problem.

Example 26.35: A company is to assign 4 jobs to 5 trainees. The expected revenue earning for each person on each job are as follows. Find the assignment of trainees to jobs that will result the maximum earning.

	Jobs			
Trainees	J_1	J_2	J_3	J_4
T_1	86	78	62	81
T_2	55	79	65	60
T_3	72	65	63	80
T_4	86	70	65	71
T_5	72	70	71	60

Solution: The given problem is of maximization type. The highest element is 86. To convert this problem to of minimization type, subtract all the elements from the highest element 86 and then to make it a square matrix add dummy column 5 consisting of zero elements, the matrix obtained is

$$\begin{bmatrix} 0 & 8 & 24 & 5 & 0 \\ 31 & 7 & 21 & 26 & 0 \\ 14 & 21 & 23 & 6 & 0 \\ 0 & 16 & 21 & 15 & 0 \\ 14 & 16 & 15 & 26 & 0 \end{bmatrix}$$

Subtract the column minima from the elements in the respective column, the resultant matrix is

$$\begin{bmatrix} 0 & 1 & 9 & 0 & 0 \\ 31 & 0 & 6 & 21 & 0 \\ 14 & 14 & 8 & 1 & 0 \\ 0 & 9 & 6 & 10 & 0 \\ 14 & 9 & 0 & 21 & 0 \end{bmatrix}$$

Start with row 1, rectangle the single zeros successively and cross all other zeros in the column so marked and then repeating the same procedure for the columns, we obtain

$$\begin{bmatrix} \cancel{0} & 1 & 9 & \boxed{0} & \cancel{0} \\ 31 & \boxed{0} & 6 & 21 & \cancel{0} \\ 14 & 14 & 8 & 1 & \boxed{0} \\ \boxed{0} & 9 & 6 & 10 & \cancel{0} \\ 14 & 9 & \boxed{0} & 21 & \cancel{0} \end{bmatrix}$$

Since each row and each column has one and only one assignment, an optimum solution is achieved and the optimum assignment is

$$T_1 \to J_4,\ T_2 \to J_2,\ t_3 \to J_5\ \ T_4 \to J_1,\ \text{and}\ T_5 \to J_3.$$

The maximum revenue earning is

$$Z = 81 + 79 + 86 + 71 = 317$$

and trainee T_3 will not be assigned any job.

EXERCISE 26.10

1. Solve the following assignment problems when the table summarizes the cost elements of the problem.

(a)

	1	2	3	4
I	2	5	7	4
II	10	8	11	10
III	5	6	12	8
IV	9	8	9	6

(b)

	1	2	3	4
I	1	4	6	3
II	9	7	10	9
III	4	5	11	7
IV	8	7	8	5

2. A CEO has four tasks to be performed by three subordinates. Following matrix gives the estimates of the time each subordinate would take to complete the task. Find the optimum allocations to minimize the total man hours.

	Subordinates		
Task	S_1	S_2	S_3
I	9	26	15
II	13	27	6
III	35	20	15
IV	18	30	20

3. A company wants to assign five salesperson to five different regions to promote a product. The expected sale (in thousand) are given below. Solve the problem to maximize the expected sale.

		Regions				
		R_1	R_2	R_3	R_4	R_5
	S_1	27	54	37	100	85
	S_2	55	66	45	80	32
Salespersons	S_3	72	58	74	80	85
	S_4	39	88	74	59	72
	S_5	72	66	45	69	85

4. The following is the cost matrix of assigning 4 persons to 4 jobs. Find the optimal assignment if person 1 can't be assigned job A. Find also the minimum total cost.

Persons	A	B	C	D
1	–	6	3	1
2	5	8	6	7
3	6	9	5	4
4	4	7	7	3

ANSWERS

Exercise 26.1 (p. 1494)

1. $z = 2x_1 + 4x_2 + 3x_3$
   $3x_1 + 4x_2 + 3x_3 \le 3600,$ $2x_1 + x_2 + 3x_3 \le 2400$
   $x_1 + 3x_2 + 3x_3 \le 4800$ $x_1, x_2,$ and $x_3 \ge 0.$
2. $z = 4x_1 + 5x_2$
   $6x_1 + 5x_2 \le 250, 4x_1 + 6x_2 \le 200,$
   $6x_1 + 5x_2 \ge 150, 9x_1 + 5x_2 \ge 130$
   $x_1, x_2 \ge 0$
3. $z = .026x_1 + .0509x_2 + .0864x_3 + .06875x_4 + .078x_5$
   $x_1 + x_2 + x_3 + x_4 + x_5 \le 12$
   $x_4 + x_5 \ge 4.8$
   $.5x_1 + .5x_2 - .5x_3 \le 0$
   $.06x_1 + .03x_2 - .01x_3 + .01x_4 - .02x_5 \le 0$
   $x_1, x_2, x_3, x_4,$ and $x_5 \ge 0.$
4. $z = 200000x_1 + 90000x_2 + 32000x_3$
   $80000x_1 + 60000x_2 + 50000x_3 \le 10{,}0000$
   $x_1 \le 10, x_2, x_3 \le 8$
   $x_1, x_2, x_3 \ge 0.$
5. $z = 240x_1 + 300x_2 + 280x_3$
   $x_1 - x_2 + 2x_3 \le 0$
   $-x_1 + x_2 \le 0$
   $x_1 + x_2 + x_3 \le 100$
   $x_1, x_2, x_3 \ge 0.$

6. $z = 8.5x_1 + 9.0x_2 + 5.0x_3$

$0.4x_1 + 3.8x_2 + 3.5x_3 \geq 12{,}800$

$0.4x_1 + 3.8x_2 + 3.5x_3 \leq 13{,}600$

$0.15x_1 + 2.41x_2 + 2.3x_3 \geq 9000$

$0.15x_1 + 2.41x_2 + 2.35x_3 \leq 9{,}400$

$x_1 + x_2 + x_3 = 4000$

$x_1, x_2, x_3 \geq 0.$

Exercise 26.2 (p. 1501)

1. $x = \dfrac{15}{7}, y = \dfrac{8}{7}, z = \dfrac{300}{7}.$
2. $x = 18, y = 12, z = 72.$
3. $x = 470.6$ lb. $y = 329.4$ lb., $z = \$437.64$ per day
4. $x = 4, y = 2, z = 160.$
5. Unbounded solution.
6. Infeasible solution
7. $x = 3, y = 1, z = 10$

   $x = 0, y = 2.5, z = 10.$
8. $x = 1/3, y = 4/3, z = -7/3.$
9. $x = 2000/11, y = 1000/11$ $z = 10{,}000.$
10. $x = 12, y = 4, z = 8{,}800.$

Exercise 26.3 (p. 1507)

1. Max $z = 3x_1 + 5x_2 + 7x_3' - 7x_3''$

   $6x_1 - 4x_2 + s_1 = 5$

   $3x_1 + 2x_2 + 5x_3' - 5x_3'' - s_2 = 11$

   $4x_1 + 3x_2 + s_3 = 2$

   $x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \geq 0.$
2. Max $(-z) = -2x_1' + 2x_1'' - 3x_2' + 3x_2''$

   $-2x_1' + 2x_1'' + 3x_2' - 3x_2'' + x_3 = 4$

   $-3x_1' + 3x_1'' - 4x_2' + 4x_2'' + x_4 = 6$

   $2x_1' - 2x_1'' + 5x_2' - 5x_2'' + x_5 = 10$

   $4x_1' - 4x_1'' - 3x_2' + 3x_2'' + x_6 = 18,$

   $x_1', x_1'', x_1', x_2'', x_3, x_4, x_5, x_6 \geq 0.$
3. $x_1 = 6/7, x_2 = 12/7, z = 48/7$
4. (c) $x_1 = 2, x_2 = 1, x_3 = 0, z = 5.$
5. $x_1 = 0, x_2 = 0, x_3 = 44/17, x_4 = 45/17, z = 28.9.$

Exercise 26.4 (p. 1515)

1. $x_1 = 5/3, x_2 = 20/3, z = \dfrac{550}{3}.$
2. $z = 6$ on the segment from $(3, 0, 0)$ to $(0, 0, 2)$
3. $x_1 = 2, x_2 = 0, z = 10.$
4. $x_1 = 89/41, x_2 = 50/41, x_3 = 62/41$ $z = 765/41$
5. $x_1 = 4, x_2 = 5, x_3 = 0, z = -11$
6. $x_1 = 60/21, x_2 = 0, x_3 = 1500/105, x_4 = 0, z = 22000/7.$

7. $x = 7/3, y = 4/3$, max. $z = 16$, $x = 1, y = 0$, min. $z = 4$.
8. $x_A = 200, x_B = 240$.
9. $x_A = 0, x_B = 380/9, x_C = 470/3, z = 3200/3$.

Exercise 26.5 (p. 1525)

1. $x_1 = 3/2, x_2 = 2, z = 5$.
3. Infinite solution lying between (6/7, 60/7) and (6, 0), $z = 60$.
7. Infeasible solution
8. $x_1 = 4.5, x_2 = 3.5, z = 17250$.

Exercise 26.6 (p. 1540)

1. $x_1 = 3, x_2 = 1, z = 7$.
2. $x_1 = 3/2, x_2 = 1/2, z = 11/2$.
3. $x_1 = 1/2, x_2 = 0, x_3 = 1/2, z = 3$.
4. $x_1 = 0, x_2 = 2, x_3 = 1, z = 19$.
5. Unbounded solution.
6. No feasible solution.
10. $x_1 = 0, x_2 = 5, z = 25$.
11. $x_1 = 55/7, x_2 = 30/7$ $z = 155/7$
12. $x_1 = 6, x_2 = 7, x_3 = 0, z = 177$.

Exercise 26.7 (p. 1544)

1. Minimize $w = 36y_1 + 60y_2 + 45y_3$

   subject to: $y_1 + 2y_2 + 2y_3 \geq 40$, $2y_1 + y_2 + 5y_3 \geq 25$, $y_1 + 4y_2 + y_3 \geq 50$ and $y_1, y_2, y_3 \geq 0$.
2. Maximize: $w = 28y_1 + 30y_2$

   subject to: $2y_1 + 3y_2 \leq 8$, $4y_1 + 5y_2 \leq 3$, $3y_1 + 6y_2 \leq 15$ and $y_1, y_2 \geq 0$.
3. Maximize: $w = 7y_1 + 12y_2 + 10y_3$

   subject to: $3y_1 + 2y_2 - 4y_3 \leq 1$, $y_1 + 4y_2 - 3y_3 \geq 3$, $-2y_1 - 8y_3 = 2$, $y_1 \leq 0$, $y_2 \geq 0$, and y_3 unrestricted.
4. Minimize $w = 2y_1 + y_2$

   subject to: $-2y_1 + 2y_2 \geq 1$, $y_1 + 3y_2 \geq -2$, $3y_1 + 4y_2 \geq 3$ and y_1, y_2 unrestricted.
5. Maximize: $w = 2y_1 + 3y_2 + 5y_3$

   subject to: $2y_1 + 3y_2 + y_3 \leq 2$, $3y_1 + y_2 + 4y_3 \leq 3$, $5y_1 + 7y_2 + 6y_3 = 4$, $y_1 \geq 0$, y_2 unrestricted, and $y_3 \leq 0$.

Exercise 26.8 (p. 1549)

1. $x_1 = 4, x_2 = 2, z = 10$.
2. $x_1 = 3/5, x_2 = 0, x_3 = 2/5, z = 19/5$
3. $x_1 = 29/5, x_2 = -2/5, z = 274/5$
4. $x_1 = 16/3, x_2 = 22/3, z = 7.4$

Exercise 26.9 (p. 1557)

1. $x_{12} = 15, x_{14} = 0, x_{23} = 15, x_{24} = 10, x_{31} = 5, x_{34} = 5, z = 475$
2. $x_{11} = 200, x_{12} = 50, x_{22} = 175, x_{24} = 125, x_{33} = 275, x_{34} = 125, z = 12{,}075$
3. $x_{12} = 4, x_{14} = 4, x_{24} = 4, x_{25} = 8, x_{31} = 4, x_{33} = 6, x_{3F} = 4, z = 80$
4. $x_{13} = 20, x_{14} = 0, x_{21} = 30, x_{2F} = 10, x_{32} = 20, x_{3F} = 10, z = 170.$
5. No. $x_{12} = 30, x_{14} = 40, x_{22} = 5, x_{23} = 50, x_{31} = 85, x_{34} = 5, z = 1160.$
6. $x_{11} = 10, x_{14} = 5, x_{22} = 0, x_{23} = 25, x_{32} = 15, x_{34} = 5, z = 285.$
7. $x_{12} = 10, x_{13} = 10, x_{2F} = 10, x_{3F} = 15, x_{41} = 5, x_{43} = 5, x_{4F} = 5, z = 55.$

Exercise 26.10 (p. 1567)

1. (a) I $\to$ 1, II $\to$ 3, III $\to$ 2, IV $\to$ 4, $z = 25.$
   (b) I $\to$ 1, II $\to$ 3, III $\to$ 2, IV $\to$ 4, $z = 21.$
2. I $\to S_1$, II $\to S_3$, III $\to S_2$, $z = 35$, Task IV will remain unassigned.
3. $S_1 \to R_4, S_2 \to R_1, S_3 \to R_3, S_4 \to R_2, S_5 \to R_5$
4. $1 \to D, 2 \to A, 3 \to C, 4 \to B$ or $1 \to D, 2 \to B, 3 \to C, 4 \to A$. $z = 18.$

Appendices

Appendix I

A.1 CYLINDRICAL AND SPHERICAL POLAR COORDINATES

Mathematical problems formulated using a particular coordinate system say cartesian, often need to be expressed in terms of different coordinate systems in order to simplify the task of finding solution. *Cylindrical coordinates* simplify the equations of cylinders while *spherical polar coordinates* simplify the equations of spheres and cones.

I Cylindrical coordinates The *cylindrical coordinates* are obtained by combining polar coordinates in the *xy*-plane with the *z*-axis. This assigns to every point in space a triplet of the form (r, θ, z) as illustrated in Fig. A1.1. The equations relating cartesian coordinates (x, y, z) and cylindrical coordinates (r, θ, z) are

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad r \geq 0 \text{ and } \quad 0 \leq \theta \leq 2\pi$$

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x.$$

The equation $r = a$ in cylindrical coordinates describes an entire cylinder about the *z*-axis. The equation of the *z*-axis is $r = 0$. The equation $\theta = \theta_0$ represents a plane containing *z*-axis, while the equation $z = k$ describes a plane perpendicular to *z*-axis.

II Spherical Coordinates: The *spherical polar coordinates* of a point *P* in space is given by (r, θ, ϕ), where r is the distance of P from the origin $O, \theta, 0 \leq \theta \leq 2\pi$ is the angle from cylindrical coordinates and $\phi, 0 \leq \phi \leq \pi$ is the angle which $\overrightarrow{OP}$ makes with the positive direction of *z*-axis as illustrated in Fig. A1.2.

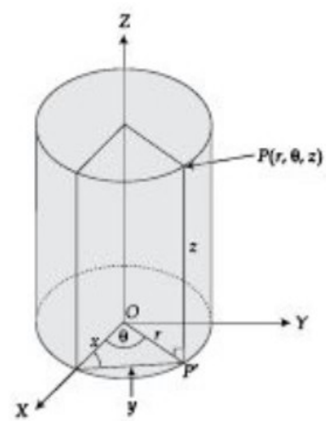


Fig. AI.I

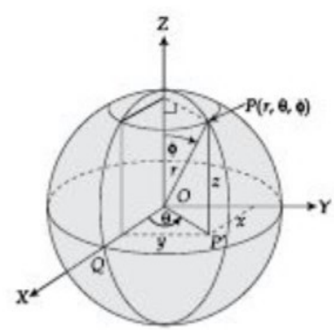


Fig. AI.2

The equations relating cartesian coordinates (x, y, z) and spherical polar coordinates (r, θ, ϕ) are

$$x = r \cos\theta \sin\phi,\ y = r \sin\theta \sin\phi,\ z = r \cos\phi,\ 0 \le \theta \le 2\pi,\ 0 \le \phi < \pi.$$

Also $\quad r = \sqrt{x^2 + y^2 + z^2}$.

The equation $r = a$ describes a sphere of radius a with centre at the origin while the equation $\phi = \phi_0$ describes a single cone with vertex at the origin and axis along z-axis, as shown in Fig. A1.3, $\phi = \pi/4$ represents a cone which in terms of cartesian coordinates is $z = \sqrt{x^2 + y^2}$. The cone $\phi = \pi/2$ gives the equation of the xy-plane. Sometimes spherical coordinates are given as (r, ϕ, θ), with role of ϕ and θ reversed. Also sometimes ρ is used instead of r in case of spherical coordinates. But we have followed the notations as introduced here.

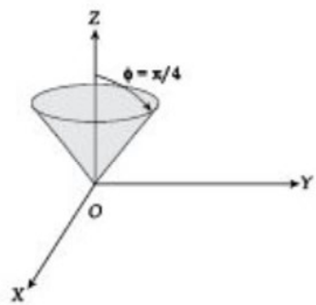


Fig. A1.3

Appendix II

HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

The exponential function e^x can be expressed as $e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$, the sum of an even and an odd function. The even part of e^x is called the *hyperbolic cosine* and the odd part of e^x is called the *hyperbolic sine*. These two functions are useful in their individual capacity and are associated with the geometry of the conic section hyperbola.

The six basic hyperbolic functions are defined as

(i) Hyperbolic sine of x: $\sinh x = \frac{e^x - e^{-x}}{2}$

(ii) Hyperbolic cosine of x: $\cosh x = \frac{e^x + e^{-x}}{2}$

(iii) Hyperbolic tangent of x: $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(iv) Hyperbolic cotangent of x: $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(v) Hyperbolic secant of x: $\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$

(vi) Hyperbolic cosecant of x: $\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$.

The notation $\sinh x$ is often read as shine x and $\cosh x$ is read as kosh x. Hyperbolic functions have number of similarities to the trigonometric functions. We observe that $\sinh x$, $\tanh x$, $\operatorname{cosech} x$ and $\coth x$ are odd functions of x. Fig. A2.1 shows the graphs of the hyperbolic functions.

Following *identities for hyperbolic functions* can be proved easily.

(i) $\tanh x = \frac{\sinh x}{\cosh x}$ (ii) $\coth x = \frac{\cosh x}{\sinh x}$

(iii) $\sinh 2x = 2\sinh x \cosh x$ (iv) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(v) $\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$ (vi) $2\cosh^2 x = \cosh 2x + 1$

(vii) $2\sinh^2 x = \cosh 2x - 1$ (viii) $\cosh^2 x - \sinh^2 x = 1$

(ix) $\tanh^2 x = 1 - \operatorname{sech}^2 x$ (x) $\coth^2 x = 1 + \operatorname{cosech}^2 x$

(xi) $\sinh 3x = 3\sinh x + 4\sinh^3 x$ (xii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(xiii) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(xiv) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(xv) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$.

Derivatives of hyperbolic functions:

(i) $\dfrac{d}{dx}(\sinh x) = \cosh x$ (ii) $\dfrac{d}{dx}(\cosh x) = \sinh x$

(iii) $\dfrac{d}{dx}(\tanh x) = \text{sech}^2 x$ (iv) $\dfrac{d}{dx}(\coth x) = -\text{cosech}^2 x$

(v) $\dfrac{d}{dx}(\text{sech}\, x) = -\text{sech}\, x \tanh x$ (vi) $\dfrac{d}{dx}(\text{cosech}\, x) = -\text{cosech}\, x \coth x$.

Also we have corresponding integrals formulae for hyperbolic functions.

Next, since, $\dfrac{d}{dx}(\sinh x) = \cosh x > 0$, thus $\sinh x$ is an increasing function of x. We define $y = \sinh^{-1} x$, $-\infty < x < \infty$, as the number whose hyperbolic sine is x.

The function $y = \cos hx$ is not one to one as is clear from Fig. A 2.1 b. In case we restrict the domain $x \geq 0$, then $y = \cosh x$ is one to one and therefore has an inverse denoted by $y = \cosh^{-1} x$.

For every value of $x \geq 1$, $y = \cosh^{-1} x$ is the number in the interval $0 \leq y < \infty$ whose hyperbolic cosine is x.

Similarly, $y = \text{sech} x$ is one to one, refer to Fig. A 2.1e, only when we restrict the domain to non-negative values of x and has an inverse denoted by $y = \text{sech}^{-1} x$, which is defined for every value of x in the interval $(0, 1)$.

The $\tanh x$, $\coth x$ and $\text{cosech}\, x$ are one to one on their domains of definitions and therefore have inverses denoted by respectively $y = \tanh^{-1} x$, $y = \coth^{-1} x$ and $y = \text{cosech}^{-1} x$.

The *inverse hyperbolic functions* can be expressed in terms of logarithmic functions as given below.

(i) $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}),\quad -\infty < x < \infty$ (ii) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}),\quad x \geq 1$

(iii) $\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{1+x}{1-x},\ |x| < 1$ (iv) $\text{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right),\quad 0 < x \leq 1$

(v) $\text{cosech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 + x^2}}{x}\right),\quad x \neq 0$ (vi) $\coth^{-1} x = \dfrac{1}{2} \ln \dfrac{x+1}{x-1},\ |x| > 1$.

To prove, $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, let $y = \sin h^{-1} x$, this gives

$x = \sin hy = \dfrac{1}{2}(e^y - e^{-y})$ or, $e^{2y} - 2x\, e^y - 1 = 0$ or, $e^y = x + \sqrt{x^2 + 1}$, or $y = \ln(x + \sqrt{x^2 + 1})$

rejecting the term corresponding to negative sign, since $e^y > 0$.

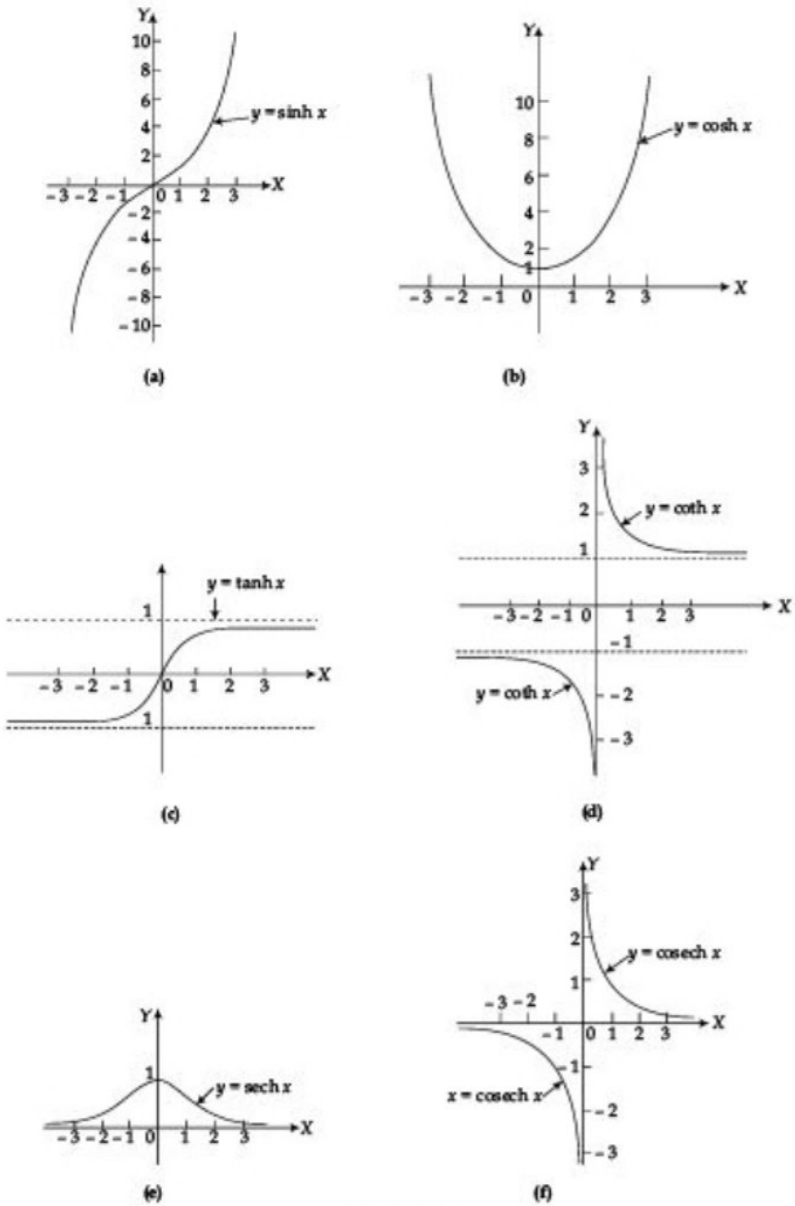

y = sinh x
(a)
y = cosh x
(b)
y = tanh x
(c)
y = coth x
y = coth x
(d)
y = sech x
(e)
y = cosech x
x = cosech x
(f)

Fig. A.2.1

Also we have the following *useful identities for the inverse hyperbolic functions:*

(i) $\operatorname{sech}^{-1} x = \cosh^{-1} \dfrac{1}{x}$ (ii) $\operatorname{cosech}^{-1} x = \sinh^{-1} \dfrac{1}{x}$

(iii) $\coth^{-1} x = \tanh^{-1} \dfrac{1}{x}$.

Derivatives of inverse hyperbolic functions:

(i) $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$ (ii) $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}, \quad |x| > 1$

(iii) $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2}, \ |x| < 1$ (iv) $\dfrac{d}{dx}(\coth^{-1} x) = \dfrac{1}{1-x^2}, \ |x| > 1$

(v) $\dfrac{d}{dx}(\operatorname{sech}^{-1} x) = -\dfrac{1}{x\sqrt{1-x^2}}, \quad 0 < x < 1,$ (vi) $\dfrac{d}{dx}(\operatorname{cosech}^{-1} x) = -\dfrac{1}{|x|\sqrt{1+x^2}}, \quad x \neq 0.$

To prove $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$, let $y = \sinh^{-1} x$. It gives $x = \sinh y$. Differentiating w.r.t. x, we have $1 = \cosh y \dfrac{dy}{dx}$, or $\dfrac{dy}{dx} = \dfrac{1}{\cosh y} = \dfrac{1}{\sqrt{1+\sinh^2 y}} = \dfrac{1}{\sqrt{1+x^2}}$.

Relations between circular functions and hyperbolic functions: We have

(i) $\sin ix = i \sinh x$ (ii) $\cos ix = \cosh x$ (iii) $\tan ix = i \tanh x$.

(iv) $\sinh ix = i \sin x$ (v) $\cosh ix = \cos x$ (vi) $\tanh ix = i \tan x$.

Series expansions for sinh x and cosh x:

Using the series expansion for

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots,$$

We have, $\quad \sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots$ and, $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots$

Appendix III

ADDITIONAL PROOFS

(i) Section 17.9 Laplace Equation in Polar Coordinates

We have

$$x = r\cos\theta,\ y = r\sin\theta. \text{ These give } r = \sqrt{(x^2+y^2)},\ \theta = \tan^{-1}(y/x)$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{(x^2+y^2)}} = \cos\theta \quad \text{and} \quad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

Thus, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$, which implies

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}. \text{ Similarly, } \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}.$$

Also, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}, \quad \text{...(i)}$$

and, $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} \quad \text{...(iii)}$$

Adding (i) and (ii), we obtain $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}$ Thus, the Laplace equation in polar coordinates is

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial\theta^2} = 0.$$

Section 17.14 Associated Legendre Functions

Equation (17.276) is $\quad \Phi'' + (\cot\phi)\,\Phi' + [n(n+1) - m^2\operatorname{cosec}^2\phi]\Phi = 0 \quad$ ...(i)

Set $\alpha = \cos\phi$, we have $\quad \frac{d\Phi}{d\phi} = \frac{d\Phi}{d\alpha}\frac{d\alpha}{d\phi} = -\sin\phi\frac{d\Phi}{d\alpha}$

$$\frac{d^2\Phi}{d\phi^2} = \frac{d}{d\phi}\left(-\sin\phi\frac{d\Phi}{d\alpha}\right) = -\cos\phi\frac{d\Phi}{d\alpha} + \sin^2\phi\frac{d^2\Phi}{d\alpha^2} = -\alpha\frac{d\Phi}{d\alpha} + (1-\alpha^2)\frac{d^2\Phi}{d\alpha^2}$$

Substituting these in (i) and simplifying, we obtain

$$(1-\alpha^2)\frac{d^2\Phi}{d\alpha^2} - 2\alpha\frac{d\Phi}{d\alpha} + \left[n(n+1) - \frac{m^2}{1-\alpha^2}\right]\Phi = 0 \qquad \text{...(ii)}$$

Again substituting $\Phi(\alpha) = (1-\alpha^2)^{m/2}v(\alpha)$, (ii) reduces to

$$(1-\alpha^2)v'' - 2(m+1)\alpha v' + [n(n+1) - m(m+1)]v = 0. \qquad \text{...(iii)}$$

Legendre's equation of parameter n is

$$(1-\alpha^2)y'' - 2\alpha y' + n(n+1)y = 0 \qquad \text{...(iv)}$$

Differentiating it m times w.r.t. α and simplifying, we obtain

$$(1-\alpha^2)\,y^{(m+2)} - 2\alpha y^{(m+1)} + [n(n+1) - m(m+1)]y^{(m)} = 0 \qquad \text{...(v)}$$

This is same as equation (iii) for $v = y^{(m)} = \dfrac{dy^m}{d\alpha^m}$ and hence solution of (iii) is $v = \dfrac{d^m P_n}{d\alpha^m}$, where P_n is Legendre function. The corresponding function

$$\Phi(\alpha) = (1-\alpha^2)^{m/2}v(\alpha) = (1-\alpha^2)^{m/2}\frac{d^m P_n}{d\alpha^m} = P_n^m(\alpha)$$

is called an *associated Legendre function.*

Appendix IV

BESSEL FUNCTIONS OF FIRST KIND OF ORDER ZERO AND ORDER ONE.

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000	3.0	−0.2601	0.3391	6.0	0.1506	−0.2767
0.1	0.9975	0.0499	3.1	−0.2921	0.3009	6.1	0.1773	−0.2559
0.2	0.9900	0.0995	3.2	−0.3202	0.2613	6.2	0.2017	−0.2329
0.3	0.9776	0.1483	3.3	−0.3443	0.2207	6.3	0.2238	−0.2081
0.4	0.9604	0.1960	3.4	−0.3643	0.1792	6.4	0.2433	−0.1816
0.5	0.9385	0.2423	3.5	−0.3801	0.1374	6.5	0.2601	−0.1538
0.6	0.9120	0.2867	3.6	−0.3918	0.0955	6.6	0.2740	−0.1250
0.7	0.8812	0.3290	3.7	−0.3992	0.0538	6.7	0.2851	−0.0953
0.8	0.8463	0.3688	3.8	−0.4026	0.0128	6.8	0.2931	−0.0652
0.9	0.8075	0.4059	3.9	−0.4018	−0.0272	6.9	0.2981	−0.0349
1.0	0.7652	0.4401	4.0	−0.3971	−0.0660	7.0	0.3001	−0.0047
1.1	0.7196	0.4709	4.1	−0.3887	−0.1033	7.1	0.2991	0.0252
1.2	0.6711	0.4983	4.2	−0.3766	−0.1386	7.2	0.2951	0.0543
1.3	0.6201	0.5220	4.3	−0.3610	−0.1719	7.3	0.2882	0.0826
1.4	0.5669	0.5419	4.4	−0.3423	−0.2028	7.4	0.2786	0.1096
1.5	0.5118	0.5579	4.5	−0.3205	−0.2311	7.5	0.2663	0.1352
1.6	0.4554	0.5699	4.6	−0.2961	−0.2566	7.6	0.2516	0.1592
1.7	0.3980	0.5778	4.7	−0.2693	−0.2791	7.7	0.2346	0.1813
1.8	0.3400	0.5815	4.8	−0.2404	−0.2985	7.8	0.2154	0.2014
1.9	0.2818	0.5812	4.9	−0.2097	−0.3147	7.9	0.1944	0.2192
2.0	0.2239	0.5 767	5.0	−0.1776	−0.3276	8.0	0.1717	0.2346
2.1	0.1666	0.5683	5.1	−0.1443	−0.3371	8.1	0.1475	0.2476
2.2	0.1104	0.5560	5.2	−0.1103	−0.3432	8.2	0.1222	0.2580
2.3	0.0555	0.5399	5.3	−0.0758	−0.3460	8.3	0.0960	0.2657
2.4	0.0025	0.5202	5.4	−0.0412	−0.3453	8.4	0.0692	0.2708
2.5	−0.0484	0.4971	5.5	−0.0068	−0.3414	8.5	0.0419	0.2731
2.6	−0.0968	0.4708	5.6	0.0270	−0.3343	8.6	0.0146	0.2728
2.7	−0.1424	0.4416	5.7	0.0599	−0.3241	8.7	−0.0125	0.2697
2.8	−0.1850	0.4097	5.8	0.0917	−0.3110	8.8	−0.0392	0.2641
2.9	−0.2243	0.3754	5.9	0.1220	−0.2951	8.9	−0.0653	0.2559

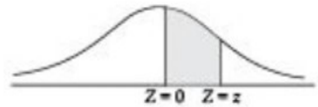
$J_0(x) = 0$ for $x = 2.405, 5.520, 8.654, 11.792, 14.931, \ldots$

$J_1(x) = 0$ for $x = 0, 3.832, 7.016, 10.173, 13.324, \ldots$

Appendix V: Statistical Tables

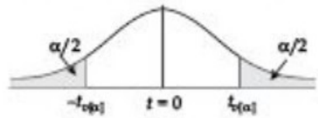
Table I: Areas Under the Standard Normal Curve

$$P(0 < Z < z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{1}{2}z^2}\, dz$$



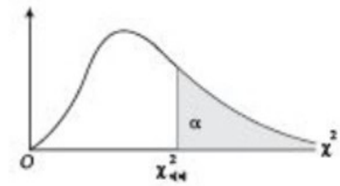
↓z→	0	1	2	3	4	5	6	7	8	9
.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0759
.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.3	1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2794	.2823	.2852
.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3655	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4637
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4678	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4959	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

Table II: Critical Values $t_{\nu[\alpha]}$ of t-Distribution (Two-Tail Areas) $P[|t| > t_{\nu[\alpha]}] = \alpha$



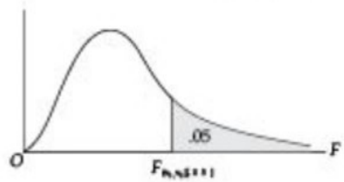
d.f. (ν)	*Level of significance* (α)					
	0.50	0.10	0.05	0.02	0.01	0.001
1	1.00	6.31	12.71	31.82	63.86	636.62
2	0.82	2.92	4.30	6.97	6.93	31.60
3	0.77	2.35	3.18	4.54	5.84	12.94
4	0.74	2.13	2.78	3.75	4.60	8.61
5	0.73	2.02	2.57	3.37	4.03	6.86
6	0.72	1.94	2.45	3.14	3.71	5.96
7	0.71	1.90	2.37	3.00	3.50	5.41
8	0.71	1.86	2.31	2.90	3.36	5.04
9	0.70	1.83	2.26	2.82	3.25	4.78
10	0.70	1.81	2.23	2.76	3.17	4.59
11	0.70	1.80	2.20	2.72	3.11	4.44
12	0.70	1.78	2.18	2.68	3.06	4.32
13	0.69	1.77	2.16	2.65	3.01	4.22
14	0.69	1.76	2.15	2.62	2.98	4.14
15	0.69	1.75	2.13	2.60	2.95	4.07
16	0.69	1.75	2.12	2.58	2.92	4.02
17	0.69	1.74	2.11	2.57	2.90	3.97
18	0.69	1.73	2.10	2.55	2.88	3.92
19	0.69	1.73	2.09	2.54	2.86	3.88
20	0.69	1.73	2.09	2.53	2.85	3.85
21	0.69	1.72	2.08	2.52	2.83	3.83
22	0.69	1.72	2.07	2.51	2.82	3.79
23	0.69	1.71	2.07	2.50	2.81	3.77
24	0.69	1.71	2.06	2.49	2.80	3.75
25	0.68	1.71	2.06	2.49	2.79	3.73
26	0.68	1.71	2.06	2.48	2.78	3.71
27	0.68	1.70	2.05	2.47	2.77	3.69
28	0.68	1.70	2.05	2.47	2.76	3.67
29	0.68	1.70	2.05	2.46	2.76	3.66
30	0.68	1.70	2.04	2.46	2.75	3.65
∞	0.67	1.65	1.96	2.33	2.58	3.29

Table III: Critical Values $\chi^2_{\upsilon[\alpha]}$ of Chi-Square Distribution (Right Tail Areas) $P[\chi^2_\nu > \chi^2_{\nu[\alpha]}] = \alpha$



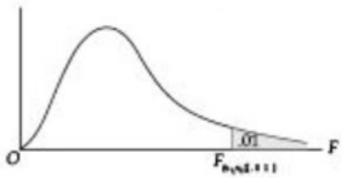
Degree of freedom	*Level of significance (α)*							
(ν)	0.995	0.99	0.975	0.95	0.05	0.025	0.01	0.005
1	0.000	0.000	0.001	0.004	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	11.070	12.832	15.086	16.750
6	0.676	0.872	1.237	1.634	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	18.360	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	22.362	24.736	24.888	29.819
14	4.075	4.660	5.629	6.571	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	33.924	36.781	40.289	42.796
23	9.260	10.196	11.688	13.091	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	36.415	39.364	42.980	45.558
25	10.520	11.524	13.120	14.611	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	43.773	46.979	50.892	53.672

Table IV A: Critical Values of The F-distribution (Right Tail Areas) $F_{(\nu_1, \nu_2)(.05)}$



ν_2 \ ν_1	1	2	3	4	5	6	8	12	24	∞
1	161.40	199.50	215.70	224.60	230.20	234.00	238.90	243.90	249.00	254.30
2	18.51	19.00	19.16	19.25	19.30	19.35	19.37	19.41	19.45	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.84	8.74	8.64	8.55
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.91	5.77	5.65
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.68	4.53	4.96
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.00	3.84	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.57	3.41	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.28	3.12	2.93
9	5.12	4.26	3.865	3.63	3.48	3.37	3.23	3.07	2.90	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.91	2.74	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	2.95	2.79	2.61	2.40
12	4.75	3.88	4.49	3.26	3.11	3.00	2.85	2.69	2.50	2.30
13	4.67	3.80	3.41	3.18	3.02	2.92	2.77	2.60	2.42	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.53	2.35	2.13
15	4.54	3.68	3.29	3.06	3.90	2.79	2.64	2.48	2.29	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.42	2.24	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.55	2.38	2.19	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.34	2.15	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.48	2.31	2.11	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.28	2.08	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.42	2.25	2.05	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.40	2.23	2.03	1.76
23	4.28	3.42	3.03	2.80	2.64	2.53	2.38	2.20	2.00	1.74
24	4.26	3.40	3.01	2.78	2.62	2.51	2.36	2.18	1.98	1.73
25	4.24	3.38	2.99	2.76	2.60	2.49	2.34	2.16	1.96	1.71
26	4.22	3.37	2.98	2.74	2.59	2.47	2.32	2.15	1.95	1.60
27	4.21	3.35	2.96	2.73	2.57	2.46	2.30	2.13	1.93	1.77
28	4.20	3.34	2.95	2.71	2.56	2.44	2.29	2.12	1.91	1.65
29	4.18	3.33	2.93	2.70	2.54	2.43	2.28	2.10	1.90	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.09	1.89	1.62

Table IV B: Critical Values of the F-distribution (Right Tail Areas) $F_{(v_1, v_2)(.01)}$



v_2 \ v_1	1	2	3	4	5	6	8	12	24	∞
1	4052	4999.5	5403	5625	5764	5859	5982	6106	6235	6366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.37	99.42	99.46	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.29	9.89	9.47	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	12.25	9.95	8.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.74	4.40	4.02	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.30	3.96	3.59	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.14	3.80	3.43	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.00	3.67	3.29	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.79	3.46	3.08	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.00	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.63	3.30	2.92	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.51	3.17	2.80	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.45	3.12	2.75	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.41	3.07	2.70	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.36	3.03	2.66	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.32	2.99	2.62	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.29	2.96	2.58	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.26	2.93	2.55	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.23	2.90	2.52	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.20	2.87	2.49	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01

Reference Textbooks

1. Bird. J. Higher Engineering Mathematics, Fourth Edition, Elsevier 2004.
2. Bolton, W. Fourier Series. Longman Scientific and Technical Publications, 1995
3. Bracewell, R. Fourier Transforms and Its Applications. McGraw-Hill, 1965
4. Braun. M. Differential Equations and Their Applications, Springer-Verlage, New York, 1975.
5. Bronson, R. Matrix Methods. An Introduction, Academic Press, 1969.
6. Churchill, R.V. Brown J.W. and Verhey, R.F. Complex Variables and Applications. McGraw-Hill, 1974
7. Churchill, R.V. Fourier Series and Boundary Value Problems. McGraw-Hill, 1963
8. Copson, E.T. An Introduction to the *Theory of Functions of a Complex Variable. Oxford University Press, 1974*
9. Cullen C.G. Linear Algebra with Applications, 2nd Edition, Addison-Wesley, Reading, MA, 1997.
10. Feller, W., An Introduction to Probability Theory and Its Applications. Vol. 1. Wiley Eastern Ltd., 1988
11. Greenberg M.D. Advanced Engineering Mathematics, 2nd Edition Pearson Education 2002.
12. Hildebrand, F.B. Advanced Calculus for Applications, Prentice-Hall, 1962.
13. Hoffman, K. and Kunze, R. Linear Algebra, Second Edition, Prentice-Hall of India, 1991.
14. Jain R.K., Iyengar SRK Advanced Engineering Mathematics, 2nd Edition Narosa 2003.
15. Johnson R.A., and Gupta C.B., Probability and Statistics for Engineers. Pearson Education Inc., 2005,
16. Kendall, M and Stuart A. The Advanced Theory of Statistics. Macmilian, New York, 1983
17. Kreyszig, E. Advanced Engineering Mathematics, Eighth Edition, John Wiley, 1999
18. Mendenhall, W., Beaver, R.J., Beaver, B.M. Introduction to Probability and Statistics. 12th ed. Thomson Learning Inc. 2007
19. O'Neil, P.V. Advanced Engineering Mathematics, Fifth Edition, Thomson Brooks, 2003
20. Pagano, M. and Gaureau, K. Principles of Biostatistics, 7th ed. Duxbury. Thomson Learning. 2007
21. Piaggio, H.T.H. An Elementary Treatise on Differential Equations and Their Applications, G. Bell and Sons, 1956.
22. Piskunov, N. Differential and Integral Calculus, Vol 1, Mir Publishers, 1974.

23. Piskunov, N. Differential and Integral Calculus, Vol 2, Mir Publishers, 1974.
24. Protter M.H. and P.E. Protter, Calculus with Analytic Geometry, 4th ed., Jones and Bartlett, Boston, 1988.
25. Ross S.L. Differential Equations, Third Edition, Wiley New York, 1984
26. Ross, S.M. Introduction to Probability and Statistics for Engineers and Scientists. 3rd ed. Academic Press 2005
27. Sneddon, IAN. Elements of Partial Differential Equations. McGraw-Hill, 1985
28. Spiegel M.R. Vector Analysis, Schaum Outline Series, McGraw-Hill, New York, 1974.
29. Staib, J.H. An Introduction to Matrices and Linear Transformations, Addison-Wesley, 1969.
30. Strang, G. Linear Algebra and its Application, 2nd Edition, Academic Press, New York, 1980.
31. Thomas Jr. G.B, and Finney, R.L. Calculus and Analytic Geometry, Ninth Edition, Pearson Education 2004.
32. Titchmarsh, E.C. The Theory of Functions. Oxford University Press, 1975
33. Walpole, R.E., Myers, R.H., Myers, S.L., Ye, K. Probability and Statistics for Engineers and Scientists. Prentice Hall, Inc., 2002
34. Watson G.N. A Treatise on the Theory of Bessel Functions, 2nd ed., Cambridge University Press, Cambridge, UK, 1966.
35. Zauderer, E. Partial Differential Equations of Applied Mathematics. 2nd ed. New York Wiley, 1989
36. Zelinsky, D. A First Course in Linear Algebra, Academic Press, 1968.
37. Zill D.G. A First Course in Differential Equations with Applications, 3rd ed., PWS, Boston, 1986.
38. Zill, D.G. and Cullen, M.R. Advanced Engineering Mathematics, Second Edition, CBS Publishers Indian Edition, 2000.

Index