$$= \lim_{\delta t \to 0} \frac{\bar{u} \times \delta\bar{v} + \delta\bar{u} \times \bar{v} + \delta\bar{u} \times \delta\bar{v}}{\delta t} = \lim_{\delta t \to 0}\left[\bar{u} \times \frac{\delta\bar{v}}{\delta t} + \frac{\delta\bar{u}}{\delta t} \times \bar{v} + \frac{\delta\bar{u}}{\delta t} \times \delta\bar{v}\right]$$

$$= \bar{u} \times \frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt} \times \bar{v}, \text{ since } \delta\bar{v} \to 0 \text{ as } \delta t \to 0.$$

We must note that since $\bar{u} \times \bar{v} \neq \bar{v} \times \bar{u}$, thus while evaluating $\frac{d}{dt}(\bar{u} \times \bar{v})$ the order of vectors $\bar{u}$ and $\bar{v}$ must be maintained. Similarly while evaluating $\frac{d}{dt}[\bar{u}\bar{v}\bar{w}]$, the cyclic order of vectors $\bar{u}$, $\bar{v}$, and $\bar{w}$ must be maintained.

Derivative of a vector function constant in magnitude, or direction only. A vector function changes if either its magnitude changes or its direction changes, or the direction and magnitude both change.

We find conditions under which a vector function will remain constant in magnitude, or in direction, or in both.

First, let $\bar{v}(t)$ be a vector with constant magnitude, say $|\bar{v}(t)| = c$. Then, we have

$$\bar{v} \cdot \bar{v} = |\bar{v}(t)|^2 = c^2 \quad ...(8.1)$$

Differentiating (8.1) w.r.t. t, we get $\frac{d\bar{v}}{dt} \cdot \bar{v} + v \cdot \frac{d\bar{v}}{dt} = 0.$

Since the dot product is commutative this gives

$$\bar{v} \cdot \frac{d\bar{v}}{dt} = 0. \quad ...(8.2)$$

Thus the derivative of a vector function $\bar{v}(t)$ of constant magnitude is either the zero vector or is perpendicular to $\bar{v}(t)$.

Next, let $\bar{v}(t)$ be a vector with constant direction and let $\bar{a}$ be a unit vector in that direction, then

$$\bar{v} = \phi\bar{a} \quad ...(8.3)$$

where $\phi = |\bar{v}|$.

From (8.3), we get $\frac{d\bar{v}}{dt} = \phi\frac{d\bar{a}}{dt} + \frac{d\phi}{dt}\bar{a}$, and thus

$$\bar{v} \times \frac{d\bar{v}}{dt} = \phi\bar{a} \times \left[\phi\frac{d\bar{a}}{dt} + \frac{d\phi}{dt}\bar{a}\right] = \phi^2\bar{a} \times \frac{d\bar{a}}{dt} \quad ...(8.4)$$

since $\bar{a} \times \bar{a} = 0$

Further since $\bar{a}$ is constant vector in magnitude as well as direction, thus $\frac{d\bar{a}}{dt} = 0$. Hence (8.4) becomes

$$\bar{v} \times \frac{d\bar{v}}{dt} = \bar{0} \qquad \text{...(8.5)}$$

Thus the derivative of a vector function $\bar{v}(t)$ of constant direction is either the zero vector or is parallel to $\bar{v}(t)$.

In fact, if $\bar{v}(t)$ is a constant vector, then $\bar{v}(t+\Delta t) = \bar{v}(t)$, for all t which gives

$$\lim_{\Delta t \to 0} \frac{\bar{v}(t+\Delta t) - \bar{v}(t)}{\Delta t} = \bar{0}. \text{ or } \frac{d\bar{v}}{dt} = \bar{0}. \qquad \text{...(8.6)}$$

In particular, we have

$$\frac{d\bar{i}}{dt} = \frac{d\bar{j}}{dt} = \frac{d\bar{k}}{dt} = \bar{0}, \qquad \text{...(8.7)}$$

where $\bar{i}$, $\bar{j}$ and $\bar{k}$ are the unit vectors along x-axis, y-axis and z-axis respectively.

8.1.3 Geometrical Interpretation of the Derivative of a Vector Function

Let $\bar{r}(t)$ be the position vector of a point P with respect to the origin of reference O. As t varies continuously P traces out a curve C as shown in Fig. 8.1. Thus a vector function $\bar{r}(t)$ represents a curve in space.

For example (i) the vector $\bar{r}(t) = a\cos t\bar{i} + b\sin t\hat{j}$ represents an ellipse in the xy-plane with center at the origin and principal axes in the direction of x and y axes, since from $\bar{r}(t)$, we have $x = a\cos t$, $y = b\sin t$, $z = 0$, which give

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0.$$

If $b = a$, then $\bar{r}(t)$ represents a circle or radius a.

(ii) The vector $\bar{r}(t) = at^2\hat{i} + 2at\hat{j}$ represents the parabola $y^2 = 4ax$, $z = 0$ in the xy-plane.

Let $\bar{r}$ and $\bar{r} + \delta\bar{r}$ be the position vectors of two neighbouring points P and Q on this curve C as shown in Fig. 8.2. Then,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (\bar{r} + \delta\bar{r}) - \bar{r} = \delta\bar{r}.$$

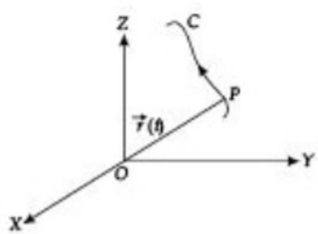


Fig. 8.1

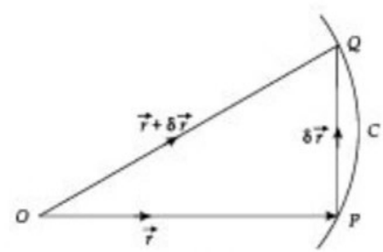


Fig. 8.2

Therefore $\frac{\delta \vec{r}}{\delta t}$ is directed along the chord PQ.

As $\delta t \to 0$, $Q \to P$, the chord PQ becomes the tangent to the curve at P. Thus $\lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} = \frac{\delta \vec{r}}{\delta t}$ is a vector along the tangent to the curve at P. If $\vec{r}'(t) \neq \vec{0}$, then $\vec{r}'(t)$ is called a *tangent vector to the curve* C at P because it has the direction of the tangent to C at point P. The corresponding unit vector is the *unit tangent vector* given by $\frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \hat{u}(t)$.

Both $\vec{r}'(t)$ and $\hat{u}(t)$ point in the direction of increasing t. Hence, their sense depends on the orientation of C.

Suppose that scalar parameter t is replaced by s, the arc length from any convenient point A on the curve up to the point P, and let $\widehat{AP} = s$, $\widehat{AQ} = s + \delta s$, so that, $\delta s = \widehat{PQ}$. In this case $\frac{d\vec{r}}{ds}$ will be a vector along the tangent at P and

$$\left|\frac{d\vec{r}}{ds}\right| = \lim_{\delta s \to 0} \left|\frac{d\vec{r}}{ds}\right| = \lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} = 1.$$

Thus $\frac{d\vec{r}}{ds}$ *is the unit vector along the tangent* at P.

8.2 VELOCITY AND ACCELERATION. TANGENTIAL AND NORMAL ACCELERATION

In this section we discuss the concepts of velocity and acceleration of a particle. The decomposition of acceleration along the tangent and normal is of great practical importance.

8.2.1 Velocity and Acceleration

If the scalar t denotes the time and $\vec{r}$ is the position vector of a moving particle P, then $\frac{d\vec{r}}{dt}$ represents the velocity vector $\vec{v}$ of the particle at P. Its direction is along the tangent at P. Further, $\frac{d^2\vec{r}}{dt}$ or $\frac{d\vec{v}}{dt}$ represents the *acceleration* $\vec{a}(t)$ of the particle at P. For example, the vector function

$$\vec{r}(t) = R\cos wt\,\hat{i} + R\sin wt\,\hat{j},\ w > 0 \qquad ...(8.8)$$

represents a circle of radius R with centre at the origin in the xy-plane. It describes the motion of a particle P in the counterclockwise sense. The velocity of the particle at P is given by

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -Rw\sin wt\,\hat{i} + Rw\cos wt\,\hat{j}.$$

Its magnitude

$$|\bar{v}(t)| = \sqrt{\frac{d\bar{r}}{dt}\cdot\frac{d\bar{r}}{dt}} = Rw \qquad ...(8.9)$$

is constant, and the direction is along the tangent to C.

The angular speed is, $\frac{Rw}{R} = w$, and the acceleration is

$$\bar{a}(t) = \frac{d\bar{v}}{dt} = -Rw^2 \cos wt\,\hat{i} - Rw^2 \sin wt\,\hat{j} = -w^2\,\bar{r} \qquad ...(8.10)$$

We observe that the acceleration is of constant magnitude, $|\bar{a}(t)| = w^2|\bar{r}| = w^2R$ and is directed towards the origin. The acceleration $\bar{a}(t)$ is called the *centripetal acceleration* and results from the fact that the velocity vector is changing its direction at a constant rate.

8.2.2 Tangential and Normal Accerlerations

Next we study the decomposition of acceleration into a component in the direction of motion, called the *tangential component*, and a component perpendicular to it, called the *normal component*.

The acceleration $\bar{a}$ is the time rate of change of the velocity $\bar{v}$. As discussed in case of motion described by Eq. (8.8), we have $|\bar{v}|$ = constant, but $|\bar{a}| \neq 0$. Thus, *magnitude of acceleration is not always the rate change of $|\bar{v}|$. The reason is that, in general, $\bar{a}$ is not along the tangent to the path C.* In fact

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds}\frac{ds}{dt} = \bar{u}(s)\frac{ds}{dt}, \qquad ...(8.11)$$

where $\bar{u}(s)$ is the unit tangent vector of C.

Further differentiating it again

$$\bar{a}(t) = \frac{d\bar{v}}{dt} = \frac{d}{dt}\left(\bar{u}(s)\frac{ds}{dt}\right) = \frac{d\bar{u}}{ds}\left(\frac{ds}{dt}\right)^2 + \bar{u}(s)\frac{d^2s}{dt^2}. \qquad ...(8.12)$$

Now $\bar{u}(s)$ is along tangent to C and of constant length one, so $\frac{d\bar{u}}{ds}$ is perpendicular to $\bar{u}(s)$. Hence, the acceleration $\bar{a}(t)$ is composed of

(i) the tangential component $\bar{u}(s)\frac{d^2s}{dt^2}$, called the *tangential acceleration;* and

(ii) the normal component $\left(\frac{d\bar{u}}{ds}\right)\left(\frac{ds}{dt}\right)^2$, called the *normal acceleration*.

Thus we observe that *if, and only if the normal acceleration is zero $|\bar{a}(t)|$ equals the time rate of change of $|\bar{v}(t)| = \frac{ds}{dt}$*, because only then from (8.12), we have

$$|\bar{a}(t)| = |\bar{u}(s)|\left|\frac{d^2s}{dt^2}\right| = \left|\frac{d^2s}{dt^2}\right|. \qquad ...(8.13)$$

8.2.3 Relative Velocity and Acceleration

Let two particles P_1 and P_2 moving along the curves C_1 and C_2 have position vectors $\vec{r}_1$ and $\vec{r}_2$ at time t, with reference to the origin O. From Fig. 8.3,

$$\vec{r} = \overrightarrow{P_1P_2} = \vec{r}_2 - \vec{r}_1.$$

Differentiating w.r.t. t, we get

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}_2}{dt} - \frac{d\vec{r}_1}{dt}. \qquad \text{...(8.14)}$$

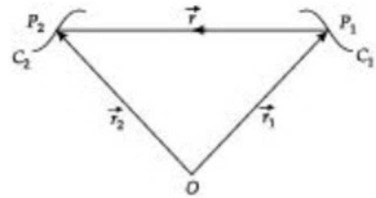


Fig. 8.3

This defines the relative velocity of P_2 w.r.t. P_1 and thus, *velocity of P_2 relative to P_1 = velocity of P_2 − velocity of P_1.*

Again differentiating (8.14) w.r.t. t, we get

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}_2}{dt^2} - \frac{d^2\vec{r}_1}{dt^2} \qquad \text{...(8.15)}$$

That is, *acceleration of P_2 relative to P_1 = acceleration of P_2 − acceleration of P_1.*

Example 8.1: Find $\frac{d\vec{w}}{dt}$ in each of the following cases:

(a) $\vec{w}(t) = (3t\hat{i} + 5t^2\hat{j} + 6\hat{k}) \cdot (t^2\hat{i} - 2t\hat{j} + t\hat{k})$ (b) $\vec{w}(t) = (t\hat{i} + e^t\hat{j} - t^2\hat{k}) \times (t^2\hat{i} + \hat{j} + t^3\hat{k})$

Solution: (a) Using $\frac{d}{dt}(\vec{u}\cdot\vec{v}) = \vec{u}\cdot\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\cdot\vec{v}$, we have

$$\frac{d\vec{w}}{dt} = (3t\hat{i} + 5t^2\hat{j} + 6\hat{k}) \cdot \frac{d}{dt}(t^2\hat{i} - 2t\hat{j} + t\hat{k}) + \frac{d}{dt}(3t\hat{i} + 5t^2\hat{j} + 6\hat{k}) \cdot (t^2\hat{i} - 2t\hat{j} + t\hat{k})$$

$$= (3t\hat{i} + 5t^2\hat{j} + 6\hat{k}) \cdot (2t\hat{i} - 2\hat{j} + \hat{k}) + (3\hat{i} + 10t\hat{j}) \cdot (t^2\hat{i} - 2t\hat{j} + t\hat{k})$$
$$= 6t^2 - 10t^2 + 6 + 3t^2 - 20t^2 = 6 - 21t^2.$$

(b) Using $\frac{d}{dt}(\vec{u}\times\vec{v}) = \vec{u}\times\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\times\vec{v}$, we have

$$\frac{d\vec{w}(t)}{dt} = (t\hat{i} + e^t\hat{j} - t^2\hat{k}) \times \frac{d}{dt}(t^2\hat{i} + \hat{j} + t^3\hat{k}) + \frac{d}{dt}(t\hat{i} + e^t\hat{j} - t^2\hat{k}) \times (t^2\hat{i} + \hat{j} + t^3\hat{k})$$

$$= (t\hat{i} + e^t\hat{j} - t^2\hat{k}) \times (2t\hat{i} + 3t^2\hat{k}) + (\hat{i} + e^t\hat{j} - 2t\hat{k}) \times (t^2\hat{i} + \hat{j} + t^3\hat{k})$$

$$= [3t^2e^t\hat{i} - 5t^3\hat{j} - 2te^t\hat{k}] + [(t^3e^t + 2t)\hat{i} - 3t^3\hat{j} + (1 - t^2e^t)\hat{k}]$$

$$= [t^2e^t(3 + t) + 2t]\hat{i} - 8t^3\hat{j} + [1 - te^t(2 + t)]\hat{k}.$$

Example 8.2: If $\bar{r}$ is a vector function of a scalar t and $\bar{a}$ is a constant vector, differentiate the following with respect to t

(a) $\dfrac{\bar{r}\times\bar{a}}{\bar{r}\cdot\bar{a}}$ (b) $\dfrac{\bar{r}+\bar{a}}{|\bar{r}|^2+|\bar{a}|^2}$.

Solution: (a) Let $\bar{R}=\dfrac{\bar{r}\times\bar{a}}{\bar{r}\cdot\bar{a}}$. Here $\bar{r}\cdot\bar{a}$ is a scalar function of t; also $\dfrac{d\bar{a}}{dt}=\bar{0}$, since $\bar{a}$ is a constant vector. Therefore,

$$\frac{d\bar{R}}{dt}=\frac{1}{\bar{r}\cdot\bar{a}}\frac{d}{dt}(\bar{r}\times\bar{a})+\left\{\frac{d}{dt}\left(\frac{1}{\bar{r}\cdot\bar{a}}\right)\right\}(\bar{r}\times\bar{a})=\frac{1}{\bar{r}\cdot\bar{a}}\left(\bar{r}\times\frac{d\bar{a}}{dt}+\frac{d\bar{r}}{dt}\times a\right)-\frac{\frac{d}{dt}(\bar{r}\cdot\bar{a})}{(\bar{r}\cdot\bar{a})^2}(\bar{r}\cdot\bar{a})$$

$$=\frac{\frac{d\bar{r}}{dt}\times\bar{a}}{\bar{r}\cdot\bar{a}}-\frac{\bar{r}\cdot\frac{d\bar{a}}{dt}+\frac{d\bar{r}}{dt}\cdot\bar{a}}{(\bar{r}\cdot\bar{a})^2}(\bar{r}\times\bar{a})=\frac{\frac{d\bar{r}}{dt}\times\bar{a}}{\bar{r}\cdot\bar{a}}-\frac{\frac{d\bar{r}}{dt}\cdot\bar{a}}{(\bar{r}\cdot\bar{a})^2}(\bar{r}\times\bar{a}).$$

(b) Let $\bar{R}=\dfrac{\bar{r}+\bar{a}}{|\bar{r}|^2+|\bar{a}|^2}$. Here $|\bar{r}|^2$ is a scalar function of t and $|\bar{a}|^2$ is a scalar independent of t. Therefore,

$$\frac{d\bar{R}}{dt}=\frac{1}{|\bar{r}|^2+|\bar{a}|^2}\frac{d}{dt}(\bar{r}+\bar{a})+\left\{\frac{d}{dt}\left(\frac{1}{|\bar{r}|^2+|\bar{a}|^2}\right)\right\}(\bar{r}+\bar{a})$$

$$=\frac{1}{|\bar{r}|^2+|\bar{a}|^2}\frac{d\bar{r}}{dt}-\left\{\frac{\frac{d}{dt}(|\bar{r}|^2+|\bar{a}|^2)}{(|\bar{r}|^2+|\bar{a}|^2)}\right\}(\bar{r}+\bar{a})=\frac{\frac{d\bar{r}}{dt}}{|\bar{r}|^2+|\bar{a}|^2}-\frac{2\bar{r}\cdot\frac{d\bar{r}}{dt}}{(|\bar{r}|^2+|\bar{a}|^2)^2}(\bar{r}+\bar{a}),$$

since, $\dfrac{d}{dt}|\bar{r}|^2=\dfrac{d}{dt}(\bar{r}\cdot\bar{r})=2\bar{r}\cdot\dfrac{d\bar{r}}{dt}$ and $\dfrac{d}{dt}|\bar{a}|^2=0$.

Example 8.3: A particle moves along a curve whose parametric equations are $x=e^{-t}$, $y=2\cos 3t$, $z=\sin 3t$. Find the velocity and acceleration at $t=0$.

Solution: Let $\bar{r}(t)$ be the position vector of the particle at any time t, then

$$\bar{r}(t)=x\hat{i}+y\hat{j}+z\hat{k}=e^{-t}\hat{i}+2\cos 3t\,\hat{j}+\sin 3t\,\hat{k}.$$

The velocity $\bar{u}(t)$ of the particle at time t is $\bar{u}(t)=\dfrac{d\bar{r}}{dt}=-e^{-t}\hat{i}-6\sin 3t\,\hat{j}+3\cos 3t\,\hat{k}$.

Thus the velocity at $t=0$ is $\left(\dfrac{d\bar{r}}{dt}\right)_{t=0}=-\hat{i}+3\hat{k}$.

Similarly the acceleration $\bar{a}(t)$ of the particle at time t is

$$\bar{a}(t) = \frac{d^2r}{dt^2} = e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 9\sin 3t\,\hat{k}.$$

Thus the acceleration at $t = 0$ is $\left(\frac{d^2r}{dt^2}\right)_{t=0} = \hat{i} - 18\hat{j}$.

Example 8.4: A particle moves along a curve whose parametric equations are $x = 3t^2$, $y = t^2 - 2t$, $z = t^3$, where the parameter t is time. Find its velocity and acceleration at $t = 2$.

Solution: Let $\bar{r}(t)$ be the position vector of the particle at time t, then

$$\bar{r}(t) = x\hat{i} + y\hat{j} + z\hat{k} = 3t^2\hat{i} + (t^2 - 2t)\hat{j} + t^3\hat{k}$$

The velocity $\bar{v}(t)$ of the particle at time t is $\bar{v}(t) = \frac{d\bar{r}}{dt} = 6t\hat{i} + (2t-2)\hat{j} + 3t^2\hat{k}$,

and the acceleration $\bar{a}(t)$ of the particle at time t is

$$\bar{a}(t) = \frac{d^2\bar{r}}{dt^2} = 6\hat{i} + 2\hat{j} + 6t\hat{k}$$

At $t = 2$, $\left(\frac{d\bar{r}}{dt}\right)_{t=2} = 12\hat{i} + 2\hat{j} + 12\hat{k}$, and $\left(\frac{d^2\bar{r}}{dt^2}\right)_{t=2} = 6\hat{i} + 2\hat{j} + 12\hat{k}$.

Example 8.5: Find the angle between the tangents to the curve $\bar{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$ at the points $t = \pm 1$.

Solution: A vector along the tangent at any point t to the given curve

$$\bar{r}(t) = t^2\hat{i} + 2t\hat{j} - t^3\hat{k} \text{ is } \frac{d\bar{r}}{dt} = 2t\hat{i} + 2\hat{j} - 3t^2\hat{k}$$

If $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$ are the vectors along the tangents at $t = 1$ and $t = -1$ respectively, then

$$\overrightarrow{T_1} = 2\hat{i} + 2\hat{j} - 3\hat{k}, \text{ and } \overrightarrow{T_2} = -2\hat{i} + 2\hat{j} - 3\hat{k}.$$

If θ is the angle between $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$, then

$$\cos\theta = \frac{\overrightarrow{T_1}\cdot\overrightarrow{T_2}}{|\overrightarrow{T_1}||\overrightarrow{T_2}|} = \frac{2(-2)+2(2)+(-3)(-3)}{\sqrt{4+4+9}\cdot\sqrt{4+4+9}} = \frac{9}{17}$$

Therefore, $\theta = \cos^{-1}(9/17)$.

Example 8.6: For the curve $x = \cos t + t \sin t$, $y = \sin t - t \cos t$, find the tangential and the normal components of the acceleration at any time t.

Solution: Let $\bar{r}(t)$ be the position vector of any point P on the curve at time t, then

$$\bar{r}(t) = (\cos t + t \sin t)\hat{i} + (\sin t - t \cos t)\hat{j}.$$

The velocity $\bar{v}(t)$ of particle P is

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = (-\sin t + \sin t + t \cos t)\hat{i} + (\cos t - \cos t + t \sin t)\hat{j} = (t \cos t)\hat{i} + (t \sin t)\hat{j}.$$

which is along the tangent to the curve at P.

The acceleration $\bar{a}(t)$ is

$$\bar{a}(t) = \frac{d^2\bar{r}}{dt^2} = (-t \sin t + \cos t)\hat{i} + (t \cos t + \sin t)\hat{j} \qquad ...(8.16)$$

Let $$\bar{a}(t) = a_T\hat{T} + a_N\hat{N} \qquad ...(8.17)$$

where a_T and a_N are the tangential and normal components of the acceleration, respectively, and $\hat{T}$ and $\hat{N}$ respectively are unit vectors along the tangent and normal at P. From (8.17),

$$\bar{a}(t)\cdot\hat{T} = a_T\hat{T}\cdot\hat{T} + a_N\hat{N}\cdot\hat{T} = a_T \qquad ...(8.18)$$

Also, $\hat{T} = \dfrac{\bar{T}}{|\bar{T}|} = \dfrac{(t\cos t)\hat{i} + (t \sin t)\hat{j}}{\sqrt{(t\cos t)^2 + (t \sin t)^2}} = \cos t\,\hat{i} + \sin t\,\hat{j}$.

Therefore, from (8.18), we obtain

$$a_T = \bar{a}\cdot\hat{T} = \{(-t\sin t + \cos t)\hat{i} + (t \cos t + \sin t)\hat{j}\}\cdot\{\cos t\,\hat{i} + \sin t\,\hat{j}\}$$
$$= \cos t\,(-t \sin t + \cos t) + \sin t\,(t \cos t + \sin t) = \cos^2 t + \sin^2 t = 1.$$

Next, from (8.17) $a_N\hat{N} = \bar{a}(t) - a_T\hat{T}$. Thus

$$a_N^2 = (a_N\hat{N})\cdot(a_N\hat{N}) = (\bar{a} - a_T\hat{T})\cdot(\bar{a} - a_T\hat{T}) = (\bar{a} - \hat{T})\cdot(\bar{a} - \hat{T}), \quad \text{since } a_T = 1$$
$$= \bar{a}\cdot\bar{a} - 2\bar{a}\cdot\hat{T} + \hat{T}\cdot\hat{T} = (-t\sin t + \cos t)^2 + (t \cos t + \sin t)^2 - 2a_T + 1$$
$$= t^2\sin^2 t + \cos^2 t - 2t \sin t \cos t + t^2 \cos^2 t + \sin^2 t + 2t \sin t \cos t - 2 + 1$$
$$= t^2 + 1 - 1 = t^2.$$

Hence, $a_N = t$.

Thus, the tangential and normal components of the acceleration are 1 and t, respectively.

Example 8.7: Obtain the tangential and normal components of the acceleration of a particle which is at a point $P(x, y)$ on the curve $x = e^t \cos t$, $y = e^t \sin t$ at any time t.

Solution: Let $\bar{r}(t)$ be the position vector of any point P on the curve at time t, then $\bar{r}(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j}$

The velocity $\bar{v}(t)$ of P at time t is $\bar{v}(t) = \dfrac{d\bar{r}(t)}{dt} = e^t(-\sin t + \cos t)\hat{i} + e^t(\cos t + \sin t)\hat{j}$

$$= e^t(\cos t - \sin t)\hat{i} + e^t(\sin t + \cos t)\hat{j}$$

which is along the tangent to the curve at P.

The acceleration $\bar{a}(t)$ is

$$\bar{a}(t) = \frac{d\bar{v}(t)}{dt} = [e^t(\cos t - \sin t) + e^t(-\sin t - \cos t)]\hat{i} + [e^t(\sin t + \cos t) + e^t(\cos t - \sin t)]\hat{j}$$

$$= -2e^t \sin t\,\hat{i} + 2e^t \cos t\,\hat{j}. \qquad ...(8.19)$$

Let $\quad \bar{a}(t) = a_T\hat{T} + a_N\hat{N} \qquad ...(8.20)$

where a_T and a_N are the tangential and normal components of the acceleration respectively and $\hat{T}$, $\hat{N}$ are the unit vectors along the tangent and the normal at point P. Next, from (8.20)

$$\bar{a}\cdot\hat{T} = a_T\hat{T}\cdot\hat{T} + a_N\hat{N}\cdot\hat{T} = a_T. \qquad ...(8.21)$$

Also, $\quad \hat{T} = \dfrac{\bar{T}}{|\bar{T}|} = \dfrac{e^t(\cos t - \sin t)\hat{i} + e^t(\sin t + \cos t)\hat{j}}{\sqrt{e^{2t}(\cos^2 t + \sin^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t)}}$

$$= \frac{1}{\sqrt{2}}[(\cos t - \sin t)\hat{i} + (\sin t + \cos t)\hat{j}].$$

Therefore, from (8.21)

$$a_T = \bar{a}\cdot\hat{T} = [-2e^t \sin t\,\hat{i} + 2e^t \cos t\,\hat{j}]\cdot\frac{1}{\sqrt{2}}[(\cos t - \sin t)\hat{i} + (\sin t + \cos t)\hat{j}]$$

$$= -\sqrt{2}\,e^t(\sin t\cos t - \sin^2 t) + \sqrt{2}\,e^t(\cos t \sin t + \cos^2 t) = \sqrt{2}\,e^t.$$

Also from (8.20), $a_N\hat{N} = \bar{a} - a_T\hat{T}$. It gives

$$a_N^2 = (a_N\hat{N})\cdot(a_N\hat{N}) = (\bar{a} - a_T\hat{T})\cdot(\bar{a} - a_T\hat{T}) = \bar{a}\cdot\bar{a} - 2a_T\bar{a}\cdot\hat{T} + a_T^2\,\hat{T}\cdot\hat{T}$$

$$= 4e^{2t}(\sin^2 t + \cos^2 t) - 2(\sqrt{2}\,e^t)(\sqrt{2}\,e^t) + 2e^{2t}.\ \frac{1}{2}[(\cos t - \sin t)^2 + (\sin t + \cos t)^2]$$

$$= 4e^{2t} - 4e^{2t} + 2e^{2t} = 2e^{2t}.$$

Hence, $a_N = \sqrt{2}\,e^t$.

Thus both the tangential and normal components of the acceleration are equal to $\sqrt{2}\,e^t$.

Example 8.8: A person going eastwards with a velocity of 4 km per hour, observes that the wind appears to blow directly from the north. He doubles his speed and the wind appears to come from north-east. Find the actual velocity of the wind.

Solution: Let the actual velocity of the wind be $\bar{v}_w = x\hat{i} + y\hat{j}$, where $\hat{i}$ and $\hat{j}$ represent velocities of 1 km per hour towards the east and the north, respectively, as shown in Fig. 8.4.

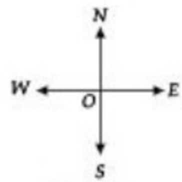


Fig. 8.4

If $\bar{v}_p$ be the velocity of the person, then $\bar{v}_p = 4\hat{i}$. Thus $\bar{v}_{wp}$ the velocity of the wind relative to that of the person is

$$\bar{v}_{wp} = \bar{v}_w - \bar{v}_p = (x\hat{i} + y\hat{j}) - 4\hat{i} = (x-4)\hat{i} + y\hat{j}$$

But it is given to be parallel to $-\hat{j}$ since it appears to blow from the north. Hence, $x = 4$.

When the velocity of the person becomes $8\hat{i}$, the velocity of the wind relative to the person is

$$\bar{v}_{wp} = (x\hat{i} + y\hat{j}) - 8\hat{i} = (x-8)\hat{i} + y\hat{j}.$$

But this is parallel to $-(\hat{i} + \hat{j})$, since it appears to come from north-east.

Therefore, $\dfrac{x-8}{y} = 1$, which by using $x = 4$ gives $y = -4$. Hence, the actual velocity of the wind is $\bar{v}_w = 4\hat{i} - 4\hat{j}$, that is, $4\sqrt{2}$ km per hour towards the south-east.

EXERCISE 8.1

In the following problems, find the indicated derivative using the differentiation rules assuming that all the given vector functions are differentiable.

1. $\bar{u}(t) = 5t^2\hat{i} + t\hat{j} + t^3\hat{k}$, $f(t) = \sin t$, find $[f(t)\bar{u}(t)]'$
2. $\bar{u}(t) = [\sin(2t)\hat{i} - \cos(2t)\hat{j} + t\hat{k}]$, $v(t) = [\cos(2t)\hat{i} - \sin(2t)\hat{j} + t^2\hat{k}]$, find $[u(t)\cdot v(t)]$
3. $\bar{u}(t) = (\cos wt)\hat{i} + (\sin wt)\hat{j}$, find $\bar{u}(t) \times \bar{u}'(t)$
4. $\bar{u}(t) = (1-t)\hat{i} + t^2\hat{j} + e^t\hat{k}$, $\bar{v}(t) = (1+t)\hat{i} + e^t\hat{j} + t\hat{k}$, find $[\bar{u}(t) \times \bar{v}(t)]$

If a and t are scalars, then find

5. $[t^2\bar{u}(t^2)]'$
6. $[\bar{u}(at) + \bar{v}(a/t)]'$
7. $[\bar{u}(t) \times \bar{u}''(t)]'$
8. $[\bar{u}(t)\cdot\bar{u}'(t) \times \bar{u}''(t)]'$

9. Verify the formula $\frac{d}{dt}(\bar{u}(t)\cdot\bar{v}(t)) = \bar{u}(t)\cdot\frac{d\bar{v}(t)}{dt} + \frac{d\bar{u}(t)}{dt}\cdot\bar{v}(t)$,

   for $\bar{u}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, and $\bar{v}(t) = \sin t\hat{i} - \cos t\hat{j}$.
10. Find the parametric equation of the tangent line to the curve $x = \sin t$, $y = \cos t$, $z = t$ at $t = \pi/4$.
11. Find the unit tangent vector at any point on the curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$.
12. Find the angle between the tangents to the curve $x = t, y = t^2, z = t^3$ at $t = \pm 1$.
13. A particle moves along the curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at $t = 0$.
14. A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of velocity and acceleration at time $t = 1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.
15. The position vector of a moving particle at a time t is $\bar{r} = t^2\hat{i} - t^3\hat{j} + t^4\hat{k}$. Find the tangential and normal components of its acceleration at $t = 1$.
16. The velocity of a boat relative to water is represented by $3\hat{i} + 4\hat{j}$ and that of water relative to earth is $\hat{i} - 3\hat{j}$. What is the velocity of the boat relative to the earth if $\hat{i}$ and $\hat{j}$ represent one km an hour east and north respectively?
17. A person travelling towards the north-east with a velocity of 6 km per hour finds that the wind appears to blow from the north, but when he doubles his speed it seems to come from a direction inclined at an angle $\tan^{-1} 2$ to the north of east. Show that the actual velocity of the wind is $3\sqrt{2}$ km. per hour towards the east.

8.3 SCALAR AND VECTOR FIELDS. GRADIENT OF A SCALAR FIELD. DIRECTIONAL DERIVATIVES.

In this section we discuss two kinds of functions: scalar and vector functions and their fields. Infact some of the vector fields can be obtained from the scalar fields by applying 'gradient'. This concept is of great practical advantage since comparatively it is easy to deal with a scalar fields, and because of this, gradient finds applications in engineering and physical sciences.

8.3.1 Scalar and Vector Point Functions

A variable quantity which depends for its value on its position only, that is, upon the co-ordinate of the points of a region, say (x, y, z) in space, is called a point function. There are two types of point functions, as explained below.

Scalar point function. *Let E be a region at each point P(x, y, z) of which a scalar $\phi = \phi(x, y, z)$ is specified, then we say that ϕ is a scalar point function, and the region E defined so, is called a scalar field.* The scalar point function does not depend upon the choice of co-ordinate system. It only depends on the point in the field. For example, the temperature distribution in a medium, the distribution of atmospheric pressure in space, density of a body are all examples of scalar point functions.

Vector point function. *Let E be a region at each point $P(x, y, z)$ of which a vector $\bar{v} = \bar{v}(x, y, z)$ is specified, then we say that $\bar{v}$ is a vector point function and the region E defined so is called a vector field.* For example, the velocity of a moving fluid at any instant, the gravitation force, or electrical intensity are all examples of vector point functions.

Level surfaces. *Let $\phi(x, y, z)$ be single valued continuous scalar point function defined at every point $P(x, y, z)$ of E. Then the surface $\phi(x, y, z) = c$, a constant, defines the equation of a surface, and is called the level surface of the function.* For example, if $\phi(x, y, z)$ represents temperature in a medium, then $\phi(x, y, z) = c$ represents a surface on which the temperature is a constant c. Such surfaces are called *isothermal surfaces*. Another example is of *equipotential surfaces*. Note that for different values of c, we get different level surfaces, no two of which intersect.

For example, the level surfaces of the scalar fields in space defined by the function $f(x, y, z) = z - \sqrt{x^2 + y^2}$ are given by $z - \sqrt{x^2 + y^2} = c$, or, $(x^2 + y^2) = (z - c)^2$ which are cones.

8.3.2 Gradient of a Scalar Field

Let $\phi(x, y, z)$ be a scalar point function defining a scalar field. To define the gradient of a scalar field, we first introduce a differential vector operator ∇, called 'del' or nabla. The differential vector operator ∇ in two and three dimensions is defined as

$$\nabla = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} \text{ and } \nabla = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$$

respectively.

The 'gradient' of a scalar field $\phi(x, y, z)$ denoted by $\nabla\phi$, or grad ϕ, is defined as

$$\nabla\phi = \frac{\partial \phi}{\partial x}\,\hat{i} + \frac{\partial \phi}{\partial y}\,\hat{j} + \frac{\partial \phi}{\partial z}\,\hat{k}$$

We observe that the operator ∇ operates on a scalar field and produces a vector field. Thus, *gradient of a scalar function is always a vector.*

8.3.3 Geometrical Interpretation of the Gradient

Let $\phi(P) = \phi(x, y, z)$ be a differentiable scalar point function. Consider the level surface through P at which the function has value ϕ and another level surface through a neighbouring point Q where the value is $\phi + \delta\phi$ as shown in Fig. 8.5. Let $\bar{r}$ and $\bar{r} + \delta\bar{r}$ be the position vectors of the points P and Q respectively with reference to the origin O, and thus, the vector here

$$\overrightarrow{PQ} = \delta\bar{r} = \hat{i}\,\delta x + \hat{j}\,\delta y + \hat{k}\,\delta z.$$

Consider, $\phi + \delta\phi = \phi(x + \delta x, y + \delta y, z + \delta z) = \phi(x, y, z) + \dfrac{\partial \phi}{\partial x}\delta x + \dfrac{\partial \phi}{\partial y}\delta y + \dfrac{\partial \phi}{\partial z}\delta z$

+ terms of second and higher orders, (by Taylor series).

Neglecting terms of second and higher order, we obtain

$$\delta\phi = \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z = \left(\hat{i}\,\frac{\partial \phi}{\partial x} + \hat{j}\,\frac{\partial \phi}{\partial y} + \hat{k}\,\frac{\partial \phi}{\partial z}\right)\cdot(\hat{i}\,\delta x + \hat{j}\,\delta y + \hat{k}\,\delta z) = \nabla\phi \cdot \delta\bar{r}$$

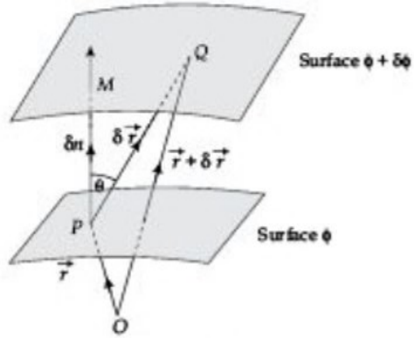


Fig. 8.5

Next, if Q approaches P, $\delta\phi \to 0$, then $\nabla\phi \cdot \delta\bar{r} = 0$. This means that $\nabla\phi$ is perpendicular to every $\delta\bar{r}$ (whose direction is along the tangent to the level surface) lying on this surface. Thus *$\nabla\phi$ is normal to the level surface $\phi(x, y, z) = c$ at the point P*. Therefore, $\nabla\phi = |\nabla\phi|\,\hat{N}$, where $\hat{N}$ is a unit normal vector to this surface. If the perpendicular distance PM between the surfaces through P and Q be δn, then the rate of change of ϕ along the normal PM to the surface through P is given by

$$\frac{\partial\phi}{\partial n} = \lim_{\delta n \to 0}\frac{\delta\phi}{\delta n} = \lim_{\delta n \to 0}\left(\Delta\phi \cdot \frac{\delta\bar{r}}{\delta n}\right) = |\nabla\phi| \lim_{\delta n \to 0}\frac{\hat{N}\cdot\delta\bar{r}}{\delta n} = |\nabla\phi|, \text{ since } \hat{N}\cdot\delta\bar{r} = |\delta\bar{r}|\cos\theta = \delta n.$$

Thus the magnitude of $\nabla\phi$ is $\dfrac{\partial\phi}{\partial n}$.

Hence the gradient of a scalar field ϕ is a vector normal to the level surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along the normal.

8.3.4 Directional Derivative

Let $\phi(x, y, z)$ be a differentiable scalar field, then $\dfrac{\partial\phi}{\partial x}$, $\dfrac{\partial\phi}{\partial y}$ and $\dfrac{\partial\phi}{\partial z}$ are the rates of change of ϕ in the directions of x, y and z axis respectively. Let $P_0(x_0, y_0, z_0)$ be any fixed point and $\hat{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be any unit vector. Then the position vector of any point Q on the line passing through P_0 in the direction of $\hat{b}$, as shown in Fig. 8.6, is given by

$$\bar{r} = (x_0 + b_1 t)\,\hat{i} + (y_0 + b_2 t)\,\hat{j} + (z_0 + b_3 t)\,\hat{k}, \qquad \text{...(8.22)}$$

where t is the parameter.

Further, since $|\hat{b}| = 1$, the distance from P_0 to Q, that is $|P_0Q|$ is equal to t.

The directional derivative of ϕ at P_0 in the direction of $\hat{b}$, that is along P_0Q, is defined as

$$\frac{\partial \phi}{\partial t} = \lim_{t \to 0} \frac{\phi(Q) - \phi(P_0)}{t}.$$

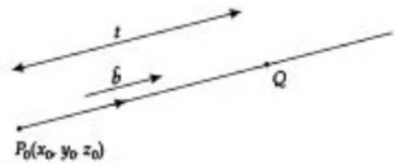


Fig. 8.6

By chain rule, $$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}$$

$$= \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right)\cdot\left(i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt}\right)$$

$$= \nabla\phi \cdot \frac{\partial \bar{r}}{\partial t}.$$

From (8.22), $\frac{\partial \bar{r}}{\partial t} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \hat{b}$. Therefore, *the directional derivative of ϕ in the direction of $\hat{b}$, denoted by $D_b(\phi)$, is*

$$D_b(\phi) = \nabla\phi \cdot \hat{b} = grad\ \phi \cdot \hat{b} \qquad ...(8.23)$$

In general, *the directional derivative of ϕ in the direction of a vector $\bar{u}$, denoted by $D_u(\phi)$, is given by*

$$D_u(\phi) = grad\ \phi \cdot \frac{\bar{u}}{|\bar{u}|}.$$

Thus the directional derivative represents the rate of change of ϕ with respect to distance at any point $P(x, y, z)$ in the direction of unit vector $\hat{b}$ and is a scalar quantity.

We observe that the directional derivative of ϕ along the positive axes are given by $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$. For example, along x-axis it is

$$\text{grad}\ \phi \cdot i = \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right)\cdot i = \frac{\partial \phi}{\partial x},$$

and similarly for others axes.

Maximum rate change of a scalar function. We have,

$$D_b(\phi) = \nabla\phi \cdot \hat{b} = |\nabla\phi|\ |\hat{b}|\ \cos\theta = |\nabla\phi|\ \cos\theta,$$

where θ is the angle between the vectors $\nabla\phi$ and $\hat{b}$. Since $-1 \le \cos\theta \le 1$, so maximum value of $D_b(\phi)$ is $\nabla\phi$ at $\theta = 0$, when $\hat{b}$ is along $\nabla\phi$, that is $\hat{b} = \hat{N}$. This direction is the direction of the normal. So *the rate of change of ϕ at a point is maximum along the normal to the surface at that point.*

When $\theta = \pi$, then $D_b(\phi) = -|\nabla\phi|$, gives the minimum value of the rate of change of ϕ. It is along the direction opposite to that of $\nabla\phi$, that is along $-\hat{N}$.

In fact we may comment that the *vector $\nabla\phi$ points in the direction in which ϕ increases most rapidly and $-\nabla\phi$ points in the direction in which ϕ decreases most rapidly.*

8.3.5 Properties of Gradients

Let ϕ and ψ be any two scalar point functions. Then,

1. $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$
2. $\nabla(c_1\phi + c_2\psi) = c_1\nabla\phi + c_2\nabla\psi$, where c_1, c_2 are two arbitrary constants.
3. $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$
4. $\nabla\left(\dfrac{\phi}{\psi}\right) = \dfrac{\psi\Delta\phi - \phi\Delta\psi}{\psi^2}$, $\psi \neq 0$.

The above properties can be proved very easily, for example to prove 3, we have, by definition

$$\nabla(\phi\psi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi\psi) = \hat{i}\,\frac{\partial(\phi\psi)}{\partial x} + \hat{j}\,\frac{\partial(\phi\psi)}{\partial y} + \hat{k}\,\frac{\partial(\phi\psi)}{\partial z}$$

$$= \hat{i}\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x}\right] + \hat{j}\left[\phi\frac{\partial\psi}{\partial y} + \psi\frac{\partial\phi}{\partial y}\right] + \hat{k}\left[\phi\frac{\partial\psi}{\partial z} + \psi\frac{\partial\phi}{\partial z}\right]$$

$$= \phi\left[\hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}\right] + \psi\left[\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right] = \phi\nabla\psi + \psi\nabla\phi$$

Conservative vector field. *A vector field $\vec{F}$ is said to be conservative, if the vector function $\vec{F}$ can be expressed as the gradient of some scalar function ϕ, that is,* $\vec{F} = \nabla\phi$. In such a field the work done in moving a particle from a point A to a point B depends only on the position of points A and B and is independent of the path along which the particle is displaced from A to B. It is useful to mention here that every vector field is not conservative. We shall discuss this concept further in Section 9.2.2.

Example 8.9: Find grad ϕ at the point (1, 2, 1) when $\phi = \ln(x^2 + y^2 + z^2)$.

Solution: $\nabla\phi = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)\ln(x^2 + y^2 + z^2)$

$$= \hat{i}\frac{\partial}{\partial x}\ln(x^2 + y^2 + z^2) + \hat{j}\frac{\partial}{\partial y}\ln(x^2 + y^2 + z^2) + \hat{k}\frac{\partial}{\partial z}\ln(x^2 + y^2 + z^2)$$

$$= \frac{2(x\hat{i} + y\hat{i} + z\hat{k})}{x^2 + y^2 + z^2} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k} \text{ at } (1, 2, 1).$$

Example 8.10: Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, –1).

Solution: Since the gradient of ϕ is normal to the surface ϕ = constant, therefore, the unit vector normal to the surface at a point $P(x, y, z)$ is $\nabla\phi/|\nabla\phi|$. We have,

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + 3xyz)$$

$$= (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + 3xy\hat{k} = 3(x^2 + yz)\hat{i} + 3(y^2 + xz)\hat{j} + 3xy\hat{k}.$$

Thus $\nabla\phi$ at $(1, 2, -1)$ is $-3\hat{i} + 9\hat{j} + 6\hat{k}$; also $|\nabla\phi|$ at $(1, 2, -1)$ is $3\sqrt{14}$.

Thus the unit vector normal to the given surface at $(1, 2, -1)$ is

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}} = \frac{1}{\sqrt{14}}(-\hat{i} + 3\hat{j} + 2\hat{k}).$$

Example 8.11: Find the directional derivative of $\phi(x, y, z) = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Solution: We have, $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial z} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = y^2\hat{i} + (2xy + z^3)\hat{j} + 3yz^2\hat{k}$

Thus, $\nabla\phi$ at $(2, -1, 1) = \hat{i} - 3\hat{j} - 3\hat{k}$.

The directional derivative of ϕ is the component of $\nabla\phi$ at the given point in the direction of the given vector $\hat{i} + 2\hat{j} + 2\hat{k}$. Thus it is equal to

$$(\hat{i} - 3\hat{j} - 3\hat{k})\cdot\frac{i + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \frac{1-6-6}{3} = -\frac{11}{3}.$$

Example 8.12: Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$, and $z = x^2 + y^2 - 3$ at $(2, -1, 2)$.

Solution: Let $\phi(x, y, z) = x^2 + y^2 + z^2$ and $\psi(x, y, z) = x^2 + y^2 - z$

Therefore, $\nabla\phi = 2(x\hat{i} + y\hat{j} + z\hat{k})$, and $\nabla\psi = 2x\hat{i} + 2y\hat{j} - \hat{k}$. Thus

$$\nabla\phi \text{ at } (2, -1, 2) = 4\hat{i} - 2\hat{j} + 4\hat{k}, \text{ and } \nabla\psi \text{ at } (2, -1, 2) = 4\hat{i} - 2\hat{j} - \hat{k}.$$

Also, $|\nabla\phi|$ at $(2, -1, 2) = \sqrt{16+4+16} = 6$, and $|\nabla\psi|$ at $(2, -1, 2) = \sqrt{16+4+1} = \sqrt{21}$

Let $\hat{N}$ and $\hat{N}'$ be the unit vectors normal to the surfaces $\phi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$, respectively at $(2, -1, 2)$. Then,

$$\hat{N} = \frac{4\hat{i} - 2\hat{j} + 4\hat{k}}{6} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}, \text{ and } \hat{N}' = \frac{4\hat{i} - 2\hat{j} - \hat{k}}{\sqrt{21}} = \frac{4}{\sqrt{21}}\hat{i} - \frac{2}{\sqrt{21}}\hat{j} - \frac{1}{\sqrt{21}}\hat{k}.$$

If θ is the angle between the two surfaces $\phi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$ at $(2, -1, 2)$, then

$$\cos\theta = \hat{N}\cdot\hat{N}' = \left(\frac{2}{3}\right)\left(\frac{4}{\sqrt{21}}\right) + \left(\frac{-1}{3}\right)\left(\frac{-2}{\sqrt{21}}\right) + \left(\frac{2}{3}\right)\left(\frac{-1}{\sqrt{21}}\right) = \frac{8+2-2}{3\sqrt{21}} = \frac{8}{3\sqrt{21}}.$$

Therefore, $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$.

Example 8.13: If $r^2 = x^2 + y^2 + z^2$, then show that $\nabla\phi(r) = \frac{\phi'(r)}{r}\bar{r}$ and hence, or otherwise, prove that

(a) $\nabla(r) = \hat{r}$ (b) $\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\bar{r}$

(c) $\nabla(r^n) = n\,r^{n-2}\,\bar{r}$ (d) $\nabla\left(\int r^n\,dr\right) = r^{n-1}\bar{r}$. Here $r = |\bar{r}|$.

Solution: By definition

$$\nabla\phi(r) = \hat{i}\,\frac{\partial}{\partial x}\phi(r) + \hat{j}\,\frac{\partial}{\partial y}\phi(r) + \hat{k}\,\frac{\partial}{\partial z}\phi(r)$$

$$= \hat{i}\,\phi'(r)\frac{\partial r}{\partial x} + \hat{j}\,\phi'(r)\frac{\partial r}{\partial y} + \hat{k}\,\phi'(r)\frac{\partial r}{\partial z} = \phi'(r)\left(\hat{i}\,\frac{\partial r}{\partial x} + \hat{j}\,\frac{\partial r}{\partial y} + \hat{k}\,\frac{\partial r}{\partial z}\right)$$

From $r^2 = x^2 + y^2 + z^2$, we have, $2r\frac{\partial r}{\partial x} = 2x$, or $\frac{\partial r}{\partial x} = \frac{x}{r}$, etc.

Hence, $\nabla\phi(r) = \phi'(r)\left[\hat{i}\,\frac{x}{r} + \hat{j}\,\frac{y}{r} + \hat{k}\,\frac{z}{r}\right] = \frac{1}{r}\,\phi'(r)\bar{r}$.

(a) $\nabla(r) = \hat{i}\,\frac{\partial r}{\partial x} + \hat{j}\,\frac{\partial r}{\partial y} + \hat{k}\,\frac{\partial r}{\partial z} = \hat{i}\,\frac{x}{r} + \hat{j}\,\frac{y}{r} + \hat{k}\,\frac{z}{r} = \frac{\bar{r}}{r} = \hat{r}$.

(b) $\nabla\left(\frac{1}{r}\right) = \hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\,\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\,\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \hat{i}\,\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} + \hat{j}\,\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y} + \hat{k}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z}$

$$= -\frac{1}{r^2}\left[\hat{i}\,\frac{x}{r} + \hat{j}\,\frac{y}{r} + \hat{k}\,\frac{z}{r}\right] = -\frac{1}{r^3}\bar{r}.$$

(c) Take $\phi(r) = r^n$, then $\nabla\phi(r) = \frac{1}{r}\,\phi'(r)\,\bar{r}$, gives $\nabla(r^n) = \frac{1}{r}\,nr^{n-1}\bar{r} = nr^{n-2}\bar{r}$.

(d) Take $\phi(r) = \int r^n\,dr$, so that $\phi'(r) = r^n$, then

$$\nabla\phi(r) = \frac{1}{r}\,\phi'(r)\bar{r}, \text{ gives } \nabla\left(\int r^n dr\right) = \frac{1}{r}\,r^n\bar{r} = r^{n-1}\bar{r}.$$

Example 8.14: If $u = x + y + z, v = x^2 + y^2 + z^2, w = xy + yz + zx$, show that ∇u, ∇v, ∇w are coplanar vectors.

Solution: By definition

$$\nabla u = \left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right) = \hat{i} + \hat{j} + \hat{k}$$

$$\nabla v = \left(\hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right) = 2(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla w = \left(\hat{i}\frac{\partial w}{\partial x} + \hat{j}\frac{\partial w}{\partial y} + \hat{k}\frac{\partial w}{\partial z}\right) = \hat{i}\,(y+z) + \hat{j}\,(z+x) + \hat{k}\,(x+y)$$

The vectors ∇u, ∇v, ∇w, will be coplanar if their scalar triple product is zero, that is, if $\nabla u \cdot (\nabla v \times \nabla w) = 0$, or if

$$\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0 \quad \text{or, if} \quad \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x+y+z & x+y+z & x+y+z \end{vmatrix} = 0, \; [R_3 \to R_3 + \tfrac{1}{2}R_2]$$

$$\text{or, if } (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

since the first and third rows in the determinant are the same. Thus, ∇u, ∇v, ∇w are coplanar.

Example 8.15: The temperature of a point in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it fly?

Solution: If $T = c$ is the level surface, then ∇T gives the direction of maximum rate of change we have $\nabla T = 2x\hat{i} + 2y\hat{j} - \hat{k}$, and ∇T at $(1, 1, 2) = 2\hat{i} + 2\hat{j} - \hat{k}$

It should move in the direction of the unit vector normal along ∇T, that is, along $(1/3)(2\hat{i} + 2\hat{j} - \hat{k})$.

Example 8.16: If f and $\vec{g}$ are respectively the scalar and vector point functions, prove that the components of the latter, normal and tangential to the surface $f = 0$, are

$$\frac{(\vec{g}\cdot\nabla f)\,\nabla f}{|\nabla f|^2} \quad \text{and} \quad \frac{\nabla f \times (\vec{g} \times \nabla f)}{|\nabla f|^2}.$$

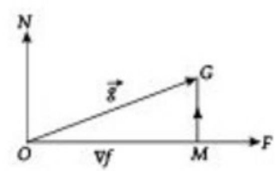


Fig. 8.7

Solution: We know that ∇f is normal to the level surface $f = 0$. Thus we are to find the components of $\vec{g}$ along and normal to ∇f.

Let O be the point of reference and F and G be two points such that $\nabla f = \overrightarrow{OF}$ and $\vec{g} = \overrightarrow{OG}$ as shown in Fig. 8.7, and let OM be the projection of $\vec{g}$ along ∇f.

The component of $\bar{g}$ along ∇f is $= OM$ times the unit vector along $\nabla f = (\bar{g}\cdot\hat{\nabla}f)\,\hat{\nabla}f = \dfrac{(\bar{g}\cdot\nabla f)\,\nabla f}{|\nabla f|^2}$.

The component of $\bar{g}$ normal to ∇f is $= \overrightarrow{MG} = \overrightarrow{OG} - \overrightarrow{OM}$

$$= \bar{g} - \frac{(\bar{g}\cdot\nabla f)\,\nabla f}{|\nabla f|^2} = \frac{(\nabla f\cdot\nabla f)\bar{g} - (\bar{g}\cdot\nabla f)\,\nabla f}{|\nabla f|^2} \quad = \frac{(\nabla f\times\bar{g})\times\nabla f}{|\nabla f|^2}.$$

Example 8.17: Determine the constant a such that at any point of intersection of the two spheres $(x-a)^2+y^2+z^2=3$, and $x^2+(y-1)^2+z^2=1$ their tangent planes are perpendicular to each other.

Solution: Let $\phi_1=(x-a)^2+y^2+z^2$ and $\phi_2=x^2+(y-1)^2+z^2$. We have

$$\nabla\phi_1 = \hat{i}\,\frac{\partial\phi_1}{\partial x} + \hat{j}\,\frac{\partial\phi_1}{\partial y} + \hat{k}\,\frac{\partial\phi_1}{\partial z} = 2(x-a)\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\nabla\phi_2 = \hat{i}\,\frac{\partial\phi_2}{\partial x} + \hat{j}\,\frac{\partial\phi_2}{\partial y} + \hat{k}\,\frac{\partial\phi_2}{\partial z} = 2x\,\hat{i} + 2(y-1)\,\hat{j} + 2z\,\hat{k}$$

The vectors $\nabla\phi_1$ and $\nabla\phi_2$ are along the normals to the two spheres at a point (x, y, z) of their intersection. The tangent planes to the two spheres at a point of intersection will be perpendicular when their normals are perpendicular to each other and for that

$$[2(x-a)\,\hat{i} + 2y\,\hat{j} + 2z\hat{k}]\cdot[2x\,\hat{i} + 2(y-1) + 2z\,\hat{k}] = 0$$

or, $4x(x-a)+4y(y-1)+4z^2=0$

or, $$x^2+y^2+z^2-ax-y=0. \qquad \text{...(8.24)}$$

Also at any point $P(x, y, z)$ of intersection of the given spheres, we have

$$x^2+y^2+z^2-2ax+a^2-3=0 \text{ and } x^2+y^2+z^2-2y=0.$$

Adding these two and dividing by 2, we obtain

$$x^2+y^2+z^2-ax-y+\frac{1}{2}a^2-\frac{3}{2}=0. \qquad \text{...(8.25)}$$

From (8.24) and (8.25), we obtain $\dfrac{a^2}{2}-\dfrac{3}{2}=0$, or $a=\sqrt{3}$.

Example 8.18: Show that the vector field defined by $\bar{F} = xyz(yz\,\hat{i} + xz\hat{j} + xy\,\hat{k})$ is conservative.

Solution: If the vector field defined by the given vector function $\bar{F}$ is conservative, then there exists some scalar function f such that $\bar{F} = \nabla f$. It gives

$$xyz(yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}) = \hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\,\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z}$$

This implies $\dfrac{\partial f}{\partial x} = xy^2z^2$, $\dfrac{\partial f}{\partial y} = yx^2z^2$, $\dfrac{\partial f}{\partial z} = zx^2y^2$. Integrating $\dfrac{\partial f}{\partial x} = xy^2z^2$ w.r.t. x, we get

$$f(x, y, z) = \frac{1}{2}x^2y^2z^2 + g(y, z) \qquad ...(8.26)$$

Substituting for f in $\frac{\partial f}{\partial y} = yx^2z^2$, we get $x^2yz^2 + \frac{\partial g}{\partial y} = x^2yz^2$, or $\frac{\partial g}{\partial y} = 0$, that is, $g = g(z)$.

Therefore, from (8.26), we have

$$f(x, y, z) = \frac{1}{2}x^2y^2z^2 + g(z) \qquad ...(8.27)$$

Substituting for f in $\frac{\partial f}{\partial z} = zx^2y^2$, we get $x^2y^2z + \frac{\partial g}{\partial z} = zx^2y^2$, or $\frac{\partial g}{\partial z} = 0$, that is, $g = c$, a constant.

Hence from (8.27), $f(x, y, z) = \frac{1}{2}x^2y^2z^2 + c$

Thus there exists a scalar function $f(x, y, z)$ such that $\vec{F} = \nabla f$.

EXERCISE 8.2

1. Find the gradient of the following scalar fields.
   (a) $\phi(x, y) = y^2 - 4xy$ at (1, 2) (b) $\phi(x, y, z) = x^2y^2 + xy^2 - z^2$ at (3, 1, 1)
2. Find a unit vector normal to the surface
   (a) $xy^3z = 2$ at (–1, –1, 2) (b) $x^2y + 2xz = 4$ at (2, – 2, 3).
3. Show that the equation of the tangent plane to the surface $z = \sqrt{x^2 + y^2}$ at the point (3, 4, 5) is $3x + 4y - 5z = 0$.
4. Show that the angle of intersection of the spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z = 47$ at (4, –3, 2) is $\theta = \cos^{-1}(19/29)$.
5. Show that the angle between the tangent planes to the surface $x \ln z = y^2 - 1, x^2y = 2 - z$ at the point (1, 1, 1) is $\cos^{-1}(-1/\sqrt{30})$.
6. Find the directional derivative of the function
   (a) $f(x, y, z) = xy^2 + yz^3$ at the point (2, – 1, 1) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.
   (b) $f(x, y, z) = 4xz^3 - 3x^2yz^2$ at the point (2, – 1, 2) along z–axis.
   (c) $f(x, y, z) = xy^2 + yz^3$ at the point (2, – 1, 1) in the direction of the normal to surface $x \ln z - y^2 + 4 = 0$ at (– 1, 2, 1).
   (d) $f(x, y) = x^2y^3 + xy$ at (2, 1) in the direction of a unit vector which makes an angle of $\pi/3$ with x-axis.
7. In what direction from (3, 1, – 2) is the directional derivative of $\phi = x^2y^2z^4$ maximum and what is its magnitude?

8. Find the constants λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point $(1, -1, 2)$.
9. If $u = u(x, y, z, t)$, $x = x(t)$, $y = y(t)$, $z = z(t)$, show that
$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \left(\frac{d\bar{r}}{dt}\cdot\nabla\right)u, \text{ where } \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$
10. If $\bar{r}$ is the position vector of the point (x, y, z) and $\bar{a}$ and $\bar{b}$ are constant vectors, prove that
$$\bar{a}\cdot\nabla\left\{\bar{b}\cdot\nabla\left(\frac{1}{r}\right)\right\} = \frac{3(\bar{a}\cdot\bar{r})(\bar{b}\cdot\bar{r})}{r^5} - \frac{\bar{a}\cdot\bar{b}}{r^3}.$$

8.4 DIVERGENCE AND CURL OF A VECTOR FIELD

The concepts of divergence and curl of a vector fields, like that of gradient, are of wide applications in engineering and physics. We introduce these two in this section.

8.4.1 Divergence of a Vector Field

Let $\bar{F}(x, y, z) = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$ be a differentiable vector point function. Then *the divergence of the vector field $\bar{F}$, denoted by div $\bar{F}$, is defined as*
$$\text{div}\bar{F} = \nabla\cdot\bar{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

We note that $\nabla\cdot\bar{F}$ is simply a notation and not a scalar product in the usual sense, since $\nabla\cdot\bar{F} \neq \bar{F}\cdot\nabla$. In fact, $\bar{F}\cdot\nabla = F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}$ is a scalar operator while $\nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ is a scalar. For example,
$$\nabla\cdot\bar{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

As another example, if $\bar{F} = 3xz^2\hat{i} + 2xy\hat{j} - y^2z^2\hat{k}$, then div $\bar{F} = \nabla\cdot\bar{F} = 3z^2 + 2x - 2y^2z$.

8.4.2 Physical Meaning of the Divergence

To give the physical interpretation to the divergence, consider the motion of a fluid in a region R having no source or sink in R, that is, there is no point in R at which the fluid is produced or disappears.

Let $\bar{v} = v_x(x, y, z)\hat{i} + v_y(x, y, z)\hat{j} + v_z(x, y, z)\hat{k}$ be the velocity of the fluid at a point $P(x, y, z)$. Consider a rectangular parallelopiped of sides δx, δy, δz in the fluid as shown in Fig. 8.8. Consider that the fluid is flowing in the positive direction of the y-axis.

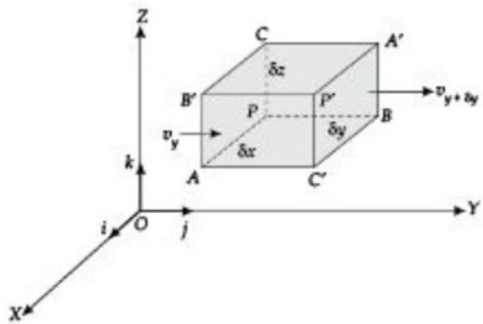


Fig. 8.8

The amount of fluid entering the face $PAB'C$ in a unit time $= v_y\,\delta z\,\delta x$.

The amount of fluid leaving the face $P'C'BA'$ in a unit time $= v_{y+\delta y}\,\delta z\,\delta x$.

Therefore the resultant fluid flow out of these two parallel faces is

$$= (v_{y+\delta y} - v_y)\,\delta z\,\delta x \approx \frac{\partial v_y}{\partial y}\,\delta x\,\delta y\,\delta z,$$

using Taylor series and neglecting the terms of second and higher orders

This is called the *flux* of the vector field $\bar{v}$ through the area $\delta z\,\delta x$.

Accounting for the resultant fluid flows across the other two pairs of faces, the total flux of $\bar{v}$ through the six faces is

$$\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\delta x\,\delta y\,\delta z.$$

Dividing by the volume $\delta x\,\delta y\,\delta z$, the flux per unit volume is given by

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

which is equal to div $\bar{v}$. *Hence divergence gives a measure of the outward flux per unit volume of the flow at a point (x, y, z).*

Similarly, if $\bar{v}$ represents an electric flux, then div $\bar{v}$ is the amount of electric flux which diverges per unit volume in unit time. If $\bar{v}$ represents the heat flux, then div $\bar{v}$ is the rate at which heat is issuing from a point per unit volume. In general the divergence of a vector point function, representing any physical quantity gives at each point the rate per unit volume at which the physical quantity is issuing from that point. This justifies the name divergence of a vector point function.

If the fluid is *incompressible*, then the balance of outflow and inflow for a given volume element is zero at any time and hence, div $\bar{v} = 0$. This equation is known as the *condition of incompressibility*. Clearly the assumption that the flow has no source or sink in the region is essential for this argument.

If for a vector point function $\bar{v}$, div $\bar{v} = 0$ everywhere, then such a point function $\bar{v}$ is called a solenoidal vector function and the field represented by such a vector function is called the solenoidal field.

8.4.3 Curl of a Vector Field

The curl of a vector $\bar{F}$, denoted by curl $\bar{F}$, is defined as

$$\text{curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} i & i & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}.$$

We note that $\nabla \times \bar{F}$ is simply a notation and is not a vector product in the usual sense, since $\nabla \times \bar{F} \neq -\bar{F} \times \nabla$. Sometimes, curl $\bar{F}$ is also written as

$$\text{curl } \bar{F} = \sum \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i},$$

where summation is over the cyclic rotation of the unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$, the components F_1, F_2, F_3 and the independent variables x, y, z. For example, if

$$\bar{F} = yz\hat{i} + 3zx\hat{j} + z\hat{k}, \text{ then curl } \bar{F} = \begin{vmatrix} i & i & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = -3x\hat{i} + y\hat{j} + 2z\hat{k}.$$

Next we give a physical interpretation to curl of a vector function.

8.4.4. Physical Interpretation of the Curl

Suppose a rigid body rotates about a fixed axis through the origin O with a uniform angular velocity $\bar{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$ and let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point $\bar{P}(x, y, z)$ on the body.

The linear velocity $\bar{v}$ of the point $P(x, y, z)$ is given by

$$\bar{v} = \bar{w} \times \bar{r} = \begin{vmatrix} i & j & k \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} = (w_2 z - w_3 y)\hat{i} + (w_3 x - w_1 z)\hat{j} + (w_1 y - w_2 x)\hat{j}.$$

Therefore $\operatorname{curl} \bar{v} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix}$

$$= (w_1 + w_1)\hat{i} + (w_2 + w_2)\hat{j} + (w_3 + w_3)\hat{k} = 2(w_1\hat{i} + w_2\hat{j} + w_3\hat{k}) = 2w,$$

or, $$\bar{w} = \frac{1}{2}\operatorname{curl} \bar{v}$$

Hence, the angular velocity of a uniformly rotating body is equal to one- half of the curl of the linear velocity.

Because of this interpretation, sometimes the word *rotation* is also used in place of curl. In fluid mechanics, if $\bar{v}$ is the velocity of a fluid, and curl $\bar{v} = \bar{0}$, then $\bar{v}$ is said to be *irrotational field* and the corresponding motion is said to be *irrotational* otherwise, *rotational*.

Example 8.19: Find div $\bar{F}$ and curl $\bar{F}$, when $\bar{F} = \operatorname{grad}(x^3 + y^3 + z^3 - 3xyz)$.

Solution: Take $u = x^3 + y^3 + z^3 - 3xyz$. Then

$$\bar{F} = \operatorname{grad} u = \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z} = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}.$$

$$\operatorname{div} \bar{F} = \nabla \cdot [(3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}]$$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy) = 6x + 6y + 6z = 6(x + y + z).$$

$$\operatorname{Curl} \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3(x^2 - yz) & 3(y^2 - xz) & 3(z^2 - xy) \end{vmatrix}$$

$$= \hat{i}\left[3\frac{\partial}{\partial y}(z^2 - xy) - 3\frac{\partial}{\partial z}(y^2 - xz)\right] + \hat{j}\left[3\frac{\partial}{\partial z}(x^2 - yz) - 3\frac{\partial}{\partial x}(z^2 - yx)\right]$$

$$+ \hat{k}\left[3\frac{\partial}{\partial x}(y^2 - zx) - 3\frac{\partial}{\partial y}(x^2 - zy)\right]$$

$$= \hat{i}(-3x + 3x) + \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z) = \bar{0}.$$

Example 8.20: If $\bar{a}$ is a constant vector and $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that curl $(\bar{a} \times \bar{r}) = 2\bar{a}$.

Solution: Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants. Then

$$\bar{a} \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\hat{i} + (a_3 x - a_1 z)\hat{j} + (a_1 y - a_2 x)\hat{k}$$

$$\text{Thus, curl } (\bar{a} \times \bar{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \hat{i}\,(a_1 + a_1) + \hat{j}\,(a_2 + a_2) + \hat{k}\,(a_3 + a_3) = 2(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = 2\bar{a}.$$

Example 8.21: Show that (a) $\nabla \cdot (r^n \bar{r}) = (n+3)r^n$, (b) $\nabla \cdot (\bar{a} \times \bar{r}) = 0$, where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\bar{a}$ is a constant vector.

Solution: (a) $$\nabla \cdot (r^n \bar{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k})$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n + xnr^{n-1}\frac{\partial r}{\partial x} + r^n + ynr^{n-1}\frac{\partial r}{\partial y} + r^n + znr^{n-1}\frac{\partial r}{\partial z}$$

$$= 3r^n + nr^{n-1}\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right) = 3r^n + nr^{n-1}\left(x\frac{x}{r} + y\frac{y}{r} + z\frac{z}{r}\right)$$

$$= 3r^n + nr^n = (n+3)r^n.$$

(b) $$\nabla \cdot (\bar{a} \times \bar{r}) = \sum \hat{i}\,\frac{\partial}{\partial x} \cdot (\bar{a} \times \bar{r}) = \sum \hat{i} \cdot \frac{\partial}{\partial x}(\bar{a} \times \bar{r}) = \sum \hat{i} \cdot \left[\frac{\partial \bar{a}}{\partial x} \times \bar{r} + \bar{a} \times \frac{\partial \bar{r}}{\partial x}\right]$$

$$= 0 + \sum \hat{i} \cdot \left[\bar{a} \times \frac{\partial \bar{r}}{\partial x}\right] = \sum \hat{i} \cdot (\bar{a} \times \hat{i}) = 0.$$

EXERCISE 8.3

In the Problems (1-2), compute div $\bar{F}$, curl $\bar{F}$ and verify that div(curl $\bar{F}$) = 0.

1. $\bar{F} = xe^{-y}\hat{i} + 2ze^{-y}\hat{j} + xy^2\hat{k}$.
2. $\bar{F} = (x^2 - y^2)\hat{i} + 4xy\hat{j} + (x^2 - xy)\hat{k}$.
3. If $\bar{F} = (x + y + 1)\hat{i} + \hat{j} - (x + y)\hat{k}$, show that $\bar{F} \cdot \text{curl } \bar{F} = 0$.
4. If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\bar{a}$ is constant vector, show that
   (i) $\bar{r}$ is irrotational (ii) curl $(\bar{a} \times \bar{r}) = 2\bar{a}$.
5. Determine the values of m and n so that the vector point function
   $\bar{F} = (xyz)^m (x^n\hat{i} + y^n\hat{j} + z^n\hat{k})$ is irrotational.
6. Show that the vector field defined by $\bar{F} = e^{x+y-2z}(\hat{i} + \hat{j} + \hat{k})$ is solenoidal.
7. If $f = x^2 + y^2 + z^2$ and $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ show that div$(f\bar{r}) = 5f$.
8. Show that the vector field $\bar{F} = 3x^2y^2z^4\hat{i} + 2x^3yz^4\hat{j} + 4x^3y^2z^3\hat{k}$ is irrotational and find a scalar function f such that $\bar{F}$ = grad f.

8.5 SOME VECTOR IDENTITIES

In this section we study two types of vector identities. First type results *when del is applied twice to a point function*, and the second type results *when del is applied to product of two point functions*. These identities contribute further in the development of the subject.

8.5.1 'Del' Applied Twice to a Point Function

Since ∇f and $\nabla \times \bar{F}$ are the vector point functions, so we can form their divergence and curl, while $\nabla \cdot \bar{F}$ is a scalar point function, only gradient can be formulated. This results in the following formulae.

1. div grad $f = \nabla \cdot \nabla f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$
2. curl grad $f = \nabla \times \nabla f = \bar{0}$
3. div curl $\bar{F} = \nabla \cdot \nabla \times \bar{F} = 0$
4. curl curl $\bar{F} = \nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$ = grad div $\bar{F} - \nabla^2 \bar{F}$
5. grad div $\bar{F} = \nabla(\nabla \cdot \bar{F}) = \nabla \times (\nabla \times \bar{F}) + \nabla^2 \bar{F}$ = curl curl $\bar{F} + \nabla^2 \bar{F}$

We note that in **1** the operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a scalar operator, called the *Laplacian operator*.

Next, we prove these results.

1. $$\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

2. $$\nabla \times \nabla f = \nabla \times \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 f}{\partial x\,\partial z} - \frac{\partial^2 f}{\partial y\,\partial z}\right) + \hat{j}\left(\frac{\partial^2 f}{\partial z\,\partial x} - \frac{\partial^2 f}{\partial z\,\partial x}\right) + \hat{k}\left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial x\,\partial y}\right) = \bar{0}$$

3. $$\nabla \cdot \nabla \times \bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[\hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \left(\frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z}\right) + \left(\frac{\partial^2 F_1}{\partial y\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x}\right) + \left(\frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y}\right) = 0$$

4. $$\nabla \times (\nabla \times \bar{F}) = \left(\sum i\frac{\partial}{\partial x}\right) \times \left[\sum i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\right] = \sum i\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\right]$$

$$= \sum i\left[\frac{\partial^2 F_2}{\partial y\,\partial x} - \frac{\partial^2 F_1}{\partial y} - \frac{\partial^2 F_1}{\partial z} + \frac{\partial^2 F_3}{\partial z\,\partial x}\right] = \sum i\left[\left(\frac{\partial^2 F_2}{\partial y\,\partial x} + \frac{\partial^2 F_3}{\partial z\,\partial x}\right) - \left(\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]$$

$$= \sum i\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]$$

$$= \left(\sum i\frac{\partial}{\partial x}\right)(\nabla \cdot \bar{F}) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\sum i F_1\right) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$$

Thus, $\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$

Also this implies $\nabla(\nabla \cdot \bar{F}) = \nabla \times (\nabla \times \bar{F}) + \nabla^2 \bar{F}$, which is **5.**

Remark. The interpretation of ∇ as a vector operator gives the justification of the results and also helps to remember the above formulae as follows.

1. $\nabla \cdot \nabla f = \nabla^2 f$; here $\nabla \cdot \nabla = \nabla^2$

2. $\nabla \times \nabla f = \bar{0}$; here $\nabla \times \nabla = \bar{0}$

3. $\nabla \cdot \nabla \times \bar{F} = 0$; here$[\nabla, \nabla, \bar{F}] = 0$, being the scalar tripple product with two vectors equal.

4. & 5. $\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla \cdot \nabla \bar{F} = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$; here expanding $\nabla \times (\nabla \times \bar{F})$ as a vector tripple product.

In continuation to the above noted results, next we consider the operation of ∇ to the product of two point functions.

8.5.2 'Del' Applied to the Product of Two Point Functions

Consider two scalar point functions f and g and two vector point functions $\bar{F}$ and $\bar{G}$. The possible forms of the product are: fg, $\bar{F} \cdot \bar{G}$, the scalar products, and $f\bar{G}$, $\bar{F} \times \bar{G}$, the vector products. When del is applied to these products, we arrive at the following formulae:

6. $\nabla(fg) = f\nabla g + g\nabla f$

7. $\nabla \cdot (f\bar{G}) = \nabla f \cdot \bar{G} + f\nabla \cdot \bar{G}$

8. $\nabla \times (f\bar{G}) = \nabla f \times \bar{G} + f\nabla \times \bar{G}$

9. $\nabla(\bar{F} \cdot \bar{G}) = (F \cdot \nabla)\bar{G} + (G \cdot \nabla)\bar{F} + \bar{F} \times (\nabla \times \bar{G}) + \bar{G} \times (\nabla \times \bar{F})$

10. $\nabla \cdot (\bar{F} \times \bar{G}) = \bar{G} \cdot (\nabla \times \bar{F}) - \bar{F} \cdot (\nabla \times \bar{G})$

11. $\nabla \times (\bar{F} \times \bar{G}) = \bar{F}(\nabla \cdot \bar{G}) - \bar{G}(\nabla \cdot \bar{F}) + (\bar{G} \cdot \nabla)\bar{F} - (\bar{F} \cdot \nabla)\bar{G}$

We note that in result **9**, the term

$$(\bar{F} \cdot \nabla)\bar{G} = \left(\hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\,\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z} \right) = F_1 \frac{\partial \bar{G}}{\partial x} + F_2 \frac{\partial \bar{G}}{\partial y} + F_3 \frac{\partial \bar{G}}{\partial z}$$

where $\bar{G} = G_1\hat{i} + G_2\hat{j} + G_3\hat{k}$; and in result **11**, the term

$$\bar{F}(\nabla \cdot \bar{G}) = \bar{F}\frac{\partial G_1}{\partial x} + \bar{F}\frac{\partial G_2}{\partial y} + \bar{F}\frac{\partial G_3}{\partial z}, \text{ where } \bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}.$$

These results can also be proved using the results on vector differentiation. The result **6** is quite obvious one.

7. $$\nabla\cdot(f\bar{G}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}\,fG_1+\hat{j}\,fG_2+\hat{k}\,fG_3) = \frac{\partial}{\partial x}(fG_1)+\frac{\partial}{\partial y}(fG_2)+\frac{\partial}{\partial z}(fG_3)$$

$$= f\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}+\frac{\partial G_3}{\partial z}\right)+\frac{\partial f}{\partial x}G_1+\frac{\partial f}{\partial y}G_2+\frac{\partial f}{\partial z}G_3 = f(\nabla\cdot\bar{G})+(\nabla f)\cdot\bar{G}.$$

8. $$\nabla\times(f\bar{G}) = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times(f\bar{G}) = \sum \bar{i}\times\frac{\partial}{\partial x}(f\bar{G})$$

$$= \sum\bar{i}\times\left\{\frac{\partial f}{\partial x}\bar{G}+f\frac{\partial\bar{G}}{\partial x}\right\} = \sum\frac{\partial f}{\partial x}\bar{i}\times\bar{G}+f\sum\bar{i}\times\frac{\partial\bar{G}}{\partial x} = \nabla f\times\bar{G}+f(\nabla\times\bar{G}).$$

9. $$\nabla(\bar{F}\cdot\bar{G}) = \sum\hat{i}\frac{\partial}{\partial x}(\bar{F}\cdot\bar{G}) = \sum\hat{i}\left[\bar{F}\cdot\frac{\partial\bar{G}}{\partial x}+\frac{\partial\bar{F}}{\partial x}\cdot\bar{G}\right]$$

$$= \sum\left(\bar{F}\cdot\frac{\partial\bar{G}}{\partial x}\right)\hat{i}+\sum\left(\frac{\partial\bar{F}}{\partial x}\cdot\bar{G}\right)i. \quad \text{...(8.28)}$$

Also, $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c}$, therefore, $(\bar{a}\cdot\bar{b})\bar{c} = (\bar{a}\cdot\bar{c})\bar{b}-\bar{a}\times(\bar{b}\times\bar{c})$

Thus, $$\left(\bar{F}\cdot\frac{\partial\bar{G}}{\partial x}\right)\hat{i} = (\bar{F}\cdot\hat{i})\frac{\partial\bar{G}}{\partial x}-\bar{F}\times\left(\frac{\partial\bar{G}}{\partial x}\times i\right) = (\bar{F}\cdot\hat{i})\frac{\partial\bar{G}}{\partial x}+\bar{F}\times\left(i\times\frac{\partial\bar{G}}{\partial x}\right)$$

and therefore, $$\sum\left(\bar{F}\cdot\frac{\partial\bar{G}}{\partial x}\right)\hat{i} = \left(\bar{F}\cdot\sum i\frac{\partial}{\partial x}\right)\bar{G}+\bar{F}\times\sum\left(i\times\frac{\partial\bar{G}}{\partial x}\right) = (\bar{F}\cdot\nabla)\bar{G}+\bar{F}\times(\nabla\times\bar{G})$$

Interchanging $\bar{F}$ and $\bar{G}$, we have

$$\sum\left(\bar{G}\cdot\frac{\partial\bar{F}}{\partial x}\right)\hat{i} = (\bar{G}\cdot\nabla)\bar{F}+\bar{G}\times(\nabla\times\bar{F})$$

Substituting the values of $\sum\left(\bar{F}\cdot\frac{\partial\bar{G}}{\partial x}\right)\hat{i}$ and $\sum\left(\bar{G}\cdot\frac{\partial\bar{F}}{\partial x}\right)\hat{i}$ in (8.28), we get the desired result.

10. $$\nabla\cdot(\bar{F}\times\bar{G}) = \sum\bar{i}\cdot\frac{\partial}{\partial x}(\bar{F}\times\bar{G}) = \sum\bar{i}\cdot\left(\frac{\partial\bar{F}}{\partial x}\times\bar{G}+\bar{F}\times\frac{\partial\bar{G}}{\partial x}\right)$$

$$= \sum\bar{i}\cdot\left(\frac{\partial\bar{F}}{\partial x}\times\bar{G}\right)-\sum i\left(\frac{\partial\bar{G}}{\partial x}\times\bar{F}\right) = \sum\left(\bar{i}\times\frac{\partial\bar{F}}{\partial x}\right)\cdot\bar{G}-\sum\left(i\times\frac{\partial\bar{G}}{\partial x}\right)\cdot\bar{F}$$

$$= (\nabla\times\bar{F})\cdot\bar{G}-(\nabla\times\bar{G}).\bar{F} = \bar{G}.(\nabla\times\bar{F})-\bar{F}\cdot(\nabla\times\bar{G}).$$

11. $\nabla \times (\bar{F} \times \bar{G}) = \sum \bar{i} \times \left[\frac{\partial}{\partial x}(\bar{F} \times \bar{G})\right] = \sum \bar{i} \times \left[\frac{\partial \bar{F}}{\partial x} \times \bar{G} + \bar{F} \times \frac{\partial \bar{G}}{\partial x}\right]$

$$= \sum \bar{i} \times \left(\frac{\partial \bar{F}}{\partial x} \times \bar{G}\right) + \sum \bar{i} \times \left(\bar{F} \times \frac{\partial \bar{G}}{\partial x}\right). \quad \text{...(8.29)}$$

Now $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$. Using this, we have

$$\sum \bar{i} \times \left(\frac{\partial \bar{F}}{\partial x} \times \bar{G}\right) = \sum (\bar{i} \cdot \bar{G}) \frac{\partial \bar{F}}{\partial x} - \sum \left(\bar{i} \cdot \frac{\partial \bar{F}}{\partial x}\right)\bar{G} = \sum (\bar{G} \cdot \bar{i}) \frac{\partial \bar{F}}{\partial x} - \sum \left(\bar{i} \cdot \frac{\partial \bar{F}}{\partial x}\right)\bar{G}$$

$$= \bar{G} \cdot \sum \left(i \frac{\partial}{\partial x}\right)\bar{F} - \left(\sum \bar{i} \cdot \frac{\partial \bar{F}}{\partial x}\right)\bar{G} = (\bar{G} \cdot \nabla)\bar{F} - (\nabla \cdot \bar{F})\bar{G}.$$

Similarly,

$$\sum \bar{i} \times \left(\bar{F} \times \frac{\partial \bar{G}}{\partial x}\right) = \sum \left(\bar{i} \cdot \frac{\partial \bar{G}}{\partial x}\right)\bar{F} - \sum (\bar{i} \cdot \bar{F}) \frac{\partial \bar{G}}{\partial x} = (\nabla \cdot \bar{G})\bar{F} - \left(\bar{F} \cdot \sum i \frac{\partial}{\partial x}\right)\bar{G} = (\nabla \cdot \bar{G})\bar{F} - (\bar{F} \cdot \nabla)\bar{G}.$$

Thus from (8.29), we have

$\nabla \times (\bar{F} \times \bar{G}) = (\bar{G} \cdot \nabla)\bar{F} - (\nabla \cdot \bar{F})\bar{G} + (\nabla \cdot \bar{G})\bar{F} - (\bar{F} \cdot \nabla)\bar{G}$, which is **11.**

Example 8.22: Prove that (a) $\nabla^2(r^n) = n(n+1)r^{n-2}$ (b) $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$.

Solution: (a) $\nabla^2 r^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n = \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x}\left(\frac{\partial r^n}{\partial x}\right)$

$$= \sum \frac{\partial}{\partial x}\left(n r^{n-1} \frac{\partial r}{\partial x}\right) = \sum \frac{\partial}{\partial x}\left(n r^{n-1} \frac{x}{r}\right) = \sum \frac{\partial}{\partial x}(n r^{n-2} x)$$

$$= \sum n\left[r^{n-2} + x(n-2)r^{n-3} \frac{\partial r}{\partial x}\right] = \sum n[r^{n-2} + (n-2)r^{n-4}x^2]$$

$$= n[3r^{n-2} + (n-2)r^{n-4}(x^2 + y^2 + z^2)] = n[3r^{n-2} + (n-2)r^{n-2}] = n(n+1)r^{n-2}$$

(b) $\nabla^2 f(r) = \nabla \cdot (\nabla f(r)) = \text{div}\,(\text{grad}\, f(r)) = \text{div}\,\{f'(r)\,\text{grad}\, r\}$

$$= \text{div}\left\{f'(r)\frac{\bar{r}}{r}\right\} = \text{div}\left\{\frac{f'(r)}{r}\bar{r}\right\} = \frac{f'(r)}{r}\,\text{div}\,\bar{r} + \bar{r} \cdot \text{grad}\left(\frac{f'(r)}{r}\right)$$

$$= \frac{3f'(r)}{r} + \bar{r} \cdot \left[\frac{1}{r}\text{grad}\, f'(r) + f'(r)\, grad\, \frac{1}{r}\right] = \frac{3f'(r)}{r} + \bar{r} \cdot \left\{\frac{1}{r} f''(r)\frac{\bar{r}}{r} + f'(r)\left(\frac{-\bar{r}}{r^3}\right)\right\}$$

$$= \frac{3}{r} f'(r) + \frac{1}{r^2} f''(r) \bar{r} \cdot \bar{r} - \frac{1}{r^3} f'(r) \bar{r} \cdot \bar{r}$$

$$= \frac{3}{r} f'(r) + f''(r) - \frac{f'(r)}{r} = f''(r) + \frac{2}{r} f'(r).$$

Example 8.23: If $f = (x^2 + y^2 + z^2)^{-n}$, find div grad f and determine n if div grad $f = 0$.

Solution: We have, $f = r^{-2n}$. Thus grad f = grad $r^{-2n} = (-2n)\, r^{-2n-2} \bar{r}$, and

$$\text{div grad } f = \nabla \cdot (-2n\, r^{-2n-2} \bar{r}) = (-2n)\, [r^{-2n-2}\, \nabla \cdot \bar{r} + \bar{r} \cdot \nabla r^{-2n-2}]$$

$$= (-2n)\, [3r^{-2n-2} + \bar{r} \cdot (-2n-2)\, r^{-2n-4} \bar{r}]$$

$$= (-2n)\, [3r^{-2n-2} + (-2n-2) r^{-2n-2}]$$

$$= (-2n)\, (-2n+1) r^{-2n-2} = (2n)\, (2n-1) r^{-2n-2}.$$

div grad $f = 0$, gives $n = 0$, or $1/2$.

Example 8.24: Prove that curl curl curl curl $\bar{F} = \nabla^4 \bar{F}$, where $\bar{F}$ is solenoidal field.

Solution: Since the vector field $\bar{F}$ is solenoidal, therefore, div $\bar{F} = 0$. Thus

curl curl $\bar{F}$ = grad div $\bar{F} - \nabla^2 \bar{F} = -\nabla^2 \bar{F} = \bar{G}$, say.

Consider, curl curl$\bar{G}$ = grad div$\bar{G} - \nabla^2 \bar{G}$

Now div$\bar{G} = \nabla \cdot (-\nabla^2 \bar{F}) = -\nabla^2 (\nabla \cdot \bar{F}) = 0$. Therefore, curl curl $\bar{G} = -\nabla^2 \bar{G}$, and thus

curl curl curl curl $\bar{F} = -\nabla^2 (-\nabla^2 \bar{F}) = \nabla^4 \bar{F}$.

Example 8.25: If $\bar{a}$ and $\bar{b}$ are constant vectors, show that

(a) $\nabla \left[\dfrac{\bar{a} \cdot \bar{r}}{r^n} \right] = \dfrac{\bar{a}}{r^n} - n \dfrac{(\bar{a} \cdot \bar{r})}{r^{n+2}} \bar{r}$ (b) $\nabla \times \{\bar{a} \times (\bar{b} \times \bar{r})\} = \bar{a} \times \bar{b}$.

Solution:

(a) $$\nabla \left[\frac{\bar{a} \cdot \bar{r}}{r^n} \right] = \nabla \left[\frac{1}{r^n} \bar{a} \cdot \bar{r} \right] = \frac{1}{r^n} \nabla (\bar{a} \cdot \bar{r}) + (\bar{a} \cdot \bar{r}) \nabla \left(\frac{1}{r^n} \right)$$

$$= \frac{1}{r^n} \bar{a} + (\bar{a} \cdot \bar{r})\, (-n) r^{-n-2} \bar{r}, \text{ since } \nabla (\bar{a} \cdot \bar{r}) = \bar{a}.$$

$$= \frac{\bar{a}}{r^n} - n \frac{(\bar{a} \cdot \bar{r})}{r^{n+2}} \bar{r}$$

(b) $$\nabla \times (\bar{a} \times (\bar{b} \times \bar{r})) = \nabla \times [(\bar{a} \cdot \bar{r}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{r}] = \nabla \times (\bar{a} \cdot \bar{r}) \bar{b} - \nabla \times (\bar{a} \cdot \bar{b}) \bar{r}$$

$$= \{\nabla (\bar{a} \cdot \bar{r}) \times \bar{b} + (\bar{a} \cdot \bar{r}) \nabla \times \bar{b}\} - \{\nabla (\bar{a} \cdot \bar{b}) \times \bar{r} + (\bar{a} \cdot \bar{b}) \nabla \times \bar{r}\}$$

$$= \bar{a} \times \bar{b}, \text{ since } \nabla (\bar{a} \cdot \bar{r}) = \bar{a},\ \nabla \times \bar{r} = \bar{0}, \nabla \times \bar{b} = \bar{0}, \text{and } \nabla (\bar{a} \cdot \bar{b}) = 0.$$

Example 8.26: If r and $\bar{r}$ have their usual meanings and $\bar{a}$ is a constant vector, prove that

$$\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = \frac{2-n}{r^n}\bar{a} + \frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}.$$

Solution:

$$\text{L.H.S.} = \nabla \times [r^{-n}(\bar{a} \times \bar{r})]$$

$$= r^{-n}[\nabla \times (\bar{a} \times \bar{r})] + \nabla r^{-n} \times (\bar{a} \times \bar{r}) \qquad ...(8.30)$$

We have, $\nabla \times (\bar{a} \times \bar{r}) = \bar{a}(\nabla\cdot\bar{r}) - \bar{r}(\nabla\cdot\bar{a}) + (\bar{r}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{r}$ ...(8.31)

Since $\bar{a}$ is a constant vector, therefore, $\nabla\cdot\bar{a}$ and $(\bar{r}\cdot\nabla)\bar{a}$ are zeros; also $\nabla\cdot\bar{r} = 3$ and $(\bar{a}\cdot\nabla)\bar{r} = \bar{a}$, thus (8.31) becomes

$$\nabla \times (\bar{a} \times \bar{r}) = 3\bar{a} - \bar{a} = 2\bar{a} \qquad ...(8.32)$$

Also $\nabla r^{-n} = -nr^{-(n+2)}\bar{r}$ ...(8.33)

Using (8.32) and (8.33) in (8.30), we have

$$\text{L.H.S.} = \frac{2\bar{a}}{r^n} - \frac{n}{r^{n+2}}[\bar{r} \times (\bar{a} \times \bar{r})] = \frac{2\bar{a}}{r^n} - \frac{n}{r^{n+2}}[(\bar{r}\cdot\bar{r})\bar{a} - (\bar{a}\cdot\bar{r})\bar{r}]$$

$$= \frac{2\bar{a}}{r^n} - \frac{n}{r^{n+2}}[r^2\bar{a} - (\bar{a}\cdot\bar{r})\bar{r}] = \frac{2-n}{r^n}\bar{a} + \frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}} = \text{R.H.S.}$$

Example 8.27: Prove that $\nabla\cdot\left(\frac{f(r)}{r}\bar{r}\right) = \frac{1}{r^2}\frac{d}{dr}(r^2 f(r))$

Solution: $\nabla\cdot\left(\frac{f(r)}{r}\bar{r}\right) = \nabla\cdot\left[\frac{f(r)}{r}(x\hat{i} + y\hat{j} + z\hat{k})\right] = \frac{\partial}{\partial x}\left[\frac{f(r)}{r}x\right] + \frac{\partial}{\partial y}\left[\frac{f(r)}{r}y\right] + \frac{\partial}{\partial z}\left[\frac{f(r)}{r}z\right]$

Now, $\frac{\partial}{\partial x}\left[\frac{f(r)}{r}x\right] = \frac{f(r)}{r} + x\frac{d}{dr}\left\{\frac{f(r)}{r}\right\}\frac{\partial r}{\partial x} = \frac{f(r)}{r} + x\left\{\frac{f'(r)}{r} - \frac{f(r)}{r^2}\right\}\frac{x}{r}$

$$= \frac{f(r)}{r} + \frac{x^2}{r^2}f'(r) - \frac{x^2}{r^3}f(r). \text{ Similarly,}$$

$$\frac{\partial}{\partial y}\left[\frac{f(r)}{r}y\right] = \frac{f(r)}{r} + \frac{y^2}{r^2}f'(r) - \frac{y^2}{r^3}f(r), \text{ and } \frac{\partial}{\partial z}\left[\frac{f(r)}{r}z\right] = \frac{f(r)}{r} + \frac{z^2}{r^2}f'(r) - \frac{z^2}{r^3}f(r)$$

Thus, $\nabla\cdot\left(\frac{f(r)}{r}\bar{r}\right) = \frac{3}{r}f(r) + \frac{1}{r^2}(x^2 + y^2 + z^2)f'(r) - \frac{1}{r^3}(x^2 + y^2 + z^2)f(r)$

$$= \frac{2}{r}f(r) + f'(r) = \frac{1}{r^2}[2rf(r) + r^2 f'(r)] = \frac{1}{r^2}\frac{d}{dr}(r^2 f(r)).$$

Example 8.28: If $\bar{A}$ and $\bar{B}$ are irrotational, prove that $\bar{A} \times \bar{B}$ is solenoidal.

Solution: Since $\bar{A}$ and $\bar{B}$ are irrotational, therefore, curl $\bar{A} = \bar{0}$, and curl $\bar{B} = \bar{0}$

We have, $\quad \text{div}\,(\bar{A} \times \bar{B}) = \bar{B} \cdot \text{curl}\bar{A} - \bar{A} \cdot \text{curl}\bar{B} = \bar{B} \cdot \bar{0} - \bar{A} \cdot \bar{0} = 0$

Thus, $\bar{A} \times \bar{B}$ is solenoidal.

EXERCISE 8.4

1. If $u = x^2 + y^2 + z^2$ and $\bar{v} = x\hat{i} + y\hat{j} + z\hat{k}$, show that div $(u\bar{v}) = 5u$.
2. If $u\bar{F} = \nabla v$, where u and v are scalars and $\bar{F}$ is a vector, show that $\bar{F} \cdot \text{curl}\ \bar{F} = 0$
3. Find the directional derivative of $\nabla \cdot (\nabla \phi)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$.
4. If $\bar{r}_1$ and $\bar{r}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z) prove that

   (a) div $(\bar{r}_1 \times \bar{r}_2) = 0$ (b) $\text{grad}(\bar{r}_1 \cdot \bar{r}_2) = \bar{r}_1 + \bar{r}_2$ (c) curl $(\bar{r}_1 \times \bar{r}_2) = 2(\bar{r}_1 - \bar{r}_2)$.
5. Show that the vector $\nabla\phi \times \nabla\psi$ is solenoidal.
6. Show that $\text{curl}\left[\hat{k} \times \text{grad}\,\frac{1}{r}\right] + \text{grad}\left[\hat{k} \cdot \text{grad}\,\frac{1}{r}\right] = \bar{0}$, where r is the distance of a point (x, y, z) from the origin and $\hat{k}$ is a unit vector in the direction of z-axis.
7. Let $f(x, y, z)$ be a solution of the Laplace equation $\nabla^2 f = 0$, show that ∇f is a vector which is both irrotational and solenoidal.
8. Let $f(x, y, z)$ be a solution of the Poisson equation $\nabla^2 f = c$, where c is a constant. If $\bar{v} = \nabla f$, show that curl $\bar{v} = 0$, but div $\bar{v} \neq 0$.
9. Show that (a) $\nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f) = f\nabla^2 g - g\nabla^2 f$ (b) $\nabla \cdot [(f\nabla g) \times (g\nabla f)] = 0$.
10. If $\nabla \cdot \bar{e} = 0, \nabla \cdot \bar{h} = 0, \nabla \times \bar{e} = -\frac{1}{c}\frac{\partial \bar{h}}{\partial t}$, and $\nabla \times \bar{h} = \frac{1}{c}\frac{\partial \bar{h}}{\partial t}$, then show that $\bar{e}$ and $\bar{h}$ satisfy the wave equation $\frac{\partial^2 f}{\partial t^2} = c^2\nabla^2 \bar{f}$, where c is a constant, and $\bar{f}$ is a vector function.

ANSWERS

Exercise 8.1 (p. 494)

1. $(5t^2 \cos t + 10t \sin t)\hat{i} + (t \cos t + \sin t)\hat{j} + (t^3 \cos t + 3t^2 \sin t)\hat{k}$
2. $4 \cos 4t + 3t^2$
3. $w\hat{k}$

4. $(3t^2 - 2e^{2t})\,\hat{i} - [(1-2t) - (2+t)e^t]\,\hat{j} - [te^t + t(2+3t)]\,\hat{k}$
5. $2t\vec{u}(t^2) + 2t^3\,\vec{u}'(t^2)$
6. $a\vec{u}'(at) - (a/t^2)\,\vec{v}'(a/t)$
7. $\vec{u}(t) \times \vec{u}'''(t) + \vec{u}'(t) \times \vec{u}''(t)$
8. $\vec{u}(t) \cdot \vec{u}'(t) \times \vec{u}'''(t)$
10. $x(t) = (1+t)/\sqrt{2}$, $y(t) = (1-t)/\sqrt{2}$, $z(t) = (\pi/4) + t$
11. $(-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k})/5$
12. $\cos^{-1}(3/7)$
13. $\sqrt{37}$, $5\sqrt{13}$
14. $8\sqrt{14}/7, -\sqrt{14}/7$
15. $70/\sqrt{29}$, $\sqrt{(436/29)}$
16. $\sqrt{17}$ m.p.h in the direction $\tan^{-1}(0.25)$ north of east

Exercise 8.2 (p. 504)

1. (a) $-8\hat{i}$ (b) $7\hat{i} + 24\hat{j} - 2\hat{k}$
2. (a) $-(\hat{i} + 3\hat{j} - \hat{k})/\sqrt{11}$ (b) $(-\hat{i} + 2\hat{j} + 2\hat{k})/3$
6. (a) $-\dfrac{11}{3}$ (b) 144 (c) $15/\sqrt{17}$ (d) $(15 + 14\sqrt{3})/2$
7. $(4\hat{i} + 3\hat{j} - 12\hat{k})/13, 1$
8. $\lambda = 2.5, \mu = 1$
10. $(xx_0/a^2) + (yy_0/b^2) + (zz_0/c^2) = 1$

Exercise 8.3 (p. 510)

5. $m = 0, n = 1$

Exercise 8.4 (p. 517)

3. $1724/\sqrt{21}$

9 Vector Integral Calculus

CHAPTER

Vector integral calculus, similar to vector differential calculus, extends the concept of integration to vector functions, enabling to generalize the idea of definite integration to curves and surfaces in three dimensions. This helps in better understanding of the physical interpretations of divergence and curl, and has applications in solid mechanics, fluid flow and heat flow problems.

9.1 INTEGRATION OF VECTOR FUNCTIONS

The integration of vector functions is defined as the reverse process of differentiation. Let $\bar{f}(t)$ and $\bar{F}(t)$ be two vector functions of a scalar variable t such that $\frac{d}{dt}\bar{F}(t) = \bar{f}(t)$, then $\bar{F}(t)$ is called the integral of $\bar{f}(t)$ with respect to t and, since $\frac{d}{dt}\left(\bar{F}(t) + \bar{c}\right) = \frac{d}{dt}\bar{F}(t)$, we write

$$\int \bar{f}(t)dt = \bar{F}(t) + \bar{c}, \qquad ...(9.1)$$

where $\bar{c}$ is any arbitrary constant vector independent of t.

$\bar{F}(t)$ is called the *indefinite integral* of $\bar{f}(t)$. The constant vector $\bar{c}$ is called the constant of integration and is determined on the basis of the initial conditions given.

The *definite integral* of $\bar{f}(t)$ between the limits $t = a$ and $t = b$ is given by

$$\int_a^b \bar{f}(t)dt = [\bar{F}(t)]_a^b = \bar{F}(b) - \bar{F}(a). \qquad ...(9.2)$$

As in case of differentiation of vectors, in order to integrate a vector function, we integrate its components, that is, if $\bar{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, then

$$\int f(t)dt = \hat{i}\int f_1(t)dt + \hat{j}\int f_2(t)dt + \hat{k}\int f_3(t)dt.$$

By considering the derivatives of suitable vector functions we obtain some standard results for integration of vector functions. For example

1. $\frac{d}{dt}(\bar{r}.\bar{s}) = \frac{d\bar{r}}{dt}.\bar{s} + \bar{r}.\frac{d\bar{s}}{dt}$, implies $\int\left(\frac{dr}{dt}.\bar{s} + \bar{r}.\frac{d\bar{s}}{dt}\right)dt = \bar{r}.\bar{s} + c$, where c is a scalar independent of t.

2. $\frac{d}{dt}(\bar{r}^2) = 2\bar{r}.\frac{d\bar{r}}{dt}$, implies $\int\left(\bar{r}.\frac{d\bar{r}}{dt}\right)dt = \frac{1}{2}\bar{r}^2 + c$, where c is a scalar independent of t.

3. $\frac{d}{dt}\left(\bar{r}\times\frac{d\bar{r}}{dt}\right) = \frac{d\bar{r}}{dt}\times\frac{d\bar{r}}{dt} + \bar{r}\times\frac{d^2\bar{r}}{dt^2} = \bar{r}\times\frac{d^2\bar{r}}{dt^2}$, implies $\int\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)dt = \bar{r}\times\frac{d\bar{r}}{dt} + \bar{c}$,

where $\bar{c}$ is a vector independent of t.

4. If $\bar{a}$ is a constant vector, then

$$\frac{d}{dt}(\bar{a}\times\bar{r}) = \frac{d\bar{a}}{dt}\times\bar{r} + \bar{a}\times\frac{d\bar{r}}{dt} = \bar{a}\times\frac{d\bar{r}}{dt}, \text{ implies} \int\left(\bar{a}\times\frac{d\bar{r}}{dt}\right)dt = \bar{a}\times\bar{r} + \bar{c},$$

where $\bar{c}$ is a vector independent of t.

5. If r and $\hat{r}$ have their usual meanings, then

$$\frac{d}{dt}(\hat{r}) = \frac{d}{dt}\left(\frac{1}{r}\bar{r}\right) = \frac{1}{r}\frac{d\bar{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\bar{r}, \text{ implies} \int\left[\frac{1}{r}\frac{d\bar{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\bar{r}\right]dt = \hat{r} + \bar{c},$$ where $\bar{c}$ is a vector independent of t.

Example 9.1: The acceleration of a particle at time t is given by

$$\bar{a}(t) = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$$

If the velocity vector $\bar{v}$ and displacement $\bar{r}$ are zero at $t = 0$, find $\bar{v}$ and $\bar{r}$ at any time t.

Solution: The acceleration is $\bar{a}(t) = \frac{d^2\bar{r}}{dt^2} = 18\cos 3t\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$

Integrating w.r.t. t, we have

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = 18\frac{\sin 3t}{3}\hat{i} + 8\frac{\cos 2t}{2}\hat{j} + 6\frac{t^2}{2}\hat{k} + \bar{c} = 6\sin 3t\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\hat{k} + \bar{c}.$$

At $t = 0$, $v = \bar{0}$, therefore, $\bar{0} = 4\hat{j} + \bar{c}$, or $\bar{c} = -4\hat{j}$, thus

$$v(t) = \frac{d\bar{r}}{dt} = 6\sin 3t\hat{i} + 4(\cos 2t - 1)\hat{j} + 3t^2\hat{k}.$$

Integrating again w.r.t. t, we have $\bar{r}(t) = -2\cos 3t\hat{i} + (2\sin 2t - 4t)\hat{j} + t^3\hat{k} + \bar{c}$

At $t = 0$, $\bar{r} = \bar{0}$, therefore, $\bar{0} = -2\hat{i} + \bar{c}$, or $\bar{c} = 2\hat{i}$, thus

$$\bar{r} = 2(1 - \cos 3t)\hat{i} + 2(\sin 2t - 2t)\hat{j} + t^3\hat{k}.$$

Example 9.2: Evaluate $\int_1^2 (\bar{a}.\bar{b}\times\bar{c})dt$, where

$$\bar{a} = t\hat{i} - 3\hat{j} + 2t\hat{k},\quad \bar{b} = \hat{i} - 2\hat{j} + 2\hat{k},\quad \bar{c} = 3\hat{i} + t\hat{j} - \hat{k}.$$

Solution: We have,

$$\bar{a}.\bar{b}\times\bar{c} = [\bar{a}\;\bar{b}\;\bar{c}] = \begin{vmatrix} t & -3 & 2t \\ 1 & -2 & 2 \\ 3 & t & -1 \end{vmatrix} = t(2-2t) + 3(-1-6) + 2t(t+6) = 7(2t-3).$$

Therefore, $\int_1^2 (\bar{a}.\bar{b}\times\bar{c})dt = 7\int_1^2 (2t-3)dt = 7\left[t^2 - 3t\right]_1^2 = 7(-2+2) = 0.$

Example 9.3: If $\bar{r}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, then prove $\int_1^2 \left[\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right]dt = -14\hat{i} + 75\hat{j} - 15\hat{k}.$

Solution: We have, $\frac{d}{dt}\left[\bar{r}\times\frac{d\bar{r}}{dt}\right] = \frac{d\bar{r}}{dt}\times\frac{d\bar{r}}{dt} + \bar{r}\times\frac{d^2\bar{r}}{dt^2} = \bar{r}\times\frac{d^2\bar{r}}{dt^2}$, therefore,

$$\int_1^2 \left[\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right]dt = \left[\bar{r}\times\frac{d\bar{r}}{dt}\right]_1^2.$$

Also, $\bar{r}\times\frac{d\bar{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix} = -2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}$. Thus

$$\int_1^2 \left[\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right]dt = \left[-2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}\right]_1^2 = (-16+2)\hat{i} + (80-5)\hat{j} - (20-5)\hat{k} = -14\hat{i} + 75\hat{j} - 15\hat{k}.$$

Example 9.4: If $r(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, then evaluate $\int_0^1 \left(\bar{r}.\frac{d\bar{r}}{dt}\right)dt.$

Solution: We have, $\frac{d}{dt}(\bar{r}^2) = 2\bar{r}\cdot\frac{d\bar{r}}{dt}$, therefore, $\int_0^1\left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right)dt = \frac{1}{2}\left[\bar{r}^2\right]_0^1$

Also, $$\bar{r}^2 = \bar{r}\cdot\bar{r} = (5t^2\hat{i} + t\hat{j} - t^3\hat{k})\cdot(5t^2\hat{i} + t\hat{j} - t^3\hat{k}) = 25t^4 + t^2 + t^6.$$

Thus, $$\int_0^1\left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right)dt = \frac{1}{2}[25t^4 + t^2 + t^6]_0^1 = \frac{1}{2}[25+1+1] = \frac{27}{2}.$$

Example 9.5: If $\frac{d^2\bar{r}}{dt^2} = 6t\hat{i} - 12t^2\hat{j} + 4\cos t\hat{k}$, then find $\bar{r}$ given that at $t = 0$,

$$\frac{d\bar{r}}{dt} = -\hat{i} - 3\hat{k} \text{ and } \bar{r} = 2\hat{i} + \hat{j}.$$

Solution: We have, $\frac{d^2\bar{r}}{dt^2} = 6t\hat{i} - 12t^2\hat{j} + 4\cos t\hat{k}$. Integrating this w.r.t t,

$\frac{d\bar{r}}{dt} = 3t^2\hat{i} - 4t^3\hat{j} + 4\sin t\hat{k} + \bar{c}$, where $\bar{c}$ is a vector independent of t.

At $t = 0$, $\frac{d\bar{r}}{dt} = -\hat{i} - 3\hat{k}$, therefore, $-\hat{i} - 3\hat{k} = \bar{c}$, and thus

$$\frac{d\bar{r}}{dt} = (3t^2 - 1)\hat{i} - 4t^3\hat{j} + (4\sin t - 3)\hat{k}.$$

Integrating w.r.t. t, $\bar{r}(t) = (t^3 - t)\hat{i} - t^4\hat{j} - (4\cos t + 3t)\hat{k} + \bar{c}$, where $\bar{c}$ is a vector independent of t.

At $t = 0$, $\bar{r} = 2\hat{i} + \hat{j}$, this gives $2\hat{i} + \hat{j} = -4\hat{k} + \bar{c}$, or $\bar{c} = 2\hat{i} + \hat{j} + 4\hat{k}$.

Thus, $\bar{r}(t) = (t^3 - t + 2)\hat{i} - (t^4 - 1)\hat{j} - (4\cos t + 3t - 4)\hat{k}$.

EXERCISE 9.1

1. Given $\bar{r}(t) = (5t^2 - 3t)\hat{i} + 6t^3\hat{j} - 7t\hat{k}$, evaluate $\int_2^4 \bar{r}(t)dt$.

2. If $\bar{r} = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\bar{s} = 2t^2\hat{i} + 6t\hat{k}$, evaluate

   (a) $\int_0^2 \bar{r}\cdot\bar{s}\,dt$ (b) $\int_0^2 \bar{r}\times\bar{s}\,dt$

3. Given that $\vec{r}(t) = 2\hat{i} - \hat{j} + 2\hat{k}$ when $t = 2$ and $\vec{r}(t) = 4\hat{i} - 2\hat{j} + 3\hat{k}$, when $t = 3$, show that

$$\int_2^3 \left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt = 10.$$

4. Find the value of $\vec{r}$ satisfying $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\,\hat{k}$, given that $\vec{r} = 2\hat{i} + \hat{j}$ and $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ at $t = 0$.

9.2 THE LINE INTEGRAL. INDEPENDENCE OF PATH

The concept of line integral is a generalization of the concept of the definite integral $\int_a^b f(x)dx$ in integral calculus. In definite integral we integrate the integrand $f(x)$ from $x = a$, along the x-axis, to $x = b$. In line integral we shall integrate the given function along a curve C in the plane or in the space.

A curve C in space can be represented by a vector function

$$\vec{r}(t) = [x(t), y(t), z(t)] = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, a \le t \le b,$$

where, x, y, z are cartesian co-ordinates. This is called a *parametric representation* of the curve C and t is called the *parameter* of the representation. To each value of t, there corresponds a point P on C, as shown in Fig. 9.1a.

We call C *the path of integration*. With A: $\vec{r}(a)$, its initial point, and B: $\vec{r}(b)$, its terminal point, the curve C is now *oriented*. The direction from A to B in which t increases is taken as the positive direction on C. The direction is indicated by an arrow. If the points A and B coincide, as in Fig. 9.1b, then C is called a *closed path*. Further, C is called a *smooth curve* if it has a unique tangent at each of its points whose direction varies continuously as we move along C. Mathematically, it means that $\vec{r}(t)$ is differentiable and the derivative $\frac{d\vec{r}}{dt}$ is continuous and different from the zero vector at every point of C.

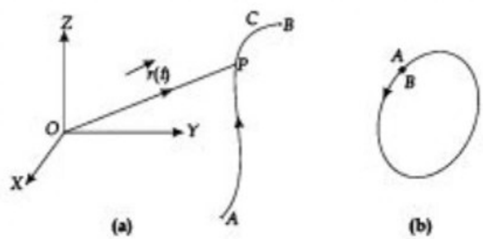


Fig. 9.1

We shall assume *every path of integration of a line integral to be piecewise smooth, that is, consisting of finitely many smooth curves.*

Now, we define line integral.

9.2.1 Line Integral of $\vec{F}$ Over C

A *line integral* of a vector function $\bar{F}(\bar{r})$ over a curve C is defined by

$$\int_C \bar{F}(\bar{r}) d\bar{r} = \int_a^b \bar{F}(\bar{r}(t)) \cdot \frac{d\bar{r}}{dt} dt \qquad ...(9.3)$$

If $\bar{F}(\bar{r}) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ and $d\bar{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$, then

$$\int_C \bar{F}(\bar{r}) d\bar{r} = \int_C (f_1 dx + f_2 dy + f_3 dz) = \int_a^b \left(f_1\frac{dx}{dt} + f_2\frac{dy}{dt} + f_3\frac{dz}{dt}\right)dt \qquad ...(9.4)$$

Here, f_1, f_2 and f_3 are functions of x, y, z, which in turn depend upon parameter t, $a \le t \le b$. When the path of integration C is a closed curve, then instead of $\int_C$ we write $\oint_C$. We note that line integral (9.4) is a scalar, not a vector. Two other types of line integrals are

$$\int_C \bar{F} \times d\bar{r} \quad \text{and} \quad \int_C f d\bar{r}$$

which are both vectors.

The line integral $\int_C \bar{F}(\bar{r}).d\bar{r}$ arises naturally in mechanics. If $\bar{F}$ represents the force acting on a particle moving along an arc AB, then the work done during the small displacement $\delta\bar{r}$ is $\bar{F}.\delta\bar{r}$. Thus *the total work done by* $\bar{F}$ *during the displacement from A to B* is given by the line integral $\int_A^B \bar{F}.d\bar{r}$.

Similarly, if $\bar{F}$ represents the velocity of a fluid particle, then the line integral $\int_C \bar{F}.d\bar{r}$ is called *the circulation of* $\bar{F}$ *around the curve C*; and *when the circulation of* $\bar{F}$ *around every closed curve C in a region E vanishes, then* $\bar{F}$ *is said to be irrotational in E.*

Example 9.6: If $\bar{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, evaluate $\int_C \bar{F}.d\bar{r}$ along the curve $C: y = x^3$ in the xy-plane from the point (1, 1) to (2, 8).

Solution: Since the particle moves in the xy-plane, therefore, $d\bar{r} = dx\hat{i} + dy\hat{j}$. Thus, we have

$$\int_C \bar{F}.d\bar{r} = \int_{C:y=x^3} [(5xy - 6x^2)dx + (2y - 4x)dy]$$

Substituting $y = x^3$, where x goes from 1 to 2, we have

$$\int_C \bar{F}.d\bar{r} = \int_1^2 [(5x^4 - 6x^2)dx + (2x^3 - 4x)\cdot 3x^2 dx] = \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)dx = \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2$$

$$= 32 - (-3) = 35.$$

Example 9.7: Using the line integral compute the work done by the force $\bar{F} = (2y+3)\hat{i} + xz\hat{j} + (yz - x)\hat{k}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2$, $y = t$, $z = t^3$.

Solution: Since the particle moves in the space, therefore, $d\bar{r} = dx\hat{i} + dy\hat{j} + dz\hat{j}$. The work W done by the force $\bar{F}$ in moving a particle from the point $A(0, 0, 0)$ to the point $B(2, 1,1)$, is given by

$$W = \int_A^B \bar{F}.d\bar{r} = \int_A^B [(2y+3)dx + xzdy + (yz - x)dz]$$

Substituting $x = 2t^2$, $y = t$ and $z = t^3$, where t goes from 0 to 1, we have

$$W = \int_0^1 [(2t+3)4t\,dt + 2t^5 dt + (t^4 - 2t^2)3t^2 dt] = \int_0^1 (8t^2 + 12t + 2t^5 + 3t^6 - 6t^4)dt$$

$$= \left[\frac{8t^3}{3} + 6t^2 + \frac{t^6}{3} + \frac{3}{7}t^7 - \frac{6}{5}t^5\right]_0^1 = \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = 8\frac{8}{35} \text{ units.}$$

Example 9.8: A vector field is given by $\bar{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2$, $z = 0$

Solution: The parametric equation of the circular path $x^2 + y^2 = a^2$, $z = 0$ are:

$x = a\cos t$, $y = a\sin t$, $z = 0$, $0 \le t \le 2\pi$. Also $d\bar{R} = dx\hat{i} + dy\hat{j}$. Therefore the line integral is

$$\oint_C \bar{F}.d\bar{R} = \oint_C [\sin y dx + x(1 + \cos y)dy] = \oint_C d(x\sin y) + \oint_C xdy$$

Substituting $x = a\cos t$, $y = a\sin t$, we have

$$\oint_C \bar{F}.d\bar{R} = \int_0^{2\pi} d[a\cos t \sin(a\sin t)] + \int_0^{2\pi} a\cos t\, a\cos t\,dt$$

$$= \left[a\cos t\sin(a\sin t)\right]_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi}(1 + \cos 2t)dt = 0 + \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \pi a^2.$$

Example 9.9: Compute the line integral $\int_C (y^2dx - x^2dy)$ about the triangle whose vertices are (1, 0), (0, 1) and (–1, 0).

Solution: Here the path is *ABCD* as shown in Fig. 9.2

Equation of line *AB* is, $x + y = 1$, of *BC* is, $-x + y = 1$, and of *CA* is, $y = 0$.

We have,

$$\int_{ABCA} (y^2dx - x^2dy) = \int_{AB} (y^2dx - x^2dy) + \int_{BC} (y^2dx - x^2dy) + \int_{CA} (y^2dx - x^2dy) \qquad ...(9.5)$$

To evaluate $\int_{AB} (y^2dx - x^2dy)$, put $y = 1 - x$, where x goes from 1 to 0. Thus

$$\int_{AB} (y^2dx - x^2dy) = \int_1^0 [(1-x)^2dx - x^2(-dx)] = \int_1^0 (1 + x^2 - 2x + x^2)dx$$

$$= \int_1^0 (1 - 2x + 2x^2)dx = \left[x - x^2 + \frac{2x^3}{3}\right]_1^0 = -\frac{2}{3}.$$

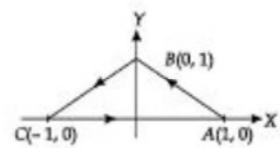


Fig. 9.2

Similarly, $\int_{BC} (y^2dx - x^2dy) = \int_0^{-1} [(1+x)^2dx - x^2dx] = \int_0^{-1} (1 + x^2 + 2x - x^2)dx$

$$= \int_0^{-1} (1 + 2x)\,dx = \left[x + x^2\right]_0^{-1} = 0.$$

and, $\int_{CA} (y^2dx - x^2dy) = \int_{-1}^{-1} 0.dx = 0$. Substituting these values in (9.5), we obtain

$$\int_{ABCA} (y^2dx - x^2dy) = -\frac{2}{3} + 0 + 0 = -\frac{2}{3}.$$

9.2.2 Independence of Path. Conservative Vector Field

An important question of interest is: *Does the value of a line integral* $\int_C \vec{F}.d\vec{r}$ *change if we integrate from the same initial point A to the same terminal point B but along another path?* The answer is *yes*, in general. Consider the following example.

Example 9.10: A vector field is given by $\bar{F} = 5z\hat{i} + xy\hat{j} + x^2z\hat{k}$. Evaluate the line integral $\int \bar{F}.d\bar{r}$ along the two different paths C_1 and C_2 with the same initial point $A(0, 0, 0)$ and the same terminal point $B(1, 1, 1)$, as given below:

(a) C_1: the straight-line segment, $x = y = z = t$, $0 \le t \le 1$.
(b) C_2: the parabolic arc, $x = y = t$, $z = t^2, 0 \le t \le 1$.

Solution: We have $\bar{F} = 5z\hat{i} + xy\hat{j} + x^2z\hat{k}$. The line integral from the initial point $A(0, 0, 0)$ to the terminal point $B(1, 1, 1)$ is

$$\int_A^B \bar{F}.d\bar{r} = \int_A^B (5z\,dx + xy\,dy + x^2z\,dz) \qquad ...(9.6)$$

(a) Along the path $C_1: x = y = z = t$, $0 \le t \le 1$. Thus, (9.6) becomes

$$\int_A^B \bar{F}.d\bar{r} = \int_0^1 (5t\,dt + t^2dt + t^3dt) = \int_0^1 (5t + t^2 + t^3)dt$$

$$= \left[\frac{5t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4}\right]_0^1 = \frac{37}{12}.$$

(b) Along the path C_2: $x = y = t$, $z = t^2$, $0 \le t \le 1$. Thus (9.6) becomes

$$\int_A^B \bar{F}.d\bar{r} = \int_0^1 (5t^2dt + t^2dt + 2t^5dt) = \int_0^1 (6t^2 + 2t^5)dt = \left[2t^3 + \frac{t^6}{3}\right]_0^1 = \frac{7}{3}.$$

We observe that the two results are different, although the end points are the same. This shows that, *the value of a line integral, in general, depends not only on $\bar{F}$ and on the endpoints A, B, of the path but also on the path along which we integrate from the initial point A to the final point B.*

Next we find condition which ensures that the value of the line integral is independent of the path, and this aspect is of great physical applications. For instance, in mechanics, independence of path may mean that we have to do the same amount of work regardless of the path to the mountain top, be it short and steep, or long and smooth. But we must note that not all forces are of this type. We have the following result.

Theorem 9.1: (Independence of Path) *A line integral $\int_C \bar{F}.d\bar{r}$ is independent of path in domain D if, and only if $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ with continuous F_1, F_2, F_3 in D is the gradient of some function f in D, that is, if $\bar{F} = \text{grad}\, f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$.*

Proof. Let $\vec{F} = \text{grad} f$ for some function f in D. Consider C to be any path in D from point A to point B, both A and B being arbitrary, given by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, where t, $a \le t \le b$, is the parameter. Consider

$$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = \int_A^B \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt = \int_a^b \frac{df}{dt}dt = [f(x(t), y(t), z(t))]_{t=a}^{t=b} = f(B) - f(A).$$

Thus the value of the integral is simply the difference of the value of f at the two endpoints of C and is therefore independent of the path C.

To prove the converse part, suppose that the line integral is independent of the path in the domain D. Choose a fixed point $A(x_0, y_0, z_0)$ in D and an arbitrary point $B(x, y, z_,)$ in D, and define function $f(x, y, z)$ as

$$f(x, y, z) = k + \int_A^B (F_1dx' + F_2dy' + F_3dz'), \qquad \text{...(9.7)}$$

where k is a constant and C' is any path from A to the arbitrary point B in D, as shown in Fig. 9.3.

Since, the line integral is independent of the path and further A is a fixed point, thus (9.7) depends only on the co-ordinates x, y, z of the point B and hence defines a function $f(x, y, z)$.

Next, because of independence of path, we may integrate along the path A to $B_1(x_1, y, z)$ and then parallel to the x-axis along B_1B as shown in the Fig. 9.3, then

$$f(x, y, z) = k + \int_A^{B_1} (F_1dx' + F_2dy' + F_3dz')$$

$$+ \int_{B_1}^{B} (F_1dx' + F_2dy' + F_3dz') \qquad \text{...(9.8)}$$

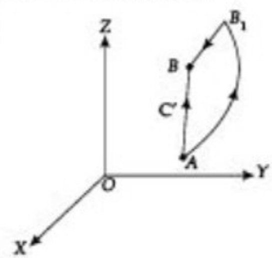


Fig. 9.3

The first integral on the right of (9.8) does not depend on x since $A(x_0, y_0, z_0)$ is fixed and $B_1(x_1, y, z)$ also does not depend on x, and the second integral does not depend on y and z, since the path B_1B is parallel to x-axis, therefore $dy' = dz' = 0$, and hence the second integral on the right of (9.8) can be written as $\int_{x_1}^{x} F_1(x', y, z)dx'$.

Taking partial derivative with respect to x on both sides of (9.8) gives

$$\frac{\partial f}{\partial x} = F_1. \qquad ...(9.9)$$

Similarly, choosing suitable paths of integration, we obtain

$$\frac{\partial f}{\partial y} = F_2, \text{ and } \frac{\partial f}{\partial z} = F_3.$$

Thus, $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \text{grad } f$, and this completes the proof.

We have seen that if the line integral $\int_C \bar{F}.d\bar{r}$ is independent of path, then $F = \text{grad } f$. In such a situation we say that the vector field $\bar{F}$ is a *gradient field* and the function f is called the *potential function for* $\bar{F}$, and such a force field where the work done by the force $\bar{F}$ in moving a particle is independent of the path and depends only on the endpoints is called a *conservative* field. We must note that if $\bar{F}$ is a conservative force field, then the work done along any simple closed path is zero, and further, to verify that the field $\bar{F}$ is conservative, we need to show that curl $\bar{F} = 0$.

Example 9.11: Show that the work done by the force $\bar{F} = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$ in moving a particle from the point $A(0, 0, 0)$ to the point $B(2, 2, 2)$ is independent of the path and find its value also.

Solution: Here, $\bar{F} = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$. We can check very easily that curl $\bar{F} = \mathbf{0}$. Hence $\bar{F}$ is a conservative field, and say $\bar{F} = \text{grad } f$, for some scalar function f. Thus we have

$$2x\hat{i} + 2y\hat{j} + 4z\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

This gives, $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 4z$.

Integrating these we obtain, $f = x^2 + \phi_1(y, z)$, $f = y^2 + \phi_2(z, x)$, and $f = 2z^2 + \phi_3(x, y)$, hence f is given by $f = x^2 + y^2 + 2z^2$.

Now since $\bar{F}$ is a conservative field, thus the work done by $\bar{F}$ in moving a particle from $A(0,0,0)$ to $B(2, 2, 2)$ is independent of the path, and is given by

$$\int_A^B \bar{F}.d\bar{r} = \int_{A(0,0,0)}^{B(2,2,2)} (2x\,dx + 2y\,dy + 4z\,dz) = \int_{A(0,0,0)}^{B(2,2,2)} d(x^2 + y^2 + 2z^2) = \left[x^2 + y^2 + 2z^2\right]_{A(0,0,0)}^{B(2,2,2)} = 16.$$

Example 9.12: Show that $\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$ is independent of any path of integration which does not pass through the origin.

Solution: Let $\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = \int_C \bar{F}\,d\bar{r}$, where $\bar{F} = \frac{x\hat{i}}{\sqrt{x^2+y^2}} + \frac{y\hat{j}}{\sqrt{x^2+y^2}}$, and $d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$.

Consider
$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad ...(9.10)$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \qquad ...(9.11)$$

Integrating (9.10) w.r.t x, we have

$$f = \sqrt{x^2+y^2} + k(y), \qquad ...(9.12)$$

where $k(y)$ is a constant depending upon y only. Differentiating (9.12) partially w.r.t. y and comparing with (9.11), we obtain $k'(y) = 0$, or $k(y) = A$, a constant.

Thus $f(x, y, z) = \sqrt{x^2+y^2} + A$. Hence $\bar{F}$ can be expressed as $\bar{F} = \text{grad } f$. Thus the vector field $\bar{F}$ is conservative and hence the given integral is independent of the path of integration which does not pass through the origin.

Example 9.13: If C is a simple closed curve in the xy-plane not enclosing the origin show that $\int_C \bar{F}\,d\bar{r} = 0$, where $\bar{F} = \frac{y\hat{i} - x\hat{j}}{x^2+y^2}$.

Solution: Here,
$$\bar{F} = \frac{y}{x^2+y^2}\hat{i} - \frac{x}{x^2+y^2}\hat{j}.$$

Consider
$$\frac{\partial f}{\partial x} = \frac{y}{x^2+y^2}, \qquad ...(9.13)$$

$$\frac{\partial f}{\partial y} = -\frac{x}{x^2+y^2}. \qquad ...(9.14)$$

Integrating (9.13) w.r.t. x, we have

$$f = \tan^{-1}\frac{x}{y} + k(y). \qquad ...(9.15)$$

Differentiating (9.15) w.r.t. y, we have

$$\frac{\partial f}{\partial y} = -\frac{x}{x^2+y^2} + k'(y).$$

Comparing it with (9.14), we obtain $k'(y) = 0$, that is, $k(y) = A$, a constant.

Thus $f(x, y, z) = \tan^{-1}\frac{x}{y} + A$. Hence, $\vec{F}$ can be expressed as $\vec{F} = \text{grad} f$. Thus, the vector field $\vec{F}$ is conservative and hence the given integral is independent of the path of integration and depends only on the endpoints. Here, since C is a simple closed curve then endpoints coincide with each other and hence $\int_C \vec{F} . d\vec{r} = 0$.

EXERCISE 9.2

1. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \vec{F} . d\vec{r}$, where C is the curve in the xy-plane $y = 2x^2$ from (0, 0) to (1, 2).
2. Evaluate the line integral $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$, where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.
3. Find the work done by the force $\vec{F} = x\hat{i} - z\hat{j} + 2y\hat{k}$ in the displacement along the closed path C consisting of the segments C_1, C_2 and C_3, where C_1: $0 \le x \le 1, y = x, z = 0$; C_2: $0 \le z \le 1, x = 1, y = 1$; C_3: $1 \ge x \ge 0, \quad y = z = x$.
4. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along, (a) the straight line from (0, 0, 0) to (2, 1, 3). (b) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.
5. If $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$, evaluate $\int_C \vec{f} \times d\vec{r}$ along the curve $x = \cos t, y = \sin t, z = 2\cos t$ from $t = 0$ to $t = \pi/2$.
6. Show that the line integral $\int_C (3x^2dx + 2yzdy + y^2dz)$ is independent of the path in any domain in space and find its value if C has the initial point $A(0, 1, 2)$ and terminal point $B(1 - 1, 7)$.
7. If $\vec{F} = y\hat{i} + x\hat{j} + xyz^2\hat{k}$, evaluate $\int_C \vec{F} . d\vec{r}$, where C is the circle $x^2 + y^2 - 2y = 2, z = 1$ going around once in the anti-clockwise direction.
8. Show that the line integral $\int_A^B [(1 - \sin x \sin y)dx + (1 + \cos x \cos y)dy]$ is independent of path of integration, and also evaluate it from $P(\pi/4, \pi/4)$ to $Q(\pi/2, 0)$.
9. Find whether the vector field $\vec{F} = \cosh(x + y)(\hat{i} + \hat{j})$ is conservative. If it is so, find the potential function.

9.3 SURFACE AND SURFACE INTEGRALS

After discussing line integrals in the preceding section, we turn to surface integrals here, in which we integrate over surface in space. We will refer *surface*, also for a *portion of a surface*, for example, a portion of a sphere, or of a cylinder, etc.

A surface S in the xyz-space is represented by $z = g(x, y)$ or $f(x, y, z) = 0$.

For example, $z = \sqrt{a^2 - x^2 - y^2}$, or $x^2 + y^2 + z^2 - a^2 = 0, z \geq 0$, represents a semi-sphere of radius a with centre at $(0, 0, 0)$.

Since the surfaces are two-dimensional, so to represent a surface parametrically we need two parameters say, u and v. Thus, a parametric representation of a surface S in space is of the form

$$\bar{R}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k},$$

where u, v belong to some region D in the uv-plane.

Further the continuous functions $u = u(t)$ and $v = v(t)$ of a real parameter t represent a curve C on the surface S.

For example, a parametric representation of the sphere $x^2 + y^2 + z^2 = a^2$ is

$$\bar{R}(u, v) = a \cos v \cos u\hat{i} + a \cos v \sin u\hat{j} + a \sin v\hat{k},$$

where $0 \leq u \leq 2\pi, -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$.

Another parametric representation of the sphere is

$$\bar{R}(u, v) = a \cos u \sin v\hat{i} + a \sin u \sin v\hat{j} + a \cos v\hat{k},$$

where $0 \leq u \leq 2\pi$, $0 \leq v \leq \pi$.

9.3.1 Surface Normal Vector

To define a surface integral, we need the concept of *surface normal vector*.

A normal vector of a surface S at a point P is a vector perpendicular to the tangent plane of S at P.

Since a surface S is given by $\bar{R} = \bar{R}(u, v)$, $u = u(t)$, $v = v(t)$, differentiating w.r.t. t, we get

$$\frac{d\bar{R}}{dt} = \frac{\partial \bar{R}}{\partial u}\frac{du}{dt} + \frac{\partial \bar{R}}{\partial v}\frac{dv}{dt}. \quad ...(9.16)$$

The vectors $\frac{\partial \bar{R}}{\partial u}$ and $\frac{\partial \bar{R}}{\partial v}$ are tangential to S at P, we assume that these are linearly independent and so determine the tangent plane of S at P. Then their cross-product gives a normal vector $\bar{N}$ of S at P, that is, $\bar{N} = \frac{\partial \bar{R}}{\partial u} \times \frac{\partial \bar{R}}{\partial v}$, and so, the corresponding unit normal vector $\hat{N}$ of S at P, is given by

$$\hat{N} = \frac{\dfrac{\partial \bar{R}}{\partial u} \times \dfrac{\partial \bar{R}}{\partial v}}{\left|\dfrac{\partial \bar{R}}{\partial u} \times \dfrac{\partial \bar{R}}{\partial v}\right|} \quad ...(9.17)$$

Also we know that if the surface S is represented by $f(x, y, z) = 0$, then $\hat{N} = \dfrac{\text{grad } f}{|\text{grad } f|}$ is the unit outward normal to S.

A surface S is called a *smooth surface* if its surface normal vector depends continuously on the points of S. Further, S is called *piecewise smooth* if it consists of finitely many smooth surfaces. For example, a sphere is a smooth surface and a cube is a piecewise smooth surface.

9.3.2 The Surface Integral

Consider a piecewise-smooth surface S given by a parametric representation $\bar{R}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$ and let $\hat{N}$ be the unit outward normal vector at a point P to S. For a given vector function $\bar{F}(\bar{R})$, we define the *surface integral* over S by

$$\iint_S \bar{F}.\bar{dS} = \iint_S \bar{F}.\hat{N}\, dS \quad ...(9.18)$$

We note that integrand in (9.18) is a scalar being the dot product $\bar{F}.\hat{N}$, the normal component of $\bar{F}$.

Next, if $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\hat{N} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$, where α, β, γ are the angles between $\hat{N}$ and the positive directions of the co-ordinate axes, then

$$\iint_S \bar{F}.\hat{N}\, dS = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dS \quad ...(9.19)$$

Two other types of surface integrals are $\iint_S \bar{F} \times \overline{dS}$, and $\iint_S f\overline{dS}$, which are both vectors.

The surface integrals $\iint_S \bar{F}.\overline{dS}$ arise naturally in flow problems. If $\bar{F} = \rho\bar{v}$, where ρ is the density of the fluid and $\bar{v}$ is the velocity of flow, then the surface integral gives *the total outward flux across the surface S*. When the flux of $\bar{F}$ across every closed surface S in a region D vanishes, then $\bar{F}$ is said to be a *solenoidal vector point function* in E.

It may be noted that $\bar{F}$ may well be taken for gravitational force, electric force, magnetic force, etc.

9.3.3 Evaluation of the Surface Integral

To evaluate surface integrals it is, in general, convenient to express them as double integrals taken over the orthogonal projection of S on one of the co-ordinate planes, say in the xy-plane, as shown in Fig. 9.4.

Let $\gamma(<\pi/2)$ be the angle which the unit normal vector $\hat{N}$ makes with the positive direction of the z-axis, then

$$|\hat{N}.\hat{k}| = |\hat{N}|\,|\hat{k}|\cos\gamma = \cos\gamma$$

Also $\delta x\,\delta y$ = projection of ΔS on the xy-plane = $\Delta S\cos\gamma$.

Thus, $$\Delta S = \frac{\delta x\,\delta y}{\cos\gamma} = \frac{\delta x\,\delta y}{|\hat{N}.\hat{k}|}.$$

Hence, $$\iint_S \vec{F}.\hat{N}\,dS = \iint \vec{F}.\hat{N}\frac{dx\,dy}{|\hat{N}.\hat{k}|}, \quad ...(9.20)$$

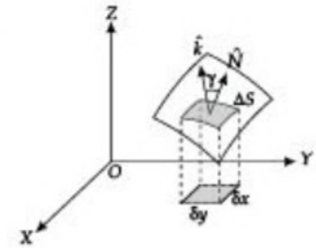


Fig. 9.4

where the integration on the right hand side of (9.20) is to be taken over the orthogonal projection of S on the xy-plane.

Another way of evaluating the surface integral is given as follows:

$$\iint_S \vec{F}.\hat{N}\,dS = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dS = \iint_S (F_1 dy\,dz + F_2 dz\,dx + F_3 dx\,dy) \quad ...(9.21)$$

using $\Delta S = \dfrac{\delta x\,\delta y}{\cos\gamma}$, etc.

Example 9.14: Evaluate $\iint_S \vec{F}.\vec{dS}$, where $\vec{F} = x\hat{i} + (z^2 - zx)\hat{j} - xy\hat{k}$ and S is the triangular surface with vertices (2, 0, 0), (0, 2, 0), and (0, 0, 4).

Solution: Equation of the triangular surface S is $\dfrac{x}{2} + \dfrac{y}{2} + \dfrac{z}{4} = 1$, or $2x + 2y + z = 4$.

A vector normal to the surface S is $\nabla(2x + 2y + z) = 2\hat{i} + 2\hat{j} + \hat{k}$.

Therefore $\hat{N}$, the unit vector normal to surface $S = \dfrac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{4+4+1}} = \dfrac{2}{3}\hat{i} + \dfrac{2}{3}\hat{j} + \dfrac{1}{3}\hat{k}$.

Also $\hat{k}.\hat{N} = \hat{k}.\left(\dfrac{2}{3}\hat{i} + \dfrac{2}{3}\hat{j} + \dfrac{1}{3}\hat{k}\right) = \dfrac{1}{3}$. Thus, $\iint_S \vec{F}.\vec{dS} = \iint_E \vec{F}.\hat{N}\dfrac{dx\,dy}{|\hat{k}.\hat{N}|}$,

where E is the projection of S on the xy-plane which is a triangle OAB bounded by x-axis, y-axis and the line $x + y = 2$, as shown in Fig. 9.5.

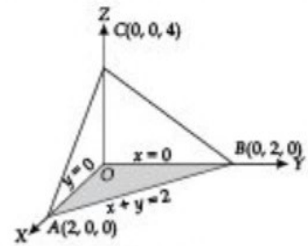


Fig. 9.5

Consider $\vec{F}.\hat{N} = [x\hat{i} + (z^2 - zx)\hat{j} - xy\hat{k}].\left(\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)$

$$= \frac{2}{3}x + \frac{2}{3}(z^2 - zx) - \frac{1}{3}xy$$

$$= \frac{2}{3}x + \frac{2}{3}[(4-2x-2y)^2 - (4-2x-2y)x] - \frac{1}{3}xy$$

(on the plane, $2x + 2y + z = 4$)

$$= \frac{1}{3}[2x + 2(4-2x-2y)(4-3x-2y) - xy]$$

$$= \frac{1}{3}[32 - 38x - 32y + 19xy + 12x^2 + 8y^2].$$ Thus

$$\iint_S \vec{F}.\hat{N}dS = \iint_E \vec{F}.\hat{N}\frac{dxdy}{|\hat{k}.\hat{N}|}$$

$$= \int_0^2\int_0^{2-x} (32 - 38x - 32y + 19xy + 12x^2 + 8y^2)dx\,dy$$

$$= \int_0^2 \left[32y - 38xy - 16y^2 + \frac{19}{2}xy^2 + 12x^2y + \frac{8}{3}y^3\right]_0^{2-x}$$

$$= \int_0^2 \left[32(2-x) - 38x(2-x) - 16(2-x)^2 + \frac{19}{2}x(2-x)^2 + 12x^2(2-x) + \frac{8}{3}(2-x)^3\right]dx$$

$$= \int_0^2 \left(\frac{64}{3} - 38x + 24x^2 - \frac{31x^3}{6}\right)dx$$

$$= \left[\frac{64}{3}x - 19x^2 + 8x^3 - \frac{31}{24}x^4\right]_0^2$$

$$= \frac{128}{3} - 76 + 64 - \frac{62}{3} = 10.$$

Example 9.15: If $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, evaluate $\iint_S \vec{F}.\hat{N}dS$, where S is the surface of the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution: A cube is a piecewise smooth surface consisting of six smooth surfaces as shown in Fig. 9.6. Therefore the given integral has to be calculated over the six faces of cube. For the face $AC'PB'$, $x = 1$, $\hat{N} = \hat{i}$ and $dS = dydz$, thus

$$\iint_{AC'PB'} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).\hat{i}\,dydz$$

$$= 4\int_0^1\int_0^1 z\,dydz = 4\left(\int_0^1 zdz\right)\int_0^1 dy = 2.$$

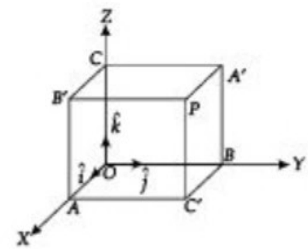


Fig. 9.6

For the face $OBA'C$, $x = 0$, $\hat{N} = -\hat{i}$ and $dS = dydz$, thus

$$\iint_{OBA'C} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 ([4(0)z\hat{i} - y^2\hat{j} + yz\hat{k}].(-\hat{i})dydz = \int_0^1\int_0^1 0dydz = 0.$$

For the face $BC'PA'$, $y = 1$, $\hat{N} = \hat{j}$ and $dS = dxdz$, thus

$$\iint_{BC'PA'} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 [4xz\hat{i} - 1^2\hat{j} + 1z\hat{k}].\hat{j}dxdz = -\int_0^1\int_0^1 dxdz = -1.$$

For the face $OAB'C$, $y = 0$, $\hat{N} = -\hat{j}$ and $dS = dxdz$, thus

$$\iint_{OAB'C} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 [4xz\hat{i} - (0)\hat{j} + (0)z\hat{k}](-\hat{j})dxdz = 0.$$

For the face $CB'PA'$, $z = 1$, $\hat{N} = \hat{k}$ and $dS = dxdy$, thus

$$\iint_{CB'PA'} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 [4x(1)\hat{i} - y^2\hat{j} + y(1)\hat{k}].\hat{k}\,dxdy = \int_0^1\int_0^1 ydxdy = \frac{1}{2}.$$

For the face $OAC'B$, $z = 0$, $\hat{N} = -\hat{k}$ and $ds = dx\,dy$, thus

$$\iint_{OAC'B} \vec{F}.\hat{N}dS = \int_0^1\int_0^1 [4x(0)\hat{i} - y^2\hat{j} + y(0)\hat{k}].(-\hat{k})dxdy = 0.$$

Adding all these, $\iint_S \vec{F}.\hat{N}dS = 2 + 0 - 1 + 0 + \frac{1}{2} + 0 = 3/2.$

Example 9.16: Evaluate $\iint_S \vec{F}.\hat{N}dS$, where $\vec{F} = z^2\hat{i} + xy\hat{j} - y^2\hat{k}$ and S is the portion of the surface of the cylinder $x^2 + y^2 = 36$, $0 \le z \le 4$ included in the first octant.

Solution: Equation of surface S is $x^2 + y^2 - 36 = 0$. A vector normal to this surface is $\nabla(x^2 + y^2 - 36)$ $= 2x\hat{i} + 2y\hat{j}$. Therefore $\hat{N}$, a unit vector normal to a point (x, y, z) of S is

$$\hat{N} = \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2+y^2}} = \frac{1}{6}(x\hat{i} + y\hat{j}), \text{ since } x^2+y^2 = 36 \text{ on } S.$$

As shown in Fig. 9.7, the projection of S on xy–plane cannot be considered. We consider the projection of S on the yz-plane, which is a rectangle with sides of lengths 6 and 4.

We have $dS = \frac{dydz}{|\hat{N}.\hat{i}|} = \frac{dydz}{x/6}$, and

$$\bar{F}.\hat{N} = \frac{1}{6}(z^2\hat{i} + xy\hat{j} - y^2\hat{k})\cdot(x\hat{i} + y\hat{j}) = \frac{1}{6}(z^2+y^2)x.$$

Thus, $$\iint_S \bar{F}.\hat{N}dS = \int_{z=0}^{4}\left[\int_{y=0}^{6}(y^2+z^2)dy\right]dz = \int_0^4 \left(\frac{y^3}{3}+yz^2\right)_0^6 dz$$

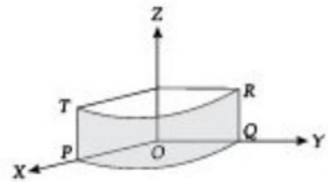


Fig. 9.7

$$= \int_0^4 (72+6z^2)dz = \left[72z+2z^3\right]_0^4 = 416.$$

Example 9.17: Evaluate $\iint_S \bar{F}.\hat{N}\,dS$, where $\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is the portion of the surface of the sphere $x^2+y^2+z^2=1$ in the first octant.

Solution: Equation of the surface S is $x^2+y^2+z^2-1=0$. A vector normal to S is $\nabla(x^2+y^2+z^2-1)$ $= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$. Therefore $\hat{N}$, a unit vector normal to any point (x, y, z) of S is

$$\hat{N} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\hat{i} + y\hat{j} + z\hat{k},$$

since $x^2+y^2+z^2=1$ on S.

The projection of S on the xy-plane is the quadrant E of the circle $x^2+y^2=1$ bounded by the lines $x=0$ and $y=0$. We have,

$$dS = \frac{dxdy}{|\hat{N}.\hat{k}|} = \frac{dxdy}{z}, \text{ and } \bar{F}.\hat{N} = (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = 3xyz.$$

Thus $$\iint_S \bar{F}.\hat{N}dS = 3\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} xy\,dxdy = 3\int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{3}{2}\int_0^1 y(1-y^2)dy = \frac{3}{2}\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1 = \frac{3}{2}\left[\frac{1}{2}-\frac{1}{4}\right] = \frac{3}{8}.$$

Example 9.18: If $\bar{r}$ denotes the position vector of any point (x, y, z) measured from the origin, then evaluate $\iint_S \frac{\bar{r}}{|\bar{r}|^3} d\bar{S}$, where S is the surface of the sphere of radius a with centre at the origin.

Solution: Equation of the surface S is $x^2+y^2+z^2-a^2=0$. A vector normal to S is $\nabla(x^2+y^2+z^2-a^2)$ $=2x\hat{i}+2y\hat{j}+2z\hat{j}$. Therefore, $\hat{N}$ a unit vector normal to a point (x, y, z) of S, is

$$\hat{N}=\frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}=\frac{1}{a}(x\hat{i}+y\hat{j}+z\hat{k}), \text{ since } x^2+y^2+z^2=a^2 \text{ on } S.$$

Also $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$, therefore, $|\bar{r}|=\sqrt{x^2+y^2+z^2}=a$, since $x^2+y^2+z^2=a^2$ on S.

Let $\bar{F}=\frac{\bar{r}}{|\bar{r}|^3}=\frac{\bar{r}}{a^3}=\frac{1}{a^3}(x\hat{i}+y\hat{j}+z\hat{k})$. Then

$$\bar{F}.\hat{N}=\frac{1}{a^3}(x\hat{i}+y\hat{j}+z\hat{k}).\frac{1}{a}(x\hat{i}+y\hat{j}+z\hat{k})=\frac{x^2+y^2+z^2}{a^4}=\frac{a^2}{a^4}=\frac{1}{a^2}.$$

Thus, $\iint_S \bar{F}.d\vec{S}=\frac{1}{a^2}\int_s\int ds=\frac{1}{a^2}4\pi a^2=4\pi$, since surface area of the sphere is $S=4\pi a^2$.

EXERCISE 9.3

1. Evaluate $\iint_S \bar{F}.\hat{N}dS$, where $\bar{F}=2x^2y\hat{i}-y^2\hat{j}+4xz^2\hat{k}$ and S is the closed surface of the region in the first octant bounded by the cylinder $y^2+z^2=9$ and the planes $x=0, x=2, y=0$ and $z=0$.
2. Evaluate $\iint_S \bar{F}.\hat{N}dS$, where $\bar{F}=6z\hat{i}-4\hat{j}+y\hat{k}$ and S is the portion of the plane $2x+3y+6z$ $=12$ in the first octant.
3. Evaluate $\iint_S \bar{F}.\hat{N}dS$, where $\bar{F}=4x\hat{i}-2y^2\hat{j}+z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4, z=0, z=3$.
4. If $\bar{r}$ is the position vector of any point (x, y, z) measured from the origin, then evaluate $\iint_S \bar{r}.d\bar{S}$ where S is that part of the surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, lying above the plane $z=0$.

5. If $\bar{F} = y\hat{i} + (x-2xz)\hat{j} - xy\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2, 0 \le x, y, z \le a$, evaluate $\iint_S \bar{F}.\hat{N}dS$, where $\hat{N}$ is a unit vector along the outward normal to the surface.

6. If $\bar{F} = y\hat{i} + (x-2xz)\hat{j} - xy\hat{k}$, evaluate $\iint_S (\nabla \times \bar{F})\cdot\hat{N}dS$, where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane.

9.4 GREEN'S THEOREM IN THE PLANE

Green's theorem provides a relationship between a double integral over a region E in a plane and the line integral over the closed curve C bounding E. This is of practical interest because it sometimes helps to make the evaluation of an integral easier. The theorem is stated as follows.

Theorem 9.2: (Green's Theorem) *Let E be a plane region in the xy-plane bounded by a closed curve C. If $f(x, y), g(x, y), \frac{\partial f}{\partial y}$ and $\frac{\partial g}{\partial x}$ are continuous on E, then*

$$\iint_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dxdy = \oint_C [f(x, y)dx + g(x, y)dy], \quad ...(9.22)$$

the integration being carried in the counter-clockwise direction of C.

Proof. We prove Green's theorem for a special region E bounded by a closed curve C which is cut by any line parallel to the axes at the most in two points.

Let E be represented by $u_1(x) \le y \le v_1(x), a \le x \le b$, as shown in Fig. 9.8a.

$$\iint_E \frac{\partial f}{\partial y}dxdy = \int_a^b \left[\int_{u_1(x)}^{v_1(x)} \frac{\partial f}{\partial y}dy\right]dx = \int_a^b [f(x, y)]_{u_1(x)}^{v_1(x)} dx$$

$$= \int_a^b [f(x, v_1(x)) - f(x, u_1(x))]dx = -\int_b^a f(x, v_1(x))dx - \int_a^b f(x, u_1(x))dx$$

Since $y = v_1(x)$ represents the curve C^{**} and $y = u_1(x)$ represents the curve C^*, thus

$$\iint_E \frac{\partial f}{\partial y}dxdy = -\int_{C^{**}} f(x, y)dx - \int_{C^*} f(x, y)dx$$

$$= -\oint_C f(x, y)dx. \quad ...(9.23)$$

Similarly, it can be shown that for the region E represented by $E: u_2(y) \le x \le u_2(y), c \le y \le d$, refer Fig. 9.8b, we have

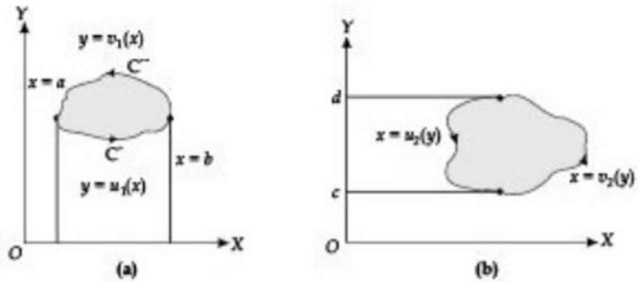


Fig. 9.8

$$\iint_E \frac{\partial g}{\partial x} dxdy = \oint_C g(x, y)dy. \quad ...(9.24)$$

(9.23) and (9.24) together give (9.22); this proves Green's theorem for the special regions.

The Green's theorem can be extended to a region E that itself is not a special region but can be subdivided into finitely many special regions E_1, E_2 etc. Such that the boundary of each is cut at the most in two points by any line parallel to either axis, refer to Figs. 9.9 (a) & (b). We apply the theorem to each subregion and then add the results; the left-hand members add up to the integral over E while the right-hand side members add up to the line integral over C plus integrals over the curves introduced for subdividing E. These additional integrals over the common boundaries cancel each other, for each is covered twice but in opposite directions and we are left with remaining line integrals which combine to form the line integral over the external curve C.

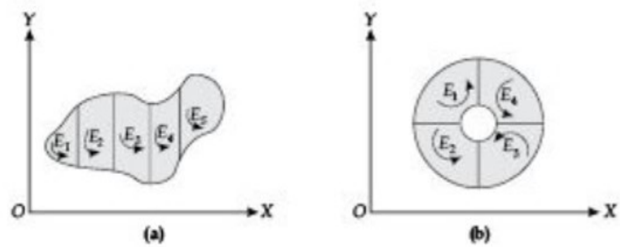


Fig. 9.9

9.4.1 Area of Plane Curves Using Green's Theorem

Green's theorem can be applied to find area of a plane region as a line integral over its boundary, e.g., for $f = 0$ and $g = x$, we have from (9.22) $\iint_E dxdy = \oint_C x\,dy$, and then, for $f = -y$, $g = 0$, again (9.22) gives $\iint_E dx\,dy = -\oint_C y\,dx$. Hence the area A of the region E is

$$A = \iint_E dx\,dy = \frac{1}{2}\oint_C (x\,dy - y\,dx), \qquad ...(9.25)$$

which has been expressed in terms of a line integral over the boundary.

For an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have $x = a\cos\theta$, $y = b\sin\theta$, and thus $dx = -a\sin\theta\,d\theta$, $dy = b\cos\theta\,d\theta$. Using (9.25) the area

$$A = \frac{1}{2}\int_0^{2\pi} [(a\cos\theta)(b\cos\theta) + (b\sin\theta)(a\sin\theta)]\,d\theta = \frac{1}{2}ab\int_0^{2\pi} d\theta = \pi ab.$$

In terms of polar co-ordinates, for $x = r\cos\theta$, $y = r\sin\theta$, we have $dx = \cos\theta\,dr - r\sin\theta\,d\theta$ and $dy = \sin\theta\,dr + r\cos\theta\,d\theta$. Substituting in (9.25), we get

$$A = \frac{1}{2}\oint_C r^2 d\theta, \qquad ...(9.26)$$

a familiar result from calculus, used to find the area enclosed by polar curves.

Example 9.19: Verify Green's theorem for $\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where C is the boundary of the region bounded by $x = 0$, $y = 0$ and $x + y = 1$.

Solution: By Green's theorem $\int_C (f dx + g dy) = \iint_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$, where E is the plane region enclosed by the closed curve C.

Here $f = 3x^2 - 8y^2$ and $g = 4y - 6xy$, and

$$\oint_C (f dx + g dy) = \int_{C_1} (f dx + g dy) + \int_{C_2} (f dx + g dy) + \int_{C_3} (f dx + g dy),$$

as shown in Fig. 9.10.

Along C_1, $y = 0$ and x varies from 0 to 1, thus $\int_{C_1} (f dx + g dy) = 3\int_0^1 x^2 dx = 1$

Along C_2, $y = 1 - x$ and x varies from 1 to 0, thus

$$\int_{C_2} (f dx + g dy) = \int_1^0 [\{3x^2 - 8(1-x)^2\}dx + \{4(1-x) - 6x(1-x)\}(-dx)]$$

$$= \int_0^1 (12 - 26x + 11x^2)dx = \left[12x - 13x^2 + \frac{11x^3}{3}\right]_0^1 = \frac{8}{3}.$$

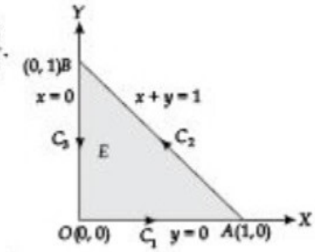


Fig. 9.10

Along C_3, $x = 0$ and y varies form 1 to 0, thus

$$\int_{C_3} (fdx + gdy) = 4\int_1^0 ydy = 4\left[\frac{y^2}{2}\right]_1^0 = -2.$$

Therefore, $$\int_C (fdx + gdy) = 1 + \frac{8}{3} - 2 = \frac{5}{3}. \qquad ...(9.27)$$

Now, $$\iint_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\,dy = \iint_E \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right]dx\,dy$$

$$= \int_0^1\int_0^{1-y} (-6y + 16y)dx\,dy = 10\int_0^1 (1-y)y\,dy = 5/3. \qquad ...(9.28)$$

Green's theorem is verified from the equality of (9.27) and (9.28).

Example 9.20: Apply Green's theorem to evaluate

$$\int_C [\sin y\,dx + x(1 + \cos y)dy], \text{ where } C \text{ is the closed path given by } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution: By Green's theorem the given line integral is equal to

$$\iint_E \left[\frac{\partial}{\partial x}x(1 + \cos y) - \frac{\partial}{\partial y}(\sin y)\right]dx\,dy,$$

where E is the region enclosed by C. Thus,

$$\int_C [\sin y\,dx + x(1 + \cos y)dy] = \iint_E (1 + \cos y - \cos y)dx\,dy = \iint_E dx\,dy = \pi ab,$$

the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Example 9.21: Find the work done by the force $\vec{F} = (x^2 - y^3)\hat{i} + (x + y)\hat{j}$ in moving a particle along the closed path C containing the curve $x + y = 0$, $x^2 + y^2 = 16$, and $y = x$ in the first and the fourth quadrants.

$$= \int_0^1 (12 - 26x + 11x^2)dx = \left[12x - 13x^2 + \frac{11x^3}{3}\right]_0^1 = \frac{8}{3}.$$

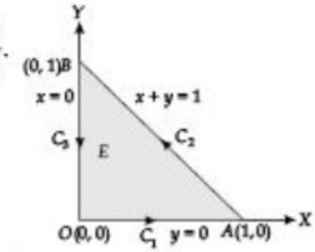


Fig. 9.10

Along C_3, $x = 0$ and y varies form 1 to 0, thus

$$\int_{C_3} (fdx + gdy) = 4\int_1^0 ydy = 4\left[\frac{y^2}{2}\right]_1^0 = -2.$$

Therefore, $$\int_C (fdx + gdy) = 1 + \frac{8}{3} - 2 = \frac{5}{3}. \qquad ...(9.27)$$

Now, $$\iint_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\,dy = \iint_E \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right]dx\,dy$$

$$= \int_0^1\int_0^{1-y} (-6y + 16y)dx\,dy = 10\int_0^1 (1-y)y\,dy = 5/3. \qquad ...(9.28)$$

Green's theorem is verified from the equality of (9.27) and (9.28).

Example 9.20: Apply Green's theorem to evaluate

$$\int_C [\sin y\,dx + x(1 + \cos y)dy], \text{ where } C \text{ is the closed path given by } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution: By Green's theorem the given line integral is equal to

$$\iint_E \left[\frac{\partial}{\partial x}x(1 + \cos y) - \frac{\partial}{\partial y}(\sin y)\right]dx\,dy,$$

where E is the region enclosed by C. Thus,

$$\int_C [\sin y\,dx + x(1 + \cos y)dy] = \iint_E (1 + \cos y - \cos y)dx\,dy = \iint_E dx\,dy = \pi ab,$$

the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Example 9.21: Find the work done by the force $\vec{F} = (x^2 - y^3)\hat{i} + (x + y)\hat{j}$ in moving a particle along the closed path C containing the curve $x + y = 0$, $x^2 + y^2 = 16$, and $y = x$ in the first and the fourth quadrants.

6. Show that the area of a polygon with vertices at (a_1, b_1), (a_2, b_2), ..., (a_n, b_n) taken in the anticlockwise direction is
$(1/2)[(a_1b_2 - a_2b_1) + (a_2b_3 - a_3b_2) + ... + (a_{n-1}b_n - a_nb_{n-1}) + (a_nb_1 - a_1b_n)]$.

7. Evaluate $\oint_C e^x(\sin y\,dx + \cos y\,dy)$, where C is the ellipse $4(x+1)^2 + 9(y-3)^2 = 36$.

9.5 STOKES' THEOREM: A GENERALIZATION OF GREEN'S THEOREM

We have seen the importance of the Green's theorem, next we consider another important theorem, the *Stokes' theorem*, which transforms line integrals into surface integrals, and conversely. Stokes' theorem generalizes Green's theorem in the sense that latter becomes a special case of the former.

The theorem is stated as follows.

Theorem 9.3: (Stokes' Theorem) *Let S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. If* $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ *be any continuously differentiable vector point function, then*

$$\oint_C \bar{F}.d\bar{r} = \iint_S curl\ \bar{F}.\hat{N}\,dS, \quad ...(9.29)$$

where $\hat{N} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$ *is a unit outward normal at any point of S, and C is traversed in positive direction.*

Proof: Writing $d\bar{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$, the Eq. (9.29) in components form is

$$\oint_C (F_1dx + F_2dy + F_3dz) = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right]dS. \quad ...(9.30)$$

We prove the result for a surface S that can be represented simultaneously in the forms

(a) $z = g(x, y)$, (b) $x = h(y, z)$,
(c) $y = k(z, x)$

where g, h, k are continuous functions and have continuous first order partial derivatives.

First we prove that

$$\oint_C F_1dx = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)dS. \quad ...(9.31)$$

Consider the case when the equation of the surface S is written in the form $z = g(x, y)$ and the projection of S on the xy-plane is the region E and the projection of C on the xy-plane is the curve C' enclosing the region E as shown in Fig. 9.12.

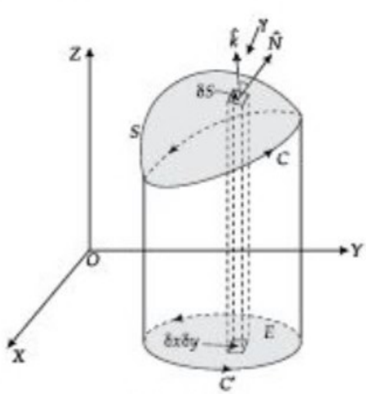


Fig. 9.12

In this case, we have

$$\oint_C F_1(x, y, z)dx = \oint_{C'} F_1[x, y, g(x, y)]dx$$
$$= \oint_{C'} (F_1[x, y, g(x, y)]dx + 0dy)$$
$$= -\iint_E \frac{\partial}{\partial y} F_1(x, y, g)dx\,dy.$$

using Green's theorem in plane.

Thus $$\int_C F_1 dx = -\iint_E \left[\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right]dx\,dy. \quad ...(9.32)$$

Next, the direction ratios of the normal to surface $z = g(x, y)$ are : $\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1$, and hence

$$\frac{\cos\alpha}{-\frac{\partial g}{\partial x}} = \frac{\cos\beta}{-\frac{\partial g}{\partial y}} = \frac{\cos\gamma}{1}, \text{ which gives, } \frac{\partial g}{\partial y} = -\frac{\cos\beta}{\cos\gamma}.$$

Also $dx\,dy$, the projection of dS on the xy-plane is $\cos\gamma dS$, and hence, (9.32) becomes

$$\oint_C F_1 dx = -\iint_S \left[\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma}\right]\cos\gamma dS = \iint_S \left[\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right]dS,$$

This proves (9.31).

Similarly, we can prove the corresponding expressions for F_2 and F_3 by assuming the representations $x = h(y, z)$ and $y = k(z, x)$ respectively for the surface S. Adding all these, we get the required result (9.30) and hence (9.29).

9.5.1 Green's Theorem as a Special Case of Stokes' Theorem

Let $\vec{F} = F_1\hat{i} + F_2\hat{j}$ be a vector function which is continuously differentiable in a domain in the xy-plane containing region S bounded by a closed curve C. Then

$$\oint_C \vec{F}.d\vec{r} = \oint_C (F_1\hat{i} + F_2\hat{j})\cdot(dx\hat{i} + dy\hat{j}) = \oint_C (F_1 dx + F_2 dy).$$

Also, curl $$\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & 0 \end{vmatrix} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}.$$

Here $\hat{N} = \hat{k}$, therefore, curl $\bar{F}.\hat{k} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$ and hence Stokes' theorem, refer to Eq. (9.29), takes the form $\oint_C (F_1 dx + F_2 dy) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$, which is Green's theorem in the plane, refer to Eq. (9.22).

Example 9.22: Verify Stokes' theorem for $\bar{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ in the rectangular region in the xy-plane given by (0, 0), $(a, 0)$, $(0, b)$ and (a, b).

Solution: Let $OACB$ be the given rectangle as shown in the Fig. 9.13.

Here, curl $\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\hat{k}$

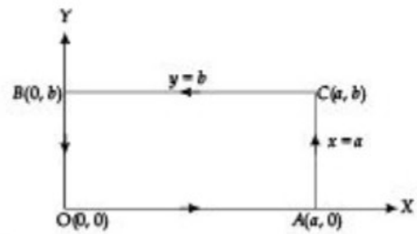


Fig. 9.13

and, curl $\bar{F}.d\bar{S} = (4y\hat{k})\cdot(\hat{k}\,dx\,dy) = 4y\,dx\,dy$. Therefore,

$$\iint_s \text{curl } \bar{F}.d\bar{S} = \int_0^a \int_0^b 4y\,dy\,dx = 2ab^2 \qquad ...(9.33)$$

We have, $\oint_{OACBO} \bar{F}.d\bar{r} = \int_{OA+AC+CB+BO} \bar{F}.d\bar{r}$, and

$$\bar{F}.d\bar{r} = [(x^2 - y^2)\hat{i} + 2xy\hat{j}]\cdot[\hat{i}\,dx + \hat{j}dy + \hat{j}\,dz] = (x^2 - y^2)dx + 2xy\,dy.$$

Thus, $\int_{OA} \bar{F}.d\bar{r} = \int_0^a x^2 dx = \frac{1}{3}a^3$, $\int_{AC} \bar{F}.d\bar{r} = \int_0^b 2ay\,dy = ab^2$

$$\int_{CB} \bar{F}.d\bar{r} = -\int_0^a (x^2 - b^2)dx = ab^2 - \frac{1}{3}a^3, \text{ and } \int_{BO} \bar{F}.d\bar{r} = 0.$$

Therefore, $\int_{OACBO} \bar{F}.d\bar{r} = \frac{1}{3}a^3 + ab^2 + ab^2 - \frac{1}{3}a^3 = 2ab^2.$ ...(9.34)

From the equality of (9.33) and (9.34), Stokes' theorem is verified.

Example 9.23: Verify Stokes' theorem for the field $\vec{F} = (2x-y)\,\hat{i} - yz^2\,\hat{j} - y^2z\,\hat{k}$ over the upper half surface of $x^2+y^2+z^2=1$ bounded by its projection on the xy-plane.

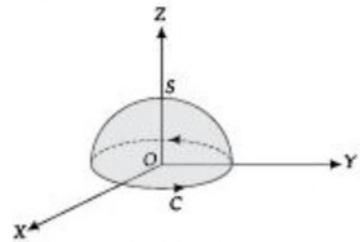


Fig. 9.14

Solution: The bounding curve C of the surface S, $x^2+y^2+z^2=1$, as shown in Fig. 9.14, in the xy-plane is $x^2+y^2=1$, $z=0$.

The equation of the curve in parametric form is

$$x=\cos\theta,\quad y=\sin\theta,\quad z=0,\quad 0\le\theta\le 2\pi.$$

Also
$$\oint_C \vec{F}.d\vec{r} = \oint_C [(2x-y)dx - yz^2dy - y^2zdz]$$

$$= \oint_C (2x-y)dx,\quad (z=0)$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)d\theta = \int_0^{2\pi} (-\sin 2\theta + \sin^2\theta)\,d\theta \qquad ...(9.35)$$

$$= \int_0^{2\pi}\left[-\sin 2\theta + \frac{1}{2}(1-\cos 2\theta)\right]d\theta = \left[\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \pi.$$

Next, curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = (-2yz+2yz)\,\hat{i} + (0-0)\,\hat{j} + (0+1)\,\hat{k} = \hat{k}.$

Therefore, curl $\vec{F}.\hat{N} = \hat{k}.\hat{N}$, where $\hat{N}$ is the unit outward normal to the surface S.

If E is the projection of S on the xy-plane then $dxdy = |\,\hat{N}.\hat{k}\,|dS$. Thus,

$$\iint_S \text{curl } \vec{F}.\hat{N}\,dS = \iint_E \hat{k}.\hat{N}\frac{dx\,dy}{|\hat{k}.\hat{N}|} = \iint_E dxdy = 4\int_0^1\int_0^{\sqrt{1-x^2}} dydx$$

$$= 4\int_0^1 \sqrt{1-x^2}\,dx = 4\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = 4\,\frac{1}{2}\,\frac{\pi}{2} = \pi. \qquad ...(9.36)$$

From the equality of (9.35) and (9.36), Stokes' theorem is verified.

Example 9.24: Evaluate $\oint_C (2y^3dx + x^3dy + zdz)$, using Stokes' theorem where C is the trace of the cone $z = \sqrt{x^2+y^2}$ intersected by the plane $z = 4$ and S is the surface of the cone below $z = 4$.

Solution: We have, $\bar{F} = 2y^3\hat{i} + x^3\hat{j} + z\hat{k}$. Thus,

$$\text{curl } \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^3 & x^3 & z \end{vmatrix} = (0)\,\hat{i} - (0)\,\hat{j} + (3x^2 - 6y^2)\,\hat{k}.$$

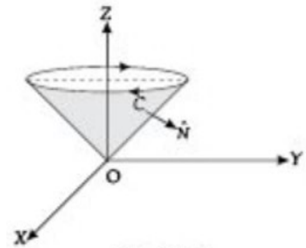


Fig. 9.15

The outward normal to the surface S points towards the downward direction as shown in the Fig. 9.15, and hence the direction of C is taken in the clockwise direction.

Let $f(x, y, z) = \sqrt{x^2+y^2} - z$ be the equation of the surface S, then the unit outward normal $\hat{N} = \dfrac{\text{grad } f}{|\text{grad } f|}$. We have, grad $f = \dfrac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}} - \hat{k} = \dfrac{x\hat{i} + y\hat{j} - z\hat{k}}{z}$, and thus

$$\hat{N} = \frac{(x\hat{i} + y\hat{j} - z\hat{k})/z}{\sqrt{(x^2+y^2+z^2)/z^2}} = \frac{x\hat{i} + y\hat{j} - z\hat{k}}{\sqrt{2}z}, \text{ using } x^2 + y^2 = z^2.$$

By Stokes' theorem

$$\oint_C (2y^3dx + x^3dy + zdz) = \iint_S \text{curl } \bar{F}.\hat{N}\, dS, \text{ where } \bar{F} = 2y^3\hat{i} + x^3\hat{j} + z\hat{k}.$$

Now, curl $\bar{F}.\hat{N} = (3x^2 - 6y^2)\,\hat{k}.\dfrac{x\hat{i} + y\hat{j} - z\hat{k}}{\sqrt{2}z} = -\dfrac{3x^2 - 6y^2}{\sqrt{2}}$.

Also, $dxdy = (\hat{N}.\hat{k})ds = -\dfrac{1}{\sqrt{2}}dS$, or $dS = -\sqrt{2}\,dxdy$.

Therefore $\iint_S \text{curl } \bar{F}.\hat{N}\, dS = \iint_E (3x^2 - 6y^2)dxdy$, where E is the region $x^2 + y^2 = 16$.

Substituting $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\iint_S \text{curl } \bar{F}.\hat{N}\, dS = \iint_E (3x^2 - 6y^2)dxdy = \int_0^4\int_{2\pi}^0 (3\cos^2\theta - 6\sin^2\theta)r^3drd\theta$$

$$= \frac{3}{2}\int_0^4\int_{2\pi}^0 [(1+\cos 2\theta) - 2(1-\cos 2\theta)]r^3 dr d\theta$$

$$= \frac{3}{2}\int_0^4\int_{2\pi}^0 [3\cos 2\theta - 1]r^3 dr d\theta = \frac{3}{2}\left[\frac{r^4}{4}\right]_0^4\left[\frac{3\sin 2\theta}{2} - \theta\right]_{2\pi}^0 = 192\pi.$$

Example 9.25: Evaluate $\iint_S \nabla\times \bar{F}\cdot d\bar{S}$ over the surface of the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$, where $\bar{F} = y\hat{i} + z\hat{j} + x\hat{k}$.

Solution: By Stokes' theorem $\iint_S \nabla\times\bar{F}\cdot d\bar{S} = \oint_C \bar{F}\cdot d\bar{r}$, where C is the closed curve binding the surface S, $z = 1 - x^2 - y^2$, $z \geq 0$; the curve C is given by $x^2 + y^2 = 1$, $z = 0$. We have $\oint_C \bar{F}\cdot d\bar{r} = \oint_C (ydx + zdy + xdz)$. Substituting $x = \cos\theta$, $y = \sin\theta$, $z = 0$, $0 \leq \theta \leq 2\pi$, we have

$$\oint_C \bar{F}\cdot d\bar{r} = -\int_0^{2\pi} \sin\theta\sin\theta d\theta = -\frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta)d\theta = -\pi.$$

Example 9.26: Show that $\oint_C \bar{r}.d\bar{r} = 0$, independently of the origin of $\bar{r}$, the position vector of a point $P(x, y, z)$.

Solution: If S is the open surface enclosed by the closed curve C, then by Stokes' theorem

$$\oint_C \bar{r}.d\bar{r} = \iint_S \text{curl } \bar{r}.d\bar{S} = \iint_S \bar{0}.d\bar{S} = 0$$

Example 9.27: If S be the surface of the sphere $x^2 + y^2 + z^2 = 9$, prove that $\int_S \text{curl } \bar{F}.d\bar{S} = 0$.

Solution: Cut open the surface of the sphere $x^2 + y^2 + z^2 = 9$, by any plane and let S_1 and S_2 denote its upper and lower portions and let C be the common curve binding both these portions. Then

$$\iint_S \text{curl } \bar{F}.d\bar{S} = \iint_{S_1} \text{curl } \bar{F}.d\bar{S} + \iint_{S_2} \text{curl } \bar{F}.d\bar{S} = \oint_C \bar{F}.d\bar{r} - \oint_C \bar{F}.d\bar{r} = 0,$$

using Stokes' theorem; the second integral is negative because it is taken in a direction opposite to that of the first.

Example 9.28: If ϕ is a scalar point function, using Stokes' theorem prove that curl (grad ϕ) = $\bar{0}$.

Solution: Let $\bar{F}$ be a vector point function such that $\bar{F} = \text{grad}\,\phi$, where ϕ is the given scalar point function. By Stokes' theorem $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\;\bar{F}\cdot d\bar{S}$, where S is the open surface bounded by the closed curve C.

Consider, $\bar{F}\cdot d\bar{r} = \text{grad}\,\phi .\, d\bar{r}$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(dx\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi.$$

Also C being a closed curve, we have $\oint_C d\phi = 0$, that is, $\oint_C \bar{F}\cdot d\bar{r} = 0$, therefore,

$\int_S \text{curl grad}\,\phi\cdot d\bar{S} = 0$. Thus curl grad $\phi = \bar{0}$.

Example 9.29: Prove that $\int_C \bar{a}\times\bar{r}\cdot d\bar{r} = 2\bar{a}.\iint_S d\bar{S}$, $\bar{a}$ being any constant vector and $\bar{r}$ being the position vector of a point $P(x, y, z)$.

Solution: If S be the open surface enclosed by the closed curve C, then by Stokes' theorem

$$\int_C \bar{a}\times\bar{r}\cdot d\bar{r} = \iint_S \text{curl}\,(\bar{a}\times\bar{r}).d\bar{S}$$

Also, $\nabla\times(\bar{a}\times\bar{r}) = \bar{a}(\nabla\cdot\bar{r}) - \bar{r}(\nabla\cdot\bar{a}) + (\bar{r}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{r}$

We have, $\nabla\cdot\bar{r} = 3$, $\nabla\cdot\bar{a} = 0$, $(\bar{r}\cdot\nabla)\bar{a} = 0$ and $(\bar{a}\cdot\nabla)\bar{r} = \bar{a}$.

Therefore, curl $(\bar{a}\times\bar{r}) = \nabla\times(\bar{a}\times\bar{r}) = 3\bar{a} - \bar{a} = 2\bar{a}$.

Hence, $$\int_C \bar{a}\times\bar{r}\cdot d\bar{r} = 2\iint_S \bar{a}\cdot d\bar{S} = 2\bar{a}\cdot\iint_S d\bar{S}.$$

EXERCISE 9.5

1. Show using Stokes' theorem that in an irrotational field $\bar{F}$, the circulation of $\bar{F}$ along every closed surface is zero.
2. Verify Stokes' theorem for $\bar{F} = (x^2 + y^2)\hat{i} - 2xy\,\hat{j}$, taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$.
3. Verify Stokes' theorem for the function $\bar{F} = x^2\hat{i} + xy\,\hat{j}$ integrated round the square of sides $x = a$, $y = a$, $x = 0$ and $y = 0$ in the plane $z = 0$.

4. Evaluate $\oint_C \bar{F}.d\bar{r}$ by Stokes' theorem, where $\bar{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).

5. Evaluate by Stokes' theorem $\oint_C (yzdx + zxdy + xydz)$, where C is the curve $x^2 + y^2 = 1, z = y^2$.

6. Evaluate by Stokes' theorem $\oint_C (\sin zdx - \cos xdy + \sin ydz)$, where C is the boundary of the rectangle $0 \le x \le \pi, 0 \le y \le 1, z = 3$.

7. Apply Stokes' theorem to evaluate $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$, where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3,0) and (0, 0, 6).

8. Evaluate $\int_C \bar{F}.d\bar{r}$ where $\bar{F} = y\hat{i} + xz^3\hat{j} - zy^3\hat{k}$ and C is the circle $x^2 + y^2 = 4, z = 1.5$.

9. Evaluate $\oint_C \bar{F}.d\bar{r}$ using the Stokes' theorem, where $\bar{F} = 3y\hat{i} + 4z\hat{j} + 2x\hat{k}$ and C is the intersection of the sphere $x^2 + y^2 + z^2 = 16, x \ge 0$ and the cylinder $y^2 + z^2 = 4$.

10. Evaluate $\oint_C \bar{F}.d\bar{r}$ using the Stokes' theorem, where $\bar{F} = x\hat{i} + z\hat{j} + y\hat{k}$ and C is the boundary of the ellipsoid $y = \sqrt{144 - 36x^2 - 9z^2}/4$ in the plane $y = 0$.

11. Evaluate the integral $\iint_S (\nabla \times \bar{F}) d\bar{S}$ by Stokes' theorem, where $\bar{F} = (x^2 - y^2)\hat{i} + (y^2 - x^2)\hat{j} + z\hat{k}$ and S is the portion of the surface $x^2 + y^2 - 2by + bz = 0$ whose boundary lies in the xy-plane; b being a constant.

9.6 VOLUME INTEGRAL GAUSS DIVERGENCE THEOREM

In this section we introduce volume integral. *Gauss divergence theorem* transforms surface integrals to volume integrals and conversely. The theorem is named so since it involves the divergence of a vector point function.

Volume Integral. To illustrate the concept of volume integral, consider a continuous vector function $\bar{F}(\bar{r})$ and a closed surface S enclosing the region (volume) E in space, for example, a solid cube, a ball, or the region between two concentric spheres. Subdivide the region E by planes parallel to the three co-ordinate planes into finite number of sub-regions $E_1, E_2, ..., E_n$. Let δV_i be the volume of the sub-region E_i enclosing an arbitrary point whose position vector is $\bar{r}_i$.

Consider the sum $\bar{V} = \sum_{i=1}^{n} \bar{F}(\bar{r}_i)\,\delta V_i$. The limit of this sum as $n \to \infty$ in such a way that $\delta V_i \to 0$, is called *the volume integral of* $\bar{F}(\bar{r})$ *over E and is* denoted by $\iiint_E \bar{F}(\bar{r})\,dV$. Under the assumption that $\bar{F}(\bar{r})$ is continuous in E and E is bounded by finitely many smooth surfaces, this limit is independent of the choice of subdivisions and the arbitrary position vector $\bar{r}_i$.

If $\bar{F}(\bar{r}) = F_1(x, y, z)\,\hat{i} + F_2(x, y, z)\,\hat{j} + F_3(x, y, z)\,\hat{k}$, then

$$\iiint_E \bar{F}\,dV = \hat{i} \iiint_E F_1(x,y,z)\,dxdydz + \hat{j} \iiint_E F_2(x,y,z)\,dxdydz + \hat{k} \iiint_E F_3(x,y,z)\,dxdydz \quad ...(9.37)$$

Next, we show that the triple integral of the divergence of a continuously differentiable vector function $\bar{F}(\bar{r})$ over a region E in space can be transformed into a surface integral of the normal component of $\bar{F}$ over the boundary surface S of E. This is executed by the Divergence theorem of Gauss, a three dimensional analog of Green's theorem in the plane.

Theorem 9.4: (Gauss Divergence Theorem) *Let E be a closed and bounded region in space whose boundary is a piecewise smooth oriented surface. Let* $\bar{F}$ *be a vector function which is continuous and has continuous first order partial derivatives in E,* then

$$\iiint_E \text{div}\bar{F}\,dV = \iint_S \bar{F}.\hat{N}\,dS, \quad ...(9.38)$$

where $\hat{N}$ is the outward unit normal vector of S.

Proof. Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and α, β, γ be the angles which the outward unit normal vector $\hat{N}$ make with the positive direction of x, y, z axes respectively, then $\hat{N} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$. Thus, the cartesian equivalent of the divergence theorem is

$$\iiint_E \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \iint_S (F_1\cos\alpha + F_3\cos\beta + F_3\cos\gamma)\,dS \quad ...(9.39)$$

To prove the divergence theorem it is sufficient to show that

$$\iiint_E \frac{\partial F_1}{\partial x}\,dxdydz = \iint_S F_1\cos\alpha\,dS \quad ...(9.40)$$

$$\iiint_E \frac{\partial F_2}{\partial y}\,dxdydz = \iint_S F_2\cos\beta\,dS \quad ...(9.41)$$

$$\iiint_E \frac{\partial F_3}{\partial z}\,dxdydz = \iint_S F_3 \cos\gamma\, dS \qquad ...(9.42)$$

We prove (9.42) only, the remaining results can be proved on similar lines.

We prove it for a special region E which is bounded by a piecewise smooth orientable surface S that has the property that any straight line parallel to z-axis cuts it in two points only, as shown in Fig. 9.16.

Let R be the orthogonal projection of S in the xy-plane; and let the bottom surface be

$$S_1 : z = h(x, y), \quad (x, y) \in R;$$

the top surface be $\quad S_2 : z = g(x, y); \quad (x, y) \in R;$

and the side surface be $S_3 : h(x, y) \le z \le g(x, y); \quad (x, y) \in R.$

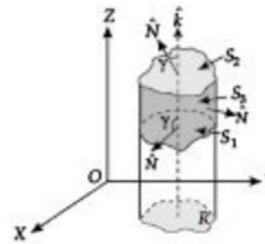


Fig. 9.16

Thus, $$\iiint_E \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R dxdy\left(\int_{h(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z}\,dz\right)$$

$$= \iint_R [F_3(x, y, g)dxdy - F_3(x, y, h)\,dx\,dy]$$

$$= \iint_R F_3(x, y, g)\,dx\,dy - \iint_R F_3(x, y, h)\,dx\,dy.$$

We show that this is equal to the R.H.S. of (9.42). On the lateral portion S_3 of S we have $\gamma = \pi/2$ and thus $\cos\gamma = 0$. Hence, this portion does not contribute to the surface integral in (9.42) refer to Fig. 9.16, and thus R.H.S. of (9.42) gives

$$\iint_S F_3 \cos\gamma\, dS = \iint_{S_1} F_3 \cos\gamma\, dS + \iint_{S_2} F_3 \cos\gamma\, dS \qquad ...(9.43)$$

On S_1 of S, the normal $\hat{N}$ to S makes an obtuse angle γ with $\hat{k}$. Therefore $dxdy = -\cos\gamma ds$. Thus,

$$\iint_R F_3(x, y, h)\,dxdy = -\iint_{S_1} F_3 \cos\gamma\, dS.$$

Also on S_2 of S, the normal $\hat{N}$ to S makes an acute angle γ with $\hat{k}$, therefore, $dxdy = \cos\gamma dS$. Thus,

$$\iint_R F_3(x, y, g)\,dxdy = \iint_{S_2} F_3 \cos\gamma\, dS.$$

Therefore,

$$\iiint_E \frac{\partial F_3}{\partial z}\,dxdydz = \iint_{S_1} F_3 \cos\gamma\, dS + \iint_{S_2} F_3 \cos\gamma\, dS$$

which is the same as (9.43). This proves (9.42).

Similarly, by considering the projections on yz and zx-planes and proceeding on parallel lines we can prove (9.40) and (9.41).

This proves the divergence theorem for special region.

For any region E which can be subdivided into finitely many special regions by means of auxiliary surfaces, the theorem follows by adding the result for each part separately. The surface integrals over the auxiliary surfaces cancel in pairs, and the sum of the remaining surface integrals is the surface integral over the whole boundary S of E. The volume integrals over the parts of E add up to give the volume integral over E.

Example 9.30: Verify divergence theorem for $\bar{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ taken over the cube bounded by $x = 0, x = 1; y = 0, y = 1; z = 0, z = 1$.

Solution: The Gauss divergence theorem is $\iiint\limits_E \text{div}\ \bar{F}\,dV = \iint\limits_S \bar{F}.\hat{N}\,ds$.

Here $\bar{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ gives $\text{div}\ \bar{F} = \dfrac{\partial}{\partial x}(x^2) + \dfrac{\partial}{\partial y}(z) + \dfrac{\partial}{\partial z}(yz) = 2x + y$.

Thus,
$$\iiint\limits_E \text{div}\ \bar{F}\,dv = \int_0^1\int_0^1\int_0^1 (2x + y)\,dxdydz = \int_0^1\int_0^1 \left[x^2 + xy\right]_0^1 dydz$$

$$= \int_0^1\int_0^1 (1 + y)\,dydz = \int_0^1 \left(y + \frac{y^2}{2}\right)_0^1 dz = \frac{3}{2}\int_0^1 dz = \frac{3}{2}. \qquad \text{...(9.44)}$$

To evaluate the surface integral, we divide the piecewise smooth closed surface S of the cuboid into six smooth surfaces, as shown in Fig. 9.17, given by

S_1 : $AC'PB'$ $\qquad$ S_2 : $OBA'C$

S_3 : $BA'PC'$ $\qquad$ S_4 : $OAB'C$

S_5 : $CA'PB'$ $\qquad$ S_6 : $OBC'A$

Thus,
$$\iint\limits_S \bar{F}.\hat{N}\,dS = \int\limits_{(S_1 + ... + S_6)}\int \bar{F}.\hat{N}\,dS \qquad \text{...(9.45)}$$

On S_1, $x = 1$, we have $\hat{N} = \hat{i}$, thus $\bar{F}.\hat{N} = x^2$, so that

$$\iint\limits_{S_1} \bar{F}.\hat{N}\,dS = \int_0^1\int_0^1 x^2 dydz = \int_0^1\int_0^1 dydz = 1.$$

On S_2, $x = 0$, $\hat{N} = -\hat{i}$, thus $\bar{F}.\hat{N} = -x^2$, so that

$$\iint\limits_{S_2} \bar{F}.\hat{N}\,dS = -\int_0^1\int_0^1 x^2 dydz = \int_0^1\int_0^1 0.dydz = 0.$$

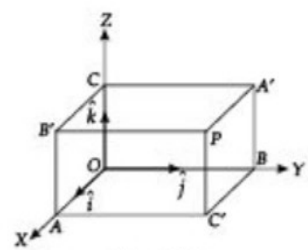


Fig. 9.17

On S_3, $y = 1$, $\hat{N} = \hat{j}$, thus $\bar{F}.\hat{N} = z$, so that

$$\iint_{S_3} \bar{F}.\hat{N}\,dS = \int_0^1\int_0^1 z\,dx\,dz = \int_0^1 \left(\frac{z^2}{2}\right)_0^1 dx = \int_0^1 \frac{dx}{2} = \frac{1}{2}.$$

On S_4, $y = 0$, $\hat{N} = -\hat{j}$, thus $\bar{F}.\hat{N} = -z$, so that

$$\iint_{S_4} \bar{F}.\hat{N}\,dS = -\int_0^1\int_0^1 z\,dx\,dz = -\int_0^1 \left(\frac{z^2}{2}\right)_0^1 dx = -\int_0^1 \frac{dx}{2} = -\frac{1}{2}.$$

On S_5, $z = 1$, $\hat{N} = \hat{k}$, thus $\bar{F}.\hat{N} = yz$, so that

$$\iint_{S_5} \bar{F}.\hat{N}\,dS = \int_0^1\int_0^1 yz\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy = \int_0^1 \left(\frac{y^2}{2}\right)_0^1 dx = \frac{1}{2}.$$

On S_6, $z = 0$, $\hat{N} = -\hat{k}$, thus $\bar{F}.\hat{N} = -yz$, so that

$$\iint_{S_6} \bar{F}.\hat{N}\,dS = -\int_0^1\int_0^1 yz\,dx\,dy = -\int_0^1\int_0^1 0.dx\,dy = 0.$$

Therefore from (9.45)

$$\iint_S \bar{F}.\hat{N}\,dS = 1 + 0 + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + 0 = 3/2. \quad \ldots(9.46)$$

The equality of (9.44) and (9.46) verifies the divergence theorem.

Example 9.31: For any closed surface S, prove that

$$\iint_S [x(y-z)\hat{i} + y(z-x)\hat{j} + z(x-y)\hat{k}].d\bar{S} = 0.$$

Solution: If E be the volume enclosed by the closed surface S, then by divergence theorem

$$\iint_S \bar{F}.\hat{N}\,dS = \iiint_E \text{div}\,\bar{F}\,dv,$$ where the symbols have their usual meanings.

Here, $\bar{F} = x(y-z)\hat{i} + y(z-x)\hat{j} + z(x-y)\hat{k}$, gives $\text{div}\,\bar{F} = \frac{\partial}{\partial x}(y-z)x + \frac{\partial}{\partial y}(z-x)y + \frac{\partial}{\partial z}(x-y)z = 0.$

Therefore, $$\iiint_E \text{div}\,\bar{F}\,dv = 0.$$

This proves the result.

Example 9.32: Using divergence theorem, prove that

(a) $\iint\limits_S \bar{r}.d\bar{S} = 3V$ (b) $\iint\limits_S \nabla r^2.d\bar{S} = 6V$

where S is any closed surface enclosing a volume V, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r^2 = x^2 + y^2 + z^2$.

Solution: (*a*) By divergence theorem $\iint\limits_S \bar{r}.d\bar{S} = \iiint\limits_V \text{div } \bar{r}\, dV = 3\iiint\limits_V dv = 3V.$

(b) Let $\bar{F} = \nabla r^2$, then by divergence theorem $\iint\limits_S \nabla r^2.d\bar{S} = \iiint\limits_V \text{div } \nabla r^2 dV.$

We have, $$\text{div } \nabla r^2 = \nabla\cdot\nabla r^2 = \nabla^2 r^2 = \frac{\partial^2}{\partial x^2}(x^2) + \frac{\partial^2}{\partial y^2}(y^2) + \frac{\partial^2}{\partial z^2}(z^2) = 6.$$

Thus, $$\iint\limits_S \nabla r^2\cdot dS = \iiint\limits_V 6dV = 6V.$$

This proves the result.

Example 9.33: Using divergence theorem evaluate $\iint\limits_S \bar{F}\, d\bar{S}$ where $\bar{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution: By divergence theorem $\iint\limits_S \bar{F}.d\bar{S} = \iiint\limits_E \text{div } \bar{F} dV$, where the symbols have their usual meanings.

We have $$\text{div } \bar{F} = \nabla\cdot(x^3\hat{i} + y^3\hat{j} + z^3\hat{k}) = 3(x^2 + y^2 + z^2) = 3a^2, \text{ thus}$$

$$\iint\limits_S \bar{F}.d\bar{S} = 3a^2\iiint\limits_E dV = 3a^2.\frac{4}{3}\pi a^3 = 4\pi a^5,$$

since the volume of $x^2 + y^2 + z^2 = a^2$ is $(4/3)\,\pi a^3$.

Example 9.34: Evaluate $\iint (xdydz + ydzdx + zdxdy)$ over the surface of a sphere of radius a.

Solution: We have $xdydz + ydzdx + zdxdy = (x\hat{i}\cdot\hat{N} + y\hat{j}\cdot\hat{N} + z\hat{k}\cdot\hat{N})dS = \bar{r}.d\bar{S}$.

By divergence theorem $\iint\limits_S \bar{r}\, d\bar{S} = \iiint\limits_E \text{div } \bar{r}\, dV = 3\iiint\limits_E dV = 3\frac{4}{3}\pi a^3 = 4\pi a^3.$

Example 9.35: Verify divergence theorem for $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

Solution: We have, $\text{div } \bar{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$. Thus

$$\iiint_E \text{div } \bar{F}\, dv = \iiint_E (4 - 4y + 2z)dxdydz = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\int_0^3 (4 - 4y + 2z)dz\right)dydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z - 4yz + z^2\right]_0^3 dydx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21 - 12y)dydx$$

$$= 21\int_{-2}^{2}\left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\right)dx, \quad \text{since} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 12y\,dy = 0$$

$$= 84\int_0^2 \sqrt{4-x^2}\, dx = 84\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = 84[2\sin^{-1}1] = 84\pi. \quad ...(9.47)$$

To evaluate the surface integral, we divide the piecewise smooth closed surface S of the cylinder into three smooth surfaces, as shown in Fig. 9.18.

S_1 : the circular base in the plane $z = 0$

S_2 : the circular top in the plane $z = 3$

S_3 : the curved surface of the cylinder given by $x^2 + y^2 = 4$; $0 \le z \le 3$.

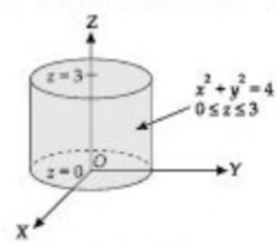


Fig. 9.18

We have, $$\iint_S \bar{F}.\hat{N}\, dS = \iint_{S_1+S_2+S_3} \bar{F}.\hat{N}\, dS \quad ...(9.48)$$

On S_1, $z = 0$, $\hat{N} = -\hat{k}$, thus $\bar{F}.\hat{N} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}).(-\hat{k}) = -z^2 = 0$.

Therefore, $$\iint_{S_1} \bar{F}.\hat{N}\, dS = 0.$$

On S_2, $z = 3$, $\hat{N} = \hat{k}$, thus $\bar{F}.\hat{N} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}).(\hat{k}) = z^2 = 9$.

Therefore, $$\iint_{S_2} \bar{F}.\hat{N}\, dS = 9\iint_{S_2} ds = 9\,(\text{area of circle } x^2 + y^2 = 4) = 36\pi.$$

On S_3, $x^2+y^2=4$, $\hat{N}=\dfrac{\nabla(x^2+y^2)}{|\nabla(x^2+y^2)|}=\dfrac{2x\hat{i}+2y\hat{j}}{2\sqrt{x^2+y^2}}=\dfrac{x\hat{i}+y\hat{j}}{2}$,

thus $$\bar{F}.\hat{N}=(4x\hat{i}-2y^2\hat{j}+z^2\hat{k}).\frac{(x\hat{i}+y\hat{j})}{2}=2x^2-y^3.$$

Therefore, $$\iint_{S_3}\bar{F}.\hat{N}\,dS=\iint_{S_3}(2x^2-y^3)dS.$$

Also on S_3, $x=2\cos\theta$, $y=2\sin\theta$ and $dS=2d\theta dz$. Thus

$$\iint_{S_3}(2x^2-y^3)dS=\int_{z=0}^{3}\int_{\theta=0}^{2\pi}(8\cos^2\theta-8\sin^3\theta)\,2d\theta dz=48\int_0^{2\pi}(\cos^2\theta-\sin^3\theta)d\theta$$

$$=48\left[\frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)d\theta-\frac{1}{4}\int_0^{2\pi}(3\sin\theta-\sin 3\theta)d\theta\right]=48\times\frac{1}{2}\times 2\pi=48\pi.$$

Thus (9.48) becomes

$$\iint_S\bar{F}.\hat{N}\,dS=0+36\pi+48\pi=84\pi. \quad ...(9.49)$$

The equality of (9.47) and (9.49) verifies the divergence theorem.

Example 9.36: Use divergence theorem to evaluate $\iint_S(y^2z^2\hat{i}+z^2x^2\hat{j}+z^2y^2\hat{k}).d\bar{S}$, where S is the closed surface of the region bounded by the hemisphere $x^2+y^2+z^2=1$, $z\ge 0$ and the plane $z=0$.

Solution: Let V be the volume enclosed by the closed surface S, then by divergence theorem

$$\iint_S(y^2z^2\hat{i}+z^2x^2\hat{j}+z^2y^2\hat{k}).d\bar{S}=\iiint_V\operatorname{div}(y^2z^2\hat{i}+z^2x^2\hat{j}+z^2y^2\hat{k})dV=2\iiint_V zy^2dV.$$

Using spherical polar co-ordinates, to find the volume enclosed by the hemisphere $x^2+y^2+z^2=1$, $z\ge 0$, we substitute $x=r\sin\phi\cos\theta$, $y=r\sin\phi\sin\theta$, $z=r\cos\phi$, $dv=r^2\sin\phi\,dr\,d\theta\,d\phi$, where $0\le r\le 1$, $0\le\theta\le 2\pi$, $0\le\phi\le\pi/2$. Thus

$$2\iiint_V zy^2dV=2\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 r\cos\phi.r^2\sin^2\phi\sin^2\theta\, r^2\sin\phi\,drd\theta d\phi$$

$$=2\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 r^5\sin^3\phi\cos\phi\sin^2\theta drd\theta d\phi$$

$$= \left(\int_0^{\pi/2} \sin^3\phi \cos\phi\, d\phi\right)\left(\int_0^{2\pi} 2\sin^2\theta\, d\theta\right)\left(\int_0^1 r^5 dr\right)$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}\left[\frac{\sin^4\phi}{4}\right]_0^{\pi/2}\left[\frac{r^6}{6}\right]_0^1 = 2\pi \times \frac{1}{4} \times \frac{1}{6} = \pi/12.$$

Example 9.37: Use the divergence theorem to evaluate $\iint_S \vec{F}.\hat{N}\, dS$, where $\vec{F} = x^2 z\hat{i} + y\hat{j} - xz^2\hat{k}$ and S is the boundary of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4y$.

Solution: Let V be volume enclosed by the closed surface S, then by divergence theorem

$$\iint_S \vec{F}.\hat{N}\, dS = \iiint_V \operatorname{div} \vec{F}\, dV$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-xz^2) = 2xz + 1 - 2xz = 1. \text{ Thus}$$

$$\iint_S \vec{F}.\hat{N}\, dS = \iiint_E \operatorname{div}\vec{F}\, dV = \int_{y=0}^{4}\int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}}\int_{z=x^2+y^2}^{4y} dz dx dy,$$

(projection of S on xy-plane is circle $x^2 + y^2 = 4y$)

$$= \int_{y=0}^{4}\int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} (4y - x^2 - y^2) dx dy = 2\int_0^4\int_0^{\sqrt{4y-y^2}} [(4y-y^2) - x^2] dx dy$$

$$= 2\int_0^4 \left[(4y-y^2)x - \frac{x^3}{3}\right]_0^{\sqrt{4y-y^2}} dy = 2\int_0^4 [(4y-y^2)]^{3/2} - \frac{1}{3}(4y-y^2)^{3/2}] dy$$

$$= \frac{4}{3}\int_0^4 (4y-y^2)^{3/2} dy = \frac{4}{3}\int_0^4 [4-(y-2)^2]^{3/2} dy$$

$$= \frac{4}{3}\int_{-\pi/2}^{\pi/2} 16\cos^4 t\, dt = \frac{128}{3}\int_0^{\pi/2} \cos^4 t\, dt, \qquad [(y-2) = 2\sin t]$$

$$= \frac{128}{3}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2} = 8\pi.$$

EXERCISE 9.6

1. Using Gauss divergence theorem show that if $\bar{F}$ defines a solenoidal field, then the flux of $\bar{F}$ around every closed surface is zero.
2. Verify divergence theorem for $\bar{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelopiped given by $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.
3. If $\bar{F} = xy^2\hat{i} + yz^2\hat{j} + zx^2\hat{k}$, evaluate $\iint_S \bar{F}.d\bar{S}$ over the sphere given by $x^2 + y^2 + z^2 = 1$.
4. Verify the divergence theorem for $\bar{F} = (2xy + z)\hat{i} + y^2\hat{j} - (x + 3y)\hat{k}$ when the surface S is that of the region bounded by the plane $2x + 2y + z = 6$ in the first octant.
5. Apply divergence theorem to evaluate $\iint (lx^2 + my^2 + nz^2)$ taken over the sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2$; l, m, n being the direction cosines of the external normal to the sphere.
6. Evaluate $\iint_S \bar{F}.\hat{N}\,dS$ over the surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, where $\bar{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$.
7. Evaluate $\iint_S (xdydz + ydzdx + zdxdy)$ using divergence theorem, where S is the surface of the sphere $(x - 2)^2 + (y - 2)^2 + (z - 2)^2 = 4$.
8. Evaluate $\iint_S (\nabla \times \bar{F}).\hat{N}\,dS$, where S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the xy-plane and $\bar{F} = (x - z)\hat{i} + (x^3 + yz)\hat{j} - 3xy^2\hat{k}$.
9. Show that $\iint_S (\bar{A}.\hat{N})\,dS = 0$, where $\bar{A}$ is a constant vector.
10. Show that $\iint_S r^n(\bar{r}.\hat{N})\,dS = (n+3)\iiint_E r^n\,dv$, $n \ne 3$, where $\bar{r}$ is the position vector of the point $\rho(x, y, z)$ and $r = |\bar{r}|$.
11. Using divergence theorem, evaluate $\iint_S (x^3dydz + x^2ydzdx + x^2zdxdy)$, where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ and the circular discs $z = 0$, and $z = b$.

ANSWERS

Exercise 9.1 (p. 522)

1. $75\frac{1}{3}\hat{i} + 360\hat{j} - 42\hat{k}$. 2. (a) 12 (b) $-24\hat{i} - \frac{40}{3}\hat{j} + \frac{64}{5}\hat{k}$.

Exercise 9.2 (p. 531)

1. $-7/6$ 2. 0 3. $3/2$ 4. (a) 16 (b) 16

5. $\left(2-\frac{\pi}{4}\right)\hat{i}-\left(\pi-\frac{1}{2}\right)\hat{j}$ 6. 6 7. 0 8. $-1/2$

9. $\sinh(x+y)$

Exercise 9.3 (p. 538)

1. 180 2. 8 3. 84π 4. $2\pi abc$

5. $\frac{2}{3}a^3 - \frac{3}{8}a^4$ 6. 0

Exercise 9.4 (p. 543)

1. $-\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$ 4. $\frac{11}{30}$ 5. $\frac{3\pi a^2}{8}$ 7. 0

Exercise 9.5 (p. 550)

2. $-4ab^2$ 4. $\frac{1}{3}$ 5. 0 6. 2

7. 21 8. $\frac{19}{2}\pi$ 9. -16π 10. 0

11. $2\pi b^3$.

Exercise 9.6 (p. 560)

3. $\frac{4}{5}\pi$ 5. $\frac{8\pi}{3}(a+b+c)\rho^3$ 6. 320π 7. 32π

8. 12π 11. $\frac{5}{4}\pi a^4 b$.

PART D

Ordinary Differential Equations and Laplace Transforms

10 First Order Ordinary Differential Equations

CHAPTER

Differential equations are of fundamental importance because they express relationships involving rate of change. Such relationships form the basis for studying phenomena in the field of science, engineering and business. Many physical laws appear mathematically in the form of differential equations. The first order ordinary differential equations are the simplest equations involving only the first derivative of the unknown function, but are capable to model many physical problems appearing in science and engineering.

10.1 BASIC CONCEPTS

The mathematical formulation of problems in engineering and science usually leads to equations involving derivatives of one or more unknown functions. Such equations are called *differential equations*. A few typical examples are:

(a) The motion of a body of mass m subjected to a force $F(t)$ along a straight line, according to Newton's second law of motion, is

$$m\frac{d^2x}{dt^2} = F(t), \qquad ...(10.1)$$

where $x(t)$ is the displacement of the mass m at time t measured from the origin.

(b) The decay of a radioactive substance is described by an equation of the form

$$\frac{dy}{dt} = -ky(t), \qquad ...(10.2)$$

where $y(t)$ is the amount of the substance at time t and $k > 0$ is a constant.

(c) The angular motion $\theta(t)$ of a pendulum of length l under the action of gravity, where g is the acceleration of gravity and t is time, is given by

$$\frac{d^2\theta}{dt^2} = \frac{-g}{l}\sin\theta. \qquad ...(10.3)$$

(d) The equation

$$\frac{d^2y}{dx^2} = c\sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad ...(10.4)$$

governs the shape of a flexible cable or string, hanging under the action of gravity, where $y(x)$ is the deflection and c is a constant that depends upon the mass density of the cable and the tension at the midpoint $x = 0$.

(e) The one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(10.5)$$

governs the small transverse vibrations of an elastic string, such as a violin string. Here, $u(x, t)$ is the deflection of the string at a distance x from the one end at time t.

Differential equations which involve only one independent variable, e.g., (10.1), (10.2), (10.3) and (10.4) are called *ordinary differential equations;* and those which involve two or more independent variables and partial differential coefficients with respect to them, e.g. (10.5), are called *partial differential equations*. In this chapter, we shall be concerned with ordinary differential equations only.

The *order* of a differential equation is the order of the highest derivative it contains, whereas the *degree* of a differential equation is the degree of the highest order differential coefficient, after the equation has been made free of radicals and fractions as far as the differential coefficients are concerned.

For example, Eq. (10.1) is of order 2 and degree 1, and Eq. (10.2) is of order 1 and degree 1, while Eq. (10.4) is of order 2 and degree 2, since it can be rationalized as

$$\left(\frac{\partial^2 y}{\partial x^2}\right)^2 = c^2\left(1+\left(\frac{\partial y}{\partial x}\right)^2\right) \qquad ...(10.6)$$

A differential equation is *linear* when the dependent variable and its derivatives occur only in the first degree and also no product term containing the dependent variable and its derivatives of various orders occur.

For example, (10.2) is a linear differential equation of order one.

The *general form of linear ordinary differential equation of order n is*

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\, y = f(x), \qquad ...(10.7)$$

where $a_i(x), a_o(x) \neq 0$, are some functions of x, or constants, called the *coefficients* of the equation. *The function $f(x)$ is called the non-homogeneous term. Sometimes it is called the forcing function because in applications it represents the influence of an external input that drives a physical system represented by the differential equation. Equation (10.7) is called homogeneous, if $f(x) = 0$.*

A differential equation which is not linear is called a *non-linear differential* equation. For example, (10.6) is a non-linear differential equation of order 2 and degree 2.

10.2 FORMATION AND SOLUTIONS

In this section, we shall consider some examples to learn how ordinary differential equations arise in the process of elimination of arbitrary constants from a relation involving variables and constants. This will give us an idea as to what kind of solution a differential equation may have. We shall discuss the types of solutions a differential equation can have. Also we consider the important question of existence and uniqueness for solution in case of first order first degree initial value differential equations.

10.2.1 Formation of Ordinary Differential Equations

We consider the following examples.

Example 10.1: Form the differential equations of all parabolas whose axis is the axis of x.

Solution: The equation of such a parabola is

$$y^2 = 4a(x - h). \qquad ...(10.8)$$

Differentiating w.r.t. x, we obtain $2y\dfrac{dy}{dx} = 4a$, or $y\dfrac{dy}{dx} = 2a$.

Differentiating this again w.r.t. x, we obtain $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0$, which is a differential equation of second order.

Example 10.2: Eliminate arbitrary constants A and α from $x = A\cos(pt - \alpha)$, the equation of simple harmonic motion. Eliminate p also.

Solution: The given equation is,

$$x = A\cos(pt - \alpha). \qquad ...(10.9)$$

Differentiating with respect to t, $\dfrac{dx}{dt} = -pA\sin(pt - \alpha)$.

Differentiating again, $\dfrac{d^2x}{dt^2} = -p^2A\cos(pt - \alpha)$. Using (10.9) it gives,

$$\frac{d^2x}{dt^2} = -p^2x \qquad ...(10.10)$$

an equation of the second order which interprets that the acceleration varies as the distance from the origin.

In case we are to eliminate p also; differentiating (10.10) again we obtain

$$\frac{d^3x}{dt^3} = -p^2\frac{dx}{dt} \qquad ...(10.11)$$

From (10.10) and (10.11), we obtain

$$\frac{d^3x}{dt^3}\Big/\frac{d^2x}{dt^2} = \frac{dx}{dt}\Big/x \text{ or, } x\frac{d^3x}{dt^3} - \frac{dx}{dt}\frac{d^2x}{dt^2} = 0,$$

an equation of the third order.

Example 10.3: Eliminate the arbitrary constants A and B from

$$y = Ae^{2x} + Be^{-2x}$$

Solution: Equation is $y = Ae^{2x} + Be^{-2x}$ ...(10.12)

Differentiating this, $\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}$

Differentiating again, $\frac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x}$ ...(10.13)

From (10.12) and (10.13), we obtain $\frac{d^2y}{dx^2} = 4y$, an equation of the second order.

10.2.2 Solution of a Differential Equation

A solution of an ordinary differential equation is a function $y = f(x)$ that when substituted into the equation satisfies it over the interval on which the differential equation is defined.

For instance, in Example (10.3) above, $y = Ae^{2x} + Be^{-2x}$ satisfies the differential equation

$$\frac{d^2y}{dx^2} = 4y \qquad ...(10.14)$$

for all x, hence it is a solution of this equation.

The general solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the equation. If the arbitrary constants in the general solution are assigned specific values, the resultant solution is called a particular solution of the differential equation.

For example, $y = e^{2x} + 2e^{-2x}$ is a particular solution of the differential Eq. (10.14), while (10.12) is the general solution of the Eq. (10.14).

A solution which cannot be obtained from the general solution for any specific choice of its arbitrary constants is called a singular solution.

Linear differential equations have no singular solutions. Non-linear equations possess sometime one or more singular solutions. But these solutions are not of much interest from the applications point of view. As an example the equation

$$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0 \qquad ...(10.15)$$

has the general solution, a family of straight lines,

$$y = ax - a^2 \quad \text{...(10.16)}$$

which may be verified by direct substitution. However, we may verify that $y = x^2/4$, a parabola, is also a solution of (10.15), which can't be obtained from (10.16) for any specific value of a, refer to Fig. 10.1.

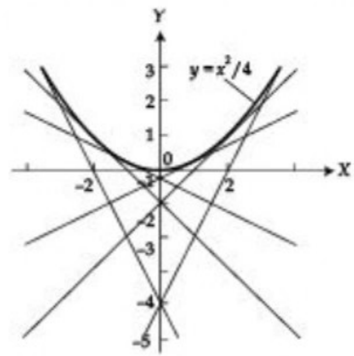


Fig. 10.1

The graph of a solution of an ordinary differential equation is called an *integral curve* of the equation.

Sometimes a solution of a differential equation appears as an implicit function given in the form $g(x, y) = 0$. For example, the differential equation

$$y\left(\frac{dy}{dx}\right) = -x, -1 < x < 1, \quad \text{...(10.17)}$$

has an implicit solution

$$x^2 + y^2 - 1 = 0, y > 0. \quad \text{...(10.18)}$$

We shall later study the conditions under which a given differential equation has solutions, but we must note that *there are differential equations which do not have solutions at all, and there are others which do not have a general solution.*

For example, the equation

$$\left(\frac{dy}{dx}\right)^2 = -1 \quad \text{...(10.19)}$$

does not have a solution for real y; and the equation

$$\left|\frac{dy}{dx}\right| + |y| = 0 \quad \text{...(10.20)}$$

has no general solution, only the trivial solution $y = 0$.

10.2.3 Initial and Boundary Value Problems

In applications of ordinary differential equation the values of the arbitrary constants in a specified problem are obtained by choosing them so that the solution satisfies the prescribed conditions identified with a particular problem.

If the required conditions are prescribed at a single point, say $x = a$, then the conditions are called the initial conditions and the differential equations together with the conditions, is called an initial value problem (i.v.p.). In an i.v.p the independent variable x often represents the time so that conditions of this type describe how the solution starts.

In case the required conditions are prescribed at two different points, say at $x = a$ and $x = b$ then the conditions are called the boundary conditions and the differential equation together with the conditions is called a boundary value problem (b.v.p.). In b.v.p., x often represents a space variable and the interval $a \le x \le b$ gives the domain of definition of the boundary value problem.

For example, the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 53y = 0 \qquad ...(10.21)$$

has the general solution

$$y(x) = Ae^{2x}\cos 7x + Be^{2x}\sin 7x.$$

The Eq. (10.21) together with the initial conditions $y(\pi) = -3, y'(\pi) = 2$ specified at the point $x = \pi$ constitute an initial value problem for y. Choosing A and B to satisfy these initial conditions, the unique solution of the i.v.p. is

$$y(x) = e^{2(x-\pi)}\left[3\cos 7x - \frac{8}{7}\sin 7x\right]. \qquad ...(10.22)$$

Similarly, the equation

$$\frac{d^2y}{dx^2} + y = 0; \quad y(0) = 0; \quad y'(\pi/3) = 3 \qquad ...(10.23)$$

constitutes a boundary value problem with a unique solution

$$y = 6\sin x, \quad 0 \le x \le \pi/3. \qquad ...(10.24)$$

However, we must note that it is possible for a boundary value problem to have a *unique solution, more than one solution*, or *no solution at all*. For example, the b.v.p.

$$\frac{d^2y}{dx^2} - y = 0, \quad x \ge 0; \quad y(0) = 0, \quad y(\pi) = 0 \qquad ...(10.25)$$

has the *non-unique solution* $y = B\sin x$, where B is arbitrary.

In case the boundary conditions are changed to $y(0) = 0$ and $y(\pi) = 1$, then the resultant boundary value problem has *no solution* at all.

10.2.4 Existence and Uniqueness for Solution of Initial Value Problems

In the theory of differential equations, the question of *existence* and *uniqueness* of the solution of an initial value problem is of fundamental importance. The question of existence and uniqueness arise even with very simple first order, first degree initial value problems. An initial value problem may have no solution, precisely one solution, or more then one solution. For example, the initial value problem $\frac{dy}{dx} = \frac{4}{3}y^{1/4}$, $y(0) = -1$ has *no solution*.

The initial value problem $\frac{dy}{dx} = x$, $y(0) = 1$ has precisely *one solution*, namely $y = \frac{1}{2}x^2 + 1$; and the initial value problem $x\frac{dy}{dx} = y - 1$, $y(0) = 1$ has *infinitely many solutions*, namely $y = 1 + cx$, where c is an arbitrary constant.

Next, we state a theorem without proof on *existence and uniqueness of solution of the first order and first degree initial value problems. The conditions are sufficient to ensure existence and uniquencess of the solutions but are not necessary one.*

Theorem 10.1: **(Existence and Uniqueness Theorem)** *If $f(x, y)$ is continuous and bounded function of x and y, such that $|f(x, y)| \le k$, in a rectangular region R: $|x - x_0| < a$, $|y - y_0| < b$, then the initial value problem*

$$\frac{dy}{dx} = f(x, y) \quad \text{with} \quad y(x_0) = y_0$$

has at least one solution for all x in the interval $|x - x_0| < h$, where h is the smaller of the two numbers a and b/k. If in addition, $\partial f/\partial y$ is continuous and bounded in R, then the solution is unique and is defined at least for all x in the interval $|x - x_0| < a$.

As already mentioned the conditions stated above are sufficient conditions rather than necessary one. In fact, the theorem holds even when the hypothesis about the continuity of $\partial f/\partial y$ is replaced by somewhat weaker condition

$$|f(x, y_2) - f(x, y_1)| \le M|y_2 - y_1|, \quad (M \text{ is a constant})$$

known as *Lipschitz condition*. However, continuity of $f(x, y)$ is not sufficient to ensure the uniqueness of the solution.

To apply the result stated above consider the initial value problem $\frac{dy}{dx} = 3y^{2/3}$; $y(0) = 0$.

The function $f(x, y) = 3y^{2/3}$ is continuous and bounded in any rectangular region R about the origin (0, 0), but $\partial f/\partial y = 2y^{-1/3}$ is unbounded at the origin. Thus, this initial value problem has a solution but it is not unique. In fact the given initial value problem has the solution $y = x^3$, and in addition to it has the singular solution $y = 0$.

10.2.5 Geometrical Interpretation of $y' = f(x, y)$

The first order and first degree differential equation in the explicit form is

$$\frac{dy}{dx} = f(x, y) \qquad ...(10.26)$$

When either no analytical solution of (10.26) is available or, if one exists, is too complicated to be useful, then the geometrical interpretation becomes of practical interest.

We know that dy/dx is the slope of the curve $y = y(x)$ and let, $m = dy/dx$ when the values of x and y are known. Let m_0 be the value of dy/dx derived from (10.26) at the point A_0 (x_0, y_0). Take a neighbouring point $A_1(x_1, y_1)$ such that the slope of A_0A_1 is m_0, and let m_1 be the corresponding value of $\frac{dy}{dx}$ derived from (10.26) at the point $A_1(x_1, y_1)$. Similarly, take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1 and so on.

In case the successive points $A_0, A_1, A_2, \ldots$ are chosen very close to one another the broken curve $A_0A_1A_2, \ldots$ as shown in Fig. 10.2 approximates to a smooth curve $y = \phi(x)$, the solution of (10.26) associated with the initial condition $y_0 = \phi(x_0)$.

A different choice of the initial point will in general give a different curve with the same property. Any such curve corresponds to a *particular solution* of the differential equation and the whole family of such curve corresponds to the *general solution* of the differential equation.

It is not possible to give such simple geometric interpretation for equations of second and higher order.

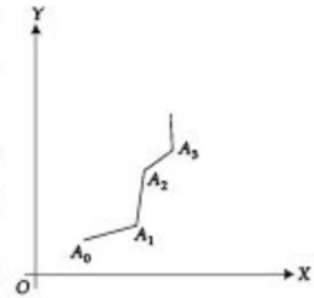


Fig. 10.2

EXERCISE 10.1

1. Determine order and degree of the following differential equations and classify them as homogeneous linear, non-homogeneous linear or non-linear.

   (a) $y'' + 2y' + 4y = x + \sin x$ (b) $(y')^2 + y = x \sin x$

   (c) $y' + 3xy = 1 + x^2$ (d) $(2 + x^2)y' + x(1 - y^2) = 0$

2. Solve the following differential equations.

   (a) $y' = x^3$ (b) $y'' = \sin 2x$ (c) $y' = 1/(1 + x)$

3. Determine whether the given function is a solution of the given differential equation in the following:

   (a) $2yy' = 1$; $f(x) = \sqrt{x - 1}$, for $x > 1$ (b) $xy' = x - y$; $f(x) = \dfrac{x^2 - 3}{2x}$, $x \neq 0$.

   (c) $(\sinh x)y' + (\cosh x)y = 0$; $f(x) = -1/\sinh x$, $x \neq 0$.

   (d) $x^2 + yy' = 0$, $x^2 + y^2 = 1$

4. Verify by implicit differentiation that the given equation implicitly defines a solution of the differential equation. Determine c so that the resulting particular solution satisfies the given initial condition

   (a) $\tan^{-1}(y/x) + x^2 = c$; $\dfrac{2x^3 + 2xy^2 - y}{x^2 + y^2} + \dfrac{x}{x^2 + y^2}y' = 0$, $y(0) = 1$

   (b) $x^2 + 4y^2 = c$, $4yy' + x = 0$, $y > 0$, $y(2) = 1$

   (c) $y = c \sec x$; $y' = y \tan x$, $y(0) = \pi/2$

   (d) $y = -\dfrac{1}{3}x + \dfrac{1}{3} + ce^{3x}$; $y' - 3y = x$, $y(0) = 1$

5. Find the differential equations corresponding to the general solution given as

   (a) $y = Ax^2 - x$ (b) $y = e^{-7x}(Ax + B)$ (c) $(x - A)^2 + (y - B)^2 = 25$

   (d) $y = Ae^{\sqrt{x}} + Be^{\sqrt[3]{x}} + x^2$ (e) $y = e^x[A \cos x + B \sin x]$

   (f) $y = Ae^{2x} + Be^{-3x} + Ce^x$

Find the differential equation in case of the following (6-10):

6. All parabolas whose axes are parallel to the axis of y.
7. All rectangular hyperbolas with axes of coordinates as asymptotes.
8. All circles of radius a.
9. All circles irrespective of their radii and position in the xy-plane.
10. All circles that pass through the origin.

10.3 VARIABLE SEPARABLE FORM

The general differential equation of *first order and first degree* is of the form

$$\frac{dy}{dx} = f(x, y), \qquad \text{...(10.26a)}$$

where $f(x, y)$ is in general function of x and y both. It is not always possible to solve the general equation of this form, we discuss some special forms of equation (10.26a) and analytic methods of solving those forms. In this section we discuss the *variable separable* from.

A differential equation is of variable separable form if it can be expressed as $g(y)\frac{dy}{dx} = f(x)$.

To solve, integrate it both sides with respect to x we obtain

$$\int g(y)\frac{dy}{dx}dx = \int f(x)dx + c, \qquad \text{...(10.27)}$$

where c is a constant of integration.

By differential calculus, $\frac{dy}{dx}dx = dy$, thus (10.27) is

$$\int g(y)\,dy = \int f(x)dx + c. \qquad \text{...(10.28)}$$

In case $g(y)$ and $f(x)$ are continuous functions, then integrals in (10.28) exist and we get the solution. Practically we write the equation in the form $g(y)dy = f(x)dx$, and integrate both sides to find the solution.

Example 10.4: Solve the differential equation, $dx/dy = x \tan y$.

Solution: The equation can be written as, $dx/x = \tan y\, dy$. Integrating, we get

$$\ln|x| = -\ln|\cos y| + c, \text{ or } \ln|x \cos y| = c, \text{ or } x\cos y = \pm e^c = \pm a,$$

where a is an arbitrary constant.

Example 10.5: Solve the differential equation

$$\frac{dy}{dx} = y^2e^{-x} \qquad \text{...(10.29)}$$

Solution: The given equation for $y \neq 0$, can be written as $dy/y^2 = e^{-x}dx$.

Integrating, we obtain $-1/y = -e^{-x} + c$, or $y = 1/(e^{-x} - c)$ ...(10.30)

as the general solution of the given equation.

We must note that (10.30) has been obtained for $y \neq 0$, but $y(x) = 0$ is also a solution of (10.29), which cannot be obtained from (10.30) for any choice of c. Thus, $y(x) = 0$ is a *singular solution* of (10.29).

Example 10.6: Solve the initial value problem $L\dfrac{dI}{dt} + RI = 0$, $I(0) = I_0$, where I, R, L are respectively the current, resistance and inductance in an electrical circuit.

Solution: For $I, L \neq 0$, the given differential equation can be written as $\dfrac{dI}{I} = -\dfrac{R}{L}dt$.

Integrating, we obtain $\ln I = -\dfrac{R}{L}t + c$, or $I = e^{-\frac{R}{L}t + c} = ke^{-\frac{R}{L}t}$ as the general solution, where k is a constant.

Applying the initial condition $I(0) = I_0$, we obtain $k = I_0$, and thus $I = I_0e^{-(R/L)t}$ is the desired solution.

Example 10.7: Solve the differential equation $(x + y + 1)^2\dfrac{dy}{dx} = 1$.

Solution: Substituting $x + y + 1 = t$, we get $1 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$, or $\dfrac{dy}{dx} = \dfrac{dt}{dx} - 1$.

The given differential equation reduces to

$$t^2\left(\frac{dt}{dx} - 1\right) = 1, \text{ or } t^2\frac{dt}{dx} = 1 + t^2, \text{ or } \frac{t^2}{1+t^2}dt = dx$$

which is of separable form. Integrating this, we obtain

$$t - \tan^{-1}t = x + c,$$

where c is a constant of integration. Substituting for t, we obtain

$$x + y + 1 - \tan^{-1}(x + y + 1) = x + c, \quad \text{or } x + y + 1 = \tan(y + 1 - c)$$

as the desired solution.

Example 10.8: Reduce the differential equation $\dfrac{d^2y}{dx^2} + e^{2y}\left(\dfrac{dy}{dx}\right)^3 = 0$ to a lower order equation and hence solve it.

Solution: Put $\frac{dy}{dx} = u$, then $\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$. The equation becomes

$$u\frac{du}{dy} + u^3 e^{2y} = 0.$$

For $u \neq 0$, this takes the form, $\frac{du}{u^2} = -e^{2y}\,dy$. Integrating it, we obtain

$$\frac{1}{u} = \frac{1}{2}e^{2y} + c, \quad \text{or} \quad \frac{dx}{dy} = \frac{1}{2}e^{2y} + c \text{ which gives } dx = \frac{1}{2}(e^{2y} + c)\,dy.$$

Integrating it, we obtain $x = \frac{1}{4}e^{2y} + cy + c'$ as the general solution of the differential equation.

Corresponding to $u = 0$, that is, $\frac{dy}{dx} = 0$, $y = \text{const.}$ is also a solution of the given differential equation.

EXERCISE 10.2

Find the general solution of the following differential equations.

1. $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$
2. $x^2\frac{dy}{dx} = 1 + y$
3. $\frac{dy}{dx} = y^2 e^{-x}$, $y(1) = 4$
4. $\frac{dy}{dx} = (4x + y + 1)^2$, $y(0) = 1$
5. $xy\frac{dy}{dx} = 1 + x + y + xy$
6. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$
7. $\frac{dy}{dx} = \frac{x(2\ln(x) - 1)}{\sin y + y\cos y}$
8. $\frac{dy}{dx}\cosh^2 x - \sin^2 y = 0$, $y(0) = \pi/2$
9. $x\frac{dy}{dx} = y + 3x^4\cos^2(y/x)$, $y(1) = 0$
10. $xy\frac{dy}{dx} = 2y^2 + 4x^2$, $y(2) = 4$
11. $\frac{dy}{dx} = \frac{1 - 2y - 4x}{1 + y + 2x}$, $y(0) = 0$
12. $\frac{y}{x}\frac{dy}{dx} + \frac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$
13. $x^4\frac{dy}{dx} + x^3 y + \operatorname{cosec}(xy) = 0$
14. $\left(x\frac{dy}{dx} - y\right)\cos(y/x) + x = 0$, $y(1) = \pi/2$
15. $2x^2 y\frac{dy}{dx} = \tan(x^2y^2) - 2xy^2$, $y(1) = \sqrt{\pi/2}$

10.4 HOMOGENEOUS EQUATIONS

A homogeneous differential equation of the first order and first degree is one which can be expressed in the form

$$\frac{dy}{dx} = f(y/x). \qquad ...(10.31)$$

The substitution $y = ux$ reduces the equation (10.31) to a separable equation involving the independent variable x and the new dependent variable u as follows.

$y = ux$, gives $\frac{dy}{dx} = u + x\frac{du}{dx}$.

Substituting in (10.31), we get $u + x\frac{du}{dx} = f(u)$ or, $\frac{du}{f(x) - u} = \frac{dx}{x}$

which is of separable form, and hence can be solved accordingly.

Example 10.9: Solve the differential equation $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$.

Solution: The given equation can be written as $\frac{dy}{dx} = \frac{(y/x)^2}{(y/x) - 1}$

Substituting $u = y/x$, this becomes $u + x\frac{du}{dx} = \frac{u^2}{u-1}$, or $x\frac{du}{dx} = \frac{u}{u-1}$.

Separating the variables, we have $\left(1 - \frac{1}{u}\right)du = \frac{dx}{x}$

Integrating, $u - \ln u = \ln x + c$, or $u - c = \ln ux$, or $ux = e^{u-c}$.

Substituting $y = ux$, it gives $y = ke^{y/x}$ as the desired solution, where $k = e^{-c}$ is constant.

Example 10.10: Solve the differential equation $(x + y)\,dy + (x - y)\,dx = 0$.

Solution: The equation can be written as $\frac{dy}{dx} = \frac{y-x}{y+x}$. Substituting $y = ux$, it becomes

$$u + x\frac{du}{dx} = \frac{u-1}{u+1}, \quad \text{or} \quad x\frac{du}{dx} = -\frac{u^2+1}{u+1}$$

Separating the variables we obtain, $-\frac{u+1}{u^2+1}du = \frac{dx}{x}$

or,

$$-\frac{u}{u^2+1}du - \frac{1}{u^2+1}du = \frac{dx}{x}.$$

Integrating, $$-\frac{1}{2}\ln(u^2+1) - \tan^{-1} u = \ln x + c'$$

or, $$2\ln x + \ln(u^2+1) + 2\tan^{-1} u + 2c' = 0$$

or, $$\ln x^2 (u^2+1) + 2\tan^{-1} u + c = 0.$$

Substituting $y = ux$, it gives $\ln(y^2+x^2) + 2\tan^{-1}\frac{y}{x} + c = 0$ as the desired solution.

An equation of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{px+qy+r} \qquad ...(10.32)$$

is called near-homogeneous, since it can be transformed to the homogeneous form as follows.

(a) If $\frac{a}{p} = \frac{b}{q}$, then substitute $px + qy = u$, the Eq. (10.32) reduces to the separable form.

(b) If $\frac{a}{p} \neq \frac{b}{q}$, then apply the transformation that shifts the origin to the point of intersection of the two lines $ax+by+c=0$ and $px+qy+r=0$. The resultant equation can be reduced to the homogeneous form immediately.

The procedure is explained in the examples given below.

Example 10.11: Solve the differential equation $\frac{dy}{dx} = \frac{y-x+1}{y-x+5}$.

Solution: Substituting $y - x = u$, the equation reduces to

$$1 + \frac{du}{dx} = \frac{u+1}{u+5}, \quad \text{or} \quad \frac{du}{dx} = -\frac{4}{u+5}.$$

Separating the variables, we obtain, $(u+5)du = -4dx$. Integrating, it gives

$$\frac{1}{2}u^2 + 5u = -4x + c', \quad \text{or,} \quad u^2 + 10u + 8x = c.$$

Substituting $u = (y-x)$, we obtain $(y-x)^2 + 10(y-x) + 8x = c$ as the desired solution.

Example 10.12: Solve the initial value problem $\frac{dy}{dx} = \frac{y+1}{x+2y}$, $y(2) = 0$.

Solution: The lines $y + 1 = 0$, and $x + 2y = 0$ intersect at the point $(2, -1)$. To transform the given equation into homogeneous form, substitute $x = X + 2$, $y = Y - 1$.

The equation becomes, $\frac{dY}{dX} = \frac{Y}{X+2Y}$. Substituting $Y = uX$, it gives

$$u + X\frac{du}{dX} = \frac{u}{1+2u}, \quad \text{or} \quad X\frac{du}{dX} = -\frac{2u^2}{1+2u}.$$

Separating the variables, we obtain $-\left(\frac{1+2u}{2u^2}\right)du = \frac{dX}{X}$.

Integrating, $\frac{1}{u} = 2\ln|cuX|$, where c is an arbitrary constant.

Setting $u = Y/X$, and $X = x-2$, $Y = y+1$, we get $x = 2 + 2(y+1)\ln|c(y+1)|$ as the general solution.

Using the initial condition $y(2) = 0$, we obtain, $c = 1$, so the solution of the initial value problem is $x = 2 + 2(y+1)\ln|y+1|$.

EXERCISE 10.3

Find the general solution of the following differential equations.

1. $(x^2 - y^2)dx - xy\,dy = 0$
2. $(y^2 - 2xy)\,dx = (x^2 - 2xy)dy$
3. $(xy - x^2)dy = y^2dx$
4. $\frac{dy}{dx} = [2x + y\cos^2(y/x)]/[x\cos^2(y/x)]$
5. $\frac{dy}{dx} = [x + y\sin^2(y/x)]/[x\sin^2(y/x)]$
6. $ye^{x/y}\,dx = (xe^{x/y} + y^2)dy$

Solve the initial value problems

7. $(3xy + y^2)dx + (x^2 + xy)dy = 0, \quad y(1) = 1$
8. $(1 + e^{x/y})\,dx + e^{x/y}(1 - x/y)dy = 0, \quad y(0) = 1$

Find the general solution of the following differential equations

9. $(y - x - 4)dy = (y + x - 2)dx$
10. $(x - 2y + 1)dy = (3x - 6y + 2)dx$
11. $(x + y + 2)dy = (y + 3)dx$
12. $(x + 2y)\,(dx - dy) = dx + dy$
13. $ydx - [x - y\cos(x/y)]dy = 0$
14. $x^2dy - xy\,dx + y^2e^{x^2/y^2}\,dy = 0$

10.5 EXACT DIFFERENTIAL EQUATIONS

The first order ordinary differential equation expressed in the form

$$M(x, y)dx + N(x, y)dy = 0 \qquad ...(10.33)$$

is said to be an exact differential equation, if there exists a function $f(x, y)$ such that the total differential,

$$d[f(x, y)] = M(x, y)dx + N(x, y)dy. \qquad ...(10.34)$$

In that case the general solution of (10.33) must be $f(x, y)$ = constant.

For example, consider the differential equation

$$\sin y \, dx + (x \cos y - 2y) \, dy = 0. \qquad ...(10.35)$$

We note that the left-hand side is the total differential of $f(x, y) = x \sin y - y^2$, thus (10.35) is simply $d(x \sin y - y^2) = 0$, which can be integrated to give $x \sin y - y^2$ = constant, as the general solution of (10.35).

Necessary and Sufficient Condition for Exactness

We consider the following important question.

Is there a test for exactness, and if, an equation is exact, is it possible to find its general solution?

The total differential of $f(x,y)$ is expressed as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \qquad ...(10.36)$$

In case of exact equation, from (10.33) and (10.36), we obtain

$$\frac{\partial f}{\partial x} = M(x, y), \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y).$$

Assuming that $M(x, y)$, $N(x, y)$ have continuous partial derivatives of second order in the region under consideration, we obtain

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}, \text{ which give}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad ...(10.37)$$

as the condition for the Eq. (10.33) to be an exact.

The condition (10.37) is not only necessary but also the sufficient condition for Eq. (10.33) to be an exact differential equation.

To prove the sufficient part, let $u = \int M dx$; here y is considered to be constant while integrating. Differentiating it partially w.r.t. x, we obtain

$$\frac{\partial u}{\partial x} = M, \text{ which gives } \frac{\partial^2 u}{\partial y \, \partial x} = \frac{\partial M}{\partial y}.$$

From (10.37), $$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial^2 x}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right).$$

Integrating w.r.t. x, taking y to be constant, we obtain

$$N = \frac{\partial u}{\partial y} + f(y), \qquad ...(10.38)$$

where $f(y)$ is a function of y alone.

Thus $Mdx + Ndy = \frac{\partial u}{\partial x}dx + \left(\frac{\partial u}{\partial y} + f(y)\right)dy = du + f(y)dy = d\left[u + \int f(y)\,dy\right]$ ...(10.39)

This proves the sufficient part.

We observe that Eq. (10.39), gives a procedure to find the general solution in case of an equation is exact. The equation $Mdx + Ndy = 0$ in case of being exact becomes

$$d\left[u + \int f(y)dy\right] = 0, \quad \text{or} \quad u + \int f(y)dy = c, \qquad ...(10.40)$$

where, $u = \int Mdx$; treating y to be constant,

$f(y)$ = terms of N not containing x, refer to Eq. (10.38), and c is an arbitrary constant.

Therefore the general solution of an exact equation $Mdx + Ndy = 0$ is

$$\int Mdx + \int (\text{terms of } N \text{ not containing } x)\, dy = c, \qquad ...(10.41)$$

where $\int Mdx$ is performed on the supposition that y is constant.

Example 10.13: Test the equation $\frac{dy}{dx} = -\frac{2xy^3 + 2}{3x^2y^2 + 8e^{4y}}$ for its exactness. If so, find its general solution.

Solution: The equation can be written as $(2xy^3 + 2)dx + (3x^2y^2 + 8e^{4y})dy = 0$.

We have $M(x, y) = 2xy^3 + 2$, and $N(x, y) = 3x^2y^2 + 8e^{4y}$. Thus

$$\frac{\partial M}{\partial y} = 6xy^2, \quad \text{and} \quad \frac{\partial N}{\partial x} = 6xy^2.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given differential equation is exact, therefore, the solution is

$$\int_{(y\ \text{const.})} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = \text{constant. This gives,}$$

$\int_{(y\ \text{const.})} (2xy^3 + 2)dx + \int 8e^{4y}dy = \text{constant}$, or $x^2y^3 + 2x + 2e^{4y} = c$ as the general solution, where c is an arbitrary constant.

Example 10.14: Solve the initial value problem

$$\{3x^2 + 2y + 2\cosh(2x + 3y)\}dx + \{2x + 2y + 3\cosh(2x + 3y)\}dy = 0, \; y(0) = 0.$$

Solution: Here, $M(x, y) = 3x^2 + 2y + 2\cosh(2x + 3y)$, and $N(x, y) = 2x + 2y + 3\cosh(2x + 3y)$

Thus, $\dfrac{\partial M}{\partial y} = 2 + 6\sinh(2x + 3y)$, and $\dfrac{\partial N}{\partial x} = 2 + 6\sinh(2x + 3y)$.

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given differential equation is exact, therefore, the solution is

$$\int_{(y\text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = \text{const.}$$

This gives, $\int [3x^2 + 2y + 2\cosh(2x + 3y)]dx + \int 2y\, dy = \text{const.}$

or, $x^3 + 2yx + \sinh(2x + 3y) + y^2 = c$

as the general solution, where c is an arbitrary constant.

Substituting $y(0) = 0$, it gives $c = 0$, therefore, $x^3 + 2yx + \sinh(2x + 3y) + y^2 = 0$ is the solution of the initial value problem.

EXERCISE 10.4

Test the following equations for their exactness, if so, find the solution.

1. $(3x^2 + 2e^y)dx + (2xe^y + 3y^2)dy = 0$
2. $\sin y\, dx + (x\cos y - 2y)dy = 0$
3. $(y^2 e^{xy^2} + 4x^3)dx + (2xy\, e^{xy^2} - 3y^2)dy = 0$
4. $(4xy + 2x^2 y)dx + (2x^2 + 3y^2)dy = 0$
5. $(\sinh x \sinh y)dx + (\cosh x \cosh y)dy = 0$
6. $yx^{y-1}\, dx + x^y \ln x\, dy = 0$

Test the following initial value problems for their exactness, if so, find the solution

7. $e^x(\cos y\, dx - \sin y\, dy) = 0; \quad y(0) = 0$
8. $(3y^4 - 1)dx + 12xy^3 dy = 0; \quad y(1) = 2$
9. $[2x - y\sin(xy)]dx + [3y^2 - x\sin(xy)]dy = 0; \quad y(0) = 2$
10. $(3x^2 \sin 2y - 2xy)dx + (2x^3 \cos 2y - x^2)\, dy = 0; \quad y(0.5) = 3.1$

Under what conditions the following differential equations are exact

11. $(ax^2 + by^2 + cxy + dx + ey + f)dx + (Ax^2 + By^2 + Cxy + Dx + Ey + F)dy = 0$
12. $(a\sinh x \cos y + b\cosh x \sin y)dx + (c\sinh x \cos y + d\cosh x \sin y)dy = 0$

10.6 EQUATIONS REDUCIBLE TO EXACT FORM. INTEGRATING FACTORS

Most differential equations of the form $Mdx + Ndy = 0$ are not exact on the domain of definition but can be made exact by multiplying with a suitable factor $\mu(x, y)$, called an *integrating factor*. For example, the differential equation

$$-ydx + xdy = 0 \quad \text{...(10.42)}$$

is not exact. But if we multiply it by $1/x^2$, we get an exact equation

$$-\frac{y}{x^2}dx + \frac{1}{x}dy = d\left(\frac{y}{x}\right) = 0$$

with the general solution $y/x = \text{const.}$

We can see very easily that $1/x^2$ is not the only integrating factor of Eq. (10.42). The other integrating factors are $1/y^2, 1/xy$ and $1/(x^2 + y^2)$.

There is no general method of finding an integrating factor. In simpler cases it can be found by inspection. However, in general, the problem can be worked out as follows.

Let the equation

$$Mdx + Ndy = 0 \quad \text{...(10.43)}$$

fails to satisfy the test for exactness and suppose it is possible to find a multiplicative factor $\mu(x, y) \neq 0$, so that

$$\mu(x, y)\, M(x, y) dx + \mu(x, y)\, N(x, y) dy = 0 \quad \text{...(10.44)}$$

is exact. Thus we need to find the function $\mu(x, y)$ such that the condition

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N), \quad \mu(x, y) \neq 0 \quad \text{...(10.45)}$$

is satisfied.

The multiplicative factor $\mu(x, y)$ satisfying (10.45) is called an *integrating factor* (I.F.) of the Eq. (10.43). From (10.45), we have

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x, \quad \text{...(10.46)}$$

a first order partial differential equation which again may not be easy to solve. To simplify the problem we consider the following two cases:

(a) *Integrating factor is a function of x alone;* $\mu = \mu(x)$. In this case $\mu_y = 0$ and $\mu_x = \frac{d\mu}{dx}$, so Eq. (10.46) becomes, $\mu M_y = \frac{d\mu}{dx}N + \mu N_x$, which gives

$$\frac{d\mu}{\mu} = \left(\frac{M_y - N_x}{N}\right)dx \quad \text{...(10.47)}$$

a differential equation in separable form, in case $(M_y - N_x)/N$ is a function of x alone. Integrating (10.47) gives

$$\mu(x) = e^{\int\left(\frac{M_y - N_x}{N}\right)dx} \quad \text{...(10.48)}$$

as the integrating factor for the Eq. (10.43), provided $(M_y - N_x)/N$ is a function of x only.

(b) *Integrating factor is a function of y alone;* $\mu = \mu(y)$. Proceeding on the similar lines as in case (a) above, we get

$$\mu(y) = e^{-\int\left(\frac{M_y - N_x}{M}\right)dy} \quad ...(10.49)$$

as the integrating factor for Eq (10.43) provided $(M_y - N_x)/M$ is a function of y only.

A few other results for finding the integrating factor are:

1. If the functions $M(x, y)$ and $N(x, y)$ in the equation $Mdx + Ndy = 0$ are homogeneous functions of degree n and $Mx + Ny \neq 0$, then $1/(Mx + Ny)$ is an integrating factor. If $Mx + Ny = 0$, then $1/(xy)$, or $1/x^2$, or $1/y^2$ are the integrating factors.

2. If $M(x, y) = f(xy)y$ and $N(x, y) = g(xy)x$ for some functions f and g, then $1/(Mx - Ny)$ is an integrating factor provided $Mx - Ny \neq 0$. In case $Mx - Ny = 0$, then $Mdx + Ndy = 0$ is an exact equation.

Example 10.15: Solve the following differential equations

(a) $xdx + ydy + 2(x^2 + y^2)dx = 0$, (b) $y(2xy + e^x)dx = e^x dy$.

Solution: (a) Multipiying the given differential equation by the integrating factor $1/(x^2 + y^2)$, we obtain

$$\frac{x\,dy + y\,dy}{x^2 + y^2} + 2dx = 0, \text{ or } d(\text{In}(x^2 + y^2)) + 4dx = 0$$

Integrating, we get $\ln(x^2 + y^2) + 4x = c$, as the general solution, where c is an arbitrary constant.

(b) Rearranging the given equation as, $ye^x dx - e^x dy + 2xy^2 dx = 0$

Multiplying by the integrating factor $1/y^2$, we obtain

$$\frac{ye^x\,dx - e^x\,dy}{y^2} + 2xdx = 0, \text{ or } d\left(\frac{e^x}{y}\right) + 2xdx = 0.$$

Integrating, we get $\dfrac{e^x}{y} + x^2 = c$, as the general solution, where c is an arbitrary constant.

Example 10.16: Solve $2\sin y^2\,dx + xy\cos y^2\,dy = 0$.

Solution: We have, $M = 2\sin y^2$ and $N = xy\cos y^2$

Since, $\dfrac{\partial M}{\partial y} = 4y\cos y^2 \neq \dfrac{\partial N}{\partial x} = y\cos y^2$, therefore, the equation is not exact.

Consider, $\dfrac{M_y - N_x}{N} = \dfrac{3y\cos y^2}{xy\cos y^2} = \dfrac{3}{x}$, is a function of x alone.

Therefore, I.F. $= e^{\int\frac{M_y - N_x}{N}dx} = e^{\int\frac{3}{x}dx} = e^{3\ln x} = x^3$.

Multiplying the given equation by x^3, the resultant equation is

$$2x^3\sin y^2\,dx + x^4 y\cos y^2\,dy = 0 \quad ...(10.50)$$

It is easy to verify that (10.50) is an exact equation. Thus, the general solution is

$$2\int_{(y\text{ const.})} x^3 \sin y^2 dx + 0 = c, \text{ which gives, } x^4 \sin y^2 = c$$

where c is an arbitrary constant.

Example 10.17: Solve the initial value problem $dx + (3x - e^{-2y})\, dy = 0, \quad y(0) = 0.$

Solution: We have, $M = 1$ and $N = (3x - e^{-2y})$.

Since, $\dfrac{\partial M}{\partial y} = 0 \neq \dfrac{\partial N}{\partial x} = 3$, therefore, the equation is not exact.

Here, $\dfrac{M_y - N_x}{M} = -3$, can be considered a function of y alone. Thus, I.F. $= e^{-\int(-3)dy} = e^{3y}$.

Multiplying the given equation by e^{3y}, the resultant equation is

$$e^{3y}dx + (3x - e^{-2y})\, e^{3y}\, dy = 0 \qquad ...(10.51)$$

It is easy to verify that (10.51) is an exact equation. Thus the general solution is

$$\int_{(y\text{ const.})} e^{3y}\, dx - \int e^{y} dy = c, \text{ which gives, } xe^{3y} - e^{y} = c,$$

where c is an arbitrary constant. Using the initial value $y(0) = 0$, it gives, $c = -1$. Thus, the solution of the given initial problem is $xe^{3y} = e^{y} - 1$.

Example 10.18: Solve $(3xy - 2ay^2)\, dx + (x^2 + 2axy)\, dy = 0$

Solution: We have, $M = 3xy - 2ay^2$ and $N = x^2 + 2axy$

Since $\dfrac{\partial M}{\partial y} = 3x - 4ay \neq \dfrac{\partial N}{\partial x} = 2x + 2ay$, therefore, the equation is not exact. We observe that M and N both are homogeneous functions of degree 2 and $Mx + Ny = 3x^2y - 2axy^2 + x^2y + 2axy^2 = 4x^2y \neq 0$. Therefore the integrating factor is

$$\mu(x, y) = \frac{1}{Mx + Ny} = \frac{1}{4x^2y}.$$

Multiplying the given equation by $\dfrac{1}{4x^2y}$, the resultant equation is

$$\left[\frac{3}{4x} - \frac{ay}{2x^2}\right] dx + \left[\frac{1}{4y} - \frac{a}{2x}\right] dy = 0. \qquad ...(10.52)$$

It is easy to verify that (10.52) is exact. Therefore the general solution is

$$\int_{(y\text{ const.})} \left(\frac{3}{4x} - \frac{ay}{2x^2}\right) dx + \int \frac{1}{4y} dy = c, \text{ or } -\frac{3}{4x^2} + \frac{ay}{x^3} + \frac{1}{4} \ln y = c, \text{ where } c \text{ is an arbitrary constant.}$$

Exmaple 10.19: Solve $(x^2y^2 + xy + 1)\, ydx + (x^2y^2 - xy + 1)\, xdy = 0$.

Solution: We have $M = (x^2y^2 + xy + 1)y$ and $N = (x^2y^2 - xy + 1)x$

Since, $\dfrac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1 \neq \dfrac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$, therefore, the equation is not exact. However, we observe that $M = f(xy)y$, and $N = g(xy)x$, where $f(xy) = x^2y^2 + xy + 1$ and $g(xy) = x^2y^2 - xy + 1$, and, since $Mx - Ny = (x^2y^2 + xy + 1)yx - (x^2y^2 - xy + 1)xy = 2x^2y^2 \neq 0$, therefore, the integrating factor is

$$\mu(x, y) = \frac{1}{M_x - N_y} = \frac{1}{2x^2y^2}.$$

Multiplying the given equation by $\dfrac{1}{2x^2y^2}$, the resultant equation is

$$\frac{1}{2}\left(1 + \frac{1}{xy} + \frac{1}{x^2y^2}\right)ydx + \frac{1}{2}\left(1 - \frac{1}{xy} + \frac{1}{x^2y^2}\right)xdy = 0$$

or,
$$\left(y - \frac{1}{x} + \frac{1}{x^2y}\right)dx + \left(x - \frac{1}{y} + \frac{1}{x^2y}\right)dy = 0. \qquad ...(10.53)$$

It is easy to verify that (10.53) is exact. Hence the general solution is

$$\int_{(y\text{ const.})}\left(y + \frac{1}{x} + \frac{1}{x^2y^2}\right)dx + \int\left(-\frac{1}{y}\right)dy = c, \text{ or } xy + \ln x - \frac{1}{xy^2} - \ln y = c, \text{ or } xy + \ln\frac{x}{y} - \frac{1}{xy^2} = c,$$

where c is an arbitrary constant.

EXERCISE 10.5

Solve the following differential equations.

1. $ydx - xdy + e^{1/x}\,dx = 0$
2. $xdy - ydx + a(x^2 + y^2)dx = 0$
3. $x(1 + y^2)dy + y(1 + x^2)dx = 0$
4. $dy/dx = (x^3 + y^3)/xy^2$
5. $(5x^3 + 12x^2 + 6y^2)dx + 6xy\,dy = 0$
6. $\cos y\, dx - [2(x - y)\sin y + \cos y]dy = 0$
7. $(3x^2y^3e^y + y^3 + y^2)dx + (x^3y^3e^y - xy)dy = 0$
8. $(3x^2\sinh 3y - 2x)dx + 3x^3\cosh 3y\, dy = 0$
9. $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$
10. $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$
11. Show that for $[p(x) + q(y)]dx + [r(x) + s(y)]dy = 0$ to be exact, it is necessary and sufficient that $q(y)dx + r(x)dy = 0$, be an exact differential equation.
12. Show that $\mu(x, y)$ is an integrating factor of $M(x, y)dx + N(x, y)dy = 0$ if, and only if,

$$\left(M\frac{\partial\mu}{\partial y} - \frac{\partial\mu}{\partial x}\right) + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = 0$$

10.7 LINEAR FORM: THE LEIBNITZ'S EQUATION

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad ...(10.54)$$

where P and Q are functions of x only, is called a first order linear differential equation in y.

If $Q(x) = 0$, for all x in the interval under consideration, the equation is said to be homogeneous, otherwise, it is said to be non-homogeneous. It is easy to see that in the homogeneous case (10.54) reduces to variable separable form.

To obtain the solution of the non-homogeneous equation, multiply Eq. (10.54) by an integrating factor, say $\mu(x)$, a function of x only. We obtain

$$\mu\frac{dy}{dx} + \mu Py = \mu Q(x) \quad ...(10.55)$$

The left hand side of Eq. (10.55) is equal to $\frac{d}{dx}(\mu y)$, when μ is chosen as such that

$$\mu P = \frac{d\mu}{dx}, \text{ or } \frac{d\mu}{\mu} = Pdx.$$

Integrating, we obtain

$$\mu = Ae^{\int Pdx}, \quad ...(10.56)$$

where A is a constant. Substituting (10.56) in (10.55), we obtain

$$Ae^{\int Pdx}\frac{dy}{dx} + Ae^{\int Pdx}Py = Ae^{\int Pdx}Q(x), \text{ or } \frac{d}{dx}\left(ye^{\int Pdx}\right) = e^{\int Pdx}Q.$$

Integrating we obtain, $\quad ye^{\int Pdx} = \int e^{\int Pdx}Qdx + c$, which gives

$$y = e^{-\int Pdx}\left[\int e^{\int Pdx}Qdx + c\right] \quad ...(10.57)$$

as the general solution of the Eq. (10.54), where c is an arbitrary function.

Thus it is important to remember that in case of Eq. (10.54),

Integrating factor is, $\mu(x) = e^{\int Pdx}$ *and, the general solution is,* $y\mu(x) = \int \mu(x).Q(x)\,dx + c.$

Also we note that the choice of the value of the constant of integration in $\int Pdx$ does not matter, since it is an integrating factor.

Example 10.20: Solve $\frac{dy}{dx} - y = e^{2x}$.

Solution: Here $P = -1$, $Q = e^{2x}$. The integrating factor, $\mu(x) = e^{\int P dx} = e^{-x}$.

Thus the general solution of the given equation is

$$ye^{-x} = \int e^{-x} . e^{2x} + c = e^{x} + c \text{ or, } y = e^{2x} + ce^{x}$$

where c is an arbitrary constant.

Example 10.21: Solve the initial value problem $\cos x \frac{dy}{dx} + y = \sin x, \quad y(0) = 2$.

Solution: The equation can be written as $\frac{dy}{dx} + \frac{1}{\cos x} y = \tan x$.

Here, $P(x) = \frac{1}{\cos x}$, $Q(x) = \tan x$. The integrating factor is

$$\mu = e^{\int P dx} = e^{\int \sec x \, dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x = \frac{1 + \sin x}{\cos x}.$$

Thus the general solution of the equation is given by

$$y \frac{1 + \sin x}{\cos x} = \int \left(\frac{1 + \sin x}{\cos x} \right) \tan x \, dx + c$$

$$= \int \sec x \tan x \, dx + \int \tan^2 x \, dx + c = \sec x + \tan x - x + c,$$

where c is an arbitrary constant. Using the initial condition $y(0) = 2$, gives $c = 1$, so the solution of the given initial value problem is $y = 1 + \frac{(1 - x)\cos x}{1 + \sin x}, \quad \sin x \neq -1$.

Example 10.22: Solve, $r \sin \theta \, d\theta + (r^3 - 2r^2 \cos \theta + \cos \theta) \, dr = 0$.

Solution: The given equation can be rewritten as

$$-\sin \theta \frac{d\theta}{dr} + \left(2r - \frac{1}{r} \right) \cos \theta = r^2 \qquad ...(10.58)$$

Put $\cos \theta = \phi$, it gives $-\sin \theta \frac{d\theta}{dr} = \frac{d\phi}{dr}$; substituting in (10.58), we obtain

$$\frac{d\theta}{dr} + \left(2r - \frac{1}{r} \right) \phi = r^2, \qquad ...(10.59)$$

a linear equation of first order in ϕ.

Here, $P(r) = \left(2r - \frac{1}{r}\right)$ and $Q(r) = r^2$. The integrating factor is

$$\mu(r) = e^{\int P dr} = e^{\int\left(2r - \frac{1}{r}\right)dr} = e^{r^2 - \ln r} = \frac{1}{r}e^{r^2}.$$

Therefore, the general solution of (10.59) is

$$\phi \cdot \frac{1}{r}e^{r^2} = \int \frac{1}{r}e^{r^2} \cdot r^2 dr + c = \int re^{r^2}\, dr + c = \frac{1}{2}e^{r^2} + c'$$

or,

$$\phi = \frac{1}{2}r\left(1 + 2c' e^{-r^2}\right).$$

Substituting $\phi = \cos\theta$, the solution of the given equation is $\cos\theta = \frac{1}{2}r\left(1 + ce^{-r^2}\right)$, where $c = 2c'$ is an arbitrary constant.

EXERCISE 10.6

Solve the following differential equations:

1. $\frac{dy}{dx} + 1 = -\frac{y}{x}$
2. $x\left(\frac{dy}{dx} + 1\right) = x^3 - 2y, \quad y(1) = 3$
3. $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$
4. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$
5. $(1 + y^2)dx = (\tan^{-1} y - x)dy$
6. $\frac{dy}{dx} + 2(1 - y)\tanh 2x = 0, \quad y(0) = 4$
7. $x\frac{dy}{dx} - 3y = x^4(e^x + \cos x) - 2x^2, \quad y(\pi) = \pi^3 e^\pi + 2\pi^2$
8. $y \ln y\, dx + (x - \ln y)dy = 0$
9. $dr + (2r\cot\theta + \sin 2\theta)d\theta = 0, \quad y(\pi/2) = 1$
10. $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$
11. $e^{-y}\sec^2 y\, dy = dx + x\,dy$
12. $\frac{dy}{dx} = \frac{y}{2y\ln y + y - x}, \quad y(0) = 1$
13. Find the integrating factor of the non-homogeneous first order linear differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ by comparing this with the equation $M(x, y)dx + N(x, y)dy = 0$, and hence solve it.

10.8 THE BERNOULLI, THE RICCATI AND THE CLAIRAUT'S EQUATIONS

The Bernoulli equation is an important first order equation with many applications, for example, it occurs in solid and fluid mechanics. It is special in the sense that it can be seen at the boundary between linear and non-linear first order differential equations. The Riccati equation is an other non-linear equation which occurs in important applications. Another equation of interest is Clairaut's equation. We discuss these three equations in this section.

10.8.1 The Bernoulli Equation

The Bernoulli equation is a non-linear first order differential equation with the standard form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n. \quad ...(10.60)$$

It is separable if, $n = 1$ and linear if, $n = 0$. A Bernoulli equation with $n \neq 1$ transforms to linear under the change of variable $u = y^{1-n}$.

Dividing (10.60) both sides by y^n, we obtain

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x).$$

The substitution $u = y^{1-n}$ reduces this to a first-order linear differential equation

$$\frac{du}{dx} + (1-n)\,P(x)u = (1-n)Q(x),$$

which can be solved by the method as discussed in Section 10.7. By replacing u by y^{1-n}, the solution for the Eq. (10.60) is obtained.

Example 10.23: Solve $\dfrac{dy}{dx} + \dfrac{1}{x}y = 3x^2y^3.$

Solution: The given equation is a Bernoulli equation with $n = 3$. Dividing both sides by y^{-3}, we obtain

$$y^{-3}\frac{dy}{dx} + \frac{1}{x}y^{-2} = 3x^2. \quad ...(10.61)$$

Substituting $u = y^{-2}$, gives $\dfrac{du}{dx} = -2y^{-3}\dfrac{dy}{dx}$, and hence the Eq. (10.61) becomes

$$\frac{du}{dx} - \frac{2}{x}u = -6x^2, \quad ...(10.62)$$

a first order linear equation with, $P(x) = -2/x$ and $Q(x) = -6x^2$. The integrating factor is

$$\mu(x) = e^{\int P dx} = e^{-\int \frac{2}{x}dx} = e^{-2\ln(x)} = 1/x^2.$$

Thus, the general solution of (10.62) is

$$u\,\frac{1}{x^2} = \int -6x^2\frac{1}{x^2}dx + c = -6x + c, \text{ or } u = x^2(c - 6x),$$

where c is an arbitrary constant. Substituting $u = 1/y^2$, we obtain, $y^2 = 1/x^2(c - 6x)$, as the general solution of the given differential equation.

Example 10.24: Solve $\cos x \dfrac{dy}{dx} + (\sin x)y = \sqrt{y \sec x}$.

Solution: The given equation can be written as

$$\frac{dy}{dx} + (\tan x)\, y = \sec^{3/2} x y^{1/2}, \qquad ...(10.63)$$

which is a Bernoulli equation with $n = 1/2$.

Dividing (10.63) throughout by $y^{-1/2}$, we obtain

$$y^{-1/2}\frac{dy}{dx} + \tan x\, y^{1/2} = \sec^{3/2} x. \qquad ...(10.64)$$

Substituting $u = y^{1/2}$ gives $\dfrac{du}{dx} = \dfrac{1}{2} y^{-\frac{1}{2}} \dfrac{dy}{dx}$; Eq. (10.64) becomes

$$\frac{du}{dx} + \frac{1}{2}\tan x . u = \frac{1}{2}\sec^{3/2} x \qquad ...(10.65)$$

a linear equation in u, with $P(x) = \dfrac{1}{2}\tan x$ and $Q(x) = \dfrac{1}{2}\sec^{3/2} x$. The integrating factor is

$$\mu(x) = e^{\int P dx} = e^{\frac{1}{2}\int \tan x\, dx} = e^{\ln \sqrt{\sec x}} = \sqrt{\sec x}.$$

The general solution of (10.65) is

$$u\sqrt{\sec x} = \int \sqrt{\sec x}\, \frac{1}{2}\sec^{3/2} x\, dx + c' = \frac{1}{2}\int \sec^2 x\, dx + c' = \frac{1}{2}\tan x + c'$$

or,
$$2u\sqrt{\sec x} = \tan x + 2c' \qquad ...(10.66)$$

where c' is an arbitrary constant.

Substituting $u = y^{1/2}$ in (10.66) and simplifying, we obtain $y = \dfrac{1}{4}(\tan x + c)^2 \cos x$, as the general solution of the given differential equation, where $c = 2c'$ is an arbitrary constant

10.8.2 The Riccati Equation

The Riccati equation is a non-linear first order differential equation with the standard form

$$\frac{dy}{dx} = P(x)\, y^2 + Q(x)y + R(x). \qquad ...(10.67)$$

If $R(x) = 0$, then it reduces to the Bernoulli equation with $n = 2$ and thus can be solved accordingly. If $P(x) = 0$, then it reduces to the linear form.

In case $R(x) \neq 0$, there does not exist any simple method for solving (10.67). However, if somehow, we can obtain one solution $v(x)$ of (10.67), then the change of variable $y = v(x) + \frac{1}{z}$ transforms the Riccati equation to a linear equation, as follows.

$y = v + \frac{1}{z}$, gives $\frac{dy}{dx} = \frac{dv}{dx} - \frac{1}{z^2}\frac{dz}{dx}$. Substituting in (10.67), we obtain

$$\frac{dv}{dx} - \frac{1}{z^2}\frac{dz}{dx} = P\left(v + \frac{1}{z}\right)^2 + Q\left(v + \frac{1}{z}\right) + R$$

$$= (Pv^2 + Qv + R) + P\left(\frac{2v}{z} + \frac{1}{z^2}\right) + \frac{Q}{z}. \quad ...(10.68)$$

Since v is a solution of (10.67), thus from (10.68), we obtain

$$-\frac{1}{z^2}\frac{dz}{dx} = P\left(\frac{2v}{z} + \frac{1}{z^2}\right) + \frac{Q}{z}.$$

Simplifying, we obtain $\frac{dz}{dx} + (2Pv + Q)z = -P$, a first order linear differential equation.

Example 10.25: Solve the Riccati equation, $\frac{dy}{dx} = \frac{1}{x}y^2 + \frac{1}{x}y - \frac{2}{x}$.

Solution: By inspection $y = 1$ is a solution of the given differential equation.

$y = 1 + \frac{1}{z}$, gives $\frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$. The given equation becomes

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{1}{x}\left(1 + \frac{1}{z}\right)^2 + \frac{1}{x}\left(1 + \frac{1}{z}\right) - \frac{2}{x}, \text{ or } \frac{dz}{dx} + \frac{3}{x}z = -\frac{1}{x}$$

which is a linear equation in z with $P(x) = \frac{3}{x}$ and $Q(x) = -\frac{1}{x}$.

The integrating factor is, $\mu(x) = e^{3\int \frac{1}{x}dx} = e^{\ln(x^3)} = x^3$. Thus the general solution is

$$zx^3 = \int x^3\left(-\frac{1}{x}\right)dx + c' = -\frac{x^3}{3} + c',$$

where c' is an arbitrary constant. Substituting, $z = 1/(y - 1)$ in it and simplifying, we get, $y = 1 + \frac{3x^3}{c - x^3}$, as the general solution of the given equation, where $c = 3c'$ is an arbitrary constant.

10.8.3 The Clairaut's Equation

The Clairaut's equation is of the form

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right), \quad ...(10.69)$$

where f is a known function of $\frac{dy}{dx}$. This equation is linear in y and x, and always has a singular solution. Substituting $p = \frac{dy}{dx}$, (10.69) takes the form

$$y = xp + f(p). \quad ...(10.70)$$

Differentiating both sides of (10.70) w.r.t. x, we have

$$\frac{dy}{dx} = p + \frac{xdp}{dx} + \frac{df}{dp}\cdot\frac{dp}{dx} \text{ or, } \left(x + \frac{df}{dp}\right)\frac{dp}{dx} = 0. \text{ Thus, } \frac{dp}{dx} = 0, \quad \text{or} \quad x + \frac{df}{dp} = 0.$$

In case $\frac{dp}{dx} = 0$, it gives $p =$ constant, say c.

Substituting $p = c$ in (10.70), the general solution of the Clairaut's equation is

$$y = xc + f(c) \quad ...(10.71)$$

a one parameter family of straight lines with c as parameter.

In case $x + \frac{df}{dp} = 0$. It gives $x = -f'(p)$.

Hence the parametric equations $x = -f'(t)$, $y = f(t) - tf'(t)$ not involving any constant give another solution of the of the Clairaut's equation. It is a singular solution and is the envelope of the one parameter family of straight lines represented by the general solution (10.71).

Example 10.26: Solve $y = xy' - (y')^2$.

Solution: The given equation is Clairaut's equation. Substituting $y' = p$, it becomes

$$y = xp - p^2 \quad ...(10.72)$$

Differentiating w.r.t. x, we have $p = p + x\frac{dp}{dx} - 2p\frac{dp}{dx}$, or $(x - 2p)\frac{dp}{dx} = 0$.

It gives $\frac{dp}{dx} = 0$, or $x - 2p = 0$, that is, $p = c$, or $x = 2p$.

When $p = c$, then the general solution from (10.72) is,

$$y = xc - c^2 \quad ...(10.73)$$

a one parameter family of straight lines.

Also, $x = 2p$, gives the singular solution as, $x = 2t$, $y = 2t^2 - t^2 = t^2$, where t is the parameter.

Eliminating t, it gives, $y = \dfrac{x^2}{4}$, a parabola, as the singular solution. The family represented by the general solution (10.73) is tangential to the singular solution, $y = x^2/4$, refer to Fig. 10.1, [p. 569].

EXERCISE 10.7

Find the general solution of the following Bernoulli equations.

1. $\dfrac{dy}{dx} - x^3y^3 + xy = 0$
2. $\dfrac{dy}{dx} - y = 2xy^{3/2}$
3. $\dfrac{dy}{dx} + \dfrac{y}{x}\ln y = \dfrac{y}{x^2}(\ln y)^2$
4. $(xy^2 - e^{y/x})\,dx - x^2y\,dy = 0$
5. $\dfrac{dy}{dx} + y\tan x = y^3\cos x$
6. $x\dfrac{dy}{dx} + y = x^2y^2\ln x$

Find the general solution of the following Riccati equations, by finding a solution by inspection.

7. $\dfrac{dy}{dx} - 2y^2 + 3y = 1$
8. $\dfrac{dy}{dx} + 2x^2y - 2xy^2 = 1$
9. $\dfrac{dy}{dx} = y^2 - (2x-1)y + (x^2 - x + 1)$
10. $\dfrac{dy}{dx} = 2e^{-x}y^2 + 3y - 4e^x$

Find the general solution and the singular solution of the following equations.

11. $y = xy' + (y')^2$
12. $y = 4xy' - 16y^3(y')^2$
13. $y = xy' - e^{-y'}$
14. $(y'-1)\,e^{3x} - (y')^3\,e^{2y} = 0$

10.9 A GEOMETRICAL APPLICATION: ORTHOGONAL TRAJECTORIES

A trajectory of a family of curves is a curve which cuts the members of the family according to a given law. An orthogonal trajectory is a curve which intersects every member of the family at right angles. Orthogonal families occur in many contexts. For example, parallels and meridians on a globe are orthogonal, as are equipotential and electric lines of force in an electric field, also the lines along which heat flows in a body are orthogonal to the isothermal surfaces. As a simple geometric example, the family consisting of circles about the origin, $x^2 + y^2 = c^2$; and the family of straight lines through the origin, $y = kx$ form orthogonal trajectories, refer to Fig. 10.3.

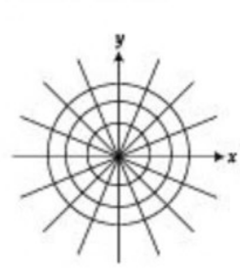


Fig. 10.3

The method of finding the orthogonal trajectory to a given family is quite general and is explained as follows.

Suppose we are given the family of curves by an equation

$$F(x, y, k) = 0, \quad ...(10.74)$$

where k is the parameter. Form its differential equation by eliminating k and let it be

$$f(x, y, y') = 0. \quad ...(10.75)$$

Using the fact that *the two lines are orthogonal if, and only if their slopes are negatives reciprocals, the corresponding differential equation for orthogonal trajectories is obtained from (10.75) by replacing y' with* $-1/y'$. It is given by

$$f(x, y, -1/y') = 0. \quad ...(10.76)$$

Solve (10.76) to obtain the family of orthogonal trajectories to the given family (10.74).

In case the family of curves is in *polar form*, $F(r, \theta, c) = 0$, then the family of orthogonal trajectories is obtained as follows.

If ψ is the angle between the tangent to the curve $F(r, \theta, c) = 0$ at the point (r, θ) and the radius vector, then

$$\tan\psi = r\frac{d\theta}{dr}, \quad \text{refer to Eq. (4.32a)}$$

If ψ' is the corresponding angle for the orthogonal family, then $\psi' = \psi \pm \pi/2$, which gives

$$\tan\psi' = -\cot\psi = -\frac{1}{r}\frac{dr}{d\theta} \quad ...(10.77)$$

Thus to obtain the differential equation of the orthogonal trajectory replace $r\frac{d\theta}{dr}$ *by* $-\frac{1}{r}\frac{dr}{d\theta}$, $\left(\textit{or}, \frac{dr}{d\theta} \text{ by } -r^2\frac{d\theta}{dr}\right)$ *in the differential equation for the family of given curve. Solve the differential equation of the orthogonal trajectories to obtain the desired result.*

In general, two families of curves such that every member of either family cuts each member of the other family at a constant angle, say α are called isogonal trajectories of each other. For orthogonal trajectories $\alpha = 90°$.

Example (10.27): Find the family of orthogonal trajectories to the family of parabolas $y = kx^2$.

Solution: The given family of curves is

$$y = kx^2 \quad ...(10.78)$$

which gives

$$dy/dx = 2kx. \quad ...(10.79)$$

Eliminating k from (10.78) and (10.79), we obtain

$$dy/dx = 2y/x \quad ...(10.80)$$

This is the differential equation of the given family (10.78). The differential equation of the family of orthogonal trajectories is obtained by replacing dy/dx with $-dx/dy$ in (10.80).

Thus we obtain, $-dx/dy = 2y/x$, or $dy/dx = -x/2y$.

This equation is of separable form and gives, $2ydy = -xdx$. Integrating, we obtain, $y^2 = -\frac{1}{2}x^2 + c$, the family of orthogonal trajectories to (10.78), which is a family of ellipse with parameter c as shown in Fig. 10.4.

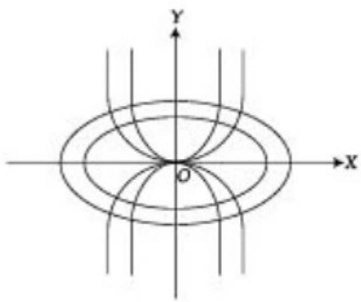


Fig. 10.4

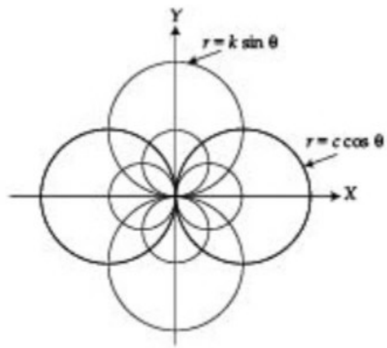


Fig. 10.5

Example 10.28: Find the orthogonal trajectories of the circles $r = k \sin\theta$, where k is a parameter.

Solution: The given family of circles is

$$r = k \sin\theta \qquad ...(10.81)$$

Differentiating (10.81) with respect to θ, we get

$$dr/d\theta = k \cos\theta \qquad ...(10.82)$$

Eliminating k from (10.81) and (10.82), we obtain

$$dr/d\theta = r \tan\theta, \qquad ...(10.83)$$

which is the differential equation for the family of circles (10.81).

Replacing $dr/d\theta$ by $-r^2 d\theta/dr$ in (10.83), we obtain

$$-r d\theta/dr = \tan\theta \qquad ...(10.84)$$

as the differential equation for the orthogonal trajectories.

Rewriting (10.84) as, $dr/r = -\cot\theta\, d\theta$, and integrating, we obtain the circles, $r = c\cos\theta$ with c as parameter, as the orthogonal trajectories to (10.81), as shown in Fig. 10.5.

EXERCISE 10.8

1. If the stream lines of a fluid flow near a corner are the rectangular hyperbola $xy = c$, c being parameter, find the orthogonal trajectories to the stream lines.
2. Find the orthogonal trajectories of the family of co-axial circles $x^2 + y^2 + 2\lambda y + c = 2$, λ being the parameter.
3. Find the orthogonal trajectories of the family of circles passing through the points (0, 2) and (0, –2).
4. Show that the family of confocal and co-axial parabolas, $y^2 = 4a(x + a)$, a being parameter is self orthogonal.
5. Find the orthogonal trajectories of the family of curves
   (a) $r = 2a/(1 + \cos\theta)$ (b) $r^2 = a^2 \cos 2\theta$ (c) $r^n = a^n \sin n\theta$
6. Find the family of curves which cut the family of curves $xy = c$, c being parameter, at an angle of 45°.

10.10 MODELLING: RADIOACTIVITY, NEWTON'S LAW OF COOLING, AND MIXING PROBLEMS

The importance of differential equations is due to the fact that many physical laws and relations appear mathematically in terms of differential equations. *The transition from the given physical problem to a corresponding mathematical model is called modelling.*

In this section we shall consider the phenomena of radioactivity, Newton's law of cooling, and mixing problems.

10.10.1 Radioactivity: Exponential Decay

Experiments show that *a radioactive substance decomposes at a rate proportional to the amount present at any given instant of time.* If $y(t)$ is the amount of substance still present at time t, then the rate of change will be dy/dt; and according to the physical law explained,

$$dy/dt = ky, \qquad ...(10.85)$$

where k is a definite constant. Since the amount of substance $y(t)$ is positive and decreases with time, dy/dt is negative and hence k is negative, say $k = -\lambda$.

It is easy to see that the solution of (10.85) is

$$y(t) = y_0 e^{-\lambda t}, \qquad ...(10.86)$$

where y_0 is the amount of the substance present at $t = 0$.

Since, $\lambda > 0$, Eq. (10.86) shows that the process is of *exponential decay* and $y(t)$ tends of zero as t tends to infinity.

The so called half-life, T_h of a radioactive substance is the time in which half of the given amount of the substance will disappear. This can be obtained by setting $y(t) = \frac{1}{2} y_0$ in Eq. (10.86) and simplifying for t, which gives

$$T_h = \frac{1}{\lambda}\ln 2. \quad ...(10.87)$$

Similar to exponential decay, *exponential growth* models are also quite important in physics, biology, etc. If relatively small population of humans, animals, bacterias, etc. are left undisturbed, they often grow at a rate proportional to the population $y(t)$ present at a time t, leading to the model (10.85) and hence the solution

$$y(t) = y_0 e^{\lambda t}, \quad ...(10.88)$$

where $\lambda > 0$ is a definite constant, and $y(0) = y_0$.

For $\lambda > 0$, Eq. (10.88) expresses exponential growth, with $y(t) \to \infty$ as $t \to \infty$, which is unrealistic because as $y(t)$ becomes sufficiently large, other factors, such as scarcity of resources, etc. will creep in and in fact λ will not really be a constant. We expect it to decrease as y increases. In such a situation the model (10.88) is non-linear one.

Example 10.29: It is given that the rate of decay of radium varies as its amount present at that time. Assuming the 'half-life' of the radium to be 1600 years, find the percentage of the amount of radium disintegrated in 200 years.

Solution: Let $y(t)$ be the amount of radium at any time t, with $y(0) = y_0$. Then we have

$$\frac{dy}{dt} = -\lambda y, \quad \lambda > 0$$

Integrating this and using the initial condition $y(0) = y_0$, we obtain

$$y(t) = y_0 e^{-\lambda t} \quad ...(10.89)$$

as the solution.

Since, $t = 1600$, when $y(t) = \frac{1}{2}y_0$, thus (10.89) gives $\lambda = \frac{1}{1600}\ln 2$

If y_1 is the amount present after 200 years, then from (10.89), $y_1 = y_0\, e^{-200\lambda}$, which gives

$$\frac{y_0}{y_1} = e^{200\lambda} = e^{\frac{1}{8}\ln 2} = e^{0.0866433} = 1.0905076$$

and hence, the required percentage is $\frac{y_0 - y_1}{y_0} \times 100 = \frac{0.0905076}{1.0905076} \times 100 = 8.3\%$ (approx.)

10.10.2 Heating Problem: Newton's Law of Cooling

Newton's law of cooling states that *the time rate of change dT/dt of the temperature T of a body is proportional to the difference between T and the temperature T_s that of the surrounding medium.*

Let k be the proportionality constant, then

$$dT/dt = k(T - T_s) \quad ...(10.90)$$

Obviously k will be negative, say $k = -\lambda$, for $\lambda > 0$, then (10.90) can be written as

$$\frac{dT}{dt} = -\lambda(T - T_s).$$

Example 10.30: A dead body is located in a room kept at a constant temperature of 68°F. An investigating officer measures the body temperature at 9:40 P.M and finds it to be 94.4°F. Again at 11:00 P.M. he measures the temperature and finds it as 89.2°F. Estimate the time of death assuming that the temperature was 'normal' 98.6°F at the time of death.

Solution: If $T(t)$ is the body temperature at time t, then according to the Newton's law of cooling

$$dT/dt = k(T - 68), \quad \text{...(10.91)}$$

where k is constant of proportionality.

Eq. (10.91) is of separable form. Rewriting it as, $dT/(T-68) = kdt$. Integrating and simplifying we obtain, $T(t) = 68 + Ae^{kt}$, where A and k are constants to be determined from the information available.

Let the time 9:40 P.M. be time 'zero', thus, $T(0) = 94.4 = 68 + A$, which gives $A = 26.4$, and hence

$$T(t) = 68 + 26.4\, e^{kt} \quad \text{...(10.92)}$$

Also the body temperature at 11:00 P.M., 80 minutes past 9:40 P.M. is 89.2°F, thus

$T(80) = 89.2 = 68 + 26.4\, e^{80k}$, which gives $e^{80k} = \dfrac{21.2}{26.4}$ or, $k = \dfrac{1}{80}\ln\left(\dfrac{5.3}{6.6}\right) = -0.002742.$

Substituting for k, (10.92) becomes

$$T(t) = 68 + 26.4\, e^{-0.002742t} \quad \text{...(10.93)}$$

To find the time of death, put $T(t) = 98.6$ in (10.93), we have $98.6 = 68 + 26.4\, e^{-0.002742t}$,

$$\text{which gives } t = -\frac{1}{0.002742}\ln\frac{30.6}{26.4} = -\frac{147636}{0.002742} = -53.8 \text{ minutes (approx.)}$$

Thus, the death occurred approximately 53.8 minutes before 9:40 P.M., which was chosen as time zero, that is, around 8:46 P.M.

10.10.3 Mixing Problems

This type of problems are specially important to chemical engineers concerning the chemical solutions. In this we consider a mixing tank with an inflow of $u(t)$ gallons per minute and an equal outflow, where t is the time. The inflow is at a constant concentration c_1 of a particular solute, and the tank is constantly stirred, so that the concentration $c(t)$ within the tank is uniform. Hence, the outflow is at concentration $c(t)$. Let v denote the volume within the tank in gallons; it is constant because the inflow and outflow rates are equal. Let $x(t)$ be the instantaneous mass of solute within the tank, then

$$\frac{dx}{dt} = u(t)c_1 - u(t)\,c(t), \quad \text{or} \quad \frac{dx}{dt} = c_1 u(t) - \frac{x(t)}{v}u(t), \quad \text{since } c(t) = \frac{x(t)}{v}$$

or,

$$\frac{dx}{dt} + \frac{u(t)}{v}x(t) = c_1 u(t), \quad \text{...(10.94)}$$

which is a first order linear differential equation in $x(t)$. Using $x(t) = vc(t)$, the Eq (10.94) becomes

$$\frac{dc(t)}{dt} + \frac{u(t)}{v} c(t) = c_1 \frac{u(t)}{v},$$

a first order linear differential equation in $c(t)$.

Example 10.31: A tank initially contains 50 gallons of fresh water. Brine containing 2 lbs. per gallon of salt flows into the tank at the rate of 2 gallons per minute and the mixture, kept uniform by stirring, runs out at the same rate. How long will it take for the quantity of salt in the tank to increase from 40 to 80 lbs.?

Solution: Let $x(t)$ be the salt content at time t, then its rate of change is $\frac{dx}{dt}$.

Now the rate of increase of salt due to inflow $= 2 \times 2 = 4$ lbs./min.

Further, If $c(t)$ is the concentration of the brine at time t, then the rate of decrease of salt due to outflow $= 2c(t)$ lbs./min. Therefore

$$dx/dt = 4 - 2\,c(t) \qquad ...(10.95)$$

Since, inflow volume of the liquid is equal to the outflow, thus the concentration $c(t) = x(t)/50$ and hence Eq. (10.95) becomes $\frac{dx}{dt} = 4 - \frac{x}{25}$ or $dt = 25 \frac{dx}{100 - x}$. Integrating this, we obtain

$$t = -25 \ln(100 - x) + k, \qquad ...(10.96)$$

where k is an arbitrary constant.

Using the initial condition, $x(0) = 0$, Eq. (10.96) gives $k = 25 \ln 100$, and hence we get

$$t = 25 \ln \frac{100}{100 - x} \qquad ...(10.97)$$

Let $t = t_1$ when $x = 40$, and $t = t_2$ when $x = 80$; (10.97) gives the required time T as

$$T = t_2 - t_1 = 25 \ln 5 - 25 \ln 5/3 = 25 \ln 3 = 27.5 \text{ min (approx.)}$$

Remark. From (10.97) we observe that

$$x(t) = 100\,(1 - e^{-t/25}). \qquad ...(10.98)$$

As t increases, the amount of salt approaches the limiting value of 100 lbs. We must note that this limiting value depends on the rate at which salt is poured into the tank but not on the initial amount of salt in the tank. The term 100 in (10.98) is called the *steady part* and the term $100\,e^{-t/25}$ the *transient part*.

EXERCISE 10.9

1. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours, in how many hours will it triple?
2. Assuming that a sphere of ice melts at a rate proportional to its surface area, retaining a spherical shape. If u_0 is the initial volume of the ice, find an expression for the volume at any time t.
3. The half-life of uranium-238 is approximately 4.5×10^9 years. How much of a 10 kilogram block of $U-238$ will be present 1 billion years from now?

4. The half life of C_6^{14} has been found to be 5730 years. What should be the C_6^{14} content, in percent of initial value, of a fossilized tree that is claimed to be 3000 years old?
5. If 30% of a radioactive substance disappeared in 10 days, how long will it take for 90% of it to disappear?
6. If the temperature of a body drops from 100°C to 60°C in one minute, when the surrounding temperature is 20°, find the time when the temperature will come down to 30°C.
7. A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C. What will be temperature of the body after 40 minutes from the original?
8. A thermometer is carried outside a house whose ambient temperature is 70°F. After five minutes, the thermometer reads 60°, and fifteen minutes after this reads 50.4°. What is the outside temperature if it is assumed to be constant?
9. A thermometer reading 5°C is brought into a room whose temperature is 22°C. One minute later the thermometer reading is 12°C. How long does it take until the reading is 21.9°C?
10. A tank contains 200 gal. of water in which 40 lb. of salt is dissolved. Five gal. of brine, each gal. containing 2 lb. of dissolved salt runs into the tank per minute, and the mixture, kept uniform by stirring runs out at the same rate. Find the amount of salt in the tank after 10 minutes. What can be the ultimate amount of salt in the tank? Explain it.
11. A tank contains 1000 gal. of brine in which 500 lb. of salt is dissolved. Fresh water runs into the tank at the rate of 10 gal./minute and the mixture kept uniform by stirring runs out at the same rate. How long will it be before only 50 lbs. of salt is left in the tank?

10.11 MODELLING: BODY FALLING IN A RESISTING MEDIUM. VELOCITY OF ESCAPE FROM EARTH. MOTION OF A ROCKET

According to *Newton's second law of motion*, the rate of change of momentum of a body is proportional to the resultant force acting on the body. In case of motion along a straight line it gives

$$F = k\frac{d}{dt}(mv). \qquad \text{...(10.99)}$$

We take $k = 1$, consistent with standard unit of measurement.

In case the mass of the moving object remains constant, (10.99) gives

$$F = m\frac{dv}{dt} = mf, \qquad \text{...(10.100)}$$

where f is the acceleration of the object along the line of motion.

If m is not constant, e.g., an aeroplane consumes fuel during its course of journey, then (10.100) gives

$$F = m\frac{dv}{dt} + v\frac{dm}{dt}. \qquad \text{...(10.101)}$$

Next, *Newton's law of gravitational attraction* states that two bodies attract each other with a force which is directly proportional to the product of their masses m_1 and m_2 and inversely proportional to the square of the distance between them. It gives

$$F = G\frac{m_1 m_2}{r^2}, \qquad ...(10.102)$$

where G is the constant of proportionality called the universal gravitational constant. If one of the objects is the earth, then

$$F = G\frac{mM}{(R+x)^2}, \qquad ...(10.103)$$

where M is the mass of the earth, R is its radius, m is the mass of the second object and x is its distance from the surface of the earth. In case $x << R$, which is about 3960 miles, (10.103) is approximately $F = \frac{GM}{R^2}m$, generally written as mg. Here $g = GM/R^2$ is the acceleration due to gravity equal to 32 ft/sec^2, or 9.8 m/sec^2.

10.11.1 Body Falling in a Resisting Medium

We consider the *motion of a body which is falling under the influence of gravity in a medium* such as water, air or oil. This medium retards the downward motion of the object, e.g., the motion of a brick dropped in a swimming pool.

Experiments show that this retarding force at an instant is proportional to the square of the velocity at that instant. Thus for a downward motion the equation of motion is

$$m\frac{dv}{dt} = mg - kv^2, \qquad ...(10.104)$$

for some constant k.

The Eq. (10.104) can be solved by variable separable method.

In case of upward resisted motion, equation corresponding to (10.104), is

$$m\frac{dv}{dt} = -mg - kv^2. \qquad ...(10.105)$$

It is clear from (10.104) that the acceleration decreases as velocity increases. In particular it vanishes when $v = \sqrt{mg/k}$. This means that *a body falling under the influence of gravity through a retarding medium with retarding force proportional to the square of the velocity will not increase in velocity indefinitely. Instead body's velocity approaches the limiting value* $\sqrt{mg/k}$, *called the terminal velocity of the object.*

Example 10.32: A moving body is opposed by a force per unit mass of value αx and resistance per unit mass of value βv^2, where x and v are the displacement and velocity at that instant. Find the velocity of the particle in terms of x, if it starts from rest.

Solution: Let u be the velocity of the body at the instant when the displacement is x, then if m denotes the mass of the body, the equation of motion is given by

$$mv\frac{dv}{dx} = -m\alpha x - m\beta v^2, \text{ or } v\frac{dv}{dx} + \beta v^2 = -\alpha x$$

which is Bernoulli's equation.

Set $v^2 = u$, this becomes

$$\frac{du}{dx} + 2\beta u = -2\alpha x. \qquad ...(10.106)$$

Solving the linear equation (10.106), we obtain

$$u = v^2 = \frac{\alpha}{2\beta^2} + ce^{-2\beta x} - \frac{\alpha x}{\beta}, \qquad ...(10.107)$$

where c is an arbitrary constant.

Using the initial condition $v(0) = 0$ in (10.107) we get $c = \alpha/2\beta^2$, and hence Eq. (10.107) gives

$$v^2 = \frac{\alpha}{2\beta^2}(1 - e^{-2\beta x}) - \frac{\alpha x}{\beta}$$

as the desired solution.

Example 10.33: A body of mass m falls from rest under gravity in a fluid whose resistance to motion at any instant is mk times its velocity, where k is a constant. Find the terminal velocity of the body and also the time taken to acquire one-half of its limiting speed.

Solution: Let v be the velocity of the body at any time t. Then the equation of motion of the body is given by

$$m\frac{dv}{dt} = mg - mkv, \text{ or } \frac{dv}{g - kv} = dt, \; v(0) = 0.$$

Solving the initial value problem, we obtain

$$t = -\frac{1}{k}\ln\left(1 - \frac{k}{g}v\right) \qquad ...(10.108)$$

Let V be the terminal velocity, it is obtained when the weight of the body exactly balances the resistance to motion. Thus mg = mkV, which gives, $V = g/k$. Substituting in (10.108), we obtain

$$t = -\frac{V}{g}\ln\left(1 - \frac{v}{V}\right). \qquad ...(10.109)$$

Let $t = T_1$ when $v = (1/2)V$, then (10.109) gives

$$T_1 = -\frac{V}{g}\ln\left(\frac{1}{2}\right) = \frac{V}{g}\ln 2.$$

10.11.2 Velocity of Escape from Earth

We consider the problem of determining the least velocity with which a body must be projected vertically upward in the radial direction from the earth so that it escapes from the earth assuming the air resistance to be neglected.

Let v be the upward velocity of the body of mass m at the instant when its distance from the centre of the earth is r, then its equation of motion is

$$mv\frac{dv}{dr} = -G\frac{Mm}{r^2}, \quad \text{...(10.110)}$$

where M is the mass of the earth and G is the universal constant of gravitation. The negative sign indicates that the velocity decreases with increase in r.

Rewriting (10.110) as

$$v\frac{dv}{dr} = -\frac{\mu}{r^2}, \quad \mu = GM. \quad \text{...(10.111)}$$

Separating the variables and integrating, we get

$$\frac{v^2}{2} = \frac{\mu}{r} + c, \quad \text{...(10.112)}$$

where c is a constant.

Let the body be projected with initial velocity v_0 from the surface of the earth with radius R. Then on the surface of the earth, $r = R$, $v = v_0$. Using this in (10.112), we obtain $c = \frac{v_0^2}{2} - \frac{\mu}{R}$ and hence (10.112) becomes

$$\frac{v^2}{2} = \frac{\mu}{r} + \left(\frac{v_0^2}{2} - \frac{\mu}{R}\right). \quad \text{...(10.113)}$$

The body will escape from the earth if v at no instant during its ascent becomes zero and then negative, that is the velocity v is always positive.

With the increase in r, $\frac{\mu}{r}$ goes on decreasing thus the velocity will remain positive if in (10.113)

$$\frac{v_0^2}{2} - \frac{\mu}{R} \geq 0, \quad \text{or} \quad v_0 \geq \sqrt{\frac{2\mu}{R}}. \quad \text{...(10.114)}$$

Now on the surface of the earth the acceleration $v\frac{dv}{dr} = -g$ and $r = R$.

Hence from (10.111) $\mu = gR^2$. Substituting the value of μ in (10.114), we get, $v_0 \geq \sqrt{2gR}$. Hence the least velocity with which the body must be projected is, $v_0 = \sqrt{2gR}$.

Taking $R = 3960$ miles and $g = 0.0060928$ miles/sec^2, we get $v_0 = 6.95$ miles/sec. (approx.)

10.11.3 Motion of a Rocket

So far we have considered that the mass of the body remains constant during the course of journey. Here we examine a case of considerable practical importance, the motion of a rocket, when the mass changes continuously during the course of its journey; the mass decreases continuously as the fuel burns out.

Consider that a rocket of mass M, included that of fuel, is fired vertically upward in a medium whose resistance varies as the velocity. Suppose that the fuel burns out at a constant rate of mass m per unit time and is ejected as an exhaust with a constant backward velocity u relative to the rocket Assuming the rocket to be initially at rest, we derive the equation of motion of the rocket and find an expression for the velocity in term of time.

Let v be the velocity of the rocket at instant t, before burn-out. The mass of the rocket at this time is $(M - mt)$ and its momentum is $(M - mt)v$.

Next, let $v + \Delta v$ be the velocity of the rocket at the instant $t + \Delta t$. The mass of the rocket at this instant is $[M - m(t + \Delta t)]$ and its momentum is $[M - m(t + \Delta t)]\,(v + \Delta v)$.

Also the linear momentum of the fuel exhausted is $m(v - u)\Delta t$.

Hence the change in momentum in Δt

$$= [M - m(t + \Delta t)]\,(v + \Delta v) - (M - mt)v + m(v - u)\Delta t = M\Delta v - m\Delta v(t + \Delta t) - mu\Delta t.$$

The rate of change of momentum as $\Delta t \to 0$ is $= M\dfrac{dv}{dt} - mt\dfrac{dv}{dt} - mu = (M - mt)\dfrac{dv}{dt} - mu.$

Using Newton's second law of motion this must be equal to the resultant external force, say F, in the upward direction at that instant. Thus, $(M - mt)\dfrac{dv}{dt} - mu = F.$

Also $F = -(M - mt)g - kv$; here kv, where k is a constant, is the resistance to motion when the velocity is v. Hence, the equation of motion of the rocket is

$$(M - mt)\frac{dv}{dt} - mu = -(M - mt)g - kv$$

or,
$$\frac{dv}{dt} + \frac{k}{M - mt}v = \frac{m}{M - mt}u - g, \qquad \text{...(10.115)}$$

which is a linear differential equation of first order with integrating factor

$\mu = e^{\int \frac{k}{M - mt}dt} = e^{-\frac{k}{m}\ln(M - mt)} = (M - mt)^{-(k/m)}$, and hence, the solution of (10.115) is given by

$$v = \frac{g}{m - k}(M - mt) + \frac{mu}{k} + c(M - mt)^{k/m}, \qquad \text{...(10.116)}$$

where c is an arbitrary constant.

Since the rocket is initially at rest, we have $v = 0$ when $t = 0$. Using this in (10.116) we obtain

$$c = -M^{-k/m}\left(\frac{g}{m - k} + \frac{mu}{k}\right) \text{ and hence (10.116)}$$

$$v = \frac{g}{m-k}(M-mt) + \frac{mu}{k} - M^{-k/m}\left(\frac{g}{m-k} + \frac{mu}{k}\right)(M-mt)^{k/m}, k \neq 0, m.$$

This equation gives the velocity of the rocket at any time t before the fuel is burnt out.

Example 10.34: Find the velocity of escape for a body which is carried by a launch vehicle and is separated from it at a distance of 100 miles from the earth's surface, when R, the radius of earth, is 3960 miles and $g = 6.1 \times 10^{-3}$ miles/sec.

Solution: If v be the velocity of the body when it is at a distance r from the surface of the earth. Then the equation of motion is $mv\frac{dv}{dr} = -\frac{GMm}{(R+r)^2}$, where m is the mass of the body, M is the mass of the earth and R is the radius of the earth.

Rewriting it as

$$vdv = -\frac{\mu}{(R+r)^2}dr, \quad \mu = GM. \qquad ...(10.117)$$

Integrating we obtain, $\frac{v^2}{2} = \frac{\mu}{R+r} + c$. If the body is projected with an initial velocity v_0 from the launch vehicle, then at $r = 100$, $v = v_0$, and thus, $c = \frac{v_0^2}{2} - \frac{\mu}{R+100}$, hence the solution of Eq. (10.117) becomes

$$\frac{v^2}{2} = \frac{\mu}{R+r} + \left(\frac{v_0^2}{2} - \frac{\mu}{R+100}\right).$$

The body will escape if $v \geq 0$. With increase in r, $\frac{\mu}{R+r}$ goes on decreasing and thus the velocity will be positive, if $\frac{v_0^2}{2} - \frac{\mu}{R+100} \geq 0$, or $v_0 \geq \sqrt{\frac{2\mu}{R+100}}$.

On the surface of the earth the acceleration, $v\frac{dv}{dr} = -g$, and also from (10.117), $v\frac{dv}{dr} = -\frac{\mu}{R^2}$, hence $\mu = gR^2$. Thus escape velocity from the rocket is $v_0 \geq \sqrt{\frac{2gR^2}{R+100}}$.

Here $R = 3960$ miles, $g = 6.1 \times 10^{-3}$ miles/sec^2, and thus

$$v_0 \geq \sqrt{\frac{2(6.1\times 10^{-3})\times(3960)^2}{3960+100}} = 6.86 \text{ miles/sec.}$$

EXERCISE 10.10

1. A 10 lb. ballast bag is dropped from a hot air balloon that is at an altitude of 342 feet and ascending at a rate of 4 ft/sec. Assuming that air resistance is not a factor, determine the maximum height attained by the bag, how long it takes to strike the ground and find the speed with which it strikes the ground.
2. A body of mass m falls from rest under gravity in a fluid whose resistance to motion at any instant is proportional to its velocity. Find the terminal velocity of the body and the time taken to acquire one-half of its limiting speed.
3. A moving body is opposed by a force per unit mass of value cx and resistance per unit mass of value bv^2, where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of the displacement, in case the body starts from rest.
4. A body of mass m falls from rest through a medium that opposes its fall with a force proportional to the square of its velocity. Find its equation of motion. If initially the body starts falling from rest, find the velocity and displacement covered after t seconds.
5. A particle moves in a straight line under a retardation αv^{n+1}, where α is a constant and v is the velocity at time t. If x is the distance traversed in time t, then show that
$$t = \frac{1}{n\alpha}\left(\frac{1}{v^n} - \frac{1}{u^n}\right), \text{ and } x = \frac{1}{(n-1)\alpha}\left(\frac{1}{v^{n-1}} - \frac{1}{u^{n-1}}\right),$$
where u is its initial velocity.
6. A rocket initially of mass M moves vertically by expelling a mass m at a constant rate c with speed u_0 relative to the rocket, where $cu_0 > Mg$. Assuming that the earth's gravitational field is constant, derive the equation of motion of the rocket and find its height when the mass of the rocket is $M/4$.
7. A rocket and its fuel has an initial mass m_0. Fuel is burnt at a constant rate $dm/dt = c$. The gases exhaust at a constant speed relative to the rocket. Neglecting air-resistance, find the speed v of the rocket at time t, when initial velocity is u.

10.12 MODELLING: SIMPLE ELECTRIC CIRCUITS

Electrical engineers often use differential equations to model electrical circuits. The electrical circuits, analogous to Newton's second law for mechanical systems, are governed by Kirchhoff's laws. In a mechanical system, comprising various elements such as masses and springs, we deal with forces and displacements, while in an electrical circuit, consisting of resistors, inductors and capacitors, we are concerned with voltages and currents. The various elements of electrical circuits along with their symbols and units are given in Fig. 10.6.

The simplest electric circuit is a series circuit having a source of electrical energy that is electromotive force such as a battery, and a resistor which uses energy, for example an electric bulb. In case the circuit is closed, a current I flows through the resistor causing a voltage drop across the two ends of the resistor. According to Ohm's law, *the voltage drop E_R across a resistor is proportional to the instantaneous current I, that is, $E_R = RI$.* The constant of proportionality R is called the resistance of the resistor. The current I is measured in *amperes*, the resistance R in ohms and the voltage E_R in volts.

Elements	Symbol	Unit
Resistor, R		ohms (Ω)
Inductor, L		henrys (H)
Capacitor, C		farads (F)
Charge, Q		coulomb
Current, I		ampere (A)
Electromotive force, E		volt (V)

Fig. 10.6

The voltage drop E_L across an inductor is proportional to the instantaneous time rate of change of the current I with respect to t, that is, $E_L = LdI/dt$, the constant of proportionality L is called the *inductance* of the inductor and is measured in henrys and the time t is in seconds.

The voltage drop E_C across a capacitor is proportional to the instantaneous electric charge Q on the capacitor, that is, $E_C = Q/C$, where C is called the *capacitance* and is measured in *farads;* the charge Q is measured in coulombs.

Further, since $\quad I(t) = \dfrac{dQ}{dt}$, thus $E_C = \dfrac{1}{C}\int I(t)dt.$

The formulation of differential equations for electric circuits results from the application of the **Kirchhoff's voltage law** which states that *the algebraic sum of all the instantaneous voltage drops around any closed circuit is zero.*

Next we consider simple electric circuits and formulate the corresponding differential equations.

10.12.1 *R-L* Series Circuit

If $I(t)$ is the current flowing in the circuit at any time t, as shown in Fig. 10.7, then by Kirchhoff's law the voltage drop around the circuit satisfies

$$L\frac{dI}{dt} + RI = E(t).$$

Dividing by L to get

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E(t)}{L} \qquad ...(10.118)$$

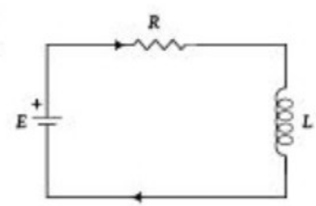


Fig. 10.7

which is a linear differential equation. The integrating factor is $e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$ and, therefore, the general solution of Eq. (10.118) is

$$I(t) = e^{-\frac{Rt}{L}}\left[\int e^{\frac{Rt}{L}}\frac{E}{L}\,dt + k\right], \quad ...(10.119)$$

where k is constant of integration.

In case of constant e.m.f. say $E = E_0$, Eq. (10.119) gives

$$I(t) = \frac{E_0}{R} + ke^{-\frac{Rt}{L}}. \quad ...(10.120)$$

The exponential form in (10.120) goes to zero as $t \to \infty$, the current approaches the limiting value E_0/R. This is the *study-state* value of the current in the circuit and we can see very easily that it is independent of the initial value $I(0)$. For $I(0) = 0$, from (10.120) we get $k = -E_0/R$, and hence,

(10.120) becomes $I(t) = \frac{E_0}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$, a particular solution of (10.118) in case of constant e.m.f.

In case of periodic e.m.f. say, $E = E_0 \sin \omega t$, (10.119) gives

$$I(t) = e^{-\frac{Rt}{L}}\left[\frac{E_0}{L}\int e^{\frac{Rt}{L}} \sin \omega t\, dt + k\right], \quad ...(10.121)$$

which on integration by parts, gives

$$I(t) = ke^{-\frac{Rt}{L}} + \frac{E_0}{R^2 + \omega^2 L^2}(R \sin \omega t - \omega L \cos \omega t)$$

$$= ke^{-\frac{Rt}{L}} + \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}}\left(\frac{R}{\sqrt{R^2 + \omega^2 L^2}} \sin \omega t - \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} \cos \omega t\right).$$

Substituting $\frac{R}{\sqrt{R^2 + \omega^2 L^2}} = \cos \theta$, and $\frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} = \sin \theta$, it becomes

$$I(t) = ke^{-\frac{Rt}{L}} + \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \theta), \ \theta = \tan^{-1}\frac{\omega L}{R} \quad ...(10.122)$$

As $t \to \infty$, the exponential term in (10.122) tends to zero and

$$I(t) \to \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \theta). \quad ...(10.123)$$

This means that after sometime $I(t)$ will execute harmonic oscillation as shown in Fig. (10.8).

In fact an electrical system is said to be in steady state when the variables describing its behaviour are periodic functions of time or constant. The steady state is not necessarily a constant. Thus for the case $I(0) = 0$, $E = E_0$, the steady state is the constant E_0/R, but for the case $I(0) = 0$ and $E(t) = E_0 \sin \omega t$, the steady state oscillation is the harmonic function given by (10.123).

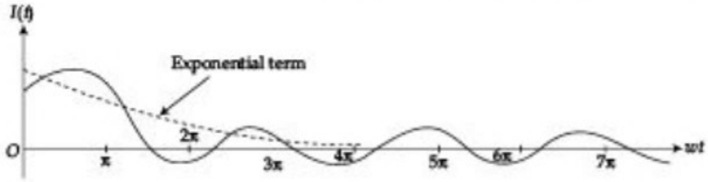


Fig. 10.8

10.12.2 *R-C* Series Circuit

Let $I(t)$ be the current flowing in the circuit at any time t as shown in Fig. 10.9. Using the Kirchhoff's law, the voltage drop around the circuit satisfies $RI + \frac{1}{C}\int I dt = E(t)$.

Differentiating this equation with respect to t, and then dividing by R, we obtain

$$\frac{dI}{dt} + \frac{1}{RC} I = \frac{1}{R}\frac{dE}{dt} \quad ...(10.124)$$

which is a Leibnitz's linear equation in I. Proceeding as in case of R-L series circuit, the general solution of the equation (10.124) is

$$I(t) = e^{-t/(RC)}\left(\frac{1}{R}\int e^{t/(RC)}\frac{dE}{dt}dt + k\right), \quad ...(10.125)$$

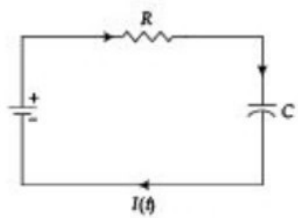


Fig. 10.9

where k is a constant of integration.

In case of constant e.m.f. say $E = E_0$, $\frac{dE}{dt} = 0$, (10.125) becomes

$$I(t) = ke^{-t/RC}.$$

In case of periodic e.m.f. say $E = E_0 \sin \omega t$, (10.125) gives

$$I(t) = e^{-\frac{t}{RC}}\left[\frac{\omega E_0}{R}\int e^{t/(RC)}\cos \omega t \, dt + k\right]. \quad ...(10.126)$$

Integrating (10.126) by parts and proceeding as in the case of R-L series circuit, we get

$$I(t) = ke^{-t/RC} + \frac{\omega E_0 C}{\sqrt{1+(\omega RC)^2}}\sin(\omega t - \theta), \quad \theta = -\tan^{-1}\left(\frac{1}{RC}\right) \quad ...(10.127)$$

as the required expression for the current.

The first term decreases steadily as t increases and the last term represents the steady state current. The behaviour is similar to as in case of R-L series circuit.

10.12.3 *R-L-C* Series Circuit

The RLC series circuit contains all three kind of elements as shown in Fig. 10.10.

If $I(t)$ is the current flowing in the circuit at any time t, then applying the Kirchhoff's law, the voltage drop around the circuit satisfies

$$RI + L\frac{dI}{dt} + \frac{1}{C}\int I dt = E(t). \quad ...(10.128)$$

Also the charge $Q(t)$ on the condenser, when the current is $I(t)$ at time t, is

$$Q(t) = \int I dt, \quad \text{that is, } I(t) = \frac{dQ}{dt}.$$

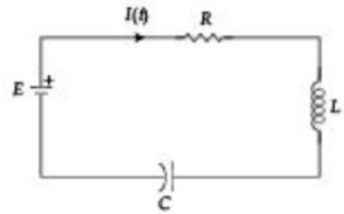


Fig 10.10

Thus the Eq. (10.128) becomes $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t)$, which is a second order linear differential equation with constant coefficients. Its solution will be discussed in Chapter 11.

EXERCISE 10.11

1. In a constant e.m.f. RL circuit show that the current at time t is given by
$$I(t) = \frac{E_0}{R}(1 - e^{-Rt/L}) + I(0)e^{-Rt/L}$$
where $I(0)$ is the current at time $t = 0$. If $I(0) = 0$, then at what time will the current reach half of its theoretical maximum value? What L should we choose with $R = 1000$ ohms, if we want the current to grow from 0 to 25% of its final value within 10^{-4} sec?
2. Solve for the current $I(t)$ in an RL circuit if $R = 2$ ohms, $L = 25$ henrys, and $E(t) = Ae^{-t}$, with $A > 0$, as a constant and $I(0) = 0$. Graph the current as a function of time.
3. Solve for current $I(t)$ in an RL series circuit having e.m.f. $E(t) = A\sin(\omega_1 t) + B\cos(\omega_2 t)$, with A, B, ω_1 and ω_2 being positive constants; and $I(0) = 0$.
4. Show that the charge $Q(t)$ in an RC circuit satisfies the linear differential equation
$$\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{1}{R}E(t).$$
(a) Solve this equation with $E(t) = E_0$ and $Q(0) = Q_0$.
(b) Find the lim $Q(t)$, $t \to \infty$, (steady state value) and show that it is independent of Q_0.
(c) Find at what time, $Q(t)$ is within 1% of its steady state value.
5. Find an expression for $I(t)$ in an RC circuit with $E(t) = E_0e^{-Rt/L}$, assuming $I(0) = 0$. Find the steady state solution.
6. Find charge $Q(t)$ in an RL circuit with $R = 10$ ohms, $C = 0.1$ farad, and e.m.f. $E(t) = 30e^{-3t}$ volts, assuming $Q(0) = 0$. At what time does $Q(t)$ reach a maximum? What is the maximum charge?

ANSWERS

Exercise 10.1 (p. 572)

1. (a) Non-homogeneous linear of order 2 and degree 1.
   (b) Non-linear of order 1 and degree 2.
   (c) Homogeneous linear of order 1 and degree 2.
   (d) Non-linear of order 1 and degree 1.

2. (a) $y = \frac{x^4}{4} + c$ (b) $y = -\frac{\sin 2x}{4} + c_1 x + c_2$ (c) $y = \ln(1+x) + c$

3. (a) Yes (b) Yes (c) Yes (d) No

5. (a) $xy' = x + 2y$ (b) $y'' + 14y' + 49y = 0$ (c) $[1 + (y')^2]^3 = 25(y'')^2$
   (d) $4xy'' + 2y' - y = 12x - x^2$ (e) $y'' - 2y' + 2y = 0$
   (f) $y''' - 7y' + 6y = 0$

6. $y''' = 0$ 7. $y' = \frac{x}{y}$ 8. $[1 + (y')^2]^3 = a^2(y'')^2$

9. $\{1 + (y')^2\}y''' = 3(y'')^2 y'$ 10. $(x^2 + y^2)y'' = 2(xy' - y)\{1 + (y')^2\}^2$

Exercise 10.2 (p. 575)

1. $y = \frac{1}{2}\ln\left[\frac{2}{3}(e^{3x} + x^3) + c\right]$ 2. $y = -1 + ce^{-1/x}$

3. $y = \dfrac{1}{e^{-x} + \frac{1}{4} - e^{-1}}$ 4. $y = -(1 + 4x) + 2\tan(2x + \pi/4)$

5. $y = x + \ln[x(1+y)] + c$ 6. $(1 - ay)(a + x) = cy$
7. $y \sin y = x^2 \ln x + c$ 8. $\cot y = -\tanh x$
9. $y = x \tan^{-1}(x^3 - 1)$ 10. $y^2 = 2x^4 - 4x^2$
11. $2(y - x) + (y + 2x)^2 = 0$ 12. $x^2 + 2y^2 - 3\ln(x^2 + y^2 + 2) + c = 0$
13. $\cos xy + \frac{1}{2x^2} = c$ 14. $\sin(y/x) + \ln|x| = 1$
15. $\sin(x^2y^2) = e^{x-1}$

Exercise 10.3 (p. 578)

1. $x^2(x^2 - 2y^2) = c$ 2. $xy(x - y) = c$ 3. $y = ce^{y/x}$

4. $-2\ln|x| + \frac{1}{2}\cos(y/x)\sin(y/x) + \frac{1}{2}(y/x) = c$

5. $-\ln|x| - (1/2)\cos(y/x)\sin(y/x) + \frac{1}{2}(y/x) = c$
6. $e^{x/y} = y + c$
7. $x^2y(2x + y) = 3$
8. $x + ye^{x/y} = 1$
9. $x^2 + 2xy - y^2 - 4x + 8y - 14 = c$
10. $\frac{1}{5}(x - 2y) + \frac{2}{25}\ln\left|x - 2y + \frac{3}{5}\right| + x = c$
11. $\ln|x-1| - \frac{x-1}{y+3} + \ln\left|\frac{y+3}{x-1}\right| = c$
12. $\ln\left|x + y + \frac{1}{3}\right| + \frac{3}{2}(y - x) = c$
13. $y[\sec(x/y) + \tan(x/y)] = c$
14. $\ln y^2 + e^{-x^2/y^2} = c$

Exercise 10.4 (p. 581)

1. $x^3 + 2xe^y + y^3 = c$
2. $x\sin y - y^2 = c$
3. $e^{xy^2} + x^4 - y^3 = c$
4. Not exact
5. $\cosh x \sinh y = c$
6. $x^y = c$
7. $e^x \cos y = 1$
8. $3xy^4 - x = 47$
9. $x^2 + y^3 + \cos(xy) = 9$
10. $x^3 \sin 2y - x^2y = -0.7854$
11. $(2A - c)x + (C - 2b)y + (D - e) = 0$
12. $a = -d, b = c$

Exercise 10.5 (p. 585)

1. $y + xe^{1/x} = cx$
2. $ax + \tan^{-1}(y/x) = c$
3. $\ln(x^2y^2) + (x^2 + y^2) = c$
4. $3\ln x - (y/x)^3 = c$
5. $x^5 + 3x^4 + 3x^2y^2 = c$
6. $(x - y)\cos^2 y = c$
7. $y(x^3e^y + x) + x = cy$
8. $x^3 \sinh 3y - x^2 = c$
9. $\frac{x}{y} - 2\ln x + 3\ln y = c$
10. $2\ln x - \ln y - 1/xy = c$

Exercise 10.6 (p. 588)

1. $y = -\frac{x}{2} + \frac{c}{x}$
2. $y = \frac{x^3}{5} - \frac{x}{3} + \frac{47}{15x^2}$
3. $y = \left(\frac{1}{3}e^{3x} + c\right)(x + 1)$
4. $y = \sec^{-1}\{\cos x\,(\sin x + c)\}$
5. $x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$
6. $y = 1 + 3\cosh 2x$
7. $y = 2x^2 + (e^x + \sin x)x^3$
8. $x = \frac{1}{2}[\ln y + c(\ln y)^{-1}]$
9. $2r = 3\,\mathrm{cosec}^2\theta - \sin^2\theta$
10. $ye^{2\sqrt{x}} = 2\sqrt{x} + c$
11. $xe^y = c + \tan y$
12. $x = y\ln y$

Exercise 10.7 (p. 593)

1. $y^2 = \dfrac{1}{\sqrt{1 + x^2 + ce^{x^2}}}$ 2. $y^{1/2} = \dfrac{1}{4 - 2x + ce^{-x/2}}$ 3. $\dfrac{1}{\ln y} = \dfrac{1}{2x} + cx$

4. $3y^2 = \dfrac{1}{2}(x^2 - 1) + ce^{-x^2}$ 5. $\cos^2 x = y^2 (c + 2 \sin x)$ 6. $y = 1/[x(c + x - x \ln x)]$

7. $y = 1 + 1/(ce^{-x} - 2)$ 8. $y = x + e^{2x^3/3} \Big/ \left\{c - 2\int xe^{2x^3/3}\, dx\right\}$

9. $y = x + 1/(ce^{-x} - 1)$ 10. $y = e^x + 3/(ce^{-7x} - e^{-x})$ 11. $y = cx + c^2,\ \ x^2 + 4y = 0$

12. $y^4 = cx - c^2,\ \ \dfrac{x^2}{4} = y^4$ 13. $y = cx + e^{-c},\ \ y = (1 - \ln x)x,\ \ x > 0$

14. $e^{y} = ce^{x} - c^3,\ \ 2y - 3x = \ln(4/9)$.

Exercise 10.8 (p. 596)

1. $x^2 - y^2 = c$ 2. $x^2 + y^2 + 2\mu x - c = 0$ 3. $x^2 + (y - 2c)^2 = 4(c^2 - 1),\ |c| > |$

5. (a) $r = 2b/(1 - \cos\theta)$ (b) $r^2 = c^2 \sin 2\theta$ (c) $r^n = b^n \cos n\theta$

6. $y^2 - x^2 + 2xy = c$

Exercise 10.9 (p. 599)

1. 3.17 hrs. 2. $v = \left[v_0^{1/3} + k\left(\dfrac{4\pi}{3}\right)^{1/3} t\right]^3$ 3. 8.57 kg

4. 69.6% 5. 64.5 days 6. 3 minutes

7. 50°C 8. 45°F 9. 9.68 minutes

10. 119.63 lb. 11. 3 hrs 50 minutes and 16 sec.

Exercise 10.10 (p. 606)

1. 342. 25 ft, 4.75 sec, 148 ft/sec

2. $\dfrac{mg}{\alpha}$, α is constant of proportionality, $\dfrac{mg}{\alpha} \ln 2$

3. $v^2 = \dfrac{c}{2b^2}(1 - e^{-2bx}) - \dfrac{cx}{b}$

4. $\sqrt{\left(\dfrac{mg}{\alpha}\right)} \tan h \sqrt{\dfrac{g\alpha}{m}};\ \ \sqrt{\dfrac{m}{\alpha}} \ln \sqrt{\dfrac{g\alpha}{m}}$, α is constant of proportionality

6. $h = \frac{Mu_0}{c}\left\{3 - \ln 4 - 9\frac{Mg}{8cu_0}\right\}$ 7. $v = (v - u)\ln\left(\frac{m_0}{m_0 - ct}\right) - gt$

Exercise 10.11 (p. 610)

1. $0.7\frac{L}{R}$, 0.348 2. $I(t) = \frac{A}{23}(e^{-2t/25} - e^{-t})$

3. $$I(t) = \frac{AL}{R^2 + L^2\omega_1^2}\left[\frac{R}{L}\sin(\omega_1 t) - \omega_1\cos(\omega_1 t)\right]$$
$$+ \frac{BL}{R^2 + L^2\omega_2^2}\left[\frac{R}{L}\cos(\omega_2 t) + \omega_2\sin(\omega_2 t)\right] + \left[\frac{AL\omega_1}{R^2 + L^2\omega_1^2} - \frac{BR}{R^2 + L^2\omega_2^2}\right]e^{-Rt/L}$$

4. (a) $Q(t) = E_0C\,(Q_0 - EC)e^{-t/RC}$ (b) E_0C (c) $-RC\ln\left(\frac{0.01\,E_0C}{Q_0 - E_0C}\right)$

5. $I(t) = \left(\frac{E_0RC}{R^2C - L}\right)(e^{-Rt/L} - e^{-t/RC})$ 6. $1.5\,(e^{-t} - e^{-3t})$; 0.549 sec; 0.577 coulomb.

11 Second and Higher Order Linear Differential Equations

CHAPTER

"The theory of linear differential equations, particularly that of with constant coefficients, is quite comprehensive. There are standard methods for solving many practically important linear differential equations. Second order linear differential equations are important, since they occur more frequently while modelling the physical situations and the techniques developed to solve these equations can be extended to higher order equations as well."

11.1 BASIC CONCEPTS

The general linear differential equation of the nth order is of the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{dy^{n-1}}{dx^{n-1}} + a_2(x)\frac{dy^{n-2}}{dx^{n-2}} + \ldots + a_n(x)y = f(x), \quad \ldots(11.1)$$

where $a_0(x)$, $a_1(x)$, $a_2(x)$ ..., $a_n(x)$ and $f(x)$ are functions of x only, and $a_0(x) \neq 0$.

In case $a_0, a_1, \ldots, a_n$ are constants, the Eq. (11.1) is called linear differential equation with constant coefficients. The characteristic feature of a linear equation is that it is linear in the unknown function y and its derivatives.

If $f(x) = 0$, then equation is called *homogeneous equation*, otherwise it is called *non-homogeneous equation*. For example, a second order homogeneous equation with constant coefficients is of the form

$$a_0\frac{d^2 y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0, \ a_0 \neq 0,$$

while a non-homogeneous second order equation is of the form

$$a_0\frac{d^2 y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = f(x), \ a_0 \neq 0.$$

Of the higher order equations, the second order equations are the simplest one and have many applications in mechanics and electric circuit theory, for example

(a) The equation $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t)$ represents the charge Q in an RLC-series circuit.

(b) The equation $m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = f(t)$ governs the motion of a mass m on a spring. Here x is the distance from a fixed point after t seconds, a is the damping factor, k is the spring stiffness and $f(t)$ is an external force.

(c) The equations

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0, \quad \text{and} \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0,$$

respectively the Legendre and Bessel equations, are important equations in applied mathematics and physics.

(d) The non-linear equation $\frac{d^2y}{dx^2} = c\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ represents the shape of a uniform flexible cable, or catenary, hanging under the action of its own weight, refer to Fig. 11.1. Here $y(x)$ is the deflection and c is a constant depending upon the mass density of the cable.

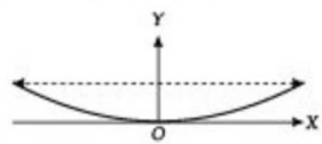


Fig. 11.1

(e) The equation, $EI\frac{d^4y}{dx^4} + ky = f(x)$, $k > 0$, occurs while studying the deflected shape $y(x)$ of a beam on an elastic foundation, under a load of $f(x)$ units per unit length, refer to Fig. 11.16 (p. 678).

In fact, there are numerous physical situations modelled by second and higher order differential equations.

11.2 SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS. LINEARLY INDEPENDENT AND DEPENDENT SOLUTIONS

In this section we shall discuss the solution of linear differential equations, particularly their linear dependence and independence. In what follows we shall assume that x varies on an interval I, may be the whole real line, which generally will not be mentioned explicitly.

11.2.1 Existence and Uniqueness of Solutions for Initial Value Problems

If $y = y_1(x)$ is a solution of the differential equation (11.1) on an interval I, then it must satisfy (11.1) indentically and hence $y_1(x)$ must be continuously differentiable $(n - 1)$ times and $y_1^{(n)}$ must be continuous on I.

We state the following result:

Theorem 11.1: *If the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ and $f(x)$ are continuous over an interval I, then there exists a unique solution to the initial value problem*

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y^{(1)} + a_n y = f(x);\ y(x_0) = k_0,\ y^{(1)}(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1}$$

where, $x_0 \in I$ and $k_0, k_1, \ldots, k_{n-1}$ are constants.

We must note the derivative $y^{(n)}(x_0)$ cannot be specified as an initial condition, because it is determined by the differential equation itself once the stated initial conditions have been given.

11.2.2 Superposition, or Linearity Principle

Theorem 11.2: *If $y_1(x)$ and $y_2(x)$ are two solutions of the linear homogeneous equation*

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y^{(1)} + a_n y = 0 \quad \ldots(11.2)$$

then $y = c_1 y_1(x) + c_2 y_2(x)$, a linear combination of y_1 and y_2, where c_1, c_2 are two arbitrary constants, is also its solution.

Proof. Substituting for $y = c_1 y_1(x) + c_2 y_2(x)$ in left-hand side of (11.2), we obtain

$$a_0(c_1 y_1 + c_2 y_2)^{(n)} + a_1(c_1 y_1 + c_2 y_2)^{(n-1)} + \ldots + a_n(c_1 y_1 + c_2 y_2)$$

$$= c_1[a_0 y_1^{(n)} + a_1 y_1^{(n-1)} + \ldots + a_n y_1] + c_2[a_0 y_2^{(n)} + a_1 y_2^{(n-1)} + \ldots + a_n y_2] = c_1(0) + c_2(0) = 0,$$

since $y_1(x)$, $y_2(x)$ are solutions of the linear homogeneous Eq. (11.2).

This result can be generalized to the case of more than two solutions.

Example 11.1: Show that $y_1 = e^x$, and $y_2 = e^{-2x}$ and their linear combination $c_1 e^x + c_2 e^{-2x}$ are solutions of the differential equation $y'' + y' - 2y = 0$.

Solution: For $y_1 = e^x$, we have $y_1' = e^x$, $y_1'' = e^x$, and thus $y'' + y' - 2y = e^x + e^x - 2e^x = 0$.

Hence, $y_1 = e^x$ is a solution of $y'' + y' - 2y = 0$.

Similarly, $y_2 = e^{-2x}$ is also a solution of $y'' + y' - 2y = 0$.

Next, $y = c_1 e^x + c_2 e^{-2x}$ gives $y' = c_1 e^x - 2c_2 e^{-2x}$ and $y'' = c_1 e^x + 4c_2 e^{-2x}$. Hence,

$$y'' + y' - 2y = (c_1 e^x + 4c_2 e^{-2x}) + (c_1 e^x - 2c_2 e^{-2x}) - 2(c_1 e^x + c_2 e^{-2x}) = c_1[0] + c_2[0] = 0.$$

Thus $y = c_1 e^x + c_2 e^{-2x}$ is also its solution.

We must note that *the linearity principle does not hold in case of non-homogeneous equations and non-linear equations.*

For example, $y_1 = 1 + \sin x$ and $y_2 = 1 + \cos x$ are solutions of the linear non-homogeneous differential equation $y'' + y = 1$ but $y_1 + y_2$, combination of y_1 and y_2 is not its solution. Similarly, $y_1 = x^2$ and $y_2 = 1$ are solutions of the non-linear differential equation $yy'' - xy' = 0$ but $y_1 + y_2$ is not a solution of this equation. We note that even $-y_1$, a simple constant multiple of y_1, is not a solution of this equation.

Next, if in $c_1 y_1 + c_2 y_2$, y_2 is simply a constant multiple of y_1, say $y_2 = ky_1$, then $c_1 y_1 + c_2 y_2 = c_1 y_1 + c_2 k y_1 = (c_1 + c_2 k) y_1$ is just another constant multiple of y_1. In such a case y_2 does not provide any additional information. We now distinguish the two cases, first, when the two solutions are constant multiple of each other; and second, when the two solutions are not constant multiple of each other.

11.2.3 Linear Independent and Dependence

Functions $y_1(x), y_2(x), \ldots, y_n(x), n > 1$ are called *linearly independent* on some interval I, where they are defined, if the equation

$$c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) = 0 \qquad \text{...(11.3)}$$

on I implies that $c_1 = c_2 = \ldots = c_n = 0$. These functions are said to be *linearly dependent* on I if the Eq. (11.3) also holds on I for some $c_1, c_2, \ldots, c_n$ not all zero. In such a case, one or more of y_i's can be expressed as a linear combination of the remaining functions. For example, if $c_1 \neq 0$, then

$$y_1(x) = -\frac{1}{c_1}[c_2y_2(x) + \ldots + c_ny_n(x)].$$

In particular, two functions y_1 and y_2 are linearly dependent on I, if there exists a constant $c \neq 0$ such that $y_1(x) = cy_2(x)$ on I; otherwise, $y_1(x)$ and $y_2(x)$ are linearly dependent on I. For example, $y_1 = \cos x$ and $y_2 = \sin x$ are two linearly independent solutions of the differential equation $y'' + y = 0$, since $y_2 = cy_1$, gives $\tan x = c$, for c, for all x, which is not true. But obviously, $y_1 = \cos x$ and $y_2 = 5 \cos x$ are two linearly dependent solutions of this equation.

It is difficult to examine independence and dependence like this in case of more than two functions. A very systematic procedure to test the linear independence and dependence of a given set of functions is the application of Wronskian. Let $y_1, y_2, \ldots, y_n$ be the given functions, then the Wronskian of these functions, denoted by $W(y_1, y_2, \ldots, y_n)$, is defined as

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W(x).$$

Obviously the Wronskian of y_i's exists only if all the y_i's are differentiable $(n-1)$ times on I. We have the following result for testing the linear dependence or independence of the solutions of the linear homogeneous differential equation

$$a_0(x)\, y_{(x)}^{(n)} + a_1(x)\, y_{(x)}^{(n-1)} + \ldots + a_{n-1}(x)\, y'(x) + a_n(x)\, y(x) = 0,\; a_0(x) \neq 0 \qquad \text{...(11.4)}$$

Theorem 11.3: If the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ of the differential Eq. (11.4) are continuous on I, then n solutions $y_1, y_2, \ldots, y_n$ of (11.4) are linearly dependent on I if and only if Wronskian $W(x) = 0$ for some $x = x_0$ on I. Further, if $W(x) = 0$ for $x = x_0$, then $W(x) = 0$ on I; hence if there is some x_1 at which $W(x) \neq 0$, then $y_1, y_2, \ldots, y_n$ are linearly independent solutions on I.

Proof Let $y_1(x), y_2(x), \ldots, y_n(x)$ be linearly dependent on I. Then there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that for all x on I,

$$c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x) = 0 \quad ...(11.5)$$

Differentiating this successively $(n-1)$ times, we obtain

$$\left.\begin{array}{l} c_1y'_1(x) + c_2y'_2(x) + ... + c_ny'_n(x) = 0 \\ \vdots \\ c_1y_1^{(n-1)}(x) + c_2y_2^{(n-1)} + ... + c_ny_n^{(n-1)}(x) = 0 \end{array}\right\} \quad ...(11.6)$$

Equations (11.5) and (11.6) is a homogeneous linear system of algebraic equations with a non-trivial solution $c_1, c_2, ..., c_n$. Hence its coefficient determinant, which is the Wronskian $W(x) = W(y_1, y_2, ..., y_n)$, must be zero for every x on I. Conversely, let $W(x) = 0$ for some fixed $x_0 \in I$. Then the system of Eqs. (11.5) and (11.6) has a solution $c_1^*, c_2^*, ..., c_n^*$ not all zero by the result just proved. Hence $y^*(x) = c_1^*y_1(x) + c_2^*y_2(x) + ... + c_n^*y_n(x)$ is a solution of the linear homogeneous Eq. (11.4). By using the system of Eqs. (11.5) and (11.6), we find that $y^*(x)$ also satisfies the initial conditions $y^*(x_0) = 0$, $(y^*)'(x_0) = 0, ..., (y^*)^{(n-1)}(x_0) = 0$. Thus $y^*(x)$ is the solution of initial value problem, and since the solution of the initial value problem is unique, refer to Theorem 11.1, thus $y^*(x) = y(x) = 0$ holds identically. That is, $c_1^* y_1(x) + c_2^*y_2(x) + ... + c_n^* y_n(x) = 0$, for all c_i's not zero and this implies the linear dependence of $y_1(x), y_2(x), ..., y_n(x)$. Thus $W(x_0) = 0$ for some $x_0 \in I$ implies $W(x) = 0$ for all $x \in I$.

From the results proved above, it follows obviously that $W(x) \neq 0$ at solve $x_1 \in I$ implies linear independence of the solutions $y_1, y_2, ..., y_n$ on I.

We have considered the solutions $y_1(x) = \cos x$, and $y_2(x) = \sin x$ of $y'' + y = 0$ for all x. In this case linear independence was obvious. Also the Wronskian of these solutions is

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x \neq 0.$$

But it is not always obvious whether the two solutions are linearly independent or dependent on an interval. For example, consider the equation $y'' + xy = 0$. This equation can be solved by power series method to be discussed in Chapter 12; the two solutions are

$$y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12960}x^9 + ...$$

and,
$$y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \frac{1}{45360}x^{10} + ...$$

Both $y_1(x)$ and $y_2(x)$ are convergent for all x on the real line. It is difficult to evaluate the Wronskian of these solutions at any non-zero x. Consider the Wronskian at $x = 0$, we have

$$W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y'_1(0) & y'_2(0) \end{vmatrix} = y_1(0)y'_2(0) - y'_1(0)y_2(0) = (1)(1) - (0)(0) = 1 \neq 0$$

and this is sufficient to ensure that $y_1(x)$ and $y_2(x)$ as defined above, are linearly independent for all x on the entire line.

Fundamental solutions: a basis. The n linearly independent solutions $y_1(x), y_2(x) ..., y_n(x)$ are called the *fundamental solutions* of the homogeneous Eq. (11.2) on I. The set $\{y_1, y_2, ..., y_n\}$ of fundamental solutions forms a *basis* of the nth order linear homogeneous Eq. (11.2).

Now we are in a position to define the general solution of the homogeneous linear equation (11.2).

The general solution If $y_1(x), y_2(x), \ldots, y_n(x)$ are n linearly independent solutions of the nth order linear homogeneous equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y^{(1)} + a_n(x) = 0$$

that is, if the set $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ forms a basis of the nth order linear homogeneous equation, then the general solution of this equation is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) \qquad \ldots(11.7)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example 11.2: Show that the set $\{1, e^x, e^{-x}\}$ forms a basis of the differential equation $y''' - y' = 0$, but $e^x, e^{-x}, \cosh x$ is not so.

Solution: It is easy to verify that each of the functions $1, e^x, e^{-x}$ satisfy the differential equation $y''' - y' = 0$. Their Wronskian is

$$W(x) = W(1, e^x, e^{-x}) = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2 \neq 0.$$

Thus the set $\{1, e^x, e^{-x}\}$ is a fundamental set and forms a basis of the differential equation $y''' - y' = 0$. Also we can verify that e^x, e^{-x} and $\cosh x$ are solutions of the differential equation $y''' - y' = 0$. But

$$W(x) = W(e^x, e^{-x}, \cosh x) = \begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix} = 0,$$

the first and third rows being equal.

Hence, the set $\{e^x, e^{-x}, \cosh x\}$ does not form a basis. In fact, we note the $\cosh x$ is a linear combination of e^x and e^{-x}.

Example 11.3: Show that the set of functions $\{x, 1/x\}$ forms a basis of the equation $x^2y'' + xy' - y = 0, 1 \le x < \infty$. Obtain a particular solution when $y(1) = 1, y'(1) = 2$.

Solution: We have $y_1 = x$, $y_1' = 1$, $y_1'' = 0$, and hence $x^2y_1'' + xy_1' - y_1 = x - x = 0$. Thus, $y_1 = x$ is a solution of $x^2y'' + xy' - y = 0$. Similarly, we can verify that $y_2 = 1/x$ is also a solution of the given homogeneous equation.

The Wronskian of y_1, y_2 is, $W(y_1, y_2) = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\dfrac{2}{x} \neq 0$. Therefore the set $\{x, 1/x\}$ forms a basis of the equation, and hence the general solution is $y(x) = c_1x + \dfrac{c_2}{x}$. It gives, $y'(x) = c_1 - \dfrac{c_2}{x^2}$.

Using the initial values, $y(1)=1, y'(1)=2$, we get respectively $c_1+c_2=1$, $c_1-c_2=2$; thus $c_1=3/2$, $c_2=-1/2$. Hence the particular solution is $y(x)=\frac{1}{2}\left(3x-\frac{1}{x}\right)$.

11.3 FINDING SECOND LINEARLY INDEPENDENT SOLUTION FROM A KNOWN SOLUTION: REDUCTION OF ORDER

Suppose we know one solution $y_1(x)$ of the homogeneous linear second order differential equation.

$$a_0(x)y''+a_1(x)y'+a_2(x)y=0,\ a_0(x)\neq 0 \qquad ...(11.8)$$

and we need to find the second linealy independent solution $y_2(x)$. The same can be obtained by the *method of reduction of order* which involves a second solution of the form

$$y_2(x)=u(x)y_1(x), \qquad ...(11.9)$$

where the function $u(x)\neq$ constant is to be determined. From (11.9), we have

$$y'_2=u'y_1+uy'_1,\ \text{and}\quad y''_2=u''y_1+2u'y'_1+uy''_1.$$

Substituting for y_2, y'_2, y''_2 in (11.8) and rearranging the terms, we obtain

$$a_0(x)y_1u''+[2a_0(x)y'_1+a_1(x)y_1]u'+[a_0(x)y''_1+a_1(x)y'_1+a_2(x)y_1]u=0 \qquad ...(11.10)$$

Using the fact that y_1 is a solution of Eq. (11.8), (11.10) reduces to

$$a_0(x)y_1u''+[2a_0(x)y'_1+a_1(x)y_1]u'=0 \qquad ...(11.11)$$

Let $v=u'$, Eq. (11.11) becomes $a_0(x)y_1v'+[2a_0(x)y'_1+a_1(x)y_1]v=0$, a first order differential equation in v. Separating the variables, we obtain $\frac{dv}{v}=-\left[\frac{2a_0(x)y'_1+a_1(x)y_1}{a_0(x)y_1}\right]dx.$

Integrating we get, $\ln v=-2\ln y_1-\int\frac{a_1(x)}{a_0(x)}dx+\text{const.}$

or,

$$v=A_1\left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2}\right]. \qquad ...(11.12)$$

where $p(x)=a_1(x)/a_0(x)$, and A_1 is a constant.

Since $v=u'$, integration of (11.12) gives, $u(x)=A_1\int\left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2}\right]dx+A_2$, ...(11.13)

where A_2 is another arbitrary constant of integration. Without any loss of generality, we can have $A_1=1$ and $A_2=0$, and thus (11.13) gives $u(x)=\int\left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2}\right]dx.$

Therefore, from (11.9), the second linearly independent solution is

$$y_2(x) = y_1(x)\int\left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2}\right]dx \qquad ...(11.14)$$

and hence the general solution of (11.8) is, $y(x) = c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are two arbitrary constants.

Example 11.4: Given that $y_1(x) = e^{-3x}$ is a solution of $y'' + 6y' + 9y = 0$, find a second linearly independent solution and hence find the general solution of the equation.

Solution: Let the second linearly independent solution be $y_2(x) = u(x)y_1(x)$, where $y_1(x) = e^{-3x}$. Here $p(x) = a_1(x)/a_0(x) = 6$. Hence,

$$u(x) = \int\left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2}\right]dx = \int\left[\frac{\exp\left\{-\int 6dx\right\}}{e^{-6x}}\right]dx = \int dx = x.$$

Thus $y_2(x) = xe^{-3x}$ and hence, the general solution is $y(x) = c_1e^{-3x} + c_2xe^{-3x}$.

Example 11.5: Given that $y_1(x) = x^2$ is a solution of $y'' - (3/x)y' + (4/x^2)y = 0, x > 0$, find a second linealy independent solution and hence find the general solution of the equation.

Solution: Let the second linearly independent solution be $y_2(x) = u(x)y_1(x)$, where $y_1(x) = x^2$.

Here $\quad p(x) = \dfrac{a_1(x)}{a_0(x)} = -3/x$. Hence,

$$u(x) = \int\left[\frac{\exp\left\{-\int p(x)dx\right\}}{y_1^2}\right]dx = \int\left[\frac{\exp\left\{-\int\frac{-3}{x}dx\right\}}{x^4}\right]dx = \ln x.$$

Thus, $y_2(x) = x^2\ln x$, and hence the general solution is $y = c_1x^2 + c_2x^2\ln x$, where c_1 and c_2 are two arbitrary constants.

EXERCISE 11.1

Verify that each of the given function is a solution of the given differential equation. Verify if the set forms a basis or not. If so, find the general solution of the differential equation.

1. $y''' - 6y'' + 11y' - 6y = 0,\ \{e^x, e^{2x}, e^{3x}\}$
2. $y''' - 6y'' + 9y' - 4y = 0,\ \{e^x, xe^x, (1-x)e^x\}$
3. $x^2y'' - xy + y = 0, x > 0,\ \{x, x\ln x\}$
4. $y^{(4)} + 2y^{(2)} + y = 0, \{\cos x, \sin x, x\cos x, x\sin x\}$

5. $x^2y'' + 4xy' + 2y = 0, \left\{\frac{1}{x}, \frac{1}{x^2}\right\}$

6. $(1 + x^2)y'' + (1+x)y' + y = 0$, $\{\cos[\ln(1 + x)], \sin[\ln(1 + x)]\}$

Verify that y_1 and y_2 are solutions of the given differential equation. Show that they form a basis. Also find the particular solution of the initial value problem.

7. $y'' - 4y = 0; y(0) = 1, y'(0) = 0, y_1 = \cosh(2x), y_2 = \sinh(2x)$

8. $y'' + 11y' + 24y = 0; y(0) = 1, y'(0) = 0, y_1 = e^{-3x}, y_2 = e^{-8x}$

9. $y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0; y(1) = 2, y'(1) = 4, y_1 = x^4, y_2 = x^4 \ln x$

10. $x^2y'' + xy' - 4y = 0; y(1) = 2, y'(1) = 6, y_1 = x^2, y_2 = \frac{1}{x^2}$

11. $4x^2y'' - 3y = 0; y(1) = 3, y'(1) = 2.5, y_1 = x^{-1/2}, y_2 = x^{3/2}$

Find the solution of the following differential equations by reducing the order, one solution is known in each case.

12. $y'' - y' - 6y = 0, y_1 = e^{-2x}$

13. $x^2y'' - xy + y = 0; x > 0, y_1 = x$

14. $x^2y'' - 5xy' + 9y = 0, y_1 = x^3$

15. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0, y_1 = x^{-1/2}\cos x$

Reduce the following equations to first order and solve

16. $y'' = y'$

17. $yy'' = 2y'^2$

18. $xy'' = \sqrt{1 + y'^2}$

19. $y^{(4)}y^{(3)} = 1$

20. A small body moves on a straight line so that the product of its velocity and acceleration is constant, say 1 m^2/sec^3. If at $t = 0$ the body's distance from the origin is 2 meter and its velocity is 2 meter/sec; then what are the distance and velocity at $t = 6$ sec?

11.4 DIFFERENTIAL OPEREATOR D. SOLUTION OF CONSTANT COEFFICIENTS HOMOGENEOUS LINEAR EQUATIONS

In this section we introduce the differential operator D and discuss methods of finding solution of linear differential equations with constant coefficients.

11.4.1 Differential Operator D

By an operator we mean a transformation that transforms a function into another function. Let D denote the differentiation with respect to x, that is, $D = \frac{d}{dx}$, and we write

$$Df(x) = Df = f' = \frac{df}{dx}.$$

Thus D transforms $f(x)$ into its derivative $f'(x)$. For example, $D(x^3) = 3x^2$, $D(\sin x) = \cos x$. Also D is a linear operator, that is, $D(af + bg) = aDf + bDg$, where a and b are constants.

Further applying D twice, we have $D(Df) = D(f') = f''$, and we simply write $D^2f = f''$, $D^3f = f'''$ etc., where f is sufficiently differentiable.

We define $D^0 = 1$, and thus $D^0f = 1f = f$.

The homogeneous linear differential equation $a_0\dfrac{d^2y}{dx^2} + a_1\dfrac{dy}{dx} + a_2y = 0$, can be written as

$$a_0D^2y + a_1Dy + a_2y = 0, \text{ or } (a_0D^2 + a_1D + a_2)y = 0$$

or,
$$F(D)y = 0, \text{ where } F(D) = a_0D^2 + a_1D + a_2$$

is the *second order differential operator.*

Similarly, linear differential equation of the nth order,

$$a_0\frac{d^ny}{dx^2} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}\frac{dy}{dx}a_ny = f(x)$$

can be expressed as $F(D)y = f(x)$,
where $F(D) = a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n$, is the *nth order differential operator.*

When a_i, $i = 0, 1, \ldots n$, are constants, the differential operator $F(D)$ can be factorized. For example, $(D^2 + 3D + 2) = (D + 1)(D + 2)$.

We must note that when $a(x)$ is a function of x, then $D[a(x)f] \neq a(x)Df$. In that case $D[a(x)f] = a(x)f' + a'(x)f$. We shall apply operator methods to solve linear differential equations with constant coefficients only. Extension of these methods to variable-coefficient equations is comparatively difficult and will not be considered here.

11.4.2 Solution of the Constant Coefficients Homogeneous Linear Equation of Second Order

Consider the second order homogeneous linear equation

$$ay'' + by' + cy = 0, \qquad \ldots(11.15)$$

where $a, b, c, a \neq 0$ are constants.

In the operator notion, Eq. (11.15) can be written as

$$(aD^2 + bD + c)y = 0. \qquad \ldots(11.16)$$

The form of the Eq. (11.16) requires that constant multiples of derivatives of $y(x)$ must sum to zero. Since the derivative of an exponential function $e^{\lambda x}$ is a constant multiple of $e^{\lambda x}$, therefore, we consider $y = e^{\lambda x}$ for solution. Substituting in (11.16), we get $(a\lambda^2 + b\lambda + c)e^{\lambda x} = 0$. Since $e^{\lambda x} \neq 0$, we obtain

$$a\lambda^2 + b\lambda + c = 0, \qquad \ldots(11.17)$$

an algebraic equation in λ.

It is called *the characteristic equation* or *the auxiliary equation* of the Eq. (11.15) and the roots of this equation $\lambda_1 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$, $\lambda_2 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ are called the *characteristic roots* of the Eq. (11.15). Thus the functions $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are the solutions of the differential equation (11.15). Depending on the sign of the discriminant b^2-4ac, the following three cases arise

Case I: *$b^2-4ac>0$, the roots are real and distinct, that is, $\lambda_1 \neq \lambda_2$.*

Case II: *$b^2-4ac=0$, the roots are real and equal, that is, $\lambda_1 = \lambda_2$.*

Case III: *$b^2-4ac<0$, the roots are complex conjugate.*

To find the complete solution in each case, we proceed as follows:

Case I: Real and distinct roots. In this case $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ form two linearly independent solutions of the differential Eq. (11.15) on any interval, since $y_1/y_2 = e^{(\lambda_1-\lambda_2)x} \neq$ constant for $\lambda_1 \neq \lambda_2$. The general solution of Eq. (11.5) is

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x},$$

where c_1 and c_2 are two arbitrary constants.

Case II: Real and equal roots. In case the discriminant $b^2-4ac=0$, the characteristic equation has the repeated root $\lambda = \lambda_1 = \lambda_2 = -b/2a$, so $y_1 = e^{-bx/2a}$ is one solution of the Eq. (11.15).

To obtain a second linearly independent solution y_2, we use the method of reduction of order as discussed in Section 11.3. Set $y_2 = uy_1$ in Eq. (11.15), we obtain

$$a(uy_1)'' + b(uy_1)' + cuy_1 = 0$$

or,
$$a(u''y_1 + 2u'y_1' + uy_1'') + b(u'y_1 + uy_1') + cuy_1 = 0.$$

Collecting terms in u'', u' and u, we obtain

$$ay_1u'' + (2ay_1' + by_1)u' + (ay_1'' + by_1' + cy_1)u = 0. \qquad \text{...(11.18)}$$

Since y_1 is a solution of Eq. (11.15), therefore, $ay_1'' + by_1' + cy_1 = 0$.

Also, $2ay_1' = 2a\left(-\dfrac{b}{2a}e^{-\frac{b}{2a}x}\right) = -be^{-\frac{b}{2a}x} = -by_1$, which gives, $(2ay_1' + by_1) = 0$.

Substituting these in (11.18), we obtain $au''y_1 = 0$, or $u'' = 0$, since $a \neq 0$ and $y_1 \neq 0$. It gives $u = Ax + B$. We can simply take $u = x$, and thus making $y_2 = xy$, as the second linearly independent solution of Eq. (11.15), for $y_2/y_1 = x$ is not a constant. Thus, in the case of real double root of the characteristic equation, the general solution of Eq. (11.15) is

$$y = (c_1 + c_2x)e^{\lambda x}, \quad \lambda = -b/2a,$$

where c_1 and c_2 are two arbitrary constants.

Case III: Complex conjugate roots. In case $b^2-4ac<0$, then the roots of the Eq.(11.17) are complex conjugates, say $p \pm iq$, where $p = -\frac{b}{2a}$ and $q = \sqrt{4ac - b^2}/2a$ are reals. Thus the two solutions of the Eq. (11.15) are $y_1 = e^{(p+iq)x}$, and $y_2 = e^{(p-iq)x}$.

These two solutions are linearly independent, since $q \neq 0$ and thus

$$\frac{y_1}{y_2} = e^{(p+iq)x}/e^{(p-iq)x} = e^{2iqx} \neq \text{constant.}$$

Hence the general solution of the equation is

$$y = Ay_1 + By_2 = Ae^{(p+iq)x} + Be^{(p-iq)x} = (Ae^{iqx} + Be^{-iqx})e^{px}$$
$$= [A(\cos px + i\sin px) + B(\cos qx - i\sin qx)]e^{px} \qquad \text{...(11.19)}$$

by Euler's formula. Rewriting (11.19), we obtain

$$y(x) = (c_1 \cos qx + c_2 \sin qx)e^{px}, \qquad \text{...(11.20)}$$

as the general solution, where $c_1 = A + B$ and $c_2 = i(A - B)$ are constants.

Hence, we have the following results to remember:

For the differential equation $ay'' + by' + cy = 0$, the characteristic equation is $a\lambda^2 + b\lambda + c = 0$. The three cases are:

I $b^2 - 4ac > 0$: *The general solution is* $y(x) = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$, *where*

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

II $b^2 - 4ac = 0$: *The general solution is* $y(x) = (c_1 + c_2x)e^{\lambda x}$, *where* $\lambda = -b/2a$.

III $b^2 - 4ac < 0$: *The general solution is* $y(x) = (c_1 \cos qx + c_2 \sin qx)e^{px}$, *where*

$$p = -\frac{b}{2a} \text{ and } q = \sqrt{4ac - b^2}/2a.$$

We note that the characteristic Eq. (11.17) for the Eq. (11.15) can be written directly replacing simply D by λ in the Eq. (11.15).

Example 11.6: Find the general solution of the differential equation $2y'' + 5y' - 3y = 0$.

Solution: The differential equation is $(2D^2 + 5D - 3)y = 0$.

The corresponding characteristic equation is given by

$$2\lambda^2 + 5\lambda - 3 = 0, \text{ or } (2\lambda - 1)(\lambda + 3) = 0.$$

It gives $\lambda = 1/2, -3$ as two distinct real roots. Hence, the general solution of the given equation is $y(x) = c_1e^{x/2} + c_2e^{-3x}$, where c_1 and c_2 are arbitrary constants.

Example 11.7: Find the general solution and solve the initial value differential equation

$$y'' + 4y' + 4y = 0; \; y(0) = 3 \text{ and } y'(0) = 1.$$

Solution: The differential equation is $(D^2 + 4D + 4)y = 0$.

The corresponding characteristic equation is given by

$$\lambda^2 + 4\lambda + 4 = 0, \text{ or } (\lambda + 2)^2 = 0,$$

which gives $\lambda = -2, -2$ as repeated root. Hence the general solution is

$$y = (c_1 + c_2 x)e^{-2x}. \qquad ...(11.21)$$

Using the initial conditions $y(0) = 3$, and $y'(0) = 1$ in (11.21), we get $c_1 = 3$ and $c_2 = 7$. So the solution of the given initial value problem is $y(x) = (3 + 7x)e^{-2x}$.

Example 11.8: Solve the initial value problem $y'' + 6y' + 13y = 0$; $y(0) = 3$, $y'(0) = 7$.

Solution: The differential equation is $(D^2 + 6D + 13)y = 0$.

The corresponding characteristic equation is given by $\lambda^2 + 6\lambda + 13 = 0$, which gives $\lambda = -3 \pm 2i$, as the two characteristic roots. Hence the general solution is

$$y(x) = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x), \qquad ...(11.22)$$

where c_1 and c_2 are arbitrary constants.

Using the initial condition $y(0) = 3$ in (11.22) gives $c_1 = 3$. Also from (11.22),

$$y'(x) = e^{-3x}[(2c_2 - 3c_1) \cos 2x - (2c_1 + 3c_2) \sin 2x].$$

Using $y'(0) = 7$, we obtain $2c_2 - 3c_1 = 7$, which gives $c_2 = 8$, since $c_1 = 3$.

Substituting for c_1 and c_2 in (11.22), the solution of the initial value problem becomes $y = e^{-3x}(3 \cos 2x + 8 \sin 2x)$.

Example 11.9: Solve the boundary value problem $y'' + w^2 y = 0$, $y(0) = 0$ and $y(e) = 0$.

Solution: The differential equation is

$$(D^2 + w^2)y = 0, \ y(0) = 0 \text{ and } y(e) = 0 \qquad ...(11.23)$$

The characteristic equation is $\lambda^2 + w^2 = 0$, which gives, $\lambda = \pm iw$, as two complex conjugate roots. Therefore the general solution is

$$y(x) = c_1 \cos wx + c_2 \sin wx \qquad ...(11.24)$$

Using the boundary conditions $y(0) = 0$, $y(e) = 0$ in Eq. (11.24), we obtain

$$c_1 = 0, \text{ and } c_2 \sin (we) = 0$$

In case $c_2 = 0$, the solution is trivial one, $y(x) = 0$.

For $c_2 \neq 0$, we get $\sin we = 0$, which gives $w = \dfrac{n\pi}{e}$, $n = 0, \pm 1, \pm 2, ...$ Therefore, the solution is

$y_n(x) = B_n\left[\sin\dfrac{n\pi x}{e}\right]$, $n = 0, \pm 1, \pm 2, ...$, where B_i's are arbitrary constants.

Since Eq. (11.23) is homogeneous, using the superposition principle, the general solution is given by

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{e}\right), \quad ...(11.25)$$

where B_i's are arbitrary constants.

Example 11.10: Solve the differential equation $9y'' - 24y' + 16y = 0$ by factorizing the differential operator and reducing it into first order equation.

Solution: The differential equation is $(9D^2 - 24D + 16)y = 0$

or,
$$(3D - 4)(3D - 4)y = 0. \quad ...(11.26)$$

Set $(3D - 4)y = u$, the Eq. (11.26) becomes $(3D - 4)u = 0$, which gives $u = ce^{4x/3}$ as its solution. Therefore,

$$(3D - 4)y = ce^{4x/3}, \text{ or } \left(D - \frac{4}{3}\right)y = \frac{c}{3}e^{4x/3},$$

which is a linear first order equation. The integrating factor is $e^{-4x/3}$, hence the solution is

$$ye^{-4x/3} = \int \frac{c}{3}dx + c_1 = \frac{cx}{3} + c_1, \text{ or } y = (c_1 + c_2x)e^{4x/3}; \quad c_2 = c/3,$$

where c_1, c_2 are two arbitrary constants.

11.4.3 Solution of the Constant Coefficients Homogeneous Linear Equations of Higher Order

The method discussed in case of second order equations can be extended in a natural way to find the solution of the higher order homogeneous linear differential equations with constant coefficients. The characteristic equation for the homogeneous linear differential equation of the nth order

$$(a_0D^n + a_1D^{n-1} + ... + a_{n-1}D + a_n)y = 0 \quad ...(11.27)$$

is written as

$$a_0\lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n = 0. \quad ...(11.28)$$

This is a polynomial of degree n and hence has n roots, say $\lambda_1, \lambda_2, ..., \lambda_n$. All the roots may be real and distinct, all or some of the roots may be equal, all or some of the roots may be in complex conjugate pairs. We discuss the following cases.

Distinct real roots. *If all the n roots $\lambda_1, \lambda_2, ..., \lambda_n$ of Eq. (11.28) are real and distinct, then the n solutions $y_1(x) = e^{\lambda_1 x}$, $y_2(x) = e^{\lambda_2 x}, ..., y_n(x) = e^{\lambda_n x}$ form a set of n linearly independent solutions of the differential equation (11.27), (The Wronskian of these being non-zero, for $\lambda_1 \neq \lambda_2 \neq ... \neq \lambda_n$), and hence the general solution of Eq. (11.28) is*

$$y(x) = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} + ... + c_ne^{\lambda_n x},$$

where $c_1, c_2, ..., c_n$ are n arbitrary constants.

Multiple real roots. If a real double root occurs, say $\lambda_1 = \lambda_2 = \lambda$, then corresponding to this root, as discussed in case of homogeneous equation of second order with constant coefficient, we take $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$ as two linearly independent solutions.

If a triple root occurs, say $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then corresponding to this root, three linearly independent solutions are $y_1 = e^{\lambda x}$, $y_2 = xe^{\lambda x}$, and $y_3 = x^2 e^{\lambda x}$.

In general, if λ is a root of multiplicity k, then the corresponding k linearly independent solutions are $y_1 = e^{\lambda x}$, $y_2 = xe^{\lambda x}$, ..., $y_k = x^{k-1}e^{\lambda x}$

The linear independece of $y_1, y_2, ..., y_k$ can be verified by finding the Wronskian and proving it to be non-zero over any open interval.

Complex roots. The coefficients in the characteristic equation (11.28) being real, the complex roots will occur only in conjugate pairs, say $p \pm iq$. The corresponding linearly independent solutions are given by $e^{px} \cos qx$ and $e^{px} \sin qx$.

In case the characteristic equation has k simple conjugate pairs of complex roots given by $q_r \pm ip_r$; $r = 1, 2, ..., k$, then the corresponding 2k linearly independent solutions are $e^{p_r x} \cos q_r x$, $e^{p_r x} \sin q_r x$, $r = 1, 2, ..., k$.

In case of multiple complex roots, say $p \pm iq$ is a pair of complex conjugate roots of multiplicity 2, then the corresponding four linearly independent solutions are: $e^{px} \cos qx$, $e^{px} \sin qx$, $xe^{px} \cos qx$ and $xe^{px} \sin qx$.

This can be extended in case roots of multiplicity of higher order occur.

Example 11.11: Find the general solution of the differential equation $y''' - 8y' + 8y = 0$.

Solution: The differential equation is $(D^3 - 8D + 8)y = 0$. The characteristic equation is

$$\lambda^3 - 8\lambda + 8 = 0 \qquad ...(11.29)$$

By inspection one root of Eq. (11.29) is $\lambda = 2$. The equation can be expressed as $(\lambda - 2)(\lambda^2 + 2\lambda - 4) = 0$. The quadratic equation $\lambda^2 + 2\lambda - 4 = 0$ has the roots $\lambda = -1 \pm \sqrt{5}$.

Hence the three roots of the characteristic Eq. (11.29) are: 2, $-1 + \sqrt{5}$ and $-1 - \sqrt{5}$. Since, the roots are distinct and real, therefore, the general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{(-1+\sqrt{5})x} + c_3 e^{(-1-\sqrt{5})x} = c_1 e^{2x} + e^{-x}[c_2 e^{\sqrt{5}x} + c_3 e^{-\sqrt{5}x}],$$

where c_1, c_2 and c_3 are arbitrary constants.

Example 11.12: Find the general solution of the equation $y'''' + ky = 0$ arising in studying the deflected shape $y(x)$ of a beam on an elastic foundation, where $k > 0$ is a constant.

solution: The differential equation is $(D^4 + k)y = 0$.

The corresponding characteristic equation is $\lambda^4 + k = 0$. Its roots are given by

$$\lambda = (-k)^{1/4} = k^{1/4}\left[\cos\frac{(2m+1)\pi}{4} + i\sin\frac{(2m+1)\pi}{4}\right], m = 0, 1, 2, 3.$$

Simplifying, the roots are $\lambda = \frac{k^{1/4}}{\sqrt{2}}(1 \pm i)$, $\frac{k^{1/4}}{\sqrt{2}}(-1 \pm i)$, two pairs of simple complex conjugates. Hence the general solution is

$$y(x) = e^{\frac{k^{1/4}}{\sqrt{2}}x}\left(c_1 \cos \frac{k^{1/4}}{\sqrt{2}}x + c_2 \sin \frac{k^{1/4}}{\sqrt{2}}x\right) + e^{-\frac{k^{1/4}}{\sqrt{2}}x}\left(c_3 \cos \frac{k^{1/4}}{\sqrt{2}}x + c_4 \sin \frac{k^{1/4}}{\sqrt{2}}x\right),$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

Example 11.13: Find the general solution of a homogeneous equation with the characteristic equation $\lambda^3(\lambda + 4)^2 (\lambda^2 + 2\lambda + 5)^2 = 0$.

Solution: In the characteristic equation $\lambda^3(\lambda + 4)^2 (\lambda^2 + 2\lambda + 5)^2 = 0$, the root $\lambda = 0$ occurs with multiplicity three, the root $\lambda = -4$ occurs with multiplicity two, and the pairs of complex conjugate roots $\lambda = -1 \pm 2i$, occur with multiplicity two.

Hence, the general solution of the differential equation with the given characteristic equation is

$$y(x) = c_1 + c_2x + c_3x^2 + (c_4 + c_5x)e^{-4x} + e^{-x}[(c_6 + c_7x)\cos 2x + (c_8 + c_9x)\sin 2x],$$

where c_i's, $i = 1, 2, \ldots, 9$ are arbitrary constants.

Example 11.14: Find the non-trivial solution of the boundary value problem $y'''' - k^4y = 0$, $y(0) = y''(0) = y(l) = y''(l) = 0$, where $k > 0$ is a constant.

Solution: The differential equation is $(D^4 - k^4)y = 0$. The corresponding characteristic equation is $\lambda^4 - k^4 = 0$. Its roots are: $\lambda = \pm k, \pm ik$, and hence the general solution is

$$y(x) = c_1'e^{kx} + c_2'e^{-kx} + c_3 \cos kx + c_4 \sin kx.$$

Since $e^{kx} = \cosh kx + \sinh kx$, and $e^{-kx} = \cosh kx - \sinh kx$, the general solution can be expressed as

$$y(x) = c_1 \cosh kx + c_2 \sinh kx + c_3 \cos kx + c_4 \sin kx, \qquad \ldots(11.30)$$

where c_i's are constants.

Using the initial condition $y(0) = 0$ in (11.30), we get $c_1 + c_3 = 0$.

Also from (11.30), $y''(x) = k^2(c_1 \cosh kx + c_2 \sinh kx - c_3 \cos kx - c_4 \sin kx)$.

Using the initial condition $y''(0) = 0$, we get $k^2(c_1 - c_3) = 0$, which gives $c_1 - c_3 = 0$, since $k \neq 0$. Solving $c_1 + c_3 = 0$, $c_1 - c_3 = 0$, we obtain, $c_1 = c_3 = 0$, and hence (11.30) becomes

$$y(x) = c_2 \sinh kx + c_4 \sin kx \qquad \ldots(11.31)$$

Next, from (11.31), we have

$$y''(x) = k^2(c_2 \sinh kx - c_4 \sin kx) \qquad \ldots(11.32)$$

Using $y(l) = 0$ in (11.31) and $y''(l) = 0$ in (11.32) we obtain respectively

$$c_2 \sinh kl + c_4 \sin kl = 0, \qquad \ldots(11.33)$$

$$\text{and, } c_2 \sinh kl - c_4 \sin kl = 0. \qquad \ldots(11.34)$$

Adding these two, we get, $c_2 \sinh kl = 0$, which gives $c_2 = 0$, since $\sinh kl \neq 0$.

Using $c_2 = 0$ in (11.33), we obtain $c_4 \sin kl = 0$. Since we are interested in non-trivial solution, taking $c_4 \neq 0$. Hence $\sin kl = 0$, which gives $k = \frac{n\pi}{l}$, $n = 1, 2, \ldots$ Thus the solutions are

$$y_n(x) = b_n \sin \frac{n\pi x}{l}, n = 1, 2, \ldots$$

Since the given equation is homogeneous, using the superposition principle, the solution is given by $y(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ where b_i's are arbitrary constants.

EXERCISE 11.2

Find the general solution of

1. $y'' + y' - 2y = 0$
2. $y'' + 2y' + 4y = 0$
3. $4y'' + 4y' + y = 0$
4. $y''' + y'' + 4y' + 4y = 0$
5. $y'''' + 4y = 0$
6. $y'''' + 8y'' + 16y = 0$

Solve the following initial value problems

7. $2y'' + 5y' - 3y = 0;\ y(0) = 4,\ y'(0) = 9$
8. $4y'' + 20y' + 125\,y = 0;\ y(0) = 3,\ y'(0) = 2.5$
9. $y'' - 6y' + 9y = 0;\ y(0) = 2,\ y'(0) = 0$
10. $y''' + 2y'' + 4y' = 0;\ y(0) = 0;\ y'(0) = 1,\ y''(0) = 0$
11. $y''' + y'' - 4y = 0;\ y(0) = 1;\ y'(0) = 1,\ y''(0) = 0$
12. $y^{iv} - y'' - 2y = 0;\ y(0) = 1;\ y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 0$

Solve the following boundary value problems

13. $y'' + 2y' + 2y = 0;\ y(0) = 1,\ y(\pi/2) = 0$
14. $y'' + 36y = 0;\ y(0) = 2,\ y(1/6) = 1/e$
15. $y'' + 2y' + 2y = 0;\ y(0) = 1,\ y(\pi/2) = e^{-\pi/2}$
16. $y''' + \pi^2 y' = 0;\ y(0) = 0,\ y(1) = 0,\ y'(0) + y'(1) = 0$
17. $y^{iv} + 13y'' + 36y = 0;\ y(0) = 0,\ y''(0) = 0,\ y(\pi/2) = -1,\ y'(\pi/2) = -4$
18. $y^{iv} + 4y''' + 8y'' + 8y' + 4y = 0;\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -2,\ y'''(0) = 2$
19. If the roots of a characteristic equation are 4, 4, 4, i and $-i$, then find the original differential equation and also its general solution.

Solve the following differential equations by factorizing the differential operator and reducing it into first order equations

20. $(D^2 + 4D + 4)y = 0$
21. $(4D^2 + 8D + 3)y = 0$
22. $(D^3 + 3D^2 - 4)y = 0$
23. Find non-trivial solution of the boundary value problem $y'' + k^2 y = 0;\quad y(0) = y(\pi) = 0.$
24. If $k > 0$, then show that the general solution of $y^{(iv)} - k^4 y = 0$ can be expressed as $y = c_1 \cos kx + c_2 \sin kx + c_3 \cosh kx + c_4 \sinh kx$.

11.5 SOLUTION OF CONSTANT COEFFICIENTS NON-HOMOGENEOUS LINEAR EQUATIONS.

In this section we discuss the method to find the solution of non-homogeneous linear differential equations of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = f(x),\ a_0 \neq 0 \qquad \dots(11.35)$$

when the general solution of the corresponding homogeneous equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0,\ a_0 \neq 0 \qquad \dots(11.36)$$

is known.

11.5.1 The General Solution of the Non-homogeneous Equation

We have the following theorem:

Theorem 11.4: *If $\{y_1(x), y_2(x), \dots, y_n(x)\}$ is a fundamental set of solution and $y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$ is the general solution of the homogeneous linear Eq. (11.36), and further, if $y_p(x)$ is any particular solution of the non-homogeneous Eq. (11.35), then the general solution of the non-homogeneous Eq. (11.35) is given by*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) + y_p(x) \qquad \dots(11.37)$$

Proof. Since $y(x)$ and $y_p(x)$ both are solutions of the non-homogeneous linear equation (11.35), thus

$$a_0[y^{(n)} - y_p^{(n)}] + a_1[y^{(n-1)} - y_p^{(n-1)}] + \dots + a_{n-1}[y' - y_p'] + a_n[y - y_p]$$
$$= (a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y) - (a_0 y_p^{(n)} + a_1 y_p^{(n-1)} + \dots + a_{n-1} y'_p + a_n y_p)$$
$$= f(x) - f(x) = 0$$

Therefore $y - y_p$ is a solution of the homogeneous Eq. (11.36). Further $\{y_1, y_2, \dots, y_n\}$ forms a fundamental set of solutions for this homogeneous equation, and thus there are constants $c_1, c_2, \dots, c_n$ such that

$$y(x) - y_p(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$$

that is,

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) + y_p(x).$$

The above result suggests the following strategy to find the general solution of the non-homogeneous linear differential equation with constant coefficients.

1. *Find the general solution of the corresponding homogeneous equation. This solution is called the complementary function and is denoted by $y_c(x)$.*
2. *Find a particular solution, a solution not containing any arbitrary constant, of the non-homogeneous equation; this solution is called the particular integral and is denoted by $y_p(x)$.*

The general solution of the non-homogeneous equation is then given as $y(x) = y_c(x) + y_p(x)$, and it contains all possible solutions on the interval.

The methods for finding $y_c(x)$ have already been discussed in the preceding section. Here, we discuss methods for finding the particular integral $y_p(x)$ of the non-homogeneous equation.

11.5.2 The Operator Method for Finding Particular Integral

The D-operator method is a concise method for finding a particular integral of a linear homogeneous equation with constant coefficients.

The general linear homogeneous equation with constant coefficients can be expressed as

$F(D)y = f(x)$, where $F(D) = a_0D^n + a_1D^{n-1} + a_2D^{n-2} + ... + a_{n-1}D + a_n$,

with a_i's, $i = 0, 1, 2, ..., n$, $a_0 \neq 0$, all being constant and $f(x)$ a function of x only.

We define $\dfrac{1}{F(D)} f(x)$, *or* $[F(D)]^{-1}(x)$ *as a that function of* x, *without any arbitrary constant, which when operated by the polynomial operator* $F(D)$ *gives* $f(x)$, *that is,* $F(D)\ [F(D)]^{-1}f(x) = f(x)$.

Thus, $[F(D)]^{-1}f(x)$ *satisfies the linear differential equation* $F(D)y = f(x)$ *and is, therefore, its particular integral* $y_p(x)$. *Hence,* $y_p(x) = [F(D)]^{-1}f(x)$.

Obviously $F(D)$ and $[F(D)]^{-1}$ respresent inverse operators.

In particular, if $F(D) = D$, then $[F(D)]^{-1}f(x) = \dfrac{1}{D}f(x) = \int f(x)dx$, for

$$y = \frac{1}{D}f(x) \Rightarrow Dy = D\left(\frac{1}{D}f(x)\right) = f(x), \text{ or } \frac{dy}{dx} = f(x) \Rightarrow y = \int f(x)dx;$$

no constant is being added since we are looking for a particular integral.

Similarly, $\dfrac{1}{D-a}f(x) = e^{ax}\int f(x)e^{-ax}dx$, for

$$y = \frac{1}{D-a}f(x) \Rightarrow (D-a)y = (D-a)\left(\frac{1}{D-a}f(x)\right) = f(x) \Rightarrow \frac{dy}{dx} - ay = f(x),$$

which is a linear differential equation of order one and degree one, with solution as $y = e^{ax}\int f(x)e^{-ax}dx$, and hence,

$$\frac{1}{D-a}f(x) = e^{ax}\int f(x)e^{-ax}dx. \qquad ...(11.38)$$

The procedure for finding the particular integral by this method depends upon the form of $f(x)$ and is described below for some specific cases.

Case I: $f(x) = e^{ax}$, *where a is a constant*. When $f(x) = e^{ax}$, we have,

$De^{ax} = ae^{ax}$, $D^2e^{ax} = a^2e^{ax}$, ..., $D^ne^{ax} = a^ne^{ax}$, and thus

$$\begin{aligned} F(D)e^{ax} &= (a_0D^n + a_1D^{n-1} + ... + a_{n-1}D + a_0)e^{ax} \\ &= a_0D^ne^{ax} + a_1D^{n-1}e^{ax} + ... + a_{n-1}De^{ax} + a_0e^{ax} \\ &= a_0a^ne^{ax} + a_1a^{n-1}e^{ax} + ... + a_{n-1}ae^{ax} + a_0e^{ax} \end{aligned}$$

$$= (a_0 a^n + a_1 a^{n-1} + \dots + a_{n-1} a + a_0) e^{ax}, \text{ which gives}$$

$$F(D)e^{ax} = F(a)e^{ax} \qquad \text{...(11.39)}$$

Operating on both sides of (11.39) by $[F(D)]^{-1}$, we have

$$[F(D)]^{-1}F(D)e^{ax} = [F(D)]^{-1}F(a)e^{ax}, \text{ or } e^{ax} = F(a)[F(D)]^{-1}e^{ax}.$$

Dividing by $F(a)$, we get

$$y_p(x) = F(D)^{-1}e^{ax} = \frac{1}{F(a)}e^{ax}, \text{ provided } F(a) \neq 0 \qquad \text{...(11.40)}$$

In case $F(a) = 0$, then $(D - a)$ is a factor of $F(D)$, and say $F(D) = (D - a)\, G(D)$, where $G(a) \neq 0$. Then

$$[F(D)]^{-1} e^{ax} = [(D-a)\, G(D)]^{-1}e^{ax} = \frac{1}{D-a}[G(D)]^{-1}e^{ax} = \frac{1}{D-a}\,\frac{1}{G(a)}e^{ax} = \frac{1}{G(a)}\,\frac{1}{(D-a)}e^{ax}$$

$$= \frac{1}{G(a)}\, e^{ax}\int e^{ax}\, e^{-ax}dx, \text{ using (11.38)}$$

$$= x\,\frac{1}{G(a)}e^{ax} = x\,\frac{1}{F'(a)}e^{ax}, \text{ provided } F'(a) \neq 0, \qquad \text{...(11.41)}$$

for $F(D) = (D - a)G(D)$ implies $F'(D) = G(D) + (D - a)G'(D)$ which in turn gives $F'(a) = G(a)$.

In case $F'(a) = 0$, then re-applying the result, to get

$$[F(D)]^{-1}e^{ax} = x^2\frac{1}{F''(a)}e^{ax}, \qquad \text{...(11.42)}$$

provided, $F''(a) \neq 0$, and so on.

Example 11.15: Find the general solution of the differential equation

$$(D^2 - 13D + 12)y = 3e^{-2x}.$$

Solution: The corresponding homogeneous equation is

$$(D^2 - 13D + 12)y = 0. \qquad \text{...(11.43)}$$

To find the complementary function, consider the characteristic equation corresponding to (11.43), which is $\lambda^2 - 13\lambda + 12 = 0$. It has roots $\lambda = 1, 12$.

Thus the complementary function is $y_c(x) = c_1e^x + c_2e^{12x}$, where c_1, c_2 are arbitrary constants.

The particular integral is

$$y_p(x) = (D^2 - 13D + 12)^{-1}(3e^{-2x}) = 3\frac{1}{(-2)^2 - 13(-2) + 12}e^{-2x} = \frac{1}{14}e^{-2x}.$$

Hence the general solution of the given differential equation is $y = c_1e^x + c_2e^{12x} + \frac{1}{14}e^{-2x}$, where c_1 and c_2 are two arbitrary constants.

Example 11.16: Find the general solution of the differential equation

$$(D^4 + 5D^3 + 6D^2 - 4D - 8)y = e^{-2x} + 2e^{-x} + 3e^x - 3$$

Solution: The corresponding homogeneous equation is

$$(D^4 + 5D^3 + 6D^2 - 4D - 8)y = 0 \quad \text{...(11.44)}$$

To find the complementary function, consider the characteristic equation corresponding to (11.44), which is $\lambda^4 + 5\lambda^3 + 6\lambda^2 - 4\lambda - 8 = 0$. It has roots $\lambda = -2, -2, -2, 1$. Thus the complementary function $y_c(x)$ is $y_c(x) = (c_1 + c_2x + c_3x^2)e^{-2x} + c_4e^x$, where c_1, c_2, c_3, c_4 are arbitrary constants.
The particular integral is

$$\begin{aligned} y_p(x) &= [(D+2)^3(D-1)]^{-1}(e^{-2x} + 2e^{-x} + 3e^x - 3) \\ &= [(D+2)^{-3}(D-1)^{-1}]e^{-2x} + 2[(D+2)^{-3}(D-1)^{-1}]e^{-x} \\ &\quad + 3[(D+2)^{-3}(D-1)^{-1}]e^x - 3[(D+2)^{-3}(D-1)^{-1}]e^0 \\ &= -\frac{1}{3}(D+2)^{-3}e^{-2x} + 2\left(-\frac{1}{2}\right)e^{-x} + 3\,\frac{1}{27}(D-1)^{-1}e^x - 3\left(-\frac{1}{8}\right) \\ &= -\frac{1}{3}\,\frac{x^3}{3!}e^{-2x} - e^{-x} + \frac{1}{9}xe^x + \frac{3}{8} = -\frac{x^3e^{-2x}}{18} - e^{-x} + \frac{1}{9}xe^x + \frac{3}{8}. \end{aligned}$$

Hence the general solution is

$$y(x) = (c_1 + c_2x + c_3x^2)e^{-2x} + c_4e^x - \frac{x^3}{18}e^{-2x} - e^{-x} + \frac{1}{9}xe^x + \frac{3}{8}$$

or,
$$y(x) = \left(c_1 + c_2x + c_3x^2 - \frac{x^3}{18}\right)e^{-2x} - e^{-x} + \left(c_4 + \frac{1}{9}x\right)e^x + \frac{3}{8}.$$

Case II: *$f(x) = \sin(ax+b)$, or $\cos(ax+b)$ where a and b are constants.*

When $f(x) = \sin(ax+b)$, we have

$D\sin(ax+b) = a\cos(ax+b)$, $D^2\sin(ax+b) = (-a^2)\sin(ax+b)$

$D^3\sin(ax+b) = -a^3\cos(ax+b)$, $D^4\sin(ax+b) = (-a^2)^2\sin(ax+b)$

Thus, for even n, say $n = 2m$, $D^n\sin(ax+b) = (-a^2)^m\sin(ax+b)$.

Hence, in case $F(D)$ contains only even powers of D,

$$F(D^2)\sin(ax+b) = F(-a^2)\sin(ax+b).$$

Operating both sides by $[F(D^2)]^{-1}$ and dividing by $F(-a^2)$, the particular integral is

$$y_p(x) = [F(D^2)]^{-1}\sin(ax+b) = \frac{1}{F(-a^2)}\sin(ax+b), \quad \text{...(11.45)}$$

provided $F(-a^2) \neq 0$.

Similarly for the case $f(x) = \cos(ax+b)$, the particular integral is

$$[F(D^2)]^{-1}\cos(ax+b) = \frac{1}{F(-a^2)}\cos(ax+b), \quad \text{...(11.46)}$$

provided $F(-a^2) \neq 0$.

When $F(D)$ contains odd powers of D also, to find the particular integral, we proceed tentatively on similar lines as described above, however, the exact procedure will be illustrated in the examples to follow.

In case $F(-a^2) = 0$, we write $\cos(ax+b) = Re(e^{i(ax+b)})$ and $\sin(ax+b) = Im.(e^{i(ax+b)})$ and apply the formulae for the *Case I, when* $f(x) = e^{ax}$, as follows,

$$y_p(x) = \frac{1}{F(D)}\cos(ax+b) = Re.\frac{1}{F(D)}e^{i(ax+b)} = Re.\frac{1}{F(ia)}e^{i(ax+b)} \quad \text{...(11.47)}$$

provided $F(ia) \neq 0$, and so on. Similarly,

$$\frac{1}{F(D)}\sin(ax+b) = Im.\frac{1}{F(D)}e^{i(ax+b)} = Im.\frac{1}{F(ia)}e^{i(ax+b)}, \quad \text{...(11.48)}$$

provided $F(ia) \neq 0$, and so on.

Example 11.17: Find the particular integral of the equation $(D^2+1)y = \cos(2x-1)$.

Solution: The particular integral is

$$y_p(x) = (D^2+1)^{-1}\cos(2x-1) = \frac{1}{-4+1}\cos(2x-1) = -\frac{1}{3}\cos(2x-1).$$

Example 11.18: Find the general solution of the differential equation $(D^3 + D^2 - D - 1)y = \sin(2x-3)$.

Solution: Characteristic equation of the corresponding homogeneous equation is $\lambda^3+\lambda^2-\lambda-1=0$. Its roots are $\lambda = -1, -1, 1$. Thus the complementary function is $y_c(x) = (c_1+c_2x)e^{-x} + c_3e^x$.

The particular integral is

$$y_p(x) = \frac{1}{D^3+D^2-D-1}\sin(2x-3)$$

$$= \frac{1}{-2^2D-2^2-D-1}\sin(2x-3), \text{ (replacing } D^2 = -2^2)$$

$$= -\frac{1}{5(D+1)}\sin(2x-3) = \frac{-(D-1)}{5(D^2-1)}\sin(2x-3)$$

$$= \frac{1}{25}(D-1)\sin(2x-3), \text{ (replacing } D^2 = -2^2)$$

$$= \frac{1}{25}[2\cos(2x-3) - \sin(2x-3)].$$

Therefore the general solution is

$$y(x) = (c_1 + c_2 x)e^{-x} + c_3 e^{x} + \frac{2}{25}\cos(2x-3) - \frac{1}{25}\sin(2x-3)$$

where c_1, c_2, c_3 are arbitrary constants.

Example 11.19: Find the particular integral of the equation $(D^2+4)y = \sin 2x$.

Solution: The particular integral is, $y_p(x) = \dfrac{1}{D^2+4}\sin 2x$.

Here, $F(-a)^2 = -a^2 + 4 = 0$ at $a = 2$, therefore, writing $y_p(x)$ as

$$y_p(x) = Im\, \frac{1}{D^2+4}e^{i2x} = Im.\, \frac{x}{2D}e^{2ix} = Im\, \frac{x}{4i}e^{2ix}$$

$$= Im\, \frac{x}{4i}(\cos 2x + i\sin 2x) = -Im.\, \frac{x}{4}(i\cos 2x - \sin 2x) = -\frac{x}{4}\cos 2x.$$

Alternatively, $$y_p(x) = \frac{1}{D^2+4}\sin 2x,$$

Here, $F(-a^2) = 0$ proceeding on similar lines as in Case I when $f(x) = e^{ax}$ and $F(a) = 0$, we have

$$y_p(x) = \frac{x}{2D}\sin 2x = \frac{x}{2}\int \sin 2x\,dx = -\frac{x}{4}\cos 2x.$$

Example 11.20: Solve the differential equation $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

Solution: Characteristic equation of the corresponding homogeneous equation is $\lambda^2 - 4\lambda + 3 = 0$. Its roots are $\lambda = 1, 3$. The complementary function is thus $y_c(x) = c_1 e^{x} + c_2 e^{3x}$, where c_1, c_2 are arbitrary constants.

The particular integral is

$$y_p(x) = \frac{1}{D^2-4D+3}(\sin 3x\cos 2x) = \frac{1}{2}\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{2}\frac{1}{D^2-4D+3}\sin x$$

$$= \frac{1}{2}\frac{1}{-25-4D+3}\sin 5x + \frac{1}{2}\frac{1}{-1-4D+3}\sin x$$

$$= -\frac{1}{4}\frac{11-2D}{121-4D^2}\sin 5x + \frac{1}{4}\frac{1+2D}{1-4D^2}\sin x = -\frac{1}{884}(11-2D)\sin 5x + \frac{1}{20}(1+2D)\sin x$$

$$= \frac{1}{884}(10\cos 5x - 11\sin x) + \frac{1}{20}(\sin x + 2\cos x).$$

Hence the general solution is

$$y_c(x) = c_1 e^{x} + c_2 e^{3x} + \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x).$$

Example 11.21: Solve the differential equation $(D^2+2D+1)y=\cosh x-\cos^2 x$.

Solution: The complementary function is $y_c(x)=(c_1+c_2x)e^{-x}$. The particular integral is

$$y_p(x)=\frac{1}{(D^2+2D+1)}[\cosh x-\cos^2 x]$$

$$=\frac{1}{2}\frac{1}{(D^2+2D+1)}(e^x+e^{-x})-\frac{1}{2}\frac{1}{(D^2+2D+1)}(1+\cos 2x)$$

$$=\frac{1}{2(D^2+2D+1)}e^x+\frac{1}{2(D^2+2D+1)}e^{-x}-\frac{1}{2(D^2+2D+1)}e^0-\frac{1}{2(D^2+2D+1)}\cos 2x$$

$$=\frac{e^x}{8}+\frac{x^2e^{-x}}{4}-\frac{1}{2}-\frac{1}{2(2D-3)}\cos 2x=\frac{e^x}{8}+\frac{x^2e^{-x}}{4}-\frac{1}{2}-\frac{(2D+3)}{2(4D^2-9)}\cos 2x$$

$$=\frac{e^x}{8}+\frac{x^2e^{-x}}{4}-\frac{1}{2}+\frac{1}{50}(2D+3)\cos 2x=\frac{e^x}{8}+\frac{x^2e^{-x}}{4}-\frac{4}{50}\sin 2x+\frac{3}{50}\cos 2x-\frac{1}{2}.$$

The general solution is

$$y(x)=(c_1+c_2x)e^{-x}+\frac{e^x}{8}+\frac{x^2e^{-x}}{4}-\frac{4}{50}\sin 2x+\frac{3}{50}\cos 2x-\frac{1}{2}.$$

Case III: $f(x)=x^m$, *$m>0$ is an integer*. When $f(x)=x^m$, the particular integral is

$$y_p(x)=[F(D)]^{-1}x^m. \qquad ...(11.49)$$

Symbolically, we expand the operator $[F(D)]^{-1}$ as an infinite series in ascending powers of D and operate on x^m. We need not to consider terms with power $m+1$ and higher, since $(m+1)$th and higher order derivatives of x^m are zeros.

Example 11.22: Find the particular integral of the differential equation

$$(D^2+2D+1)y=2x+x^3.$$

Solution: The particular integral is

$$y_p(x)=\frac{1}{D^2+2D+1}(2x+x^3)=(1+D)^{-2}(2x+x^3)$$

$$=(1-2D+3D^2-4D^3+...)(2x+x^3)$$

$$=2x+x^3-2(2+3x^3)+3(6x)-24=x^3-6x^2+20x-28.$$

Case IV: $f(x)=e^{ax}g(x)$, *$g(x)$ being some function of x.*

In case $f(x)=e^{ax}g(x)$,

$$Df(x)=D[e^{ax}g(x)]=e^{ax}Dg(x)+ae^{ax}g(x)=e^{ax}(D+a)g(x)$$

$$D^2f(x)=D^2[e^{ax}g(x)]=D[e^{ax}Dg(x)+ae^{ax}g(x)]$$

$$=e^{ax}D^2g(x)+2ae^{ax}Dg(x)+a^2e^{ax}g(x)=e^{ax}(D+a)^2g(x).$$

In general, $D^n(e^{ax}g(x))=e^{ax}(D+a)^ng(x)$.

Therefore, $F(D)e^{ax}g(x) = e^{ax}F(D+a)g(x)$ and hence the particular integral is

$$y_p(x) = [F(D)]^{-1}e^{ax}g(x) = e^{ax}[F(D+a)]^{-1}g(x) \qquad \text{...(11.50)}$$

and $[F(D+a)]^{-1}g(x)$ can be evaluated for a specific value of $g(x)$.

Example 11.23: Solve the differential equation $(D^2-4)y = x\sinh x$.

Solution: The complementary function is $y_c(x) = c_1e^{2x} + c_2e^{-2x}$. The particular integral is

$$y_p(x) = \frac{1}{D^2-4}\, x\sinh x = \frac{1}{2}\frac{1}{D^2-4}x(e^x - e^{-x})$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}xe^x - \frac{1}{D^2-4}xe^{-x}\right] = \frac{1}{2}\left[e^x\frac{1}{(D+1)^2-4}x - e^{-x}\frac{1}{(D-1)^2-4}\right]$$

$$= \frac{1}{2}\left[e^x\frac{1}{D^2+2D-3}x - e^{-x}\frac{1}{D^2-2D-3}x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1}x - \frac{e^{-x}}{-3}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1}x\right]$$

$$= -\frac{1}{6}\left[e^x\left\{1+\left(\frac{2D}{3}+\frac{D^2}{3}\right)+\dots\right\}x - e^{-x}\left\{1-\left(\frac{2D}{3}-\frac{D^2}{3}\right)+\dots\right\}x\right]$$

$$= -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right) - e^{-x}\left(x+\frac{2}{3}\right)\right] = -\frac{x}{3}\left(\frac{e^x-e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x.$$

Hence the general solution is $y(x) = c_1e^{2x} + c_2e^{-2x} - \dfrac{x}{3}\sinh x - \dfrac{2}{9}\cosh x$.

Example 11.24: Solve the differential equation $(D^2+4)y = x\sin x$.

Solution: The complementary function is $y_c(x) = c_1\cos 2x + c_2\sin 2x$. Particular integral is

$$y_p(x) = \frac{1}{D^2+4}x\sin x = Im\frac{1}{D^2+4}xe^{ix} = Im\, e^{ix}\frac{1}{(D+i)^2+4}x = Im\, e^{ix}\frac{1}{D^2+2iD+3}x$$

$$= Im\frac{e^{ix}}{3}\left[1+\frac{2iD+D^2}{3}\right]^{-1}x = Im\frac{e^{ix}}{3}\left[1-\frac{2iD+D^2}{3}+\dots\right]x = Im\frac{e^{ix}}{3}\left[x-\frac{2i}{3}\right]$$

$$= \frac{1}{3}Im(\cos x + i\sin x)\left(x-\frac{2i}{3}\right) = \frac{x}{3}\sin x - \frac{2}{9}\cos x.$$

Hence, the general solution is $y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{3}\sin x - \frac{2}{9}\cos x$.

So far have discussed the procedure to find particular integral using operator method for some specific forms of $f(x)$. In general, if $f(x)$ is any function of x, then we can proceed to find the particular integral as follow. By definition $y_p(x) = \frac{1}{F(D)} f(x)$.

If $F(D) = a_0(D - m_1)(D - m_2) \ldots (D - m_n)$, resolving $\frac{1}{F(D)}$ into partial fractions, we have

$$\frac{1}{F(D)} = \frac{c_1}{D - m_1} + \frac{c_2}{D - m_2} + \ldots \frac{c_n}{D - m_n} = \sum_{i=1}^{n} \frac{c_i}{D - m_i}.$$

Thus,
$$y_p(x) = \frac{1}{F(D)} f(x) = \sum_{i=1}^{n} \frac{c_i}{D - m_i} f(x)$$

$$= \sum_{i=1}^{n} \left(c_i e^{m_i x} \int f(x) e^{-m_i x} dx \right), \qquad \ldots(11.51)$$

using (11.38). In case of repeated roots the result can be modified accordingly.

Example 11.25: Solve the differential equation $(D^2 + 3D + 2)y = e^{e^x}$.

Solution: The compliementary function is $y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$, where c_1, c_2 are arbitrary constants.

The particular integral is

$$y_p(x) = \frac{1}{(D+1)(D+2)} e^{e^x} = \left(\frac{1}{D+1} - \frac{1}{D+2} \right) e^{e^x}$$

$$= \frac{1}{D+1} e^{e^x} - \frac{1}{D+2} e^{e^x} = e^{-x} \int e^{e^x} e^x dx - e^{-2x} \int e^{e^x} e^{2x} dx. \qquad \ldots(11.52)$$

Substitute $e^x = t$, (11.52) gives

$$y_p(x) = \frac{1}{t} \int e^t dt - \frac{1}{t^2} \int t e^t dt = \frac{e^t}{t} - \frac{1}{t^2}(te^t - e^t) = \frac{e^t}{t} - \frac{e^t}{t} - \frac{e^t}{t^2} = \frac{e^t}{t^2} = \frac{e^{e^x}}{e^{2x}} = e^{(e^x - 2x)}.$$

Hence, the general solution is $y(x) = c_1 e^{-x} + c_2 e^{-2x} + e^{(e^x - 2x)}$.

Example 11.26: Solve the differential equation $(D^2 + a^2)\, y = \tan ax$.

Solution: The complementary function is $y_c(x) = c_1 \cos ax + c_2 \sin ax$. The particular integral is

$$y_p(x) = \frac{1}{D^2+a^2}\tan ax = \frac{1}{(D+ai)(D-ai)}\tan ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right]\tan ax$$

$$= \frac{1}{2ai}\frac{1}{D-ai}\tan ax - \frac{1}{2ai}\frac{1}{D+ai}\tan ax. \qquad ...(11.53)$$

We have, $\frac{1}{D-ai}\tan ax = e^{aix}\int e^{-aix}\tan ax\,dx$ using (11.38)

$$= e^{aix}\int [\cos ax - i\sin ax]\tan ax\,dx = e^{aix}\int\left[\sin ax - i\frac{(1-\cos^2 ax)}{\cos ax}\right]dx$$

$$= e^{aix}\int (\sin ax + i\cos ax - i\sec ax)dx$$

$$= e^{aix}\left[-\frac{\cos ax}{a} + \frac{i\sin ax}{a} - \frac{i}{a}\ln|\sec ax + \tan ax|\right] \qquad ...(11.54)$$

Replacing i by $-i$ in (11.54), we have

$$\frac{1}{D+ai}\tan ax = e^{-aix}\left[-\frac{1}{a}\cos ax - \frac{i}{a}\sin ax + \frac{i}{a}\ln|\sec ax + \tan ax|\right]. \qquad ...(11.55)$$

Using (11.54) and (11.55) in (11.53), we have

$$y_p(x) = \frac{1}{2a^2 i}\left[-(e^{aix} - e^{-aix})\cos ax + i(e^{aix} + e^{-aix})\sin ax - i(e^{aix} + e^{-aix})\ln|\sec ax + \tan ax|\right]$$

$$= \frac{1}{2a^2 i}\left[-2i\cos ax\sin ax + 2i\sin ax\cos ax - 2i\cos ax\ln|\sec ax + \tan ax|\right]$$

$$= -\frac{1}{a^2}\cos ax\ln|\sec ax + \tan ax|.$$

Hence the general solution is $y(x) = c_1\cos ax + c_2\sin ax - \frac{1}{a^2}\cos ax\ln|\sec ax + \tan ax|$.

EXERCISE 11.3

Find the general solution of the following equations by the operator method

1. $(2D^2 + 5D - 3)y = 6$
2. $(D^2 - D - 6)y = 2e^x$
3. $(2D^2 - D - 3)y = 5e^{3/2x}$
4. $(D^3 + 5D)y = \sinh 2x$

5. $(D^2-4)y=x^2e^{3x}$
6. $(D^2-2D+1)y=xe^x\cos x$
7. $(D^2+D)y=(1+e^x)^{-1}$
8. $(D^2+D)y=x^2+2x+4$
9. $(D^2+a^2)y=\sec ax$
10. $(D^4-1)y=\cos x\cosh x$
11. $(D^2-4)y=\cosh(2x-1)+3^x$
12. $(D^2+5D+6)y=e^{-2x}\sec^2 x(1+2\tan x)$

Solve the following equations

13. $(D^2+n^2)y=k\sin px$; k, n and p are constants, $n^2\neq p^2$; $y(0)=y'(0)=0$.
14. $(D^2-7D+10)y=e^{2x}+20$; $y(0)=0$, $y'(0)=-1/3$.
15. $(2D^2-D-6)\,y=6e^x\cos x$; $y(0)=-21/29$, $y'(0)=-194/29$
16. $(D^2+D)=2+2x+x^2$; $y(0)=8$, $y'(0)=-1$.

11.6 EQUATIONS REDUCIBLE TO LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

In this section we shall study two such forms of linear differential equations with variable coefficients which can be reduced to linear differential equations with constant coefficients by suitable substitutions.

11.6.1 Cauchy's Homogeneous Linear Equation

An equation of the form

$$a_0x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}x\frac{dy}{dx}+a_ny=f(x), \qquad \text{...(11.56)}$$

where $a_0, a_1, \dots, a_n$, $a_0\neq 0$ are constants and $f(x)$ is a function of x only, is called Cauchy's homogeneous linear equation. This can be reduced to linear differential equation with constant coefficients if we substitute $x=e^t$, or $t=\ln x$. Then, if $D=\dfrac{d}{dt}$, we have

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\cdot\frac{1}{x}, \text{ which gives, } x\frac{dy}{dx}=\frac{dy}{dt}=Dy.$$

Next, $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d}{dx}\left(\dfrac{1}{x}\dfrac{dy}{dt}\right)=-\dfrac{1}{x^2}\dfrac{dy}{dt}+\dfrac{1}{x^2}\dfrac{d^2y}{dt^2}$, which gives, $x^2\dfrac{d^2y}{dx^2}=D(D-1)y$.

Similarly, $x^3\dfrac{d^3y}{dx^3}=D(D-1)(D-2)y$, and so on.

Substituting these in (11.56) and simplifying, we get a linear equation with constant coefficients. Solving the equation obtained, we get the solution in terms of t. Substituting $t=\ln x$, we obtain the solution of the given equation.

11.6.2 Legendre's Homogeneous Linear Equation

An equation of the form

$$a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1}\frac{d^n y}{dx^n} + \ldots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = f(x), \quad \ldots(11.57)$$

where $a_0, a_1, \ldots, a_n, a_0 \neq 0$ are constants and $f(x)$ is a function of x only, is called Legendre's homogeneous linear equation. It reduces to linear differential equation with constant coefficient if we substitute $ax + b = e^t$, or $t = \ln(ax+b)$. Proceeding on the similar lines as above, we obtain

$$(ax+b)\frac{dy}{dx} = aDy, \quad (ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y,$$

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y, \text{ and so on; here } D = \frac{d}{dt}.$$

Substituting these in Eq. (11.57), the resultant equation is linear with constant coefficients and thus can be solved accordingly.

Example 11.27: Solve the differential equation $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x\ln x.$

Solution: Substituting $x = e^t$, the equation becomes

$$[D(D-1)(D-2) + 3D(D-1) + D + 1]y = te^t, \text{ where } D = \frac{d}{dt}.$$

Simplifying it gives, $(D^3+1)y = te^t$. The characteristic equation is $\lambda^3 + 1 = 0$, which has the roots $\lambda = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}$ and, therefore, the complimentary function is

$$y_c(t) = c_1e^{-t} + e^{t/2}\left(c_2\cos\frac{\sqrt{3}}{2}t + c_3\sin\frac{\sqrt{3}}{2}t\right).$$

The particular integral is

$$y_p(t) = \frac{1}{(D^3+1)}te^t = e^t\frac{1}{(D+1)^3+1}t = e^t\frac{1}{D^3+3D^2+3D+2}t$$

$$= \frac{e^t}{2}\left[1+\frac{3}{2}D+\frac{3}{2}D^2+\frac{1}{2}D^3\right]^{-1}t = \frac{e^t}{2}\left[1-\left(\frac{3}{2}D+\frac{3}{2}D^2+\frac{1}{2}D^3\right)+\ldots\right]t = \frac{e^t}{2}\left(t-\frac{3}{2}\right).$$

Therefore, the complete solution is

$$y(t) = c_1e^{-t} + e^{t/2}\left(c_2\cos\frac{\sqrt{3}}{2}t + c_3\sin\frac{\sqrt{3}}{2}t\right) + \frac{1}{2}e^t\left(t-\frac{3}{2}\right).$$

Substituting $t = \ln x$ gives

$$y = \frac{c_1}{x} + \sqrt{x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right] + \frac{x}{2}\left[\ln x - \frac{3}{2}\right]$$

as the general solution for the given equation.

Example 11.28: Solve the differential equation $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \ln x.\left[\sin(\ln x) + \frac{1}{x}\right]$

Solution: Substituting $x = e^t$, the equation becomes

$$[D(D-1) - 3D + 1]y = t\sin t + te^{-t}, \text{ where } D = \frac{d}{dt}.$$

Simplifying it gives, $(D^2 - 4D + 1)y = te^{-t} + t\sin t$. The characteristic equation is $\lambda^2 - 4\lambda + 1 = 0$ which has roots $\lambda = 2 \pm \sqrt{3}$. Thus, the complementary function is $y_c(t) = (c_1e^{\sqrt{3}t} + c_2e^{-\sqrt{3}t})e^{2t}$. The particular integer is

$$y_p(t) = \frac{1}{D^2 - 4D + 1}te^{-t} + \frac{1}{(D^2 - 4D + 1)}t\sin t$$

$$= e^{-t}\frac{1}{(D-1)^2 - 4(D-1) + 1}t + Im.\frac{1}{D^2 - 4D + 1}te^{it}$$

$$= e^{-t}\frac{1}{D^2 - 6D + 6}t + Im.\,e^{it}\frac{1}{(D+i)^2 - 4(D+i) + 1}t$$

$$= \frac{e^{-t}}{6}\left\{1 - \left(D - \frac{D^2}{6}\right)\right\}^{-1}t + Im.e^{it}\frac{1}{D^2 + 2(i-2)D - 4i}t$$

$$= \frac{e^{-t}}{6}\left\{1 + \left(D - \frac{D^2}{6}\right) + \dots\right\}t + Im.e^{it}\left[-4i\left\{1 - \frac{2(1+2i)D - iD^2}{4}\right\}^{-1}t\right]$$

$$= \frac{(t+1)e^{-t}}{6} + Im.e^{it}\left[-4i\left\{1 + \frac{2(1+2i)D - iD^2}{4} + \dots\right\}t\right]$$

$$= \frac{(t+1)e^{-t}}{6} + Im.(\cos t + \sin t)\left[-4i\left\{t + \frac{1}{2}(1+2i)\right\}\right]$$

$$= \frac{(t+1)e^{-t}}{6} - 2(2t\cos t + \cos t - 2\sin t).$$

Therefore, the general solution is

$$y(t) = (c_1\, e^{\sqrt{3}t} + c_2\, e^{-\sqrt{3}t})\, e^{2t} + \frac{(t+1)}{6} e^{-t} - 2(2t \cos t + \cos t - 2 \sin t).$$

Substituting $t = \ln x$, we obtain

$$y = \left(c_1 x^{\sqrt{3}} + \frac{c_2}{x^{\sqrt{3}}}\right) x^2 + \frac{(\ln x + 1)}{6x} - 2(2 \ln x \cos (\ln x) + \cos (\ln x) - 2 \sin (\ln x)]$$

$$= c_1 x^{2+\sqrt{3}} + c_2 x^{2-\sqrt{3}} + \frac{(1 + \ln x)}{6x} - 2[(2 \ln x + 1) \cos (\ln x) - 2 \sin (\ln x)],$$

as the general solution of the given differential equation.

Example 11.29: Solve the differential equation $(2x+3)^2 \frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x.$

Solution: Substituting $2x + 3 = e^t$, the equation becomes

$$[4D(D-1) - 2D - 12]y = 3(e^t - 3), \quad D = \frac{d}{dt}.$$

Simplifying it gives, $2(2D^2 - 3D - 6)y = 3e^t - 9$. The characteristic equation is, $2\lambda^2 - 3\lambda - 6 = 0$, with roots $\lambda = \frac{3}{4} \pm \frac{\sqrt{57}}{4}$. Thus the complementary function is $y_c(t) = \left(c_1 e^{\frac{\sqrt{57}}{4}t} + c_2 e^{-\frac{\sqrt{57}}{4}t}\right)(e^{3t/4})$. The particular integral is

$$y_p(t) = \frac{1}{2(2D^2 - 3D - 6)}(3e^t - 9) = \frac{3}{2}\left[\frac{1}{2D^2 - 3D - 6} e^t - 3 \frac{1}{2D^2 - 3D - 6} e^0\right]$$

$$= \frac{3}{2}\left[\frac{e^t}{2-3-6} - 3.\frac{1}{-6}\right] = \frac{3}{2}\left[-\frac{1}{7}e^t + \frac{1}{2}\right] = -\frac{3}{14}e^t + \frac{3}{4}.$$

Hence the general solution is

$$y(t) = \left(c_1 e^{\frac{\sqrt{57}}{4}t} + c_2 e^{-\frac{\sqrt{57}}{4}t}\right) e^{3t/4} - \frac{3}{14}e^t + \frac{3}{4}.$$

Substituting $t = \ln(2x+3)$, we obtain

$$y = \left[c_1(2x+3)^{\sqrt{57}/4} + c_2(2x+3)^{-\sqrt{57}/4}\right](2x+3)^{3/2} - \frac{3}{14} + (2x+3) + \frac{3}{4},$$

as the general solution for the given differential equation.

EXERCISE 11.4

Solve the differential equations:

1. $(x^2D^2 - xD + 1)y = \ln x$
2. $(x^2D^2 - xD - 3)y = x^2(\ln x)^2$
3. $\left(xD^2 - \frac{2}{x}\right)y = x + \frac{1}{x^2}$
4. $(x^2D^2 + 3xD + 1)y = \frac{1}{(1-x)^2}$
5. $(x^3D^3 + 3x^2D^2 + xD + 8)y = 65 \cos(\ln x)$
6. $(x^4D^4 + 2x^3D^3 + x^2D^2 - xD + 1)y = \ln x$
7. $[(1+x)^2D^2 + (1+x)D + 1]y = 4 \cos \ln(1+x)$
8. $[(3x+2)^2D^2 + 3(3x+2)D - 36]y = 3x^2 + 4x + 1$
9. $(4x^2D^2 + 1)y = \ln x,\ x > 0;\ y(1) = 0,\ y(e) = 5$
10. $(x^2D^2 + 3xD + 10)y = 9x^2;\ y(1) = 5/2,\ y'(1) = 8$
11. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u + kr^3 = 0$, where k is a constant. Solve the equation under the conditions, $u(0) = u(a) = 0$.

11.7 METHOD OF VARIATION OF PARAMETERS. METHOD OF UNDETERMINED COEFFICIENTS

So far we have discussed the operator method for finding the particular integral which is easily applicable in some specific forms of $f(x)$. In this section we discuss two general methods for finding the particular integral of a non-homogeneous differential equation, whenever the complementary function is known. The methods are:

I. *Method of variation of parameters*

II. *Method of undetermined coefficients.*

First method is applicable to both constant coefficients and variable-coefficients non-homogeneous differential equations, while the second is applicable only to constant coefficients one.

11.7.1 Method of Variation of Parameters

Consider the non-homogeneous differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x),\ a_0(x) \neq 0. \qquad ...(11.58)$$

If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \qquad ...(11.59)$$

then the complementary function is

$$y(x) = c_1y_1(x) + c_2y_2(x), \qquad ...(11.60)$$

where c_1 and c_2 are two arbitrary constants.

The method of variation of parameters consists of finding a particular solution of the non-homogeneous equation by replacing the constants $c_1$and c_2 in (11.60) with functions of x, that is, we find functions $u(x)$ and $v(x)$ such that

$$y_p(x) = u(x)\, y_1(x) + v(x) y_2(x) \qquad \text{...(11.61)}$$

is a particular solution of Eq. (11.58). To determine $u(x)$ and $v(x)$ we need two equations. These are obtained as follows.

We compute

$$y'_p = uy'_1 + vy'_2 + u'y_1 + v'y_2 \qquad \text{...(11.62)}$$

To simplify this expression we find u and v such that

$$u'y_1 + v'y_2 = 0. \qquad \text{...(11.63)}$$

Thus (11.62) becomes

$$y'_p = uy'_1 + vy'_2 \qquad \text{...(11.64)}$$

Next we compute

$$y''_p = uy''_1 + vy''_2 + u'y'_1 + v'y'_2 \qquad \text{...(11.65)}$$

Substituting expressions for y_p, y'_p and y''_p from (11.61), (11.64) and (11.65) in Eq. (11.58), we obtain

$$a_0(x)\, [uy''_1 + vy''_2 + u'y'_1 + v'y'_2] + a_1(x)\, [uy'_1 + vy'_2] + a_2(x)\, [uy_1 + vy_2] = f(x)$$

Rearranging the terms to obtain

$$u[a_0(x)y''_1 + a_1(x)y'_1 + a_2(x)y_1] + v[a_0(x)y''_2 + a_1(x)\, y'_2 + a_2(x)y_2] + a_0(x)\, [u'y'_1 + v'y'_2] = f(x) \qquad \text{...(11.66)}$$

Using the fact that y_1 and y_2 are two solutions of the homogeneous Eq. (11.59), the Eq. (11.66) reduces to

$$a_0(x)\, (u'y'_1 + v'y'_2) = f(x)$$

or,

$$u'y'_1 + v'y'_2 = f(x)/a_0(x) = g(x), \text{ say.} \qquad \text{...(11.67)}$$

Solving Eqs. (11.63) and (11.67) for u' and v' we obtain

$$u'(x) = \frac{W_1(x)}{W(x)} \text{ and } v'(x) = \frac{W_2(x)}{W(x)}, \qquad \text{...(11.68)}$$

where $W(x) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \neq 0$; $W_1(x) = \begin{vmatrix} 0 & y_2 \\ g(x) & y'_2 \end{vmatrix}$, and $W_2(x) = \begin{vmatrix} y_1 & 0 \\ y'_1 & g(x) \end{vmatrix}$.

Here $W(x)$, the Wronskian of y_1, y_2 is non-zero, since y_1 and y_2 are two linearly independent solutions of the homogeneous Eq. (11.59).

Integrating (11.68), we obtain u and v to get y_p. The arbitrary constants of integrations, are taken zeros since our interest is to find y_p only. In fact, if we consider the constants of integration to be non-zeros, then $y_p(x)$ gives the general solution of the non-homogeneous differential equation. The method of variation of parameter is applicable to equations of higher order also.

For example, consider the third order equation

$$a_0(x)y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = f(x),\ a_0(x) \neq 0,$$

If y_1, y_2, y_3 are three linearly independent solutions of the associated homogeneous equation, then the complementary function is $y_c(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x)$, where c_1, c_2, c_3 are three arbitrary constants. Consider

$$y_p(x) = u(x)y_1(x) + v(x)\,y_2(x) + w(x)y_3(x)$$

as the particular integral and proceed on the similar lines as above. The required conditions to determine $u(x)$, $v(x)$ and $w(x)$ are:

$$\left.\begin{aligned} u'y_1 + v'y_2 + w'y_3 &= 0 \\ u'y_1' + v'y_2' + w'y_3' &= 0 \\ u'y_1'' + v'y_2'' + w'y_3'' &= \frac{f(x)}{a_0(x)} = g(x) \end{aligned}\right\} \qquad \text{...(11.69)}$$

If $\quad W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0$, is the Wronskian of y_1, y_2, y_3 and

$$W_1(x) = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ g(x) & y_2'' & y_3'' \end{vmatrix},\quad W_2(x) = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & g(x) & y_3'' \end{vmatrix},\quad W_3(x) = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & g(x) \end{vmatrix},$$

then, $\quad u'(x) = \dfrac{W_1(x)}{W(x)},\quad v'(x) = \dfrac{W_2(x)}{W(x)}$, and $\quad w'(x) = \dfrac{W_3(x)}{W(x)}$. ...(11.70)

Hence, the particular integral is

$$y_p(x) = \left(\int \frac{W_1(x)}{W(x)}dx\right)y_1 + \left(\int \frac{W_2(x)}{W(x)}dx\right)y_2 + \left(\int \frac{W_3(x)}{W(x)}dx\right)y_3. \qquad \text{...(11.71)}$$

The result can be generalized to the equation of the nth order.

Example 11.30: Solve by the method of variation of parameters the differential equation $(D^2 + 4)y = \sec x$.

Solution: The characteristic equation of the associated homogeneous equation is

$$\lambda^2 + 4 = 0, \text{ with roots } \lambda = \pm 2i.$$

Thus, two independent solutions are $y_1(x) = \cos 2x$ and $y_2 = \sin 2x$.

The Wronskian of y_1 and y_2 is $W(x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2(\cos^2 2x + \sin^2 2x) = 2.$

Also, $\quad W_1(x) = \begin{vmatrix} 0 & \sin 2x \\ \sec x & 2\cos 2x \end{vmatrix} = -\sin 2x \sec x = -2\sin x,$

and, $W_2(x) = \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sec x \end{vmatrix} = \cos 2x \sec x.$

Hence, $u'(x) = W_1(x)/W(x) = -\sin x$, which gives $u(x) = \cos x$.

Similarly, $v'(x) = \dfrac{w_2(x)}{w(x)} = \dfrac{1}{2}\cos 2x \sec x = \cos x - \dfrac{1}{2}\sec x$, which gives

$$v(x) = \sin x - \frac{1}{2}\ln|\sec x + \tan x|.$$

Thus the particular integral is

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) = \cos x \cos 2x + (\sin x - \frac{1}{2}\ln|\sec x + \tan x|)\sin 2x.$$

Therefore, the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \cos x \cos 2x + (\sin x - \frac{1}{2}\ln|\sec x + \tan x|)\sin 2x.$$

Example 11.31: Solve by the method of variation of parameter the differential equation $(x^2D^2 - 3xD + 4)y = \ln x,\ x > 0.$

Solution: It is homogeneous differential equation of the Euler's form, we can see easily that the two linearly independent solutions of the associated homogeneous differential equation are

$$y_1 = x^2, \text{ and } y_2 = x^2 \ln x.$$

The Wronskian of y_1 and y_2 is

$$W(x) = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & 2x\ln x + x \end{vmatrix} = 2x^3\ln x + x^3 - 2x^3\ln x = x^3 \neq 0, \text{ since } x > 0.$$

Also here $g(x) = f(x)/a_0(x) = \ln x/x^2$; and hence

$$W_1(x) = \begin{vmatrix} 0 & x^2\ln x \\ \ln x/x^2 & 2x\ln x + x \end{vmatrix} = -(\ln x)^2, \text{ and } W_2(x) = \begin{vmatrix} x^2 & 0 \\ 2x & \ln x/x^2 \end{vmatrix} = \ln x.$$

Thus, $u'(x) = -(\ln x)^2/x^3$, and $v'(x) = \ln x/x^3$. Integrating, we obtain

$$u(x) = \frac{1}{2}\frac{(\ln x)^2}{x^2} + \frac{1}{2}\frac{\ln x}{x^2} + \frac{1}{4x^2}, \text{ and } v(x) = -\frac{1}{2}\frac{\ln}{x^2} - \frac{1}{4x^2}.$$

Thus the particular integral is

$$\begin{aligned} y_p(x) &= u(x)y_1(x) + v(x)y_2(x) \\ &= \frac{1}{2}\left(\frac{(\ln x)^2}{x^2} + \frac{\ln x}{x^2} + \frac{1}{2x^2}\right)x^2 - \frac{1}{2}\left(\frac{\ln x}{x^2} + \frac{1}{2x^2}\right)x^2\ln x = \frac{1}{4}\ln x + \frac{1}{4}. \end{aligned}$$

Therefore, the general solution is $y(x) = c_1x^2 + c_2x^2 \ln x + \frac{1}{4} \ln x + \frac{1}{4}$.

Example 11.32: It is given that $y_1 = \frac{1}{x}$, $y_2 = x$ and $y_3 = x^2$ are three linearly independent solutions of the associated homogeneous equation $x^3y''' + x^2y'' - 2xy' + 2y = \frac{2}{x}$, $x > 0$. Find a particular integral to this equation using the method of variation of parameters.

Solution: The Wronskian of y_1, y_2, y_3 is $W(x) = \begin{vmatrix} 1/x & x & x^2 \\ -1/x^2 & 1 & 2x \\ 2/x^3 & 0 & 2 \end{vmatrix} = \frac{6}{x}$, after simplification.

We have $\quad g(x) = f(x)/a_0(x) = 2/x^4$.

Thus, $\quad W_1(x) = \begin{vmatrix} 0 & x & x^2 \\ 0 & 1 & 2x \\ 2/x^4 & 0 & 2 \end{vmatrix} = 2/x^2, \quad W_2(x) = \begin{vmatrix} 1/x & 0 & x^2 \\ -1/x^2 & 0 & 2x \\ 2/x^3 & 2/x^4 & 2 \end{vmatrix} = -6/x^4,$

and, $\quad W_3(x) = \begin{vmatrix} 1/x & x & 0 \\ -1/x^2 & 1 & 0 \\ 2/x^3 & 0 & 2/x^4 \end{vmatrix} = 4/x^5$. This gives

$$u'(x) = \frac{W_1}{W} = \frac{1}{3x}, \quad v'(x) = \frac{W_2}{W} = \frac{1}{x^3}, \text{ and } \quad w'(x) = \frac{W_3}{W} = \frac{1}{3x^4}. \text{ Hence,}$$

$$u(x) = \frac{1}{3} \ln x, \; v(x) = 1/2x^2, \text{ and } \quad w(x) = -\frac{2}{9x^3}$$

Thus, particular integral is

$$y_p(x) = u(x)y_1 + v(x)y_2 + w(x)y_3 = \frac{\ln x}{3x} + \frac{1}{2x} - \frac{2}{9x} = \frac{\ln x}{3x} + \frac{5}{18x}.$$

We note that the term $5/18x$ can be dropped from $y_p(x)$, since it will appear in the C.F. because of the solution $y_1(x) = 1/x$. Hence, $y_p(x)$ can be taken as $y_p = \ln x/3x$. The same can be verified by direct substitution in the given non-homogeneous equation.

11.7.2 Method of Undetermined Coefficients

This method of finding particular integral is applicable only to linear differential equations with constant coefficients. When the right hand side $f(x)$ is of a special form, say containing polynomials, exponentials, cosine and sine function, sum or product of these functions, then the form of $y_p(x)$ can be guessed. By substituting this in the differential equation, the undetermined constants in $y_p(x)$ are determined.

For example, if $f(x) = x^m$, then its derivatives contain the terms $x^m, x^{m-1}, \ldots, x, 1$ and hence $y_p(x)$ can be chosen as

$$y_p(x) = c_0 x^m + c_1 x^{m-1} + \ldots + c_{m-1} x + c_m,$$

where c_i's are constants.

Similarly, if $f(x) = e^{ax} \cos bx$, or $e^{ax} \sin bx$, then $y_p(x)$ can be chosen as

$$y_p(x) = e^{ax} (c_1 \cos bx + c_2 \sin bx)$$

However, if any term in the choice of the particular integral is also a solution of the associated homogeneous equation, then we multiply this term by x^m, where m is the multiplicity of the root. Also this method is applicable only if the repeated differentiation of $f(x)$, the right hand side of the non-homogeneous equation, produces only a finite number of linearly independent terms.

For example, in case $f(x) = x^2 e^x$. The sequence consisting of this term and successive derivatives is

$$\{x^2e^x, x^2e^x + 2xe^x, x^2e^x + 4xe^x + 2e^x, \ldots\}$$

We observe that sequence consists of only three linearly independent functions x^2e^x, xe^x and e^x. However, in case of $f(x) = \ln x$, the sequence consists of

$$\left\{\ln x, \frac{1}{x}, -\frac{1}{x^2}, +\frac{2}{x^3}, \ldots\right\},$$

an infinite number of linearly independent terms. Thus method of undetermined coefficients cannot be applied in this case. Similarly this method fails in case of $f(x) = \tan x$ or $\sec x$.

Example 11.33: Solve by the method of undetermined coefficients the differential equation $(D^2 - 4)y = 8x^2 - 2x$.

Solution: The characteristic equation is $\lambda^2 - 4 = 0$. Its roots are $\lambda = \pm 2$. Hence, the complementary function is $y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$.

Here $f(x) = x^2 - 2x$; assuming the particular integral as

$$y_p(x) = a_0 x^2 + a_1 x + a_2$$

where a_0, a_1 and a_2 are undetermined constants. It gives

$$y'_p(x) = 2a_0 x + a_1, \text{ and } y_p''(x) = 2a_0.$$

Substituting these in the given equation to obtain

$$2a_0 - 4(a_0 x^2 + a_1 x + a_2) = 8x^2 - 2x$$

Rearranging it as $-4a_0 x^2 - 4a_1 x + (2a_0 - 4a_2) = 8x^2 - 2x$.

Comparing on both sides, we obtain $-4a_0 = 8$, $-4a_1 = -2$, and $2(a_0 - 2a_2) = 0$, which gives $a_0 = -2$, $a_1 = \frac{1}{2}$, $a_2 = -1$. Hence, the particular integral is, $y_p(x) = -2x^2 + \frac{1}{2}x - 1$.

Thus, the general solution is, $y(x) = c_1 e^{2x} + c_2 e^{-2x} - 2x^2 + \frac{1}{2}x - 1$.

Example 11.34: Solve $y''(x) + y(x) = \sin x$ by the method of undetermined coefficient.

Solution: It is easy to see that the complementary function is $y_c(x) = c_1 \cos x + c_2 \sin x$.

Here $f(x) = \sin x$, and as a normal guess the particular integral would have been, $a_1 \cos x + a_2 \sin x$, but since, $\cos x$ and $\sin x$ have already appeared in $y_c(x)$, we choose particular integral as

$$y_p(x) = x(a_1 \cos x + a_2 \sin x),$$

where, a_1, a_2 are undetermined constants. This gives

$$y_p'(x) = (a_1 + a_2x) \cos x + (a_2 - a_1x) \sin x, \text{ and } y_p''(x) = (2a_2 - a_1x) \cos x - (2a_1 + a_2x) \sin x.$$

Substituting these in the given equation we obtain

$$(2a_2 - a_1x) \cos x - (2a_1 + a_2x) \sin x + x(a_1 \cos x + a_2 \sin x) = \sin x.$$

Simplifying it gives, $2a_2 \cos x - 2a_1 \sin x = \sin x$. Comparing coefficients of $\sin x$ and $\cos x$ on both sides, we get $a_1 = (-1/2)$, $a_2 = 0$. Hence, the particular integral is $y_p(x) = (-1/2)x \cos x$.

Thus the general solution is $y(x) = c_1 \cos x + c_2 \sin x - (1/2)x \cos x$.

Example 11.35: Solve, $y''' - 5y'' + 6y' = x^2 + \sin x$, by the method of undetermined coefficients.

Solution: The characteristic equation is $\lambda^3 - 5\lambda + 6\lambda = 0$. It gives $\lambda_1 = 0, \lambda_2 = 2$ and $\lambda_3 = 3$. So the complementary function is $y_c(x) = c_1 + c_2e^{2x} + c_3e^{3x}$, where c_1, c_2 and c_3 are arbitrary constants.

Here $f(x) = x^2 + \sin x$. As a normal guess the form of particular integral corresponding to the term x^2 in $f(x)$ is $a_0x^2 + a_1x + a_2$, but since a constant term is already there in $y_c(x)$, so we modify this to as $a_0x^3 + a_1x^2 + a_2x$. Next, the form corresponding to the term $\sin x$ in $f(x)$ is $a_3 \sin x + a_4 \cos x$. Combining these two, we choose $y_p(x)$ as

$$y_p(x) = a_0x^3 + a_1x^2 + a_2x + a_3\sin x + a_4 \cos x$$

where a_i's are undetermined coefficients to be determined. It gives

$$y_p'(x) = 3a_0x^2 + 2a_1x + a_2 + a_3 \cos x - a_4 \sin x$$

$$y_p''(x) = 6a_0x + 2a_1 - a_3 \sin x - a_4 \cos x, \text{ and } y_p'''(x) = 6a_0 - a_3 \cos x + a_4 \sin x.$$

Substituting these in the given differential equation, we obtain

$$(6a_0 - a_3 \cos x + a_4 \sin x) - 5(6\, a_0x + 2a_1 - a_3 \sin x - a_4 \cos x)$$
$$+ 6(3a_0x^2 + 2a_1x + a_2 + a_3 \cos x - a_4 \sin x) = x^2 + \sin x.$$

Rewriting it as

$$2(3a_0 - 5a_1 + 3a_2) - 6(5a_0 - 2a_1)\, x + 18a_1\, x^2 + 5(a_3 + a_4) \cos x + 5(a_3 - a_4) \sin x = x^2 + \sin x.$$

Comparing the coefficients of like terms on both sides, we obtain

$$3a_0 - 5a_1 + 3a_2 = 0, \quad 5a_0 - 2a_1 = 0, \quad 18a_1 = 1, \quad a_3 + a_4 = 0, \quad \text{and } 5(a_3 - a_4) = 1.$$

Solving for a_0, a_1, a_2, a_3 and a_4, we have $a_0 = 1/45$, $a_1 = 1/18$, $a_2 = 19/270$, $a_3 = 1/10$ and $a_4 = -1/10$.

Thus the particular integral is $y_p(x) = \frac{1}{45}x^3 + \frac{1}{18}x^2 + \frac{19}{270}x + \frac{1}{10}(\sin x - \cos x)$.

Hence the general solution is

$$y(x) = c_1 + c_2e^{2x} + c_3e^{3x} + \frac{1}{45}x^3 + \frac{1}{18}x^2 + \frac{19}{270}x + \frac{1}{10}(\sin x - \cos x).$$

Remark. Though the method of undetermined coefficients is applicable only to linear differential equations with constant coefficients, yet, whenever the differential equation with variable coefficients is reducible to a constant coefficients one, this method can be applied, provided the repeated differentiation of the right hand side of the reduced equation generate only finite number of linearly independent functions.

Example 11.36: Solve $x^2y'' - 5xy' + 8y = 2\ln x$ using the method of undetermined coefficients.

Solution: It is Euler's homogeneous equation. Applying the transformation $x = e^t$, the equation reduces to

$$y''(t) - 6y'(t) + 8y(t) = 2t \qquad \text{...(11.72)}$$

a linear differential equation with constant coefficients.

Its complementary function is $y_c(t) = c_1e^{2t} + c_2e^{4t}$.

Also we can find easily by applying the method of undermined coefficients, that its particular integral as $y_p(t) = t/4 + 3/16$.

Hence the general solution for Eq. (11.72) is $y(t) = c_1e^{2t} + c_2e^{4t} + t/4 + 3/16$.

Using $t = \ln x$, the general solution for the given equation is

$$y(x) = c_1x^2 + c_2x^4 + (\ln x)/4 + 3/16.$$

EXERCISE 11.5

For the following equations find the general solution by the method of variation of parameter.

1. $y'' + 2y' + y = xe^x$
2. $y'' + y = \operatorname{cosec} x,\ x \neq n\pi$
3. $y'' + 16y = 32\sec 2x,\ x \neq \left(n + \frac{1}{2}\right)\frac{\pi}{2}$
4. $y'' + 3y' + 2y = 3/(1 + e^x)$
5. $y'' + y = \sec^2 x$
6. $y'' + 4y' + 5y = xe^{-2x}\cos x$
7. $y''' + 6y'' + 11y' - 6y = e^{-x}$
8. $y''' + 4y' = \sec 2x$
9. $x^2y'' + xy' - 4y = x^2\ln x$
10. $x^2y'' - 2xy' + 2y = x^3 + x$

For the following equations verify that the functions $y_1(x), y_2(x)$ are linearly independent solutions of the associated homogeneous equation. Using these find a particular integral and general solution of the given equation

11. $x^2y'' + xy' - y = x,\ x \neq 0;\ y_1 = x,\ y_2 = 1/x$
12. $y'' + 4y' + 8y = 16e^{-2x}\operatorname{cosec}^2 2x,\ x \neq \frac{n\pi}{2};\ y_1 = e^{-2x}\cos 2x,\ y_2 = e^{-2x}\sin 2x$
13. $x^2y'' + 3xy' - 3y = \sqrt{x};\ y_1 = x,\ y_2 = 1/x^3$
14. $(1 - x^2)y'' - xy' + 4y = x;\ y_1 = 2x^2 - 1,\ y_2 = x\sqrt{x^2 - 1}$
15. $y''' + y'' - y' - y = x;\ y_1 = e^x,\ y_2 = e^{-x},\ y_3 = xe^{-x}$

For the following equations find the general solution by the method of undetermined coefficients.

16. $y'' + 2y' + 4y = 2x^2 + 3e^{-x}$
17. $y'' + 5y' + 6y = 4e^{-x} + 5\sin x$
18. $y'' + y' - 12y = e^{3x}$
19. $y'' - y = e^{3x}\cos 2x - e^{2x}\sin 3x$
20. $y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x}$
21. $x^2y'' + xy' + 4y = \sin(2\ln x)$
22. $x^2y'' + 3xy' + y = 9x^2 + 8x + 5$

11.8 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

So far we have considered the case where there is a single dependent variable such as the current $I(t)$ in a circuit at the time t. However, many problems involve two or more dependent variables, but only one independent variable. For example, in a chemical reaction two substances with specific concentrations at an instant t, react to form a third substance with some concentration. Since the concentrations are interrelated the governing differential equations are coupled and lead to system of simultaneous differential equations. As another example, governing of various currents in an electric circuit, comprising two or more loops, leads to a system of simultaneous linear differential equations.

We shall consider the solution of a system of two linear first order equations in two dependent variables x and y and one independent variable t with constant coefficients only. For example, the equations

$$\frac{dx}{dt} - \frac{dy}{dt} - y = e^{-t},\ \frac{dy}{dt} + x - y = e^{2t}$$

represent a system of linear constant coefficients first order differential equations. These can be expressed in operator form as

$$Dx - (D+1)y = e^{-t},\ x + (D-1)y = e^{2t};\ D \equiv \frac{d}{dt}$$

The method of solving these simultaneous equations consists of eliminating one of the variables x or y, solving the resulting differential equation and substituting the value of that variable in one of the two equations to get the value of the second variable. The complete solution consists of the two equations giving expressions for x and y in term of the independent variable t. This method can be extended to a system of simultaneous equations with more than two dependent variables and one independent variable.

Example 11.37: Solve the system of differential equations

$$\frac{dx}{dt} - 2x + y = 4 - t^2,\ \frac{dy}{dt} + x - 2y = 1$$

Solution: The equations can be expressed as

$$(D-2)x + y = 4 - t^2 \quad \text{...(11.73)}$$

$$x + (D-2)y = 1 \quad \text{...(11.74)}$$

Operating (11.73) by $(D-2)$, and then subtracting (11.74) from the resultant equation, we obtain

$$(D-2)^2x - x = (D-2)(4-t^2) - 1$$

or, $$(D^2 - 4D + 3)x = 2t^2 - 2t - 9. \quad ...(11.75)$$

Solving Eq. (11.75) by operator method, the general solution is

$$x(t) = c_1e^{3t} + c_2e^t + \frac{2}{3}t^2 + \frac{10}{9}t - \frac{53}{27}, \quad ...(11.76)$$

where c_1 and c_2 are two arbitrary constants.

To find $y(t)$, from (11.73) we have $y(t) = -(D-2)\,x(t) + 4 - t^2$.

Using (11.76) in it, gives

$$y(t) = -c_1e^{3t} + c_2e^t + \frac{1}{3}t^2 + \frac{8}{9}t - \frac{28}{27}. \quad ...(11.77)$$

The Eqs. (11.76) and (11.77) form the general solution for the given system of differential equations.

Example 11.38: Solve the system of differential equations

$$(3D+1)x + 3Dy = 3t + 1 \quad ...(11.78)$$
$$(D-3)x + Dy = 2t \quad ...(11.79)$$

Solution: To eliminate y, multiply Eq. (11.79) by 3 and subtract from Eq. (11.78) and simplify we obtain

$$x(t) = (1-3t)/10 \quad ...(11.80)$$

To find $y(t)$ from Eq. (11.79) we have

$$Dy = 2t - (D-3)x = 2t - \frac{1}{10}(D-3)\,(1-3t), \quad \text{using (11.80)}$$

$$= \frac{1}{10}(11t + 6), \text{ a linear first order equation in } y.$$

Integrating, we obtain

$$y(t) = \frac{11}{20}t^2 + \frac{3}{5}t + c, \quad ...(11.81)$$

where c is an arbitrary constant.

The Eqs. (11.80) and (11.81) form the general solution for the given system of differential equations.

Example 11.39: Solve the system of differential equations

$$(D^2-3)x - 4y = 0 \quad ...(11.82)$$
$$x + (D^2+1)y = 0 \quad ...(11.83)$$

Solution: To eliminate y, multiply Eq. (11.82) by (D^2+1) and Eq. (11.83) by 4 and add, we get

$$[(D^2+1)\,(D^2-3) + 4]x = 0$$

or, $$(D^4 - 2D^2 + 1)x = 0 \quad ...(11.84)$$

The general solution of Eq. (11.84) is

$$x = (c_1 + c_2 t)e^t + (c_3 + c_4 t)e^{-t}, \quad \text{...(11.85)}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

To find y, from (11.82) we have $y = \frac{1}{4}(D^2 - 3)x$.

Substituting for x from (11.85) and simplifying, we obtain

$$y = \frac{1}{2}(c_2 - c_1 - c_2 t)e^t - \frac{1}{4}(c_2 + c_4 + c_4 t)e^{-t}. \quad \text{...(11.86)}$$

Eqs. (11.85) and (11.86) form the general solution for the given system of differential equations.

Example 11.40: The small oscillations of a certain system with two degrees of freedom is given by

$$(D^2 + 2)x - y = 0 \quad \text{...(11.87)}$$

$$-x + (D^2 + 2)y = 0 \quad \text{...(11.88)}$$

Find x and y as a function of t.

Solution: To eliminate y, multiply (11.87) by $(D^2 + 2)$ and add to (11.88), we obtain

$$(D^4 + 4D^2 + 3)x = 0 \quad \text{...(11.89)}$$

Similarly eliminating x from (11.87) and (11.88), we get

$$(D^4 + 4D^2 + 3)y = 0 \quad \text{...(11.90)}$$

The general solution for Eqs. (11.89) and (11.90) are respectively

$$x(t) = c_1 \cos t + c_2 \sin t + c_3 \cos \sqrt{3}\, t + c_4 \sin \sqrt{3}\, t \quad \text{...(11.91)}$$

and, $$y(t) = c'_1 \cos t + c'_2 \sin t + c'_3 \cos \sqrt{3}\, t + c'_4 \sin \sqrt{3}\, t. \quad \text{...(11.92)}$$

To find any relation among the constants c_i's and c'_i's, we use the fact that (11.91) and (11.92) are solutions of the given simultaneous equations. Substituting these in, say Eq. (11.87), we obtain

$$(c_1 - c'_1)\cos t + (c_2 - c'_2)\sin t - (c_3 + c'_3)\cos\sqrt{3}\, t - (c_4 + c'_4)\sin\sqrt{3}\, t = 0, \text{ for all } t$$

which gives, $c_1 = c'_1$, $c_2 = c'_2$, $c_3 = -c'_3$ and $c_4 = -c'_4$.

Hence (11.92) becomes

$$y(t) = c_1 \cos t + c_2 \sin t - c_3 \cos\sqrt{3}\, t - c_4 \sin\sqrt{3}\, t \quad \text{...(11.93)}$$

The Eqs. (11.91) and (11.93) form the general solution for the given system of equations.

Example 11.41: Solve the system of differential equations

$$Dx = y + z,\ Dy = x + z, \text{ and } Dz = x + y.$$

Solution: Equations are

$$Dx = y + z \quad \text{...(11.94)}$$

$$Dy = x + z \quad \text{...(11.95)}$$

$$Dz = x + y \qquad \text{...(11.96)}$$

Operating (11.94) with D and substituting for Dy and Dz respectively from (11.95) and (11.96), we obtain $(D^2 - D - 2)x = 0$, which is a second order linear homogeneous equation with solution

$$x(t) = c_1e^{-t} + c_2e^{2t}, \qquad \text{...(11.97)}$$

where c_1 and c_2 are two arbitrary constants.

Next, operating (11.95) with D and substituting for Dz from (11.96) in it, we obtain $(D^2 - 1)y = (D + 1)x$, or $(D^2 - 1)y = 3\, c_2\, e^{2t}$, using (11.97). It is a second order non-homogeneous linear equation with constant coefficients. Its general solution is

$$y(t) = c_2e^{2t} + c_3e^{-t} + c_4e^{t}. \qquad \text{...(11.98)}$$

We note that while solving these linear differential equations, the four constants have already appeared, which is not desirable. To rectify this, from (11.94) and (11.95) we have

$$Dx - Dy = y - x. \qquad \text{...(11.99)}$$

Substituting from (11.97) and (11.98) in (11.99) and simplifying, we get $-c_4e^t = c_4e^t$, which gives, $c_4 = 0$, since $e^t \neq 0$. Hence, (11.98) becomes

$$y(t) = c_2e^{2t} + c_3e^{-t} \qquad \text{...(11.100)}$$

To find z, from (11.94), we have $z = Dx - y$.

Substituting for x and y from (11.97) and (11.100) respectively and simplifying we get

$$z(t) = -(c_1 + c_3)e^{-t} + c_2e^{2t} \qquad \text{...(11.101)}$$

The Eqs. (11.97), (11.100) and (11.101) form the solution of the given system of equations.

Remark. We have been solving system of differential equations like that of system of linear algebraic equations. As in case of system of algebraic equations, we have system of differential equations with no solution, that is, an inconsistent system, or a system with infinite number of solutions, a redundant system. For example, the system of equations

$$2Dx + Dy = 1, \quad 4Dx + 2Dy = 4$$

has no solution, since the process of elimination of variable x or y leads to $2 = 4$, which is not possible. But if we modify the system to

$$2Dx + Dy = 2, \quad 4Dx + 2Dy = 4$$

then it has infinite solutions. Indeed then the second equation is merely twice the first and thus can be discarded, leaving the single equation $2Dx + Dy = 2$ in two unknowns $x(t)$ and $y(t)$. Choosing one of these arbitrarily and solving for the second, so there are infinitely many linearly independent solutions.

EXERCISE 11.6

Solve the following system of differential equations

1. $(D + 2)x + 3y = 0,\ 3x + (D + 2)y = 2e^{2t}$ 2. $(D + 2)x - y = 1 + e^{-t},\ x + (D + 2)y = 3$
3. $2(D - 2)x + (3D + 5)y = 3t + 2,\ (D - 2)x + (D + 1)y = t$
4. $(D + a)x - ay = 0,\ ax + (D + a)y = 0,\ a \neq 0$

5. $(D-2)x+12y=0,\ 3x-(2D+8)y=0;\ x(0)=0$ and $y(0)=1$
6. $(D^2-3)x-y=0,\ 2x+Dy=0$
7. $D^2x=y,\ D^2y=x$
8. $(D^2+3)x-2y=0,\ (D^2-3)x+(D^2+5)y=0;\ x(0)=y(0)=0,\ Dx(0)=3$ and $Dy(0)=2$
9. $D^2x+y=\sin t,\ x+D^2y=\cos t$
10. $Dx=2y,\ Dy=2x,\ Dz=2x.$

11.9 MODELLING SIMPLE HARMONIC MOTION

A simple harmonic motion (S.H.M.) is a periodic motion in which the acceleration of a particle is proportional to its displacement from a fixed point called the centre and is always directed towards it.

Let x be the displacement of the particle P at any time t from the fixed point O as shown in Fig. 11.2, then by definition of S.H.M.

$$\frac{d^2x}{dt^2}=-\mu^2x, \qquad ...(11.102)$$

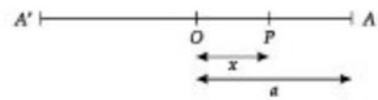


Fig. 11.2

μ^2 is the constant of proportionality negative sign is taken since the force acting on the particle is directed towards the fixed point, a direction of decreasing x.

Rewriting Eq. (11.102) as $(D^2+\mu^2)x=0$, which is a linear differential equation of second order with constant coefficients with complete solution as

$$x=c_1\cos\mu t+c_2\sin\mu t. \qquad ...(11.103)$$

In case the particle starts from rest at A, where $OA=a$, then

$$x(0)=a, \text{ and } x'(0)=0. \qquad ...(11.104)$$

Using (11.104) in (11.103) gives, $c_1=a$ and $c_2=0$ and hence, Eq. (11.103) becomes

$$x=a\cos\mu t, \text{ which gives, } \frac{dx}{dt}=-a\mu\sin\mu t=-\mu\sqrt{(a^2-x^2)}.$$

These respectively are the expressions for the *displacement* and the *velocity* of particle P at any time t. The maximum displacement, from the centre a is called the *amplitude* and the time of complete oscillation, $\frac{2\pi}{\mu}$ is called the *periodic time* and $\frac{1}{(\text{Periodic time})}=\frac{\mu}{2\pi}$ is called the *frequency*, the number of oscillations per second.

Also we note that the general solution (11.103), can be expressed as

$$x=a\cos(\mu t-\alpha), \text{ where } \alpha=\tan^{-1}(c_2/c_1)$$

The quantity α is called the *starting phase or the epoch of the motion* and the quantity $(\mu t-\alpha)$ is called the *argument of the motion* at time t.

Example 11.42: A particle is moving linearly with the speed v given by the relation $v^2=a+2bx-cx^2$, where x is the displacement of the particle from a fixed point on the path, and a, b, c are constants, with $c>0$. Show that the motion is simple harmonic and find its period and amplitude.

Solution: The velocity v of the particle is given by

$$v^2 = a + 2bx - cx^2 \quad \text{...(11.105)}$$

Differentiating both sides w.r.t. x, we get, $2v\dfrac{dv}{dx} = 2b - 2cx$

or,
$$\frac{d^2x}{dt^2} = b - cx = -c\left(x - \frac{b}{c}\right) \quad \text{...(11.106)}$$

Since $c > 0$, the Eq. (11.106) represents a S.H.M. directed towards the point $x = b/c$ and period $\dfrac{2\pi}{\mu} = \dfrac{2\pi}{\sqrt{c}}$. To find amplitude put $v = 0$ in (11.105), we obtain, $x = \dfrac{b \pm \sqrt{b^2 + ac}}{c}$. Thus the distances of two positions of instantaneous rest from the fixed point O, $\left(x = \dfrac{b}{c}\right)$, are

$$OA = \frac{b + \sqrt{b^2 + ac}}{c}, \text{ and } OA' = \frac{b - \sqrt{b^2 + ac}}{c}.$$

Hence the amplitude of the motion is $\left|\dfrac{b \pm \sqrt{b^2 + ac}}{c} - \dfrac{b}{c}\right| = \dfrac{\sqrt{b^2 + ac}}{c}$

Example 11.43: A particle of mass m executes simple harmonic motion in the line joining the points A and A' on a smooth table and is connected with these points by elastic strings. If T is tension in equilibrium and l, l' are the extensions of the strings beyond their natural lengths, find the periodic time.

Solution: Let O be the position of equilibrium of the particle, so that $OA = a + l$ and $A'O = a' + l'$ where a and a' are the natural lengths of the strings, refer Fig. 11.3. The tension T in the position of equilibrium is given by

$$T = \lambda l / a = \lambda' l' / a'.$$

Let P be the position of the particle at instant t during its motion such that $OP = x$ and T_1, T_2 be the tensions in the two portions, then

$$T_1 = \lambda \frac{l + x}{a} \text{ and } T_2 = \lambda' \frac{l' - x}{a'}.$$

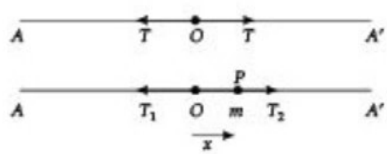


Fig. 11.3

Hence the equation of motion is

$$m\frac{d^2x}{dt^2} = T_2 - T_1 = \lambda' \frac{l' - x}{a'} - \lambda \frac{l + x}{a} = \left(\frac{\lambda' l'}{a'} - \frac{\lambda l}{a}\right) - \left(\frac{\lambda'}{a'} + \frac{\lambda}{a}\right)x$$

$$= (T - T) - T\left(\frac{1}{l'} + \frac{1}{l}\right)x = -\frac{T(l + l')}{ll'} x \text{ or,}$$

$\frac{d^2x}{dt^2} = -\frac{T(l+l')}{mll'}x = -\mu x$. The periodic time is given by $T = \frac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\frac{mll'}{(l+l')T}}$.

EXERCISE 11.7

1. A horizontal shelf with a body of mass m placed on it is moving up and down in a simple harmonic motion of period 1 sec. Find its greatest amplitude so that the body placed on it is not thrown off.
2. Find the time required for a particle in simple harmonic motion with amplitude 20 cm and period 4 seconds, in passing between two points which are at distances 15 cm and 5 cm from the origin.
3. An elastic string of natural length $2a$ and modulus λ is stretched between two points A and B, distant $4a$ apart on a smooth horizontal table. A particle of mass m is attached to the middle of the string. Show that it can vibrate in line AB with period $\sqrt{(2am/\lambda)}\ \pi$.
4. If x_1, x_2, x_3 are the positions of a particle at the end of Ist, 2nd, 3rd second of its motion in S.H.M., then show that the time period is

$$2\pi \Big/ \cos^{-1}\left(\frac{x_1 + x_2}{x_3}\right).$$

11.10 MODELLING MASS-SPRING SYSTEM: FREE AND FORCED OSCILLATIONS

Consider a spring with unstretched length l and spring modulus k, a measure of the stiffness of the spring. The spring is suspended vertically from a fixed support. Let a body of mass m is attached at the lower end of the spring, (assuming m to be large so that the mass of the spring is neglected), which stretches the spring by d units over its natural length before coming to rest in its equilibrium position. Next, let us suppose that the body is then displaced vertically (up, or down) by distance y_0 units and is released possibly with an initial velocity. We want to study the motion of this spring-mass system, refer to Fig. 11.4.

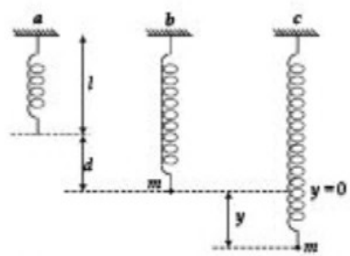


Fig. 11.4 (a) Unstretched; (b) Static equilibrium; (c) System in motion

11.10.1 The Spring Equation

Let $y(t)$ be the displacement of the object at time t from the equilibrium position, say $y = 0$ and select the downward direction to be positive. Consider the forces acting on the body of mass m at time t.

The force due to gravity which pulls it downward is of magnitude mg.

The force the spring exerts on the body at time t, due to Hooke's law, is of magnitude ky. At equilibrium position the force of the spring is of magnitude $-kd$, sign is negative since the force acts upward. If the body is displaced downward by a distance y from the equilibrium position, then an additional force $-ky$ is exerted on it at time t.

Thus, the total force on the body due to gravity and the spring is

$$mg - kd - ky.$$

At equilibrium position ($y = 0$) this force is zero, and hence $mg = kd$. Thus the net force acting on the object is therefore just $F_1 = -ky$, an upward force. It is a *restoring force* and has the tendency to restore the system, that is, pull the body back to the equilibrium position $y = 0$.

Next, every system is subjected to some damping or retarding forces, which may be air resistance or the viscosity of the medium if the body is suspended in some fluid such as oil, etc. In case these forces are not negligible then we need to take the corresponding damping into account. Experiments show that *the magnitude of the damping forces at any instant t is proportional to the velocity $y'(t)$, and direction is opposite to the instantaneous motion*. Thus, the *damping force* is $F_2 = -cy'$, where $c > 0$ is some constant, called the *damping constant*.

Also there may be a driving force of magnitude $F_3 = f(t)$ acting on the body, and then the total external force on the body has the magnitude

$$F_1 + F_2 + F_3 = -ky - cy' + f(t). \quad \text{...(11.107)}$$

Using Newton's second law of motion, *the equation of the mass-spring system* is,

$$my'' = -ky - cy' + f(t)$$

$$\text{or, } my'' + cy' + ky = f(t). \quad \text{...(11.108)}$$

This is called *the spring equation*.

In the absence of the external force, that is $f(t) = 0$, the equation (11.108) becomes homogeneous one given by $my'' + cy' + ky = 0$, and the motion of the mass-spring system are called the *free motions*.

In the presence of the external force the motions of the spring-mass system are called the *forced motion*. The external force $f(t)$ is called the *input*, or *the deriving force*; and the corresponding solution is called an *output* or a *response* of the system to the deriving force.

Next we analyze the motion described by solutions of the spring Eq. (11.108) under various conditions.

11.10.2 Mass-Spring System: Free Motions

In case of free motions the deriving force $f(t) = 0$, hence the spring equation is

$$my'' + cy' + ky = 0. \quad \text{...(11.109)}$$

We study the following cases of practical interest.

A ***Undamped system*** In case the damping forces are negligible, (at least it can be so when the velocity is small during the initial phase of the motion), then $c = 0$, the Eq. (11.109) becomes

$$my'' + ky = 0. \quad \text{...(11.110)}$$

The general solution of (11.110) is

$$y(t) = c_1 \cos wt + c_2 \sin wt,\ w = \sqrt{k/m} \quad \text{...(11.111)}.$$

In case the body is first pulled to a point at a distance a units from the position of static equilibrium and is released, then $y(0) = a$ and $y'(0) = 0$.

Using these initial conditions in (11.111) gives $c_1 = a$ and $c_2 = 0$, and hence (11.111) becomes $y(t) = a\cos wt$. The motion is simple harmonic motion with period $\frac{2\pi}{w} = 2\pi\sqrt{\frac{m}{k}}$ with amplitude a as shown in Fig. 11.5. The curve touches the line $y = \pm a$ when wt is an integral multiple of π.

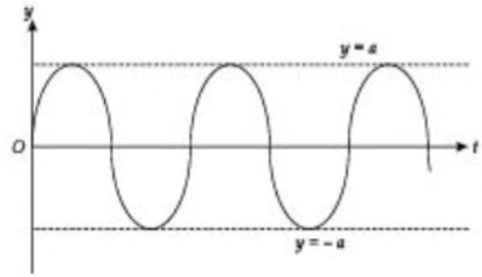


Fig. 11.5

B ***Damped system*** If the damping forces are not negligible, then $c \neq 0$ and the spring equation is

$$my'' + cy' + ky = 0. \qquad \text{...(11.112)}$$

The corresponding characteristic equations is $\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$, with roots

$$\lambda_1 = -\alpha + \beta, \text{ and } \lambda_2 = -\alpha - \beta, \text{ where } \alpha = \frac{c}{2m} \text{ and } \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

As is evident, the form of the solution of (11.112) will depend on the mass m, the amount of damping and the stiffness of the spring. We have the following three cases:

Case I: $c^2 > 4mk$: Two distinct real roots λ_1, λ_2 *(over damping)*,

Case II: $c^2 = 4mk$: Two equal and real roots *(critical damping)*,

Case III: $c^2 < 4mk$: Complex conjugate roots *(under damping)*.

Case I: Over damping. If the damping constant c is so large that $c^2 > 4mk$, then λ_1 and λ_2 are two real and distinct roots and the general solution of the equation (11.112) is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}. \qquad \text{...(11.113)}$$

Obviously $\lambda_2 = -\alpha - \beta$ is negative, and also

$$\lambda_1 = -\alpha + \beta = -\frac{c}{2m} + \frac{\sqrt{c^2 - 4mk}}{2m} < -\frac{c}{2m} + \frac{c}{2m} = 0.$$

Since both λ_1 and λ_2 are negative, therefore, the terms in (11.113) tend to be zero as t approaches infinity. Thus the body does not oscillate and after a sufficiently long time the mass will be at rest at its static equilibrium position $y = 0$, a case of over damping.

Case II: Critical damping. If $c^2 = 4mk$, then $\beta = 0$ and the two roots of the equation (11.112) are $\lambda_1 = \lambda_2 = -\dfrac{c}{2m} = -\alpha$. The general solution is

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}. \qquad \text{...(11.114)}$$

Here also $y(t) \to 0$ as $t \to \infty$, as in the case of over damping. This case marks the boundary between the over damped behaviour discussed above and the oscillatory behaviour to be discussed next.

Case III: Under damping. If the damping coefficient is so small that $c^2 < 4mk$, then the roots of the Eq. (11.114) are complex conjugate, say $\lambda_1 = -\alpha + i\beta^*$ and $\lambda_2 = -\alpha - i\beta^*$, where

$\alpha = \dfrac{c}{2m}$, and $\beta^* = \dfrac{1}{2m}\sqrt{4km - c^2}$. The general solution of Eq. (11.112) is

$$y(t) = e^{-\alpha t}(c_1 \cos \beta^* t + c_2 \sin \beta^* t). \qquad \text{...(11.115)}$$

Since $\alpha > 0$, thus $y(t) \to 0$ as $t \to \infty$. However, the motion is now oscillatory because of the sine and cosine terms in the solution. But it is not the periodic one because of the exponential factors which causes the amplitude of the oscillations to decay to zero as t becomes sufficiently large. The Eq. (11.115) can be expressed as

$$y(t) = ce^{-\alpha t}\cos(\beta^* t - \theta), \text{ where } c = \pm\sqrt{c_1^2 + c_2^2}, \text{ and } \theta = \tan^{-1}\frac{c_2}{c_1}.$$

Thus the solution curve lies between $\pm ce^{-\alpha t}$, touching these curves when $(\beta^* t - \theta)$ is an integral multiple of π, as shown in Fig. 11.6.

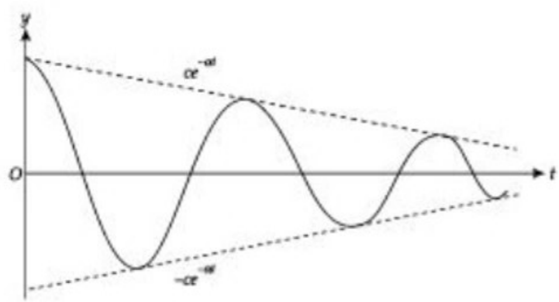


Fig. 11.6

Example 11.44: A weight of about 89.0 Newton stretches a spring by 10.0 cm. How many cycles per second will this mass-spring system execute? What will be its equation of motion in case the weight is pulled down by 15.00 cm from its position of static equilibrium and then released, ignore the damping forces. How does the motion will change if the system has damping given by

(i) $c = 200.0$ kg/sec (ii) $c = 179.81$ kg/sec (iii) $c = 100.0$ kg/sec?

Solution: If k is the coefficient of stiffness for the spring, then using Hookes' law, we have

$89.0 = 0.1\,k$, (10.0 cm = 0.1 meter), hence $k = 890$ N/meter. Also mass,

$m = w/g = 89.0/9.80 = 9.082$ kg. Thus frequency, $\dfrac{w}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{890}{9.082}} = 1.576.$

If $y(t)$ denotes the displacement of the body at any instant from its position of static equilibrium, then ignoring the damped forces, motion is described by the initial value problem

$$y'' + w^2 y = 0,\ y(0) = 0.15,\ y'(0) = 0;\ w = \sqrt{k/m} = 9.899. \qquad \text{...(11.116)}$$

We can check that the solution of (11.116) is $y(t) = 0.1500 \cos 9.899t$.

In case the system has damping, then the spring equation is $my'' + cy' + ky = 0$.

(i) When $c = 200.0$, the spring equation for $m = 9.082$ and $k = 890$ becomes

$$9.082y'' + 200.0y' + 890.0y = 0, \qquad \text{...(11.117)}$$

The characteristic equation has the roots $\lambda = -11.01 \pm 4.82 = -6.190, -15.83$.

The general solution is

$$y(t) = c_1 e^{-6.19t} + c_2 e^{-15.83t} \qquad \text{...(11.118)}$$

Using the initial conditions $y(0) = 0.15$, and $y'(0) = 0$, Eq. (11.118) becomes

$$y(t) = 0.2463e^{-6.190t} - 0.0963e^{-15.83t}.$$

Thus, $y(t)$ tends to zero as $t \to \infty$. It is the case of *over damping*.

(ii) When $c = 179.81$, then $c^2 = 4mk$ and hence the characteristic equation has the double root $\lambda = -9.899$. Therefore the solution is

$$y(t) = (c_1 + c_2 t)e^{-9.899t}. \qquad \text{...(11.119)}$$

Using the initial conditions $y(0) = 0.15$, and $y'(0) = 0$, (11.119) becomes

$$y(t) = (0.150 + 1.48t)e^{-9.899t}.$$

In this case also $y(t) \to 0$ as $t \to \infty$. It is the case of *critical damping*

(iii) When $c = 100$, the roots of the characteristic equation are $\lambda = -5.506 \pm 8.227i$, the complex conjugate. The general solution is

$$y(t) = e^{-5.506t}(c_1 \cos 8.227t + c_2 \sin 8.227\,t). \qquad \text{...(11.1120)}$$

Using the initial conditions $y(0) = 0.15$ and $y'(0) = 0$, (11.120) becomes

$$y(t) = e^{-5.506t}(0.1500 \cos 8.227t + 0.1004 \sin 8.227t) = 0.1805e^{-5.506t} \cos (8.227t - 0.981).$$

It is a case of *damped oscillation* with frequency, $w/2\pi = \dfrac{8.227}{2\pi} = 1.309$.

11.10.3 *Mass-Spring System: Forced Motions*

Now suppose that an external deriving force of magnitude $f(t)$ acts on the body. Different forces will cause different kinds of motions. Of practical interest are periodic deriving force of the type $f(t) = A \cos wt$, where $A > 0$ and $w > 0$ are constants. Then the spring equation (11.108) becomes

$$my'' + cy' + ky = A \cos wt. \qquad \text{...(11.121)}$$

A general solution of the non-homogeneous Eq. (11.121) is the sum of the complementary function $y_c(t)$, a solution of the corresponding homogeneous equation given by

$$y_c(t) = \begin{cases} e^{-\frac{ct}{2m}\left[c_1 \cos\sqrt{w_0^2 - \left(\frac{c}{m}\right)^2}\, t\right]}, & c^2 < 4mk \text{ (underdamped)} \\ e^{-\frac{ct}{2m}[c_1 + c_2 t]}, & c^2 = 4mk \text{ (critically damped)} \\ e^{-\frac{ct}{2m}\left[c_1 \cosh\sqrt{\left(\frac{c}{2m}\right)^2 - w_0^2}\, t\right]}, & c^2 > 4mk \text{ (overdamped)} \end{cases} \qquad \text{...(11.122)}$$

here $w_0 = \sqrt{k/m}$; and a particular solution $y_p(t)$ of the Eq. (11.121), which we derive next.

To determine $y_p(t)$ we use the method of undetermined coefficients and hence try a solution of the from

$$y_p(t) = a \cos wt + b \sin wt, \qquad \text{...(11.123)}$$

where a and b are the coefficients to be determined.

Substituting this in (11.121) and comparing the coefficients of $\cos wt$ and $\sin wt$ on both sides, we have $(k - mw^2)a + wcb = A$, and $-wca + (k - mw^2)b = 0$. Solving these for a and b, we obtain

$$a = \frac{A(k - mw^2)}{(k - mw^2)^2 + w^2c^2} \text{ and } b = \frac{Awc}{(k - mw^2)^2 + w^2c^2}, \text{ provided } (k - mw^2)^2 + w^2c^2 \neq 0.$$

Set $w_0 = \sqrt{k/m}$. Then particular solution is

$$y_p(t) = \frac{mA(w_0^2 - w^2)}{m^2(w_0^2 - w^2)^2 + w^2c^2} \cos wt + \frac{Awc}{m^2(w_0^2 - w^2)^2 + w^2c^2} \sin wt, \qquad \text{...(11.124)}$$

provided $w \neq w_0$, or $c \neq 0$.

We shall now discuss the behaviour of the spring mass system with this deriving force, distinguishing between the two cases $c = 0$, the undamped system, and $c > 0$, the damped system.

A. Damped forced motion

The particular solution (11.124) can be expressed in the form

$$y_p = B\cos(wt - \theta), \qquad \text{...(11.125)}$$

where the amplitude B and angle θ are given by

$$B = \frac{A}{\sqrt{m^2(w_0^2 - w^2)^2 + w^2c^2}}, \text{ and } \theta = \tan^{-1}\frac{wc}{m(w_0^2 - w^2)},\ 0 < \theta < \pi.$$

We observe that $y_c(t)$ part of the solution tends to zero as t goes to infinity because of the exp.$(-ct/2m)$ factor as long as $c > 0$, no matter how small it is. Practically it is zero after a sufficient long time. Thus, $y_c(t)$ is the *transient part* of the solution. The part $y_p(t)$ is the *steady-state part* since $y(t) \to B\cos(wt - \theta)$ as $t \to \infty$. Hence, after a sufficiently long time the output corresponding to a purely sinusoidal input will practically be a harmonic oscillation whose frequency is that of the input, and this is what happens in almost all cases since damped forces are never zero.

B. Undamped forced motion (resonance)

In the absence of the damping forces, $c = 0$. The spring equation becomes

$$my'' + ky = A\cos wt. \qquad \text{...(11.126)}$$

The general solution of this equation is

$$y(t) = c_1\cos w_0t + c_2\sin w_0t + \frac{A\cos wt}{m(w_0^2 - w^2)}, \qquad \text{...(11.127)}$$

where $w_0 = \sqrt{k/m}$ is called the *natural frequency* of the mass-spring system, and w is the *input frequency* to the system. The general solution (11.127) assumes that $w \neq w_0$.

The output represents a superposition of two harmonic oscillations with frequency $w_0/2\pi$ of the system and the frequency $w/2\pi$ of the input.

Consider the case when the input frequency matches the natural frequency of the system, that is, $w = w_0$ then the spring equation becomes

$$my'' + ky = A\cos w_0t, \qquad \text{...(11.128)}$$

and (11.127) is no longer its solution.

The complementary function of (11.128) is $y_c(t) = c_1\cos w_0t + c_2\sin w_0t$.

To find particular solution we use method of undetermined coefficients. Since the right side of Eq. (11.128) contains a term appearing in the complementary function, we attempt a function of the form

$$y_p(t) = at\cos w_0t + bt\sin w_0t. \qquad \text{...(11.129)}$$

Substituting (11.129) in (11.128) and comparing the coefficients of $\sin w_0t$ and $\cos w_0t$ on both sides, we get $a = 0$, and $b = A/2mw_0$, and thus, (11.129) becomes $y_p(t) = \dfrac{A}{2mw_0} t \sin w_0 t$, and hence the general solution of Eq. (11.128) is

$$y(t) = c_1 \cos w_0 t + c_2 \sin w_0 t + \frac{A}{2mw_0} t \sin w_0 t. \qquad \text{...(11.130)}$$

In this special case the response of $y_p(t)$ is not harmonic oscillation but t times a harmonic function which causes the magnitude to tend to infinity as $t \to \infty$, as shown in Fig. 11.7.

We observe that $y_p(t)$ becomes larger and larger. In practice this means that systems with very little damping may undergo large vibrations. This phenomena is called *resonance*.

Resonance is sometimes desirable and sometimes undesirable. For example, we may wish to amplify a given input when we tune a radio circuit to a desired broadcast frequency. But we may wish to suppress inputs from a bumpy road to an automobile. Practically we can never be equal to w_0. It may therefore be interesting to observe the case when w approaches to w_0.

Let the initial conditions be $y(0) = 0$ and $y'(0) = 0$. Using these conditions in (11.127) to evaluate c_1 and c_2, we obtain

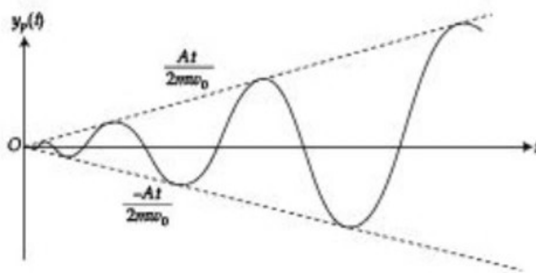


Fig. 11.7

$$y(t) = \frac{A}{m(w_0^2 - w^2)} (\cos wt - \cos w_0 t).$$

$$= \frac{2A}{m(w_0^2 - w^2)} \sin\left(\frac{w_0 + w}{2} t\right) \sin\left(\frac{w_0 - w}{2} t\right). \qquad \text{...(11.131)}$$

Since the difference $(w_0 - w)$ is small, so the period of the second sinusoid in (11. 131) is large. This results in a periodic variation of amplitude in the outcome $y(t)$ depending on the relative sizes of $w_0 + w$ and $w_0 - w$, as shown in Fig. 11.8. This periodic variation is called a '*beat*' and this is what which interests musicians while tuning their instruments.

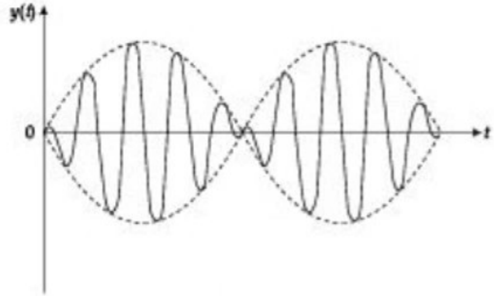


Fig. 11.8

Example 11.45: An 8 lb weight is placed at one end of a spring suspended from the ceiling. The weight is raised to 5 inches above the static equilibrium position and is released. Assuming the spring constant 12 lb./ft, find the equation of motion, displacement $y(t)$, amplitude, period and maximun velocity.

Solution: If $y(t)$ is the displacement of the mass with respect to its position of static equilibrium, then the equation of motion is

$$m\frac{d^2y}{dt^2} + ky = 0, \qquad \text{...(11.132)}$$

where m is the mass suspended and k is the spring constant.

Here, $m = \frac{8}{32} = \frac{1}{4}$, $k = 12$. The solution of Eq. (11.132) is

$$y(t) = c_1 \cos wt + c_2 \sin wt \qquad \text{...(11.133)}$$

where $w = \sqrt{k/m} = \sqrt{48} = 4\sqrt{3}$.

Using the initial conditions, $y(0) = -5/12$ and $y'(0) = 0$, Eq. (11.133) becomes

$$y(t) = -\frac{5}{12}\cos 4\sqrt{3}\,t = \frac{5}{12}\sin\left(4\sqrt{3}t - \frac{\pi}{2}\right).$$

Thus amplitude $A = \frac{5}{12}$ ft, period $T = \frac{2\pi}{w} = \frac{2\pi}{4\sqrt{3}} = \frac{\pi\sqrt{3}}{6}$ sec., and the velocity

$y'(t) = \frac{5\sqrt{3}}{3}\cos\left(4\sqrt{3}t - \frac{\pi}{2}\right)$; the maximum velocity is $\frac{5\sqrt{3}}{3}$ ft/sec.

Example 11.46: A weight of 980 gm is suspended at the lower end of a spring which is fixed at its upper end. The weight is pulled down 1/4 cm below its static equilibrium position and then released. In case the resistance (in gm. wt) to the motion of the weight is 1/10 of the velocity of the weight in cm/sec, write the equation of motion, displacement $y(t)$ and also the time it takes the damping factor to drop to 1/10 of its initial value. Assume that spring constant is 20 gm/cm.

Solution: If $y(t)$ is the displacement of the body at any time t from its position of static equilibrium, then the equation of motion is $my'' + cy' + ky = 0$.

Here, $m = \dfrac{980}{g} = \dfrac{980}{980} = 1$ gm, $c = \dfrac{1}{10}$, $k = 20$ gm/cm. Thus the equation becomes

$$10y'' + y' + 200y = 0. \qquad ...(11.134)$$

Solution of (11.134) is $y(t) = e^{-0.05t}\,[c_1 \cos(4.5)t + c_2 \sin(4.5)t]$.

Using the initial conditions $y(0) = \dfrac{1}{4}$, $y'(0) = 0$, we obtain

$$\begin{aligned} y(t) &= e^{-0.05t}\,[0.25 \cos(4.5)t + 0.003 \sin(4.5)t] \\ &= 0.25e^{-0.05t} \cos(4.5t - \theta), \text{ where } \theta = \tan^{-1}(0.012),\ 0 < \theta < \pi/2. \end{aligned} \qquad ...(11.135)$$

Since $\cos(4.5t - \theta)$ lies between ± 1, thus the displacement $y(t)$ lies between the curves $y = \pm 0.25\, e^{-0.05t}$. Also the damping factor in (11.135) is $0.25e^{-0.05t}$. At $t = 0$ initial value is 0.25. If t is the time the damping factor reduces to its (1/10)th of its value, then

$$0.25e^{-0.05t} = 0.025, \text{ which gives, } t = 20 \ln 10 = 46 \text{ sec.}$$

Example 11.47: A weight of 16 lb. is suspended from a spring with spring constant 5 lb/ft. The system is subjected to an external force $24 \sin 10t$ and a damping force equal to 5 times the velocity of the weight. Find the displacement of the weight at any time t, if initially the weight is at rest at its equilibrium position. Describe the transient and steady state solutions.

Solution: If $y(t)$ is the displacement of the body at any time t from its position of static equilibrium, then the equation of motion is $my'' + cy' + ky = f(t)$.

Here, $m = w/g = 16/32 = (1/2)$ lb, $c = 5$, $k = 5$ lb/ft. and $f(t) = 24 \sin 10t$.

Substituting these values, equation of motion becomes

$$y'' + 10y' + 10y = 48 \sin 10t. \qquad ...(11.136)$$

The complementary function is $y_c(t) = c_1e^{-1.13t} + c_2e^{-8.87t}$.

The particular integral is $y_p(x) = \dfrac{1}{D^2 + 10D + 10}\, 48 \sin 10t = -\dfrac{240}{905} \cos 10t - \dfrac{216}{905} \sin 10t$

$$= -(0.265 \cos 10t + 0.238 \sin 10t).$$

Thus the complete solution of Eq (11.136) is

$$y(t) = c_1 e^{-1.13t} + c_2 e^{-8.87t} - (0.265\cos 10t + 0.238\sin 10t).$$

Using the initial conditions $y(0) = y'(0) = 0$, it gives $c_1 = 0.663$ and $c_2 = -0.398$. Thus

$$y_c(t) = 0.663e^{-1.13t} - 0.398e^{-8.87t} \quad ...(11.137)$$

Further the particular integral $y_p(t)$ can be expressed as

$$y_p(t) = 0.356\sin(10t + 3.982). \quad ...(11.138)$$

Thus, the displacement of the weight at any time t is $y(t) = y_c(t) + y_p(t)$ where $y_c(t)$ and $y_p(t)$ are given by (11.137) and (11.138) respectively. We observe that $y_c(t)$ represents the transient solution and tends to zero as $t \to \infty$, while $y_p(t)$ represents the steady state solution and is a harmonic oscillation with amplitude 0.356 and time period $2\pi/10 = \pi/5$ sec. Since $y_c(t) \to 0$ as $t \to \infty$, thus after a sufficiently long time, the output becomes the sinusoidal wave $0.356\sin(10t + 3.982)$.

Example 11.48: A weight of 6 lb is suspended from a spring with spring constant 12 lb./ft. and an external force $3\cos 8t$ acts on the weight. Describe the motion under the assumption of no damping force when initially the body is at rest in its position of static equilibrium.

Solution: Let $y(t)$ be displacement of the body at time t, then motion is described by the equation

$$\frac{6}{32}y'' + 12y = 3\cos 8t \text{ or, } y'' + 64y = 16\cos 8t \quad ...(11.139)$$

with initial conditions $y(0) = y'(0) = 0$.

The C.F. of (11.139) is $y_c(t) = c_1\cos 8t + c_2\sin 8t$.

To find the particular integral we use the method of undetermined coefficients and let $y_p(t)$ be

$$y_p(t) = at\cos 8t + bt\sin 8t, \quad ...(11.140)$$

where a and b are coefficients to be determined. Substituting (11.140) in (11.139), and comparing the coefficients of the corresponding terms, we get $a = 0$, $b = 1$, thus $y_p(t) = t\sin 8t$. Hence the complete solution is

$$y(t) = c_1\cos 8t + c_2\sin 8t + t\sin 8t.$$

Using the initial conditions $y(0) = y'(0) = 0$, we obtain $c_1 = c_2 = 0$, and hence the displacement is given by $y(t) = t\sin 8t$.

Thus the displacement is t times the harmonic oscillation $\sin 8t$ which causes the magnitude to tend to infinity as $t \to \infty$, as shown in Fig. 11.9, and hence, the resonance occurs.

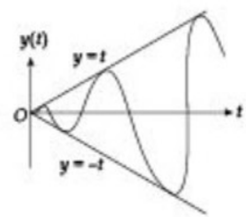


Fig. 11.9

EXERCISE 11.8

1. Show that the frequency of a harmonic oscillation of a body on a spring is $\left(\sqrt{g/l}\right)/2\pi$, where l is the elongation.
2. A body weight 10 kg is suspended from a spring with spring modulus 200 kg/metre. The body is pulled down to 0.20 metres below its position of static equilibrium and then released. Find the displacement of the body from its position of equilibrium at any time t, the maximum velocity and the period of oscillation.
3. A light elastic string of natural length l has one extremity fixed at a point A and other end attached to a stone, the weight of which in equilibrium extends the string to a depth l_1. Show that if the stone be dropped from rest at A, it will come to instantaneous rest at a depth $\sqrt{(l_1^2 - l^2)}$ below the equilibrium position.
4. A 2 lb. weight suspended from one end of a spring stretches it to (1/2)ft. A velocity of 5 ft/sec^2 upward is imparted to the weight at its position of equilibrium. In case the damping force is c, $(c > 0)$, times the velocity of the weight at time t, find the displacment of the weight at any time t. Find the amplitude, period and maximum velocity of the motion. Find the values of c for which the system is damped, overdamped or underdamped.
5. A mass m_1 is attached to a spring and allowed to vibrate with undamped motion having period p. At some later time, a second mass m_2 is instantaneously fused with m_1. Prove that the new object having mass $(m_1 + m_2)$ exhibits simple harmonic motion with period $p/\sqrt{1 + m_2/m_1}$.
6. Let $y(t)$ be the solution of $y'' + w_0{}^2 y = (A/m)\cos wt$, with $y(0) = y'(0) = 0$. Assuming that $w \neq w_0$, find $\lim\limits_{w \to w_0} y(t)$. How does this limit compare with the solution of $y'' + w_0^2 y = (A/m)\cos(w_0 t)$, with $y(0) = y'(0) = 0$?
7. Consider damped force motion governed by $my'' + cy' + ky = A\cos wt$. Show that the maximum amplitude of the steady-state solution is achieved, if $w = \dfrac{1}{m\sqrt{2}}\sqrt{2km - c^2}$.
8. Show that the damped forced motion of a mass m on a spring with forcing function $A\cos wt$ is always bounded in magnitude.
9. A body weighting 16 lb. is suspended by a spring in a fluid whose resistance in lb. wt. is twice the velocity of the body in ft./sec. A pull of 25 lb. wt would stretch the spring by 3 inches. The body is drawn 3 inches below the equilibrium position in the position and then released. Find the period of oscillation and the time required for the damping force to be reduced to (1/10)th of its initial value.
10. A body weighing 4 lb. hangs at rest at the lower end of a spring producing an extension of 1 ft. The upper end of the spring is subjected to a driving force $f(t) = \sin 4t$. If the body is subjected to a damping force (1/4) times its velocity in ft/sec, find the expression for the displacement of the body at time t, when t is large and explain it.

11.11 MODELLING R-L-C ELECTRICAL CIRCUIT: ANALOGY WITH MASS-SPRING SYSTEM

In Chapter 10, we have considered the applications of the linear first order differential equations to an R-L series circuit and an R-C series circuit. If a circuit contains a resistance R, inductance L, capacitance C, an electromotive force $E(t)$, and if, $I(t)$ is the current flowing in the circuit at any time t, then the voltage drop around the circuit, refer Fig. 10.10, by Kirchoff's law, satisfies

$$LI'(t) + RI(t) + \frac{1}{C}\int I\,dt = E(t). \quad \text{...(11.141)}$$

If $Q(t)$ is the charge, and since $Q'(t) = I(t)$, the Eq. (11.141) can be written as

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t). \quad \text{...(11.142)}$$

If L, R, and C are constants, then (11.142) is a second order linear differential equation of the type which we have already solved for various choices of $E(t)$.

11.11.1 An Analogy with Mass-Spring System

It is interesting to observe that Eq. (11.142) is of exactly the same form as the Eq. (11.108) for the displacement $y(t)$ of an object of mass m units suspended to a spring with spring modulus k, which is

$$my'' + cy' + ky = f(t). \quad \text{...(11.143)}$$

Here c is the damping force and $f(t)$ is the driving force to which the mass-spring system is subjected.

This provides an appropriate example of the important mathematical fact that entirely different physical systems may lead to the same mathematical model and thus can be solved by the same methods. This analogy between mechanical and electrical systems facilitates us to construct an electric circuit whose current will give the exact values of the displacement in the mechanical system when suitable scale factors are introduced. This may be of practical importance due to the fact that electric circuits are far easy to assemble and observe as compared to the mechanical systems.

The forms of the two Eqs. (11.142) and (11.143) suggest the following analogy between the two systems:

Displacement function	$y(t)$	$\leftrightarrow$	charge	$Q(t)$
Velocity	$y'(t)$	$\leftrightarrow$	current	$I(t)$
Driving force	$f(t)$	$\leftrightarrow$	e.m.f.	$E(t)$
Mass	m	$\leftrightarrow$	inductance	L
Damping force	c	$\leftrightarrow$	resistance	R
Spring modulus	k	$\leftrightarrow$	reciprocal of the capacitance	$1/C$,

Further the Eq. (11.141) can be written as

$$L\,I''(t) + RI'(t) + \frac{1}{C}I = \frac{dE}{dt}. \quad ...(11.144)$$

This equation is used more often since in most practical problems we need to find current $I(t)$ instead of the charge $Q(t)$. Further an RLC-circuit reduces to an RL-circuit in the absence of a capacitor and to an RC-circuit in the absence of an inductor. Generally the e.m.f. is of the form of harmonic oscillations $E_0 \cos wt$ or $E_0 \sin wt$.

Next we consider the following different cases.

11.11.2 Free Oscillations in an Electric Circuit

LC-Circuit. Consider an electrical circuit containing a condenser of capacity C and an inductance L as shown in Fig. 11.10. In case no e.m.f. is employed in the circuit, then Eq. (11.142) gives

$$LQ''(t) + \frac{1}{C}Q(t) = 0 \text{ or, } Q''(t) + w^2Q(t) = 0 \quad ...(11.145)$$

where $w^2 = \dfrac{1}{LC}$.

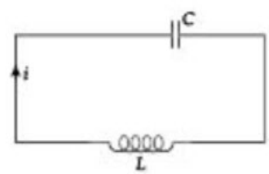


Fig. 11.10

This equation is similar to the Eq. (11.110) in case of mass-spring system, simply replacing the displacement $y(t)$ by the charge $Q(t)$. It represents free electrical oscillations. Thus, the discharging of a condenser through an inductance L is the same as the motion of an object of mass m suspended at the end of a spring.

LCR-circuit. Next consider the discharge of a condenser C through an inductance L and the resistance R as shown in Fig. 11.11.

In case no e.m.f. is employed, then Eq. (11.142) gives

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = 0. \quad ...(11.146)$$

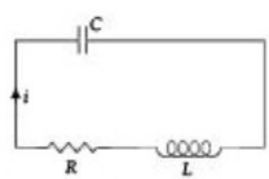


Fig. 11.11

The equation is similar to the Eq. (11.112) of free motions of a mass-spring system when the system is damped and hence has the same solution as for the object of mass m suspended at the end of a spring with spring modulus k and damping c.

Similarly, in case of *LC-circuit with e.m.f.* $E_0 \cos wt$, the equation giving the charge $Q(t)$ at time t is

$$LQ''(t) + \frac{1}{C}Q(t) = E_0 \cos wt. \quad ...(11.147)$$

It is the same as Eq. (11.126), a case of undamped forced motion of mass-spring system and, therefore, has the same solution as for the motion of a mass m suspended at the end of a spring with a driving force $E_0 \cos wt$ acting on it.

Also in the case of *LCR-circuit with e.m.f.* $E_0 \cos wt$, equation giving the charge $Q(t)$ at time t

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E_0 \cos wt \qquad ...(11.148)$$

is the same as Eq. (11.121), a case of damped forced motion as of mass-spring system and hence can be solved accordingly.

Example 11.49: For the circuit as shown in Fig. 11.12, find the charge $Q(t)$ and the current $I(t)$ at time t, if at time $t = 0$ the current is zero and the charge on the capacitor is $1/1000$ coulomb.

Solution: If $Q(t)$ is the charge on the capacitor for $t > 0$, then by Kirchoff's law,

$$10\,Q'' + 120Q' + 1000Q = 17 \sin 2t. \qquad ...(11.149)$$

The C. F. is $Q_c(t) = e^{-6t}[c_1 \cos 8t + c_2 \sin 8t]$.

The P.I. is,

$$Q_p(t) = \frac{1}{10D^2 + 120D + 1000}(17 \sin 2t) = 17\frac{1}{120D + 960}\sin 2t$$

$$= -\frac{1}{480}(D - 8)\sin 2t = -\frac{1}{240}(\cos 2t - 4\sin 2t)$$

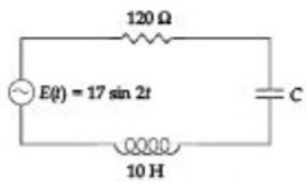


Fig. 11.12

Hence the solution of Eq. (11.149) is

$$Q(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) - \frac{1}{240}(\cos 2t - 4 \sin 2t).$$

Using the initial conditions, $Q(0) = \frac{1}{2000}$ and $Q'(0) = 0$, we obtain $c_1 = \frac{7}{1500}$, $c_2 = \frac{-1}{1500}$, and hence the charge is

$$Q(t) = \frac{1}{1500}e^{-6t}[7\cos 8t - \sin 8t] - \frac{1}{240}[\cos 2t - 4\sin 2t]. \qquad ...(11.150)$$

The current, $I(t) = Q'(t) = -\frac{1}{30}e^{-6t}[\cos(8t) + \sin(8t)] + \frac{1}{120}[4\cos 2t + \sin 2t]$. ...(11.151)

We observe that the current as given by (11.151) is a sum of a transient part, $-\frac{1}{30}e^{-6t}[\cos(8t) + \sin(8t)]$, which decays to zero as t increase, and a steady-part, $\frac{1}{120}[4\cos 2t + \sin 2t]$, which is periodic.

Thus, after a long time, the output will be harmonic oscillation given by the steady part.

Example 11.50: Determine the current $I(t)$ in an RLC-circuit with e.m.f. $E(t) = E_0 \sin wt$, in case the circuit is tuned to resonance so that $w^2 = 1/LC$ and R/L is so small that second and higher order terms can be rejected. Assuming that at $t = 0$, $I(0) = I'(0) = 0$.

Solution: The differential equation giving the current in an RLC-circuit with e.m.f. $E_0 \sin wt$ is

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=\frac{dE}{dt}=wE_0\cos wt. \quad ...(11.152)$$

The roots of the auxiliary equation $L\lambda^2+R\lambda+\frac{1}{C}=0$, are

$$\lambda=-\frac{R}{2L}\pm\frac{1}{2L}\sqrt{R^2-\frac{4L}{C}}=-\frac{R}{2L}\pm\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\approx-\frac{R}{2L}\pm i\frac{1}{\sqrt{LC}}=-\frac{R}{2L}\pm iw.$$

Hence, the C.F. is $I_c(t)=e^{-\frac{R}{2L}t}(c_1\cos wt+c_2\sin wt)\simeq\left(1-\frac{R}{2L}t\right)(c_1\cos wt+c_2\sin wt)$,

neglecting terms of second and higher order in $\frac{R}{L}$.

The particular integral is

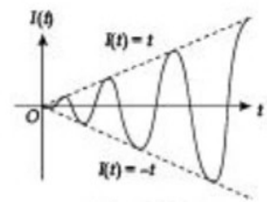


Fig. 11.13

$$I_p(t)=\frac{1}{LD^2+RD+\frac{1}{C}}wE_0\cos wt=\frac{wE_0}{-Lw^2+RD+\frac{1}{C}}\cos wt$$

$$=\frac{wE_0}{RD}\cos wt,\text{ since } w^2=\frac{1}{LC}$$

$$=\frac{wE_0}{R}\int\cos wt\,dt=\frac{E_0}{R}\sin wt.$$

Hence the general solution of Eq. (11.152) is

$$I(t)=\left(1-\frac{R}{2L}t\right)(c_1\cos wt+c_2\sin wt)+\frac{E_0}{R}\sin wt. \quad ...(11.153)$$

The initial conditions $I(0)=0$, and $I'(0)=0$ gives $c_1=0$, and $c_2=-\frac{E_0}{R}$. Substituting in (11.153), we obtain $I(t)=\frac{E_0}{2L}t\sin wt$, which is t times the harmonic oscillation $\frac{E_0}{2L}\sin wt$, which causes the current to increase indefinitely as t increases as shown in Fig. 11.13 and hence resonance occurs.

EXERCISE 11.9

1. Show that the frequency of free vibrations in a closed electrical circuit with inductance L and capacitance C in series is $\frac{30}{\pi\sqrt{LC}}$ per minute.

2. An e.m.f. $E_0 \sin wt$ is applied at $t = 0$ to a circuit containing a capacitance C and inductance L. If $w^2 = 1/LC$ and initially the current I and the charge Q are zero, then show that the current at time t is $(E_0 t/2L) \sin wt$.
3. A circuit consists of an inductance of 0.05 H, a resistance of 5 Ω and a condensor of 4×10^{-4} C. If $Q(0) = I(0) = 0$, find $Q(t)$ and $I(t)$ when, (a) there is a constant e.m.f. of 110 V, (b) there is an alternating e.m.f. of 200 cos 100t; also find the study state solution in this case.
4. Find the current $I(t)$ in the LC circuit, assuming zero initial current and charge, with the following data:

   (a) $L = 10$ H, $C = 0.1$ F, $E = 10t$ V

   (b) $L = 2$ H, $C = 0.005$ F, $E = 220 \sin 4t$ V
5. What RLC circuit with $L = 1$ H is the analog of the mass-spring system with mass 2 kg, damping constant 20 kg/sec, spring constant 58 kg/sec^2, and driving force 110 cos 5t N?
6. Find the current in the RLC circuit, assuming zero initial current and capacitor charge, with the following data:

   (a) $R = 400\,\Omega$, $L = 0.12$ H, $C = 0.04$ F, $E(t) = 120 \sin 2t$ V

   (b) $R = 450\,\Omega$, $L = 0.95$ H, $C = 0.007$ F, $E(t) = e^{-t} \sin^2 3t$ V

11.12 MODELLING: BENDING OF ELASTIC BEAMS

In this section we consider an application of fourth-order differential equation in case of bending of an elastic beam such as that of wood or iron girder in a building or a bridge.

Consider a beam B of uniform elastic material with length L and of constant, say rectangular cross-section. Let the beam B is subjected to a uniform load in a vertical plane through the x-axis, the axis of symmetry, and as a result the beam is bent and its axis is curved into the elastic curve $y = y(x)$, as shown in Fig. 11.13.

Using the theory of elasticity it can be shown that the bending moment $M(x)$ is proportional to the curvature of the curve $y = y(x)$, that is,

$$M(x) = EI \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}, \qquad \text{...(11.154)}$$

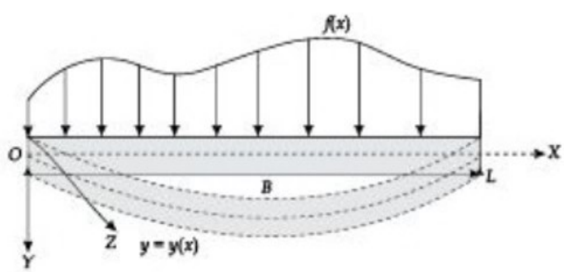


Fig. 11.14

where EI is the constant of proportionality called the *flexural rigidity of the beam*. Here E is the Young's modulus of the beam material and I is the moment of inertia of cross-section about the z-axis.

If $f(x)$ is the load per unit length acting along the beam creating a total load of $\int_a^b f(x)\,dx$ on the segment $a \le x \le b$, then it can be shown that

$$\frac{d^2M}{dx^2} = f(x), \qquad ...(11.155)$$

and hence, from (11.154)

$$\frac{d^2}{dx^2}\left\{\frac{EI\,d^2y/dx^2}{[1+(dy/dx)^2]^{3/2}}\right\} = f(x). \qquad ...(11.156)$$

In case of small bending the term $\left(\frac{dy}{dx}\right)^2$ can be neglected, and hence, (11.155) reduces to

$$EI\frac{d^4y}{dx^4} = f(x). \qquad ...(11.157)$$

It is the *equation of deflection for the uniform elastic beam* under the load density of $f(x)$ units.

Also in case of small bending, from (11.154), the *bending moment* is

$$M(x) = EI\frac{d^2y}{dx^2}. \qquad ...(11.158)$$

Differentiating, we get

$$\frac{dM}{dx} = EI\frac{d^3y}{dx^3}, \qquad ...(11.159)$$

as the *shear force*, and differentiating it again, we get

$$\frac{d^2M}{dx^2} = EI\frac{d^4y}{dx^4}, \qquad ...(11.160)$$

as the *intensity of loading* of the beam

Next we mention a few important supports and the corresponding boundary conditions:

(a) *Simply supported ends*

In this case there is no displacement and no bending moment at the points $x = 0$ and $x = L$.

Thus, $y(0) = y(L) = 0$, and $y''(0) = y''(L) = 0$.

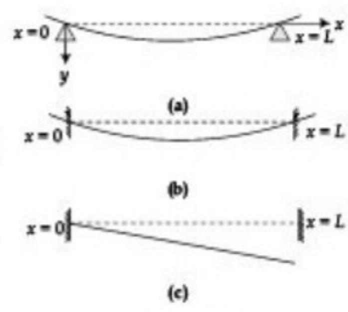


Fig. 11.15

(b) *Clamped at both ends*

In this case there is no deflection and no slope at the points $x = 0$ and $x = L$. Thus,

$$y(0) = y(L) = 0 \quad \text{and} \quad y'(0) = y'(L) = 0.$$

(c) *Clamped at $x = 0$ and free at $x = L$ (Cantilever)*

In this case there is displacement and no slope at the point $x = 0$ and no bending moment and no shear force at the point $x = L$. Thus,

$y(0) = y'(0) = 0$, and $y''(L) = y'''(L) = 0$.

As another example of the applications of higher order equations, the differential equation

$$EI\frac{d^4y}{dx^4} + ky = f(x) \qquad \text{...(11.161)}$$

governs the deflection $y(x)$ of a beam that rests upon an elastic foundation under a load density of $f(x)$ units, as shown in Fig. 11.16.

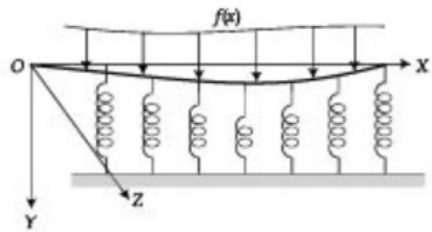


Fig. 11.16

Here, E is the Young's modulus of the beam material, I is moment of inertia of cross-section of the beam about the z-axis, and k is the *spring modulus* of the elastic foundation, called the *foundation modulus*.

Example 11.51: An elastic beam of length π, simply supported at two ends, is uniformly loaded with the load density $f(x) = c \sin x$, $0 \le x \le \pi$, $c > 0$ is a constant. Find an expression for the deflection of the beam and also show that the maximum deflection occurs in the middle of the beam.

Solution: Let E be the Young's modulus of the material of the beam and I be the moment of inertia of the cross-section about the z-axis and let the downward deflection be taken as positive, as shown in Fig. 11.17.

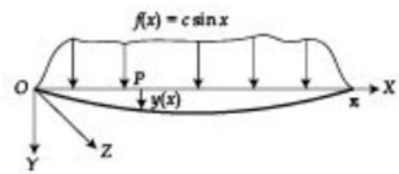


Fig. 11.17

If $y(x)$ is the deflection at a point P, x distance from the origin, then we have

$$EI\frac{d^4y}{dx^4} = c \sin x. \qquad \text{...(11.162)}$$

Since there is no deflection or bending moment at the points $x = 0, \pi$, thus $y(0) = y(\pi) = 0$, $y''(0) = y''(\pi) = 0$.

Rewriting (11.162) as, $y^{iv}(x) = k \sin x$, where $k = \dfrac{c}{EI}$.

Integrating twice, $y''(x) = -k \sin x + c_1 x + c_2$

Applying $y''(0) = y''(\pi) = 0$, we obtain $c_1 = c_2 = 0$. Thus, $y''(x) = -k \sin x$.

Again integrating twice, $y(x) = k \sin x + c_3 x + c_4$.

Applying $y(0) = y(\pi) = 0$, we obtain $c_3 = c_4 = 0$. Thus, $y(x) = k \sin x$.

Obviously the maximum deflection occurs at the point $x = \pi/2$, and it is $y_{max} = k = \frac{c}{EI}$.

Example 11.52: The deflection y of a beam of length L with one end built in and other end subjected to the end thrust P satisfies the equation $\frac{d^2y}{dx^2} + a^2y = \frac{a^2R}{P}(L - x)$.

Find the equation of the deflection curve, when y is the deflection of the beam at a distance x from the built-in-end, as shown in Fig. 11.18.

Solution: Since the end at $x = 0$ is built in, thus $y(0) = y'(0) = 0$

Hence, the initial value problem is

$$(D^2 + a^2)y = \frac{a^2R}{P}(L - x),\ y(0) = y'(0) = 0. \quad ...(11.163)$$

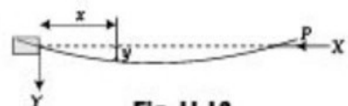


Fig. 11.18

Its complementary function is, $y_c = c_1 \cos ax + c_2 \sin ax$, and particular integral is,

$$y_p = \frac{1}{D^2 + a^2}\frac{a^2R}{P}(L - x) = \frac{R}{P}\left[1 + \frac{D^2}{a^2}\right]^{-1}(l - x)$$

$$= \frac{R}{P}\left[1 - \frac{D^2}{a^2} + \ldots\right](l - x) = \frac{R}{P}(l - x).$$

Thus, the general solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{R}{P}(l - x). \quad ...(11.164)$$

Using the initial conditions, $y(0) = y'(0) = 0$, we obtain, $c_1 = \frac{-Rl}{P}$ and $c_2 = \frac{R}{aP}$. Hence, (11.164) gives, $y = \frac{R}{P}\left[\frac{1}{a}\sin ax - l\cos ax + l - x\right]$, as the desired deflection curve.

EXERCISE 11.10

1. An elastic beam of length L simply supported at two ends is uniformly loaded with the load density f_0 = const. Find an expression for the deflection of the beam and show that the maximum deflection is in the middle at $x = L/2$ and it is equal to $5f_0L^4/(16.24\ EI)$, where E and I have usual meanings.
2. A cantilever beam of length L and weighing f_0 lb. per unit length is subjected to a horizontal compressive force P applied at the free end. With origin at the free end and y-axis upwards, differential equation governing the deflection of the beam is $EI\frac{d^2y}{dx^2} + Py = -\frac{f_0x^2}{2}$.

   Find the maximum deflection of the beam.

3. The shape of a strut of length L subjected to an end thrust P and lateral load density f_0 units, when the ends are built-in, is governed by $EI\frac{d^2y}{dx^2} + Py = \frac{f_0x^2}{2} - \frac{f_0Lx}{2} + M, y(0) = y'(0) = 0$, where M is the moment at a fixed end. Find y in terms of x.
4. A long column of length L fixed at one end and hinged at the other end, is under the action of axial load P. If a force F is applied laterally at the hinge to prevent lateral movement, then the equation of deflection is given by $\frac{d^2y}{dx^2} + w^2y = \frac{Fw^2}{P}(L - x)$. Find the equation of the deflection curve.

11.13 MODELLING: APPLICATIONS OF SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

So far we have considered the modelling in the situations where there is a single dependent variable such as the current $I(t)$ in a circuit, the time t is the independent variable. However, many problems, in engineering and science, involve two or more dependent variables of the single independent variable.

For example, an RL-circuit comprising two loops as shown in Fig. 11.19, using Kirchhoff's law, leads to following two simultaneous differential equations in two variables $I_1(t)$ and $I_2(t)$, given by

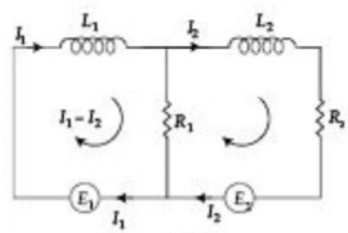


Fig. 11.19

$$L_1I_1' + R_1I_1 - R_1I_2 = E_1,\ L_2I_2' - R_1I_1 + (R_1 + R_2)\,I_2 = E_2$$

As another example, in the combustion of the fossil fuels there are many interacting chemical species whose generation and demise, as a function of time, are governed by a large set of differential equations. The mathematical formulation of such problems results into a system of simultaneous differential equations equal in number to the number of dependent variables. In the examples to follow, we shall elaborate the applications of simultaneous differential equations in modelling of mixing problems, mass-spring system and induction coils, etc.

Example 11.53: Initially a tank A contains 100 liters of brine and 50 kg of salt and another tank B contains 50 litres of fresh water. Fresh water runs in tank A, brine runs out from tank A in tank B and is taken out from tank B at a uniform rate of 2 litres per minute. If both the tanks A and B are kept stirring uniformly, find the amounts of salt present in tanks at time t.

Solution: Let x and y be the amounts of salt present in kg. in tanks A and B respectively, at time t. Under the given rates the volumes of brine solution in tanks A and B remains unchanged.

Thus for tank A, $\frac{dx}{dt} = -\frac{2x}{100} = -\frac{x}{50}$, which gives

$$(50D+1)x = 0, \quad D = \frac{d}{dt} \qquad ...(11.165)$$

For tank B, $\frac{dy}{dt} = \frac{2x}{100} - \frac{2y}{50} = \frac{x}{50} - \frac{y}{25}$, which gives

$$-x + (50D+2)y = 0. \qquad ...(11.166)$$

Solving (11.165) and (11.166), we obtain $x = c_1 e^{-t/50}$, and $y = c_1 e^{-t/50} + c_2 e^{-t/25}$.

Using the initial conditions $x(0) = 50$ and $y(0) = 0$, we obtain $c_1 = 50$ and $c_2 = -50$. Thus,

$x = 50e^{-t/50}$, and $y = 50\,(e^{-t/50} - e^{-t/25})$ are the desired amounts.

Example 11.54: Two bodies each of mass 10 gm are suspended from two springs of same spring modulus $\frac{1}{10}$ gm/cm as shown in Fig. 11.20. After the system attains its static equilibrium, the lower mass is pulled 5 cm downwards and released. Discuss their motion assuming the mass of springs to be negligible.

Solution: Let $x(t)$ and $y(t)$ denote the displacement of the upper and lower masses at time t, from their respective positions of state equilibrium. Thus, elongation for the upper spring is x and for the lower spring is $(y-x)$; and hence the restoring force acting on the upper mass $= -kx + k(y-k) = k(y-2x)$, and the restoring force for the lower mass $= -k(y-x)$, where k is the common spring modulus. The equations of motion are, therefore,

$$mx'' = k(y-2x) \quad \text{and} \quad my'' = -k(y-x)$$

or,

$$(mD^2 + 2k)x - ky = 0, \quad \text{and} \quad -kx + (mD^2 + k)\,y = 0.$$

Here, $m = 10$ $k = \frac{1}{10}$, the equations become

$$(100\,D^2 + 2)x - y = 0 \qquad ...(11.167)$$

$$-x + (100D^2 + 1)y = 0 \qquad ...(11.168)$$

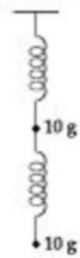


Fig. 11.20

Operating (11.167) by $(100D^2+1)$ and adding to (11.168) we obtain

$$(1000D^4 + 300D^2 + 1)\,x = 0.$$

Auxiliary equation is $1000\lambda^4 + 300\lambda^2 + 1 = 0$, which gives, $\lambda = \pm 0.162i, \pm 0.0618i$ as its roots, and hence

$$x(t) = c_1\cos(0.162t) + c_2\sin(0.162t) + c_3\cos(0.0618t) + c_4\sin(0.0618t). \qquad ...(11.169)$$

Using this in (11.167) and simplifying, we obtain

$$y(t) = -0.624\,[c_1\cos(0.162t) + c_2\sin(0.162t)] + 1.618\,[c_3\cos(0.0618t) + c_4\sin(0.0618t)]. \qquad ...(11.170)$$

Using the initial conditions $x(0) = y(0) = 5$ and $x'(0) = y'(0) = 0$, from (11.167) and (11.168), we obtain

$$c_1 + c_3 = 5, \; -0.624\,c_1 + 1.618\,c_3 = 5, \; 0.162c_2 + 0.0618c_4 = 0, \text{ and, } -0.101c_2 + .0999c_4 = 0.$$

Solving for c_1, c_2, c_3 and c_4, we have $c_1 = 1.378$, $c_2 = 0$, $c_3 = 3.622$, and $c_4 = 0$.

Hence, (11.167) and (11.168) give respectively

$x(t) = 1.378\cos(0.162t) + 3.622\cos(0.0618t)$ and $y(t) = -0.859\cos(0.162t) + 5.860\cos(0.0618t)$,
as the desired displacements.

Thus the motion of the spring is a combination of two simple harmonic motions of period $2\pi/(0.162) = 38.8$ sec, and $2\pi/(0.0618) = 101.7$ sec.

Example 11.55: The two coils of an induction coil are identical with resistance R, inductance L, mutual inductance M, and a battery with e.m.f. E inserted in the primary coil. Determine the currents in the coils at any instant, assuming that initially there is no current in the either coil.

Solution: Let I_1 and I_2 be the currents flowing through the primary and the secondary coil at any instant as shown in Fig. 11.21.

Then using the Kirchhoff's law, we have

$$(R + LD)\,I_1 + MDI_2 = E \qquad ...(11.171)$$

for the primary coil, and

$$(R + LD)\,I_2 + MDI_1 = 0 \qquad ...(11.172)$$

for the secondary coil.

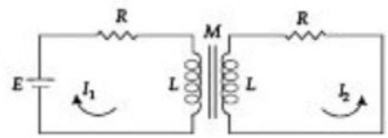


Fig. 11.21

Eliminating I_2 from (11.171) and (11.172), we obtain

$$[(L^2 - M^2)\,D^2 + 2LRD + R^2]\,I_1 = RE. \qquad ...(11.173)$$

The complementary function of (11.173) is

$$I_c = c_1 e^{-\frac{Rt}{L+M}} + c_2 e^{-\frac{Rt}{L-M}}$$

where, c_1 and c_2 are two arbitrary constants.

The particular integral of (11.173) is, $I_p = \dfrac{1}{(L^2 - M^2)D^2 + 2LRD + R^2}\,RE = \dfrac{E}{R}$.

Hence, the general solution for (11.173) is

$$I_1(t) = c_1 e^{-\frac{Rt}{L+M}} + c_2 e^{-\frac{Rt}{L-M}} + \frac{E}{R}. \qquad ...(11.174)$$

From (11.172), $I_2 = -\dfrac{MD}{LD+R} I_1$. Substituting for I_1 from (11.174), we have

$$I_2 = -\frac{MD}{LD+R}\left(c_1 e^{-\frac{Rt}{L+M}}\right) - \frac{MD}{LD+R}\left(c_2 e^{-\frac{Rt}{L+M}}\right) - \frac{MD}{LD+R}\frac{E}{R}$$

$$= -\frac{c_1 M}{L\left(\frac{-R}{L+M}\right)+R} De^{-\frac{R}{L+M}t} - \frac{c_2 M}{L\left(\frac{-R}{L-M}\right)+R} De^{-\frac{R}{L-M}t} - \frac{M}{R} D\left(\frac{E}{R}\right)$$

$$= c_1 e^{-\frac{R}{L-M}} - c_2 e^{-\frac{R}{L-M}}. \qquad ...(11.175)$$

Using the initial conditions $I_1(0) = I_2(0) = 0$ in (11.174) and (11.175), we obtain $c_1 + c_2 = -\frac{E}{R}$, $c_1 - c_2 = 0$, which gives $c_1 = c_2 = -E/2R$. Substituting in (11.174) and (11.175), we obtain

$$I_1(t) = \frac{E}{2R}\left[2 - e^{-\frac{Rt}{L+M}} - e^{-\frac{Rt}{L-M}}\right], \text{ and } I_2(t) = \frac{E}{2R}\left[e^{-\frac{Rt}{L-M}} - e^{-\frac{Rt}{L+M}}\right]$$

as the currents at any instant t.

EXERCISE 11.11

1. Initially a tank T_1 contains 400 litres of brine and 100 kg of salt and another tank T_2 contains 200 litres of fresh water. Brine from tank T_1 runs into tank T_2 at 12 litres per minute and from T_2 to T_1 at 8 litres per minute. If each tank is kept well stirred, find how much salt does tank T_1 contains after 50 minutes?
2. Initially a tank T_1 contains 100 gal. of pure water and another tank T_2 contains 100 gal. of water in which 150 lb. of fertilizer is dissolved. Liquid circulates through the tanks at a constant rate of 2 gal/min, and the mixture is kept uniform by stirring. Prove that the ultimately amount of fertilizer in each tank will be equal.
3. Find the currents $I_1(t)$ and $I_2(t)$ in the network shown in Fig. 11.22, assuming that all charges and currents are zero at $t = 0$.

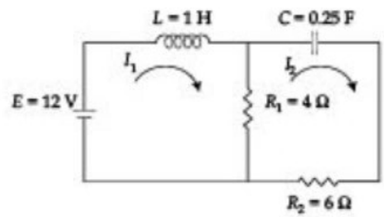


Fig. 11.22

4. The rate of increase of y with respect to x is $4z$ and that of z is $3y$. If y is 1000, and z is 500 when $x = 0$, find the value of x and y when z is 1000.
5. For the network as shown in Fig. 11.23, show that the currents I_1 and I_2 diminish numerically as t increases provided $L_1L_2 > M^2$.

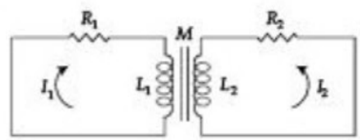


Fig. 11.23

6. The currents I_1 and I_2 in a mesh are given by the differential equations:

$$I'_1 - w_2 I_2 = a\cos pt, \; I'_2 + wI_1 = a\sin pt.$$

Find I_1 and I_2 assuming $I_1(0) = I_2(0) = 0$.

7. Under certain conditions the motion of an electron is given by the equations

$$m\frac{d^2x}{dt^2} + eH\frac{dy}{dt} = eE, \text{ and } m\frac{d^2y}{dt^2} - eH\frac{dx}{dt} = 0.$$

Find the path of the electron, if it started from rest at the origin.

8. Two particles each of mass m are suspended from two vertical springs of same stiffness k. After the system comes to rest the lower mass is pulled downward by 1 metre and then released. Show that the motion is combination of two simple harmonic motions.

9. A system consists of springs A, B, C and two objects D and E attached in a straight line on a frictionless horizontal table, with end of the springs A and B attached to fixed points P and Q as shown in Fig. 11.24. The system is set into vibration by holding the object D in place, moving the object E to the right through a distance $a > 0$ and then releasing the both. Assuming the masses of the two objects to be equal to m and spring coefficients of all the springs to be K, find the two equations of motion for the system, and show that system oscillations are combination of two simple harmonic oscillations.

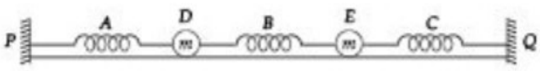


Fig. 11.24

10. Two substances with concentrations $x(t)$ and $y(t)$, react to form a third substance with concentration $z(t)$. The reaction is governed by the system $x' + \alpha x = 0$, $z' = \beta y$ and $x + y + z = \gamma$, where, α, β, γ are known positive constants. Solve for $x(t)$, $y(t)$, $z(t)$ subject to the initial conditions $z(0) = z'(0) = 0$ for the cases (i) $\alpha \neq \beta$ (ii) $\alpha = \beta$.

ANSWERS

Exercise 11.1 (p. 622)

1. $W(x) = 2e^{6x} \neq 0$, forms a basis $y = c_1e + c_2e^{2x} + c_3e^{3x}$
2. $W(x) = 0$, not a basis
3. Forms a basis, $y = c_1x + c_2x \ln x$
4. Forms a basis, $y = (c_1 + c_2x) \cos x + (c_2 + c_3x) \sin x$
5. Forms a basis $y = c_1/x + c_2/x^2$
6. Forms a basis $y = c_1 \cos \ln (1 + x) + c_2 \sin \ln (1 + x)$
7. $y = \cos h\,(2x)$
8. $y = (12/5)e^{-3x} - (7/5)e^{-8x}$
9. $y = 2x^4 - 4x^4 \ln |x|$
10. $y = (1/2)(5x^2 - 1/x^2)$
11. $y = x^{-1/2} + 2x^{3/2}$
12. $y = c_1e^{3x} + c_2e^{-2x}$
13. $y = c_1x + c_2x \ln x$
14. $y = c_1x^3 + c_2x^3 \ln x$
15. $y = c_1x^{-1/2} \cos x + c_2x^{-1/2} \sin x$
16. $y = c_1e^x + c_2$
17. $y = (c_1x + c_2)^{-1}$
18. $y = \dfrac{1}{2}\left[\dfrac{c_1x^2}{2} - \dfrac{1}{c_2} \ln x\right] + c_2$

19. $y = \frac{8\sqrt{2}}{105}(x + c_1)^{7/2} + \frac{1}{2}c_2x^2 + c_3x + c_4$

20. $y = \frac{1}{3}[(2t+4)^{3/2} - 2], \quad y(6) = 62/3, \quad y'(6) = 4$

Exercise 11.2 (p. 631)

1. $y = c_1e^x + c_2e^{-2x}$
2. $y = e^{-x}[c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x]$
3. $y = (c_1 + c_2x)e^{-x/2}$
4. $y = c_1e^{-x} + c_2 \cos 2x + c_3 \sin 2x$
5. $y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^x(c_3 \cos x + c_4 \sin x)$
6. $y = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$
7. $y = (1/6)e^{x/2} - 2e^{-3x}$
8. $y = e^{5x/2}(3 \cos 5x + 2 \sin 5x)$
9. $y = 2(1 - 3x)e^{3x}$
10. $y = \frac{1}{2} + e^{-x}\left(\frac{\sqrt{3}}{6}\sin\sqrt{3x} - \frac{1}{2}\cos\sqrt{3}\,x\right)$
11. $y = \frac{3}{4} + \left(\frac{1}{68}\right)\left[9\sqrt{17}\sinh\left(\frac{\sqrt{17}\,x}{2}\right) + 17\cosh\left(\frac{\sqrt{17}\,x}{2}\right)\right]e^{-x/2}$
12. $y = \left(\frac{1}{3}\right)\cosh(\sqrt{2}\,x) + \frac{2}{3}\cos x$
13. $y = e^{-x}\cos x$
14. $y = [(2e^2 - 1)e^{-6x} - e^{6x}]/(e^2 - 1)$
15. $y = \cos 5x + c \sin 5x$
16. $y = c \sin \pi x$
17. $y = 2 \sin 2x + \sin 3x$
18. $y = (1 + x)e^{-x}\cos x$
19. $y^{(v)} - 12y^{(iv)} + 49y''' - 76y'' + 48y' - 64y = 0$
    $y = (c_1 + c_2x + c_3x^2)e^{4x} + c_4 \cos x + c_5 \sin x$
20. $y = (c_1 + c_2x)e^{-2x}$
21. $y = c_1e^{-x/2} + c_2e^{-3x/2}$
22. $y = c_1e^x + (c_2 + c_3x)e^{-2x}$
23. $y(x) = \sum_{n=1} B_n \sin nx$

Exercise 11.3 (p. 641)

1. $y = c_1 e^{\frac{1}{2}x} + c_2 e^{-3x} - 2$
2. $y = c_1e^{3x} + c_2e^{-2x} - \frac{1}{3}e^x$
3. $y = c_1 e^{\frac{3}{2}x} + c_2 e^{-x} + xe^{\frac{3}{2}x}$
4. $y = c_1 + c_2 \cos\sqrt{5}x + c_3 \sin\sqrt{5}x + (1/18)\cos h\,2x$

5. $y = c_1e^{2x} + c_2e^{-2x} + (1/125)(25x^2 - 60x + 62)e^{3x}$
6. $y = (c_1 + c_2x)e^x + e^x(2\sin x - x\cos x)$ 7. $y = c_1 + c_2e^{-x} + x - (1 + e^{-x})\ln(1 + e^x)$
8. $y = c_1 + c_2e^{-x} + 4x + x^3/3$
9. $y = (c_1\cos ax + c_2\sin ax) + (1/a)x\sin ax + (1/a^2)\cos ax\ln\cos ax$
10. $y = c_1e^x + c_2e^{-x} - (1/2)(x\sin x + \cos x) + xe^x(2x^2 - 3x + 9)/12$
11. $y = c_1e^{2x} + c_2e^{-2x} + (c_3\cos x + c_4\sin x) + x/4\sinh(2x - 1) + 3^x/[(\ln 3)^2 - 4]$
12. $y = (\tan x + c_1)e^{-2x} + c_2e^{-3x}$ 13. $y = \dfrac{k}{n^2 - p^2}\left(\sin pt - \dfrac{p}{n}\sin nt\right)$
14. $y = (4/3)e^{5x} - (10/3)e^{2x} - (1/3)xe^{2x} + 2$ 15. $y = 2e^{-3x/2} - 2e^{2x} + 3e^x(3\sin x - 7\cos x)/29$
16. $y = 3e^{-x} + 5 + 2x + (1/3)x^2$

Exercise 11.4 (p. 646)

1. $y = (c_1 + c_2\ln x)x + \ln x + 2$
2. $y = \dfrac{c_1}{x} + c_2x^3 - (14/27)x^2 - (4/9)x^2\ln x - x^2(\ln x)^{2/3}$
3. $y = c_1x^2 + c_2/x + (1/3)(x^2 - 1/x)\ln x$ 4. $y = c_1/x + c_2(\ln x)/2 - \dfrac{\ln(1 - x)}{x}$
5. $y = c_1/x + \{c_2\cos(\ln x) + c_3\sin(\ln x)\}x + 5x + (10\ln x)/x$
6. $y = c_1x + c_2x\ln x + c_3x(\ln x)^2 + c_4x(\ln x)^3 + \ln x + 4$
7. $y = c_1\cos\{\ln(1 + x) - c_2\} + 2\ln(1 + x)\sin\ln(1 + x)$
8. $y = c_1(3x + 2)^2 + c_2(3x + 2)^{-2} + (1/108)[(3x + 2)^2\ln(3x + 2)]$
9. $y = 4(\ln x - 1)\sqrt{x} + \ln x + 4$ 10. $y = 1/x[2\cos(3\ln x) + 3\sin(3\ln x)] + x^2/2$
11. $u = (kr/8)(a^2 - r^2)$

Exercise 11.5 (p. 653)

1. $y = c_1e^{-x} + c_2xe^{-x} + \dfrac{1}{6}x^3e^{-x}$
2. $y = c_1\cos x + c_2\sin x - x\cos x + \sin x\ln|\sin x|$
3. $y = c_1\cos 4x + c_2\sin 4x + 8\cos 2x - 4\sin 4x\ln|\sec 2x + \tan 2x|$
4. $y = c_1e^{-2x} + c_2e^{-x} + 3e^{-2x}\ln(1 + e^x) + 3e^{-x}\ln(1 + e^x)$
5. $y = c_1\cos x + c_2\sin x - 1 - \cos x + 2\tanh^{-1}[\sin x/(1 + \cos x)]\sin x$
6. $y = c_1e^{-2x}\cos x + c_2e^{-2x}\sin x + (1/4)xe^{-2x}\cos x + (1/4)x^2e^{-2x}\sin x$
7. $y = c_1e^x + c_2e^{2x} + c_3e^{3x} - (1/24)e^{-x}$
8. $y = c_1 + c_2\cos 2x + c_3\sin 2x - (1/4)x\cos 2x + (1/8)\sin 2x\ln|\cos 2x| + (1/8)\ln|\sec 2x + \tan 2x|$

9. $y = c_1x^2 + c_2/x^2 + x^2[8(\ln x)^2 - 4\ln x + 1]$
10. $y = c_1x + c_2x^2 + x^{3/2} - x(1 + \ln x)$
11. $y = c_1x + c_2/x + (x\ln x/2) - x/4$
12. $y = e^{-2x}(c_1\cos 2x + c_2\sin 2x) + 4e^{-2x}\cos 2x \ln|\operatorname{cosec} 2x + \cot 2x| - 4e^{-2x}$
13. $y = c_1x + c_2/x^3 - 4\sqrt{x}/7$
14. $y = c_1(2x^2 - 1) + c_2x(x^2 - 1)^{1/2} + x/3$
15. $y = c_1e^x + (c_2 + xc_3)e^{-x} + 1 - x$
16. $y = e^{-x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x) + 1x^2/2 - x/2 + e^{-x}$
17. $y = c_1e^{-2x} + c_2e^{-3x} + 2e^{-x} + (\sin x - \cos x)/2$
18. $y = c_1e^{3x} + c_2e^{-4x} + xe^{3x}/7$
19. $y = c_1e^x + c_2e^{-x} + e^{2x}(2\cos 3x + \sin 3x)/30 + e^{3x}(\cos 2x + 3\sin 2x)/40.$
20. $y = (c_1x^2 + c_2x + c_3)e^{2x} + 2x^3e^{2x} - e^{-2x}$
21. $y = c_1\cos(2\ln x) + c_2\sin(2\ln x) - [\cos(2\ln x)\ln x]/4$
22. $y = c_1x^{-1} + c_2x^{-1}\ln x + x^2 + 2x + 5$

Exercise 11.6 (p. 657)

1. $x(t) = c_1e^{-5t} + c_2e^t - (6/7)e^{2t};\; y(t) = c_1e^{-5t} + c_2e^t + (8/7)e^{2t}$
2. $x(t) = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t + 1 + (1/2)e^{-t}$
   $y(t) = -c_1e^{-2t}\sin t + c_2e^{-2t}\cos t + 1 - (1/2)e^{-t}$
3. $x(t) = c_2e^{2t} - \frac{2}{5}c_1e^{-3t} - \frac{t}{3} + \frac{5}{18};\; y(t) = c_1e^{-3t} + \frac{t}{3} + \frac{5}{9}.$
4. $x(t) = e^{-at}(c_1\cos at + c_2\sin at);\; y(t) = e^{-at}(-c_1\sin at + c_2\cos at)$
5. $x(t) = -6e^{2t} + 6e^{-t};\quad y(t) = 4e^{2t} - 3e^{-t}$
6. $x(t) = (c_1 + c_2t)e^t + c_3e^{-2t};\; y(t) = 2(c_2 - c_1 - c_2t)e^t + c_2e^{-2t}$
7. $x(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t;\; y(t) = c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t$
8. $x(t) = (1/4)(11\sin t + 1/3\sin 3t);\; y(t) = (1/4)(11\sin t - \sin t)$
9. $x(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t - (t\cos t)/4 + (t\sin t)/4$
   $y(t) = -c_1e^t - c_2e^{-t} + c_3\cos t + c_4\sin t + (2 + t)(\sin t - \cos t)/4$
10. $x(t) = c_1e^{2t} + c_2e^{-t}\cos(\sqrt{3}t - c_3),\; y(t) = c_1e^{2t} + c_2e^{-t}\cos\left(\sqrt{3}t - c_3 + 2\pi/3\right)$

    $z(t) = c_1e^{2t} + c_2e^{-t}\cos\left(\sqrt{3}t - c_3 + 4\pi/3\right)$

Exercise 11.7 (p. 660)

1. $\frac{1}{4}m$
2. $\frac{2}{\pi}\left[\cos^{-1}\frac{3}{4} - \cos^{-1}\frac{1}{4}\right]$

Exercise 11.8 (p. 671)

1. $0.2 \cos 14\, t$, 2.8 m/sec, 0.45 sec

4. $-5e^{-8ct}\left(\dfrac{1}{8\sqrt{1-c^2}}\right)\sin 8\sqrt{1-c^2}\, t$, $\dfrac{5}{8\sqrt{1-c^2}}e^{-8ct}$, $\dfrac{\pi}{4\sqrt{1-c^2}}$, $5e^{-8\sqrt{1-c^2}\,t}$,

   overdamped, damped, underdamped for $c^2 \gtreqless 1$.

6. $\dfrac{A}{2\ln w_0} t \sin (w_0 t)$

9. 0.45 sec; 1.15 sec

10. $0.8(2 \sin 4t - \cos 4t)$

Exercise 11.9 (p. 675)

3. (a) $Q(t) = -e^{-50t}\left(\dfrac{11}{250}\cos 50\sqrt{19}t + \dfrac{11\sqrt{19}}{4750}\sin 50\sqrt{19}\right) + \dfrac{11}{250}$,

   $$I(t) = \frac{44}{\sqrt{19}} e^{-50t} \sin 50\sqrt{19}t$$

   (b) $Q(t) = -e^{-50t}\left[\dfrac{16}{170}\cos wt + \dfrac{12\sqrt{19}}{1615}\sin wt\right] + \dfrac{4}{170}[4 \cos 100\, t + \sin 100\, t]$

   $I(t) = -40e^{-50t} + \left(\dfrac{1}{17}\cos wt - \dfrac{410\sqrt{19}}{323}\sin wt\right) + \dfrac{40}{17}[\cos 100\, t - 4 \sin 100\, t]$; $w = 50\sqrt{19}$

   The steady solutions:

   $Q(t) = \dfrac{4}{170}[4 \cos 100\, t + \sin 100\, t]$, $I(t) = \dfrac{40}{17}[\cos 100\, t - 4 \sin 100\, t]$

4. (a) $1 - \cos t$ (b) $\dfrac{110}{21}(\cos 4t - \cos 10t)$

5. $R = 10\,\Omega$, $C = 1/29$ F, $E = 11 \sin 5t$

6. (a) $I(t) = .015\, e^{-0.0625t} - 5.4 \times 10^{-7} e^{-3333.27t} + 0.015 \cos 20\, t - 0.00043 \sin 20\, t$

   (b) $I(t) = 0.001633e^{-t} + 0.00161e^{-0.3177t} + 0.0000\ 23e^{-t} \cos 6t - 0.000183e^{-t} \sin 6t$

Exercise 11.10 (p. 679)

1. $y = \dfrac{f_0}{24EI}(x^4 - 2Lx^3 + L^3x)$

2. $y_{max} = \frac{f_0}{2Pw^2}\left[1 - \frac{L^2w^2}{2} - \sec wL + wl \tan wL\right]$

3. $y = \frac{f_0 l}{2Pw} \operatorname{cosec} \frac{wL}{2} \cos\left(wx - \frac{wL}{2}\right) - \frac{f_0 l}{2wP} \cot \frac{wL}{2} + \frac{f_0}{2P}(x^2 - Lx),\ w = P/EI$

4. $y = \frac{F}{P}(w \sin wx - L \cos wx + L - x)$

Exercise 11.11 (p. 683)

1. 34.32 kg
3. $I_1(t) = -8e^{-2t} + 5e^{-0.8t} + 3, \quad I_2(t) = -4e^{-2t} + 4e^{-0.8t}$
4. $x = 1408, \quad y = 1415$
6. $I_1 = \frac{a}{p+w} \sin pt, \quad I_2 = \frac{a}{p+w} \cos pt$
7. $x = \frac{E}{Hw}(1 - \cos wt), \quad y = \frac{E}{Hw}(wt - \sin wt),\ w = \frac{eH}{m}$
9. $m\frac{d^2x}{dt^2} + 2kx - ky = 0, \quad m\frac{d^2y}{dt^2} + 2ky - kx = 0$

   Motion is sum of two S.H.M. with period $2\pi\sqrt{\frac{m}{K}}$ and $\frac{2\pi}{\sqrt{3}}\sqrt{\frac{m}{K}}$.

12 Series Solutions of Differential Equations and Special Functions

CHAPTER

A series solution is a standard strategy for solving an initial value problem with variable coefficients when the standard methods do not yield solution in the closed form. The series forms are sometimes more informative about the behaviour of the solution. The methods are broadly classified as power series method and general series method. Equations like that of Legendre and Bessel and their solutions play a basic role in applied mathematics. The study of their solutions constitutes the theory of special functions.

12.1 INTRODUCTION

We have seen in the preceding chapter that a homogeneous linear differential equation with constant coefficients can be solved by algebraic methods and its solutions are elementary functions from calculus. Also some special types of linear differential equations with variable coefficients can be reduced to equations with constant coefficients by applying change of variables and can be solved. However, many differential equations arising in practical applications are linear with variable coefficients but can't be reduced to equations with constant coefficients, and thus are need to be solved by some other methods. For example, Legendre and Bessel equations are two such important equations. Their solutions are also not usual functions from calculus but are *special functions* having applications in engineering and applied mathematics. The study of these solutions is called the *theory of special functions*. The power series method is the standard strategy to solve linear differential equations with variable coefficients. Such a solution being explicit one, giving $y(x)$ as an infinite series in x, may also give important information about the nature of the solution like passing through origin, even or odd, increasing or decreasing on a given interval, etc.

We begin with power series solutions for differential equations admitting such solutions, and then will study an extension of it, called the *Frobenius method* for the general problems which do not admit power series solutions about a particular point.

12.2 POWER SERIES SOLUTIONS

A *power series* in x about a point x_0 is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots$$

where $a_0, a_1, a_2, \ldots$ are constants, called the *coefficients* of the series and x_0 is a constant, called the *center* of the series. In particular, if x_0 is zero, then the power series is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

If a function $f(x)$ has a power series representation in some open interval about x_0, that is, if $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, for $x \in (x_0 - h, x_0 + h)$, $h > 0$, then $f(x)$ is said to be analytic at $x = x_0$.

For example, $f(x) = \dfrac{1}{1-x}$ is analytic at $x = 0$, since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $-1 < x < 1$.

Similarly $\sin x$ is analytic at $x = 0$, since $\sin x = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} x^{2n+1}$, for all x.

A necessary condition for a function $f(x)$ to be analytic at a point $x = x_0$ is that $f(x)$ is infinitely differentiable at $x = x_0$.

12.2.1 Power Series Solution of First Order Initial Value Problems

Consider the first-order initial value differential equation of the form

$$y' + p(x)y = r(x);\ y(x_0) = y_0 \qquad \ldots(12.1)$$

where the functions $p(x)$ and $r(x)$ are analytic at x_0.

We state without proof that *an initial value problem whose coefficients are analytic at x_0 has an analytic solution at x_0.*

Thus, we say that the initial value problem (12.1) has the expanded solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n, \quad \text{where } a_n = \frac{1}{n!} y^{(n)}(x_0). \qquad \ldots(12.2)$$

We illustrate this approach by considering the following examples.

Example 12.1: Find the power series solution of

$$y' + (1+x^2)y = \sin x;\quad y(0) = 1. \qquad \ldots(12.3)$$

Solution: The coefficients of the initial value problem are analytic at the point $x = 0$, hence the solution is of the form

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)}(0)x^n = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \ldots \qquad ...(12.4)$$

Initial condition is $y(0) = 1$. Put $x = 0$ in Eq. (12.3) to obtain

$$y'(0) + y(0) = 0, \text{ or } \quad y'(0) = -y(0) = -1.$$

To determine $y''(0)$, differentiate Eq. (12.3) to obtain

$$y'' + (1 + x^2)y' + 2xy = \cos x. \qquad ...(12.5)$$

Put $x = 0$, $y''(0) + y'(0) = 1$, or $\quad y''(0) = 1 - y'(0) = 2$.

To determine $y'''(0)$, differentiate Eq. (12.5) to obtain

$$y''' + (1 + x^2)y'' + 4xy' + 2y = -\sin x.$$

Put $x = 0$, $y'''(0) + y''(0) + 2y(0) = 0$, or $\quad y'''(0) = -(y''(0) + 2(y(0)) = -4$, and so on.

Substituting for $y(0)$, $y'(0)$, $y''(0)$ etc. in (12.4), we obtain

$$y(x) = 1 - x + x^2 - \frac{2}{3}x^3 + \ldots$$

as the required series solution.

Example 12.2: Find the power series solution of

$$y' + (\sin x)\, y = 1 - x; \quad y(\pi) = -3. \qquad ...(12.6)$$

Solution: The coefficients $\sin x$ and $1 - x$ are analytic for all x but, since the initial condition is at $x = \pi$, we obtain the power series solution about $x = \pi$, of the form

$$y(x) = y(\pi) + y'(\pi)(x - \pi) + \frac{y''(\pi)}{2!}(x - \pi)^2 + \frac{y'''(\pi)}{3!}(x - \pi)^3 + \ldots \qquad ...(12.7)$$

Initial condition is $y(\pi) = -3$. Put $x = \pi$ in Eq. (12.6) to obtain $y'(\pi) = 1 - \pi$.

To determine $y''(\pi)$, differentiate Eq. (12.6) to obtain

$$y'' + (\sin x)y' + (\cos x)y = -1. \qquad ...(12.8)$$

Put $x = \pi$, $y''(\pi) - y(\pi) = -1$, or $\quad y''(\pi) = y(\pi) - 1 = -4$.

To determine $y'''(\pi)$, differentiate (12.8) to obtain

$$y''' + \sin xy'' + 2\cos xy' - \sin xy = 0.$$

Put $x = \pi$, $y'''(\pi) - 2y'(\pi)$, or $\quad y'''(\pi) = 2y'(\pi) = 2(1 - \pi)$.

Substituting for $y(\pi)$, $y'(\pi)$, $y''(\pi)$ etc. in (12.7), we obtain

$$y(x) = -3 + (1 - \pi)\,(x - \pi) - 2(x - \pi)^2 + \frac{(1 - \pi)}{3}(x - \pi)^3 + \ldots$$

as the required series solution.

12.2.2 Power Series Solution for Second and Higher Order Equations

The method for obtaining series solution of a first order initial value problem can be extended to second and higher order equations. For a second order differential equation of the form

$$y'' + p(x)y' + q(x)y = f(x); \quad y(x_0) = y_0, \quad y'(x_0) = y_1. \qquad ...(12.9)$$

We have the following result:

If $p(x)$, $q(x)$ and $f(x)$ are analytic at x_0, then the initial value problem (12.9) has a unique solution which is also analytic at x_0.

Example 12.3: Find the power series solution of the initial value problem

$$y'' - e^x y' + 2y = 1; \quad y(0) = -3, y'(0) = 1. \qquad ...(12.10)$$

Solution: The coefficients e^x, 2 and 1 are analytic at $x = 0$; let the solution be

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + ... \qquad ...(12.11)$$

Initial conditions are $y(0) = -3, y'(0) = 1.$

Put $x = 0$ in (12.10) to obtain $y''(0) - y'(0) + 2y(0) = 1$, or $y''(0) = 1 + y'(0) - 2y(0) = 8.$

To obtain $y'''(0)$, differentiate (12.10), we have

$$y''' - e^x y'' - (e^x - 2)y' = 0 \qquad ...(12.12)$$

Put $x = 0$, $y'''(0) - y''(0) + y'(0) = 0$, or $y'''(0) = y''(0) - y'(0) = 7.$

To obtain $y^{(iv)}(0)$, differentiate (12.12) we have

$$y^{(iv)} - e^x y''' - 2(e^x - 1)y'' - e^x y' = 0. \qquad ...(12.13)$$

Put $x = 0$, $y^{(iv)}(0) - y'''(0) - y'(0) = 0$, or $y^{(iv)}(0) = y'(0) + y'''(0) = 8.$

Substituting for $y(0), y'(0), y''(0), y'''(0)$ etc. in (12.11), we have

$$y(x) = -3 + x + 4x^2 + \frac{7}{6}x^3 + \frac{1}{3}x^4 + ...$$

as the required series solution.

Example 12.4: Find the power series solution of $y'' + \cos x\, y' + 4y = 2x - 1$ about $x = 0.$

Solution: We rewrite the given differential equation as the initial value problem, say

$$y'' + \cos x\, y' + 4y = 2x - 1; \quad y(0) = a, \quad y'(0) = b, \qquad ...(12.14)$$

with a and b as two arbitrary constants.

The coefficients $\cos x$, 4 and $(2x - 1)$ are analytic for all real values of x. Let the solution be

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + ... \qquad ...(12.15)$$

Initial conditions are $y(0) = a$ and $y'(0) = b.$

To find $y''(0)$, put $x = 0$ in Eq. (12.14), we obtain

$y''(0) + y'(0) + 4y(0) = -1$, or $y''(0) = -1 - 4y(0) - y'(0) = -1 - 4a - b$.

To find $y'''(0)$, differentiate (12.14), we obtain

$$y''' + \cos x y'' + (-\sin x + 4)y' = 2. \quad ...(12.16)$$

Put $x = 0$ to obtain $y'''(0) = 2 - 4y'(0) - y''(0) = 2 - 4b + 1 + 4a + b = 4a - 3b + 3$.

Thus, from (12.15), we obtain

$$y(x) = a + bx - \frac{(1 + 4a + b)}{2!}x^2 + \frac{(3 - 3b + 4a)}{3!}x^3 + ...$$

as the desired general solution.

Example 12.5: Find the power series solution of

$$(1 - x^2)y'' - 2xy' + 2y = 0 \quad ...(12.17)$$

about $x = 0$.

Solution: Let the initial conditions be $y(0) = a$ and $y'(0) = b$, where a and b are two arbitrary constants. Substituting $x = 0$ in (12.17), we have $y''(0) + 2y(0) = 0$, which gives, $y'' = -2y(0) = -2a$.

To find $y'''(0)$ differentiate (12.17) w.r.t. x to obtain

$$(1 - x^2)y''' - 4xy'' = 0. \quad ...(12.18)$$

Put $x = 0$ in (12.18) to obtain $y'''(0) = 0$. To find $y^{iv}(0)$ differentiate (12.18) to obtain

$$(1 - x^2)y^{iv} - 6xy''' - 4y'' = 0. \quad ...(12.19)$$

Put $x = 0$ it gives, $y^{iv}(0) = 4y''(0) = -8a$.

Hence, the power series solution for the given differential equation is

$$y = a + bx - ax^2 - \frac{1}{3}ax^4 + ...$$

where a and b are two arbitrary constants.

In the above example, we observe that coefficients $-2x/(1 - x^2)$ and $2/(1 - x^2)$ of the differential equation (12.17) are analytic at $x = 0$, and therefore the power series solution exists. However, these coefficients are not analytic at $x = \pm 1$. Thus, the power series solution cannot exist in any interval I which contains 1, or –1. Therefore, (–1, 1) is the *interval of convergence* and 1 is the *radius of convergence* for the series solution obtained.

12.2.3 Power Series Solution by Developing the Recurrence Relation

So far we have obtained the coefficients of the power series by differentiating the given differential equation step by step. Practically this procedure suits when we need to find only a few coefficients. Otherwise the coefficients can be obtained in a more systematic way by developing a recurrence relation between them, which allows us to generate coefficients once some preceding coefficients are known. We illustrate the procedure through a few examples given next.

Example 12.6: Find the power series solution of

$$y'' + 2xy' + (1 + x^2)y = 0; \quad y(0) = 3; \quad y'(0) = -1 \qquad ...(12.20)$$

about $x = 0$ by developing the recurrence relation between the coefficients.

Solution: Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad ...(12.21)$$

be the solution of Eq. (12.20). We have $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

Substituting expressions for y, y' and y'' in (12.20), we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + (1 + x^2)\sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\text{or, } \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} + 2\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0. \qquad ...(12.22)$$

We shift the indices in the first and last summation to make the index of x in each series in (12.22) same. For the first summation on left side of Eq. (12.22), replace $(n-2)$ by n

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \text{ becomes } \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Similarly, for the last summation on left side of Eq. (12.22), replace $n+2$ by n

$$\sum_{n=0}^{\infty} a_n x^{n+2} \text{ becomes } \sum_{n=2}^{\infty} a_{n-2}x^n.$$

Thus (12.22) reduces to

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0. \qquad ...(12.23)$$

Rewriting it as

$$(2a_2 + a_0) + (6a_3 + 3a_1)x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + (2n+1)a_n + a_{n-2}]x^n = 0 \qquad ...(12.24)$$

The condition for Eq. (12.24) to hold for all x in some open interval about zero is that the coefficient of each power of x on the right side of it must be zero. Therefore,

$2a_2 + a_0 = 0$, $6a_3 + 3a_1 = 0$, and $(n+2)(n+1)a_{n+2} + (2n+1)a_n + a_{n-2} = 0$, for $n \geq 2$.

From the first two conditions, we obtain $a_2 = -\frac{1}{2}a_0$ and $a_3 = -\frac{1}{2}a_1$.

The third condition is the recurrence relation and holds for $n = 2, 3, 4, \ldots$. Rewriting this as

$$a_{n+2} = -\frac{[(2n+1)a_n + a_{n-2}]}{(n+1)(n+2)} \qquad \ldots(12.25)$$

For $n = 2$, $\quad a_4 = -\frac{(5a_2 + a_0)}{12} = \frac{a_0}{8}$, using $a_2 = -\frac{a_0}{2}$

For $n = 3$, $\quad a_5 = -\frac{(7a_3 + a_1)}{20} = \frac{a_1}{8}$, using $a_3 = -\frac{1}{2}a_1$.

Similarly, $\quad a_6 = -\frac{a_0}{48}$, $a_7 = -\frac{a_1}{48} \ldots$

Substituting for these coefficients in (12.21) and grouping the terms with coefficient a_0 and a_1, we obtain

$$y(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7 + \ldots\right) \qquad \ldots(12.26)$$

Here a_0 and a_1 are arbitrary constants and the functions

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \ldots$$

$$y_2(x) = x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7 + \ldots$$

are two linearly independent solutions of the given differential equation.

To find a_0 and a_1, we use the initial conditions $y(0) = 3$ and $y'(0) = -1$ in (12.26), which gives $a_0 = 3$ and $a_1 = -1$. Hence, the required power series solution is

$$y(x) = 3 - x - \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^4 - \frac{1}{8}x^5 \ldots - \ldots$$

Example 12.7: Find the power series solution for $y'' + xy' - y = e^{3x}$ about $x = 0$.

Solution: Substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$, and $e^{3x} = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n$ in the given equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n. \qquad \ldots(12.27)$$

Shifting the indices in the first summation on left side of Eq. (12.27) by replacing $n-2$ with n, (12.27) becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\frac{3^n}{n!}x^n.$$ Rewriting it as

$$(2a_2 - a_0) + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + (n-1)a_n]x^n = 1 + \sum_{n=1}^{\infty}\frac{3^n}{n!}x^n. \quad ...(12.28)$$

Equating coefficients of like powers of x on both sides of Eq. (12.28), we obtain

$$2a_2 - a_0 = 1, \text{ or } a_2 = \frac{(1+a_0)}{2}, \text{ and, the recurrence relation}$$

$$(n+2)(n+1)a_{n+2} + (n-1)a_n = \frac{3^n}{n!}, \text{ for } n = 1, 2, ...$$

which can be rewritten as

$$a_{n+2} = \frac{(3^n/n!) - (n-1)a_n}{(n+2)(n+1)}, \text{ for } n = 1, 2, ...$$

It gives, $a_3 = \frac{1}{2}$, $a_4 = (9 - 2a_2)/24 = (8 - a_0)/24$ etc.

Hence the desired series solution is

$$y(x) = a_0 + a_1 x + \frac{(1+a_0)}{2}x^2 + \frac{1}{2}x^3 + \left(\frac{1}{3} - \frac{a_0}{24}\right)x^4 + ...$$

where a_0 and a_1 are two arbitrary constants.

EXERCISE 12.1

Find the power series solution of the following initial value problems about the point where the initial conditions are given

1. $y' + e^x y = x^2; \quad y(0) = 4$
2. $y' + (1 + x^2)y = \sin x; \quad y(0) = a$
3. $y' + 4xy = 3e^{x-1}; \quad y(1) = 1$
4. $y' + x(1 - 2x)y = 1; \quad y(0) = 1$
5. $(x+1)y' - (x+2)y = 0; \quad y(-2) = 1$
6. $y'' - xy' + e^x y = 4; \quad y(0) = 1, \quad y'(0) = 4$
7. $y'' - xy = 2x; \quad y(1) = 3, \quad y'(1) = 0$
8. $y'' + \frac{1}{x+2}y' - xy = 0; \quad y(0) = y'(0) = 1$
9. $y'' + \frac{1}{x-1}y' + \frac{1}{x+2}y = 2; \quad y(0) = y'(0) = 3.$
10. $(2 + x^2)y'' - 2xy' + 3y = 0; \quad y(1) = 1, \quad y'(1) = -1.$

Find the Maclaurin's series expansion solution of the following differential equations:

11. $y'' - 2xy' + x^2y = 0;\quad y(0) = a,\quad y'(0) = b.$

12. $y'' + xy' + (1 - x^2)y = x;\quad y(0) = a,\quad y'(0) = b.$

13. $(4 + x^2)y'' - 6xy' + 8y = 0;\quad y(0) = a,\quad y'(0) = b.$

14. $(1 + x^2)y'' + 3xy' + (1 - x^2)y = 0;\quad y(0) = a,\quad y'(0) = b.$

15. $2y'' + 3x^2y' + (1 - x^2)y = 2x;\quad y(0) = a,\quad y'(0) = b.$

Using the power series method, find the solution for the following differential equations about $x = 0$. Find also the recurrence relation between the coefficients.

16. $y'' + w^2y = 0,\quad w > 0$ is constant.

17. $y'' + x^2y = 0$

18. $y'' + x^2y' + 4y = 1 - x^2$

19. $y'' + xy' + (1 + x)y = 0;\quad y(0) = -1,\quad y'(0) = 0$

20. $(1 + 2x)y'' - y' + y = 0;\quad y(0) = 0,\quad y'(0) = 1.$

12.3 LEGENDRE EQUATION. LEGENDRE POLYNOMIALS. FOURIER-LEGENDRE SERIES

In this section, we apply the power series method to solve an important differential equation of applied mathematics, the *Legendre equation* given by

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \qquad \text{...(12.29)}$$

where $\alpha \geq 0$ is a real parameter. This equation arises in a variety of applications, particular in connection with physical problems with spherical symmetry.

The coefficients of the Legendre equation are analytic at the origin and the leading coefficient $(1 - x^2)$ only vanishes at $x = \pm 1$, so a power series solution of the Legendre equation exists about the point $x = 0$ in the interval $-1 < x < 1$. Solutions of Eq. (12.29) are called *Legendre functions*, and are examples of *special functions*. In case $\alpha = n$, a non-negative integer, then one of the solution of Legendre Eq. (12.29) is a polynomial of degree n. These polynomials, multiplied by some suitable constants, are called *Legendre polynomials* which are of great practical importance because of their important properties, particularly orthogonality.

12.3.1 Solution of Legendre Equation

Let the solution be

$$y = \sum_{n=0}^{\infty} a_n x^n. \qquad \text{...(12.30)}$$

Substitute y and its derivatives in Eq. (12.29), we obtain

$$(1 - x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha(\alpha + 1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

This can be rewritten as

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} na_n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

or, $$[2a_2 + \alpha(\alpha+1)a_0] + [6a_3 - 2a_1 + \alpha(\alpha+1)a_1]x$$

$$+ \sum_{n=2}^{\infty}[(n+2)(n+1)\,a_{n+2} - n(n+1)\,a_n + \alpha(\alpha+1)\,a_n]\,x^n = 0. \qquad ...(12.31)$$

Equating the coefficients of constant terms on the left side of (12.31) to zero, we obtain

$$2a_2 + \alpha(\alpha+1)a_0 = 0. \qquad ...(12.32)$$

Similarly, equating the coefficients of terms containing x to zero, we obtain

$$6a_3 - 2a_1 + \alpha(\alpha+1)a_1 = 0. \qquad ...(12.33)$$

Equating the coefficients of the general term x^n, $n \geq 2$ to zero, we have

$$(n+2)(n+1)a_{n+2} + \{\alpha(\alpha+1) - n(n+1)\}a_n = 0. \qquad ...(12.34)$$

From (12.32), $a_2 = \dfrac{-\alpha(\alpha+1)}{2}\,a_0 = \dfrac{-\alpha(\alpha+1)}{2!}\,a_0$

From (12.33), $a_3 = \dfrac{[2-\alpha(\alpha+1)]}{6}\,a_1 = \dfrac{-(\alpha-1)(\alpha+2)}{3!}\,a_1$

From (12.34), we obtain the recurrence relation

$$a_{n+2} = -\frac{(\alpha-n)(\alpha+n+1)}{(n+2)(n+1)}a_n, \quad n \geq 2. \qquad ...(12.35)$$

It gives, $a_4 = -\dfrac{(\alpha-2)(\alpha+3)}{4.3}\,a_2 = \dfrac{(\alpha-2)\alpha(\alpha+1)(\alpha+3)}{4!}\,a_0$

$$a_5 = -\frac{(\alpha-3)(\alpha+4)}{5.4}\,a_3 = \frac{(\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)}{5!}\,a_1$$

$$a_6 = -\frac{(\alpha-4)(\alpha-2)\alpha(\alpha+1)(\alpha+3)(\alpha+5)}{6!}\,a_0, \text{ and so on.}$$

Substituting for these coefficients in Eq. (12.30), we obtain

$$y(x) = a_0\left[1 - \alpha(\alpha+1)\frac{x^2}{2!} + (\alpha-2)\alpha(\alpha+1)(\alpha+3)\frac{x^4}{4!} - \ldots\right]$$

$$+ a_1\left[x - (\alpha-1)(\alpha+2)\frac{x^3}{3!} + (\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)\frac{x^5}{5!} - \ldots\right] = a_0y_1 + a_1y_2(x),$$

where $$y_1(x) = 1 - \alpha(\alpha+1)\frac{x^2}{2!} + (\alpha-2)\alpha(\alpha+1)(\alpha+3)\frac{x^4}{4!} - \ldots \quad \ldots(12.36)$$

and, $$y_2(x) = x - (\alpha-1)(\alpha+2)\frac{x^3}{3!} + (\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)\frac{x^5}{5!} - \ldots \quad \ldots(12.37)$$

The series for $y_1(x)$ and $y_2(x)$ converge for $-1 < x < 1$, and further since $y_1(x)$ contains even powers of x only and $y_2(x)$ contains odd powers of x only, hence these are the two linearly independent solutions of the Legendre Eq. (12.29).

12.3.2 Legendre Polynomials

From the form of the solutions (12.36) and (12.37), we observe that if $\alpha = n$ is even, the series $y_1(x)$ will reduce to a polynomial of degree n in even powers of n, whereas if $\alpha = n$ is odd, the series $y_2(x)$ will reduce to a polynomial of degree n in odd powers of x. For example, for $n = 0$,

$$y_1(x) = 1; \quad n = 2, \quad y_1(x) = 1 - 3x^2; \quad n = 4, \quad y_1(x) = 1 - 10x^2 + \frac{35}{3}x^4,$$

and, for $n = 1$,

$$y_2(x) = x; \quad n = 3, \quad y_2(x) = x - \frac{5}{3}x^3; \quad n = 5, \quad y_2(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5.$$

These polynomials multiplied with suitable constants are called Legendre polynomials and are denoted by $P_n(x)$, where n denotes the degree of the polynomial. The constants are selected such that $P_n(1) = 1$, for $n = 0, 1, 2, \ldots$. Thus, for $\alpha = n$, one of the linearly independent solutions of the Legendre Eq. (12.29) is a Legendre polynomial of degree n and the second independent solution is an infinite series denoted by $Q_n(x)$.

The first five Legendre polynomials obtained by setting the condition $P_n(1) = 1$ are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x), \text{ and } P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

It is a little bit cumbersome to obtain $P_n(x)$ like this. These polynomials are explicitly given by the formula known as *Rodrigue's formula*, proved next.

12.3.3 Rodrigue's Formula

The Rodrigue's formula is

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}[(x^2-1)^n]. \quad \ldots(12.38)$$

To prove it, let $u = (x^2 - 1)^n$. Differentiating with respect to x, we obtain

$$u_1 = 2nx(x^2-1)^{n-1} = \frac{2nxu}{(x^2-1)}, \text{ or } (1-x^2)u_1 + 2nxu = 0, \text{ where } u_1 = \frac{du}{dx}.$$

Therefore, $z = n\pi + (-1)^n\left(\frac{\pi}{2} - 4i\right)$, since $\sin\theta = \sin\alpha$ implies $\theta = n\pi + (-1)^n\alpha$

(b) The equation is, $i = \sinh z = \frac{e^z - e^{-z}}{2}$. It gives $e^{2z} - 2ie^z - 1 = 0$ or, $(e^z - i)^2 = 0$ or, $e^z = i$

or,
$$z = \ln i = Ln 1 + i\left(\frac{\pi}{2} - 2n\pi\right) = i\left(2n + \frac{1}{2}\right)\pi$$

EXERCISE 18.3

1. Verify the following:
   (a) $e^{z_1}e^{z_2} = e^{z_1+z_2}$ (b) $\sin^2 z + \cos^2 z = 1$
   (c) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$ (d) $\cos iz = \cosh z$
   (e) $\sin iz = i \sinh z$
2. Compute
   (a) $\cos(2 + i)$ (b) 1^i (c) $(1 + i)^i$ (d) i^{-i}
3. Find all the values of z which satisfy the following
   (a) $\cos z = i$ (b) $\tan z = 2$
   (c) $\tanh z = -2$ (d) $\cosh z + \sinh z = \alpha$, α is complex constant.
4. Find all solutions of the following equations:
   (a) $e^z + 1 = 0$ (b) $\sin z - 2 = 0$ (c) $\cos z - 1 = 0$
5. If $\cot(\theta + i\phi) = e^{i\alpha}$, then show that
   $$\theta = (2n + 1)\frac{\pi}{4} \text{ and } \phi = \frac{1}{2}\ln\left[\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right], \text{ where } n \text{ is an integer.}$$
6. If $\sin(\alpha + i\beta) = x + iy$, prove that
   $$(x^2/\cosh^2\beta) + (y^2/\sinh^2\beta) = 1, \quad (x^2/\sin^2\alpha) - (y^2/\cos^2\alpha) = 1$$
7. Separate $\ln\cos(x + iy)$ into real and imaginary parts.
8. If $i^{\alpha + i\beta} = \alpha + i\beta$, then prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$.
9. If $(a_1 + ib_1)(a_2 + ib_2) \ldots (a_n + ib_n) = A + iB$, prove that
   (a) $\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right) \ldots \left(a_n^2 + b_n^2\right) = A^2 + B^2$
   (b) $\tan^{-1}\frac{b_1}{a_1} + \tan^{-1}\frac{b_2}{a_2} + \ldots + \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{B}{A}$
10. If $\sin^{-1}(u + iv) = \alpha + i\beta$, prove that $\sin^2\alpha$ and $\cosh^2\beta$ are the roots of the equation $x^2 - (1 + u^2 + v^2)x + u^2 = 0$.

18.5 ANALYTIC FUNCTIONS. CAUCHY-RIEMANN EQUATIONS

In complex analysis we are interested in the functions which are differentiable in some domain, called the analytic functions. A large variety of functions of complex variables which are useful for applications purpose are analytic. In this section we introduce analytic functions and discuss the necessary and sufficient conditions for the analyticity of a function.

18.5.1 Analytic Function

A function $f(z)$ is said to be analytic at a point z_0, if it is differentiable at z_0 and, in addition, it is differentiable throughout some neighbourhood of z_0.

Thus the analyticity at a point z_0 means that the function is analytic in a neighbourhood of z_0. Further a function $f(z)$ is said to be *analytic in a domain D* if $f(z)$ is defined and differentiable at all points of D. In fact, *analyticity is a global property while differentiability is a local property.*

The terms *regular* and *holomorphic* are also used in place of analytic.

We note that the polynomial functions $f(z) = a_0z^n + a_1z^{n-1} + a_2z^{n-2} + ... + a_{n-1}z + a_n$ are differentiable everywhere, and hence analytic everywhere in the complex plane. However, the function $f(z) = |z|^2$ is differentiable only at the point $z = 0$, refer to Example 18.16, hence this function is analytic nowhere.

A function $f(z)$ is said to be analytic at $z = \infty$, if the function $f(1/z)$ is analytic at $z = 0$.

If a function $f(z)$ ceases to be analytic at a point $z = z_0$, then z_0 is called a *singular point* of the function $f(z)$. For example $z = 0$ is a singular point of the function $f(z) = \dfrac{1}{z}$.

Further, if functions $f(z)$ and $g(z)$ are analytic in D, then the functions $f(z) \pm g(z)$, $f(z)g(z)$ are also analytic in D. The rational function $f(z)/g(z)$ is also analytic in D, except at the points where $g(z) = 0$; here we assume that the common factors of f and g have been cancelled. Also, the composition of two analytic functions is also analytic.

Next we discuss the necessary and sufficient conditions for a function to be analytic.

18.5.2 Cauchy-Riemann Equations

Cauchy-Riemann equations provide a criterion for the analyticity of a complex function $f(z) = u(x, y) + iv(x, y)$.

Theorem 18.2: ***(Necessary conditions for a function to be analytic)*** *If $f(z) = u(x, y) + iv(x, y)$ is continuous in some neighbourhood of a point $z = x + iy$ and is differentiable at z, then the first order partial derivatives of $u(x, y)$ and $v(x, y)$ exist and satisfy the Cauchy-Riemann equations*

$$u_x = v_y \text{ and } u_y = -v_x \qquad ...(18.35)$$

at the point $z = x + iy$.

That is, if $f(z)$ is analytic in a domain D, then partial derivatives exist and satisfy the C.R. = ns at all points of D.

Proof. Since $f(z)$ is differentiable at z, we have

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\}\{u(x,y)+iv(x,y)\}}{\Delta x + i\Delta y} \quad ...(18.36)$$

and the limit on the right of (18.36) must be independent of the path along which $\Delta z \to 0$.

First set $\Delta y = 0$ in $\Delta z = \Delta x + i\Delta y$ so that $\Delta z = \Delta x$, that is, Δz tends to zero parallel to x-axis. Thus the limit on the right side of (18.36) becomes

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad ...(18.37)$$

Next, set $\Delta x = 0$ in $\Delta z = \Delta x + i\Delta y$ so that $\Delta z = i\Delta y$, that is, $\Delta z \to 0$ parallel to the y-axis. Then the limit on the right side of (18.36) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad ...(18.38)$$

Since $f(z)$ is differentiable the value of the limits attained along two different paths, refer to Fig. 18.14, in (18.27) and (18.38) must be equal. Therefore

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Comparing the real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at the point $z = (x, y)$.

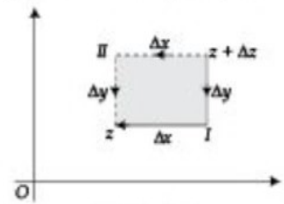


Fig. 18.14

These are known as the *Cauchy-Riemann equations*. Satisfaction of these equations is necessary for differentiability and analyticity of the function $f(z)$ at a given point. Thus, if a function $f(z)$ does not satisfy the Cauchy-Reimann equations at a point, it is not differentiable and hence not analytic at that point. But these conditions are not sufficient since these have been obtained by considering only two possible paths of approach to the point z. For example, consider the function

$$f(z) = \begin{cases} \dfrac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}. \text{ Here, } u = \frac{x^3 - y^3}{x^2 + y^2} \text{ and } v = \frac{x^3 + y^3}{x^2 + y^2}.$$

By definition, $\left.\dfrac{\partial u}{\partial x}\right|_{(0,0)} = \lim\limits_{x \to 0} \dfrac{u(x,0) - u(0,0)}{x} = \lim\limits_{x \to 0} \dfrac{u(x,0)}{x} = \dfrac{x^3}{x^3} = 1.$

Similarly, $\left.\dfrac{\partial u}{\partial y}\right|_{(0,0)} = -1, \quad \left.\dfrac{\partial v}{\partial x}\right|_{(0,0)} = 1, \text{ and } \left.\dfrac{\partial v}{\partial y}\right|_{(0,0)} = 1$

Hence C.R. equations (18.35) are satisfied at (0,0). Now consider

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}.$$

Substituting $y = mx$ and taking $x \to 0$, we have $f'(0) = \dfrac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)}$ which depends upon m and thus $f'(z)$ does not exist at $z = 0$, though C.R. equations are satisfied at (0,0).

Next, we give the sufficient conditions for a function to be analytic.

Theorem 18.3: ***(Sufficient conditions for a function to be analytic)*** *If the real and imaginary parts $u(x, y)$ and $v(x, y)$ of the function $f(z) = u(x, y) + iv(x, y)$ have continuous first order partial derivatives that satisfy the Cauchy-Riemann equations (18.35) at all points in D, then the function $f(z)$ is analytic in D.*

Proof. Let $P(x, y)$ be any fixed point in D and let $Q(x+\Delta x, y+\Delta y)$ be a point in its neighbourhood such that the straight line segment PQ is in D. Then, since u_x, u_y, v_x, v_y exist and are continuous, applying mean value theorem for functions of two variables, we have

$$u(x+\Delta x, y+\Delta y) - u(x, y) = u_x\Delta x + u_y\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

and
$$v(x+\Delta x, y+\Delta y) - v(x, y) = v_x\Delta x + v_y\Delta y + \epsilon_3\Delta x + \epsilon_4\Delta y,$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ and $\epsilon_4 \to 0$ as $\Delta z \to 0$. Thus

$$\begin{aligned} f(z+\Delta z) - f(z) &= \{u(x+\Delta x, y+\Delta y) - u(x, y)\} + i\{v(x+\Delta x, y+\Delta y) - v(x, y)\} \\ &= (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + (\epsilon_1 + i\epsilon_3)\Delta x + (\epsilon_2 + i\epsilon_4)\Delta y \\ &= (u_x + iv_x)\Delta x + (-v_x + iu_x)\Delta y + (\epsilon_1 + i\epsilon_3)\Delta x + (\epsilon_2 + i\epsilon_4)\,\Delta y, \text{ using (18.35)} \\ &= (u_x + iv_x)(\Delta x + i\Delta y) + (\epsilon_1 + i\epsilon_3)\Delta x + (\epsilon_2 + i\epsilon_4)\Delta y \\ &= (u_x + iv_x)\Delta z + w, \text{ where } w = (\epsilon_1 + i\epsilon_3)\Delta x + (\epsilon_2 + i\epsilon_4)\Delta y. \text{ Hence,} \end{aligned}$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = u_x + iv_x + \frac{w}{\Delta z}$$

Now,
$$\left|\frac{w}{\Delta z}\right| \le (|\epsilon_1| + |\epsilon_3|)\left|\frac{\Delta x}{\Delta z}\right| + (|\epsilon_2| + |\epsilon_4|)\left|\frac{\Delta y}{\Delta z}\right|$$

$$\le |\epsilon_1| + |\epsilon_3| + |\epsilon_2| + |\epsilon_4|, \text{ since } \left|\frac{\Delta x}{\Delta z}\right| \le 1 \text{ and } \left|\frac{\Delta y}{\Delta z}\right| \le 1.$$

Thus, $\displaystyle\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = u_x + iv_x$, that is, $f'(z) = u_x + iv_x$

Therefore $f(z)$ is differentiable at an arbitrary point z in D, that is, $f(z)$ is analytic in D. This proves the sufficiency part, and also in case $f(z)$ is analytic, then $f'(z)$ is given by (18.37) or (18.38).

Example 18.25: Using the Cauchy-Riemann equations, show that

(a) $f(z) = z^3$ is analytic everywhere,

(b) $f(z) = |z|^2$ is analytic nowhere,

(c) $f(z) = \sin z$ is analytic everywhere,

(d) $f(z) = 1/z,\ z \neq 0$, is analytic at all points except at the point $z = 0$.

Solution: (a) We have, $f(z) = z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$

Here $u = x^3 - 3xy^2, \quad v = 3x^2y - y^3$

Thus $u_x = 3x^2 - 3y^2, \quad v_x = 6xy, \quad u_y = -6xy, \quad v_y = 3x^2 - 3y^2$

We observe that u_x, u_y, v_x, v_y are continuous everywhere, and $u_x = v_y,\ u_y = -v_x$ at all points. Hence $f(z) = z^3$ is analytic for every z and further

$$f'(z) = u_x + iv_x = 3x^2 - 3y^2 + i6xy = 3[x^2 + (iy)^2 + 2ixy] = 3(x + iy)^2 = 3z^2.$$

(b) We have $f(z) = |z|^2 = z\bar{z} = x^2 + y^2$. Here $u = x^2 + y^2, v = 0$.

Thus $u_x = 2x, v_x = 0,\ u_y = 2y,\ v_y = 0$. We observe that u_x, u_y, v_x, v_y are continuous everywhere. Moreover $u_x = v_y$ is satisfied at all points on $x = 0$, (the imaginary axis) and $u_y = -v_x$ is satisfied at all points on $y = 0$, (the real axis) so both Cauchy-Riemann equations are satisfied only at (0, 0). Thus $f(z) = |z|^2$ is differentiable only at (0, 0) and hence it is analytic nowhere.

(c) We have, $f(z) = \sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

Here $u = \sin x \cosh y, \quad v = \cos x \sinh y$. Thus

$$u_x = \cos x \cosh y, \quad v_x = -\sin x \sinh y, \quad u_y = -\sin x \sinh y, \quad v_y = \cos x \cosh y.$$

We observe that u_x, u_y, v_x, v_y are continuous everywhere and $u_x = v_y, \quad u_y = -v_x$ at all points. Hence $f(z) = \sin z$ is analytic everywhere.

Further $f'(z) = u_x + iv_x = \cos x \cosh y - i \sin x \sinh y$

$$= \cos x \cos(iy) - \sin x \sin(iy) = \cos(x + iy) = \cos z.$$

(d) We have, $$f(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Here $$u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}$$

$$u_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad v_x = \frac{2xy}{(x^2 + y^2)^2}, \quad u_y = \frac{-2xy}{(x^2 + y^2)^2}, \quad v_y = \frac{-(x^2 - y^2)}{(x^2 + y^2)^2}.$$

We observe that u_x, u_y, v_x, v_y are continuous everywhere except at the point $z = 0$, and $u_x = v_y, u_y = -v_x$ at all points except at $z = 0$. Hence $f(z) = 1/z$ is analytic everywhere except at $z = 0$. Further,

$$f'(z) = u_x + iv_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{2ixy}{(x^2 + y^2)^2}$$

$$= \frac{-[x^2 + (iy)^2 - 2ixy]}{(x^2 + y^2)^2} = \frac{-(x - xy)^2}{(x^2 + y^2)^2} = \frac{-(\bar{z})^2}{(z\bar{z})^2} = -\frac{1}{z^2}, z \neq 0.$$

Example 18.26: Show that the function $f(z) = \sqrt{xy}$ is not analytic at the origin even though Cauchy-Reimann equations are satisfied there.

Solution: We have $f(z) = \sqrt{xy}$. Thus $u = \sqrt{xy}$ and $v = 0$.

By definition $$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0.$$

Similarly, $$\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = 0, \quad \left.\frac{\partial v}{\partial x}\right|_{(0,0)} = 0, \text{ and } \left.\frac{\partial v}{\partial y}\right|_{(0,0)} = 0.$$

Hence C.R. equations are satisfied at the origin. But

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{\substack{y=mx\\ x\to 0}} \frac{\sqrt{xy}-0}{x+iy} = \lim_{x\to 0} \frac{x\sqrt{m}}{x(1+im)} = \frac{\sqrt{m}}{(1+im)}$$

depends on m and hence is not unique.

Thus $f'(0)$ does not exist and hence function is not analytic at (0, 0).

Example 18.27: If $f(z) = u + iv$ is an analytic function of z, and $u - v = (x - y)(x^2 + 4xy + y^2)$, then find $f(z)$.

Solution: We have, $u - v = (x - y)(x^2 + 4xy + y^2)$

Differentiating it partially w.r.t. x and y we have, respectively

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = (x^2 + 4xy + y^2) + (x - y)(2x + 4y)$$

and, $$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y)$$

Adding these two and using the C.R. equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we obtain

$$2\frac{\partial u}{\partial y} = 6(x^2 - y^2), \quad \text{or} \quad \frac{\partial u}{\partial y} = 3(x^2 - y^2) \qquad ...(18.39)$$

Integrating w.r.t. y, we have

$$u = 3x^2y - y^3 + c_1(x), \qquad ...(18.40)$$

where $c_1(x)$ is arbitrary function of x only.

Rewriting (18.39) using C.R. equations, as $-\frac{\partial v}{\partial x} = 3(x^2 - y^2)$, or $\frac{\partial v}{\partial x} = 3(y^2 - x^2)$

Integrating w.r.t. x, we have

$$v = 3y^2x - x^3 + c_2(y), \qquad ...(18.41)$$

where $c_2(y)$ is an arbitrary function of y only.

From (18.40) and (18.41) we have $u - v = 3xy(x - y) + (x^3 - y^3) + c_1(x) - c_2(y)$

Comparing it with the given expression, we obtain

$$c_1(x) - c_2(y) = 0, \quad \text{or} \quad c_1(x) = c_2(y) = \text{constant, say } a.$$

Hence, $f(z) = 3x^2y - y^3 + i(3y^2x - x^3) + A$, where $A = a + ia$ is constant.

Example 18.28: If $f(z)$ is an analytic function of z, then show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$

Solution: Let $f(z) = u + iv$, then $|f(z)| = (u^2 + v^2)^{\frac{1}{2}}$. Consider

$$\frac{\partial}{\partial x}|f(z)| = \frac{\partial}{\partial x}(u^2+v^2)^{\frac{1}{2}} = \frac{1}{2}(u^2+v^2)^{-\frac{1}{2}}.\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right) = \frac{u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x}}{(u^2+v^2)^{\frac{1}{2}}}$$

or,

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 = \frac{\left[u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x}\right]^2}{u^2+v^2}. \qquad ...(18.42)$$

Similarly, $$\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = \frac{\left\{u\dfrac{\partial u}{\partial x} - v\dfrac{\partial v}{\partial x}\right\}^2}{u^2+v^2}.$$

Since $f(z)$ is analytic, using C.R. equations, it becomes

$$\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = \frac{\left\{v\dfrac{\partial u}{\partial x} - u\dfrac{\partial v}{\partial x}\right\}^2}{u^2+v^2} \qquad ...(18.43)$$

From (18.42) and (18.43), we have

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = \frac{u^2\left(\dfrac{\partial u}{\partial x}\right)^2 + v^2\left(\dfrac{\partial v}{\partial x}\right)^2 + u^2\left(\dfrac{\partial v}{\partial x}\right)^2 + v^2\left(\dfrac{\partial u}{\partial x}\right)^2}{u^2+v^2}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2$$

18.5.3 Polar Form of the Cauchy-Riemann Equations

Sometimes it is convenient to express $f(z)$ in terms of the polar coordinates (r, θ) as $f(z) = u(r, \theta) + iv(r, \theta)$, where $z = re^{i\theta}$, which gives

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

By chain rule of differentiation, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = (\cos\theta)\frac{\partial u}{\partial x} + (\sin\theta)\frac{\partial u}{\partial y} \qquad \text{...(18.44)}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial u}{\partial x} + (r\cos\theta)\frac{\partial u}{\partial y} \qquad \text{...(18.45)}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = (\cos\theta)\frac{\partial v}{\partial x} + (\sin\theta)\frac{\partial v}{\partial y} \qquad \text{...(18.46)}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial v}{\partial x} + (r\cos\theta)\frac{\partial v}{\partial y} \qquad \text{...(18.47)}$$

Using C-R equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ in (18.46), we have

$$\frac{\partial v}{\partial r} = -(\cos\theta)\frac{\partial u}{\partial y} + (\sin\theta)\frac{\partial u}{\partial x} = -\frac{1}{r}\left[(-r\sin\theta)\frac{\partial u}{\partial x} + (r\cos\theta)\frac{\partial u}{\partial y}\right] = -\frac{1}{r}\frac{\partial u}{\partial \theta}, \quad \text{using (18.45).}$$

Similarly from (18.47), we have

$$\frac{\partial v}{\partial \theta} = (-r\sin\theta)\left(-\frac{\partial u}{\partial y}\right) + (r\cos\theta)\frac{\partial u}{\partial x} = r\left[(\cos\theta)\frac{\partial u}{\partial x} + (\sin\theta)\frac{\partial u}{\partial y}\right] = r\frac{\partial u}{\partial r}, \text{ using (18.44).}$$

Therefore, the C.R. equations in polar coordinates are

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \textit{ and } \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}. \qquad \text{...(18.48)}$$

Further, $f(z) = u + iv$, gives $f'(z)e^{i\theta} = u_r + iv_r$, (using $z = re^{i\theta}$). Thus

or,

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$= \frac{1}{r}e^{-i\theta}(v_\theta - iu_\theta)$$

$$= e^{-i\theta}\left(u_r - \frac{i}{r}u_\theta\right)$$

$$= e^{-i\theta}\left(\frac{i}{r}v_\theta + iv_r\right)$$

using C.R. equations in polar coordinates.

All these are the various expressions for the derivatives of $f(z)$ in terms of the polar coordinates.

18.6 HARMONIC FUNCTIONS. LAPLACE EQUATION

A real valued function $\phi(x, y)$ of two variables x and y that has continuous second order partial derivatives and satisfy the Laplace equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad ...(18.49)$$

is called a harmonic function.

The Laplace equation (18.49) is of great practical importance and occurs frequently in the study of fluid flow, heat conduction, gravitation and electrostatic and the *real and imaginary parts of an analytic function are harmonic functions and thus satisfy the Laplace equation.* This is one of the main reasons for the importance of complex analysis in engineering mathematics.

We have the following result:

Theorem 18.4: *If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v satisfy the Laplace equation (18.36) in D and have continuous second order partial derivatives in D.*

Proof. Since $f(z) = u + iv$ is analytic in D, thus $u_x = v_y$ and $u_y = -v_x$.

Differentiating $u_x = v_y$ with respect to x and $u_y = -v_x$ with respect to y, we get

$$u_{xx} = v_{yx} \text{ and } u_{yy} = -v_{xy} \qquad ...(18.50)$$

Since u and v have continuous partial derivatives of all orders, so mixed second order derivatives are equal, that is $v_{yx} = v_{xy}$. Thus, from (18.50), we obtain $u_{xx} + u_{yy} = 0$

Similarly, differentiating $v_y = u_x$ with respect to y and $v_x = -u_y$ with respect to x, we get $v_{yy} = u_{xy}$ and $v_{xx} = -u_{yx}$. Adding these two, we get $v_{xx} + v_{yy} = 0$.

Thus, if $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its real and imaginary parts are harmonic functions in D.

Also we observe that, if $f(z) = u(x,y) + iv(x, y)$ is analytic in D, then the harmonic functions u and v are a related pair, since they satisfy the Cauchy-Reimann equations. The functions $u(x, y)$ and $v(x, y)$ are called the *conjugate harmonic functions* of each other in D. Given one harmonic function, the conjugate harmonic function can be obtained by using the Cauchy-Reimann equations. *A conjugate of a given harmonic function is uniquely determined upto an arbitrary real additive constant.*

Remark. If $u(x, y)$ and $v(x, y)$ are any two harmonic functions in D, then $f(z) = u + iv$ need not be analytic in D. For example, $u = \sinh x \cos y$ and $v = \cosh x \cos y$ are harmonic functions but $f(z) = u + iv$ is not an analytic function since it does not satisfy the C.R. equations.

Next, we give another interesting result concerning the analytic functions.

Theorem 18.5: *If $f(z) = u + iv$ is an analytic function, then the two families of level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system.*

Proof. Differentiating $u(x, y) = c_1$ partially w.r.t. x, we obtain

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}=0, \quad \text{or} \quad \frac{dy}{dx}=-\frac{\partial u/\partial x}{\partial u/\partial y}=\frac{\partial v/\partial y}{\partial v/\partial x}=m_1, \quad \text{say}$$

Similarly, differentiating $v(x, y) = c_2$ partially w.r.t. x, we obtain

$$\frac{dy}{dx}=-\frac{\partial v/\partial y}{\partial v/\partial x}=m_2, \quad \text{say}$$

Here $m_1 m_2 = -1$, thus $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system of curves.

18.6.1 Polar Form of the Laplace Equation

In polar system the real and imaginary parts of an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ satisfy the Laplace equation in the form $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$, *and* $\frac{\partial^2 v}{r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}=0$ *respectively.*

Functions $u(r, \theta)$ and $v(r, \theta)$ satisfy the C.R. Eqs. (18.48), given by

$$\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}, \text{ and } \frac{\partial v}{\partial \theta}=r\frac{\partial u}{\partial r}.$$

Consider the equation, $\frac{\partial v}{\partial \theta}=r\frac{\partial u}{\partial r}$. Differentiating w.r.t. r, we get

$$\frac{\partial^2 v}{\partial r\,\partial \theta}=\frac{\partial u}{\partial r}+r\frac{\partial^2 u}{\partial r^2} \qquad ...(18.51)$$

Similarly considering, $\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$, and differentiating w.r.t. θ, we get

$$\frac{\partial^2 v}{\partial \theta\,\partial r}=-\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \qquad ...(18.52)$$

Assuming that $v(r, \theta)$ has continuous second order partial derivatives, so that $\frac{\partial^2 v}{\partial r\,\partial \theta}=\frac{\partial^2 v}{\partial \theta\,\partial r}$.

Thus from (18.51) and (18.52), we obtain

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0. \qquad ...(18.53)$$

On similar lines the corresponding equation for $v(r, \theta)$ can be obtained.

The real valued polar functions $u(r, \theta)$ of two variables r, θ that has continuous second order partial derivatives and satisfy the Laplace Eq. (18.53) are called 'harmonic functions'.

Example 18.29: If $f(z)$ is an analytic function with constant modulus, then $f(z)$ is constant.

Solution: Let $f(z) = u + iv$, then $|f(z)| = \sqrt{u^2 + v^2} =$ constant

Differentiating $u^2 + v^2 =$ constant w.r.t. x, we have $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x} = 0.$

Differentiating it again w.r.t. x, we have, $\left(\dfrac{\partial u}{\partial x}\right)^2 + u\dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{\partial v}{\partial x}\right)^2 + v\dfrac{\partial^2 v}{\partial x^2} = 0.$

Similarly, $$\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} = 0.$$

Adding these two and using $\nabla^2 u = 0$ and $\nabla^2 v = 0$, (since u and v are harmonic, being real and imaginary parts of an analytic function $f(z) = u + iv$), we obtain

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = 0, \text{ which implies, } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0.$$

Thus u and v are independent of both x and y and hence $f(z)$ is constant.

Example 18.30: If $f(z)$ is an analytic function of z, then prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Re} f(z)|^2 = 2|f'(z)|^2.$$

Solution: Let $f(z) = u + iv$, then $\text{Re} f(z) = u$. Consider

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Re} f(z)|^2 = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right)$$

$$= 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial y}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2u\nabla^2 u$$

$= 2|f'(z)|^2$, since $f'(z) = u_x - iu_y$, and $\nabla^2 u = 0$, u being harmonic function.

Example 18.31: Show that the function $v(x, y) = \ln(x^2 + y^2) + x - 2y$ is harmonic. Find its conjugate harmonic function $u(x, y)$ and the corresponding analytic function $f(z)$.

Solution: We have $v(x, y) = \ln(x^2 + y^2) + x - 2y$. This gives

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2 + y^2} + 1, \quad \frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2} - 2, \quad \frac{\partial^2 v}{\partial x^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}, \quad \frac{\partial^2 v}{\partial y^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}.$$

Since $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, thus the function $v(x, y)$ is harmonic.

From the Cauchy-Reimann equation $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$, we get $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} - 2.$

Integrating w.r.t. x, we get $u(x, y) = 2\tan^{-1}\frac{x}{y} - 2x + \phi(y)$, where $\phi(y)$ is an arbitrary function of y.

Differentiating w.r.t. y it gives, $\frac{\partial u}{\partial y} = \frac{-2x}{x^2+y^2} + \phi'(y)$

Next, using the Cauchy-Riemann equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we get

$$-\frac{2y}{x^2+y^2} + \phi'(y) = -\frac{2y}{x^2+y^2} - 1, \text{ or } \phi'(y) = -1.$$

Integrating with respect to y, we obtain $\phi(y) = -y + c$, where c is any arbitrary constant.

Hence, $u(x, y) = 2\tan^{-1}\frac{x}{y} - 2x - y + c$, and thus

$$f(z) = \left(2\tan^{-1}\frac{x}{y} - 2x - y\right) + i\{\ln(x^2+y^2) + x - 2y\} + c$$

$$= 2i\left[\frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\frac{y}{x}\right] - (2-i)(x+iy) + c = 2i\ln z - (2-i)z + c.$$

Milne-Thomson method. This method can be alternatively applied to find the analytic function $f(z)$, when only real part or imaginary part is given.

We have, $\quad z = x + iy$, and $\bar{z} = x - iy$, thus $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

Therefore, $\quad f(z) = u(x, y) + iv(x, y)$ ...(18.54)

$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Considering this as a formal identity in two independent variables z, $\bar{z}$ and setting $\bar{z} = z$, we get

$$f(z) = u(z, 0) + iv(z, 0). \quad ...(18.55)$$

We observe that (18.55) is the same as obtained from (18.54) by replacing x by z and y by 0. Thus Milne-Thomson method consists of obtaining function in terms of z by replacing x with z and y with 0. It provides an elegant method of finding $f(z)$ when its real part or the imaginary part is given. To illustrate it we apply this method to solve Example 18.31 alternatively. We have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\,\frac{\partial v}{\partial x} = \left(\frac{2y}{x^2+y^2} - 2\right) + i\left(\frac{2x}{x^2+y^2} + 1\right).$$

Replace x by z and y by 0, we obtain $f'(z) = i\,\dfrac{2}{z} + (i-2)$.

Integrating w.r.t. z, we have $f(z) = 2i \ln z + (i-2)z + c$, where c is an arbitrary real constant as obtained earlier.

Example 18.32: Show that the function $v(x, y) = e^x \sin y$ is harmonic. Find its conjugate harmonic $u(x, y)$ and the corresponding analytic function.

Solution: We have $v(x, y) = e^x \sin y$. This gives

$$\frac{\partial v}{\partial x} = e^x \sin y, \qquad \frac{\partial v}{\partial y} = e^x \cos y, \qquad \frac{\partial^2 v}{\partial x^2} = e^x \sin y, \qquad \frac{\partial^2 v}{\partial y^2} = -e^x \sin y.$$

Since $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$, thus the given function $v(x, y)$ is harmonic.

Next, let $f(z) = u + iv$. Thus $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} + i\dfrac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y$.

Using Milne-Thomson method, replace x by z and y by zero, we get $f'(z) = e^z$

Integrating w.r.t. z, we have $f(z) = e^z + c$, where c is an arbitrary real constant.

Example 18.33: Determine the analytic function $f(z) = u + iv$, if $v = \ln[(x-1)^2 + (y-2)^2]$.

Solution: We have $v = \ln[(x-1)^2 + (y-2)^2]$. This gives

$$\frac{\partial v}{\partial x} = \frac{2(x-1)}{(x-1)+(y-2)^2}, \text{ and } \frac{\partial v}{\partial y} = \frac{2(y-2)}{(x-1)+(y-2)^2}$$

From the Cauchy-Reimann equation $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial y}$, we get $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = \dfrac{2(y-2)}{(x-1)+(y-2)^2}$.

Integrating w.r.t. x, we get $u(x, y) = 2\tan^{-1}\dfrac{x-1}{y-2} + \phi(y)$, where ϕ is an arbitrary function of y.

Differentiating it w.r.t. y, we obtain

$$\frac{\partial u}{\partial y} = \frac{-2(x-1)}{(x-1)^2+(y-2)^2} + \phi'(y)$$

Using the Cauchy-Riemann equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we get

$$\frac{-2(x-1)}{(x-1)^2+(y-2)^2} + \phi'(y) = \frac{-2(x-1)}{(x-1)^2+(y-2)^2}$$

which gives $\phi'(y)=0$, or $\phi(y)=c$, where c is a constant.

Hence, $$u(x, y) = 2\tan^{-1}\frac{x-1}{y-2} + c = -2\tan^{-1}\frac{y-2}{x-1} + c.$$

Thus, $$f(z) = -2\tan^{-1}\frac{y-2}{x-1} + i\ln[(x-1)^2 + (y-2)^2] + c$$

$$= 2i\left[\ln|(x-1)+i(y-2)| + i\tan^{-1}\frac{y-2}{x-1}\right] + c$$

$$= 2i\ln\{(x-1)+i(y-2)\} + c = 2i\ln(z-1-2i) + c,$$

is the corresponding analytic function.

Example 18.34: Two concentric circular cylinders of radii a and b, $(a < b)$, are kept at potentials ϕ_1 and ϕ_2 respectively. Using complex function $w = A\ln z + c$, prove that the capacitance per unit length of the capacitor formed by them in vacuum is $2\pi/\ln(b/a)$.

Solution: Let $w = \phi + i\psi$, where ϕ is the potential function and ψ is the corresponding flux function. Thus, $\phi + i\psi = A\ln z + c = (A\ln r + c) + iA\theta$

Therefore, $\phi = A\ln r + c$, and $\psi = A\theta$

So that $\phi_1 = A\ln a + c$ and $\phi_2 = A\ln b + c$, and thus the potential difference $\phi_2 - \phi_1 = A\ln(b/a)$

Also the total charge Q is given by $Q = \int_0^{2\pi} d\psi = A\int_0^{2\pi} d\theta = 2\pi A$.

Hence the capacitance, that is, the charge required to maintain a unit potential difference

$$= \frac{Q}{\phi_2 - \phi_1} = \frac{2\pi A}{A\ln(b/a)} = \frac{2\pi}{\ln(b/a)}.$$

Example 18.35: Show that the function $v(r, \theta) = r^2\cos 2\theta - r\cos\theta + 2$ is harmonic. Find its conjugate harmonic function and the corresponding analytic function $f(z)$.

Solution: We have, $v(r, \theta) = r^2\cos 2\theta - r\cos\theta + 2$. This gives

$v_r = 2r\cos 2\theta - \cos\theta$, $v_{rr} = 2\cos 2\theta$, $v_\theta = -2r^2\sin 2\theta + r\sin\theta$, $v_{\theta\theta} = -4r^2\cos 2\theta + r\cos\theta$.

Since, $v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 2\cos 2\theta + 2\cos 2\theta - \frac{1}{r}\cos\theta - 4\cos 2\theta + \frac{1}{r}\cos\theta = 0$, thus

the function $v(r, \theta)$ satisfies the Laplace equation and, therefore, is harmonic.

From the Cauchy-Riemann equations in polar coordinates, refer to to (18.48), $ru_r = v_\theta = -2r^2 \sin 2\theta + r \sin\theta$, we have

$$u_r = -2r \sin 2\theta + \sin\theta \qquad ...(18.56)$$

Similarly, $-\dfrac{1}{r} u_\theta = v_r = 2r \cos 2\theta - \cos\theta$, gives

$$u_\theta = -2r^2 \cos 2\theta + r \cos\theta \qquad ...(18.57)$$

Integrating (18.56) with respect to r, we get $u = -r^2 \sin 2\theta + r \sin\theta + \phi(\theta)$, where $\phi(\theta)$ is an arbitrary function of θ.

Differentiating it partially w.r.t. θ, we have

$$u_\theta = -2r^2 \cos 2\theta + r \cos\theta + \phi'(\theta). \qquad ...(18.58)$$

From (18.57) and (18.58), we get $\phi'(\theta) = 0$, or $\phi(\theta) = c$. Thus, $u = -r^2 \sin 2\theta + r \sin\theta + c$.

Hence, $f(z) = u + iv = (-r^2 \sin 2\theta + r \sin\theta + c) + i(r^2 \cos 2\theta - r\cos\theta + 2)$

$$= r^2(-\sin 2\theta + i\cos 2\theta) + r(\sin\theta - i\cos\theta) + c + 2i = i(r^2 e^{2i\theta} - re^{i\theta}) + c + 2i.$$

EXERCISE 18.4

1. Determine which of the following functions are analytic:

   (a) z^6 (b) $z\bar{z}$ (c) $2xy + i(x^2 - y^2)$

   (d) $\dfrac{(x - iy)}{x^2 + y^2}$ (e) $Re z/ Im z$ (f) $Re(z^3)$

2. Show that

   (a) $f(z) = xy + iy$ is everywhere continuous but is not analytic.

   (b) $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.

3. Prove that an analytic function $f(z)$ with $Re. f(z) =$ const. is constant.

4. Determine whether the following functions are harmonic. If answer is yes, find a corresponding analytic function $f(z) = u + iv$

   (a) $u = x/(x^2 + y^2)$ (b) $u = -e^{-x} \sin y$

   (c) $u = e^{2x}(x \cos 2y - y \sin 2y)$ (d) $u = x \sin x \cosh y - y \cos x \sinh y$

   (e) $v = e^{-x}(x \sin y - y \cos y)$ (f) $u = \sin 2x/(\cosh 2y - \cos 2x)$

5. Determine the analytic function $f(z) = u + iv$, if

   (a) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$, $f(\pi/2) = 0$

   (b) $u - v = e^{-x}[(x - y)\sin y - (x + y)\cos y]$, $f(0) = 0$

6. Determine whether the following functions are harmonic. If yes, find a corresponding analytic function $f(z) = u(r, \theta) + iv(r, \theta)$

(a) $v(r, \theta) = r^2 \cos 2\theta$ (b) $u(r, \theta) = \left(r + \frac{1}{r}\right)\cos\theta,\ r \neq 0$

7. If u is a harmonic function, then show that u^2 is not a harmonic function unless u is a constant.
8. If $f(z)$ is an analytic function of z, then show that

$$\nabla^2[\,|f(z)|^p] = p^2\,|f(z)|^{p-2}|f'(z)\,|^2,$$

where ∇^2 is the Laplacian operator and p is a real greater than 1.

9. If $f(z)$ is an analytic function of z, prove that

(a) $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 = |f'(z)|^2$

(b) $\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\} \ln |f(z)| = 0.$

10. Consider the analytic function $f(z) = e^z$, $z \neq 0$. Find its level curves. Show that these curves are mutually orthogonal.

18.7 GEOMETRIC ASPECTS OF ANALYTIC FUNCTIONS

So far in this chapter we have discussed the analytical aspects, such as the values of the function, differentially and analyticity of the function of a complex variable. We have observed that a function $w = f(z)$ is actually a mapping from a given region in the z-plane to a corresponding one in the w-plane. To get a better understanding of the properties of function $f(z)$, in this section we pay our attention to the geometrical aspects of the mapping $w = f(z)$.

A relationship $u = f(x, y, t)$ containing three independent real variables x, y and t requires a four-dimensional space for geometric representation. Similar difficulties arise when one attempts to represent graphically complex function $w = f(z) = u(x, y) + iv(x, y)$, with $z = x + iy$. For to each pair of values (x, y), there correspond two values (u, v) and in order to plot a quadruplet of real values (u, v, x, y) we need a four-dimensional space.

However, we visualize $w = f(z)$ as a relationship which assigns to each point z in its domain of definition D, the corresponding point $w = f(z)$ in the w-plane. Thus we need two planes, the z-plane in which we plot values of z and the w-plane in which we plot the corresponding value $w = f(z)$. We say that f defines a mapping of region D in the $z = x + iy$ plane to another region D' defined by $w = f(z)$ in the $w = u + iv$ plane. On separating $w = f(z)$ into real and imaginary parts, we obtain two real functions $u = u(x, y)$, $v = v(x, y)$ which can be viewed as the equations of a transformation that maps a specified set of points (x, y) in the xy-plane into another set of points (u, v) in the uv-plane. We adopt this mode of studying complex functions and introduce a few standard transformations in this section.

18.7.1 Translation: $w = z + c$

Set $z = x + iy$, $w = u + iv$ and $c = h + ik$ in $w = z + c$, we get

$$u + iv = x + iy + h + ik = (x + h) + i(y + k). \text{ Hence, } u = x + h, \text{ and } v = y + k.$$

Thus under translation the point $P(x, y)$ in the z-plane is mapped onto the point $P'(x + h, y + k)$ in the w-plane. Thus this transformation maps a region in the z-plane into a region in the w-plane of the same shape and size, where each point is moved h units in the direction of the x-axis and k units in the direction of y-axis.

For example, the rectangle $OABC$ with the vertices $O(0, 0)$, $A(2, 0)$, $B(2, 1)$ and $C(0, 1)$ in the z-plane, refer to Fig. 18.15a, is transformed to rectangle $O'A'B'C'$ with vertices $O'(1, 2)$, $A'(3, 2)$, $B'(3, 3)$ and $C'(1, 3)$ in the w-plane under the transformation $w = z + (1 + 2i)$, refer to Fig. 18.15b.

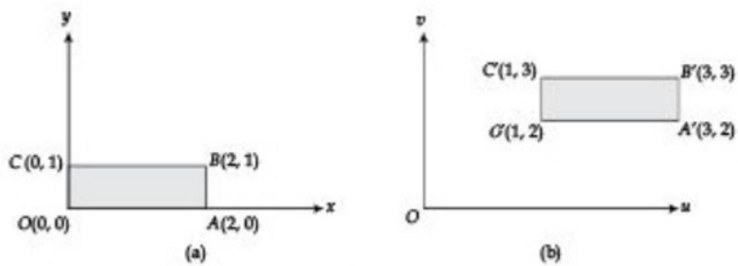


Fig. 18.15

18.7.2 Scaling and Rotation: $w = az$

To study this transformation it is convenient to use polar-coordinates.

Set $z = re^{i\theta}$, $w = \rho e^{i\phi}$ and $a = Ae^{i\alpha}$, $w = az$ becomes, $\rho e^{i\phi} = Are^{i(\alpha+\theta)}$

so that, $\rho = Ar$, $\phi = \alpha + \theta$.

Thus the point $P(r, \theta)$ in the z-plane is mapped onto the point, $P'(Ar, \alpha + \theta)$ in the w-plane. Hence the transformation results in *magnification*, if $|a| = A > 1$, or *contraction*, if $|a| = A < 1$, accompanied by a rotation through an angle α. Thus a square in the z-plane is transformed into a square, a circle of radius R is transformed into a circle of radius AR. If $A = 1$, we have a pure rotation through an angle α.

The same conclusion can be reached by setting $w = u + iv$, $z = x + iy$ and $a = h + ik$ in $w = az$ and deriving the transformation equations $u = hx - ky$, $v = kx + hy$ in cartesian co-ordinates.

For example, a rectangular region $ABCD$ with vertices at $A(2, 1)$, $B(3, 1)$, $C(3, 3)$ and $D(2, 3)$ in the z-plane, refer to Fig. 18.16a, is transformed into the rectangle $A'B'C'D'$, with vertices at $A'(1/\sqrt{2}, 3/\sqrt{2})$, $B'(2/\sqrt{2}, 4/\sqrt{2})$, $C'(0, 6/\sqrt{2})$ and $D'(-1/\sqrt{2}, 5/\sqrt{2})$ in the w-plane under the transformation $w = (1 + i)z/\sqrt{2}$, as shown in Fig. 18.16b.

The rectangular region $ABCD$ has been rotated by an angle $\pi/4$ but the length of the sides remains unchanged since $|w| = |z|$.

18.7.3 Inversion and Reflection: $w = \dfrac{1}{z}, z \neq 0$

To study this relationship we again use polar co-ordinates. On setting $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, we get

$$\rho e^{i\phi} = \left(\frac{1}{r}\right)e^{-i\theta}, \text{ so that } \rho = \frac{1}{r} \text{ and } \phi = -\theta.$$

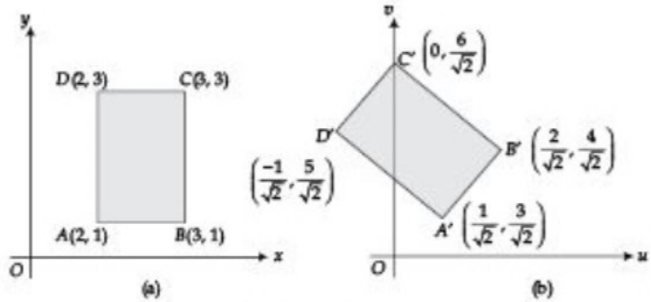


Fig. 18.16

Thus the point $P(r, \theta)$ in the z-plane is mapped onto the point $P'(1/r, -\theta)$ in the w-plane. The unit circle $|z| = 1$ is transformed into the unit circle $|w| = 1$ in the w-plane. As the point P traces out the circle $|z| = 1$ in the clockwise direction the corresponding point P' in the w-plane traces out the circle $|w| = 1$ in the counter-clockwise direction. Points in the interior of $|z| = 1$ are mapped into points in the exterior of $|w| = 1$, and the exterior of $|z| = 1$ is mapped into the interior of $|w| = 1$. In particular, the origin $z = 0$ corresponds to the improper point $w = \infty$, called the *point at infinity* and if we consider the inverse transformation $z = 1/w$, we observe that $w = 0$ corresponds to $z = \infty$.

The equations of transformation defined by $w = 1/z$ in cartesian co-ordinates have the form

$$u = \frac{x}{x^2 + y^2}, v = -\frac{y}{x^2 + y^2},$$

with the inverse transformation equations as

$$x = \frac{u}{u^2 + v^2}, y = -\frac{v}{u^2 + v^2}. \quad ...(18.59)$$

Next, we state an important result concerning the inversion transformation.

The inversion transformation $w = 1/z$ maps a circle onto a circle or to a straight line if the circle passes through the origin.

To prove it consider the general equation of a circle in the z-plane given by

$$x^2 + y^2 + 2gx + 2fy + c + 0 \quad ...(18.60)$$

Using (18.59), (18.60) becomes

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+2g\frac{u}{u^2+v^2}+2f\frac{-v}{u^2+v^2}+c=0$$

or, $$c(u^2+v^2)+2gu-2fv+1=0, \quad ...(18.61)$$

which is the equation of a circle in the w-plane. If $c=0$, the circle (18.60) passes through the origin and its image (18.61) reduces to the straight line $2gu-2fv+1=0$.

18.7.4 Bilinear Transformation

The transformation

$$w=\frac{az+b}{cz+d}, \quad ad-bc\neq 0, \quad ...(18.62)$$

where a, b, c and d are complex constants, is called the *bilinear transformation*, or *Mölius transformation*, or *linear fractional transformation*. The condition $ad-bc\neq 0$ ensures that $w(z)$ is not merely a constant or $0/0$.

Since $w'=\dfrac{ad-bc}{(cz+d)^2}$ exists for all z except $z=-\dfrac{d}{c}$, we observe that the bilinear transformation is *conformal*, (refer to Section (18.8)) everywhere except at $z=-d/c$, a point at which w is also not defined. The point $z=-d/c$ is called the *pole* of the transfomation and the image of this point can be regarded as the *point at infinity* in the w-plane denoted as $w=\infty$. The resultant w-plane is called the *extended complex plane*.

The corresponding inverse mapping is $z=-\dfrac{dw-b}{cw-a}$, $w\neq\dfrac{a}{c}$

which gives a unique z for each w. Hence besides being conformal, the bilinear transformation is also *one-to-one* on the extended z and w-planes.

Next if $a\neq 0$, $c\neq 0$ then we can rewrite (18.62) as $w=\dfrac{a}{c}+\dfrac{bc-ad}{c^2}\left[\dfrac{1}{z+(d/c)}\right]$,

which can be considered a composition of following mappings

(a) Translation: $w_1=z+d/c$

(b) Inversion: $w_2=1/w_1$

(c) Scaling and rotation: $w_3=\dfrac{bc-ad}{c^2}w_2$; and then again

(d) Translation: $w_4=\dfrac{a}{c}+w_3$

Therefore the general bilinear transformation can be considered a composition of the translation, inversion, scaling and rotation, and since under each of these the totatily of the circles and straight lines in the z-plane are mapped as the totality of circles and straight lines in the w-plane, thus we have the following theorem:

Theorem 18.6: *Every bilinear transformation of the form (18.62) maps every circle or straight line in the z-plane onto either a circle or a straight line in the w-plane.*

18.7.5 Fixed Points or Invariant Points

Fixed points of a mapping $w = f(z)$ are the points that are mapped onto themselves under the mapping, thus they are obtained from $w = f(z) = z$.

For example, the identity mapping $w = z$ has every point as a fixed point. The mapping $w = \bar{z}$ has points on the whole of the real axis as fixed points while the inversion mapping $w = 1/z$ has two fixed points $z = \pm 1$.

For the bilinear transformation (18.62), fixed points are given by

$$z = (az + b)/(cz + d), \text{ or } \quad cz^2 + (d - a)z - b = 0$$

This is quadratic in z whose coefficients all vanish if, and only if the mapping is the identity mapping $w = z$ and in this case, $a = d \neq 0$, $b = c = 0$. Hence we have the following theorem:

Theorem 18.7: *A bilinear transformation, not the identity has at most two fixed points. If it is known to have three or more fixed points it must be the identity mapping $w = z$.*

18.7.6 Procedure to Find Bilinear Transformations

It appears that a bilinear transformation of the from (18.62) is determined by the four constants a, b, c, d, but by dividing the numerator and the denominator on the right of (18.62) by one of the four constants, it is clear that one constant can be dropped and thus the three conditions determine a unique bilinear transformation. Thus we have the following theorem:

Theorem 18.8: *Three given distinct points z_1, z_2, z_3 can always be mapped onto three prescribed distinct points w_1, w_2, w_3 by one, and only one bilinear transformation and this transformation is given by*

$$\frac{w - w_1}{w - w_2} \cdot \frac{w_3 - w_2}{w_3 - w_1} = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}$$

If one of these points is the point at infinity, the quotient of the two differences containing these points must be replaced by 1.

Proof. Let the required transformation be $w = \dfrac{az + b}{cz + d}, \quad ad - bc \neq 0$

It is given that $\qquad w_j = \dfrac{az_j + b}{cz_j + d}, \quad j = 1, 2, 3, z_i \neq z_j$, for $i \neq j$.

For $j = 1$, $\qquad w - w_1 = \dfrac{az_1 + b}{cz_1 + d} - \dfrac{az_1 + b}{cz_1 + d} = \dfrac{(ad - bc)(z - z_1)}{(cz + d)(cz_1 + d)}$ Similarly,

$$w - w_2 = \frac{(ad - bc)(z - z_2)}{(cz + d)(cz_2 + d)}, \; w_3 - w_1 = \frac{(ad - bc)(z_3 - z_1)}{(cz_3 + d)(cz_1 + d)} \; w_3 - w_2 = \frac{(ad - bc)(z_3 - z_2)}{(cz_3 + d)(cz_2 + d)}.$$

Hence, we have

$$\frac{w-w_1}{w-w_2}=\frac{(z-z_1)(cz_2+d)}{(z-z_2)(cz_1+d)} \text{ and } \frac{w_3-w_1}{w_3-w_2}=\frac{(z_3-z_1)(cz_2+d)}{(z_3-z_2)(cz_1+d)}.$$

Thus,
$$\frac{w-w_1}{w-w_2}\cdot\frac{w_3-w_2}{w_3-w_1}=\frac{z-z_1}{z-z_2}\cdot\frac{z_3-z_2}{z_3-z_1}. \qquad \text{...(18.63)}$$

This proves the theorem.

If one of the point is infinity, say $w_1=\infty$, then

$$\lim_{w_1\to\infty}\frac{w-w_1}{w_3-w_1}=\lim_{w_1\to\infty}\left[\frac{(w/w_1)-1}{(w_3/w_1)-1}\right]=\frac{-1}{-1}=1.$$

For example, the bilinear transformation that maps $z_1=0, z_2=1, z_3=\infty$ onto $w_1=-1, w_2=-i$, $w_3=1$ respectively, is obtained from (18.50) by substituting the respective values. We obtain,

$$\frac{w-(-1)}{w-(-i)}\cdot\frac{1-(-i)}{1-(-1)}=\frac{z-0}{z-1}\cdot\frac{\infty-1}{\infty-0}.$$

Replacing $\frac{\infty-1}{\infty}$ by 1 and simplifying, we get $w=(z-i)/(z+i)$ as the desired mapping.

Remark. The property (18.63) is also stated as that, *a bilinear transformation preserves cross ratio of four points.*

Example 18.36: Find the image of the infinite strip $1/4 \le y \le 1/2$ under the transformation $w=1/z$.

Solution: If $z=x+iy$, $w=u+iv$, then under the given transformation $w=1/z$, we have

$$x=\frac{u}{u^2+v^2} \text{ and } y=\frac{-v}{u^2+v^2}$$

Thus, $y=\frac{1}{4}$ gives $\frac{-v}{u^2+v^2}=\frac{1}{4}$, that is, $u^2+v^2+4v=0$, or $u^2+(v+2)^2-4=0$,

and $y=\frac{1}{2}$ gives $\frac{-v}{u^2+v^2}=\frac{1}{2}$, that is, $u^2+v^2+2v=0$, or $u^2+(v+1)^2-1=0$.

Hence the infinite strip $1/4 \le y \le 1/2$ in the z-plane, as shown in Fig. 18.17a, is transformed into the region $u^2+(v+2)^2 \le 4$, $u^2+(v+1)^2 \ge 1$ in the w-plane, that is, the region between the two circles both passing through (0, 0). Circles are: $u^2+(v+2)^2=4$ with center (0, −2) and radius 2 and, $u^2+(v+1)^2=1$ with center (0, −1) and radius 1 as shown in Fig. 18.17b.

Example 18.37: Find the image of the region $|z-i|<2$ under the mapping $w=(1+i)/(z+i)$.

Solution: The given mapping is $w=\frac{1+i}{z+i}$, or $z=\frac{1+i(1-w)}{w}$.

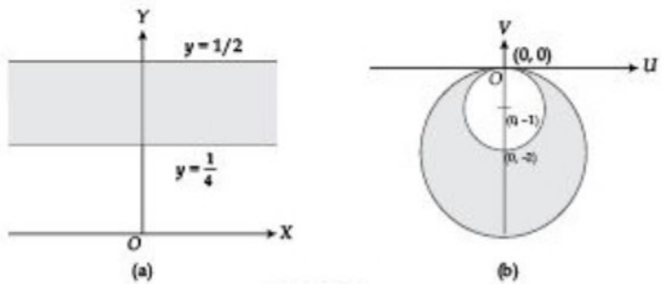


Fig. 18.17

Hence, $$|z-i| = \left|\frac{1+i(1-w)}{w} - i\right| = \left|\frac{1+i(1-2w)}{w}\right|$$

Thus, $|z-i| < 2$ gives $|1+i(1-2w)| < 2|w|$

or, $|(1+2v)+i(1-2u)| < 2\,|u+iv|$, where $w = u+iv$

or, $(1+2v)^2+(1-2u)^2 < 4(u^2+v^2)$ or, $2v-2u+1<0$.

Therefore, the interior of the circular disk $|z-i| < 2$ as shown in Fig. 18.18a is mapped as the half plane below the line $2v-2u+1=0$ and the boundary circle $|z-i|=2$ is mapped as the line $2v-2u+1=0$, as shown in the Fig. 18.18b.

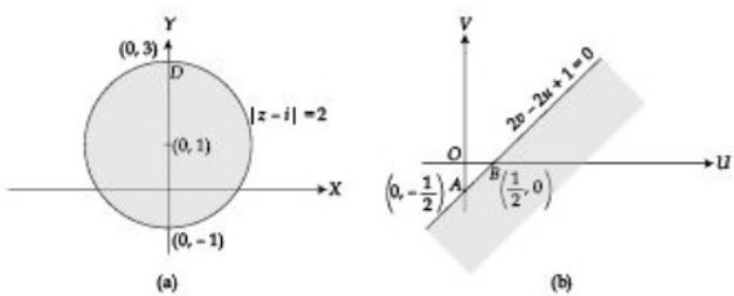


Fig. 18.18

Example 18.38: Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$. Hence, find the image of $|z| < 1$.

Solution: Let the required bilinear transformation be $w = \dfrac{az+b}{cz+d}$.

Substituting the corresponding values of w and z, we get

$$i = \frac{a+b}{c+d}, \quad 0 = \frac{ai+b}{ci+d}, \quad -i = \frac{a(-1)+b}{c(-1)+d},$$

which give respectively $(a+b) - i(c+d) = 0$, $\quad b + ia = 0$, and $(-a+b) + i(-c+d) = 0$.

Solving these equations in term of a, we get $b = -ia$, $c = -a$, $d = -ia$.

Substituting for b, c and d, the required transformation becomes

$$w = \frac{az - ia}{-az - ia}, \text{ or } w = \frac{i-z}{i+z}, \text{ or } z = \frac{i(1-w)}{1+w}.$$

Thus the region $|z| < 1$ is mapped onto the region

$$\left|\frac{i(1-w)}{i+w}\right| < 1, \text{ or } \frac{|i||1-w|}{|i+w|} < 1, \text{ or } |1-w| < |1+w|$$

or, $\quad |1 - u - iv| < |1 + u + iv|$, $\quad$ where $w = u + iv$

or, $\quad (1-u)^2 + v^2 < (1+u)^2 + v^2$, or $u > 0$.

Hence, the interior of the circle $x^2 + y^2 = 1$ in the z-plane is mapped into the entire positive half plane $u > 0$ in the w-plane. Further we note that the circle $|z| = 1$ in the z-plane is mapped into the imaginary axis $u = 0$ in the w-plane.

EXERCISE 18.5

1. Find the image of the triangle with vertices at i, $1+i$, $1-i$ in the z-plane under the transfomation $w = 3z + 4 - 2i$.
2. Find the image of the rectangular region with vertices at (0, 0), (1, 0) (1, 2) and (0, 2) under the transformation $w = \sqrt{2}\, e^{-i\pi/4}\,[z + (1-i)]$.
3. Find the image of $|z - 2i| = 2$ under the transformation $w = 1/z$.
4. Show that every circle in the z-plane maps by the transformation $w = 1/z$ into a circle in the w-plane if one considers straight lines as the limiting cases of circles.
5. Find the invariant points of the transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$.
6. Show that if a is complex with $|a| < 1$, then $w = (z-a)/(1-\bar{a}z)$ maps $|z| < 1$ onto $|w| < 1$, with 'a' being mapped to the origin.
7. Find the bilinear transformation which maps
   (a) The points $z = 1, i, -1$ into the points $w = 0, 1, \infty$
   (b) The points $z = 0, -1, \infty$ into the points $w = -1, -2-i, i$
   (c) $\text{Re}(z) > 0$ into interior of unit circle so that $z = \infty, i, 0$ map into $w = -1, -i, 1$.
8. Find the image of the closed half disk $|z| \le 1$, $\text{Im}(z) \ge 0$ under the bilinear transformation $w = z/(z+1)$.
9. Find all bilinear transformations whose fixed points are
   (a) -1 and 1 $\qquad$ (b) i and $-i$

10. Show that the image of the half plane $x + y > 0$ under the bilinear transformation $w = (z-1)/(z+i)$ is the interior of the unit disk $|w| = 1$.

18.8 CONFORMAL MAPPING

In this section we consider the most important geometrical property of the mappings defined by analytic functions, namely, *the conformality* or *the angle preserving property*. Also, we shall discuss a few specific conformal mappings like z^2, e^z, $z+1/z$, $\sin z$, $\cos z$, $\cosh z$ etc.

18.8.1 The Conformal Mapping

A mapping in the plane is said to be 'angle-preserving', or 'conformal', if it preserves angles between oriented curves in magnitude as well as in sense.

The angle between two oriented curves is the angle α, $(0 \le \alpha \le \pi)$, between their oriented tangents at their point of intersection, as shown in Fig. 18.19.

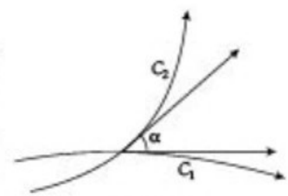


Fig. 18.19

Angle preserving property of analytic functions. We have the following result.

Theorem 18.9: *The mapping defined by an analytic function $f(z)$ is conformal except at points where the derivative $f'(z) = 0$, called the critical points.*

Proof. Let $w = f(z)$ be an analytic function and defines a mapping from a region D in the z-plane onto a region D' in the w-plane and let C be a continuous curve in the z-plane passing through a point $z_0 \in D$, with the parametric representation $z(t) = x(t) + iy(t)$, $a \le t \le b$

The increasing value of t is taken as the positive direction of the curve C. We assume $z(t)$ differentiable, $\dot{z}(t) \ne 0$ and continuous on C. Therefore C is a smooth curve as shown in Fig. 18.20a.

Let C^* be the image curve of C in the w-plane as shown in Fig. 18.20b. Then,

$$w = f[z(t)], \quad a \le t \le b$$

is the parametric representation of the curve C^* in the w-plane under the analytic mapping $w = f(z)$.

Let $z_1 = z(t_1)$ be a point in the nbd. of the point $z_0 = z(t_0)$ on C and $\Delta t = t_1 - t_0$. Now $\dot{z}(t_0)$ is the tangent vector to the curve C at the point t_0, given by

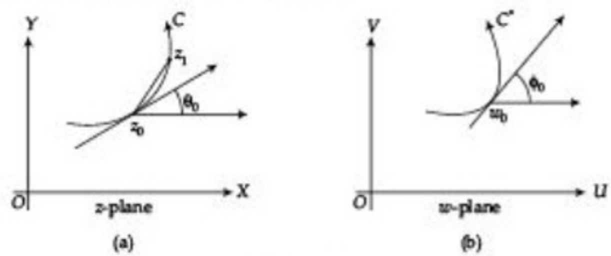


Fig 18.20

$$\dot{z}(t_0) = \left.\frac{dz}{dt}\right|_{t_0} = \lim_{\Delta t \to 0} \frac{z_1 - z_0}{\Delta t} = \lim_{\Delta t \to 0} \frac{z(t_0 + \Delta t) - z(t_0)}{\Delta t}.$$

The angle between this vector and the positive x-axis is $\arg(\dot{z}(t_0))$ and let $\theta_0 = \arg(\dot{z}(t_0))$.

Next, the point $z_0 = z(t_0)$ on C corresponds to the point $w(t_0) = w_0$ of C^* and $\dot{w}(t_0)$ represents a tangent vector to C^* at the point t_0. By chain rule

$$\frac{dw}{dt} = \frac{df}{dz}\frac{dz}{dt} \text{ or, } \dot{w}(t_0) = f'(z_0)\,\dot{z}(t_0). \qquad \text{...(18.64)}$$

Thus, if $f'(z_0) \neq 0$, then $\dot{w}(t_0) \neq 0$ and C^* has a unique tangent at $w(t_0)$, also the angle between the tangent vector $\dot{w}(t_0)$ and the positive u-axis is given by $\arg(\dot{w}(t_0))$. Since the argument of a product equals the sum of the arguments of the factors, thus from (18.64) we have

$$\arg \dot{w}(t_0) = \arg f'(z_0) + \arg \dot{z}(t_0)$$

or,
$$\arg \dot{w}(t_0) - \arg \dot{z}(t_0) = \arg f'(z_0).$$

Thus, under the mapping $w = f(z)$ the tangent to C at z_0 is rotated through an angle $\arg f'(z_0)$, which is independent of the choice of C. That is, the transformation $w = f(z)$ rotates the tangents of all the curves through z_0 by the same angle $\arg f'(z_0)$. *Hence, the curves C_1 and C_2 through z_0 which form a certain angle at z_0 are mapped upon curves C_1^* and C_2^* forming the same angle in sense as well as in magnitude at the image point w_0 of z_0.*

This proves the angle-preserving property or conformality of mapping by analytic functions.

A mapping that preserves the magnitude of the angle but not necessarily the direction is called an isogonal mapping.

Remarks 1. If $w = f(z) = u + iv$ defines a conformal mapping, then u and v must satisfy C–R equations

and, therefore, $$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{-\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2.$$

Hence, in a conformal transformation infinitesimal areas are magnified by the factor $J\left(\frac{u, v}{x, y}\right)$ and also *the condition of a conformal mapping is* $J\left(\frac{u, v}{x, y}\right) \neq 0.$

2. The practical importance of conformal mapping results from the fact that harmonic functions of two real variables remains harmonic under a change of variables arising a conformal transformation. This has important consequences. Suppose that it is required to solve a boundary value problem in connection with a two-dimensional potential, that is, to determine a solution of Laplace equation in two independent variables in a given region D which assumes given value on the boundary of D. It may be possible to find a conformal mapping which transforms D into some simpler region D^* such as a circular disk or a half-plane. Thus, we may solve Laplace's equation subject to the transformed boundary conditions in D^*. The resulting solution when carried back to D by the use of that mapping will be the solution of the original problem.

18.8.2 Some Special Conformal Transformations

(a) Transformation: $w = z^2$

It is conformal everywhere except where $f'(z) = 2z = 0$.

We have $\quad u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$

It gives $\quad u = x^2 - y^2$ and $v = 2xy$

If $u = a$ is constant, then $x^2 - y^2 = a$. Similarly if $v = b$ is constant, then $xy = b/2$.

Both represent pair of rectangular hyperbolas. Hence, a pair of lines $u = a$, $v = b$ parallel to the axes in the w-plane map into pair of orthogonal rectangular hyperbolas in the z-plane as shown in Fig. 18.21a and 18.21b.

Similarly we can show that the pair of lines $x = c$ and $y = d$ parallel to the axes in the z-plane map into orthrogonal parabolas in the w-plane, as shown in Fig. 18.22a and 18.22b.

Also $\dfrac{dw}{dz} = 0$ gives $z = 0$, therefore, it is a critical point of the mapping.

Taking $z = re^{i\theta}$, and $w = \mathrm{Re}^{i\phi}$, the transformation $w = z^2$ becomes $\mathrm{Re}^{i\phi} = r^2 e^{2i\theta}$, which gives

$$R = r^2, \text{ and } \phi = 2\theta.$$

Hence circles $r = r_0$ are mapped onto circles $R = r_0^2$ and rays $\theta = \theta_0$ onto rays $\phi = 2\theta_0$, as shown in Fig. 18.23a and 18.24b. In fact, the upper half of the z-plane $0 \le \theta \le \pi$ transforms into the entire w-plane $0 \le \phi \le 2\pi$. The same holds for the lower half. On the similar lines we can explain that the transformation $w = z^n$, n being a positive integer is conformal mapping of the z-plane everywhere except at $z = 0$. The sector $0 \le \theta \le \pi/n$ is mapped by $w = z^n$ onto the upper half-plane $v \ge 0$.

(b) *Transformation: $w = e^z$*

Since $\dfrac{dw}{dz} = e^z \neq 0$ for any z, the mapping is conformal at every point z. Setting $z = x + iy$ and $w = \mathrm{Re}^{i\phi}$, we have $\mathrm{Re}^{i\phi} = e^{x+iy} = e^x e^{iy}$, which gives $R = e^x$, and $\phi = y$.

Thus the lines parallel to y-axis, that is, x = constant in the z-plane maps into the circles R = constant in the w-plane, as shown in Fig. 18.24a and 18.24b. Their radii being less than or greater than 1 according as x is less than or greater than 0. Similarly the lines parallel to x-axis that is, y = constant in the z-plane maps into the radial lines ϕ = constant of the w-plane. For example, the rectangular region $ABCD$ given by $0 \le x \le 1$ and $0.5 \le y \le 1$ in the z-plane, as shown in Fig. 18.25a, is mapped onto the circular region $A'B'C'D'$ given by $1 \le R \le e$, and $0.5 \le \phi \le 1$ as shown in Fig. 18.25b.

Also we can easily verify that under $w = e^z$ any strip of height 2π in the z-plane will cover once the entire w-plane.

(c) *Joukowski transformation: $w = z + 1/z$*

In this case $\dfrac{dw}{dz} = 1 - \dfrac{1}{z^2}$, the mapping is conformal except at the points $1 - \dfrac{1}{z^2} = 0$, or $z = \pm 1$, which corresponds to the points $w = 2$ and $w = -2$ of the w-plane. Hence the points $z = \pm 1$ are the critical points. Changing to polar co-ordinates we have

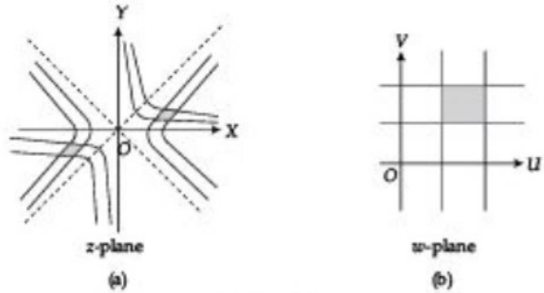


Fig. 18.21

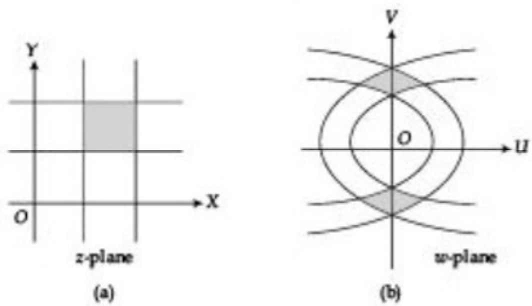


Fig. 18.22

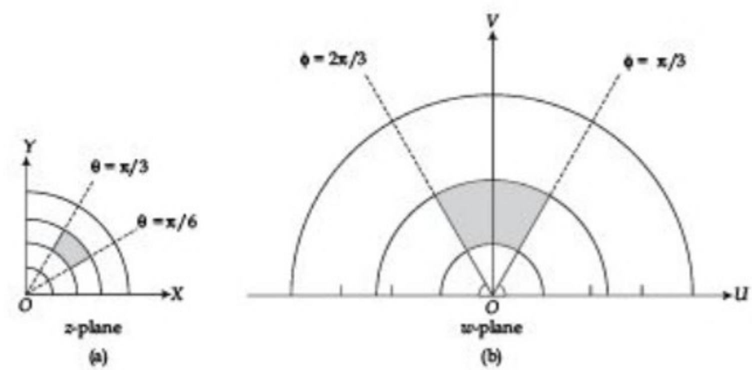


Fig. 18.23

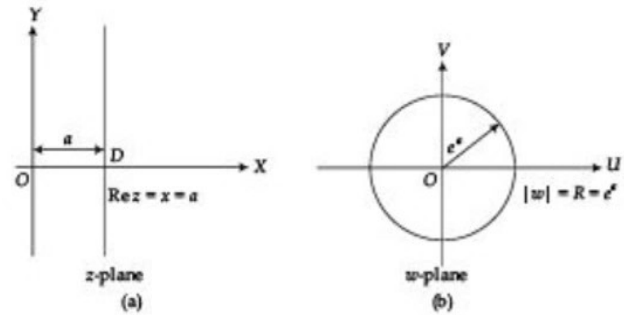


Fig. 18.24

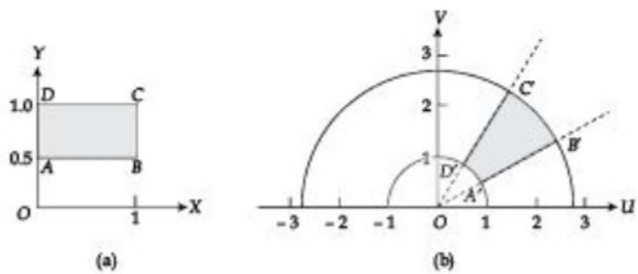


Fig. 18.25

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Therefore, $\quad u = \left(r + \frac{1}{r}\right)\cos\theta$ and $v = \left(r - \frac{1}{r}\right)\sin\theta$ ...(18.65)

Eliminating θ from these we obtain

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1 \qquad \text{...(18.66)}$$

From (18.66) it follows that family of circles r = constant in the z-plane maps onto the family of the ellipses in the w-plane. These ellipse are *confocal* since $\left(r + \frac{1}{r}\right)^2 + \left(r - \frac{1}{r}\right)^2 = 4$, which is independent of r, refer to Figs. 18.26a and 18.26b.

In particular, the unit circle $r = 1$ in the z-plane maps into the segment $u = -2$ to $u = 2$ of the real axis in the w-plane.

Next eliminating r from (18.65) we have

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2 = 4 \text{ or, } \frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1. \quad ...(18.67)$$

From (18.67) it follows that the radial line θ = constant in the z-plane transforms into a family of hyperbolas which are also confocal. The transformations are shown in Figs. 18.26a and 18.26b.

(d) *Bilinear transformation*: $w = (az + b)/(cz + d)$, $ad - bc \neq 0$.

This is conformal everywhere except at $z = -d/c$ and $w = a/c$, since $\frac{dw}{dz} = \frac{(ad - bc)}{(cz + d)^2} \neq 0$ because of the condition $ad - bc \neq 0$.

The points $z = -d/c$ and $w = a/c, c \neq 0$ are called the critical points of the transformation. Also it is conformal at $z = \infty$, since for $z = 1/\xi$, we get $w = \frac{b\xi + a}{b\xi + c}$ and it is easy to check that $\frac{dw}{d\xi} \neq 0$ as $\xi \to 0$.

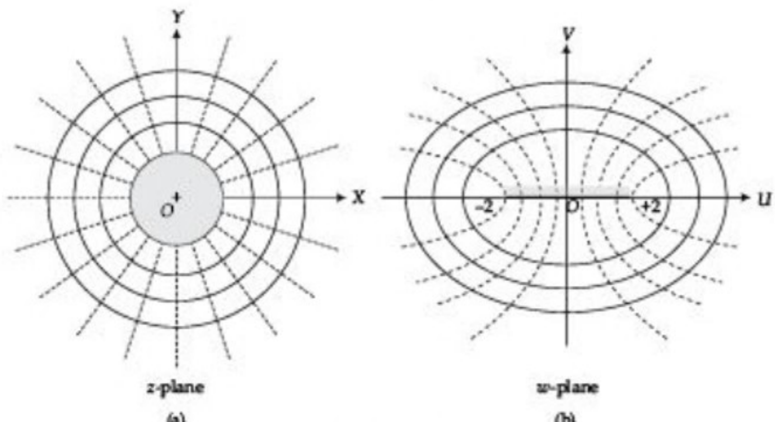


Fig. 18.26

(e) *Transformations*: $w = \sin z, \cos z, \sinh z, \cosh z, \tan z$ and $\tanh z$.

We have, $\quad u + iv = \sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

Thus $u = \sin x \cosh y$, and $v = \cos x \sinh y$. Eliminating y from these, we obtain

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1 \quad ...(18.68)$$

Again eliminating x, we obtain

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1 \quad ...(18.69)$$

The Eqs. (18.68) and (18.69) show that the rectangular net of straight lines x = constant and y = constant is mapped onto a net of hyperbolas, (image of x = constant) and ellipses, (image of y = constant). Exceptions are the vertical lines $x = \pm\pi/2$ which are transformed onto $u \le -1$ and $u \ge +1$, with $v = 0$, respectively. The mapping is shown in Figs. 18.27a and 18.27b.

Further, the upper semi-infinite strip $-\pi/2 \le x \le \pi/2$, $y \ge 0$ in the z-plane under $w = \sin z$ transforms to the upper half plane in the w-plan, refer to Fig. 18.28a and 18.28b and the lower strip $-\pi/2 \le x \le \pi/2$, $y \le 0$ maps to the lower half-plane of the w-plane.

The *mapping* $w = \cos z$ can be discussed on the similar lines as $\sin z$. Alternatively, we can view $\cos z$ as $\cos z = \sin(z + \pi/2)$. The mapping is a translation to the right through $\pi/2$ units given as $z_1 = z + \pi/2$ followed by the sine mapping $w = \sin z_1$.

The mapping $w = \sinh z$ can be expressed as $w = \sinh z = -i \sin(iz)$. This can be interpreted as an anticlockwise rotation $z_1 = iz$ through $\pi/2$, followed by the sine mapping $z_2 = \sin z$, and then followed by a clockwise rotation through $\pi/2$ given as $w = -iz_2$.

The *mapping* $w = \cosh z = \cos(iz)$ can be explained in terms of anticlockwise rotation through $\pi/2$ followed by the cosine mapping. However, to describe the transformation independently express it is $u + iv = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$, which gives

$$u = \cosh x \cos y, \quad \text{and} \quad v = \sinh x \sin y.$$

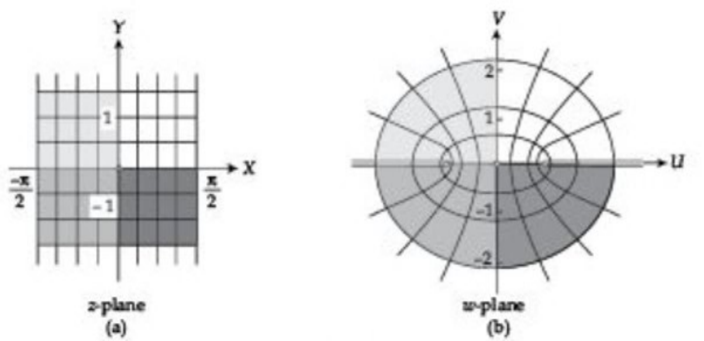


Fig. 18.27

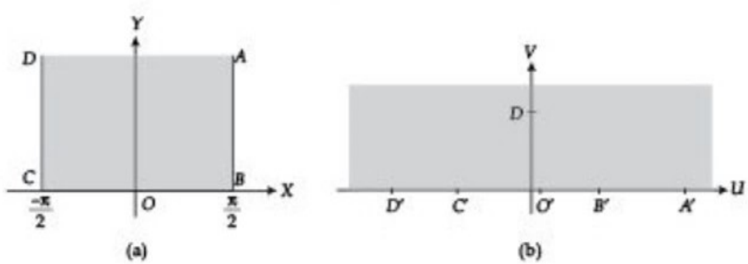


Fig. 18.28

Eliminating x and y from these equations give respectively

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1 \quad ...(18.70)$$

and

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1. \quad ...(18.71)$$

Equation (18.70) shows that the line y = constant in the z-plane maps to a hyperbola in the w-plane, and the Eq. (18.71) shows that the line x = constant maps to ellipse in the w-plane. The rectangular region $a < x < b$, $a < y < d$ in the z-plane transforms into the shaded region in the w-plane as shown in Fig. 18.29a and 18.29b.

Next, the mapping $w = \tan z$ can be expressed as

$$w = \tan z = \frac{\sin z}{\cos z} = \frac{(e^{iz} - e^{-iz})/i}{e^{iz} + e^{-iz}} = \frac{-i(e^{2iz} - 1)}{e^{2iz} + 1}. \quad ...(18.72)$$

In case we set $z_1 = e^{2iz}$ and $z_2 = \frac{z_1 - 1}{z_1 + 1}$, then (18.72) becomes

$$w = \tan z = \frac{-i(z_1 - 1)}{(z_1 + 1)} = -iz_2.$$

Thus the mapping $w = \tan z$ can be viewed as a bilinear transformation preceded by an exponential mapping and proceeded by a clockwise rotation through $\pi/2$.

Similarly, the mapping $w = \tanh z$ is expressed as

$$w = \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^z}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1} = \frac{z_1 - 1}{z_2 - 1},$$

where $z_1 = e^{2z}$ and thus can be explained accordingly.

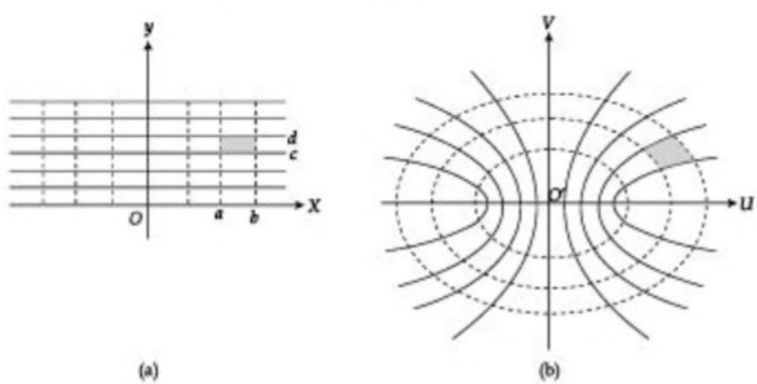


Fig. 18.29

Example 18.39: Determine the angle of rotation at the point $z = (1 + i)/2$ under the mapping $w = z^2$. Find its scale factor also.

Solution: The angle of rotation is given by $\psi_0 = \arg[f'(z_0)]$

Here $f(z) = z^2$ and $z_0 = (1+i)/2$. Thus, $f'(z_0) = f'[(1+i)/2] = (1+i)$.

Hence, $\psi_0 = \arg(1+i) = \tan^{-1}(1) = \pi/4$.

Also the scale factor is, $a = |f'(z_0)| = |1+i| = \sqrt{2}$.

Example 18.40: Find the image of the infinite strip $0 \le x \le \pi/2$ in the z-plane under the mapping $w = \tan^2 z/2$.

Solution: We have $w = \tan^2(z/2) = \dfrac{\sin^2(z/2)}{\cos^2(z/2)} = \dfrac{1-\cos z}{1+\cos z}$.

Also, $\cos z = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$.

At $x = 0$, $\cos z = \cosh y$, hence $w = \dfrac{1-\cosh y}{1+\cosh y}$ is purely real at $x = 0$. Thus $u = \dfrac{1-\cosh y}{1+\cosh y}$.

At $y = 0$, $u = 0$ and for $0 < y < \infty$, u varies from 0 to -1. Also as $y \to -\infty$, u again tends to -1.

At $x = \pi/2$, $\cos z = -i \sinh y$, and hence, $w = \dfrac{1+i\sinh y}{1-i\sinh y}$. It gives

$$|w| = \frac{|1+i\sinh y|}{|1-i\sinh y|} = \frac{\sqrt{1+\sinh^2 y}}{\sqrt{1-\sinh^2 y}} = 1, \text{ for all } y.$$

Thus the line $x = \pi/2$ in the z-plane is mapped into the unit circle $|w| = 1$ in the w-plane. *For any line* $x = a$, when $0 < a < \pi/2$, we have

$$|w| = \left|\frac{1-\cos(a+iy)}{1+\cos(a+iy)}\right| = \frac{\sqrt{(1-\cos a\cosh y)^2 + (\sin a \sinh y)^2}}{\sqrt{(1+\cos a\cosh y)^2 + (\sin a\sinh y)^2}} < 1, \text{ for all } y.$$

Thus as x goes from 0 to a, $0 < a < \pi/2$, the interior of the circle $|w| = 1$ is mapped. The mapping is shown in Fig. 18.30a and 18.30b.

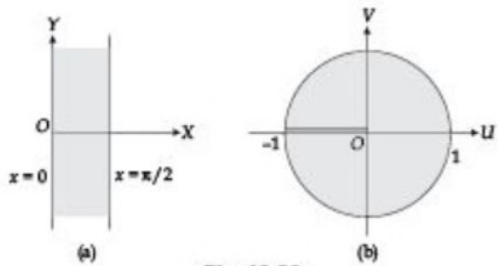


Fig. 18.30

18.9 SCHWARZ-CHRISTOFFEL TRANSFORMATION

The Schwarz-Christoffel transformation maps the interior of a polygon P which can be a triangle, rectangle, or other polygon, bounded or unbounded into the upper-half of the z-plane and conversely. The boundary of the polygon is mapped into the real axis.

Let the polygon have vertices $w_1, w_2, ..., w_n$ in the w-plane and let $\alpha_1, \alpha_2, ..., \alpha_n$ be its interior angles as shown in Fig. 18.31a. Then the Schwarz-Christoffel transformation is given by

$$w = a\int (z-x_1)^{\frac{\alpha_1}{\pi}-1}(z-x_2)^{\frac{\alpha_2}{\pi}-1}\ldots(z-x_n)^{\frac{\alpha_n}{\pi}-1}dz + b, \quad ...(18.73)$$

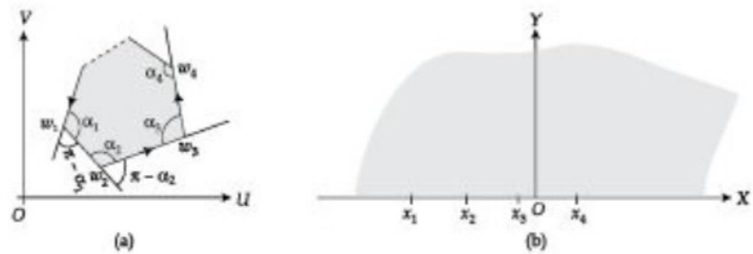


Fig. 18.31

where $x_1, x_2, ..., x_n$ are the points on the real axis corresponding to the vertices $w_1, w_2, ..., w_n$ of the polygon P; a and b are constants and integral in (18.73) is taken over any path from z_0 to z in the upper half-plane.

From (18.73), we have $$\frac{dw}{dz} = a(z-x_1)^{\frac{\alpha_1}{\pi}-1}(z-x_2)^{\frac{\alpha_2}{\pi}-1}\ldots(z-x_n)^{\frac{\alpha_n}{\pi}-1}. \quad ...(18.74)$$

Thus,

$$\arg\left(\frac{dw}{dz}\right) = \arg(a) + \left(\frac{\alpha_1}{\pi}-1\right)\arg(z-x_1) + \left(\frac{\alpha_2}{\pi}-1\right)\arg(z-x_2) + \ldots + \left(\frac{\alpha_n}{\pi}-1\right)\arg(z-x_n). \quad ...(18.75)$$

Now imagine z moving from left to right along the real axis, refer to Fig. 18.31b. When z moves from $-\infty$ to x_1, suppose that w moves along the side $w_n w_1$ of the polygon P towards w_1 and hence no change in the angle. As z passes over x_1 from left to right then in (18.75), the $\arg(z-x_1)$ changes from π to 0 but all other terms remain unaffected. Hence the arg $\frac{dw}{dz}$ decreases by $\left(\frac{\alpha_1}{\pi}-1\right)\pi = \alpha_1 - \pi$, or increases by $\pi - \alpha_1$ in the anticlockwise direction. Thus the direction at w_1 turns by the angle $(\pi - \alpha_1)$ in the positive direction along w_1w_2. Next this angle remains unchanged as z moves from x_1 towards x_2 but as it passes over x_2 the argument increases by $\pi - \alpha_2$ in the anticlockwise direction along w_2w_3 and so on. Thus, we observe that as z moves along x-axis, w traces the polygon $w_1w_2 \ldots w_n$ and the real axis is mapped to a polygon with exterior angles $\pi-\alpha_1, \pi-\alpha_2, ..., \pi-\alpha_n$, where

$$\sum_{j=1}^{n} (\pi - \alpha_j) = 2\pi, \quad \text{or} \quad \sum_{j=1}^{n} \alpha_j = (n-2)\pi.$$

In the case of an unbounded polygon, the vertex w_n may be considered as a point at ∞. Then the transformation (18.73) is modified to

$$w = a\int (z - x_1)^{\frac{\alpha_1}{\pi}-1} (z - x_2)^{\frac{\alpha_2}{\pi}-1} \ldots (z - x_{n-1})^{\frac{\alpha_{n-1}}{\pi}-1} dz + b. \qquad \text{...(18.76)}$$

Example 18.41: Find the transformation which maps the semi-infinite strip of breadth d in the w-plane ($u \geq 0$) into the upper half of the z-plane.

Solution: The semi-infinite strip $ABCD$ as shown in Fig. 18.32a can be considered as the limiting case of a triangle with two vertices at B and C and the third vertex A (or D) at infinity.

Let the vertices B and C map into the points $B'(-1, 0)$ and $C'(1, 0)$ of the z-plane as shown in Fig. 18.32b. The interior angles at B and C are $\pi/2$, hence the Schwarz-Christoffel transformation becomes

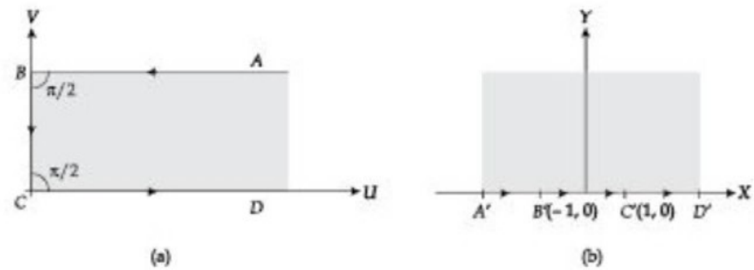


Fig. 18.32

$$w = a\int (z+1)^{\frac{\pi/2}{\pi}-1} (z-1)^{\frac{\pi/2}{\pi}-1} dz + b = a\int \frac{dz}{\sqrt{z^2-1}} + b = a\cosh^{-1} z + b. \qquad \text{...(18.77)}$$

When $z = 1$, then $w = 0$. Thus $0 = a\cosh^{-1}(1) + b$, which gives $b = 0$, since $\cosh^{-1}(1) = 0$.

When $z = -1$, then $w = id$. Thus $id = a\cosh^{-1}(-1)$, or, $\cosh(id/a) = -1$, or, $\cos(d/a) = \cos\pi$, or $a = d/\pi$. Substituting for a and b in (18.77), the requisite transformation is

$$w = \frac{d}{\pi}\cosh^{-1} z, \quad \text{or} \quad z = \cosh\frac{\pi w}{d}.$$

Example 18.42: Find the transformation which maps the strip given by Im $(w) \geq 0$, $-c < \text{Re}(w) < c$ in the w-plane into the upper half of the z-plane.

Solution: The strip $\text{Im}(w) \geq 0, -c < \text{Re}(w) < c$ is shown in Fig. 18.33a. This can be considered a polygon with vertices at $A(-c, 0)$, $B(c, 0)$ and ∞.

Let the vertices A and B map into the points $A'(-1, 0)$ and $B'(1, 0)$ of the z-plane as shown in Fig. 18.33b. The interior angles at A and B are $\pi/2$. The Schwarz-Christoffel transformation becomes

$$w = a\int (z+1)^{\frac{\pi/2}{\pi}-1}(z-1)^{\frac{\pi/2}{\pi}-1}dz + b = -ia\int \frac{1}{\sqrt{1-z^2}}dz + b = -ia\sin^{-1} z + b. \quad ...(18.78)$$

When $z = -1$, then $w = -c$. Thus (18.78) gives, $-c = -ia\sin^{-1}(-1) + b$, or $-c = ia\,\pi/2 + b$

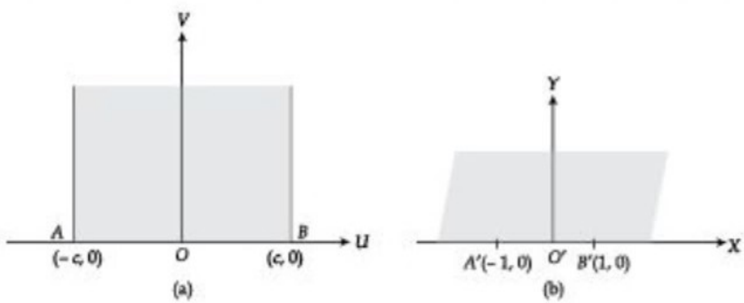


Fig. 18.33

Similarly, at $z = 1$, $w = c$ and we obtain, $c = -ia\,\pi/2 + b$.

These give $b = 0$ and $-ia = \dfrac{2c}{\pi}$. Substituting in (18.78), the requisite transformations is

$$w = \frac{2c}{\pi}\sin^{-1}(z), \text{ or } z = \sin\frac{\pi w}{2c}.$$

EXERCISE 18.6

1. Show that the bilinear transformation $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ is conformal at $z = \infty$.
2. Discuss the transformation $w = \sqrt{z}$. Is it conformal at the origin?
3. Under the transformation $w = 1/z$, show that the image of the hyperbola $x^2 - y^2 = 1$ is a lemniscate.
4. (a) Show that transformation $w = z + (1/z)$ converts the straight line $\arg z = \alpha$, $|\alpha| < \pi/2$ into a branch of hyperbola of eccentricity $\sec\alpha$.

   (b) Show that $w = (z + 1/z)/2$ maps $|z| = r$ to an ellipse if $0 < r \leq 1$.

5. Show that transformation $w = \sin z$ maps the upper semi-infinite strip $-\pi/2 \le x \le \pi/2, y \ge 0$ to the upper half-plane in the w-plane and the lower semi-infinite strip $-\pi/2 \le x \le \pi/2, y \le 0$ to the lower half-plane.
6. Find and sketch the image of $2 \le |z| \le 3, \pi/4 \le \theta \le \pi/2$ under the mapping $w = Ln(z)$.
7. Discuss the transformation $w = e^z$ and show that it transforms the region $0 \le y \le \pi$ into the upper-half of the w-plane.
8. Determine the points where the following mappings are not conformal
   (a) $\bar{z}$ (b) $\cos z$ (c) $\sin \pi z$ (d) $\cosh z$
9. The interior of a square with vertices at 0, 1, $1 + i$ and i in the z-plane is mapped onto a region R under the mapping $w = z + 2 - 3i$. Show that the mapping is conformal and the interior angles of the mapped region R are at right angle.
10. Find the images of the circle $(x - 3)^2 + (y - 2)^2 = 2$ and the line $x + 2y = 8$ under the transformation $w = 1/z$. Show that the images of the circle and the line intersect at the same angle in the w-plane as in the z-plane.
11. Show that under the transformation $w = \cos z$, the infinite strip given by $c \le x \le d$ when $c, d \in (0, \pi/2)$ is mapped into the region between the two branches of the hyperbola lying in the right half of the w-plane.
12. Show that under the conformal mapping $w = u + iv = f(z)$, $f(z)$ being analytic, any harmonic function $\phi(x, y)$ is transformed into another harmonic function.
13. Find the transformation which maps the semi-infinite strip of width π bounded by the lines $v = 0$, $v = \pi$ and $u = 0$ onto the upper-half of the z-plane.
14. Find the transformation which will map the interior of the infinite strip bounded by the lines $v = 0$, $v = \pi$ onto the upper-half of the plane.
15. Show that the Schwarz-Christoffel transformation $f(z) = 2i\int_0^z (z+1)^{-1/2}(z-1)^{-1/2} z^{-1/2}\, dz$

    maps the upper-half-plane onto the rectangle with vertices o, c, $c + ic$ and ic, where $c = \Gamma(1/2)\,\Gamma(1/4)\,\Gamma(3/4)$. Here Γ is the gamma function.

ANSWERS

Exercise 18.1 (p. 1034)

1. (a) $x = 5/13, \quad y = 14/13$ (b) $x = -1, \; y = 3$ (c) $x = 6/19, \quad y = 11/19$
2. (a) $4e^{2\pi i/3}$ (b) $1e^{2\pi i/3}$
3. (a) $\sec \alpha, \quad \alpha - \pi/2$ (b) $2 \sin \alpha/2, z = \dfrac{\pi - \alpha}{2}$
   (c) $\cos(\alpha/2 - \pi/4), \quad \alpha/2 - \pi/4$
6. $z = 6 + 8i$ and $z = 6 + 17i$
10. $\dfrac{1 \pm i}{\sqrt{2}}, \; \dfrac{-1 \pm i}{\sqrt{2}}$ 12. $-1, \dfrac{1 \pm i}{\sqrt{2}}, \dfrac{-1 \pm i}{\sqrt{2}}, \dfrac{1 \pm i\sqrt{3}}{\sqrt{2}}$

16. $(-1 \pm i)/\sqrt{2}$

18. $-\dfrac{1}{2048}(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta)$

Exercise 18.2 (p. 1043)

1. (a) Closed disk, center $-2-5i$, radius $1/2$, connected, not a domain, bounded.
   (b) $y \geq 0$, connected, unbounded, not open, not a domain.
   (c) $x \leq y < \infty$, connected, unbounded, not open, not a domain,
   (d) Between the branches of the hyperbola, connected, unbounded, not open, not a domain
   (e) Horizontal infinite strip of width 2π, connected, unbounded, open, defines a domain.
   (f) $|z| < 1/2$, open disk, center $(0, 0)$, radius $1/2$, connected, bounded, not a domain.
2. (a) Complex plane, except $z = i, -1$
   (b) Complex plane, except $z = \cos(k\pi/2) + i\sin(k\pi/2)$, $k = 0, 1, 2, 3$
   (c) Complex plane, except the circle $|z| = 1$
3. (a) $2 < u < 3, 1 < v < 2.$ (b) $\dfrac{u^2}{4} - 1 < v < 1 - \dfrac{u^2}{4}, -2 < u < 0.$
4. (a) $-3/20, 1/20$ (b) $-1/5, 7/5$ (c) $1, 8.$
6. (a) $-4i$ (b) $-i/2$ (c) does not exist (d) does not exist
7. No.
8. (a) discontinuous (b) discontinuous (c) continuous
10. (a) $-1/(z+1)^2$ (b) $-2/z^3$ (c) not differentiable (d) $2/(1-z)^2$

Exercise 18.3 (p. 1053)

2. (a) $\dfrac{1}{2}(e^{-1} + e)\cos 2,$ (b) $e^{-2n\pi}$ (c) $e^{i\ln\sqrt{2} - (\pi/4 + 2n\pi)}$ (d) $e^{\pi/2 + 2n\pi}$
3. (a) $z = -i[\ln|1 \pm \sqrt{2}| + i\{(\pm\pi/2) + 2n\pi\}]$ (b) $z = [2n\pi - \tan^{-1}(4/3)]/2$
   (c) $z = [\ln(1/3) + i(2n+1)\pi/2]$ (d) $z = \ln|\alpha| + i[2n\pi + \text{Arg}(\alpha)]$
4. (a) $(\pi + 2\pi n)i,$ (b) $\pi/2 + 2\pi k - i\ln(2 \pm \sqrt{3})$ (c) $2n\pi$
7. $\sqrt{(\cos 2x + \cosh 2y)/2} - i\tan^{-1}(\tan x \tanh y).$

Exercise 18.4 (p. 1067)

1. (a) Analytic (b) Not analytic (c) Not analytic
   (d) Not analytic (e) Not analytic (f) Not analytic
4. (a) $1/z + ic$ (b) $e^{-x}(\cos y - i\sin y) + c$ (c) $ze^{2z} + ic$

(d) $z \sin z$ (e) $\bar{z}e^{-\bar{z}+c}$ (f) $\cot z + ic$

5. (a) $f(z) = (1 - \cot(z/2))/2$ (b) ize^{-z}
6. (a) $f(z) = z^2 + c$ (b) $f(z) = z + (1/z) + ic$

Exercise 18.5 (p. 1075)

1. A triangle with vertices $(4, -1)$, $(7, 1)$ and $(7, -3)$
2. A rectangle with vertices $(0, -2)$, $(1, -3)$, $(3, -1)$ and $(2, 0)$
3. A straight line $1 + 4v = 0$ in the w-plane.
5. $w = (1 + iz)/(1 - iz)$ invariant points are $z = -(1 + i \pm \sqrt{6}i)/2$
7. (a) $w = i(1 - z)/(1 + z)$; (b) $w = (iz - 2)/(z + 2)$ (c) $w = (1 - z)/(1 + z)$
8. Image is $u \le 1/2$ and $v \ge 0$.
9. (i) $w = (az + b)/(bz + a)$, a, b, arbitrary
   (ii) $w = (az - c)/(cz + a)$, a, c arbitrary
10. If z_0 is the upper-half of the z-plane show that the bilinear transformation $w = e^{i\alpha}(z - z_0)/(z - \bar{z}_0)$ maps the upper-half of the z-plane into the interior of the unit circle at the origin in the w-plane.

Exercise 18.6 (p. 1087)

6. $\ln 2 \le u \le \ln 3$, $\pi/4 \le v \le \pi/2$.
8. (a) nowhere (b) $z = n\pi$, $n = 0, \pm 1, \ldots$
   (c) $z = (2n + 1)/2$, $n = 0, \pm 1, \ldots$ (d) $z = n\pi i$, $n = 0, \pm 1, \ldots$
10. $11(u^2 + v^2) - 6u + 4v + 1 = 0$ and $8(u^2 + v^2) - u + 2v = 0$; the common angle of intersection is $\tan^{-1}\sqrt{3}$.
13. $z = \cosh w$ 14. $w = Ln\,(z)$

19 Complex Integration

CHAPTER

The concept of definite integral of real functions does not directly extend to the case of complex functions, since real functions are usually integrated over intervals and complex functions are integrated over curves. Surprisingly, complex integrations are not so complex to evaluate, oftenly simpler than the evaluation of real integrations. Some real integrals which are otherwise difficult to evaluate can be evaluated easily by complex integration, and moreover, some basic properties of analytic functions are established by complex integration only.

19.1 LINE INTEGRAL IN THE COMPLEX PLANE

The concept of definite integral $\int_a^b f(x)dx$, as studied in calculus of a real valued function f on a real variable x, was generalized to line integral as applied to vector field in Chapter 9. Here we extend the concept once more and consider the line integral of a complex function. As in calculus of a real variable, here also we distinguish between definite integrals and indefinite integrals. Complex definite integrals are called the *line integrals* and are written as $\int_C f(z)dz$.

The integrand $f(z)$ is integrated over a given curve C in the complex plane called the *path of integration* normally represented by a parametric representation, $z(t) = x(t) + iy(t)$, $a \le t \le b$. The sense of increasing t is called the positive sense on C. The curve C is assumed to be *smooth* curve, that is, it has continuous and non-zero derivative at each $t \in (a, b)$. In case the initial point and terminal point of a curve coincide, that is $z(a) = z(b)$, the curve is said to be closed one.

19.1.1 The Complex Line Integral

The definition of the complex line integral is similar to that of definite integral in calculus of a real variable.

Consider a smooth curve C in the complex plane given by $z(t) = x(t) + iy(t)$, $a \le t \le b$ and $f(z)$ a continuous function defined at each point of C. Divide C into n parts at the points $z_0, z_1, \ldots, z_{n-1}, z_n$ corresponding to the partition $t_0(=a), t_1, \ldots, t_{n-1}, t_n(=b)$, such that $t_0 < t_1 < \ldots < t_n$ of the interval $a \le t \le b$, as shown in Fig. 19.1.

Let $\Delta z_i = z_i - z_{i-1}$ and ξ_i be any point on the arc $z_{i-1} z_i$ between z_{i-1} and z_i

Consider the sum $S_n = \sum_{i=1}^{n} f(\xi_i)\Delta z_i$. *The limit of the sum S_n as $n \to \infty$ in such a way that the length of the chord Δz_i approaches zero is called the line integral of $f(z)$ taken along the curve C oriented from z_0 to z_n and is denoted by*

$$I = \int_C f(z)\,dz. \qquad \ldots(19.1)$$

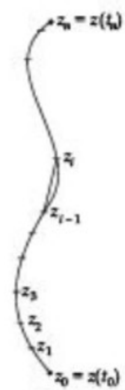


Fig. 19.1

In case the path of integral C is a closed curve, then the integration is denoted by $\oint_C f(z)\,dz$ and the integral along a closed curve is sometimes called the *contour integral*.

In general, the path of integration for complex line integrals are assumed to be *piecewise smooth*, that is, consisting of finitely many smooth curves joined end to end.

Next, writing $f(z) = u(x, y) + iv(x, y)$ and $dz = dx + idy$, the line integral (19.1) can be expressed as

$$\int_C f(z)\,dz = \int_C (u + iv)(dx + idy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy). \qquad \ldots(19.2)$$

Thus the line integral of a complex function can be evaluated in terms of two line integrals of real functions.

19.1.2 Basic Properties of Line Integrals

A few basic properties of the line integrals of a complex function which follow directly from the definition are given below:

1. *Linearity*: $\int_C [k_1 f_1(z) + k_2 f_2(z)]\,dz = k_1\int_C f_1(z)\,dz + k_2\int_C f_2(z)\,dz$, where k_1 and k_2 are two constants.

2. *Sense reversal*: $\int_{z_0}^{z} f(z)\,dz = -\int_{z}^{z_0} f(z)\,dz$

3. *Partitioning of path*: $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$ where the curve C consists of two smooth curves C_1 and C_2 joined end to end as shown in Fig. 19.2.

4. *ML-inequality*: $\left|\int_C f(z)\,dz\right| \le ML$ where M is a constant such that $|f(z)| \le M$ everywhere on C and L is the length of the curve.

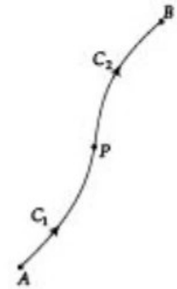


Fig. 19.2

Example 19.1: Evaluate $\int_C z^2 dz$, where C is the straight line joining the origin O to the point $P(2, 1)$ in the complex plane.

Solution: The equation of the line OP is $x = 2y, 0 \le y \le 1$. Thus,

$$dz = dx + i\,dy = 2dy + i\,dy = (2 + i)dy, \text{ and}$$
$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy = 3y^2 + 4iy^2.$$

Hence, $$\int_C z^2 dz = \int_0^1 (3 + 4i)y^2(2 + i)dy = (2 + 11i)\int_0^1 y^2 dy = \frac{1}{3}(2 + 11i).$$

Example 19.2: Evaluate $\oint_C (z - a)^n dz$, where 'a' is a given complex number, n is any integer and C is a circle of radius R centered at 'a' and oriented anticlockwise.

Solution: It is convenient here to use parametric equation of the circle in the form

$$C : z - a = \mathrm{Re}^{i\theta}, 0 \le \theta \le 2\pi, \text{ so } dz = i\mathrm{Re}^{i\theta}\, d\theta.$$

Thus, $$\oint_C (z - a)^n dz = \int_0^{2\pi} R^n e^{in\theta}\, iRe^{i\theta}\, d\theta = iR^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\, d\theta$$

$$= R^{n+1}\left|\frac{e^{i(n+1)\theta}}{n+1}\right|_0^{2\pi} = \frac{R^{n+1}}{n+1}\,[e^{2(n+1)i\pi} - 1] = 0, \text{ provided } n \ne -1.$$

For $n = -1$, we have $\oint_C \frac{dz}{z - a} = \int_0^{2\pi} \frac{1}{\mathrm{Re}^{i\theta}}\, i\mathrm{Re}^{i\theta}\, d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$

Example 19.3: Evaluate the integral $\int_0^{1+i} (x - y + ix^2)dz$

(a) along the straight line from $z = 0$ to $z = 1 + i$

(b) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + i$.

Solution: (a) The equation of the straight line OP, refer to Fig. 19.3, is $y = x$. Thus along the line OP, $z = x + iy = x + ix = (1 + i)x$, which gives $dz = (1 + i)dx, 0 \le x \le 1$, and hence

$$\int_0^{1+i} (x - y + ix^2)dz = \int_0^1 (x - x + ix^2)(1 + i)dx$$

$$= i(1 + i)\int_0^1 x^2dx = -\frac{1}{3}(1 - i).$$

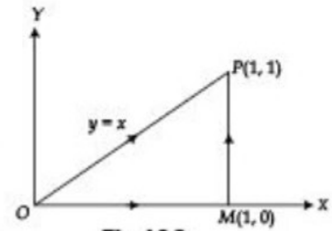


Fig. 19.3

(b) Along the path OM, we have $y = 0$ and thus $z = x + iy = x$ and hence $dz = dx$, $0 \le x \le 1$. Also, along the path MP, we have $x = 1$ and thus $z = x + iy = 1 + iy$, and hence $dz = idy$, $0 \le y \le 1$.

Therefore, the line integral

$$\int_0^{1+i} (x - y + ix^2)dz = \int_0^1 (x + ix^2)dx + \int_0^1 (1 - y + i)(idy).$$

$$= \left[\frac{x^2}{2} + \frac{ix^3}{3}\right]_0^1 + \left[(i-1)y - \frac{iy^2}{2}\right]_0^1 = \frac{1}{2} + \frac{i}{3} + (i - 1) - \frac{i}{2} = -\frac{1}{2} + \frac{5}{6}i.$$

Example 19.4: Evaluate $\oint_C \ln z\, dz$, C is the unit circle $|z| = 1$ taken in counter clockwise sense.

Solution: Any point on the unit circle $|z| = 1$ in parametric form is $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, which gives $dz = ie^{i\theta}d\theta$. Thus the line integral becomes

$$\oint_C \ln z dz = \int_0^{2\pi} \ln e^{i\theta}.ie^{i\theta}d\theta = -\int_0^{2\pi} \theta e^{i\theta}d\theta = -\left[\theta\frac{e^{i\theta}}{i} - 1\frac{e^{i\theta}}{i^2}\right]_0^{2\pi} = -\left[\frac{2\pi e^{2\pi i}}{i} + e^{2\pi i} - 1\right] = -\frac{2\pi}{i} = 2\pi i.$$

Example 19.5: Evaluate $\oint_C |z|^2dz$ around the square with vertices at (0, 0), (1, 0), (1, 1), (0, 1).

Solution: The contour of integration C is $OABCO$ as shown in Fig. 19.4.

We have, $|z|^2 = (x^2 + y^2)$, and also along

OA: $y = 0$, $0 \le x \le 1$, $dz = dx$, $|z|^2 = x^2$

AB: $x = 1$, $0 \le y \le 1$, $dz = idy$, $|z|^2 = 1 + y^2$

BC: $y = 1$, x goes from 1 to 0, $dz = dx$, $|z|^2 = 1 + y^2$

CO: $x = 0$, y goes from 1 to 0, $dz = idy$, $|z|^2 = y^2$

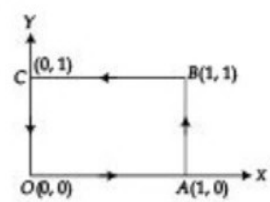


Fig. 19.4

Thus, $$\oint_C |z|^2dz = \int_0^1 x^2dx + i\int_0^1 (1 + y^2)dy + \int_1^0 (1 + x^2)dx + i\int_1^0 y^2dy$$

$$= \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = -1 + i.$$

Example 19.6: Find an upper bound to the integral $I = \int_C \frac{e^z}{z^2} dz$, where C is the straight line from (0, 1) to (2, 0) in the complex plane.

Solution: The path C is the line segment AB as shown in Fig. 19.5. Consider

$$|f(z)| = \left|\frac{e^z}{z^2}\right| = \frac{|e^{x+iy}|}{|x+iy|^2} = \frac{|e^x||e^{iy}|}{x^2+y^2} = \frac{e^x}{x^2+y^2} \quad ...(19.3)$$

On C, e^x is maximum at $x = 2$, so maximum value of e^x is e^2.

Next the minimum value of $x^2 + y^2$ on C is the square of OP, the perpendicular distance from O to the line AB given by $x + 2y - 2 = 0$. This is $(2/\sqrt{5})^2 = 4/5$.

Thus from (19.3) we have, $|f(z)| \le \frac{5e^2}{4}$. Also L, the length $|AB| = \sqrt{5}$. Using the ML-inequality, we have

$$\left|\int_C \frac{e^z}{z^2} dz\right| \le \frac{5e^2}{4}(\sqrt{5}) = 20.65.$$

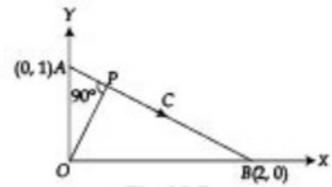


Fig. 19.5

EXERCISE 19.1

1. Evaluate $\int_C z^2 dz$, where C is the curve given by

   (a) $z(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ 2 + i(t-1), & 1 \le t \le 2 \end{cases}$

   (b) the straight line joining the point (1, 1) to the point (2, 4) on the complex plane.

   (c) the parabola $x = 4 - y^2$ as y goes from 2 to −2.

2. Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along, (a) line $y = x/2$,

   (b) real axis to 2 and then vertically to $2 + i$.

3. Evaluate $\int_C (z - z^2) dz$, where C is the upper half of the circle $|z - 2| = 3$.

4. Evaluate $\int_C z \,\mathrm{Re}(z) dz$, where C: $z(t) = t - it^2$ for $0 \le t \le 2$.

5. Evaluate $\int_C \sin^2 z\, dz$, where C: $|z| = \pi$, Re $z \geq 0$, from $-\pi i$ to πi in the right half-plane.

6. Evaluate $\int_C \cos z\, dz$, where C is the semicircle $|z| = \pi$, $x \geq 0$ from $-\pi i$ to πi.

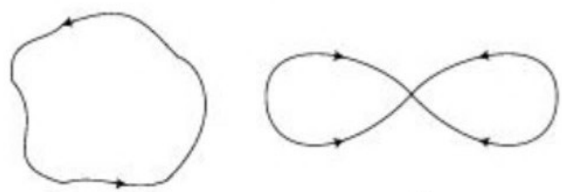
7. Find an upper bound to the integral $\int_C \frac{\sin z}{z(z^2+9)} dz$, where C: $|z| = 5$.

8. If C is a straignt line from $z = 2i$ to $z = 3$, show that $\left|\int_C \frac{\cos z}{z} dz\right| \leq \frac{13}{6} \cosh 2$.

19.2 CAUCHY'S INTEGRAL THEOREM. INDEPENDENCE OF PATH

Cauchy's integral theorem is considered the fundamental theorem of complex integration because of its important consequences. This theorem extends the idea of contour integration further. Before introducing this theorem we need to introduce a few concepts concerning the types of domains.

Simple closed path. A *simple closed path* is a closed path that does not intersect or touch itself. For example, an ellipse, or a circle are simple closed paths. But the self-intersecting an 8-shaped curve is not a simple closed path, refer to Fig. 19.6a and 19.6b.



(a) (b)

Fig. 19.6

Simply connected domain. A *simply connected domain* D in the complex plane is a domain such that every simple closed path in D encloses only points of D. A domain that is not simply connected is called *multiply connected*. For example, interior of an ellipse, or of a circle are examples of simply connected domains while interior of an annulus, for example $1 < |z| < 2$, is boubly connected domain. Figures 19.7a, 19.7b and 19.7c represent respectively simply, doubly and triply connected domains.

Now we are in a position to state the Cauchy's integral theorem.

Theorem 19.1: (Cauchy's integral theorem) *If $f(z)$ is analytic and $f'(z)$ is continuous in a simply connected domain D, then for every piecewise smooth closed curve C in D the contour integral*

$$\oint_C f(z)dz = 0 \quad \text{...(19.4)}$$

(a) (b) (c)

Fig. 19.7

Proof. Writing $f(z) = u + iv$ and $dz = dx + idy$, we have

$$\oint_C f(z)dz = \oint_C (u + iv)(dx + idy) = \int_C (udx - vdy) + i\int_C (vdx + udy) \quad ...(19.5)$$

Since $f'(z)$ is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in D, and hence in the region enclosed by C. Thus Green's theorem, refer to Section 9.4, is applicable to the right side of (19.5) and hence it becomes

$$\oint_C f(z)dz = -\iint_E \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + i\iint_E \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy, \quad ...(19.6)$$

where E is the region bounded by the closed curve C, refer to Fig. 19.7a.

Since $f(z)$ is analytic, u and v satisfy the Cauchy-Riemann equations (18.35), and thus the integrands of the two double integrals on the right side of (19.6) are identically zero and hence we obtain (19.4).

We note that *analyticity of $f(z)$ is only sufficient but not a necessary condition for (19.4) to be true.* We can check very easily that $\oint_C \frac{dz}{z^2} = 0$, where C is the unit circle, refer Example (19.2) for $a = 0$ and $n = -2$, but this result does not follow from Cauchy's theorem since $f(z) = 1/z^2$ is not analytic in $|z| < 1$, zero being the point of singularity.

On the other hand, *simple connectedness of the domain is essential one.* For example, $\oint_C \frac{dz}{z} = 2\pi i$, where C is the unit circle lying in the annulus $1/2 < |z| < 3/2$, refer to Example 19.2. Here, $f(z) = 1/z$ is analytic in the given domain but this domain is not simply connected so Cauchy theorem is not applicable.

Example 19.7: Evaluate the following integrals by applying Cauchy's integral theorem, in case applicable

(a) $\oint_C \cos z\, dz$ (b) $\oint_C \sec z\, dz$ (c) $\oint_C \frac{dz}{z^2-5z+6} dz$ (d) $\oint_C \bar{z}\, dz$

where C is the unit circle $|z| = 1$.

Solution: (a) The integrand $f(z) = \cos z$ is analytic for all z and also $f'(z) = \sin z$ is continuous everywhere, and hence on and inside C also. Thus by Cauchy's theorem $\oint_C \cos z\, dz = 0$.

(b) The integrand $f(z) = \sec z = \frac{1}{\cos z}$ is not analytic at the points $z = \pm\pi/2, \pm 3\pi/2, \ldots$ but all these points lie outside the unit circle $|z| = 1$. Hence $f(z)$ is analytic and $f'(z)$ is continuous in and on C, and thus $\oint_C \sec z\, dz = 0$.

(c) The integrand $f(z) = \frac{1}{z^2 - 5z + 6} = \frac{1}{(z-2)(z-3)}$ is analytic everywhere except at $z = 2, 3$, the points which lie outside the circle $|z| = 1$, hence by Cauchy's theorem $\oint_C \frac{1}{z^2-5z+6} dz = 0$.

(d) The integrand $f(z) = \bar{z}$ is not analytic and hence the Cauchy's theorem is not applicable. In fact, about C: $|z| = 1$, we have $\oint_C \bar{z} dz = \int_0^{2\pi} e^{-i\theta} ie^{i\theta}\, d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$

19.2.1 Independence of Path

In the preceding section, we have noted that a line integral of a function $f(z)$ depends not merely on the end points of the path but also the path itself, refer to Example 19.3. We say that an integral of $f(z)$ is *independent of path in a domain D*, if for every z_1, z_2 in D the value of $\int_{z_1}^{z_2} f(z)dz$ depends only on the end points z_1 and z_2 and not on the choice of the path C joining z_1 to z_2. An important consequence of Cauchy's theorem is to look for the situations when the line integral is independent of path in a domain D. We have the following result:

Theorem 19.2: **(Independence of path)** *If $f(z)$ is analytic in a simply connected domain D, then $\int_C f(z)dz$ is independent of the path for every piecewise smooth curve C lying entirely within D.*

Proof. Let $P(z_1)$ and $Q(z_2)$ be any two points in D and let C_1 and C_2 be two arbitrary paths in D from P to Q intersecting each other only at the end points P and Q, as shown in Fig. 19.8a. Consider the

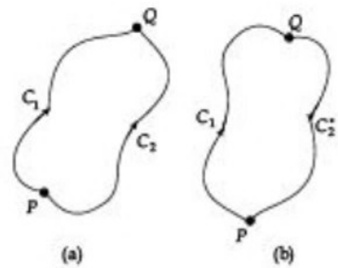


Fig. 19.8

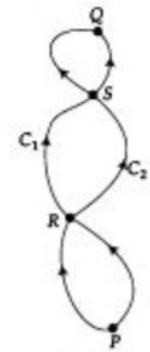


Fig. 19.9

curve C_2^* same as C_2 but with reverse orientation as shown in Fig. 19.8b. We observe that $C_1 \cup C_2^*$ is as piecewise smooth simple closed curve in D, and so according to Cauchy's integral theorem

$$\int_{C_1 \cup C_2^*} f(z)dz = 0, \text{ which gives } \int_{C_1} f(z)dz = -\int_{C_2^*} f(z)dz, \text{ or } \int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

The minus sign disappears in case we integrate in the reverse direction.

This proves the theorem.

In case the two paths have finitely many points in common as shown in Fig. 19.9, then the independence of path can be proved by applying the argument to each loop separately.

19.2.2 Deformation of Path

It is useful to consider path independence in terms of process of *path deformation*. We can visualize deforming C_1 continuously into C_2, refer to Fig. 19.10, keeping the end points P and Q fixed. If f is analytic on C_1 and C_2 and we cross no singular point in the process of deformation from C_1 to C_2, then the line integral $\int_{C_1} f(z)dz$ is equal to $\int_{C_2} f(z)dz$, and the latter may be more easily evaluated than the former.

The path deformation can be applied to closed paths as well and this gives the *extension of Cauchy's theorem to doubly connected regions*.

Consider $f(z)$ to be analytic on and between the two closed paths C_1 and C_2 as shown in Fig. 19.11a. Slit the region enclosed by C_1 and C_2 by introducing a piecewise smooth curve connecting a point A on C_1 with a point B on C_2. Denote the curve A to B as C' and from B to A as C'' as shown in Fig. 19.11b. Consider the curve C as $C = C_1 + C' + C_2^* + C''$

The function $f(z)$ is analytic inside and on C, thus from Cauchy's theorem $\oint_C f(z)dz = 0$, which gives

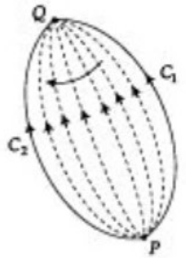


Fig. 19.10

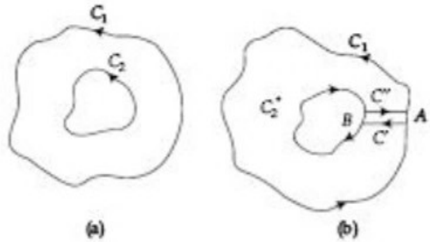


Fig. 19.11

$$\oint_{C_1} f(z)dz + \int_{C'} f(z)dz + \oint_{C_2^*} f(z)dz + \int_{C''} f(z)dz = 0 \text{ or, } \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

since, $$\oint_{C_2^*} f(z)dz = -\oint_{C_2} f(z)dz \quad \text{and} \quad \int_{C''} f(z)dz = -\int_{C'} f(z)dz.$$

Thus, we have the following theorem

Theorem 19.3 (Extension of the Cauchy's integral theorem): *If $f(z)$ is analytic on and between two closed paths C_1 and C_2, then*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz \qquad \text{...(19.7)}$$

This result can be extended to multiply connected regions also as shown in Fig. 19.12. The result is as follows:

Theorem 19.4: *If $f(z)$ is analytic on and between the region included in the closed curves C, C_1, C_2, C_3 etc., then*

$$\int_{C} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \dots \qquad \text{...(19.8)}$$

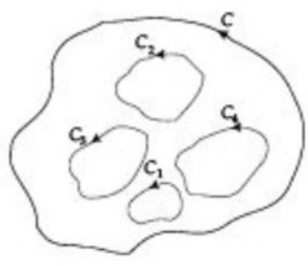


Fig. 19.12

Example 19.8: Verify that the line integral $I = \int_C z^2 dz$ is the same in each of the following cases

(a) C is the straight line OP joining the points $O(0, 0)$ and $P(1, 2)$.
(b) C is the straight line from $O(0, 0)$ to $A(1, 0)$ and then from $A(1, 0)$ to $P(1, 2)$.
(c) C is the parabolic path $y = 2x^2$.

Solution: The three paths are shown in the Fig. 19.13.

(a) Equation of the line OP is $y = 2x, 0 \le x \le 1$.

Thus, $z^2 = (x + iy)^2 = (1 + 2i)^2 x^2$ and $dz = dx + i dy = (1 + 2i)dx$

Therefore, $I = \int_C z^2 dz = \int_0^1 (1+2i)^3 x^2 dx = \frac{1}{3}(1+2i)^3 = -\frac{1}{3}(11+2i)$

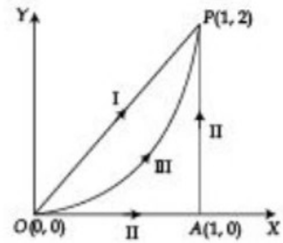


Fig. 19.13

(b) Along OA; $y = 0, 0 \le x \le 1, z = x$. Thus $z^2 = (x+iy)^2 = x^2$, and $dz = dx$.
Along AP; $x = 1, 0 \le y \le 2$, thus $z = 1 + iy, z^2 = (1+iy)^2$, and $dz = idy$.

Therefore,
$$I = \int_0^1 x^2 dx + i\int_0^2 (1+iy)^2 dy = \frac{1}{3} + \frac{(1+2i)^3}{3} - \frac{1}{3}$$
$$= -\frac{1}{3}(11+2i).$$

(c) Along the curve $y = 2x^2, 0 \le x \le 1$, we have, $z = x + iy = x + 2ix^2$, thus $z^2 = (1+2ix)^2 x^2$ and $dz = dx + 4ix\, dx = (1+4ix)dx$

Therefore,
$$I = \int_C z^2 dz = \int_0^1 (1+2ix)^2 x^2 (1+4ix)dx = \int_0^1 (x^2 - 4x^4 + 4ix^3)(1+4ix)dx$$
$$= \int_0^1 [(x^2 - 20x^4) + i(8x^3 - 16x^5)]dx = -\frac{1}{3}(11+2i)$$

Thus along all the three paths the value of integral is the same. In fact integrand z^2 is analytic in the entire complex plane, the value of the line integral I depends only on the end points.

Example 19.9: Evaluate $\oint_\Gamma \frac{1}{z-a} dz$ over any closed path enclosing the given point a.

Solution: Figure 19.14 shows a typical such path but it cannot be parameterized, since we do not know the contour Γ specifically. Let C be a circle of radius r with centre a. Since the function $f(z)$ is analytic on and between Γ and C, thus by principle of deformation of path

$$\int_\Gamma \frac{1}{z-a} dz = \int_C \frac{1}{z-a} dz.$$

The contour C can be parametrized as $z = a + re^{i\theta}, 0 \le \theta \le 2\pi$, and thus

$$\int_\Gamma \frac{1}{z-a} dz = \int_C \frac{1}{z-a} dz = \int_0^{2\pi} \frac{e^{-i\theta}}{r} . ire^{i\theta} d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$

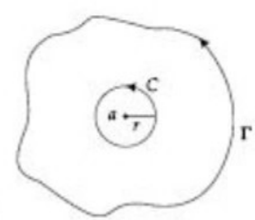


Fig. 19.14

In general, we note that for any closed anticlockwise contour Γ about a point a, we have

$$\oint_\Gamma (z-a)^n dz = \begin{cases} 2\pi i, & n = -1 \\ 0, & n \ne -1 \end{cases} \qquad ...(19.9)$$

This result follows from the principle of deformation and Example 19.2.

Example 19.10: Evaluate $I = \oint_C \frac{dz}{z^2(z-2)(z-4)}$, where C is the rectangle joining the points $(-1,-1)$, $(3,-1)$, $(3,1)$ and $(-1,1)$ in the complex plane.

Solution: The curve C is the rectangle $ABCD$ as shown in Fig. 19.15. Expanding the integrand in the partial fractions, we obtain

$$I = \frac{3}{32}\oint_C \frac{dz}{z} + \frac{1}{8}\oint_C \frac{dz}{z^2} - \frac{1}{8}\oint_C \frac{dz}{z-2} + \frac{1}{32}\oint_C \frac{dz}{z-4} \quad ...(19.10)$$

$$= \frac{3}{32}(2\pi i) + \frac{1}{8}(0) - \frac{1}{8}(2\pi i) + \frac{1}{32}(0) = -\frac{\pi i}{16}$$

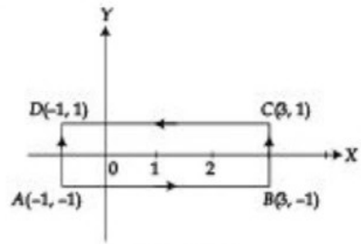


Fig. 19.15

The first three integrals on the right side of (19.10) are evaluated by using (19.9), and the last integral is zero by Cauchy's integral theorem.

Example 19.11: Evaluate the integral $\oint_C \frac{dz}{z(z+2)}$, where C is any rectangle containing the points $z=0$ and $z=-2$ inside it.

Solution: The integrand $f(z)$ is analytic everywhere except at the points $z=0$ and $z=-2$, both the points lying inside the rectangle C. Draw circles C_1 and C_2 respectively enclosing the points $z=0$ and $z=-2$ as shown in Fig. 19.16. The function $f(z)$ is analytic on and between the curves C, C_1 and C_2 and hence by the extension of Cauchy's theorem we have

$$\oint_C \frac{dz}{z(z+2)} = \oint_{C_1} \frac{dz}{z(z+2)} + \oint_{C_2} \frac{dz}{z(z+2)} = \frac{1}{2}\left[\oint_{C_1} \frac{dz}{z} - \oint_{C_1} \frac{dz}{z+2} + \oint_{C_2} \frac{dz}{z} - \oint_{C_2} \frac{dz}{z+2}\right] \quad ...(19.11)$$

By Cauchy's theorem, $\oint_{C_1} \frac{dz}{z+2} = 0$, $\int_{C_2} \frac{dz}{z} = 0$.

Also, $\int_{C_1} \frac{dz}{z} = 2\pi i$ and $\int_{C_2} \frac{dz}{z+2} = 2\pi i$,

refer to Example 19.9. Therefore, from (19.11), we obtain

$$\oint_C \frac{dz}{z(z+2)} = \frac{1}{2}(2\pi i - 2\pi i) = 0.$$

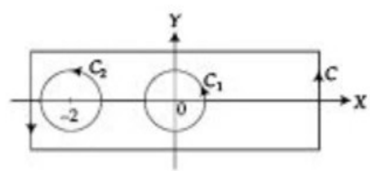


Fig. 19.16

19.3 EXISTENCE OF INDEFINITE INTEGRAL: FUNDAMENTAL THEOREM OF THE COMPLEX INTEGRAL CALCULUS

In this section, we discuss the existence of the indefinite integral of a function $f(z)$ and give the *fundamental theorem of the complex integral calculus*, a result analogous to the fundamental theorem of integral calculus. The theorem is useful for evaluating the integrals for which an antiderivative can be found simply by inspection. The theorem is stated as follows.

Theorem 19.5 (Fundamental theorem of complex integral calculus): *If $f(z)$ is analytic in a simply connected domain D and z_0 be any fixed point in D, then $F(z) = \int_{z_0}^{z} f(z^*)dz^*$ is analytic in D given by $F'(z) = f(z)$, and*

$$\int_{z_0}^{z} f(z^*)dz^* = F(z) - F(z_0). \quad \text{...(19.12)}$$

Proof. Since the conditions of Cauchy's integral theorem are satisfied hence the line integral of $f(z)$ from point z_0 in D to any point z in D is independent of path in D. Keeping z_0 fixed, this integral becomes a function of z, say $F(z)$. Thus, $F(z) = \int_{z_0}^{z} f(z^*)dz^*$, which is uniquely determined.

Next we show that $F(z)$ is analytic in D and $F'(z) = f(z)$. For this consider

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(z^*)dz^* - \int_{z_0}^{z} f(z^*)dz^*\right]$$

$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^*)dz^* = \frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(z) + f(z^*) - f(z)]dz^*$$

$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)dz^* + \frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(z^*) - f(z)]dz^*$$

$$= f(z) + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^*) - f(z)]dz^*$$

or, $$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(z^*) - f(z)]dz^* \quad \text{...(19.13)}$$

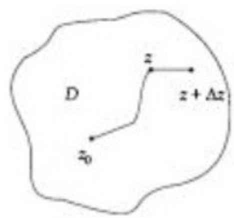


Fig. 19.17

The points z_0, z, $z + \Delta z$ are shown in Fig. 19.17. Because of path independence principle, without loss of generality, the path from z to $z + \Delta z$ may be taken as straight line.

Using the ML-inequality on the right side of (19.13), we obtain

$$\left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(z^*) - f(z)]dz^*\right| \le \frac{1}{\Delta z} M|\Delta z| = M,$$

where $M = \max. |f(z^*) - f(z)|$ on the straight line from z to $z + \Delta z$. Since $f(z)$ is analytic and therefore, continuous also, thus $M \to 0$ as $\Delta z \to 0$, and, hence letting $\Delta z \to 0$ in (19.13) gives

$$F'(z) = f(z) \qquad ...(19.14)$$

A function $F(z)$ satisfying $F'(z) = f(z)$ is called an indefinite integral or primitive of f.

Next, we show that any two primitives of a given function f differ at most by an arbitrary additive constant. For this, let $G(z)$ be an other primitive of $f(z)$, therefore, $G'(z) = f(z)$, and then

$$G'(z) - F'(z) = f(z) - f(z) = 0, \text{ and hence } G(z) - F(z) = c \text{ in } D,$$

where c is an arbitrary constant.

Thus, in general, for a specific primitive $G(z)$ of f, we have

$$G(z) = \int_{z_0}^{z} f(z^*)\, dz^* = F(z) + c.$$

To evaluate c put $z = z_0$, we obtain $0 = F(z_0) + c$ which gives, $c = -F(z_0)$,

and so $\int_{z_0}^{z} f(z^*)dz^* = F(z) - F(z_0)$. This completes the proof.

Example 19.12: Evaluate the following integrals

(a) $\int_{2i}^{3} \sin z\, dz$ (b) $\int_{0}^{1} z^2 e^{z^3} dz$ (c) $\int_{0}^{2i} \sinh z\, dz$

Solution: The functions $\sin z, z^2e^{z^3}$ and $\sinh z$ are analytic everywhere, therefore, the integrals can be evaluated by applying indefinite integration.

(a) $F(z) = \int_{2i}^{3} \sin z\, dz = [-\cos z]_{2i}^{3} = -\cos 3 + \cos 2i = \cosh 2 - \cos 3.$

(b) $F(z) = \int_{0}^{1} z^2 e^{z^3} dz = \frac{1}{3}\int_{0}^{1} e^t dt = \frac{1}{3}\left[e^t\right]_0^1 = \frac{1}{3}(e-1)$

(c) $F(z) = \int_{0}^{2i} \sinh z\, dz = [\cosh z]_0^{2i} = \cosh 2i - 1 = \cos 2 - 1.$

EXERCISE 19.2

1. Verify Cauchy's integral theorem for the integral of z^2 over the boundary of the square with vertices $1 + i, -1 + i, -1 - i$, and $1 - i$ taken counterclockwise.
2. Can the Cauchy's integral theorem be applied for evaluating the following integrals? If so, evaluate, if not, evaluate otherwise

   (a) $\oint_C e^{\sin z^2}\, dz$; C: $|z| = 1$ (b) $\oint_C \frac{e^z}{z^2+9}\, dz$; C: $|z| = 2$

   (c) $\oint_C \frac{3z+5}{z(z+2)}\, dz$; C: $|z| = 1$ (d) $\oint_C \frac{dz}{z^2}$; C: $|z| = \frac{1}{2}$

3. Evaluate $\oint_\Gamma \left(\frac{4}{z-1} - \frac{5}{z+4}\right) dz$, Γ is any square of side 3 units with its centre at origin.
4. Evaluate the following integrals around the unit circle C: $|z| = 1$ indicating whether Cauchy's integral theorem applies

   (a) $\oint_C e^{-z^2}\, dz$ (b) $\oint_C \frac{dz}{|z|^2}$ (c) $\oint_C \text{Im}\, z\, dz$

5. Evaluate the following integrals, C taken in counterclockwise sense

   (a) $\oint_C \frac{e^z}{z^2 - 5iz - 6}\, dz$; C: $|z| = 1$ (b) $\oint_C \left(z + \frac{3}{z^2}\right) dz$; C: $|z| = 1$

   (c) $\oint_C \frac{\cosh^2 2z}{(z+3i)(z^2+16)}\, dz$; C: $|z| = 2$

6. Evaluate the following integrals using the extension of the Cauchy's integral theorem to multiply connected domains

   (a) $\oint_C \frac{2z-3}{z^2-3z-18}\, dz$; C: $|z| = 8$ (b) $\oint_C \frac{2z^3+z^2+4}{z^4+4z^2}\, dz$; C: $|z-2| = 4$

   (c) $\oint_C \frac{dz}{(z-1)(z-2)(z-3)}$; C: $|z| = 4$

7. By evaluating $\oint_C e^z dz$, C: $|z| = 1$, show that $\int_0^{2\pi} e^{\cos\theta} \cos(\theta + \sin\theta) d\theta = 0$ and $\int_0^{2\pi} e^{\cos\theta} \sin(\theta + \sin\theta) d\theta = 0$.

8. Prove that $\int_C (z^2+2)^2 dz = 8\pi a(12\pi^4 a^4 + 20\pi^2 a^2 + 15)/15$, where C is the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ joining the points $(0, 0)$ and $(2\pi a, 0)$.

9. Show that the integral $\int_C e^{-2z}\, dz$, where C is the path joining the points $z = 1 + 2\pi i$ and $z = 3 + 4\pi i$ is independent of the path of integration. Evaluate it by taking a suitable path.

10. Use the fundamental theorem to evaluate the following integrals

(a) $\int_i^0 \cos 3z\, dz$ (b) $\int_0^{3i} ze^{z^2}\, dz$ (c) $\int_0^{1+2i} z\sin(z^2) dz$

(d) $\int_0^{1+\pi i} (z^2 + \cosh 2z) dz$ (e) $\int_0^1 \frac{\tan^{-1} z}{1+z^2} dz$ (f) $\int_{-i}^{i} z\cosh^2 z\, dz$

19.4 CAUCHY'S INTEGRAL FORMULA. DERIVATIVES OF AN ANALYTIC FUNCTION

Cauchy's integral formula is an important consequence of Cauchy's integral theorem. This gives a representation of an analytic function $f(z)$ at any interior point z_0 of a simply connected domain D as a contour integral evaluated along the boundary of a simple closed curve C which lies inside D and encloses the point z_0. The result is of fundamental importance and is stated as follows.

Theorem 19.6 (Cauchy's integral formula): *Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0*

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz, \qquad \text{...(19.15)}$$

the integration being taken counter-clockwise.

Proof. Let C_1 be a circle with centre z_0 and radius r lying entirely within C. The function $\frac{f(z)}{z-z_0}$ is analytic on and within the closed curves C and C_1 as shown in Fig. 19.18, thus by the extension of Cauchy's integral theorem,

$$\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_1} \frac{f(z)}{z-z_0} dz = \oint_{C_1} \frac{[f(z_0) + f(z) - f(z_0)]}{z-z_0} dz$$

$$= f(z_0) \oint_{C_1} \frac{dz}{z-z_0} + \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0} dz. \qquad \text{...(19.16)}$$

Consider the first integral on the right side of (19.16). Put $z - z_0 = re^{i\theta}$, we have $dz = ire^{i\theta}\,d\theta$, and hence

$$\oint_{C_1} \frac{dz}{z - z_0} = \int_0^{2\pi} i\,d\theta = 2\pi i.$$

Next, if I denotes the second integral on the right side of (19.16), then

$$|I| = \left|\oint_{C_1} \frac{f(z) - f(z_0)}{z - z_0}\,dz\right| \le \oint_{C_1} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| |dz| = \oint_{C_1} \frac{|f(z) - f(z_0)|}{|z - z_0|} |dz|. \qquad \text{...(19.17)}$$

Since $f(z)$ is continuous in D, (for it is analytic in D), thus for a given $\epsilon > 0$, there exists a number $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$, wherever $|z - z_0| < \delta$.

Choosing the radius r of the circle C_1 such that $r < \delta$ and hence from (19.17), we have

$$|I| \le \oint_{C_1} \frac{|f(z) - f(z_0)|}{|z - z_0|} |dz| < \oint_{C_1} \frac{\epsilon}{r} |dz| = \frac{\epsilon}{r} 2\pi r = 2\pi\epsilon.$$

Since $\epsilon > 0$ can be chosen arbitrary small, thus $|I|$ can be made arbitrary small tending to zero, and thus Eq. (19.16) becomes

$$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0), \text{ or } f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\,dz, \text{ which is (19.15).}$$

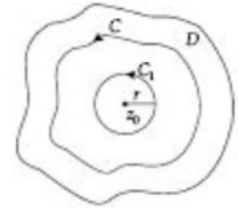


Fig. 19.18

Example 19.13: Evaluate the integral $\oint_C \frac{z^2 + 1}{z^2 - 1}\,dz$, C: $|z - 1| = 1$.

Solution: Writing the integrand as $\frac{z^2 + 1}{z^2 - 1} = \frac{(z^2 + 1)/(z + 1)}{z - 1}$.

We observe that $f(z) = (z^2 + 1)/(z + 1)$ is analytic on and inside C, and here $z_0 = 1$, as shown in Fig. 19.19. Hence by Cauchy's integral formula

$$\oint_C \frac{z^2 + 1}{z^2 - 1}\,dz = 2\pi i f(1) = 2\pi i.$$

Example 19.14: Evaluate the integral

$$\oint_C \frac{z^2 + 1}{z(2z - 1)}\,dz, \text{ C: } |z| = 1.$$

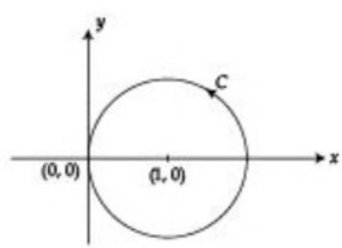


Fig. 19.19

Solution: Let $I = \oint_C \frac{z^2+1}{z(2z-1)} dz$. The integrand $(z^2+1)/z(2z-1)$ is not analytic at the point $z = 0$ and $z = 1/2$ both of which lie inside C. Writing it as

$$\frac{z^2+1}{z(2z-1)} = (z^2+1)\left[\frac{1}{(z-1/2)} - \frac{1}{z}\right].$$

Therefore, $I = \oint_C \frac{z^2+1}{z-1/2} dz - \oint_C \frac{z^2+1}{z} dz = 2\pi i\,[z^2+1]_{z=1/2} - 2\pi i\,[z^2+1]_{z=0} = \frac{5\pi i}{2} - 2\pi i = \frac{\pi i}{2}$,

using the Cauchy's integral formula.

Example 19.15: Evaluate the integral $\oint_C \frac{dz}{(z-z_0)(z-z_1)}$, where the points z_0 and z_1 lie inside the simple closed curve C and integration is taken in counter-clockwise sense.

Solution: Let C_0 and C_1 be two small simple closed non-intersecting curves surrounding z_0 and z_1 respectively and lying entirely within C. Then by the extension of the Cauchy's integral theorem,

$$\oint_C \frac{dz}{(z-z_0)(z-z_1)} = \oint_{C_0} \frac{dz}{(z-z_0)(z-z_1)} + \oint_{C_1} \frac{dz}{(z-z_0)(z-z_1)}. \quad \text{...(19.18)}$$

Consider the first integral on the right side of (19.18), we have

$$\oint_{C_0} \frac{dz}{(z-z_0)(z-z_1)} = \oint_{C_0} \frac{dz/(z-z_1)}{z-z_0} = \frac{2\pi i}{(z_0-z_1)}, \text{ using Cauchy's formula.}$$

Similarly, $\oint_{C_1} \frac{dz}{(z-z_0)(z-z_1)} = \frac{2\pi i}{z_1-z_0}$. Hence (19.18) becomes

$$\oint_C \frac{dz}{(z-z_0)(z-z_1)} = \frac{2\pi i}{z_0-z_1} + \frac{2\pi i}{z_1-z_0} = 0.$$

Example 19.16: Evaluate the integral $\oint_C \frac{\tan z}{z^2-1} dz$, C: $|z| = 3/2$, where integration is taken counter-clockwise.

Solution: The function $\tan z$ is not analytic at $z = \pm\pi/2, \pm 3\pi/2, \ldots$ but all these points lie outside the curve C: $|z| = 3/2$. Further $(z^2-1)^{-1}$ is not analytic at $z = 1$ and $z = -1$, both of these lie inside C.

Writing the integrand as $\frac{\tan z}{z^2-1} = \frac{1}{2}\left[\frac{\tan z}{z-1} - \frac{\tan z}{z+1}\right]$

and thus,

$$I = \oint_C \frac{\tan z}{z^2-1}dz = \frac{1}{2}\left[\oint_C \frac{\tan z}{z-1}dz - \oint_C \frac{\tan z}{z+1}dz\right] = \frac{2\pi i}{2}[\tan^{-1}(1) - \tan^{-1}(-1)] = 2\pi i \tan^{-1}(1) = \frac{\pi^2 i}{2}.$$

19.4.1 Derivatives of an Analytic Function

We now apply Cauchy's integral formula to show that *if a complex function f(z) is analytic in a domain D, then its derivatives of all orders exist and are also analytic in D.* This result is an important departure when compared to the real functions. A real function that is differentiable need not have a second derivative, and if it has a second, it need not have a third, and so on. In case of the derivatives of analytic functions we have the following result.

Theorem 19.7 **(Generalized Cauchy's integral formula):** *If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D which are also analytic in D and the values of these derivatives at a point z_0 in D are given by*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz,\ n = 1, 2, \ldots, \qquad \ldots(19.19)$$

where C is any simple closed path in D taken in counter-clockwise sense.

Proof. The Cauchy's integral formula is $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz.$

Differentiating it under the integral sign w.r.t. z_0, we obtain

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz. \qquad \ldots(19.20)$$

Similarly, $f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}dz$ and, in general, $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz.$

This completes the proof.

Example 19.17: Evaluate the integral $\oint_C \frac{e^z}{z^3}dz$, C: $|z| = 1$, taken in counter-clockwise sense.

Solution: Let $I = \oint_C \frac{e^z}{z^3}dz$. Here $f(z) = e^z$ is analytic in the region bounded by the simple closed curve $|z| = 1$. The singular point $z = 0$ of $1/z^3$ lies inside $|z| = 1$. Hence, applying the generalized Cauchy's integral formula $I = \oint_C \frac{e^z}{z^3}dz = \frac{2\pi i}{2!}\frac{d^2}{dz^2}(e^z)\Big|_{z=0} = \pi i.$

Example 19.18: Evaluate $\oint_C \frac{(z+1)}{z(z-2)(z-4)^3}dz$, C: $|z-3| = 2$ in the counterclockwise sense.

Solution: Let $I = \oint_C \frac{(z+1)}{z(z-2)(z-4)^3} dz.$

The integrand has singularities at $z = 0, 2,$ and 4, out of these $z = 2$ and 4 lie inside C.

Consider two non-intersecting closed contour C_1 and C_2, as shown in Fig. 19.20, lying completely within C, respectively about the point $z = 2$ and $z = 4$. Applying the principle of deformation the integral I becomes

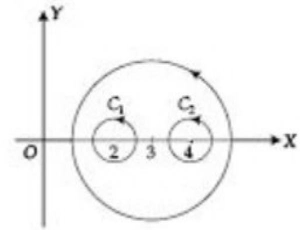


Fig. 19.20

$$I = \oint_C \frac{z+1}{z(z-2)(z-4)^3} dz$$

$$= \oint_{C_1} \left[\frac{z+1}{z(z-4)^3}\right] \frac{dz}{z-2} + \oint_{C_2} \left[\frac{z+1}{z(z-2)}\right] \frac{dz}{(z-4)^3} = I_1 + I_2, \text{ say.}$$

Now, $I_1 = \oint_{C_1} \left[\frac{z+1}{z(z-4)^3}\right] \frac{dz}{z-2} = 2\pi i \left[\frac{z+1}{z(z-4)^3}\right]_{z=2} = -\frac{3\pi i}{8}$, using Cauchy's integral formula.

Similarly, $I_2 = \oint_{C_2} \left[\frac{z+1}{z(z-2)}\right] \frac{dz}{(z-4)^3} = \frac{2\pi i}{2!} \frac{d^2}{dz^2} \left[\frac{z+1}{z(z-2)}\right]_{z=4} = \frac{23\pi i}{64}$

Therefore, $I = \frac{-3\pi i}{8} + \frac{23\pi i}{64} = -\frac{\pi i}{64}.$

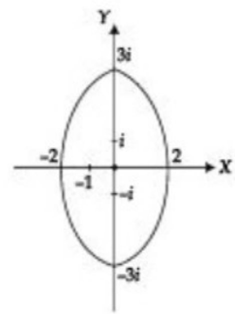


Fig. 19.21

Example 19.19: If $F(a) = \oint_C \frac{4z^2 + z + 5}{z - a} dz$, where C: $(x/2)^2 + (y/3)^2 = 1$, taken in counter-clockwise sense, then find $F(3.5)$, $F(i)$, $F'(-1)$ and $F''(-i)$.

Solution: We have, $F(3.5) = \oint_C \frac{4z^2 + z + 5}{z - 3.5} dz$

The integrand $\frac{4z^2 + z + 5}{z - 3.5}$ is analytic everywhere except at the point $z = 3.5$ which lies outside the ellipse $(x/2)^2 + (y/3)^2 = 1$, as shown in Fig. 19.21. Therefore, it is analytic everywhere within C and hence by Cauchy's integral theorem $F(3.5) = 0$.

Next the numerator $f(z) = 4z^2 + z + 5$ of the integrand is analytic everywhere in C and $a = i, -1$ and $-i$ all lie within C. Therefore by Cauchy's integral theorem, $f(a) = \frac{1}{2\pi i} \oint_C \frac{4z^2 + z + 5}{z - a} dz$, which gives

$$\oint_C \frac{4z^2+z+5}{z-a}dz = 2\pi i f(a) = 2\pi i[4a^2 + a + 5]$$

Hence $F(a) = 2\pi i[4a^2 + a + 5]$, which implies $F'(a) = 2\pi i[8a + 1]$ and $F''(a) = 16\pi i$.

Thus, $$F(i) = 2\pi(i+1),\ F'(-1) = -14\pi i \text{ and } F''(-i) = 16\pi i.$$

19.5 CONVERSE OF CAUCHY'S INTEGRAL THEOREM: MORERA'S THEOREM. CAUCHY'S INEQUALITY. LIOUVILLE'S THEOREM

In this section, we prove the converse of the Cauchy's integral theorem called the Morera's theorem. Two other results, Cauchy's inequality and Lioville's theorem associated with the complex integration are also discussed.

Theorem 19.8 (**Morera's theorem**): *If $f(z)$ is continuous in a simply connected domain D and if $\oint_C f(z)dz = 0$ for every closed path C in D, then $f(z)$ is analytic in D.*

Proof. Let z be an arbitrary point in D and z_0 be a fixed point in D. If $f(z)$ is continuous and its integral around any closed path C in D is zero, then the definite integral of $f(z)$, refer to Theorem 19.5, is defined by $F(z) = \int_{z_0}^{z} f(z^*)dz^*$ and further $F(z)$ is analytic in D with $F'(z) = f(z)$. Now the analyticity of $F(z)$ in D implies the analyticity of $F'(z)$ and hence that of $f(z)$ in D. This proves the converse of Cauchy's theorem.

Next, using the generalized Cauchy's integral formula (19.19), we find a bound for $f^{(n)}(z_0)$; a result known as *Cauchy's inequality* given as follows:

Theorem 19.9 (**Cauchy's inequality**): *If $f(z)$ is analytic within and on a circle C: $|z - z_0| = r$ and $|f(z)| \le M$ on C, then*

$$|f^{(n)}(z_0)| \le \frac{M(n!)}{r^n} \qquad ...(19.21)$$

Proof. To prove this, choose the contour C in the generalized Cauchy's integral formula (19.19), a circle $|z - z_0| = r$ and apply the *ML*-inequality with $|f(z)| \le M$ on C, we obtain

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \le \frac{n!}{2\pi}M.\frac{1}{r^{n+1}}2\pi r = \frac{n!M}{r^n},$$

This proves the Cauchy's inequality (19.21).

For $n = 0$, the Cauchy's inequality gives $|f(z_0)| \le M$, where z_0 is the centre of circle C and M is the bound of $f(z)$ on C. From this, there follows another important result called the *Maximum-minimum principle* stated next.

Theorem 19.10 (**Maximum-minimum principal**): *If $f(z)$ is analytic within and on a simple closed curve C and $f(z)$ is not a constant, then the maximum and minimum value of $|f(z)|$ occurs on the boundary of C.*

Next using the Cauchy's inequality we prove another important result on *entire functions, the functions which are analytic everywhere*, called the Liouville's theorem.

Theorem 19.11 (Liouville's theorem): *If $f(z)$ is entire and bounded for all z in the complex plane, then $f(z)$ must be constant.*

Proof. Since $f(z)$ is bounded, say $|f(z)| \le M$ for all z. The Cauchy's inequality (19.21) for $n = 1$, gives

$$|f'(z)| \le M/r \quad \text{...(19.22)}$$

Further $f(z)$ is analytic for every z in the complex plane, so we can take r in (19.22) as large as we please and conclude that $|f'(z_0)| \to 0$. Since z_0 is arbitrary, thus $f'(z) = 0$ for all z and therefore $f(z)$ is constant.

Example 19.20: If $f(z) = e^z$, then find a bound on $f^{(n)}(0)$.

Solution: The function $f(z)$ is analytic for all z in the finite complex plane. Consider a unit circle $|z| = 1$ about the point $z = 0$, then $|f(z)| = |e^z| = e^x \le e$ for all z on C and hence by Cauchy's inequality $|f^{(n)}(0)| \le \dfrac{n!M}{r} = \dfrac{(n!)e}{1} = (n!)e$.

Example 19.21: By direct calculations verify the maximum-minimum principle for $f(z) = \sin z$ in the domain D defined by $0 \le x \le \pi$ and $0 \le y \le 1$ and find bounds on $|\sin z|$ inside D.

Solution: The function $f(z) = \sin z$ is analytic for all z and the domain D is bounded. We have

$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \text{ and thus,}$$

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y.$$

Differentiating it w.r.t. x and y, we obtain respectively

$$\frac{\partial}{\partial x}|\sin z|^2 = 2\sin x \cos x \cosh^2 y - 2\cos x \sin x \sinh^2 y$$

$$= \sin 2x(\cosh^2 y - \sinh^2 y) = \sin 2x,$$

and, $$\frac{\partial}{\partial y}|\sin z|^2 = 2\sin^2 x \cosh y \sinh y + 2\cos^2 x \sinh y \cosh y$$

$$= \sinh 2y[\sin^2 x + \cos^2 x] = \sinh 2y.$$

The extreme values of $|\sin z|^2$, and hence that of $|\sin z|$ will occur at those points of D where both of these derivatives vanish simultaneously.

Now $\sin 2x = 0 \Rightarrow x = \pi/2$, but $\sinh 2y \ne 0$ for $0 < y < 1$, so $|\sin z|^2$ and hence $|\sin z|$ has neither maxima nor minima in D.

On the boundary $x = 0$ of D, as shown in Fig. 19.22,

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} = \sinh y.$$

It has a minimum value 0 at (0, 0) and a maximum value sinh1 at (0, 1).

Next on the boundary $x = \pi$ of D

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} = \sinh y$$

has a minimum value 0 at $(\pi, 0)$ and a maximum value sinh1 at $(\pi, 1)$.

Proceeding similarly, we find that on the boundary $y = 0$ of D, $|\sin z|$ has two minima of 0 at $(0, 0)$ and $(\pi, 0)$ and a maximum of 1 at $(\pi/2, 0)$, and on the boundary $y = 1$ of D, $|\sin z|$ has two minima equal to sinh 1 at $(0, 1)$ and $(\pi, 1)$ and a maximum of $\sqrt{1+\sinh^2 1}$ at $(\pi/2, 1)$.

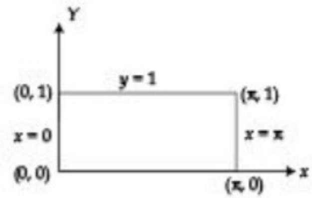


Fig. 19.22

Thus the smallest value of $|\sin z|$ on the boundary of D is zero, and the largest value is $\sqrt{1+\sinh^2 1}$. This verifies maximum-minimum principle, and hence $0 < |\sin z| < \sqrt{1+\sinh^2 1}$ for all $z = x + iy$ in D.

EXERCISE 19.3

1. Evaluate the integral $\oint_C \frac{dz}{z^2+9}$, where C is

   (a) $|z-3i| = 4$ (b) $|z+3i| = 2$ (c) $|z| = 5$

   taken in counter-clockwise sense.

2. Evaluate the following integrals, the contour C being taken in counter-clockwise sense

   (a) $\oint_C \frac{z^2+1}{z(2z+1)} dz$; C: $|z| = 1$ (b) $\oint_C \frac{e^{2z}}{(z-1)(z-2)} dz$; C: $|z| = 3$

   (c) $\oint_C \frac{\cos \pi z}{z^2-1} dz$; C the rectangle with vertices $2 \pm i, -2 \pm i$

   (d) $\oint_C \frac{\cosh(z^2-\pi i)}{z-\pi i} dz$; C the rectangle with vertices ± 4 and $\pm 1 + 4i$

   (e) $\oint_C \frac{z-3}{z^3+z} dz$; C: $|z| = 2$

   (f) $\oint_C \frac{\sin(iz)}{z^2+1} dz$; C the triangle with vertices at $1-2i, -1-2i$ and $2i$.

3. Evaluate the following integrals, the contour C being taken in counter-clockwise sense

   (a) $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3} dz$; C: $|z| = 1$ (b) $\oint_C \frac{e^z}{(z^2+\pi^2)^2} dz$; C: $|z| = 4$

(c) $\oint_C \frac{1}{(z^2+4)^2}dz$; C: $|z-i|=2$ (d) $\oint_C \frac{e^z}{z^2(z+1)^3}dz$; C: $|z|=2$

(e) $\oint_C \left[\frac{e^{iz}}{z^3}+\frac{z^4}{(z+i)^2}\right]dz$; C: $|z|=2$ (f) $\oint_C \frac{\sin z}{z^m}dz$; C: $|z|=1, m=2,3,4,\ldots$

4. Evaluate the contour integral of $f(z)=\frac{(2+i)\sin z^4}{(z+4)^2}$ over Γ, any closed path enclosing -4.

5. Let f be differentiable on a domain D containing a closed path Γ and all points enclosed by Γ. Prove that $\oint_\Gamma \frac{f'(z)}{z-a}dz=\oint_\Gamma \frac{f(z)}{(z-a)^2}dz$ for any 'a' enclosed by Γ.

6. If the function $f(z)$ is analytic inside and on a simple closed curve C containing the point $z=a$ inside it, then show that

$$f^{(n)}(a)=\frac{n!}{2\pi}\int_0^{2\pi} e^{-in\theta} f(a+e^{i\theta})d\theta,\ n=0,1,2,\ldots$$

7. Using the Cauchy's inequality find a bound on $f^{(n)}(0)$, when $f(z)=e^{3z}$ and C: $|z|=1$.

8. Verify the maximum/minimum principle for the function $f(z)=e^z$ in the domain $-1\le x\le 1$, $-2\le y\le 2$ and place bounds on $|e^z|$ inside the given domain.

ANSWERS

Exercise 19.1 (p. 1095)

1. (a) $\frac{1}{3}(2+11)i$ (b) $\frac{-86}{3}-6i$ (c) $\frac{16}{3}i$

2. (a) $\frac{5}{2}(2-i)$ (b) $\frac{1}{3}(14+11i)$ 3. 30 4. $\frac{-152}{15}-12i$

5. $\left(\pi-\frac{1}{2}\sinh 2\pi\right)i$ 6. $2i\sinh\pi$.

Exercise 19.2 (p. 1105)

2. (a) Yes !, 0 (b) Yes !, 0 (c) No !, $5\pi i$ (d) No!, 0

3. $8\pi i$

4. (a) Yes!, 0 (b) No, 0 (c) No, $-\pi$

5. (a) 0, (b) 0 (c) 0

6. (a) $4\pi i$ (b) $4\pi i$ (c) 0

9. $(e^{-2} - e^{-6})/2$

10. (a) $-\frac{i}{3}\sinh 3$ (b) $(e^{-9} - 1)/2$ (c) $[1 - \cos(-3 + 4i)]/2$

(d) $\frac{1}{3}(\pi i + 1)^3 + \frac{1}{2}\sinh 2$ (e) $\pi^2/32$ (f) 0

Exercise 19.3 (p. 1113)

1. (a) $\pi/3$ (b) $-\pi/3$ (c) 0

2. (a) $-\frac{\pi i}{2}$ (b) $2\pi i(e^4 - e^2)$ (c) 0

(d) $-2\pi i \cosh \pi^2$ (e) 0 (f) $-2\pi \sin 1$

3. (a) πi (b) i/π (c) $\pi/16$

(d) $(11e^{-1} - 4)\pi i$ (e) $-\pi(8 + i)$ (f) $\frac{2\pi i}{(m-1)!}\sin\left[(m-1)\frac{\pi}{2}\right]$

4. $-512\pi(1 - 2i)\cos(256)$ 7. $e^3(n!)$ 8. $\frac{1}{e} < |e^z| < e.$

20 Taylor Series, Laurent Series and The Residue Theorem

CHAPTER

Power series in general, and Taylor series in particular, are direct generalizations of the power and Taylor series in reals. The Laurent series, a series of positive and negative powers of $(z - z_0)$, represents an extension of the Taylor series which is no longer applicable when an expansion of $f(z)$ is required about a singularity z_0 and is used to classify points at which $f(z)$ is not analytic. Residue of $f(z)$ at z_0 is the coefficient of $(z - z_0)^{-1}$ in the Laurent series expansion of $f(z)$ about $z = z_0$. Residues are used to compute contour integration and certain complicated real integrals as well.

20.1 COMPLEX SERIES AND CONVERGENCE TESTS

In this section we define complex series and discuss their convergence and divergence. Also we consider the concept of absolute, conditional and that of uniform convergence.

20.1.1 Complex Series and Their Convergence

By a *complex series* we mean a series of the form

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \dots, \qquad \dots(20.1)$$

where the z_n's are complex numbers. As in the real case, we say that the series (20.1) *converges* if the limit of the sequence of *partial sums*, $S_n = z_1 + z_2 + \dots + z_n$ exists as $n \to \infty$. That is, the series converges to S if to each number $\epsilon > 0$, no matter how small, there exists an integer $N(\epsilon)$ such that $|S_n - S| < \epsilon$ for all $n > N$; S is called the *sum* of the series.

A series that is not convergent is called a *divergent series*.

Further the series Σz_n can be expressed in terms of two real series for $z_n = x_n + iy_n$. Thus, Σz_n converges and has the sum $S = U + iV$ if, and only if the series Σx_n converges and has the sum U and the series Σy_n converges and has the sum V.

Convergence or divergence tests in complex series are practically the same as in case of reals.

Cauchy's convergence principle for complex series. *An infinite series Σz_n is convergent if, and only if for every $\epsilon > 0$, no matter how small, we can find $N(\epsilon)$ such that*

$$|z_{n+1} + z_{n+2} + ... + z_{n+m}| < \epsilon, \text{ for every } n > N, \ m = 1, 2, ... \quad ...(20.2)$$

In case we set $m = 1$, (20.2) gives that to each $\epsilon > 0$, no matter how small, there corresponds an integer $N(\epsilon)$ such that $|z_{n+1}| < \epsilon$ for all $n > N$, which is equivalent to saying that $z_{n+1} \to 0$ (or, for that matter $z_n \to 0$) as $n \to \infty$. Thus, we have the following result:

Theorem 20.1 (A necessary condition for convergence): *If an infinite series Σz_n converges, then $\lim_{n\to\infty} z_n = 0$.*

We must note that *it is only the necessary condition for convergence but not sufficient.* For example, the series $1 + 1/2 + 1/3 + ...$ satisfies this condition but is divergent one. Thus this result can be applied only to prove the non-convergence of the series.

Absolute convergence. *A series Σz_n is said to be 'absolutely convergent' if the series of the absolute values of the terms $\Sigma|z_n| = |z_1| + |z_2| + ...$ is convergent.*

If the series Σz_n converges but $\Sigma|z_n|$ diverges, then the series Σz_n is called *conditionally convergent.* For example, the series $1 - 1/2 + 1/3 - 1/4 + ...$ is only conditionally convergent.

If a series is absolutely convergent, then it is convergent also. But the divergence of the series $\Sigma|z_n|$ does not imply the divergence of the series Σz_n.

20.1.2 Tests for Convergence and Divergence

We discuss a few tests for convergence of the series.

I. Comparison test: *For a given series Σz_n if we can find a convergent series Σb_n of non-negative real terms such that $|z_n| \le b_n$, $n = 1, 2, ...,$ then the series Σz_n is also convergent, even absolutely.*

The result follows from the Cauchy's convergence principle for series.

For comparison, normally we use the geometric series $\Sigma x^n = 1 + x + x^2 + ...$, which converges to the sum $1/(1 - x)$ for $|x| < 1$ and diverges for $|x| \ge 1$.

II. Ratio test: *If for a series Σz_n, $z_n \ne 0$, $(n = 1, 2,...)$, $\lim_{n\to\infty} |z_{n+1}/z_n| = l$, then the series Σz_n converges absolutely, if $l < 1$ and diverges absolutely, if $l > 1$. In case $l = 1$, or if the limit does not exist, then no conclusion is drawn.*

Another powerful test to test the convergence of the complex series is the root test.

III. Root test: *If for a series Σz_n, $\lim_{n\to\infty} (z_n)^{1/n} = l$, then the series Σz_n converges absolutely, if $l < 1$ and diverges absolutely, if $l > 1$. In case $l = 1$, or if the limit does not exist, then no conclusion is drawn.*

Example 20.1: Test for convergence or divergence of the series

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} = 1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + ...$$

Solution: Consider $|z_n| = \left|\frac{i^n}{n!}\right| = \frac{1}{n!} < \frac{1}{2^n}$ for all $n \geq 4$. Further the series $\sum_{n=0}^{\infty} 1/2^n$ is a geometric series with common ratio $r = 1/2 < 1$, and hence, is a convergent series. Thus it follows from the comparison test that the given series is also convergent.

Example 20.2: Test for convergence or divergence of the series $\sum_{n=0}^{\infty} \frac{(100+75i)^n}{n!}$.

Solution: Consider $\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(100+75i)^{n+1}}{(n+1)!}\frac{n!}{(100+75i)^n}\right| = \left|\frac{100+75i}{n+1}\right| = \frac{125}{n+1}$ which tends to zero as $n \to \infty$. Hence by ratio test the series is convergent.

Example 20.3: Test for convergence or divergence of the series $\Sigma e^{-(2+3i)n}$

Solution: Consider $\left|\frac{z_{n+1}}{z_n}\right| = |e^{-(2+3i)}| = |e^{-2}.e^{-3i}| = e^{-2} < 1$.

Hence by ratio test the series is convergent.

Example 20.4: Test for convergence or divergence of the series $\sum_{n=1}^{\infty} \left(\frac{3n-2}{np+1}\right)^n (3-4i)^n$.

Solution: Here, $|z_n| = \left(\frac{3n-2}{np+1}\right)^n |3-4i|^n = \left(\frac{3n-2}{np+1}\right)^n 5^n = \left(\frac{15n-10}{np+1}\right)^n$.

Consider $\lim_{n\to\infty} |z_n|^{1/n} = \lim_{n\to\infty} \frac{15n-10}{np+1} = \frac{15}{p}$.

Hence by root test the given series is convergent for $p > 15$ and divergent for $p < 15$. For $p = 15$ we can verify that $\lim_{n\to\infty} z_n \neq 0$ and hence the series is divergent for $p = 15$.

20.1.3 Uniform Convergence of Series of Functions

In general, the terms of an infinite series may be functions of z rather than constants, that is, $\Sigma f_n(z) = f_1(z) + f_2(z) + \ldots$. Then the set of all points in the z-plane for which the series converges is called the *region of convergence* of the series.

Let $\Sigma f_n(z)$ be a series of single valued complex functions defined in a domain D and $S_n(z) = f_1(z) + \ldots + f_n(z)$ be the nth partial sum. Consider the sequence $\{S_n(z)\}$ of partial sums. If at a point $z = z_0$ in D, the sequence $\{S_n(z)\}$ of partial sums converges to $f(z_0)$, then we say that the series $\Sigma f_n(z_0)$ converges to $f(z_0)$. This convergence is called *'pointwise convergence'*.

If for each $z \in D$, the sequence $\{S_n(z)\}$ of partial sums converges to $f(z)$, then we say that the series $\Sigma f_n(z)$ *converges uniformly* to $f(z)$, that is, for every $\epsilon > 0$, no matter how small, there exists an $N(\epsilon)$ independent of z, such that $|S_n(z) - f(z)| < \epsilon$, for all $n > N(\epsilon)$.

A series which is uniformly convergent is also pointwise convergent.

Next we state the sufficient condition for a given series to be uniformly convergent.

Weierstrass' M-test. *Let $\Sigma f_n(z)$ be an infinite series of single valued complex functions defined in a domain D and let $\{M_n\}$ be a sequence of positive terms, where $|f_n(z)| \le M_n$ for $n = 1, 2, \ldots$ and for all $z \in D$. If the series ΣM_n is convergent, then the series $\Sigma f_n(z)$ is uniformly and absolutely convergent.*

Example 20.5: Show that $\Sigma \dfrac{z^n - 1}{n^2 + |z|^2}$ converges uniformly for $|z| < 1$.

Solution: We have, $|f_n(z)| = \left|\dfrac{z^n - 1}{n^2 + |z|^2}\right| \le \dfrac{|z|^n + 1}{n^2 + |z|^2} < \dfrac{2}{n^2}$, for all z in $|z| < 1$. Since, the series $\Sigma \dfrac{1}{n^2}$ is convergent, the given series is uniformly convergent by Weierstrass's M-test.

Remark: There is no relation between absolute and uniform convergence. There are series that converge absolutely but not uniformly and other that converge uniformly but not absolutely.

EXERCISE 20.1

Test for convergence or divergence of the following series:

1. $\displaystyle\sum_{n=1}^{\infty} n^2 \left(\frac{i}{2}\right)^n$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(3i)^n n!}{n^n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n - 3i)}{n^p}$
4. $\displaystyle\sum \left(\frac{3}{4}i\right)^n n^3$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2+i)^n}$
6. $\displaystyle\sum_{n=1}^{\infty} e^{-in}$
7. Show that the series $\Sigma (\sin nz)/n^2$ is uniformly convergent in $|z| \le 1$.
8. Show that the geometric series $1 + z + z^2 + \ldots$ is (a) uniformly convergent in any closed disk $|z| \le r < 1$, (b) not uniformly convergent in its whole disk of convergence $|z| < 1$.
9. Show that the series $x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \ldots$ converges absolutely but not uniformly.
10. Show that the series $\dfrac{1}{x^2+1} - \dfrac{1}{x^2+2} + \dfrac{1}{x^2+3} - \ldots$ converges uniformly but not absolutely.

20.2 POWER SERIES REPRESENTATIONS

Power series are the most important series in the study of complex analysis since we shall observe that their sums are analytic functions and also every analytic function can be represented by a power series. In this section we discuss power series, their convergence and power series representations of the analytic functions.

20.2.1 Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \qquad \text{...(20.3)}$$

is called a *power series* about $z = z_0$. Here $a_0, a_1, \dots$ are real or complex constants called the *coefficients* of the series, and z_0 again a constant, real or complex, is called the *center of the series*. For $z_0 = 0$, we obtain a power series in powers of z given as $\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots$.

Convergence of a power series We note that a power series always converges at the point $z = z_0$, since for $z = z_0$ the series reduces to a constant a_0. In general the series (20.3) may converge in a disk with center z_0, or in the whole z-plane, or only at z_0. For example, refer to Example 20.6,

(a) the geometric series Σz^n converges absolutely if $|z| < 1$;

(b) the power series $\Sigma\, z^n/n! = 1 + z + z^2/2! + z^3/3! + \dots$ converges absolutely for every z; while

(c) the series $\Sigma n! z^n = 1 + z + 2z^2 + 6z^3 + \dots$, converges only at $z = 0$.

We have the following result for the convergence of series of the form (20.3).

Theorem 20.2 **(Convergence of a power series):**

(a) Every power series converges at its center z_0.

(b) If the power series converges at a point $z = z_1 \neq z_0$, then it converges absolutely for all z in the disk $|z - z_0| < |z_1 - z_0|$.

(c) If the power series diverges at a point $z = z_2$, then it diverges for all z farther away from z_0 than z_2, that is, for all z such that $|z - z_0| > |z_2 - z_0|$.

Proof.

(a) Obviously for $z = z_0$ the series (20.3) sums to simply a_0.

(b) The convergence at the fixed point $z = z_1$ implies that $a_n(z_1 - z_0)^n \to 0$ as $n \to \infty$. Thus, there exists a finite positive constant M such that $|a_n(z_1 - z_0)^n| \le M$ for every $n = 0, 1, 2, \dots$

Now $|a_n(z-z_0)^n| = |a_n||z-z_0|^n = |a_n||z_1 - z_0|^n \cdot \left|\dfrac{z-z_0}{z_1-z_0}\right|^n \le M\left|\dfrac{z-z_0}{z_1-z_0}\right|^n$

For $|z - z_0| < |z_1 - z_0|$, the series $M\sum \left|\dfrac{z-z_0}{z_1-z_0}\right|^n$ is a convergent geometric series with common ratio less than 1. Therefore the series $\Sigma a_n(z-z_0)^n$ converges absolutely for all z in $|z - z_0| < |z_1 - z_0|$.

(c) The result follows simply from (b), since convergence at a point z_3, farther away from z_0 than z_2, would imply convergence at z_2 which is a contradiction.

Radius of convergence: Let R be the radius of the smallest circle with center at z_0 that contains all the points at which the power series (20.3) is convergent. Then the series is convergent for all z for which $|z - z_0| < R$ and diverges for all z for which $|z - z_0| > R$. The real number R is called the *radius of convergence* and the circle $|z - z_0| = R$ is called the *circle of convergence* of the power series (20.3).

If $R = 0$ the series is convergent only at the point $z_0 = 0$, and if $R = \infty$ the series is convergent for all z.

On the circle of convergence, the series may converge at some, or all, or none of the points. For example, in the three cases listed below the radius of convergence is $R = 1$ but

(a) The series $\Sigma z^n/n^2$ converges everywhere on R, since $\Sigma z^n/n^2 = \Sigma 1/n^2$ on $|z| = 1$;

(b) The series $\Sigma z_n/n$ converges at $z = -1$ but diverges at $z = 1$;

(c) The series Σz^n diverges everywhere on $|z| = 1$.

20.2.2 Test for the Convergence of a Power Series

The convergence of a power series $\Sigma a_n(z-z_0)^n$ can be determined by the application of the ratio test. Let z_n denote the nth term of the power series, then we obtain

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{a_{n+1}}{a_n}(z - z_0)\right| = \left|\frac{a_{n+1}}{a_n}\right| |z - z_0| \text{ and, therefore,}$$

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L^*|z - z_0|, \text{ where } L^* = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Thus by ratio test, the series converges, if $L^*|z-z_0| < 1$, or $|z-z_0| < 1/L^*$, provided $L^* \neq 0$, and diverges, if $|z - z_0| > 1/L^*$, and the *radius of convergence* of the power series (20.3) is

$$R = \frac{1}{L^*} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

We can also use the Cauchy's root test and write $L^* = \lim_{n\to\infty} |a_n|^{1/n}$.

Example 20.6: Find the region of convergence for the following power series:

(a) $\sum_{n=0}^{\infty} z^n$ (b) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ (c) $\sum_{n=0}^{\infty} n!z^n$

Solution:

(a) Here $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = |z|$. Hence by ratio test the series converges for all $|z| < 1$ and radius of convergence is $R = 1$.

(b) Here, $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty} \left|\frac{z}{n+1}\right| = 0 < 1$. Hence by ratio test the series converges for all z and radius of converges is $R = \infty$.

(c) Here, $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}|(n+1)z| \to \infty$. Thus the series is divergent for all z except at $z = 0$, its center, hence the radius of convergence is $R = 0$.

Example 20.7: Find the radius of convergence, region of convergence and the circle of convergence of the following power series

(a) $\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}(z-3i)^n$ (b) $\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^{n^2}z^n$ (c) $\sum_{n=1}^{\infty}\frac{n(5+2i)^n}{3^n}(z-1)^n$

Solution: (a) We have $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n)!}{(n!)^2}\frac{[(n+1)!]^2}{(2n+2)!}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}$

Thus the series converges in the region $|z-3i| < 1/4$. The circle of convergence is $|z-3i| = 1/4$ with center $z_0 = 3i$.

(b) $R = \frac{1}{\lim_{n\to\infty}|a_n|^{1/n}} = \lim_{n\to\infty}\left[1+\frac{2}{n}\right]^{-n} = \lim_{n\to\infty}\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^{-2} = e^{-2}$. Thus, the series converges in the region $|z| < e^{-2}$. The circle of convergence is $|z| = e^{-2}$ with center $z = 0$.

(c) $R = \frac{1}{\lim_{n\to\infty}|a_n|^{1/n}} = \lim_{n\to\infty}\frac{3}{|5+2i|}\cdot\frac{1}{n^{1/n}} = \frac{3}{|5+2i|}\lim_{n\to\infty}(1/n^{1/n}) = \frac{3}{\sqrt{29}}$. Thus, the series converges in the region $|z-1| < 3/\sqrt{29}$. The circle of convergence is $|z| = 3/\sqrt{29}$ with center $z = 0$.

20.2.3 Function Represented by Power Series

Using Cauchy's convergence principle for series we can check very easily that the power series is uniformly convergent within its circle of convergence. Let the power series $\Sigma a_n(z-z_0)^n$ converge to the function $f(z)$. To simplify the formulae, without loss of generality, we can take $z_0 = 0$. Thus, if the power series $\Sigma a_n z^n$ has a non-zero radius of convergence R with sum, say $f(z)$, then we write

$$f(z) = \Sigma a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots,\ |z| < R$$

and we say that $f(z)$ is represented by the power series. In fact, *a power series $\Sigma a_n z^n$ represents an analytic function within its circle of convergence.*

For example, the geometric series Σz^n represents the function $1/(1-z)$ which is analytic within its circle of convergence $|z| = 1$.

Further, this power series representation is unique in the sense that a function $f(z)$ cannot be represented by two different power series with the same center.

20.2.4 Operations on Power Series

We define the following operations:

(a) **Termwise addition or subtraction:** If $f(z) = \Sigma a_n z^n$ and $g(z) = \Sigma b_n z^n$ are two power series with radii of convergence R_1 and R_2 respectively, then the termwise addition or subtraction of the series is defined as

$$f(z) \pm g(z) = \Sigma\,(a_n \pm b_n) z^n$$

The radius of convergence is equal to the minima of R_1 and R_2.

(b) **Termwise multiplication:** If $f(z) = \Sigma a_n z^n$ and $g(z) = \Sigma b_n z^n$ are two power series with radii of convergence R_1 and R_2 respectively, then the termwise multiplication, called the *Cauchy product*, of these series is defined as

$$f(z)g(z) = \sum_n \left(\sum_{r=0}^{n} a_r b_{n-r} \right) z^n = \Sigma\, c_n z^n, \text{ where } c_n = \sum_{r=0}^{n} a_r b_{n-r}\,.$$

The radius of convergence is equal to the minima of R_1 and R_2.

(c) **Termwise differentiation:** Termwise differentiation of a power series is permissible within its circle of convergence and the resultant series is called the *derived series*. It is given as

$$f'(z) = \sum_{n=1}^{\infty} n\, a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \ldots$$

The radius of convergence of the derived series is the same as the radius of convergence of the original series.

(d) **Termwise integration:** Termwise integration of a power series is permissible within its circle of convergence. It is given as

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \ldots\,.$$

The radius of convergence of the integrated series is the same as the radius of convergence of the original series.

EXERCISE 20.2

Find the center and radius of convergence of the following power series:

1. $\sum_{n=0}^{\infty} (n+1)(z+1)^n$
2. $\sum_{n=1}^{\infty} n(z+i\sqrt{2})^n$
3. $\sum_{n=0}^{\infty} (n+2i)^n z^n$
4. $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n}(z+2-i)^n$
5. $\sum_{n=1}^{\infty} \left(1-\frac{\pi}{n}\right)^{n^2} z^n$
6. $\sum_{n=0}^{\infty} z^{n!}$

7. $\sum\limits_{n=4}^{\infty} e^n(z+i)^n$ 8. $\sum\limits_{n=0}^{\infty} e^{in} z^n$ 9. $\sum\limits_{n=1}^{\infty} \frac{2^{n-1} z^{2n-1}}{(4n-3)^2}$

10. $\sum\limits_{n=1}^{\infty} \frac{n}{n+1}\left(\frac{z}{2}\right)^n$ 11. $\sum\limits_{n=1}^{\infty} i^n z^n$ 12. $\sum\limits_{n=1}^{\infty} \left(\frac{1+2ni}{n+2i}\right)^n z^n$

13. Using Cauchy's product of two power series show that if $f(z) = \Sigma$ for all z, then $[f(z)]^2 = f(2z)$.

20.3 TAYLOR AND MACLAURIN SERIES

In the preceding section we have observed that the sum of a power series with positive radius of convergence is an analytic function. Here we will explore that the converse of this result is also true, that is, every given analytic function $f(z)$ which is analytic inside the region $|z - z_0| < R$ can be expressed as a power series inside this region. This series is called the *Taylor series* of the function $f(z)$, and is the complex analogous of the real Taylor series.

Theorem 20.3 (Taylor series): *A function $f(z)$ which is analytic inside a circle $|z - z_0| = R$ may be represented inside the circle as a convergent power series*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n; \quad a_n = \frac{1}{n!} f^{(n)}(z_0),$$

called the Taylor expansion of $f(z)$ about the point $z = z_0$.

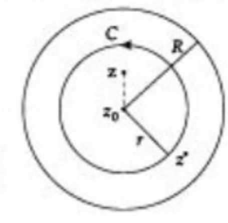


Fig. 20.1

Proof. Let z be any arbitrary point inside the circle $|z - z_0| = R$. Draw a circle with centre at z_0 of radius $r < R$ enclosing the point z and let z^* be any point on the circle $|z - z_0| = r$, as shown in Fig. 20.1. Consider

$$\frac{1}{z^* - z} = \frac{1}{(z^* - z_0) - (z - z_0)} = \frac{1}{(z^* - z_0)}\left[1 - \frac{z - z_0}{z^* - z_0}\right]^{-1}. \qquad ...(20.4)$$

Since $\left|\frac{z - z_0}{z^* - z_0}\right| < 1$, expanding the righthand side of (20.4) in binomial series, we obtain

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0}\left[1 + \frac{z - z_0}{z^* - z_0} + \left(\frac{z - z_0}{z^* - z_0}\right)^2 + \ldots + \left(\frac{z - z_0}{z^* - z_0}\right)^n\right] + \frac{1}{z^* - z}\left(\frac{z - z_0}{z^* - z_0}\right)^{n+1} \qquad ...(20.5)$$

Now, since $f(z)$ is analytic inside the region $|z^* - z_0| = r$ and z is an interior point of this, by Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^* - z} dz^*, \qquad ...(20.6)$$

where C: $|z^* - z_0| = r < R$, taken in counter-clockwise sense.

Inserting (20.5) in (20.6) yields

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^* - z_0} dz^* + \frac{z - z_0}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^2} dz^*$$

$$+ \frac{(z - z_0)^2}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^3} dz^* + \ldots + \frac{(z - z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^* + R_n(z), \quad \ldots(20.7)$$

where $$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} dz^*$$

is the *remainder term*.

Using Cauchy's generalized integral formula for derivatives we obtain from (20.7),

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \ldots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z) \quad \ldots(20.8)$$

This is called *Taylor formula with remainder* $R_n(z)$.

Since $f(z)$ is analytic, it has derivatives of all orders, so n can be taken as large as desired. Taking $n \to \infty$, Eq. (20.8) becomes

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \ldots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + \ldots .$$

or, $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \ldots(20.9)$$

where $a_n = \frac{1}{n!} f^{(n)}(z_0)$, provided $\lim_{n \to \infty} R_n(z) = 0$.

The series given by (20.9) is called the *Taylor series expansion of* $f(z)$ *about the point* z_0. Obviously the series, (20.9) will converge and represent $f(z)$ if, and only if $\lim_{n \to \infty} R_n(z) = 0$, which we prove as follows.

Since $|z^* - z| \neq 0$ and $f(z)$ is analytic inside and on C, so it is bounded, and thus so is the function $\left|\frac{f(z^*)}{(z^* - z)}\right|$, say $\left|\frac{f(z^*)}{(z^* - z)}\right| \le M$, for all z^* on C. Also C has the radius $|z^* - z_0| = r$ and length $2\pi r$.

Hence, by ML-inequality, we have

$$|R_n(z)| = \frac{|z - z_0|^{n+1}}{2\pi}\left|\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} dz^*\right| \le \frac{|z - z_0|^{n+1}}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = Mr\left|\frac{z - z_0}{r}\right|^{n+1} . \quad \ldots(20.10)$$

Now z lies inside C, thus $\left|\frac{z-z_0}{r}\right| < 1$, so the right side of (20.10) approaches zero as n tends to infinity. This proves the convergence of the Taylor series.

Since z is chosen as an arbitrary point inside C, so the convergence holds for every z inside C. Hence, the *convergence is uniform*.

Further the expansion (20.9) is unique, since otherwise, consider that

$$f(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n, \qquad ...(20.11)$$

where at least one of the coefficients $b_n \neq a_n$. Now, the power sereis (20.11) converges to $f(z)$ in the circular region $|z - z_0| < R$, and thus $b_n = \frac{1}{n!} f^{(n)}(z_0)$ which is same as a_n. This proves the uniqueness.

Remarks 1. The radius of convergence of Taylor series expansion of a function $f(z)$ is the distance between z_0 and the nearest singular point of $f(z)$. For example $1/(1 - z)$ has singularity at $z = 1$, its expansion, the geometric series has radius of convergence 1.

2. Since the Taylor's series representation of an analytic function is unique, we can obtain this representation directly expanding the function using binomial theorem for any index, whenever possible. For example, the expansion of $f(z) = 1/(1 - z)$ can be obtained simply as $(1 - z)^{-1} = 1 + z + z^2 + z^3 + ..., |z| < 1$.

3. Taylor's series are power series and conversly also, a power series with non-zero radius of convergence is the Taylor series of its sum.

4. The complex analytic functions can always be represented by power series of the form $\Sigma a_n(z - z_0)^n$, which is not true in general for real functions. There are real functions with derivatives of all orders but cannot be represented by a power series e.g., $f(x) = e^{-1/x^2} \neq 0$, and $f(0) = 0$ about $x = 0$.

Maclaurin's series: *A Maclaurin's series is a Taylor series with center $z_0 = 0$. Thus Maclaurin's series representation of $f(z)$ is*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \textit{where} \quad a_n = \frac{f^{(n)}(0)}{n!} \qquad ...(20.12)$$

Next, we list some important power series expansions, which can be derived as in case of calculus of a real variable simply replacing x by z, since the coefficient formulae are the same.

(a) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... + \frac{z^n}{n!} + ..., |z| < \infty$

(b) $\frac{1}{1-z} = 1 + z + z^2 + z^3 + ... + z^n + ..., |z| < 1$

(c) $\frac{1}{1+z} = 1 - z + z^2 - z^3 + ... + (-1)^n z^n + ..., |z| < 1$

(d) $\operatorname{Ln}(1+z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \ldots + (-1)^{n+1}\dfrac{z^n}{n} \ldots, \ |z| < 1$

(e) $\operatorname{Ln}(1-z) = -z - \dfrac{z^2}{2} - \dfrac{z^3}{3} - \ldots - \dfrac{z^n}{n} - \ldots, \ |z| < 1$

(f) $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \ldots + (-1)^{n+1}\dfrac{z^{2n-1}}{(2n-1)!} + \ldots, \ |z| < \infty$

(g) $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \ldots + (-1)^n\dfrac{z^{2n}}{(2n)!} + \ldots, \ |z| < \infty$

(h) $\sinh z = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \ldots + \dfrac{z^{2n+1}}{(2n+1)!} + \ldots, \ |z| < \infty$

(i) $\cosh z = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \ldots + \dfrac{z^{2n}}{(2n)!} + \ldots, \ |z| < \infty$

Example 20.8: Expand $f(z) = 1/z$ about $z = 1$ in Taylor series. Obtain its radius of convergence.

Solution: For $z_0 = 1$, the Taylor series is $f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n$, where $a_n = \dfrac{f^{(n)}(1)}{n!}$

Here, $f(z) = \dfrac{1}{z}$. Thus $f'(z) = -\dfrac{1}{z^2}, f''(z) = \dfrac{(-1)^2 2!}{z^3}, \ldots, f^{(n)}(z) = \dfrac{(-1)^n n!}{z^{n+1}}$

$f^{(n)}(1) = (-1)^n n!$, and hence $a_n = (-1)^n$. This gives

$$\frac{1}{z} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \ldots + (-1)^n (z-1)^n \ldots$$

as the required series.

The radius of convergence is $R = \lim\limits_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty} |-1| = 1.$

Alternatively, $\dfrac{1}{z} = \dfrac{1}{1+(z-1)} = [1+(z-1)]^{-1} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \ldots,$

provided $|z-1| < 1$.

Example 20.9: Expand $f(z) = 2i/(4+iz)$ about $z = -3i$ in Taylor series.

Solution: Consider

$$f(z) = \frac{2i}{4+iz} = \frac{2i}{4+i(z+3i)+3} = \frac{2i}{7+i(z+3i)} = \frac{2i}{7}\frac{1}{\left[1+\frac{i}{7}(z+3i)\right]} = \frac{2i}{7}\left[1+\frac{i}{7}(z+3i)\right]^{-1}$$

$$= \frac{2i}{7}\sum_{n=0}^{\infty}(-1)^n\left[\frac{i}{7}(z+3i)\right]^n, \text{ provided } \left|\frac{i}{7}(z+3i)\right| < 1, \text{ or } |z+3i| < 7.$$

It is the Taylor series expansion of $f(z) = 2i/(4+iz)$ about the point $z_0 = -3i$ which converges in the region $|z+3i| < 7$. Hence the radius of convergence is 7.

Example 20.10: Find the Taylor series expansion of $f(z) = 1/(1-z)^3$ about the origin.

Solution: We have $g(z) = \dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n, \ |z| < 1.$

Differentiating it termwise, we obtain $g'(z) = \dfrac{1}{(1-z)^2} = \displaystyle\sum_{n=1}^{\infty} nz^{n-1}.$

Differentiating again $g''(z) = \dfrac{2}{(1-z)^3} = \displaystyle\sum_{n=2}^{\infty} n(n-1)z^{n-2} = \sum_{n=0}^{\infty}(n+1)(n+2)z^n$

Hence, $f(z) = \dfrac{1}{(1-z)^3} = \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}(n+1)(n+2)z^n, \ |z| < 1$ is the Taylor series expansion about $z = 0$.

Alternatively, $\dfrac{1}{(1-z)^3} = (1-z)^{-3} = 1 + 3z + \dfrac{3(4)}{2!}z^2 + \dfrac{3(4)(5)}{3!}z^3 + \ldots, \ |z| < 1$

$$= \frac{1}{2}[2 + (2)(3)z + (3)(4)z^2 + (4)(5)z^3 + \ldots] = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)z^n, \ |z| < 1.$$

Example 20.11: Obtain the series expansion of $f(z) = \dfrac{1}{z^2+(1+2i)z+2i}$ about $z = 0$

Solution: We have $f(z) = \dfrac{1}{z^2+(1+2i)z+2i} = \dfrac{1}{(z+2i)(z+1)} = \dfrac{1}{(1-2i)}\left[\dfrac{1}{z+2i} - \dfrac{1}{z+1}\right]$

Consider, $\dfrac{1}{z+2i} = \dfrac{1}{2i}\left[1+\dfrac{z}{2i}\right]^{-1} = \dfrac{1}{2i}\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{z}{2i}\right)^n; \ \left|\dfrac{z}{2i}\right| < 1, \text{ or } |z| < 2.$

Also, $\dfrac{1}{z+1} = [1+z]^{-1} = \sum_{n=0}^{\infty} (-1)^n z^n,\ |z|<1$. Hence,

$$f(z) = \frac{1}{1-2i}\left[\frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2i}\right)^n - \sum_{n=0}^{\infty}(-1)^n z^n\right] = \frac{1}{1-2i}\left[\sum_{n=0}^{\infty}(-1)^n\left\{\left(\frac{1}{2i}\right)^{n+1}-1\right\}z^n\right],$$

with radius of convergence, $R = \min.\{1, 2\} = 1$.

Example 20.12: Find Taylor series expansion of $f(z) = \dfrac{2z^2+9z+5}{z^3+z^2-8z-12}$ about $z = 1$.

Solution: We have, $f(z) = \dfrac{2z^2+9z+5}{z^3+z^2-8z-12} = \dfrac{2z^2+9z+5}{(z+2)^2(z-3)} = \dfrac{1}{(z+2)^2} + \dfrac{2}{(z-3)}$ ...(20.13)

Consider, $\dfrac{1}{(z+2)^2} = \dfrac{1}{[3+(z-1)]^2} = \dfrac{1}{9}\left[1+\dfrac{z-1}{3}\right]^{-2}$

$$= \frac{1}{9}\left[1+(-2)\left(\frac{z-1}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{z-1}{3}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{z-1}{3}\right)^2+\ldots\right],\ |z-1|<3,$$

$$= \frac{1}{9}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z-1}{3}\right)^n,\ |z-1|<3.$$

Also, $\dfrac{1}{z-3} = \dfrac{1}{-2+(z-1)} = -\dfrac{1}{2}\left[1-\dfrac{z-1}{2}\right]^{-1} = -\dfrac{1}{2}\sum_{n=0}^{\infty}\left[\dfrac{(z-1)}{2}\right]^n,\ |z-1|<2$

Substituting these in (20.13) yields

$$f(z) = \left[\frac{1}{9}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n\right] = \sum_{n=0}^{\infty}\left[\binom{-2}{n}\frac{1}{3^{n+2}}-\frac{1}{2^n}\right](z-1)^n,$$

with radius of convergence $R = \min\{2, 3\} = 2$.

Example 20.13: Find Taylor series expansion of the function $f(z) = Ln\, z$ about the point $z = (-1+i)$. Obtain its radius of convergence.

Solution: We have, $f(z) = Ln\, z = Ln\, [(-1+i) + \{z-(-1+i)\}] = Ln(-1+i) + Ln\left[1+\dfrac{z-(-1+i)}{-1+i}\right]$

Using $$Ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{z} + ...$$

We obtain, $$f(z) = Ln(-1+i) + \frac{z-(-1+i)}{-1+i} - \frac{1}{2}\left(\frac{z-(-1+i)}{-1+i}\right)^2 + \frac{1}{3}\left(\frac{z-(-1+i)}{-1+i}\right)^3 + ...$$

as the Taylor series expansion about $(-1+i)$.

The series is convergent for $\left|\frac{z-(-1+i)}{-1+i}\right| < 1$, or $|z-(-1+i)| < \sqrt{2}$. Hence the radius of convergence is $\sqrt{2}$.

EXERCISE 20.3

Develop the following functions in a Taylor series about the given point as center. Find the radius of convergence.

1. $1/z, 2$
2. e^z, a
3. $\sin z, \pi/2$
4. $\sinh z, \pi i/2$
5. $\cosh z, \pi i$
6. $Ln(2+iz), i$

Develop the follwing functions in a Maclaurin's series. Find the radius of convergence.

7. $1/(1+z^2)$
8. $\tan^{-1} z$
9. $\sin^2 z$
10. $\frac{z+2}{1-z^2}$
11. $e^z \cos z$
12. $\cos z^2 - \sin z$
13. Suppose f is differentiable in an open disk about zero and satisfies $f''(z) = 2f(z) + 1$. If $f(0) = 1$ and $f'(0) = i$, find the Maclaurin expansion of $f(z)$.
14. Develop the Maclaurin expansion of $1/\sqrt{1-z^2}$ and integrating the same show that
$$\sin^{-1} z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1.3}{2.4}\right)\frac{z^5}{5} + \left(\frac{1.3.5}{2.4.6}\right)\frac{z^7}{7} + ..., \; |z| < 1.$$
15. Using $\sin z = \tan z \cos z$ and the Maclaurin series of $\sin z$ and $\cos z$, find the first four non-zero terms of the Maclaurin series of $\tan z$.
16. Find the Maclaurin's series expansion of $[(1-z)(1+z^2)^2]^{-1}$ and its radius of convergence.
17. Obtain the terms upto z^4 in the Maclaurin's series expansion of $f(z) = (z^2 + \sin^2 z)/(1-\cos z)$.
18. Using the suitable series expansions prove that
(a) $\frac{d}{dz}(\sinh z) = \cosh z$ (b) $\frac{d}{dz}(\sin z) = \cos z$.
19. Using the suitable series expansions prove that

(a) $\sin\left(z+\frac{1}{2}\pi\right)=\cos z$ (b) $\frac{1}{2}(e^{iz}+e^{-iz})=\cos z.$

20. If the Maclaurin series of sec z is $\sec z = E_0 - \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 - \ldots$, then show that the numbers E_{2n}, (called the Euler's numbers), satisfy the relation
$E_0 = 1$ and ${}^{2n}C_0E_0 + {}^{2n}C_2E_2 + \ldots + {}^{2n}C_{2n}E_{2n} = 0.$

21. Using the Maclaurin series expansion of the integrand evaluate the following

(a) $erf.z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$ (error function)

(b) $Si(z) = \int_0^z \frac{\sin t}{t}dt$ (sine integral)

(c) $S(z) = \int_0^z \sin t^2\,dt$ (Fresnel integral)

20.4 LAURENT SERIES

In the preceding section we have seen that how a function $f(z)$ which is analytic at a point z_0 can be expanded as a Taylor series about z_0. Although Taylor series expansions are sufficient for many applications but sometimes it becomes necessary to expand a function $f(z)$ about a point where it is not analytic. Then we can no longer use Taylor series and require a more general form of series called Laurent series.

A Laurent series is of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad \ldots(20.14)$$

where z_0 is a fixed point in the complex plane and the coefficients a_n may be real or complex. The first series on the right side of (20.14) containing only negative powers of $(z - z_0)$ is called the *principal part* of the Laurent series and the second series containing only positive powers of $(z-z_0)$ is called its *regular part*.

A Laurent series will converge in a domain common to the domains of convergence of the principal part and the regular part. In general, the common domain of convergence is an annulus $r < |z-z_0| < R$, where $0 < r < R$ and the sum of the Laurent series is the sum of the individual sums of the principal part and the regular part.

A simple example of the Laurent series is obtained by considering the function $f(z) = \frac{\cos z}{z}$ and expanding cos z as a Maclaurin series. We obtain

$$\frac{\cos z}{z} = \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \ldots \qquad \ldots(20.15)$$

The principal part of the series is $\frac{1}{z}$, which is convergent for all $z \neq 0$.

The regular part of the series is $-\frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \ldots$.

Using ratio test, it can be verified that it converges for all z. Hence, the annulus in which the Laurent series (20.15) converges is the complex plane except the single point at the origin.

Next, we state and prove the main result on Laurent series.

Theorem 20.4 (Laurent series): *A function $f(z)$ analytic inside the annulus $r < |z - z_0| < R$ and on the bounding circles C_1: $|z - z_0| = R$ and C_2: $|z - z_0| = r$, can be expressed as a unique Laurent series given as*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad \ldots(20.16)$$

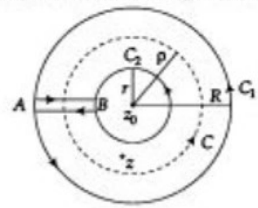


Fig. 20.2

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*, \; n = 0, \pm 1, \pm 2, \ldots;$ $\qquad \ldots(20.17)$

and the circles C_1, C_2 and C: $|z - z_0| = \rho$, $r < \rho < R$ are positively oriented.

Proof. Let the annulus $r < |z - z_0| < R$ be the one as shown in Fig. 20.2 with its center at z_0, outer boundary a circle C_1 of radius R, inner boundary a circle C_2 of radius r and C is the circle $|z - z_0| = \rho$, $r < \rho < R$, inside the annulus. Let z be any arbitrary fixed point inside the annulus $r < |z - z_0| < R$ and inside C.

Introduce a slit AB in the region enclosed by C_1 and C_2, the function $f(z)$ is analytic in the region D bounded by C_1 (anticlockwise), AB, C_2 (clockwise) and then BA, refer to Fig.(20.2). Thus, if z is any point in D, we have

$$f(z) = \frac{1}{2\pi i}\left[\int_{C_1} \frac{f(z^*)}{z^* - z} dz^* + \int_{AB} \frac{f(z^*)}{z^* - z} dz^* + \int_{C_2 \text{(clockwise)}} \frac{f(z^*)}{z^* - z} dz^* + \int_{BA} \frac{f(z^*)}{z^* - z} dz^*\right]$$

$$= \frac{1}{2\pi i} \int_{C_1} \frac{f(z^*)}{z^* - z} dz^* - \frac{1}{2\pi i} \int_{C_2} \frac{f(z^*)}{z^* - z} dz^*, \qquad \ldots(20.18)$$

where both C_1 and C_2 are traversed in anticlockwise sense.

The first integral in (20.18) is precisely the same as in case of Taylor series, refer to Eq. (20.6), so expanding $\frac{1}{z^* - z}$ on the same lines, we get

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(z^*)}{z^*-z}dz^* = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad ...(20.19)$$

where $$a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*. \qquad ...(20.20)$$

Using the principle of deformation of path we can replace C_1 by C in (20.20) since the point z_0 where the integrand in (20.20) is not analytic, does not lie in the annulus, and hence a_n in (20.20) is same as (20.17) for $n = 0, 1, 2, \ldots$.

For the second integral in (20.18), z^* lies on C_2. We write

$$\frac{1}{z^*-z} = \frac{1}{(z^*-z_0)-(z-z_0)} = \frac{-1}{(z-z_0)}\left[1-\frac{z^*-z_0}{z-z_0}\right]^{-1} \qquad ...(20.21)$$

Since $\left|\frac{z^*-z_0}{z-z_0}\right| < 1$, expanding the right side of (20.21) we obtain

$$\frac{1}{z^*-z} = -\frac{1}{z-z_0}\left[1+\frac{z^*-z_0}{z-z_0}+\left(\frac{z^*-z_0}{z-z_0}\right)^2+\ldots+\left(\frac{z^*-z_0}{z-z_0}\right)^n\right]-\frac{1}{z-z^*}\left(\frac{z^*-z_0}{z-z_0}\right)^{n+1} \qquad ...(20.22)$$

Multiplying both sides of Eq. (20.22) by $-f(z^*)/2\pi i$ and integrating over C_2, we get

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(z^*)}{z^*-z}dz^* = \frac{1}{2\pi i}\oint_{C_2}\frac{f(z^*)}{(z-z_0)}dz^* + \frac{1}{2\pi i}\oint_{C_2}\frac{(z^*-z_0)f(z^*)}{(z-z_0)^2}dz^* + \ldots + \frac{1}{2\pi i}\oint_{C_2}\frac{(z^*-z_0)^n}{(z-z_0)^{n+1}}f(z^*)\,dz^* + R_n(z), \qquad ...(20.23)$$

where $R_n(z)$ is the remainder term given by

$$R_n(z) = \frac{1}{2\pi i}\oint_{C_2}\frac{(z^*-z_0)^{n+1}}{(z-z_0)^{n+1}}\frac{f(z^*)}{z-z^*}dz^*.$$

Further since, $f(z)$ is analytic in the annulus $r < |z - z_0| < R$ and $z - z^* \neq 0$ it follows that $f(z^*)/(z - z^*)$ is continuous and hence bounded on C_2. Thus, there exists a real positive number M, such that $|f(z)/(z - z^*)| \leq M$ for all z^* on C_2. Hence

$$|R_n(z)| = \left|\frac{1}{2\pi i}\oint_{C_2}\frac{(z^*-z_0)^{n+1}}{(z-z_0)^{n+1}}\frac{f(z^*)}{z-z^*}dz^*\right| \leq \frac{M}{2\pi}\left|\left(\frac{r}{z-z_0}\right)^{n+1}\right|.2\pi r \to 0 \text{ as } n\to\infty,$$

since $|r/(z-z_0)| < 1$.

Using principle of deformation of path we can replace C_2 by C, hence as $n \to \infty$, (20.23) yields

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(z^*)}{z^*-z}dz^* = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z-z_0)}dz^* + \frac{1}{2\pi i}\oint_C \frac{(z^*-z_0)}{(z-z_0)^2}f(z^*)dz^* + \ldots + \frac{1}{2\pi i}\oint_C \frac{(z^*-z_0)^{n-1}}{(z-z_0)^n}f(z^*)dz^* + \ldots$$

$$= \sum_{n=1}^{\infty}\frac{a_{-n}}{(z-z_0)^n}, \qquad \ldots(20.24)$$

where $a_{-n} = \frac{1}{2\pi i}\oint_C (z^*-z_0)^{n-1} f(z^*)\,dz^*$ is (20.17) when n is replaced by $-n$.

From Eqs. (20.18), (20.19) and (20.24), we obtain (20.16), the Laurent expansion.

Remarks

1. The Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique. However, $f(z)$ may have different Laurent series in two annuli with the same center.

2. Since $f(z)$ is not given to be analytic inside the closed contour C, the coefficients of the positive powers of $(z-z_0)^n$ in the Laurent series that is, $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$ can not be replaced by $\frac{f^{(n)}(z_0)}{n!}$. However, if $r \to 0$ and $f(z)$ is analytic at z_0 also, then $a_{-n} = 0$ and $a_n = \frac{f^{(n)}(z_0)}{n!}$ and the Laurent series of $f(z)$ reduces to Taylor series.

3. As in case of Taylor series, to obtain Laurent series whenever applicable we simply expand $f(z)$ by binomial theorem instead of finding coefficients a_n by complex integration.

Example 20.14: Find all Taylor and Laurent series expansion of $f(z) = \frac{1}{6-z-z^2}$ with center 0.

Solution: The function $f(z) = \frac{1}{6-z-z^2} = \frac{1}{(2-z)(z+3)}$ has singularities at the points $z = 2, -3$. So we find the expansions of $f(z)$ with center 0 in the regions

(a) $|z| < 2$, (b) $2 < |z| < 3$, and (c) $|z| > 3$.

We have, $f(z) = \frac{1}{6-z-z^2} = \frac{1}{(2-z)(z+3)} = \frac{1}{5}\left[\frac{1}{2-z} + \frac{1}{z+3}\right]$.

(a) For $|z| < 2$, since $|z/2| < 1$ and $|z/3| < 1$, write

$$f(z) = \frac{1}{5}\left[\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right] = \frac{1}{5}\left[\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n\right] = \sum_{n=0}^{\infty}\frac{1}{5}\left[\frac{1}{2^{n+1}} + \frac{(-1)^n}{3^{n+1}}\right]z^n.$$

This expansion contains no principal part and since $f(z)$ is analytic in $|z| < 2$, the expansion is just the Maclaurin series expension of $f(z)$ in $|z| < 2$.

(b) For $2 < |z| < 3$, since $|2/z| < 1$ and $|z/3| < 1$, write

$$f(z) = \frac{1}{5}\left[-\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} + \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right] = \frac{1}{5}\left[-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n\right]$$

$$= \frac{1}{5}\left[-\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n} + \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}\right] = \sum_{n=1}^{\infty}\left(-\frac{2^{n-1}}{5}\right)\frac{1}{z^n} + \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{5.3^{n+1}}\right)z^n$$

The first summation represents the principal part and the second summation the regular part of the Laurent series expansion of $f(z)$ in the domain $2 < |z| < 3$.

(c) For $|z| > 3$, since $|2/z| < 1$ and $|3/z| < 1$, write

$$f(z) = \frac{1}{5}\left[-\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} + \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1}\right] = \frac{1}{5}\left[-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n\right]$$

$$= \left[\sum_{n=1}^{\infty}\left(-\frac{2^{n-1}}{5}\right) + \left(\frac{(-1)^{n-1}3^{n-1}}{5}\right)\right]\frac{1}{z^n}.$$

This expansion contains only principal part.

Example 20.15: Expand $f(z) = \exp\left(z+\frac{1}{z}\right)$ as a Laurent series about the origin.

Solution: The function $f(z)$ is analytic everywhere except at the origin which is a singular point and hence $f(z)$ can be expanded about the origin in Laurent series of the form

$$\exp\left(z+\frac{1}{z}\right) = \sum_{n=0}^{\infty}a_n z^n + \sum_{n=1}^{\infty}a_{-n}\frac{1}{z^n}, \text{ for } |z| > 0. \qquad ...(20.25)$$

To determine the coefficients $a_{\pm n}$, we write

$$f(z) = (\exp z)\left(\exp\frac{1}{z}\right) = \left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+...\right)\left(1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\frac{1}{4!z^4}+...\right) \qquad ...(20.26)$$

The constant term in (20.26) is $a_0 = 1 + 1 + \frac{1}{(2!)^2} + \frac{1}{(3!)^2} + \frac{1}{(4!)^2} + ... = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}$.

Also from (20.26) we observe that the coefficients a_n and a_{-n} are equal so it is sufficient to determine a_n only. Now a_n the coefficient of z^n in (20.26) is

$$a_n = \frac{1}{n!} + \frac{1}{1!(n+1)!} + \frac{1}{2!(n+2)!} + \frac{1}{3!(n+3)!} + ... = \sum_{k=0}^{\infty}\frac{1}{k!(n+k)!}.$$

Hence, Laurent expansion of $f(z)$ is (20.25), where $a_n = a_{-n} = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}$, and the region of convergence is $|z| > 0$.

Example 20.16: Expand $f(z) = \frac{7z-2}{z(z+1)(z-2)}$ as a Laurent series in the region $1 < |z+1| < 3$.

Solution: Set $z + 1 = w$, the function $f(z)$ becomes

$$f(z) = \frac{7(w-1)-2}{w(w-1)(w-3)} = \frac{7w-9}{w(w-1)(w-3)} = -\frac{3}{w} + \frac{1}{w-1} + \frac{2}{w-3}.$$

The region is $1 < |w| < 3$. Since $|1/w| < 1$ and $|w/3| < 1$, write

$$f(z) = -\frac{3}{w} + \frac{1}{w}\left(1-\frac{1}{w}\right)^{-1} - \frac{2}{3}\left(1-\frac{w}{3}\right)^{-1} = -\frac{3}{w} + \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{1}{w}\right)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{w}{3}\right)^n$$

$$= -\frac{2}{w} + \sum_{n=2}^{\infty}\left(\frac{1}{w}\right)^n - \sum_{n=0}^{\infty}\left(\frac{2}{3^{n+1}}\right)w^n$$

or, $f(z) = -\frac{2}{1+z} + \sum_{n=2}^{\infty}\frac{1}{(1+z)^n} - \sum_{n=0}^{\infty}\frac{2}{3^{n+1}}(1+z)^n$, which is valid in the region $1 < |z+1| < 3$.

Example 20.17: Show that the function $f(z) = \text{Ln}[z/(z-1)]$ is analytic in the region $|z| > 1$. Obtain its Laurent series expansion about $z = 0$ valid in this region.

Solution: The function $f(z) = \text{Ln}[z/(z-1)]$ is not analytic in the region where

$$\text{Im}[z/(z-1)] = 0 \text{ and } \text{Re}[z/(z-1)] \le 0.$$

Consider $\frac{z}{z-1} = \frac{(x+iy)}{(x-1)+(iy)} = \frac{x(x-1)+y^2-iy}{(x-1)^2+y^2} = \frac{x(x-1)+y^2}{(x-1)^2+y^2} - i\frac{y}{(x-1)^2+y^2}$

Now $\text{Im}[z/(z-1)] = 0$ gives $y = 0$ and $\text{Re}[z/(z-1)] \le 0$ gives $x(x-1) + y^2 \le 0$, or, $x(x-1) \le 0$, or $0 \le x \le 1$. Thus $f(z)$ is not analytic in $R = \{(x, y): 0 \le x \le 1, y = 0\}$.

Hence, it is analytic in the region $|z| > 1$.

Next, for $|z| > 1$, we have $|1/z| < 1$, consider

$$\frac{1}{z} - \frac{1}{z-1} = \frac{1}{z} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} = \frac{1}{z} - \frac{1}{z}\left[\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right] = -\sum_{n=2}^{\infty}\frac{1}{z^n}.$$

Integrating both sides of this term by term we obtain $\text{Ln}\left(\frac{z}{z-1}\right) = \sum_{n=1}^{\infty}\frac{1}{nz^n}$, valid in the region $|z| > 1$.

EXERCISE 20.4

Find the Laurent series that converges for $0 < |z| < R$ and determine the specific region of convergence for the following functions:

1. $z^2 e^{1/z}$
2. $\cos z/z^4$
3. e^{z^2}/z^3
4. $1/(e^z - 1)$
5. $z^3 \cosh(1/z)$
6. $\sinh z/z^5$
7. Find all possible Taylor and Laurent series expansions of the function

$$f(z) = 1/[(z+1)(z+2)^2]$$

about the point $z = 1$. Find the Laurent series that converges for $0 < |z - z_0| < R$ and determine the specific region of convergence for the Problems (8 - 11)

8. $e^{2z}/(z-1)^3, \quad z_0 = 1$
9. $z^4/(z+2i)^4, \quad z_0 = -2i$
10. $\cosh z/(z+\pi i)^2, \quad z_0 = -\pi i$
11. $1/(1+z^2), \quad z_0 = -i.$
12. The series expansion of the functions $1/(1-z)$ and $1/(z-1)$ are

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots, \qquad \text{and} \qquad \frac{1}{z-1} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right)$$

Adding we get $(1 + z + z^2 + \ldots) + \dfrac{1}{z}\left(1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \ldots\right) = 0.$

Is this result true? Justify your answer.

13. Show that $e^{\frac{u}{2}\left(z - \frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} C_n z^n, \ |z| > 0$, where $C_n = J_n(u) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos(n\theta - u\sin\theta)\, d\theta$ is the Bessel function of the first kind.

14. Expand (a) $\dfrac{1}{z^2}\displaystyle\int_0^z \frac{e^t - 1}{t}\, dt$ (b) $\dfrac{1}{z^3}\displaystyle\int_0^z \frac{\sin t}{t}\, dt$

in a Laurent series that converges for $|z| > 0$.

20.5 SINGULARITIES AND ZEROS

In this section we discuss singularities and zeros of complex functions. There are various types of singularities which can be classified using the Laurent series. The zeros of an analytic function can be classified using the Taylor series.

20.5.1 Singular Points of a Function

A singular point of a function $f(z)$ is a point at which it ceases to be analytic. For example, $1/z$ is singular at $z = 0$ and $\sin z/(z - \pi)$ is singular at $z = \pi$. The points $z = 0$ and $z = \pi$ are called the *singularities* of respectively $1/z$ and $\sin z/(z - \pi)$.

Classifications of singularities: *If $z = z_0$ is a singularity of $f(z)$ such that there exists a neighbourhood of z_0 which has no singularity of $f(z)$ other than z_0, then $z = z_0$ is called an 'isolated singularity'.*

For example, $\tan z$ has isolated singularities at $z = \pm\pi/2, \pm 3\pi/2, \ldots$, while $\tan(1/z)$ has a *non-isolated singularity* at 0. Since, $\tan\left(\frac{1}{z}\right) = \frac{\sin(1/z)}{\cos(1/z)}$ has singular points when $\cos(1/z) = 0$, that is, when $1/z = (2n+1)\pi/2$, or $z = 2/[\pi(2n+1)]$, $n = 0, \pm 1, \pm 2, \ldots$. Each one of these points is an isolated singularity. Since $\tan(1/z)$ is not defined at $z = 0$, thus the point $z = 0$ is also a singular point, and we observe that $\lim_{n\to\infty} \frac{2}{\pi(2n+1)} = 0$. Therefore, the neighbourhood of the point $z = 0$, that is, $|z| < \epsilon$, how so small ϵ may be, contains many singular points of $f(z)$ other than $z = 0$ and thus $z = 0$ is not an isolated singularity of $f(z) = \tan(1/z)$.

Next, a function $f(z)$ can be expanded as a Laurent's series about an isolated singularity z_0, as

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \qquad \ldots(20.27)$$

By means of the terms appearing on the right side of Eq. (20.27) we classify the isolated singularities of $f(z)$ as follows.

*If the coefficients of all the negative powers of $(z-z_0)$ in (20.27) are zeros, then $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and hence the singularity of $f(z)$ can be removed by defining $f(z)$ at $z = z_0$ in such a way that it becomes analytic at $z = z_0$. Such a singularity is called a **removable singularity**.*

For example, the Laurent's series expansion of $f(z)$ given by

$$f(z) = \frac{1-\cos z}{z} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n-1}, \quad 0 < |z| < \infty$$

is simply a power series having the value 0 at $z = 0$. In case we define

$$g(z) = \begin{cases} f(z), & \text{for } z \neq 0 \\ 0, & \text{for } z = 0 \end{cases}$$

then $g(z)$ is differentiable at $z = 0$, (since it has power series expansion about $z = 0$); thus it has been possible to extend $f(z)$ to a function $g(z)$ which is differentiable at zero, we call zero a removable singularity of f.

To test whether the singularity $z = z_0$ is removable we simply find $\lim_{z\to z_0} f(z)$. If the limit exists and is finite, then $z = z_0$ is removable singularity.

In case the principal part of (20.27) has only finitely many terms, that is, it is of form

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots + \frac{b_m}{(z-z_0)^m}, \quad (b_m \neq 0),$$

*then the singularity $z = z_0$ of $f(z)$ is called a **pole** and m is called its **order**. Poles of order one are also known as simple poles. A pole of order two is called a double pole.*

In case the principal part of (20.27) has infinitely many terms, then the singularity $z = z_0$ of $f(z)$ is called ***an essential singularity***. For example, the function

$$f(z) = \frac{\sin(z)}{z^3} = \frac{1}{z^2} - \frac{1}{6} + \frac{1}{120}z^2 - \frac{1}{5040}z^4 + \ldots,\ z \neq 0 \text{ has a double pole at } z = 0;$$

and the function $f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n}$ has $z = 0$ as an essential singularity.

We observe that if $f(z)$ has a pole at $z = z_0$, then $\lim_{z \to z_0} f(z) = \infty$. In case f is analytic in the region $0 < |z - z_0| < R$, then f has a pole of order m at z_0 if, and only if $\lim_{z \to z_0} (z - z_0)^m f(z)$ exists and is non-zero.

20.5.2 Zeros of an Analytic Function

A zero of an analytic function $f(z)$ is a z for which $f(z) = 0$. A zero is said to be of order n if, not only f but also the derivatives $f', f'', \ldots, f^{(n-1)}$ are all 0 at that z, but $f^{(n)}(z) \neq 0$. A zero of order one is called a *simple zero*; and a second order zero is called a *double zero*.

For example, the function $f(z) = \sin z$ has simple zeros at $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, and the function $f(z) = (1 + z^2)^2$ has double zeros at $z = \pm i$.

At nth order zero $z = z_0$ of $f(z)$, Taylor series is of the form

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \ldots$$

The zeros of an analytic function $f(z)$ ($\neq 0$) are isolated, that is, each of them has a neighbourhood that contains no further zeros of $f(z)$. Also in case $f(z)$ is analytic at $z = z_0$ and has a zero of nth order at $z = z_0$, then $1/f(z)$ has a pole of order n at $z = z_0$. Further, we say that $f(z)$ has an nth-order zero at infinity if $g(z)$, defined as $g(z) = f(1/z)$, has such a zero at $z = 0$. A similar result holds for poles also.

Following are a few useful results regarding the poles and zeros in terms of quotients and products of two functions. The results can be proved very easily.

1. *Let $f(z) = h(z)/g(z)$, where h and g are analytic in some open disk about z_0. Suppose $h(z_0) \neq 0$ but g has a zero of order n at z_0, then f has a pole of order n at z_0.*

For example, $f(z) = (1 + 4z^3)/\sin^6 z$ has a pole of order 6 at 0, because the numerator does not vanish at 0, and the denominator has a zero of order 6 at 0. On the same argument $f(z)$ has a pole of order 6 at $n\pi$ for any integer n.

2. *Let $f(z) = h(z)/g(z)$ and suppose h and g are analytic in some open disk about z_0. Let h have a zero of order k at z_0 and g a zero of order n at z_0, with $n > k$, then f has a pole of order $n - k$ at z_0.*

For example, $f(z) = (z - 3\pi/2)^4/\cos^7 z$. Obviously the numerator has a zero of order 4 at $3\pi/2$ and the denominator has a zero of order 7 at $3\pi/2$. Thus, $f(z)$ has a pole of order 3 at $3\pi/2$.

3. *Let f has a pole of order m at z_0 and let g has a pole of order n at z_0. Then fg has a pole of order $m + n$ at z_0.*

For example let, $f(z) = 1/(z^2 \sin z)$. Now $1/z^2$ has a double pole at $z = 0$ and $1/\sin z$ has a simple pole at $z = 0$, thus $f(z)$, which is the product of these two functions has a pole of order 3 at $z = 0$.

EXERCISE 20.5

Determine the location and type of singularities of the following functions, including those at ∞

1. $z^2 - \frac{1}{z^2}$
2. $\frac{1}{(z^2+a^2)^2}$
3. $\frac{\sin^2 z}{z^2}$
4. $\frac{\sinh z}{(z-\pi i)^2}$
5. $\frac{z-\sin z}{z^2}$
6. $\frac{1}{\cos z - \sin z}$
7. $(z+1)\sin\frac{1}{z-2}$
8. $\frac{e^{2z}}{(z-1)^4}$

Determine the location and order of the zeros of the following functions

9. $\cos^2(z/2)$
10. $(z^2+1)(e^z-1)$
11. $(z^4-z^2-6)^3$
12. $\frac{z^4}{\sin z}$
13. $\sin\frac{1}{z}$
14. $\frac{1-\cot z}{z}$
15. $\cos z^3$
16. Locate and classify all singularities of $f(z) = \frac{(\pi-z)(z^4-3z^2)}{\sin^2 z}$

20.6 THE RESIDUE THEOREM

Consider the evaluation of the integral $I = \oint_C f(z)dz$ taken around a simple closed path C. If $f(z)$ is analytic everywhere on and inside C, then $I = 0$ by Cauchy's integral theorem and the problem is over. If $f(z)$ has a singularity at a point $z = z_0$ inside C but is otherwise analytic on and inside C, then $f(z)$ has a Laurent series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots$ which converges in some domain $0 < |z - z_0| < R$.

Let C be a simple closed path in this domain enclosing z_0, then the Laurent's coefficients are given by Eq. (20.17) and in particular the coefficient of $\frac{1}{z-z_0}$ is

$$b_1 = a_{-1} = \frac{1}{2\pi i}\oint_C f(z)dz, \qquad \ldots(20.28)$$

which implies

$$\oint_C f(z)dz = 2\pi i\, b_1, \qquad \ldots(20.29)$$

where C is taken in counter-clockwise sense.

The coefficient b_1 is called the residue of $f(z)$ at $z = z_0$ and we denote it by $b_1 = \operatorname*{Res}_{z=z_0} f(z)$.

Generally we obtain Laurent series expansion of a function $f(z)$ without actually using the integral formulae for the coefficients, thus can find b_1 easily. The knowledge of b_1 is used to evaluate the integration of $f(z)$ around the closed curve C by using (20.29). This method of evaluating the integral I is called *the residue method of contour integration*.

Example 20.18: Evaluate by residue method the integral $\oint_C z^{-4} \sin z \, dz$; C: $|z| = 1$ taken in counter-clockwise sense.

Solution: We have, $f(z) = \dfrac{\sin z}{z^4} = \dfrac{1}{z^4}\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \dots\right) = \dfrac{1}{z^3} - \dfrac{1}{3!z} + \dfrac{z}{5!} - \dfrac{z^3}{7!} + \dots$

This series is convergent for all $|z| > 0$ and $f(z)$ has a pole of third order at $z = 0$ and the residue of $f(z)$ at $z = 0$ is $b_1 = -1/3!$. Thus by (20.29) $\oint_C \dfrac{\sin z}{z^4} dz = 2\pi i b = -\dfrac{\pi i}{3}$.

Next, we extend the idea of evaluating contour integration to include the case when the simple closed curve encloses any finite number of singularities of the function $f(z)$. We state the following result.

Theorem 20.5 (**Residue theorem**): *If $f(z)$ is analytic in a closed curve C except at a finite number of singular points z_k, $k = 1, 2, \dots$, within C, then* $\oint_C f(z)dz = 2\pi i \sum_k \operatorname*{Res}_{z=z_k} f(z)$,

that is, value of this integral is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities of $f(z)$ enclosed by C.

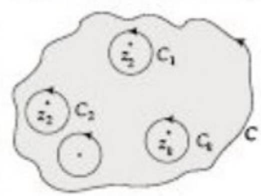


Fig. 20.3

Proof. Enclose each singularity z_k within a closed contour C_k as shown in Fig. (20.3) so that each C_k is in the interior of C and encloses exactly one singularity and does not intersect any other C_j. By the application of Cauchy's integral theorem to the multiply connected region, refer Theorem 15.3, we obtain $\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$, the desired result.

To apply residue theorem, if we need to write the Laurent expansion of $f(z)$ about each singularity z_k to find the coefficient of $1/(z - z_k)$ term, then the theorem would have not been so advantageous to apply in many instances but, there are efficient ways of calculating residues and this makes the residue theorem quite useful.

20.6.1 Calculation of Residues

We will now develop some methods for the calculation of residues.

(a) *If $f(z)$ has a simple pole at $z = z_0$, then* $\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$.

Since, in case of simple pole at $z = z_0$ the Laurent series expansion about z_0 is

$$f(z) = \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n, \; 0 < |z - z_0| < R.$$

This gives, $(z - z_0) f(z) = b_1 + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1}$, which implies $\lim_{z \to z_0} (z - z_0) f(z) = b_1$.

Thus $\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$. ...(20.30)

Next, we give somewhat simpler formula for the residue at a simple pole for functions of the form $f(z) = p(z)/q(z)$, both p, q analytic, $p(z_0) \neq 0$ and $q(z)$ has a simple zero at z_0, so that $f(z)$ has a simple pole at z_0.

Since $q(z)$ has a simple zero at z_0, the Taylor series of $q(z)$ about z_0 is

$$q(z) = (z - z_0) q'(z_0) + \frac{(z - z_0)^2}{2!} q''(z_0) + \ldots$$

Consider, $\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} (z - z_0) \frac{p(z)}{q(z)}$

$$= \lim_{z \to z_0} \frac{(z - z_0) p(z)}{(z - z_0)\left[q'(z_0) + \frac{(z - z_0)}{2!} q''(z_0) + \ldots\right]} = \frac{p(z_0)}{q'(z_0)}.$$

Thus, $\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$. ...(20.31)

(b) *If $f(z)$ has a pole of order m at z_0, then* $\operatorname*{Res}_{z=0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$.

Since in case of pole of order m at $z = z_0$, the Laurent series expansion about z_0 is

$$f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \ldots + \frac{b_1}{(z - z_0)} + \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

and thus $(z - z_0)^m f(z) = b_m + b_{m-1}(z - z_0) + \ldots + b_1 (z - z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m}$.

Differentiating both sides $(m - 1)$ times and taking limit $z \to z_0$, we obtain

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] = (m - 1)! \, b_1.$$

Hence, $\operatorname*{Res}_{z=z_0} f(z) = b_1 = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$. ...(20.32)

Example 20.19: Determine the poles of function $f(z) = \dfrac{e^z}{(z+2i)^3(z+i)}$ and the residue at each pole. Hence calculate $\oint_C f(z)dz$; C: $|z| = 2.5$, taken in counter-clockwise sense.

Solution: Since $\lim_{z\to -i}[(z+i)f(z)] = \lim_{z\to -i}\dfrac{e^z}{(z+2i)^3} = ie^{-i}$ is finite and non-zero, thus $f(z)$ has a simple pole at $z = -i$ and $\operatorname*{Res}_{z=-i} f(z) = ie^{-i}$.

Since $\lim_{z\to -2i}[(z+2i)^3 f(z)] = \lim_{z\to -2i}\dfrac{e^z}{z+i} = \dfrac{e^{-2i}}{-i} = ie^{-2i}$ is finite and non-zero, thus $f(z)$ has a pole of order 3 at $z = -2i$, and

$$\operatorname*{Res}_{z=-2i} f(z) = \frac{1}{2!}\lim_{z\to -2i}\frac{d^2}{dz^2}\left(\frac{e^z}{z+i}\right) = \frac{1}{2}\lim_{z\to -2i}[e^z\{(z+i)^{-1} - 2(z+i)^{-2} + 2(z+i)^{-3}\}] = \frac{1}{2}e^{-2i}(2-i).$$

To evaluate the integral $\oint_C f(z)dz$; C: $|z| = 2.5$, we observe that $f(z)$ is analytic on and inside $|z| = 2.5$ at all points except at the poles $z = -i, 2i$ and hence by residue theorem

$$\oint_C f(z)dz = 2\pi i\left[\operatorname*{Res}_{z=-i} f(z) + \operatorname*{Res}_{z=-2i} f(z)\right] = 2\pi i\left[ie^{-i} + \frac{1}{2}e^{-2i}(2-i)\right].$$

Example 20.20: Evaluate $\oint_C \tan z\,dz$; C: $|z| = 2$, taken counter-clockwise.

Solution: The poles of $f(z) = \sin z/\cos z$ are given by $\cos z = 0$, which gives $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$.

Out of these only $z = \pm\pi/2$ are within the circle C: $|z| = 2$, using (20.31), we

$$\operatorname*{Res}_{z=\pi/2} f(z) = \lim_{z\to\pi/2}\frac{\sin z}{(\cos z)'} = \lim_{z\to\pi/2}\frac{\sin z}{(-\sin z)} = -1.$$

Similarly, $\operatorname*{Res}_{z=-\pi/2} f(z) = \lim_{z\to -\pi/2}\dfrac{\sin z}{(\cos z)'} = \lim_{z\to -\pi/2}\dfrac{\sin z}{(-\sin z)} = -1.$

Hence by residue theorem $\oint_C f(z)dz = 2\pi i\,[-1-1] = -4\pi i.$

Example 20.21: Evaluate $\oint_C \dfrac{\tan z}{z^2-1}dz$; C: $|z| = 3/2$, taken counter-clockwise.

Solution: Poles of $f(z) = \dfrac{\tan z}{z^2 - 1}$ are at $z = \pm 1$ and at $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$.

Out of these only $z = \pm 1$ lie within the circle $|z| = 3/2$. Now

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} \frac{\tan z}{z+1} = \frac{1}{2} \tan 1;$$

and,
$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} \frac{\tan z}{z-1} = \frac{1}{2} \tan 1.$$

Thus, $$\oint_C \frac{\tan z}{z^2 - 1} dz = 2\pi i \left[\frac{1}{2} \tan 1 + \frac{1}{2} \tan 1 \right] = 2\pi i \tan 1.$$

Example 20.22: Evaluate $\oint_C \left(\dfrac{z e^{\pi z}}{z^4 - 16} + z e^{\pi/z} \right) dz$, C is ellipse $9x^2 + y^2 = 9$ taken counter-clockwise.

Solution: The integrand is $f(z) = \dfrac{z e^{\pi z}}{z^4 - 16} + z e^{\pi/z}$. The first term has simple poles at $\pm 2i$ and ± 2. Out of these $\pm 2i$ lie inside the ellipse $9x^2 + y^2 = 9$ and ± 2 lie outside it. Using (20.31), we have

$$\operatorname*{Res}_{z=2i} \left[\frac{z e^{\pi z}}{z^4 - 16} \right] = \lim_{z=2i} \left[\frac{z e^{\pi z}}{4z^3} \right] = \lim_{z=2i} \left[\frac{e^{\pi z}}{4z^2} \right] = -\frac{1}{16}, \text{ since } e^{2\pi i} = 1.$$

$$\operatorname*{Res}_{z=-2i} \left[\frac{z e^{\pi z}}{z^4 - 16} \right] = \lim_{z=-2i} \left[\frac{z e^{\pi z}}{4z^3} \right] = \lim_{z=-2i} \left[\frac{e^{\pi z}}{4z^2} \right] = -\frac{1}{16}.$$

Next, considering the second term of the integrand $f(z)$, we have

$$z e^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2! z^2} + \frac{\pi^3}{3! z^3} + \ldots\right) = z + \pi + \frac{\pi^2}{2! z} + \frac{\pi^3}{3! z^2} + \ldots$$

Here $z = 0$ is the essential singularity with residue $\pi^2/2$. Hence, by residue theorem

$$\oint_C f(z) dz = 2\pi i \left[-\frac{1}{16} - \frac{1}{16} \right] + 2\pi i \left[\frac{\pi^2}{2} \right] = \pi\left(\pi^2 - \frac{1}{4}\right) i.$$

EXERCISE 20.6

Find the residue of the following functions at each pole

1. $\dfrac{z^2 + 1}{z^2 - 2z}$ 2. $\dfrac{z^2 - 2z}{(z+1)^2 (z^2 + 1)}$ 3. $\dfrac{\sin 2z}{z^6}$ 4. $\dfrac{1}{1 - e^z}$

5. $\cot \pi z$ 6. $\dfrac{\sin z}{z^2(z^2+4)}$ 7. $e^{1/z}$ 8. $\dfrac{\cos z}{(z+i)^3}$

Evaluate the following integrals taken counter-clockwise

9. $\oint_C \dfrac{z-3}{z^2+2z+5}dz$, where C is the circle

(i) $|z|=1$ (ii) $|z+1-i|=2$, (iii) $|z+1+i|=2$

10. $\oint_C \tan \pi z dz$; C: $|z|=1$

11. $\oint_C \dfrac{z+1}{z^4-2z^3}dz$; C: $|z|=\dfrac{1}{2}$

12. $\oint_C \dfrac{e^{-z}}{z^2}dz$; C: $|z|=1$

13. $\oint_C z^2 e^{\frac{1}{z}} dz$; C: $|z|=1$

14. $\oint_C \dfrac{z\sec z}{(1-z)^2}dz$; C: $|z|=3$

15. $\oint_C \dfrac{z\cos z}{(z-\pi/2)^3}dz$; C: $|z-1|=1$

16. $\oint_C \dfrac{z\cosh \pi z}{z^4+13z^2+36}dz$; C: $|z+i|=\pi$

17. $\oint_C \dfrac{z-\cos(4iz)}{(1+z^2)(z^2-1)}dz$; C: $|z+i|=1$

18. Obtain Laurent expansion for the function $f(z) = \dfrac{1}{z^2 \sinh z}$ and hence evaluate

$$\oint_C \frac{dz}{z^2 \sinh z}; \quad C: |z-1|=2.$$

20.7 APPLICATIONS OF THE RESIDUE THEOREM TO THE EVALUATION OF REAL DEFINITE INTEGRALS

In this section we will illustrate how the residue theorem is used in evaluating certain classes of real definite integrals which are otherwise difficult to solve.

20.7.1 Integrals of the Type $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$

Here $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ and is finite over the interval of integration. To evaluate this integral, let C be the unit circle taken counter-clockwise, then any point on this curve is $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. So we obtain

$$\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}\left(z+\frac{1}{z}\right) \text{ and } \sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

Also, $dz = ie^{i\theta}d\theta = iz\, d\theta$, which gives, $d\theta = dz/iz$.

Substituting for $\cos\theta$, $\sin\theta$ and $d\theta$, the given integral takes the form

$$I = \oint_C f(z)dz; \quad C: |z| = 1,$$

where $f(z)$ is a rational function of z. By residue throrem this integral is equal to $2\pi i$ times the sum of the residues at those poles of $f(z)$ which lie within C.

Example 20.23: Show that $\int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \frac{2\pi}{\sqrt{3}}$.

Solution: Put $z = e^{i\theta}$, we obtain $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$.

Thus, the given integral becomes

$$I = \oint_C \frac{1}{2 - \frac{z^2-1}{2iz}}\frac{dz}{iz} = -2\oint_C \frac{dz}{z^2 - 4iz - 1} = -2\oint_C f(z)dz; \quad C: |z| = 1, \qquad ...(20.33)$$

where $f(z) = \frac{1}{z^2 - 4iz - 1} = \frac{1}{[z-(2+\sqrt{3})i][z-(2-\sqrt{3})i]}$.

The function $f(z)$ has simple poles at $z = (2 \pm \sqrt{3})i$. Out of these $z_1 = (2+\sqrt{3})i$ lies outside C and only $z_2 = (2-\sqrt{3})i$ lies inside C. Also

$$\operatorname*{Res}_{z=z_2} f(z) = \lim_{z\to z_2} [[(z-(2-\sqrt{3})i]f(z)] = \left[\frac{1}{z-(2+\sqrt{3})i}\right]_{z=(2-\sqrt{3})i} = \frac{1}{-2\sqrt{3}i}.$$

By residue theorem, $\oint_C f(z)dz = (2\pi i)\left(\frac{-1}{2\sqrt{3}i}\right) = -\frac{\pi}{\sqrt{3}}$, and hence from (20.33), $I = \frac{2\pi}{\sqrt{3}}$.

Example 20.24: Apply residue theorem to evaluate the integral

$$\int_0^{\pi} \frac{\cos 2\theta}{1 - 2p\cos\theta + p^2}\, d\theta, \quad -1 < p < 1.$$

Solution: Let $I = \int_0^{\pi} \frac{\cos 2\theta}{1 - 2p\cos\theta + p^2}\, d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{1-2p\cos\theta+p^2}\,d\theta, \text{ since } \int_0^{2a} f(\theta)d\theta = 2\int_0^{a} f(\theta)d\theta, \text{ when } f(2a-\theta)=f(\theta).$$

Put $z = e^{i\theta}$, we obtain $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$, $\cos 2\theta = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right)$ and $d\theta = \frac{dz}{iz}$. Thus

$$I = \frac{1}{2}\oint_C \frac{\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)}{1-p\left(z+\frac{1}{z}\right)+p^2}\frac{dz}{iz} = -\frac{1}{4i}\oint_C \frac{z^4+1}{z^2(z-p)(pz-1)}dz = -\frac{1}{4i}\oint_C f(z)\,dz, \qquad \text{...(20.34)}$$

where $\quad f(z) = \frac{z^4+1}{z^2(z-p)(pz-1)} \quad$ and $\quad C: |z| = 1.$

The function $f(z)$ has simple poles at $z = p$ and $\frac{1}{p}$ and a double pole at $z = 0$. Out of these 0 and p, since $|p| < 1$, are within the unit circle C. Also

$$\operatorname*{Res}_{z=p} f(z) = \lim_{z\to p}(z-p)f(z) = \left[\frac{z^4+1}{z^2(pz-1)}\right]_{z=p} = \frac{p^4+1}{p^2(p^2-1)}$$

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0}\frac{d}{dz}[z^2 f(z)] = \lim_{z\to 0}\frac{d}{dz}\left[\frac{z^4+1}{(z-p)(pz-1)}\right] = \frac{d}{dz}\left[\frac{z^4+1}{pz^2-(p^2+1)z+p}\right]_{z=0}$$

$$= \left[\frac{[pz^2-(p^2+1)z+p](4z^3)-(z^4+1)[2pz-(p^2+1)]}{[pz^2-(p^2+1)z+p]^2}\right]_{z=0} = \frac{p^2+1}{p^2}.$$

By residue theorem, $\oint_C f(z)dz = 2\pi i\left[\frac{p^4+1}{p^2(p^2-1)}+\frac{p^2+1}{p^2}\right] = -\frac{4\pi i p^2}{1-p^2}$; hence from (20.34)

$$I = -\frac{1}{4i}\left(-\frac{4\pi i p^2}{1-p^2}\right) = \frac{\pi p^2}{1-p^2}.$$

Example 20.25: Prove that $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$, when $0 < b < a$.

Solution: Let $I = \int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, since $\int_0^{2a} f(\theta)d\theta = 2\int_0^{a} f(\theta)d\theta$, when $f(2a-\theta) = f(\theta)$.

Put $z = e^{i\theta}$, we obtain $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$. Thus I becomes

$$I = \frac{1}{2}\oint_C \frac{dz/iz}{a + b\frac{1}{2}(z + 1/z)} = \frac{2}{i}\oint_C \frac{dz}{bz^2 + 2az + b} = \frac{2}{i}\oint_C f(z)\,dz, \qquad \text{...(20.35)}$$

Fig. 20.4

where $\quad f(z) = \frac{1}{(bz^2 + 2az + b)}$ and C: $|z| = 1$.

The poles of $f(z)$ are at the roots of equation $bz^2 + 2az + b = 0$, which are

$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$. Let $\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}$, and $\beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$.

It is easy to verify that $|\beta| > 1$ and since $\alpha\beta = 1$, thus $|\alpha| < 1$. So $z = \alpha$ is the only simple pole inside the unit circle $|z| = 1$.

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z \to \alpha}(z - \alpha)f(z) = \left.\frac{1}{b(z - \beta)}\right|_{z=\alpha} = \frac{1}{b(\alpha - \beta)} = \frac{1}{2\sqrt{a^2 - b^2}}$$

By residue theorem $\oint_C f(z)dz = 2\pi i\left(\frac{1}{2\sqrt{a^2 - b^2}}\right) = \frac{\pi i}{\sqrt{a^2 - b^2}}$; hence from (20.35)

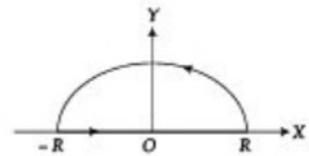
$$I = \frac{2}{i}\left(\frac{\pi i}{\sqrt{a^2 - b^2}}\right) = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

20.7.2 Improper Integrals of the Type $\int_{-\infty}^{\infty} f(x)dx$

When $f(x)$ is a real rational function whose denominator is non-zero for all real x and is of degree at least two units higher than the degree of the numerator, then the improper integral of type under discussion converges and we can write $\int_{-\infty}^{\infty} f(x)dx = \lim_{R \to \infty}\int_{-R}^{R} f(x)dx$.

To evaluate this integral, we consider the corresponding contour integral $\oint_C f(z)dz$, where C is a closed path from $-R$ to R and then R to $-R$ along C_R, as shown in Fig. 20.4.

Since $f(x)$ is rational thus $f(z)$ has finitely many poles in the upper half-plane and if we choose R large enough such that C includes all these poles, then by residue theorem, under the assumption that $f(z)$ has no singular point on the real axis, we have



$$\oint_C f(z)dz = \oint_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i\, \Sigma \text{Res}\, f(z), \qquad ...(20.36)$$

where the summation is over all the residues of $f(z)$ corresponding to the poles of $f(z)$ in the upper half-plane.

Finally, taking $R \to \infty$, we obtain from (20.36)

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i\, \Sigma \text{Res}\, f(z), \qquad ...(20.37)$$

provided $\oint_{C_R} f(z)dz \to 0$.

To prove this, set $z = Re^{i\theta}$, then C_R is $|z| = R, 0 \le \theta \le \pi$, and under the assumption that the degree of the denominator of $f(z)$ is at least two units higher than that of the numerator, we have $|f(z)| < M/R^2$, for sufficiently large M. Thus by ML-inequality

$$\left|\int_{C_R} f(z)dz\right| < \frac{M}{R^2}\pi R = \frac{M\pi}{R} \qquad ...(20.38)$$

which tends to 0 as $R \to \infty$. Thus (20.37) holds.

Example 20.26: Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^6 + 64} dx$.

Solution: Consider the contour integral $\oint_C \frac{1}{z^6+64} dz = \int_C f(z)dz$,

where C is the contour consisting of the semicircle C_R of radius R and the segment of the real axis from $-R$ to R as in Fig. 20.4. The integrand $f(z)$ has six simple poles at the points given by

$$z = (-64)^{1/6} = 2e^{i(\pi + 2n\pi)/6}, \quad n = 0, 1, 2, 3, 4, 5.$$

These are: $z_1 = 2e^{\pi i/6}$, $z_2 = 2e^{3\pi i/6}$, $z_3 = 2e^{5\pi i/6}$, $z_4 = 2e^{7\pi i/6}$, $z_5 = 2e^{9\pi i/6}$, $z_6 = 2e^{11\pi i/6}$.

Out of these z_1, z_2 and z_3 lies only in the upper half-plane as shown in Fig. 20.5. We have

$$\underset{z=z_1}{\text{Res}}\, f(z) = \underset{z=2e^{\pi i/6}}{\text{Res}} \left(\frac{1}{z^6+64}\right) = \left[\frac{1}{6z^5}\right]_{z=2e^{\pi i/6}} = \frac{1}{192}e^{-5\pi i/6},$$

Similarly, $\underset{z=z_2}{\text{Res}}\, f(z) = \underset{z=2e^{3\pi i/6}}{\text{Res}} \left(\frac{1}{z^6+64}\right) = \frac{1}{192}e^{-15\pi i/6} = \frac{1}{192}e^{-3\pi i/6}$

and, $\operatorname*{Res}_{z=z_1} f(z) = \operatorname*{Res}_{z=2e^{\pi i/6}}\left(\frac{1}{z^6+64}\right) = \frac{1}{192}e^{-25\pi i/6} = \frac{1}{192}e^{-\pi i/6}$.

Thus, by residue theorem

$$\int_C \frac{1}{z^6+64}dz = \frac{2\pi i}{192}\left[\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6} + \cos\frac{3\pi}{6} - i\sin\frac{3\pi}{6} + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]$$

$$= \frac{\pi i}{96}\left[\left(\cos\frac{5\pi}{6} + \cos\frac{\pi}{6}\right) - i\left(\sin\frac{5\pi}{6} + \sin\frac{\pi}{6}\right) - i\right].$$

Using $\cos\frac{5\pi}{6} + \cos\frac{\pi}{6} = 0$ and $\sin\frac{5\pi}{6} + \sin\frac{\pi}{6} = 1$, this gives

$$\oint_C \frac{1}{z^6+64}dz = \frac{\pi}{48} \qquad ...(20.39)$$

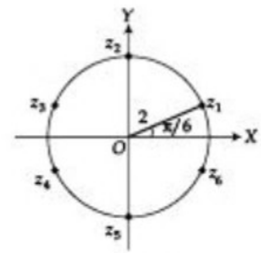


Fig. 20.5

Also, $$\oint_C \frac{1}{z^6+64}dz = \int_{-R}^{R}\frac{1}{x^6+64}dx + \int_{C_R}\frac{1}{z^6+64}dz \qquad ...(20.40)$$

Now as $R \to \infty$, for any point on C_R, $|z| \to \infty$ and consequently the integrand in the second integral on the right side of (20.40) tends to zero and hence it vanishes and thus, from (20.39) and (20.40), we have $\int_{-\infty}^{\infty} \frac{1}{x^6+64}dx = \frac{\pi}{48}$.

Example 20.27: Evaluate the integral $\int_0^{\infty} \frac{x^2dx}{(x^2+a^2)^2}, a > 0.$

Solution: Let $I = \int_0^{\infty} \frac{x^2dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2}dx.$

Consider the contour integral $\oint_C f(z)dz$, where $f(z) = \frac{z^2}{2(z^2+a^2)^2}$ and C is the closed contour consisting of the semicircle C_R of radius R and the segment of the real axis from $-R$ to R as in Fig. (20.4). The integrand $f(z)$ has poles of order 2 at $z = \pm ia$. Out of these only $z = +ia$ lies in the upper half-plane. Also

$$\operatorname*{Res}_{z=ia} f(z) = \lim_{z\to ia}\frac{d}{dz}[(z-ia)^2 f(z)] = \lim_{z\to ia}\frac{d}{dz}\left[\frac{z^2}{2(z+ia)^2}\right]$$

$$= \frac{1}{2}\left[\frac{2z(z+ia)^2 - 2z^2(z+ia)}{(z+ia)^4}\right]_{z=ia} = -\frac{i}{8a}.$$

Hence by the residue theorem $\oint_C f(z)dz = 2\pi i\left(-\frac{i}{8a}\right) = \frac{\pi}{4a}$. ...(20.41)

Also, $\oint_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz.$ ...(20.42)

When $R \to \infty$, then for any point on C_R $|z| \to \infty$ and

$f(z) = \frac{z^2}{2(z^2+a^2)^2} = \frac{1}{2z^2}\frac{1}{\left(1+\frac{a^2}{z^2}\right)^2}$ tends to 0 and consequently, the second integral on the right side of (20.42) tends to zero and hence from (20.41) and (20.42), we have

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2}dx = \frac{\pi}{4a}.$$

20.7.3 Improper Real Integrals of the types $\int_{-\infty}^{\infty} f(x) \cos ax\, dx$ and $\int_{-\infty}^{\infty} f(x) \sin ax\, dx$ (Fourier Integrals).

We can express these integrals as

$$\int_{-\infty}^{\infty} f(x)\cos ax\, dx = \text{Re}\int_{-\infty}^{\infty} f(x)e^{iax}dx, \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\sin ax\, dx = \text{Im}\int_{-\infty}^{\infty} f(x)e^{iax}dx,$$

where a is real and positive.

Consider the contour integral $\oint_C f(z)e^{iaz}dz$, where C is the closed contour given by $C = C_R \cup [-R, R]$, as shown in Fig. (20.4). Using the residue theorem, we get

$$\oint_C f(z)e^{iaz}dz = 2\pi i\, \Sigma \text{Res}\,[f(z)e^{iaz}],$$

or, $\int_{C_R} f(z)e^{iaz}dz + \int_{-R}^{R} f(x)e^{iax}dx = 2\pi i\, \Sigma\, \text{Res}[f(z)e^{iaz}]$ ...(20.43)

where the sum on the right side is over the residues of $f(z)e^{iaz}$ at its poles in the upper half-plane.

Consider the integral $\int_{C_R} f(z)e^{iaz}dz$, we have

$$\left|\int_{C_R} f(z)e^{iaz}\,dz\right| \le \int_{C_R} |e^{iaz}f(z)|\,|dz| = \int_{C_R} |e^{-ay}f(z)|\,|dz| \le \int_{C_R} |f(z)|\,|dz|, \qquad \text{...(20.44)}$$

since $a > 0$. Now, as $R \to \infty$, if the $\int_{C_R} |f(z)|\,|dz|$ tends to zero, then from (20.44) the integral, $\int_{C_R} f(z)e^{iaz}dz$ tends to zero and hence from (20.43). We have

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i\,\Sigma\,\text{Res}\,[f(z)e^{iaz}]. \qquad \text{...(20.45)}$$

Comparing the real and imaginary parts on both sides of (20.45) we get the requisite results.

Example 20.28: Evaluate the integrals

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+\alpha^2)(x^2+\beta^2)}dx$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ax}{(x^2+\alpha^2)(x^2+\beta^2)}dx$

where a, α, and β are positive numbers and $\alpha \ne \beta$.

Solution: Consider the contour integral $I = \displaystyle\oint_C \frac{e^{iaz}}{(z^2+\alpha^2)(z^2+\beta^2)}dz = \oint_C f(z)e^{iaz}dz$,

where $f(z) = \dfrac{1}{(z^2+\alpha^2)(z^2+\beta^2)}$ and C is the path $C_R \cup [-R, R]$ as in Fig. (20.4).

The integrand $f(z)$ has simple poles at $z = \pm i\alpha$ and $\pm i\beta$. Out of these $i\alpha$ and $i\beta$ lie in the upper-half plane. Also

$$\underset{z=i\alpha}{\text{Res}}\,[e^{iaz} f(z)] = \lim_{z\to i\alpha}\left[\frac{(z-i\alpha)e^{iaz}}{(z^2+\alpha^2)(z^2+\beta^2)}\right] = \frac{e^{-\alpha a}}{2\alpha i(\beta^2-\alpha^2)}.$$

Similarly, $\underset{z=i\beta}{\text{Res}}\,[e^{iaz} f(z)] = \dfrac{e^{-\beta a}}{2\beta i(\alpha^2-\beta^2)}$. Hence, by residue theorem

$$\oint_C e^{iaz} f(z)dz = \frac{\pi}{\beta^2-\alpha^2}\left[\frac{e^{-\alpha a}}{\alpha} - \frac{e^{-\beta a}}{\beta}\right] \qquad \text{...(20.46)}$$

Now, $$\oint_C e^{iaz} f(z)dz = \int_{-R}^{R} e^{iax} f(x)dx + \int_{C_R} e^{iaz} f(z)dz. \quad ...(20.47)$$

We have, $\left|\int_{C_R} e^{iaz} f(z)dz\right| \le \int_{C_R} |f(z)|\,|dz|$, and here

$$f(z) = \frac{1}{(z^2+\alpha^2)(z^2+\beta^2)} = \frac{1}{z^4\left(1+\frac{\alpha^2}{z^2}\right)\left(1+\frac{\beta^2}{z^2}\right)}.$$

When $R \to \infty$, then for any point on C_R, $|z| \to \infty$ and hence $f(z) \to 0$.

Thus as $R \to \infty$, the second integral on the right side of Eq. (20.47) tends to zero and hence from (20.46) and (20.47), we obtain

$$\int_{-\infty}^{\infty} e^{iax} f(x)dz = \frac{\pi}{\beta^2-\alpha^2}\left[\frac{e^{-\alpha a}}{\alpha} - \frac{e^{-\beta a}}{\beta}\right] \quad ...(20.48)$$

Comparing the real and imaginary parts on both sides of (20.48), we obtain

$$\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+\alpha^2)(x^2+\beta^2)} dx = \frac{\pi}{\beta^2-\alpha^2}\left[\frac{e^{-\alpha a}}{\alpha} - \frac{e^{-\beta a}}{\beta}\right], \text{ and, } \int_{-\infty}^{\infty} \frac{\sin ax}{(x^2+\alpha^2)(x^2+\beta^2)} dx = 0.$$

Example 20.29: Evaluate the integral $\int_0^{\infty} \frac{\sin ax \sin bx}{x^2+\alpha^2} dx$, $0 < a < b, \alpha > 0$.

Solution: Since, $\sin ax \sin bx = \frac{1}{2}[\cos(b-a)x - \cos(b+a)x]$, given integral can be expressed as

$$I = \frac{1}{2}\int_0^{\infty} \frac{\cos(b-a)x}{x^2+\alpha^2} dx - \frac{1}{2}\int_0^{\infty} \frac{\cos(b+a)x}{x^2+\alpha^2} dx = \frac{1}{4}\int_{-\infty}^{\infty} \frac{\cos(b-a)x}{x^2+\alpha^2} dx - \frac{1}{4}\int_{-\infty}^{\infty} \frac{\cos(b+a)x}{x^2+\alpha^2} dx$$

$$= \frac{1}{4}\text{Re}\int_{-\infty}^{\infty} \frac{e^{(b-a)ix}}{x^2+\alpha^2} dx - \frac{1}{4}\text{Re}\int_{-\infty}^{\infty} \frac{e^{(b+a)ix}}{x^2+\alpha^2} dx. \quad ...(20.49)$$

We consider the contour integrals

$$I_1 = \oint_C \frac{e^{(b-a)iz}}{z^2+\alpha^2} dz = \oint_C e^{(b-a)iz} f(z)dz \quad ...(20.50)$$

and, $$I_2 = \oint_C \frac{e^{(b+a)iz}}{z^2+\alpha^2} dz = \oint_C e^{(b+a)iz} f(z)dz, \quad ...(20.51)$$

where $f(z) = \dfrac{1}{z^2+\alpha^2}$ and C is contour given by $C_R \cup [-R, R]$ as in Fig. 20.4.

The function $f(z)$ has two simple poles at $z = \pm i\alpha$ and between these two, only $z = +i\alpha$ lies in the upper half-plane. Also

$$\operatorname*{Res}_{z=i\alpha}\,[e^{(b-a)iz}f(z)] = \lim_{z\to i\alpha}\left[(z-i\alpha)\frac{e^{(b-a)iz}}{z^2+\alpha^2}\right] = \frac{e^{-(b-a)\alpha}}{2\alpha i},$$

and $$\operatorname*{Res}_{z=i\alpha}\,[e^{(b+a)iz}f(z)] = \lim_{z\to i\alpha}\left[(z-i\alpha)\frac{e^{(b+a)iz}}{z^2+\alpha^2}\right] = \frac{e^{-(b+a)\alpha}}{2\alpha i}.$$

Hence, by residue theorem

$$I_1 = 2\pi i\left(\frac{e^{-(b-a)\alpha}}{2\alpha i}\right) = \frac{\pi}{\alpha}e^{-(b-a)\alpha} \text{ and } I_2 = 2\pi i\left(\frac{e^{-(b+a)\alpha}}{2\alpha i}\right) = \frac{\pi}{\alpha}e^{-(b+a)\alpha}.$$

Further, since $|e^{(b-a)iz}| = e^{-(b-a)y} \le 1$, $(b > a)$, and $|e^{(b+a)iz}| = e^{-(b+a)ay} \le 1$, thus we have

$$\left|\int_{C_R}\frac{e^{(b-a)iz}}{z^2+\alpha^2}dz\right| \le \int_{C_R}\left|\frac{1}{z^2+\alpha^2}\right||dz| \quad \text{...(20.52)}$$

and, $$\left|\int_{C_R}\frac{e^{(b+a)iz}}{z^2+\alpha^2}dz\right| \le \int_{C_R}\left|\frac{1}{z^2+\alpha^2}\right||dz|. \quad \text{...(20.53)}$$

When R tends to ∞, then for every point on C_R, $|z|$ tends to ∞ and hence $\dfrac{1}{z^2+\alpha^2} = \dfrac{1}{z^2}\cdot\dfrac{1}{1+\dfrac{\alpha^2}{z^2}}$ tends to 0, thus both the integrations on C_R in (20.52) and (20.53) tends to zero as $R\to\infty$. Also since

$$\oint_C e^{(b\pm a)iz}f(z)dz = \int_{-R}^{R} e^{(b\pm a)ix}f(x)dx + \int_{C_R} e^{(b\pm a)iz}f(z)dz,$$

thus from (20.50) and (20.51) we obtain respectively

$$\int_{-\infty}^{\infty} e^{(b-a)ix}f(x)dx = \frac{\pi}{\alpha}e^{-(b-a)\alpha} \text{ and } \int_{-\infty}^{\infty} e^{(b+a)ix}f(x)dx = \frac{\pi}{\alpha}e^{-(b+a)\alpha}.$$

In fact, we have $\operatorname{Re}\displaystyle\int_{-\infty}^{\infty}\frac{e^{(b-a)ix}}{x^2+\alpha^2}dx = \frac{\pi}{\alpha}e^{-(b-a)\alpha}$, and $\operatorname{Re}\displaystyle\int_{-\infty}^{\infty}\frac{e^{(b+a)ix}}{x^2+\alpha^2}dx = \frac{\pi}{\alpha}e^{-(b+a)\alpha}$.

Thus from (20.49), we obtain $I = \frac{\pi}{4\alpha} e^{-b\alpha}[e^{a\alpha} - e^{-a\alpha}] = \frac{\pi}{2\alpha} e^{-b\alpha} \sinh(a\alpha)$.

20.7.4 Improper Integrals with Singular Points on the Real Axis

We consider an improper integral $I = \int_A^B f(x)dx$, where the integrand $f(x)$ becomes infinite at some point a in the interval of integration, that is, $\lim_{x \to a} |f(x)| = \infty$. We define the Cauchy principal value ($p_r.v.$) of the integral I as $p_r.v.\,(I) = \lim_{\epsilon \to 0} \left[\int_A^{a-\epsilon} f(x)dx + \int_{a+\epsilon}^{B} f(x)dx \right]$.

The principal value of an integral may exist, although the integral may be meaningless, refer to Section 7.8. We shall assume here that the principal value of such an improper integral exists. Further, we shall assume $f(x)$ is rational function with degree of denominator at least two units higher than that of numerator; also the corresponding function $f(z)$ of the complex variable z is having finite number of simple poles $a, b, c, \ldots$ on the real axis and is analytic in the upper half-plane except possibly at finite number of points, $z_1, z_2, \ldots$.

However for simplicity, in the result to be proved, we shall consider that $f(z)$ has only one simple pole $z = a$ on the real axis. The result can be generalized to finite number of poles a, b, c, etc.

Enclose this pole with semicircle C_1 with center at a and of small radius r and let C_R be the semicircle with center at the origin and radius R large enough to include all the poles of $f(z)$ in the upper half-plane.

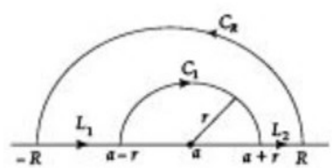


Fig. 20.6

We consider the '*indented contour*' C which is the union of the semicircle C_R, path L_1 from $-R$ to $a - r$, semicircle C_1, and then path L_2 from $a + r$ to R as per the orientations in Fig. 20.6. Thus by the residue theorem, we have

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{L_1} f(z)dz + \int_{C_1} f(z)dz + \int_{L_2} f(z)dz = 2\pi i \sum_{z=z_i} \text{Res } f(z), \quad ...(20.54)$$

where summation on the right is taken over all the poles, $z_1, z_2, \ldots$ of $f(z)$ in the upper-half plane.

When $r \to 0$, the semicircle C_1 contracts to a point and the line segments L_1 and L_2 expand to cover the line segment $-R$ to R on the real axis.

Further $f(z)$ has a simple pole $z = a$ on the real axis, its Laurent series expansion about $z = a$ should be of the form

$$f(z) = \frac{b_1}{z-a} + g(z);\ b_1 = \text{Res}_{z=a} f(z), \quad ...(20.55)$$

where $g(z)$ is analytic on the semicircle C_1: $z = a + re^{i\theta}$, θ varing from π to 0, (note orientation of C_1 in Fig. 20.6), and for all z between C_1 and the x-axis.

Integrating (20.55) over C_1, we obtain

$$\int_{C_1} f(z)dz = b_1 \int_{\pi}^{0} \frac{ire^{i\theta}}{re^{i\theta}} d\theta + \int_{C_1} g(z)dz = -\pi i b_1 + \int_{C_1} g(z)dz \qquad ...(20.56)$$

Now, $\left|\int_{C_1} g(z)dz\right| \le \int_{C_1} |g(z)||dz| \le M\pi r$, using the ML inequality. Thus I tends to zero as $r \to 0$ and hence from (20.56), for $r \to 0$ we have

$$\int_{C_1} f(z)dz = -\pi i \operatorname*{Res}_{z=a} f(z) \qquad ...(20.57)$$

Also, under the assumption that the degree of denomiator in $f(z)$ is two higher than the numerator, then as shown earlier, refer to Eq. (20.38), when $R \to \infty$

$$\left|\int_{C_R} f(z)dz\right| \le \int_{C_R} |f(z)||dz| \to 0. \qquad ...(20.58)$$

Thus, as $R \to \infty$ and $r \to 0$, we obtain from (20.54), (20.57) and (20.58),

$$\oint_C f(z)dz = p_r.v \int_{-\infty}^{\infty} f(x)dx - \pi i \operatorname*{Res}_{z=a} f(z) = 2\pi i \sum_{z=z_k} \operatorname{Res} f(z)$$

or,
$$p_r.v \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{z=z_k} \operatorname{Res} f(z) + \pi i \operatorname*{Res}_{z=a} f(z)$$

In case of more than one simple poles on the real axis, this result is generalized to

$$p_r.v \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{z=z_k} \operatorname{Res} f(z) + \pi i \sum_{z=a} \operatorname{Res} f(z), \qquad ...(20.59)$$

where the first sum extends over all the poles to $f(z)$ in the upper half-plane and second extends over all the poles to $f(z)$ on the real axis.

Example 20.30: Find principal value of $\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)}$ using the residue theorem.

Solution: Consider the function $f(z) = \frac{1}{(z^2-3z+2)(z^2+1)}$. It has simple poles at $z = 1, 2, i$ and $-i$.

Out of these 1 and 2 lie on real axis, i lies on the upper-half plane and $-i$ lies on the lower-half plane and is thus of no interest. We have

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}(z-1)\frac{1}{(z^2-3z+2)(z^2+1)} = \lim_{z\to 1}\frac{1}{(z-2)(z^2+1)} = -\frac{1}{2},$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2}(z-2)\frac{1}{(z^2-3z+2)(z^2+1)} = \lim_{z\to 2}\frac{1}{(z-1)(z^2+1)} = \frac{1}{5},$$

and $\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i}(z-i)\frac{1}{(z^2-3z+2)(z^2+1)} = \lim_{z\to i}\frac{1}{(z^2-3z+2)(z+i)} = \frac{1}{6+2i} = \frac{3-i}{20}.$

Also we note that degree of denominator in $f(z)$ is at least two more than the degree of denominator hence the line integral $\int_{C_R} f(z)dz$, where C_R is semicircular path of radius R in the upper half-plane, tends to zero as R tends to infinity. Thus using (20.58), we have

$$p.v\int_{-\infty}^{\infty}\frac{dx}{(x^2-3x+2)(x^2+1)} = 2\pi i\left(\frac{3-i}{20}\right) + \pi i\left(-\frac{1}{2}+\frac{1}{5}\right) = \frac{\pi}{10}.$$

Example 20.31: Show using the contour integration that

(a) $\int_0^{\infty}\frac{\sin mx}{x}dx = \frac{\pi}{2}$ (b) $\int_0^{\infty}\frac{\cos mx}{x}dx = 0$

Solution: Since $e^{imx} = \cos mx + i\sin mx$, we consider the contour integration $\oint_C \frac{e^{imz}}{z}dz = \oint_C f(z)dz$, where C is the 'indented contour' given by $C = C_R \cup [-R,-r] \cup C_r \cup [r, R]$ as per the orientation shown in Fig. 20.7.

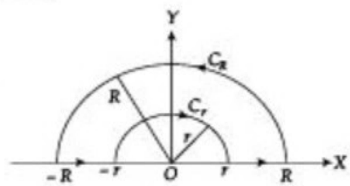


Fig. 20.7

The only singularity of $f(z) = \frac{e^{imz}}{z}$ is $z = 0$ which lies outside C and hence by Cauchy's integral theorem.

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(z)dz + \int_r^R f(x)dx = 0 \quad ...(20.60)$$

Since on C_R, we have $z = Re^{i\theta}$, thus

$$\left|\int_{C_R} f(z)dz\right| = \left|\int_0^{\pi} \frac{e^{imR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} iRe^{i\theta} d\theta\right| \le \int_0^{\pi} \left|e^{imR(\cos\theta + i\sin\theta)} d\theta\right| \le \int_0^{\pi} e^{-mR\sin\theta} d\theta$$

$$= 2\int_0^{\pi/2} e^{-mR\sin\theta} d\theta, \quad \text{...(20.61)}$$

since $\int_0^{2a} f(\theta)d\theta = 2\int_0^{a} f(\theta)d\theta$, when $f(2a-\theta) = f(\theta)$.

For $0 \le \theta \le \pi/2, 1 \ge \frac{\sin\theta}{\theta} \ge \frac{2}{\pi}$ and therefore, $e^{-mR\sin\theta} \le e^{-2mR\theta/\pi}$, hence from (20.60),

$$\left|\int_{C_R} f(z)dz\right| \le 2\int_0^{\pi/2} e^{-2mR\theta/\pi} d\theta = \frac{\pi}{mR}(1 - e^{-mR}), \quad \text{...(20.62)}$$

which tends to zero as R tends to infinity.

Also as $r \to 0$, $\int_{C_r} f(z)dz = -\pi i \operatorname*{Res}_{z=0}(e^{iz}/z) = -\pi i \lim_{z\to 0} e^{iz} = -\pi i.$ ...(20.63)

Hence, as $R \to \infty$ and $r \to 0$, using (20.62) and (20.63), (20.60) gives

$$p.v. \int_{-\infty}^{\infty} f(x)dx = i\pi \text{ or, } p.v. \int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = i\pi.$$

Equating imaginary and real parts on both sides we get respectively

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi \text{ and } \int_{-\infty}^{\infty} \frac{\cos mx}{x} dx = 0$$

or,
$$\int_0^{\infty} \frac{\sin mx}{x} dx = \pi/2 \text{ and } \int_0^{\infty} \frac{\cos mx}{x} dx = 0.$$

Example 20.32: Evaluate the integeral $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx.$

Solution: We have $\frac{\sin^2 x}{x^2} = \frac{1 - \cos 2x}{2x^2} = \operatorname{Re}\left[\frac{1 - e^{2ix}}{2x^2}\right].$

Consider the contour integral $\oint_C \left(\frac{1-e^{2iz}}{2z^2}\right) dz = \oint_C f(z)dz$, where C is the indented contour given by $C = C_R \cup [-R, -r] \cup C_r \cup [r, R]$, as shown in Fig 20.7.

The only singularity of $f(z) = \frac{1-e^{2iz}}{2z^2}$ is at $z = 0$, a simple pole which lies outside C, and hence by Cauchy's integration theorem

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(x)dx + \int_r^R f(z)dz = 0 \qquad ...(20.64)$$

On C_R, as $R \to \infty$, we have

$$\left|\int_{C_R} f(z)dz\right| \le \int_{C_R} \left|\frac{1-e^{2iz}}{2z^2}\right| |dz| \le \int_{C_R} \frac{1+|e^{2iz}|}{2|z|^2} |dz| \le \frac{2}{2R^2}(\pi R) = \frac{\pi}{R} \to 0, \qquad ...(20.65)$$

since $|e^{2iz}| = e^{-2y} < 1$, for $y \ge 0$.

Also on C_r as $r \to 0$, we have

$$\int_{C_r} f(z)dz = -\pi i \operatorname*{Res}_{z=0} f(z) = -\pi i \lim_{z\to 0}\left[\frac{1-e^{2iz}}{2z}\right] = -\pi i \lim_{z\to 0}\left[\frac{-2ie^{2iz}}{2}\right] = -\pi. \qquad ...(20.66)$$

Hence, as $R \to \infty$ and $r \to 0$, using (20.65) and (20.66), we obtain from (20.64)

$$p.v. \int_{-\infty}^{\infty} \frac{1-e^{2ix}}{2x^2} dx = \pi. \qquad ...(20.67)$$

Comparing the real parts on both sides of (20.67), we get

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi \quad \text{or,} \int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

Example 20.33: Evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2-3x+2)} dx$.

Solution: We have $\frac{\cos x}{(x^2+1)(x^2-3x+2)} = \text{Re.} \frac{e^{ix}}{(x^2+1)(x^2-3x+2)}$.

Consider the contour integral $\oint_C \frac{e^{iz}}{(z^2+1)(z^2-3z+2)} dz = \oint_C f(z)dz$, where C is the indented contour given by

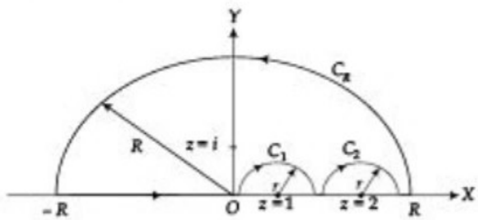


Fig. 20.8

$$C = C_R \cup [-R, 1-r] \cup C_1 \cup [1+r, 2-r] \cup C_2 \cup [2+r, R]$$

as shown in Fig. 20.8.

Here C_1 and C_2 are both semicircles with radius r. The function $f(z) = \dfrac{e^{iz}}{(z^2+1)(z^2-3z+2)}$ has simple poles at $z = \pm i, 1, 2$. Out of these $z = i$ lies inside C, $z = 1$ and 2 lie on the real line and $z = -i$ lies in the lower-half plane. Hence, by residue theorem

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{1-r} f(z)dz + \int_{C_1} f(z)dz + \int_{1+r}^{2-r} f(z)dz + \int_{C_2} f(z)dz + \int_{2+r}^{R} f(z)dz = 2\pi i \operatorname*{Res}_{z=i} f(z). \quad ...(20.68)$$

Now, $$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i}\left[(z-i)\frac{e^{iz}}{(z^2+i)(z^2-3z+2)}\right] = \frac{e^{-1}}{2i(1-3i)} = \frac{1}{2e(3+i)} = \frac{3-i}{20e}. \quad ...(20.69)$$

Also, $$\left|\int_{C_R} f(z)dz\right| \le \int_{C_R}\left|\frac{e^{iz}}{(z^2+1)(z^2-3z+2)}\right| |dz| \le \frac{\pi R}{(R^2+1)(R^2-3R+2)} \to 0, \text{ as } R\to\infty, \quad ...(20.70)$$

since $|e^{iz}| = e^{-y} \le 1$ for $y \ge 0$.

Also when $r \to 0$, $$\int_{C_1} f(z)dz = -\pi i \operatorname*{Res}_{z=1} f(z) = -\pi i \lim_{z\to 1}(z-1)f(z)$$

$$= -\pi i \lim_{z\to 1}\frac{e^{iz}}{(z^2+1)(z-2)} = -\pi i\frac{e^{i}}{-2} = \frac{i\pi}{2}e^{i}. \quad ...(20.71)$$

and, $$\int_{C_2} f(z)dz = -\pi i \operatorname*{Res}_{z=2} f(z) = -\pi i \lim_{z\to 2}(z-2)f(z)$$

$$= -\pi i \lim_{z\to 2}\frac{e^{iz}}{(z^2+1)(z-1)} = -\pi i\frac{e^{2i}}{5.1} = -\frac{i\pi}{5}e^{2i}. \quad ...(20.72)$$

Thus, as $R \to \infty$ and $r \to 0$, using (20.68), (20.70), (20.71) and (20.72), we obtain from (20.68),

$$pr.v \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)(x^2-3x+2)} dx = \frac{\pi(1-3i)}{10e} - \frac{i\pi e^i}{2} + \frac{i\pi}{5} e^{2i}$$

$$= \frac{\pi(1-3i)}{10e} - \frac{i\pi}{2} [\cos(1) + i \sin(1)] + \frac{i\pi}{5} [\cos(2) + i \sin(2)].$$

Equating the real parts on each side of this equation we obtain

$$p_r v. \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2-3x+2)} dx = \frac{\pi}{10} \left(\frac{1}{e} + 5 \sin(1) - 2 \sin(2)\right).$$

20.7.5 Solutions of a few more Improper Real Integrals using Residues

We consider the following examples.

Example 20.34: Evaluate the integral $\int_0^{\infty} \frac{x^{1/3}}{(x+1)^2} dx.$

Solution: Consider the contour integral $I = \oint_C \frac{z^{1/3}}{(z+1)^2} dz = \oint_C f(z)dz,$

where $f(z) = \frac{z^{1/3}}{(z+1)^2}$ and C is a suitably chosen contour as shown in Fig. 20.9. The outer circle is of radius R and the inner circle is of radius r. The contour consists of four parts : AB, BC, CD, DA. Let the outer circle taken counter clockwise sense be denoted by C_R and inner taken clockwise sense be denoted by C_r. We intend to make $R \to \infty$ and $r \to 0$.

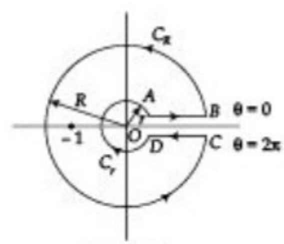


Fig. 20.9

Also in $f(z)$, the term $z^{1/3}$ is multivalued, a branch cut to the right, see Fig. 20.9, at $\theta = 0$ makes the function $f(z)$ to be single-valued and hence ensures the applicability of the residue theorem to $f(z)$.

We note that function $f(z)$ is analytic inside and on C, except for a double pole at $z = -1$, thus

$$\oint_C f(z)dz = \int_{AB} f(z)dz + \int_{C_R} f(z)dz + \int_{C_r} f(z)dz + \int_{CD} f(z)dz = 2\pi i \operatorname*{Res}_{z=-1} [f(z)] \quad ...(20.73)$$

Any point on C_R is $z = Re^{i\theta}$, thus

$$|f(z)| = \left|\frac{z^{1/3}}{(z+1)^2}\right| = \frac{|R^{1/3}e^{i\theta/3}|}{|z^2+2z+1|} \le \frac{R^{1/3}}{|z|^2-2|z|+1} = \frac{R^{1/3}}{(R-1)^2}.$$

This gives $\left|\int\limits_{C_R} f(z)dz\right| \le \frac{R^{1/3}}{(R-1)^2}.2\pi R \simeq \frac{2\pi}{R^{2/3}} \to 0$, as $R \to \infty$. ...(20.74)

Similarly, $\left|\int\limits_{C_r} f(z)dz\right| \le \frac{r^{1/3}}{(1-r)^2}.2\pi r \simeq 2\pi r^{4/3} \to 0$, as $r \to 0$. ...(20.75)

Also, $\underset{z=-1}{\text{Res}}\,[f(z)] = \frac{1}{1!}\lim_{z\to-1}\frac{d}{dz}[(z+1)^2 f(z)] = \lim_{z\to-1}\left(\frac{1}{3}z^{-2/3}\right) = \frac{1}{3}(-1)^{-2/3} = \frac{1}{3}(e^{\pi i})^{-2/3} = \frac{1}{3}e^{-2\pi i/3}$

Further on AB, we have $z = xe^{i0}$, so $z^{1/3} = x^{1/3}$, hence $f(z) = \frac{x^{1/3}}{(x+1)^2}$; and on CD, $z = xe^{2\pi i}$, so $z^{1/3} = x^{1/3}\,e^{2\pi i/3}$, hence $f(z) = \frac{x^{1/3}e^{2\pi i/3}}{(x+1)^2}$. Using these, taking $R \to \infty$, $r \to 0$ and using (20.74) and (20.75) in (20.73), we obtain

$$\int_0^\infty \frac{x^{1/3}}{(x+1)^2}dx + \int_\infty^0 \frac{x^{1/3}e^{2\pi i/3}}{(x+1)^2}dx = 2\pi i\left(\frac{e^{-2\pi i/3}}{3}\right)$$

or,

$$(1-e^{2\pi i/3})\int_0^\infty \frac{x^{1/3}}{(x+1)^2}dx = 2\pi i\left(\frac{e^{-2\pi i/3}}{3}\right)$$

or, $$\int_0^\infty \frac{x^{1/3}}{(x+1)^2}dx = \frac{2\pi i}{3}\,\frac{e^{-2\pi i/3}}{1-e^{2\pi i/3}} = \frac{-2\pi i}{3}\,\frac{(1+i\sqrt{3})}{(3-i\sqrt{3})} = \frac{2\pi}{3\sqrt{3}}.$$

Example 20.35: Evaluate the integral $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx$, $0 < a < 1$.

Solution: Consider the contour integral $\oint\limits_C \frac{e^{az}}{e^z+1}dz$ $= \oint\limits_C f(z)dz$, where $f(z) = \frac{e^{az}}{e^z+1}$ has finite poles given by $e^z = -1$ $= e^{(2n+1)\pi i}$, or $z = (2n+1)\pi i$, $n = 0, \pm1, \pm2, \ldots$ and C is rectangular contour $ABCD$ with vertices at $A(R, 0)$, $B(R, 2\pi)$, $C(-R, 2\pi)$ and $D(-R, 0)$, as shown in Fig. 20.10.

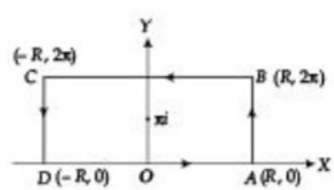


Fig. 20.10

The only pole inside the contour C is πi. Therefore by residue theorem

$$\oint_C f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz = 2\pi i \operatorname*{Res}_{z=\pi i} f(z). \quad ...(20.76)$$

We have,
$$\operatorname*{Res}_{z=\pi i} f(z) = \operatorname*{Res}_{z=\pi i}\left[\frac{e^{az}}{e^z+1}\right] = \left[\frac{e^{az}}{(e^z+1)'}\right] = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i} \quad ...(20.77)$$

We observe that along

$$\begin{aligned} AB &: z = R + iy; & 0 \le y \le 2\pi; \\ BC &: z = x + 2\pi i; & x \to R \text{ to } -R; \\ CD &: z = -R + iy; & y \to 2\pi \text{ to } 0; \\ DA &: z = x; & -R \le x \le R \end{aligned}$$

using these and (20.77), (20.76) becomes

$$\int_0^{2\pi} i f(R+iy)\,dy - \int_{-R}^{R} f(x+2\pi i)dx - \int_0^{2\pi} i f(-R+iy)dy + \int_{-R}^{R} f(x)dx = -2\pi i\, e^{a\pi i} \quad ...(20.78)$$

Consider the first integral $\int_0^{2\pi} i f(R+iy)dy$, on the right side of (20.78) we have

$$\left|\int_0^{2\pi} i f(R+iy)dy\right| \le \int_0^{2\pi} |f(R+iy)|\,|dy| = \int_0^{2\pi} \left|\frac{e^{a(R+iy)}}{e^{R+iy}+1}\right| |dy| \le \frac{e^{aR}}{e^R-1}2\pi \to 0 \quad ...(20.79)$$

as $R \to \infty$, since $a < 1$.

Similarly, consider the third integral $\int_0^{2\pi} i f(-R+iy)dy$, we have

$$\left|\int_0^{2\pi} i f(-R+iy)dy\right| \le \int_0^{2\pi} |f(-R+iy)|\,|dy| = \int_0^{2\pi} \left|\frac{e^{a(-R+iy)}}{e^{-R+iy}+1}\right| |dy| \le \frac{e^{-aR}}{e^{-R}-1}2\pi \to 0 \quad ...(20.80)$$

as $R \to \infty$, since $a > 0$.

Thus making $R \to \infty$ and using (20.79) and (20.80), (20.78) becomes

$$-e^{2a\pi i}\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = -2\pi i e^{a\pi i}$$

or, $$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^{x}+1}\,dx = \frac{2\pi i e^{a\pi i}}{e^{2a\pi i}-1} = \frac{2\pi i}{e^{a\pi i}-e^{-a\pi i}} = \frac{\pi}{\sin a\pi}.$$

Example 20.36: Prove that $\int_0^{\infty} \sin x^2 dx = \int_0^{\infty} \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Solution: Consider the contour integral $\oint_C e^{-z^2}\,dz$, where C consists of the real axis from O to A, the circular arc AB of radius R and the line $\theta = \pi/4$ from B to O as shown in Fig. 20.11.

The function $f(z) = e^{-z^2}$ is analytic everywhere inside C and hence by Cauchy's integral theorem

$$\int_{OA} e^{-z^2}\,dz + \int_{AB} e^{-z^2}\,dz + \int_{BO} e^{-z^2}\,dz = 0 \qquad ...(20.81)$$

We observe that along

$$OA: z = x, \qquad 0 \le x \le R$$

$$AB: z = Re^{i\theta}, \qquad 0 \le \theta \le \pi/4$$

$$BO: z = re^{i\pi/4}, \qquad r \to R \text{ to } 0,$$

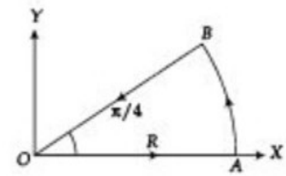


Fig. 20.11

Hence (20.81) becomes

$$\int_0^R e^{-x^2}dx + \int_0^{\pi/4} e^{-R^2e^{2i\theta}}\, iRe^{i\theta}d\theta - \int_0^R e^{-r^2e^{i\pi/2}}\, e^{i\pi/4}\,dr = 0$$

or, $$\int_0^R e^{-x^2}dx + i\int_0^{\pi/4} Re^{-R^2(\cos 2\theta + i\sin 2\theta)}e^{i\theta}d\theta - \left(\frac{1+i}{\sqrt{2}}\right)\int_0^R e^{-ix^2}\,dx = 0 \qquad ...(20.82)$$

As $R \to \infty$, then $\int_0^R e^{-x^2}dx \to \int_0^{\infty} e^{-x^2}\,dx = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$, refer to (7.54).

Also $\int_0^{\pi/4} Re^{-R^2[\cos 2\theta + i\sin 2\theta]}\, e^{i\theta}d\theta \to 0$ or $R \to \infty$, since the integrend tends to zero.

Thus when $R \to \infty$, (20.82) becomes $\frac{\sqrt{\pi}}{2} - \frac{1+i}{\sqrt{2}}\int_0^{\infty} [\cos x^2 - i\sin x^2]dx = 0$

or, $$\int_0^{\infty} (\cos x^2 - i\sin x^2)\,dx = \frac{\sqrt{\pi}}{\sqrt{2}(1+i)} = \frac{(1-i)\sqrt{\pi}}{2\sqrt{2}}.$$

Equating the real and imaginary parts, we obtain

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Example 20.37: Evaluate $\int_0^\infty e^{-x^2} \cos 2bx\, dx,$

Solution: Consider the contour integral $\oint_C e^{-z^2} dz$ where C is the rectangular contour $ABCD$ with vertices $A(a, 0)$, $B(a, b)$, $C(-a, b)$ and $D(-a, 0)$, as shown in Fig. 20.12.

The function $f(z) = e^{-z^2}$ is analytic everywhere on and inside C and hence by Cauchy's integral theorem

$$\oint_C f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz = 0 \quad ...(20.83)$$

We note that along

$AB: z = a + iy;\quad 0 \le y \le b$

$BC: z = x + bi;\quad x \to a \text{ to } -a$

$CD: z = -a + iy;\quad y \to b \text{ to } 0$

$DA: z = x;\quad -a < x < a$

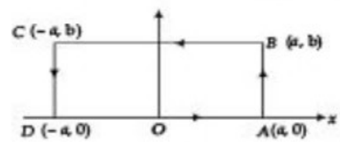


Fig. 20.12

Hence, (20.83) becomes

$$\int_0^b ie^{-(a+iy)^2} dy - \int_{-a}^{a} e^{-(x+ib)^2} dx - \int_0^b ie^{-(-a+iy)^2} dy + \int_{-a}^{a} e^{-x^2} dx = 0$$

or, $$\int_{-a}^{a} e^{-x^2} dx - \int_{-a}^{a} e^{-x^2-2ibx+b^2} dx + \int_0^b ie^{-a^2-2iay+y^2} dy - \int_0^b ie^{-a^2+2iay+y^2} dy = 0$$

or, $$\int_{-a}^{a} e^{-x^2} dx - e^{b^2}\int_{-a}^{a} e^{-x^2}.e^{-2ibx} dx + 2e^{-a^2}\int_0^b e^{y^2} \sin 2ay\, dy = 0, \quad ...(20.84)$$

since $\sin 2ay = (e^{2iay} - e^{-2iay})/2i$.

When $a \to \infty$, $e^{-a^2} \to 0$ and $\sin 2ay$ remains between –1 and 1, hence (20.84) becomes

$$\int_{-\infty}^{\infty} e^{-x^2} dx - e^{b^2}\int_{-\infty}^{\infty} e^{-x^2}\, e^{-2ibx} dx = 0$$

or, $\int_{-\infty}^{\infty} e^{-x^2} [\cos 2bx - i \sin 2bx] dx = e^{-b^2} \int_{-\infty}^{\infty} e^{-x^2} dx = 2e^{-b^2} \int_{0}^{\infty} e^{-x^2} dx = e^{-b^2} \Gamma(1/2) = \sqrt{\pi}\, e^{-b^2}.$

Equating the real parts on both sides we obtain $\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx\, dx = \sqrt{\pi}\, e^{-b^2}.$

Example 20.38: Determine the inverse Fourier transform of $F(w) = \dfrac{1}{w^2+1}$, defined by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{w^2+1} e^{iwx}\, dw.$$

Solution: Consider the contour integral $I = \dfrac{1}{2\pi} \oint_C \dfrac{e^{ixw}}{w^2+1} dw$ in the complex w plane where C is the closed contour as shown in Fig. 20.13a, for $x > 0$ and the closed contour as shown in Fig. 20.13b, for $x < 0$.

For $x > 0$, refer to Fig. 20.13a, the function $f(w) = \dfrac{1}{2\pi} \dfrac{e^{ixw}}{w^2+1}$ has singularity $w = i$ inside C, thus by residue theorem

$$\frac{1}{2\pi} \oint_C \frac{e^{ixw}}{w^2+1} dw = \frac{1}{2\pi} \int_{-R}^{R} \frac{e^{ixw}}{w^2+1} dw + \frac{1}{2\pi} \int_{C_R} \frac{e^{ixw}}{w^2+1} dw = 2\pi i \operatorname*{Res}_{w=i} \left[\frac{1}{2\pi} \frac{e^{ixw}}{w^2+1} \right], \quad ...(20.85)$$

where C_R is the semicircular part of C in Fig. 20.13a. On C_R, $w = Re^{i\theta}$, thus

$$\left| \frac{1}{2\pi} \int_{C_R} \frac{e^{ixw}}{w^2+1} dw \right| \le \frac{1}{2\pi} \int_{C_R} \left| \frac{e^{ixw}}{w^2+1} \right| |dw| \le \frac{1}{2\pi} \frac{1}{R^2-1} 2\pi R \to 0 \text{ as } R \to \infty.$$

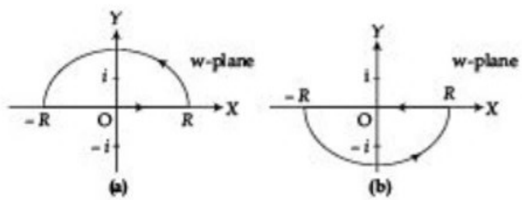


Fig. 20.13

Also $2\pi i \operatorname*{Res}_{w=i}\left[\frac{1}{2\pi}\frac{e^{ixw}}{w^2+1}\right] = 2\pi i.\frac{1}{2\pi}\frac{e^{ix(i)}}{2i} = \frac{e^{-x}}{2}$.

Hence, as $R \to \infty$ (20.85) gives $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixw}}{w^2+1}dw = \frac{e^{-x}}{2}, x > 0.$ ...(20.86)

Next, for $x < 0$, refer to Fig (20.13b), the function $f(w) = \frac{1}{2\pi}\frac{e^{ixw}}{w^2+1}$ has singularity $w = -i$ inside C, and thus by residue theorem

$$\frac{1}{2\pi}\oint_C \frac{e^{ixw}}{w^2+1}dw = \frac{1}{2\pi}\int_R^{-R}\frac{e^{ixw}}{w^2+1}dw + \frac{1}{2\pi}\int_{C_R}\frac{e^{ixw}}{w^2+1}dw = 2\pi i \operatorname*{Res}_{w=-i}\left[\frac{1}{2\pi}\frac{e^{ixw}}{w^2+1}\right] \quad ...(20.87)$$

Also $2\pi i \operatorname*{Res}_{w=-i}\left[\frac{1}{2\pi}.\frac{e^{ixw}}{w^2+1}\right] = 2\pi i \frac{1}{2\pi}\frac{e^{ix(-i)}}{-2i} = -\frac{e^{x}}{2}$.

On the same lines, as above $\frac{1}{2\pi}\int_{C_R}\frac{e^{ixw}}{w^2+1}dw \to 0$ when $R \to \infty$, and hence (20.86) gives

$$\frac{1}{2\pi}\int_{\infty}^{-\infty}\frac{e^{ixw}}{w^2+1}dw = -\frac{e^{x}}{2} \text{ or, } \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixw}}{w^2+1}dw = \frac{e^{x}}{2}, x < 0. \quad ...(20.88)$$

Hence, from (20.87) and (20.88) we obtain $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixw}}{w^2+1}dw = \frac{e^{-|x|}}{2}$.

EXERCISE 20.7

Evaluate the following integrals

1. $\int_0^{\pi}\frac{d\theta}{k+\cos\theta}, (k > 1)$
2. $\int_0^{\pi}\sin^4\theta\, d\theta$
3. $\int_0^{2\pi}\frac{d\theta}{1-2p\sin\theta+p^2}, |p| < 1$
4. $\int_0^{2\pi}\frac{\cos\theta}{13-12\cos 2\theta}d\theta$
5. $\int_0^{\pi}\frac{\cos^2 3\theta}{5-4\cos 2\theta}d\theta$
6. $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta$

7. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$

8. $\int_{0}^{\infty} \frac{dx}{1+x^4}$

9. $\int_{0}^{\infty} \frac{dx}{(a^2+x^2)^2}$

10. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2(x^2+4)} dx$

11. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3}$

12. $\int_{-\infty}^{\infty} \frac{x^{2m}}{1+x^{2n}} dx;$
$m \neq n$ are positive integer ≤ 2.

13. $\int_{0}^{\infty} \frac{\cos ax}{(x^2+a^2)^2} dx,\ a>0$

14. $\int_{-\infty}^{\infty} \frac{\sin^2 2x}{1+x^2} dx$

15. $\int_{0}^{\infty} \frac{x^3 \sin mx}{x^4+a^4} dx,\ a>0$

16. $\int_{-\infty}^{\infty} \frac{x^3 \sin mx}{(x^2+a^2)(x^2+b^2)} dx,$
(m positive integer)

17. $\int_{-\infty}^{\infty} \frac{x}{8-x^3} dx$

18. $\int_{0}^{\infty} \frac{\sin \pi x}{x(1-x^2)} dx$

19. $\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} dx,\ (a, b \geq 0)$

20. $\int_{0}^{\infty} \frac{\sin^3 x}{x^3} dx$

21. $\int_{0}^{\infty} \frac{\cos ax}{1-x^4} dx,\ (a>0)$

22. Find a condition on α for which the integral $\int_{0}^{\infty} \frac{x^{\alpha-1}}{x^2+1} dx$ exists, and evaluate the integral subject to this condition.

23. Show that $\int_{0}^{\infty} \frac{\ln x}{x^2+a^2} dx = \frac{\pi}{2a} \ln a,\ (a>0)$

24. Evalute $\int_{0}^{\infty} \frac{x^{1/2}}{x^3+1} dx$

25. By considering integration round a suitable rectangular contour, show that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x} dx = \sec \frac{a}{2}, \text{ and hence } \int_{0}^{\infty} \frac{\cosh ax}{\cosh \pi x} dx = \frac{1}{2} \sec \frac{a}{2}, (-\pi < a < \pi).$$

26. Using the inversion formula and the residue theorem evaluate the inverse of the following Fourier transforms as per the definition given in Example (20.38),

(a) $\dfrac{1}{(1+w^2)^2}$ (b) $\dfrac{1}{(1+iw)^2}$

27. By considering the contour integral $\oint_C \exp(iz^2)dz$ around a suitable contour C, show that

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

ANSWERS

Exercise 20.1 (p. 1119)

1. Convergent
2. Divergent
3. Converges $p > 1$, diverges $0 < p \le 1$.
4. Converges
5. Converges
6. Divergent

Exercise 20.2 (p. 1123)

1. $-1, 1$
2. $-i\sqrt{2}, 1$
3. $0, 0$
4. $-2+i, \sqrt{2}$
5. $0, e^x$
6. $0, 1$
7. $-i, 1/e$
8. $0, 1$
9. $0, 1/\sqrt{2}$
10. $0, 2$
11. $0, 1$
12. $0, 1/2$

Exercise 20.3 (p. 1130)

1. $\frac{1}{2} - \frac{1}{4}(z-2) + \frac{1}{8}(z-2)^2 - \frac{1}{16}(z-2)^3 + \ldots;\ R = 2.$

2. $e^a\left[1+(z-a)+\frac{1}{2!}(z-a)^2+\frac{1}{3!}(z-a)^3+\ldots\right];\ R=\infty$

3. $1-\frac{1}{2!}\left(z-\frac{\pi}{2}\right)^2+\frac{1}{4!}\left(z-\frac{\pi}{2}\right)^4-\ldots;\ R=\infty$

4. $i\left[1+\frac{1}{2!}\left(z-\frac{\pi i}{2}\right)^2+\frac{1}{4!}\left(z-\frac{\pi i}{2}\right)^4+\ldots\right];\ R=\infty$

5. $-\left[1+\frac{1}{2!}(z-\pi i)^2+\frac{1}{4!}(z-\pi i)^2+\ldots\right];\ R=\infty$

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} i^n (z-i)^n}{n}$; $R = 1$

7. $\sum_{n=0}^{\infty} (-1)^n z^{2n}$; $R = 1$

8. $z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots$; $R = 1$

9. $z^2 - \frac{1}{3}z^4 + \frac{2}{45}z^6 - \frac{1}{315}z^8 + \ldots$; $R = \infty$

10. $2 + z + 2z^2 + z^3 + 2z^4 + \ldots$; $R = 1$

11. $\sum_{n=0}^{\infty} \frac{(1+i)^n + (1-i)^n}{2n!} z^n$; $R = \infty$

12. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{4n} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$; $R = \infty$

13. $1 + iz + \sum_{n=1}^{\infty} \left(\frac{2^n + 2^{n-1}}{(2n)!} z^{2n} + i \frac{2^n}{(2n+1)!} z^{2n+1} \right)$; $R = \infty$

15. $1 + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \ldots$; $R = \pi/2$

16. $1 + z - z^2 - z^3 + 2z^4 + 2z^5 - \ldots$; $R = 1$

17. $4 - \frac{z^2}{3} + \frac{1}{20}z^4 + \ldots$; $R = 2\pi$

21. (a) $\left(2/\sqrt{\pi}\right)\left(z - z^3/3 + z^5/2!\,5 - z^7/3!\,7 + \ldots\right)$; $R = \infty$
 (b) $z - z^3/3!\,3 + z^5/5!\,5 - z^7/7!\,7 + \ldots$; $R = \infty$
 (c) $z^3/1!\,3 - z^7/3!\,7 + z^{11}/5!\,11 - \ldots$; $R = \infty$

Exercise 20.4 (p. 1137)

1. $\frac{1}{2} + z + z^2 + \sum_{n=1}^{\infty} \frac{1}{(n+2)!\,z^n}$; $|z| > 0$

2. $\frac{1}{z^4} - \frac{1}{2z^2} + \frac{1}{24} - \frac{1}{720}z^2 + \ldots$; $|z| > 0$

3. $\frac{1}{z^3} + \frac{1}{z} + \frac{1}{2}z + \frac{1}{6}z^3 + \ldots$; $|z| > 0$

4. $\frac{1}{z} - \frac{1}{2} + \frac{z}{12}$; $0 < |z| < 2\pi$

5. $z^3 + \frac{1}{2}z + \frac{1}{24z} + \frac{1}{720z^3} + \ldots$; $|z| > 0$

6. $\frac{1}{z^4} + \frac{1}{3!}\frac{1}{z^2} + \frac{1}{5!} + \frac{1}{7!}z^2 + \ldots$; $|z| > 0$

7. (i) $\sum_{n=0}^{\infty} (-1)^n \left[2^{-n-1} - (n+4)3^{-n-2}\right](z-1)^n$, $|z-1| < 2$

(ii) $\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(z-1)^{n+1}} - \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n (n+4)\left(\frac{z-1}{3}\right)^n, 2 < |z-1| < 3$

(iii) $\sum_{n=0}^{\infty} (-1)^n \left[\frac{2^n - 3^n}{(z-1)^{n+1}} - \frac{3^n(n+1)}{(z-1)^{n+2}}\right], |z-1| > 3$

8. $e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2}{3}(z-1) + \dots\right]; |z-1| > 0$

9. $1 - \frac{8i}{(z+2i)} - \frac{24}{(z+2i)^2} + \frac{32i}{(z+2i)^3} + \dots; |z+2i| > 0$

10. $\frac{1}{-(z+\pi i)^2} - \frac{1}{2} - \frac{1}{24}(z+\pi i)^2 + \dots; |z+\pi i| > 0$

11. $\frac{i}{2}\frac{1}{z+i} + \frac{i}{2}\sum_{n=0}^{\infty}\frac{1}{(2i)^{n+1}}(z+i)^n;\ 0 < |z+i| < 2$

12. No; the first is valid for $|z| < 1$ and the second is valid for $|z| > 1$; there are no common points between the two regions.

Exercise 20.5 (p. 1140)

1. $0, \infty$ (second order poles)
2. $\pm ia$, second order poles
3. ∞, essential singularity
4. πi, simple pole; ∞ (essential singularity)
5. 0, removable singularity
6. $\pi/4$, simple pole
7. 2, essential singularity
8. 1, pole of order 4
9. $(2n+1)\pi, n = 0, \pm 1, \pm 3$ (second order)
10. $\pm i,\ 0,\ \pm 2\pi i,\ \pm 4\pi i; \dots$ simple
11. $\pm\sqrt{3} \pm i\sqrt{2}$; third order
12. 0, third order
13. $\pm\frac{1}{n\pi}, n = 1, 2, 3, \dots$
14. $\frac{\pi}{4} + n\pi, n = 0, \pm 1, \pm 2 \dots$ simple
15. $z = [(2k+1)\pi/2]^{1/3}$ and $z = \frac{1}{2}\sqrt{(2k+1)\pi/2}\left(1 \pm i\sqrt{3}\right), k = 0, \pm 1, \pm 2, \dots$ simple
16. First order pole at π, second order poles at $z = n\pi, n = -1, \pm 2, \pm 3, \dots$

Exercise 20.6 (p. 1144)

1. $-\frac{1}{2}$ (at 0), $\frac{5}{2}$ (at 2)
2. $\frac{1}{2}$ (at -1), $\frac{1}{4}(1+2i)$ (at i), $\frac{1}{4}(1-2i)$ (at $-i$)

3. $4/15$ (at 0), 4. -1, (at $\pm 2n\pi i$) 5. $1/\pi$ (at $0, \pm 1, \ldots$)

6. $\frac{1}{4}$ (at $+0$), $\frac{i}{16}\sin(2i)$, (at $2i$),

7. 1, (at 0) 8. $-\frac{1}{2}\cos i$, (at $-i$) 9. (i) 0, (ii) $\pi(i-2)$, (iii) $\pi(2+i)$

10. $-4i$ 11. $-\frac{3}{4}\pi i$ 12. $-2\pi i$

13. $8\pi i/3e^2$ 14. $2\pi i \sec 1\,(1+\tan 1)$ 15. $-2\pi i$

16. $4\pi i/5$ 17. $\frac{\pi}{2}(-i-\cos 4)$

18. $\frac{1}{z^3}-\frac{1}{6z}+\frac{7z}{360}-\frac{31z^6}{15120}+\ldots; -\frac{1}{3}\pi i$

Exercise 20.7 (p. 1167)

1. $\frac{\pi}{\sqrt{k^2-1}}$ 2. $\frac{3\pi}{8}$ 3. $\frac{2\pi}{1-p^2}$ 4. 0

5. $\frac{3\pi}{16}$ 6. 2π 7. $\pi/3$ 8. $\frac{\pi}{2\sqrt{2}}$

9. $\frac{\pi}{4a^3}$ 10. $\pi/18$ 11. $\frac{3\pi}{8a^5}$

12. $\frac{\pi}{[n\sin\{(2m+1)\pi/2n\}]}$ 13. $\frac{\pi(1+a)}{4a^3e^a}$ 14. $\frac{\pi}{2}(1-e^{-4})$

15. $\frac{\pi}{2}e^{-ma/\sqrt{2}}\cos\left(ma\sqrt{2}\right)$ 16. $\frac{\pi[a^2e^{-ma}-b^2e^{-mb}]}{a^2-b^2}$ 17. $-\sqrt{3}\pi/6$

18. π 19. $\pi(b-a)/2$ 20. $3\pi/8$ 21. $\frac{\pi}{4}[e^{-a}+\sin a]$

22. $\frac{\pi\cos(\alpha\pi/2)}{\sin\alpha\pi}, 0<\alpha<2, \alpha\neq 1$ 24. $\pi/3$

PART G

Numerical Methods

Numbers which represent the exact number to a certain degree of accuracy are approximate numbers. For examples, 3.1416 is an approximate value of π; a still better approximation is 3.141592. But we cannot write the exact value of π since it is non-terminating. Similarly $\sqrt{2}$ = 1.414213 ..., 1/3 = 0.3333 ..., are non-terminatings but 1/2 = 0.50 is terminating.

The digits used to express a number are called *significant digits* or *significant figures*.

Thus each of the number 3.1416, 0.33333, 45.292 contains five significant digits while the number 0.000015 contains two significant digits, since the zeros are being applied to fix the position of the decimal point.

In numerical computations we come across numbers with a large number of digits and it becomes necessary to limit such numbers to desirable number of digits. For example, we may like to take the approximate value of π as 3.14 or 3.141. This process is called *rounding off*. The rounding off a number to n significant digits is done according to the following rules:

Discard all digits to the right of nth digit, and if this discarded number is, (a) less than half a unit in the nth place, leave the nth digit unchanged; (b) greater than half a unit in the nth place, increase the nth digit by unity; (c) exactly half a unit in the nth place, increase the nth digit by unity if it is odd otherwise leave it unchanged.

The number thus rounded off is said to be correct to n significant digits.

For example, following numbers are rounded off to five significant digits.

1.62139	to	1.6214
1.62173	to	1.6217
1.62115	to	1.6212
1.62125	to	1.6212

Types of errors

In numerical computation we usually come across the following types of errors.

(i) Inherent errors: These are the errors involved in the statement of a problem. Such errors arise either due to the given data being approximate or due to the limitations of the computing aids: mathematical tables, calculators or the digital computers. Due to this limitation, numbers are to be rounded off, causing *rounding errors*.

Inherent errors can be minimized by obtaining better data and higher precision computing aids. In particular rounding errors can be minimized by retaining at least one more significant digit at each step than that given in the data and rounding off at the last step. Also this error can be minimized by avoiding subtraction of nearly equal numbers or division by a small number.

(ii) Truncation errors: These are the errors introduced by using approximate formulae, such as replacing an infinite series in x after 'truncating' it at a certain stage.

For example, if $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$ may be replaced by $x - \frac{x^3}{3!} + \frac{x^5}{5!}$, then the truncation error creeps in. As an other example, if we are using a computing aid having a fixed word length of say 8 digits, then rounding off of 12.65786372 gives 12.657864 where as truncation gives 12.657863.

Truncation error in a problem can be evaluated and it is desirable to minimize it.

Absolute, relative and percentage errors

If X is the true value of a quantity and X′ is its approximate value, then error, E is given by

$$E = X - X'.$$

The absolute value of the error E, that is $|X - X'|$ is called the *absolute error* E_A. Thus

$$E_A = |X - X'|.$$

The *relative error* denoted by E_R, is $E_R = \left|\dfrac{X - X'}{X}\right|$,

and the *percentage error* denoted by E_P, is $E_P = 100\,E_R = \left|\dfrac{X - X'}{X}\right| \times 100.$

The relative and percentage errors are independent of the units while the absolute error depends upon the units being used. For example if $\pi = 22/7$ is approximated as 3.14, then

absolute error, $\quad E_A = \left|\dfrac{22}{7} - 3.14\right| = \left|\dfrac{22 - 21.98}{7}\right| = \left|\dfrac{0.02}{7}\right| = 0.0028571$

relative error, $\quad E_R = \left|\dfrac{0.0028571}{22/7}\right| = 0.000909.$

percentage error, $\quad E_P = E_R \times 100 = 0.0909\%.$

21.2 SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Mathematical formulations for a variety of problems in science and engineering reduce to the form

$$f(x) = 0. \qquad ...(21.1)$$

Examples are: $x^2 - 5x + 6 = 0$, $x^3 + x - 1 = 0$, $\log x = x^2$, $\tan x = 3x$, $\sinh x = \cos x$, etc. which can be written in the form (21.1). The first two are *algebraic equations* and the remaining three are *transcendental equations*. A transcendental equation may have a finite or an infinite number of real roots or may not have real roots at all. Except in a few simple cases, there are practically no formulae to solve equations of the form (21.1).

In the absence of a direct analytical method to solve Eq. (21.1) we use an approximation method, in particular an *iteration method*. An iterative technique usually begins with an approximate value of the root, known as the *initial guess*, which is then successively improved iteration by iteration. Iteration methods are easy to program because the computational operations are the same at each iteration. Next, we discuss a few iterative methods to solve equations of the form (21.1).

21.2.1 Bisection Method

The bisection method for finding a solution of $f(x) = 0$ with continuous f is based on *intermediate value theorem* which states that *if a continuous function f has opposite signs at some $x = a$ and $x = b$ ($> a$), then f must be zero at least for one $x \in (a, b)$.*

Define $x_1 = \dfrac{a+b}{2}$ as the mid-point between a and b. There exist the following three possible cases:

1. If $f(x_1) = 0$, then the root is at $x = x_1$.
2. If $f(a)\,f(x_1) < 0$, then a root is between a and x_1.
3. If $f(x_1)\,f(b) < 0$, then a root is between x_1 and b.

Thus, by testing the sign of f at the mid-point, we can identify subinterval which contains the root. We can further divide this subinterval into two halves to locate a new subinterval containing the root. In general, the process continues until the interval containing the root is small to the desired level.

Convergence of the bisection method. The bisection method is guaranteed to converge but convergence is slower. In case the process is repeated n times, then the interval containing the root is reduced to the size $I/2^n$, where $I = b - a$ is the length of the interval $[a, b]$. Thus, the error bound after the nth iteration is $E_n = \left|\dfrac{I}{2^n}\right|$

Further, $E_{n+1} = E_n/2$ and also as $n \to \infty$, $E_n \to 0$, thus, the *convergence is linear but guaranteed in case of bisection method.*

Example 21.1: Find the root of $f(x) = x^3 + x - 1 = 0$ lying in the interval $(0, 1)$ using bisection method up to four iterations. Also find the maximum possible error in the root computed.

Solution: We have, $f(x) = x^3 + x - 1$.

Since $f(0) = -1$ is negative and $f(1) = 1$ is positive, thus a root of $f(x) = 0$ lies between 0 and 1.

The 1st approximation to the root is $x_1 = \dfrac{0+1}{2} = 0.5$

We have, $f(0.5) = (0.5)^3 + (0.5) - 1 = -0.375 < 0$, thus, the root lies between 0.5 and 1.

The 2nd approximation to the root is $x_2 = \dfrac{0.5+1}{2} = 0.75$.

We have, $f(0.75) = (0.75)^3 + (0.75) - 1 = 0.1719 > 0$, thus, the root lies between 0.5 and 0.75.

The 3rd approximation to the root is $x_3 = \dfrac{0.5+0.75}{2} = 0.625$.

We have, $f(0.625) = (0.625)^3 + (0.625) - 1 = -0.1309 < 0$, thus, the root lies between 0.625 and 0.75.

The 4th approximation to the root is $x_4 = \dfrac{0.625+0.75}{2} = 0.6875$.

The maximum possible error after the 4th iteration is $E_4 = \pm\dfrac{1-0}{2^4} = \pm\dfrac{1}{16} = \pm 0.0625$.

Example 21.2: Find an approximate root of the equation $\sin x = 1/x$, that lies between $x = 1$ and $x = 1.5$ radians using bisection method up to seven iterations.

Solution: Let $f(x) = x \sin x - 1$.

Since $f(1) = 1 \times \sin(1) - 1 = -0.15849$ and, $f(1.5) = 1.5 \times \sin(1.5) - 1 = 0.49625$; a root lies between 1 and 1.5.

The first approximation to the root is $x_1 = \frac{1}{2}(1 + 1.5) = 1.25$.

Since $f(x_1) = (1.25) \sin(1.25) - 1 = 0.18627$ and $f(1) < 0$, thus, a root lies between 1 and $x_1 = 1.25$.

The second approximation to the root is $x_2 = \frac{1}{2}(1+1.25) = 1.125$.

Since $f(x_2) = 1.125 \sin(1.125) - 1 = 0.01509$ and $f(1) < 0$, thus, a root lies between 1 and $x_2 = 1.125$.

The third approximation to the root is $x_3 = \frac{1}{2}(1 + 1.125) = 1.0625$.

Since $f(x_3) = 1.0625 \sin(1.0625) - 1 = -0.0718 < 0$ and $f(x_2) > 0$, thus, the root lies between $x_3 = 1.0625$ and $x_2 = 1.125$.

The fourth approximation to the root is $x_4 = \frac{1}{2}(1.0625 + 1.125) = 1.09375$.

Since $f(x_4) = -0.02836 < 0$ and $f(x_2) > 0$, thus, the root lies between $x_4 = 1.09375$ and $x_2 = 1.125$.

The fifth approximation to the root is $x_5 = \frac{1}{2}(1.09375 + 1.125) = 1.10937$.

Since $f(x_5) = -0.00664 < 0$ and $f(x_2) > 0$, thus, the root lies between $x_5 = 1.10937$ and $x_2 = 1.125$.

The sixth approximation to the root is $x_6 = \frac{1}{2}(1.109337 + 1.125) = 1.11719$.

Since $f(x_6) = 0.00421 > 0$. But $f(x_5) < 0$, thus the root lies between $x_5 = 1.10937$ and $x_6 = 1.11719$.

The seventh approximation to the root is $x_7 = \frac{1}{2}(1.10937 + 1.11719) = 1.11328$.

Hence, the root to the desired approximation is 1.11328.

21.2.2 False Position Method (or, Regula-Falsi Method)

In bisection method the initial interval $[x_0, x_1]$, (taking x_0 for a and x_1 for b), is divided into two equal subintervals irrespective of the location of the root. However, the root may be closer to one end than the other; as shown in Fig. 21.1, root is closer to x_0. Here we join the points $A\ (x_0, f(x_0))$ and $B(x_1, f(x_1))$ by a straight line. The point of intersection of the line AB with the x-axis, say x_2 gives an estimate of the root and is called the false position of the root. In the next iteration, the point B is replaced by the point $C(x_2, f(x_2))$ and the process is repeated getting the intersection of AC with x-axis as the improved estimate. The iteration process continues till the root is found to the desired accuracy.

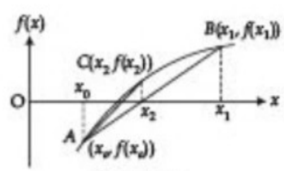


Fig. 21.1

The equation of the chord AB is $y - f(x_0) = \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x - x_0)$.

When $x = x_2$, $y = 0$, it gives $x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$. ...(21.2)

The expression (21.2) is known as the *false position formula*; here x_2 is obtained by applying a correction to x_0. After calculating the first approximation to the root, this process is repeated for the new interval at the end points of which $f(x)$ has the opposite sign, and so on.

The process of iteration in case of false position method *converges linearly and also the convergence is guaranteed.*

Example 21.3: Apply false position method to find a root of $f(x) = x^2 - x - 2 = 0$ in the interval (1, 3) up to three iterations.

Solution: Here $x_0 = 1$, $x_1 = 3$ and $f(x) = x^2 - x - 2$.

We have, $f(1) = -2$ and $f(3) = 4$, thus, a root of $f(x) = 0$ lies between 1 and 3.

The first approximation is $x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 1 - \dfrac{3-1}{4-(-2)}(-2) = 1.667$.

We have, $f(1.667) = -0.8889 < 0$, thus, the root lies between 1.667 and 3.

The second approximation is $x_3 = 1.667 - \dfrac{3 - 1.667}{4-(-0.8889)}(-0.8889) = 1.909$.

We have, $f(1.909) = -0.2647$, thus, the root lies between 1.909 and 3. The third approximation is

$$x_4 = 1.909 - \frac{3 - 1.909}{4-(-0.2647)}(-0.2647) = 1.909 + \frac{1.091}{4.2647}(0.2647) = 1.9767.$$

Example 21.4: Find a real root of the equation $x \log_{10} x = 1.2$ by Regula-Falsi method correct to four decimal places.

Solution: Let $f(x) = x \log_{10} x - 1.2$, so that, $f(1) = -\text{ve}$, $f(2) = -\text{ve}$ and $f(3) = +\text{ve}$, thus, a root lies between 2 and 3. Taking $x_0 = 2$ and $x_1 = 3$, $f(x_0) = -0.59794$ and $f(x_1) = 0.23136$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.72102$$

Now, $f(x_2) = f(2.72102) = -0.01709$, thus, the root lies between 2.72102 and 3.

Taking $x_0 = 2.72102$, $x_1 = 3$, $f(x_0) = -0.01709$ and $f(x_1) = 0.23136$, we get

$$x_3 = 2.72102 + \frac{0.27898}{0.23136 + 0.01709}(0.01709) = 2.74021$$

Repeating this process, the next approximation is $x_4 = 2.7402$.

Hence, the root is 2.7402, correct to 4 decimal places.

Example 21.5: Use the method of false position, to find the fourth root of 32 correct to three decimal places.

Solution: Let $x = (32)^{1/4}$, so that, $x^4 - 32 = 0$. Take $f(x) = x^4 - 32$. Then $f(2) = -16$ and $f(3) = 49$, thus, a root lies between 2 and 3. Taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -16$, $f(x_1) = 49$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{16}{65} = 2.2462.$$

Now, $f(x_2) = f(2.2462) = -6.5438$, thus, the root lies between 2.2462 and 3. Next taking $x_0 = 2.2462$, $x_1 = 3$, $f(x_0) = -6.5438$, $f(x_1) = 49$, we get

$$x_3 = 2.2462 - \frac{3 - 2.2462}{49 + 6.5438}(-6.5438) = 2.335.$$

Now, $f(x_3) = f(2.335) = -2.2732$, thus, the root lies between 2.335 and 3. Next taking $x_0 = 2.335$ and $x_1 = 3$, $f(x_0) = -2.2732$ and $f(x_1) = 49$, we obtain

$$x_4 = 2.335 - \frac{3 - 2.335}{49 + 2.2732}(-2.2732) = 2.3645.$$

Repeating this process, the successive approximations are $x_5 = 2.3770$, $x_6 = 2.3779$, and since $x_5 = x_6$ upto 3 decimal places, we take $(32)^{1/4} = 2.378$.

21.2.3 Secant Method

This method is an improvement over the method of false position since it does not require the condition that $f(x)$ has the opposite signs at the end points of the interval $[x_1, x_2]$. Here also the function $y = f(x)$ is approximated by the chord AB, refer Fig. 21.1, and the successive approximations are obtained by using two recent preceding approximations.

As in case of false position method, the first approximate x_2, when the initial interval is $[x_0, x_1]$, is given by $x_2 = x_1 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$.

Thus, the general formula for the successive approximations is, therefore, given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n),\ n \geq 1 \quad ...(21.3)$$

The convergence of the secant method is not necessary but once it is so, the rate of convergence is faster than that of the method of false position.

Example 21.6: Find a positive solution of $f(x) = x - 2\sin x = 0$ by the secant method starting from $x_0 = 2$ and $x_1 = 1.9$ (in radians) up to three iterations.

Solution: Here $f(x) = x - 2\sin x$. We have,

$f(x_0) = f(2) = 0.1814051$ and $f(x_1) = f(1.9) = 0.0073998$. By secant method,

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 1.9 - \frac{1.9 - 2}{0.0073998 - 0.1814051}(.0073998) = 1.895747$$

This gives, $f(x_2) = .0004141$. Thus,

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 1.895747 - \frac{1.895747 - 1.9}{0.0004141 - 0.0073998}(.0004141) = 1.895495$$

This gives, $f(x_3) = 0.0000012$. Thus,

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 1.895495 - \frac{1.895495 - 1.895747}{0.0000012 - 0.0004141}(.0004141) = 1.895464.$$

Hence, the root of $x - 2\sin x = 0$ up to three iterations is 1.895464.

21.2.4 Newton-Raphson Method

This method for solving equations $f(x) = 0$ assumes that f has continuous derivative f'. Here the graph of the function $y = f(x)$ is approximated by a suitable tangent.

Let x be an approximate root of $f(x) = 0$. Draw a tangent to the curve $y = f(x)$ at $x = x_0$, and let x_1 be the point of intersection of this tangent with the x-axis, as shown in Fig. 21.2. Then the slope $\tan\alpha$ is given by $\tan\alpha = f'(x_0) = \dfrac{f(x_0)}{x_0 - x_1}$.

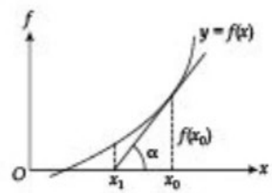


Fig. 21.2

This gives, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

as the improved approximation to the root of $y = f(x)$.

The next approximation would be $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$,

and, in general, the $(n+1)$th approximation is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad ...(21.4)$$

The process continues until the difference between two successive approximations is within the prescribed limits. The expression (21.4) is called the *Newton-Raphson formula*.

This method is useful in cases of large value of $f'(x)$, that is, when the graph of $f(x)$ while crossing the x-axis is nearly vertical.

Convergence of Newton-Raphson method. Newton method converges conditionally under the condition $|f(x)\,f''(x)| < |f'(x)|^2$ in the interval under consideration. But when it is so, the convergence is faster as compared to the false position method. In fact, *Newton-Raphson method has quadratic convergence* as shown below.

Let a be the actual root of $f(x) = 0$ and let x_n, the nth approximation differs from a by a small quantity, say ε_n. Thus, $x_n = a + \varepsilon_n$ and $x_{n+1} = a + \varepsilon_{n+1}$ so the expression (21.4) becomes

$$a + \varepsilon_{n+1} = a + \varepsilon_n - \frac{f(a+\varepsilon_n)}{f'(a+\varepsilon_n)}$$

or, $\varepsilon_{n+1} = \varepsilon_n - \dfrac{f(a+\varepsilon_n)}{f'(a+\varepsilon_n)} = \varepsilon_n - \dfrac{f(a) + \varepsilon_n f'(a) + \frac{1}{2}\varepsilon_n^2 f''(a) + \dots}{f'(a) + \varepsilon_n f''(a) + \dots}$. (By Taylor's expansion)

Using $f(a) = 0$, taking LCM on the right-hand side and simplifying, we get

$\dfrac{\varepsilon_{n+1}}{\varepsilon_n^2} = \dfrac{\frac{1}{2}f''(a) + \frac{1}{3}\varepsilon_n f'''(a) + \dots}{f'(a) + \varepsilon_n f''(a) + \dots}$ This gives, $\lim_{n\to\infty}\left|\dfrac{\varepsilon_{n+1}}{\varepsilon_n^2}\right| = \dfrac{1}{2}\dfrac{f''(a)}{f'(a)}$ = constant, $c \neq 0$.

This proves the quadratic convergence of the Newton-Raphson method.

Remark: We have derived the Newton-Raphson formula (21.4) using the geometrical approach. Alternatively, it can be derived by Taylor series expansion also as follows.

Let x_0 be an approximation root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ is the exact root, then

$$0 = f(x_1) = f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \dots \quad ...(21.5)$$

Considering h to be so small such that its square and higher powers are neglected, then from (21.5), we get $h \approx -\dfrac{f(x_0)}{f'(x_0)}$.

Thus, a closer approximation to the root is given by $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

Similarly, starting with x_1, a still better approximation x_2 is given by $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

and, in general, we obtain $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, \dots$, the same as (21.4).

Example 21.7: By applying Newton's method up to two iterations, find the real root near to 2 for the equation $x^4 - 12x + 7 = 0$.

Solution: Let $f(x) = x^4 - 12x + 7$. This gives, $f'(x) = 4x^3 - 12$.

Here $x_0 = 2$. Therefore, $f(x_0) = f(2) = 2^4 - 12.2 + 7 = -1$, and $f'(x_0) = f'(2) = 4(2)^3 - 12 = 20$.

Using Newton-Raphson formula, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{(-1)}{20} = \frac{41}{20} = 2.05.$$

and, further $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.05 - \frac{(2.05)^4 - 12(2.05) + 7}{4(2.05)^2 - 12} = 2.6706$$

Thus, the root of the equation is 2.6706.

Example 21.8: Find using Newton's method, the root of the equation $e^x = 4x$, near to 2, correct to three places of decimals.

Solution: We have, $f(x) = e^x - 4x$.

$f(2) = e^2 - 8 = 7.389056 - 8 = -0.610944 < 0$, and

$f(3) = e^3 - 12 = 20.085537 - 12 = 8.085537 > 0$.

Since $f(2)f(3) < 0$, thus, $f(x) = 0$ has a root between 2 and 3.

Let $x_0 = 2.1$ be the approximate value of the root of the equation given.

Now, $f(x) = e^x - 4x$ gives $f'(x) = e^x - 4$

Therefore, $f(x_0) = e^{2.1} - 4(2.1) = 8.16617 - 8.4 = -0.23383$, and $f'(x_0) = e^{2.1} - 4 = 4.16617$.

Let x_1 be the first approximation of the root, then by Newton-Raphson formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.1 - \frac{(-0.23383)}{4.16617} = 2.1 + 0.0561258 = 2.1561 \quad \text{(approximately)}.$$

If x_2 denotes the second approximation, then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1561 - \frac{[e^{2.1561} - 4(2.1561)]}{[e^{2.1561} - 4]} = 2.1561 - \frac{0.0129861}{4.6373861} \approx 2.1533.$$

We have $f(x_2) = f(2.1533) = -0.0013484$ and, $f'(x_2) = f'(2.1533) = 4.6106516$.

If x_3 denotes the third approximation to the root, then

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.1533 - \frac{(-0.0013484)}{(4.6106516)} \approx 2.1532.$$

Thus, the value of the root correct to three decimal places is 2.153.

Example 21.9: Solve $x^4 - 5x^3 + 20x^2 - 40x + 60 = 0$ by Newton-Raphson method given that all the roots of the given equation are complex.

Solution: Let $f(x) = x^4 - 5x^3 + 20x^2 - 40x + 60$ so that $f'(x) = 4x^3 - 15x^2 + 40x - 40$.

The given equation is $f(x) = 0$.

Using Newton-Raphson method, we obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 5x_n^3 + 20x_n^2 - 40x_n + 60}{4x_n^3 - 15x_n^2 + 40x_n - 40} = \frac{3x_n^4 - 10x_n^3 + 20x_n^2 - 60}{4x_n^3 - 15x_n^2 + 40x_n - 40}.$$

Set $n = 0$ and, take $x_0 = 2(1+i)$ by trial, we get

$$x_1 = \frac{3x_0^4 - 10x_0^3 + 20x_0^2 - 60}{4x_0^3 - 15x_0^2 + 40x_0 - 40} = \frac{3(2+2i)^4 - 10(2+2i)^3 + 20(2+2i)^2 - 60}{4(2+2i)^3 - 15(2+2i)^2 + 40(2+2i) - 40} = 1.92 + 1.92i$$

Similarly, $x_2 = \dfrac{3(1.92+1.92i)^4 - 10(1.92+1.92i)^3 + 20(1.92+1.92i) - 60}{4(1.92+1.92i)^3 - 15(1.92+1.92i)^2 + 40(1.92+1.92i) - 40} = 1.915 + 1.908i.$

Thus, $1.915 + 1.908i$ is a root of the given equation.

Imaginary roots appear in pairs, therefore, $1.915 - 1.908i$ is also a root of the equation.

Since, $f(x) = 0$ is a biquadratic equation, the number of roots of the equation is four. Let us assume that $\alpha + i\beta$ and $\alpha - i\beta$ is the other pair of roots of the given equation.

From the given equation, sum of the roots = 5.

Thus, $(1.915 + 1.908i) + (1.915 - 1.908i) + (\alpha + i\beta) + (\alpha - i\beta) = 5.$

This gives, $2\alpha + 3.83 = 5$ or $\alpha = \dfrac{5 - 3.83}{3} = 0.585.$

Also the product of roots is = 60, thus

$$(\alpha + i\beta)(\alpha - i\beta)\ (1.915 + 1.908i)(1.915 - 1.908i) = 60$$

This gives, $(\alpha^2 + \beta^2)\,[(1.915)^2 + (1.908i)^2] = 60$

or, $((0.585)^2 + \beta^2)(7.307689) = 60$

Simplifying, we obtain $\beta = \sqrt{7.8703045} = 2.805$

Therefore, the other two roots are $0.585 \pm 2.805i$.

Hence, the root of the given equation are $1.915 \pm 1.908i$ and $0.585 \pm 2.805i$.

Some useful iterative formulae. If N is a positive integer, then *Newton-Raphson* formula (21.4) can be applied to derive the following useful iterative formulae

(1) Iterative formula to find $1/N$: $x_{n+1} = x_n(2 - Nx_n)$ ...(21.6)

(2) Iterative formula to find $\sqrt{N}$: $x_{n+1} = \dfrac{1}{2}(x_n + N/x_n)$ ...(21.7)

(3) Iterative formula to find $1/\sqrt{N}$: $x_{n+1} = \dfrac{1}{2}(x_n + 1/Nx_n)$ ...(21.8)

(4) Iterative formula to find $\sqrt[k]{N}$: $x_{n+1} = \dfrac{1}{k}[(k-1)x_n + N/x_n^{k-1})]$ ...(21.9)

Proof: (1) Let $x = 1/N$, this gives, $1/x - N = 0$. Taking $f(x) = 1/x - N$, we have, $f'(x) = -1/x^{-2}$.

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-1/x_n^{-2}} = x_n + \left(\frac{1}{x_n} - N\right)x_n^2 = x_n + x_n - Nx_n^2 = x_n(2 - Nx_n).$$

(2) Let $x = \sqrt{N}$, this gives, $x^2 - N = 0$. Taking $f(x) = x^2 - N$, we have, $f'(x) = 2x$.

Then Newton's formula gives $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^2 - N}{2x_n} = \dfrac{1}{2}(x_n + N/x_n)$.

(3) Let $x = \dfrac{1}{\sqrt{N}}$, this gives $x^2 - \dfrac{1}{N} = 0$. Taking $f(x) = x^2 - 1/N$, we have, $f'(x) = 2x$.

Then Newton's formula gives $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^2 - 1/N}{2x_n} = \dfrac{1}{2}\left(x_n + \dfrac{1}{Nx_n}\right)$.

(4) Let $x = \sqrt[k]{N}$, this gives $x^k - N = 0$. Taking $f(x) = x^k - N$, we have, $f'(x) = kx^{k-1}$.

Then Newton's formula gives $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^k - N}{kx_n^{k-1}} = \dfrac{1}{k}\left[(k-1)x_n + \dfrac{N}{x_n^{k-1}}\right]$.

Example 21.10: Evaluate the following, correct to four decimal places by Newton's iteration method

(i) $1/31$ (ii) $\sqrt{28}$

(iii) $1/\sqrt{14}$ (iv) $\sqrt[3]{24}$

(v) $(30)^{-1/5}$

Solution: (i) Taking $N = 31$, (21.6) becomes $x_{n+1} = x_n(2 - 31x_n)$.

Since an approximate value of $1/31$ is 0.03, we take $x_0 = 0.03$. Thus,

$$x_1 = x_0(2 - 31x_0) = 0.03(2 - 31 \times 0.03) = 0.0321$$
$$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31 \times 0.0321) = 0.032257$$
$$x_3 = x_2(2 - 31x_2) = 0.032257(2 - 31 \times 0.032257) = 0.03226$$

Since $x_2 = x_3$ upto 4 decimal places, thus, we have $1/31 = 0.0321$.

(ii) Taking $N = 28$, (21.7) becomes $x_{n+1} = \dfrac{1}{2}(x_n + 28/x_n)$.

Since an approximate value of $\sqrt{28}$ is 5, we take $x_0 = 5$. Thus,

$$x_1 = \frac{1}{2}(x_0 + 28/x_0) = \frac{1}{2}(5 + 28/5) = 5.3$$

$$x_2 = \frac{1}{2}(x_1 + 28/x_1) = \frac{1}{2}(5.3 + 28/5.3) = 5.29151$$

$$x_3 = \frac{1}{2}(x_2 + 28/x_2) = \frac{1}{2}(5.29151 + 28/5.29151) = 5.29150$$

Since $x_2 = x_3$ upto 4 decimal places, we have, $\sqrt{28} = 5.2915$.

(iii) Taking $N = 14$, (21.8) becomes $x_{n+1} = \dfrac{1}{2}[x_n + 1/(14x_n)]$

Since an approximate value of $1/\sqrt{14}$ is given by $1/\sqrt{16} = \frac{1}{4} = 0.25$, we take $x_0 = 0.25$. Thus,

$$x_1 = \frac{1}{2}[x_0 + (14x_0)^{-1}] = \frac{1}{2}[0.25 + (14 \times 0.25)^{-1}] = 0.26785$$

$$x_2 = \frac{1}{2}[x_1 + (14x_1)^{-1}] = \frac{1}{2}[0.26785 + (14 \times 0.26785)^{-1}] = 0.2672618$$

$$x_3 = \frac{1}{2}[x_2 + (14x_2)^{-1}] = \frac{1}{2}[0.2672618 + (14 \times 0.2672618)^{-1}] = 0.2672612$$

Since $x_2 = x_3$ upto 4 decimal places, we take $1/\sqrt{14} = 0.2673$.

(iv) Taking $N = 24$ and $k = 3$, (21.9) becomes $x_{n+1} = \frac{1}{3}[2x_n + 24/x_n^2]$.

Since an approximate value of $(24)^{1/3}$ is given by $(27)^{1/3} = 3$, we take $x_0 = 3$. Thus,

$$x_1 = \frac{1}{3}(2x_0 + 24x_0^2) = \frac{1}{3}(6 + 24/9) = 2.88889$$

$$x_2 = \frac{1}{3}(2x_1 + 24/x_1^2) = \frac{1}{3}[(2 \times 2.88889) + 24/(2.88889)^2] = 2.88451$$

$$x_3 = \frac{1}{3}(2x_2 + 24/x_2^2) = \frac{1}{3}[2 \times 2.88451 + 24/(2.88451)^2] = 2.8845$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(24)^{1/3} = 2.8845$.

(v) Taking $N = 30$ and $k = -5$, (21.9) becomes $x_{n+1} = \frac{1}{-5}(6x_n + 30/x_n^{-6}) = \frac{x_n}{5}(6 - 30x_n^5)$

Since an approximate value of $(30)^{-1/5}$ is given by $(32)^{-1/5} = 1/2$, we take $x_0 = 1/2$. Thus,

$$x_1 = \frac{x_0}{5}(6 - 30x_0^5) = \frac{1}{10}(6 - 30/2^5) = 0.50625$$

$$x_2 = \frac{x_1}{5}(6 - 30x_1^5) = \frac{0.50625}{5}[6 - 30(0.50625)^5] = 0.506495$$

$$x_3 = \frac{x_2}{5}(6 - 30x_2^5) = \frac{0.506495}{5}[6 - 30(0.506495)^5] = 0.506496$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(30)^{-1/5} = 0.5065$.

21.2.5 Fixed-Point Method

This is a general iteration method in which to solve the equation $f(x) = 0$ by successive approximations, we rewrite it in the equivalent form $x = g(x)$. A solution of this equation is called the fixed point of g justifying the name of the method.

Starting with a suitable initial value x_0 close to the expected root, we compute the successive approximations as $x_1 = g(x_0)$, $x_2 = g(x_1)$, $x_3 = g(x_2)$ ..., and, in general,

$$x_{n+1} = g(x_n) \quad ...(21.10)$$

The sequence $\{x_n\}$ of numbers x_n converges to a under certain conditions and this a is the desired root.

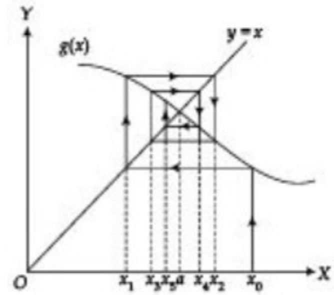


Fig. 21.3

Geometrically interpreting, the roots of $f(x) = 0$ are the same as the point of intersection of the straight line $y = x$ and the curve $y = g(x)$. Figure 21.3 illustrates the working of this iteration method. Further, we must note that, we may get several different forms of $x = g(x)$ from $f(x) = 0$. The behaviour of the corresponding iterative sequences $\{x_n\}$ may differ, in particular, with respect to their speed of convergence and, in fact, some of those may not converge at all, e.g. refer to Examples (21.11 and 21.13). We are to select the initial approximation x_0 suitably so that the successive approximations x_1, x_2 ... converge to the root 'a'. A sufficient condition for convergence is given in the following theorem.

Theorem 21.1 (Convergence of fixed-point method): *If $x = a$ is a root of $x = g(x)$ and if g has continuous derivative in some interval I containing a and if $|g'(x)| \le k < 1$ for all $x \in I$, then the iteration process defined by $x_{n+1} = g(x_n)$ converges for any $x_0 \in I$.*

Proof: By Cauchy's mean value theorem for some $\xi \in (a, x)$, we have

$$g(x) - g(a) = g'(\xi)(x - a). \quad ...(21.11)$$

Since a is a root of $x = g(x)$, we have, $a = g(a)$ and also $x_1 = g(x_0)$, $x_2 = g(x_1)$. For $x \in I$ using the condition $|g'(x)| \le k$, from (21.11), we have

$$|x_n - a| = |g(x_{n-1}) - g(a)| = |g'(\xi)|\,|x_{n-1} - a|$$
$$\le k|g(x_{n-2}) - a| \le k^2|x_{n-2} - a| \;\; \ldots \;\; \le k^n|x_0 - a|$$

As $n \to \infty$, the right-hand side tends to zero since $k < 1$. Hence, $\{x_n\} \to a$ as $n \to \infty$.

This proves the convergence condition. Further, since $|x_n - a| \le k|x_{n-1} - a|$, $k < 1$,

thus, the *rate of convergence is linear*.

This method of iteration is particularly useful for finding the real roots of an equation given in the form of an infinite series.

Example 21.11: Find a root of the equation $x^3 - x - 1 = 0$ close to 1 correct to four decimal places by applying the fixed-point iteration method.

Solution: We have, $f(x) = x^3 - x - 1$. Now, $f(0) = -1$, $f(1) = -1$, $f(2) = 5$.

Thus, a root lies in the interval [1, 2], therefore, take $I = [1, 2]$.

Writing $f(x) = 0$ as $x = x^3 - 1 = g(x)$, we observe that $g'(x) = 3x^2 > 1$ for all $x \in [1, 2]$ and, hence, $g(x)$ does not satisfy the convergence theorem condition and thus is not suitable.

Rewriting $f(x) = 0$ as $x = (x+1)^{1/3} = g(x)$. This gives, $g'(x) = \dfrac{1}{3(x+1)^{2/3}}$

and obviously $|g'(x)| < 1, \forall\, x \in [1, 2]$ and thus the convergence theorem condition is satisfied.

Taking $x_0 = 1$, we have

$x_1 = (1+1)^{1/3} = 1.259921,$ $\quad x_2 = (1.259921 + 1)^{1/3} = 1.3122938,$

$x_3 = (1.3122938 + 1)^{1/3} = 1.3223547,$ $\quad x_4 = (1.3223547 + 1)^{1/3} = 1.3242689,$

$x_5 = (1.3242689 + 1)^{1/3} = 1.3246327,$ $\quad x_6 = (1.3246327 + 1)^{1/3} = 1.3247018$

$x_7 = (1.3247018 + 1)^{1/3} = 1.3247149,$

Hence, the root of $x^3 - x - 1 = 0$ correct to four decimal places is 1.3247.

Example 21.12: Using fixed-point iteration method find the smallest positive root of $x - \tan x = 0$ correct to three decimal places.

Solution: The roots of $x - \tan x = 0$. are given by the intersection $y = x$ and $y = \tan x$. The smallest positive root thus lies in the interval $I = [\pi, 3\pi/2]$, as shown in Fig. 21.4.

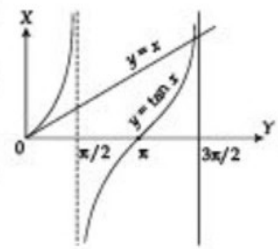


Fig. 21.4

Taking $\qquad g(x) = \pi + \tan^{-1} x, \quad$ for $\quad x \in [\pi, 3\pi/2]$.

Therefore, $\qquad g'(x) = \dfrac{1}{1+x^2}$, and obviously $|g'(x)| < 1$ for all $x \in [\pi, 3\pi/2]$.

Taking $x_0 = \pi = 3.1416$, we have

$x_1 = 3.1416 + \tan^{-1}(3.1416) = 4.4042,$ $\quad x_2 = 3.1416 + \tan^{-1}(4.4042) = 4.4891,$

$x_3 = 3.1416 + \tan^{-1}(4.4891) = 4.4932,$ $\quad x_4 = 3.1416 + \tan^{-1}(4.4932) = 4.4934.$

Hence, the root of $x - \tan x = 0$, correct to three decimal places is 4.493.

Example 21.13: Find the square root of 5 using fixed point iteration method, correct to three decimal places.

Solution: Let $x = \sqrt{5}$. This gives $f(x) = x^2 - 5 = 0$.

Consider $g(x) = 5/x$ and assume $x_0 = 2$. Then $x_1 = 2.5, x_2 = 2, x_3 = 2.5, \ldots$

The process is oscillatory divergent and does not converge to a definite value.
Consider $g(x) = x^2 + x - 5$ and assume $x_0 = 3$. Then $x_1 = 7, x_2 = 51, x_3 = 2647, \ldots$

The process diverges monotonically to $+\infty$.

Next, consider $g(x) = \dfrac{x + (5/x)}{2}$ and assume $x_0 = 2$. Then

$x_1 = 2.25,\; x_2 = 2.3611,\; x_3 = 2.2394,\; x_4 = 2.2361,\; x_5 = 2.2361, \ldots$

Hence, the square root of 5 correct to three decimal points is 2.2361.

Example 21.14: Find the smallest root of the equation

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \dots = 0 \text{ correct to two decimal places.}$$

Solution: Rewriting the given equation as

$$x = 1 + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \dots = g(x).$$

Omitting x^2 and higher powers of x, we get $x = 1$ as an approximation to the given equation. Taking $x_0 = 1$, we obtain

$$x_1 = g(x_0) = 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} - \frac{1}{(5!)^2} + \dots = 1.224$$

$$x_2 = g(x_1) = 1 + \frac{(1.224)^2}{(2!)^2} - \frac{(1.224)^3}{(3!)^2} + \frac{(1.224)^4}{(4!)^2} - \frac{(1.224)^5}{(5!)^2} + \dots = 1.326$$

Similarly, $x_3 = 1.381, x_4 = 1.409, x_5 = 1.425, x_6 = 1.434, x_7 = 1.439, x_8 = 1.442.$

The values of x_7 and x_8 indicate that the root is 1.44 correct to two decimal places.

EXERCISE 21.1

1. Find a root of $e^x + x^4 + x = 2$ lying in the interval (0, 1) using bisection method correct upto five decimal points.
2. Solve $e^{-x} = \ln x$ using bisection method.
3. Solve $x = \cos x$ by bisection method correct upto five decimal points.
4. Find a root of the equation $x^3 - 3x + 1 = 0$ in the interval (0, 0.5) by applying false position method up to four iterations.
5. Find the root of the equation $xe^x = \cos x$ using the false position method up to five iterations.
6. Find the root of the equation $xe^x = \cos x$ using the secant method correct to four decimal places taking $x_0 = 0$ and $x_1 = 1$ as the initial approximations.
7. Find the solution of $\cos x \cos hx = 1$ using the secant method upto five iterations taking $x_0 = 4$ and $x_1 = 5$ as the initial approximations.
8. Find the root of $x^3 - 8x - 4 = 0$ in the interval (3, 4) by Newton-Raphson method correct upto four decimal points.

9. Find a positive root of $10^x + x - 4 = 0$ by Newton-Raphson method correct to five significant figures.
10. Use Newton-Raphson method to find the smallest positive root of $\tan x = x$ correct upto three decimal points.
11. Using Newton-Raphson method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.
12. Develop a Newton iteration formula to find $\sqrt[k]{N}, N > 0$ and use it to compute $\sqrt{3}, \sqrt[3]{3}, \sqrt[4]{3}$ and $\sqrt[5]{3}$ upto three iterations each.
13. Solve $x^4 - x - 0.12 = 0$ by starting from $x_0 = 1$ using fixed-point iteration.
14. Find a real root of $\cos x = 3x - 1$ correct to three decimal places using fixed-point iterations.
15. Find by fixed-point iteration method the root near to 3.8 of the equation $2x - \log_{10} x = 7$ correct to four decimal places.

21.3 SOLUTION OF LINEAR SYSTEM OF EQUATIONS

A system of n linear equations in n variables is generally represented as

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array}\right\} \qquad \dots(21.12)$$

In matrix notation the system (21.12) can be expressed as

$$AX = B \qquad \dots(21.13)$$

where $A = [a_{ij}]_{n \times n}$ is the coefficient matrix, $B = [b_1, b_2 \dots, b_n]'$, and $X = [x_1, x_2 \dots x_n]'$ is a vector of n unknowns.

The methods for solving system of linear equations (21.12), can be classified as

1. *Direct methods*
2. *Iterative methods*

Direct methods, e.g., matrix inversion method, (as studied in Chapter 2), becomes tedious when system involves large number of variables. In such cases, iterative approach is the best suited one. In this section, first we shall discuss some direct methods and then iterative methods of solving a system of linear equations.

21.3.1 Direct Methods

The following three direct methods are discussed.

1. *Gauss-elimination method*
2. *Gauss-Jorden method*
3. *Crout's triangularization method (or, LU-factorization method)*

We explain these methods by considering a system of three equations in three variables.

1. Gauss-elimination method. Consider the system of equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad \text{...(21.14)}$$

The method consists of a systematic process of elimination that reduces the linear system (21.14) to *upper triangular form* and then solving by *back substitution*.

In the first step, assuming $a_1 \neq 0$, we eliminate x from the second equation in (21.14) by subtracting (a_2/a_1) times the first equation from the second equation. Similarly, we eliminate x from the third equation in (21.14) by subtracting (a_3/a_1) times of the first from the third equation. The resultant system is

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ b_3'y + c_3'z &= d_3' \end{aligned} \quad \text{...(21.15)}$$

The first equation is called the *pivot equation* and a_1 is called the *first pivot*.

In the second step, assuming $b_2' \neq 0$, we eliminate y from the third equation of the system (21.15) by subtracting (b_3'/b_2') times of the second equation from the third equation. The resultant system is

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \quad \text{...(21.16)}$$

Here the second equation is called the pivot equation and b_2' is called the *second pivot*.

The value of the unknowns x, y, z are obtained from the system (21.16) by back substitution.

Remarks:

1. In case any one of the pivots a_1, b_2' or c_3'' is zero, then we rewrite the equations in a different order to make the pivots non-zero.
2. In general, equation with the numerically largest coefficient of x is chosen as the first equation. Then from the remaining ones, the equation with the numerically largest coefficient of y is chosen as the second equation, and so on. This procedure is called *partial pivoting* and helps in reducing the round off error.
3. If we are not particular about the elimination of x, y, z in this specific order, then we can choose at each stage the numerically largest coefficient among all the coefficients and can rearrange the equations and the variables accordingly. This procedure is called *complete pivoting*.

Example 21.15: Solve the following system of equations by Gauss-elimination method

$$\begin{aligned} 3x + 2y + z &= 10 \\ 2x + 3y + 2z &= 14 \\ x + 2y + 3z &= 14 \end{aligned}$$

Solution: To eliminate x from the second equation, we multiply first by (2/3) and subtract from the second equation. This gives, $5y + 4z = 22$.

Similarly, multiplying first by (1/3) and subtracting from third equation gives $y + 2z = 8$.

Thus, the new system of equations is

$$\begin{aligned} 3x + 2y + z &= 10 \\ 5y + 4z &= 22 \\ y + 2z &= 8 \end{aligned}$$

To eliminate y from the third equation in the new system, multiply second equation by (1/5) and subtract from the third equation, we obtain $6z = 18$.

Thus, the system reduced to the upper triangular form is

$$\begin{aligned} 3x + 2y + z &= 10 \\ 5y + 4z &= 22 \\ 6z &= 18 \end{aligned}$$

Solving this system by back substitution process as follows:

From the third equation, $z = 18/6 = 3$

Then from second equation $5y + 4(3) = 22$, this gives, $y = 2$,

and, then from third equation $3x + 2(2) + 3 = 10$, this gives, $x = 1$.

Hence, the solution is $x = 1$, $y = 2$ and $z = 3$.

Example 21.16: By Gauss-elimination method using partial pivoting solve the following system of equations $2x + 2y + z = 6$, $4x + 2y + 3z = 4$, $x - y + z = 0$.

Solution: To apply partial pivoting, rearrange the system of equations, as

$$\begin{aligned} 4x + 2y + 3z &= 4 \\ 2x + 2y + z &= 6 \\ x - y + z &= 0 \end{aligned}$$

Eliminating x from the second and third equation using the first equation, as in Example 21.15, we obtain the resultant system as

$$\begin{aligned} 4x + 2y + 3z &= 4 \\ y - (1/2)z &= 4 \\ (-3/2)y + (1/4)z &= -1 \end{aligned}$$

Again, rearranging the system as

$$\begin{aligned} 4x + 2y + 3z &= 4 \\ (-3/2)y + (1/4)z &= -1 \\ y - (1/2)z &= 4 \end{aligned}$$

Eliminating y from the third equation using the second equation, we obtain the resultant system as

$$4x + 2y + 3z = 4$$
$$(-3/2)y + (1/4)z = -1$$
$$(-1/3)z = 10/3$$

By back substitution, we obtain the solution as $x = 9$, $y = -1$, and $z = -10$.

2. *Gauss-Jorden method.* This method is a modification over the Gauss-elimination method in which back substitution is avoided by reducing the matrix to the diagonal form instead of the upper triangular form. Since the labour of back substitution is saved at the cost of additional computations only, thus, as such, method is of not much advantageous for solving linear system of equations.

Example 21.17: Apply Gauss-Jorden method to solve the system of equations:

$$x + y + z = 9 \quad \text{...(i)}$$
$$2x - 3y + 4z = 13 \quad \text{...(ii)}$$
$$3x + 4y + 5z = 40 \quad \text{...(iii)}$$

Solution: To eliminate x from (ii) and (iii), perform (ii) -2 (i) and (iii) -3 (i), the resultant system is

$$x + y + z = 9 \quad \text{...(iv)}$$
$$-5y + 2z = -5 \quad \text{...(v)}$$
$$y + 2z = 13 \quad \text{...(vi)}$$

To eliminate y from (iv) and (vi), perform (iv) $+ \frac{1}{5}$ (v) and (vi) $+ \frac{1}{5}$ (v), the resultant system is

$$x + \frac{7}{5}z = 8 \quad \text{... (vii)}$$
$$-5y + 2z = -5 \quad \text{...(viii)}$$
$$\frac{12}{5}z = 12 \quad \text{...(ix)}$$

To eliminate z from (iii) and (viii), perform (vii) $-\frac{7}{12}$ (ix), and (viii) $-\frac{5}{6}$ (ix), we obtain

$$x = 1,\ -5y = -15,\ \frac{12}{5}z = 12.$$

Thus, the solution is $x = 1$, $y = 3$, and $z = 5$.

3. *Crouts triangularization method (or, LU-factorization method).* In this method, the coefficient matrix A of the system of equations $AX = B$ is factorized into the product of lower triangular matrix L and an upper triangular matrix U. We write the matrix A as $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Thus, the system of equations $\boldsymbol{AX} = \boldsymbol{B}$ becomes

$$\boldsymbol{LUX} = \boldsymbol{B} \qquad \ldots(21.17)$$

Writing $\boldsymbol{UX} = \boldsymbol{V}$, where obviously $\boldsymbol{V}$ is a column matrix, (21.17) becomes

$$\boldsymbol{LV} = \boldsymbol{B}$$

This gives, $\quad v_1 = b_1, \quad l_{21}v_1 + v_2 = b_2, \quad l_{31}v_1 + l_{32}v_2 + v_3 = b_3$

Solving these for v_1, v_2 and v_3, we can formulate for $\boldsymbol{V}$ and then we can calculate x_1, x_2, x_3 from $\boldsymbol{UX} = \boldsymbol{V}$.

To compute the matrices $\boldsymbol{L}$ and $\boldsymbol{U}$, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{22} & a_{33} \end{bmatrix}$$

By comparing the corresponding elements on both sides, we get a set of nine equations and from these equations we can compute all the nine unknown elements appearing in $\boldsymbol{L}$ and $\boldsymbol{U}$. However, to simplify the computations, normally we proceed in the following set order:

(i) First row of $\boldsymbol{U}$
(ii) First column of $\boldsymbol{L}$
(iii) Second row of $\boldsymbol{U}$
(iv) Second column of $\boldsymbol{L}$
(v) Third row of $\boldsymbol{U}$.

Remarks:

(1) A square matrix A can be expressed in the form $A = LU$, only if, when A is a *positive definite* matrix, that is, when all the principal minors of A are non-singular. Thus, LU factorization method fails in case the matrix A is not positive definite.

(2) Another name for the LU-factorization method is Cholesky's method.

Example 21.18: Solve the system of equations

$$\begin{aligned} 2x + y + z - 2w &= -10 \\ 4x + 2z + w &= 8 \\ 3x + 2y + 2z &= 7 \\ x + y + 2z - w &= -5 \end{aligned}$$

using the LU-factorization method. Also find the inverse of the coefficient matrix.

Solution: We write the coefficient matrix, say A, as

$$\begin{bmatrix} 2 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

Comparing the corresponding elements on both sides, we get

First row: $u_{11}=2, u_{12}=1, u_{13}=1, u_{14}=-2,$

First column: $l_{21}=2, l_{31}=3/2, l_{41}=1/2,$

Second row: $l_{21}u_{12}+u_{22}=0$, or $u_{22}=-2,$

$l_{21}u_{13}+u_{23}=2$, or $u_{23}=0,$

$l_{21}u_{14}+u_{24}=1$, or $u_{24}=5,$

Second column: $l_{31}u_{12}+l_{32}u_{22}=2$, or $l_{32}=-1/4,$

$l_{41}u_{12}+l_{42}u_{22}=3$, or $l_{42}=-5/4,$

Third row: $l_{31}u_{13}+l_{31}u_{23}+u_{33}=2$, or $u_{33}=1/2$

$l_{31}u_{14}+l_{32}u_{24}+u_{34}=0$, or $u_{34}=17/4,$

Third column: $l_{41}u_{13}+l_{42}u_{23}+l_{43}u_{23}=2$, or $l_{43}=3,$

Fourth row: $l_{41}u_{14}+l_{42}u_{24}+l_{43}u_{34}+u_{44}=-1$ or $u_{44}=-13/2.$

Thus, we obtain

$$L=\begin{bmatrix}1&0&0&0\\2&1&0&0\\3/2&-1/4&1&0\\1/2&-5/4&3&1\end{bmatrix},\quad U=\begin{bmatrix}2&1&1&-2\\0&-2&0&5\\0&0&1/2&17/4\\0&0&0&-13/2\end{bmatrix}$$

We now write the system of equations $AX=B$ as $LUX=B$ or, $LV=B$, where $UX=V$. Consider $LV=B$, we get

$$\begin{bmatrix}1&0&0&0\\2&1&0&0\\3/2&-1/4&1&0\\1/2&-5/4&3&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}=\begin{bmatrix}-10\\8\\7\\-5\end{bmatrix}$$

Using forward substitution, we obtain

$$v_1=-10,$$

$$v_2=8-2v_1=28,$$

$$v_3=7-\frac{3}{2}v_1+\frac{1}{4}v_2=29,$$

$$\text{and, } v_4=-5-\frac{1}{2}v_1+\frac{5}{4}v_2-3v_3=-52.$$

Thus, $UX=V$ gives

$$\begin{bmatrix}2&1&1&-2\\0&-2&0&5\\0&0&1/2&17/4\\0&0&0&-13/2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-10\\28\\29\\-52\end{bmatrix}$$

Using backward substitution, we obtain

$$x_4 = 8,$$

$$x_3 = 2\left[29 - \frac{17}{4}x_4\right] = -10,$$

$$x_2 = -\frac{1}{2}[28 - 5x_4] = 6,$$

$$\text{and, } x_1 = \frac{1}{2}[-10 - x_2 - x_3 + 2x_4] = 5.$$

Therefore, the solution vector is $[5, 6, -10, 8]^T$. Also, we have

$$\mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -2 & 1/4 & 1 & 0 \\ 3 & 1/2 & -3 & 1 \end{bmatrix} \text{ and } \mathbf{U}^{-1} = \frac{1}{13}\begin{bmatrix} 13/2 & 13/4 & -13 & -8 \\ 0 & -13/2 & 0 & -5 \\ 0 & 0 & 26 & 17 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Hence,

$$\mathbf{A}^{-1} = (\mathbf{LU})^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1} = \frac{1}{13}\begin{bmatrix} 2 & -4 & 11 & -8 \\ -2 & -9 & 15 & -5 \\ -1 & 15 & -25 & 17 \\ -6 & -1 & 6 & -2 \end{bmatrix}$$

21.3.2 Iterative Methods

In case when the number of unknowns is large, direct methods are not only time consuming but the accuracy of the solution is affected due to round off errors. In such case iterative methods provide the better option. In an iterative method, we start from an approximate solution and, if feasible, approach towards the better and better approximation from a repetitive computational process, repeated till the desired accuracy is achieved. However, like any other iterative process, these methods introduce truncation errors. We apply iterative method if the convergence is fast and it is so, only if, the coefficient matrices have numerically large principal diagonal elements. This saves the computational efforts when compared to the direct method. Further, the iterative methods are self-corrective process; any error committed during the computation process is self-corrected in successive iterations.

We discuss the following three iterative methods:

1. *Jacobi iteration method*
2. *Gauss-Seidel iteration method*
3. *Relaxation method.*

As in case of direct methods, here also, we consider a system of three linear equations in three unknowns to explain these methods.

1. Jacobi iteration method. Consider the system of equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad \text{...(21.18)}$$

Assuming a_1, b_2, c_3 are numerically large (if not so, we apply partial or complete pivoting) as compared to other coefficients, solve for x, y and z respectively and write the system (21.18) as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \qquad \text{...(21.19)}$$

We start with an initial approximation x_0, y_0, z_0 for the true solution. Substitute these values in the right-hand sides of the equations in (21.19) and obtain x_1, y_1, z_1 as the first approximation. This again we substitute in the system (21.19) and obtain x_2, y_2, z_2 as the second approximation and the process is continued till the solution upto desired accuracy is achieved. In the absence of any better initial approximation, x_0, y_0, z_0 are taken as $x_0 = 0, y_0 = 0, z = 0$.

Example 21.19: Solve the following system of linear equations using Jacobi iteration method

$$2x + y + z = 5$$

$$3x + 5y + 2z = 15$$

$$2x + y + 4z = 8$$

upto four iterations.

Solution: Solving the equations respectively for x, y and z, we have

$$\left.\begin{aligned} x &= \frac{1}{2}(5 - y - z) \\ y &= \frac{1}{5}(15 - 3x - 2z) \\ z &= \frac{1}{4}(8 - 2x - y) \end{aligned}\right\} \qquad \text{...(21.20)}$$

Taking $x_0 = 0$, $y_0 = 0$ and $z_0 = 0$ as the initial approximation and substituting these values in the right-hand side of equations in (21.20), we obtain $x_1 = 2.5$, $y_1 = 3$ and $z_1 = 2$ as the first approximation. Then substituting the first approximation, we obtain x_2, y_2, z_2 as the second approximation, and so on. The results obtained are tabulated in the form of the Table 21.1.

Table 21.1

Iteration (n)	x	y	z
1	2.5	3.0	2.0
2	0.0	0.7	0.0
3	2.15	3.0	1.825
4	0.0875	1.225	0.175

Hence, the solution upto the fourth iteration is $x = 0.0875,\ y = 1.225,\ z = 0.175$.

2. Gauss-Seidel iteration method. The method is an improved version of the Jacobi-iteration method. Here as soon as a new approximation is found for an unknown variable it is immediately used in the next step as explained below. As in case of Jacobi iteration method, we start with the initial approximation x_0, y_0, z_0, substitute for $y = y_0$ and $z = z_0$ in the right side of the first equation of the set (21.19) and obtain x_1. In the second equation of the set (21.19), we substitute $x = x_1$ and $z = z_0$ to obtain y_1; and then in the third equation, we substitute $x = x_1$, $y = y_1$ to obtain z_1, and so on. Thus, we utilize the most recent information obtained. This process is repeated till the solution up to the desired degree of accuracy is obtained. *The convergence in this method is twice as fast as in Jacobi iteration method.*

Remarks:

1. Jacobi method is also called *the method of simultaneous displacements* and Gauss-Seidel method as *the method of successive displacements.*
2. The Jacobi and Gauss-Seidel methods converge for any choice of the initial approximation x_0, y_0, z_0 if for each equation the absolute value of the diagonal element is greater than the sum of the absolute values of the monodiagonal elements, that is,

$$|a_{ii}| > \sum_{j=1}^{3} |a_{ij}|,\ i \neq j,\ i = 1, 2, 3. \qquad \text{...(21.21)}$$

This condition is sufficient for convergence but not necessary. Systems of equations which satisfy the condition (21.21) are called *diagonally dominant systems.*

Example 21.20: Solve by Gauss-Seidel iteration method the system of equations

$$10x + y + z = 12,\ 2x + 10y + z = 13,\ 2x + 2y + 10z = 14.$$

Solution: Solving the equations respectively for x, y and z, we have

$$x = \frac{1}{10}(12 - y - z) \qquad \text{...(i)}$$

$$y = \frac{1}{10}(13 - 2x - z) \qquad \text{...(ii)}$$

$$z = \frac{1}{10}(14 - 2x - 2y) \qquad \text{...(iii)}$$

For first iteration, substituting $y = 0, z = 0$ in (i), we obtain $x = 1.2$

Substituting $x = 1.2$, $z = 0$ in (ii), we obtain $y = 1.06$

Substituting $x = 1.2$, $y = 1.06$ in (iii), we obtain $z = 0.948$

Thus, the first approximation is $x_1 = 1.2$, $y_1 = 1.06$, and $z_1 = 0.948$.

Similarly, we can proceed for successive iterations. Results obtained are tabulated in Table 21.2.

Table 21.2

Iteration (n)	*x*	*y*	*z*
1	1.2	1.06	0.948
2	0.9992	1.0054	0.9991
3	0.9996	1.001	1.001
4	1.000	1.000	1.000
5	1.000	1.000	1.000

Therefore, the solution is $x = 1.0$, $y = 1.0$, and $z = 1.0$.

3. ***Relaxation method.*** Relaxation method is a modified version of the Gauss-Seidel method which results in a faster convergence. Consider the system of linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \quad \text{...(21.22)}$$

In relaxation method initial approximation x_0, y_0, z_0 normally taken as zeros, are improved by reducing the so called *residuals* R_x, R_y, R_z defined by the relations

$$\left.\begin{aligned} R_x &= d_1 - a_1x - b_1y - c_1z \\ R_y &= d_2 - a_2x - b_2y - c_2z \\ R_z &= d_3 - a_3x - b_3y - c_3z \end{aligned}\right\} \quad \text{...(21.23)}$$

Residuals are reduced in steps by giving suitable increments to the variables. At each step, the numerically largest residual is reduced to almost zero. To facilitate it, we construct an operation table given below as Table 21.3 in context with the system of residuals (21.23).

Table 21.3

Increment	*Change in Residuals*		
	R_x	R_y	R_z
$\Delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\Delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\Delta z = 1$	$-c_1$	$-c_2$	$-c_3$

From the operation table, we observe that a unit increase in the variable x decrease the residuals R_x, R_y and R_z by a_1, a_2 and a_3 units respectively. The respective changes in the residual for unit

Differentiating it $(n+1)$ times using the Leibnitz theorem, we obtain

$$[(1-x^2)u_{n+2} + (n+1)(-2x)u_{n+1} + \frac{1}{2!}(n+1)(n)(-2)u_n] + 2n[xu_{n+1} + (n+1)u_n] = 0$$

or, $$(1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0, \qquad ...(12.39)$$

where $u_n' = \frac{du_n}{dx}$ and $u_n'' = \frac{d^2u_n}{dx^2}$.

Eq. (12.39) is Legendre differential equation in $y = cu_n$, where c is an arbitrary constant. Since $P_n(x)$ is the finite series solution of the Legendre equation, therefore,

$$P_n(x) = cu_n = c\frac{d^n}{dx^n}[(x^2-1)^n]. \qquad ...(12.40)$$

The constant c is determined by setting $P_n(1) = 1$. From (12.40), we have

$$P_n(1) = c\frac{d^n}{dx^n}[(x^2-1)^n]_{x=1} = c\frac{d^n}{dx^n}[(x-1)^n(x+1)^n]_{x=1}$$

$$= c[n!(x+1)^n + \text{terms containing } (x-1) \text{ and its higher powers}]_{x=1}$$

$$= cn!2^n. \text{ Thus, } c = \frac{1}{n!2^n}.$$

Substituting this value of c in (12.40) we obtain (12.38), the Rodrigue's formula.

From (12.38), we have

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x^2),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \text{ and, in general,}$$

$$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}, \qquad ...(12.41)$$

which is proved as follows. By Binomial theorem

$$(x^2-1)^n = \sum_{r=0}^{n} c_r^n(x^2)^{n-r}(-1)^r = \sum_{r=0}^{n}(-1)^r\frac{n!}{r!(n-r)!}x^{2n-2r}$$

and thus, from (12.38)

$$P_n(x) = \frac{1}{n!2^n}\sum_{r=0}^{n}(-1)^r\frac{n!}{r!(n-r)!}\frac{d^n}{dx^n}(x^{2n-2r}) = \sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r},$$

which is (12.41). Here $N = \frac{n}{2}$, or $\frac{(n-1)}{2}$, whichever is integer, that is, $N = \left\lfloor \frac{n}{2} \right\rfloor$, so that the power of x for the last term in the series (12.41) is either 0 or 1.

The graphs of some of the even and the odd polynomials are shown in Figs. 12.1a & b.

$P_n(x)$ is of degree n, and contains only even powers of x if n is even, and only odd powers of x if n is odd. These polynomials are defined for all real x, but the relevant interval for the Legendre differential equation is $-1 < x < 1$. The Legendre polynomials belong to an important class of polynomials called the *orthogonal polynomials*, to be discussed in Section 12.3.6.

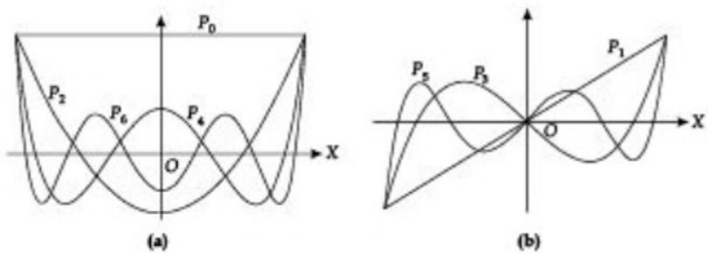


Fig. 12.1

12.3.4 Generating Function for Legendre Polynomials

Many properties of Legendre polynomials can be derived by using the concept of generating function. We claim that the generating function of $P_n(x)$ can be given by

$$L(x, t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \qquad ...(12.42)$$

that is, if $L(x, t)$ is expanded as a power series in t, then the coefficient of t^n is $P_n(x)$, the Legendre polynomial of degree n.

To prove this, write the Maclaurin series for $(1-u)^{-1/2}$, $-1 < u < 1$, we have

$$(1-u)^{\frac{-1}{2}} = 1 + \frac{1}{2}u + \frac{(1/2)(3/2)}{2!}u^2 + \frac{(1/2)(3/2)(5/2)}{3!}u^3 + ...$$

$$= 1 + \frac{2!}{(1!)^2 2^2}u + \frac{4!}{(2!)^2 2^4}u^2 + ... + \frac{(2n)!}{(n!)^2 2^{2n}}u^n + ... \qquad ...(12.43)$$

Setting $u = (2x - t)t$, (12.43) becomes

$$(1-2xt+t^2)^{-1/2} = 1 + \frac{2!}{(1!)^2 2^2}(2x-t)t + \frac{4!}{(2!)^2 2^4}(2x-t)^2 t^2 + ...$$

$$+\frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}}(2x-t)^{n-r}t^{n-r}+\ldots+\frac{(2n)!}{[n!]^2 2^{2n}}(2x-t)^n t^n+\ldots$$

Now, the term in t^n from the term containing $(2x-t)^{n-r}t^{n-r}$

$$=\frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}}\cdot c_r^{n-r}\cdot(2x)^{n-2r}(-t)^r t^{n-r}$$

$$=\frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}}\cdot\frac{(n-r)!}{r!(n-2r)!}(-1)^r 2^{n-2r}x^{n-2r}t^n=\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}t^n$$

Collecting all such terms in t^n which will occur, till the term containing $(2x-t)^n t^n$, we get the term in t^n

$$=\sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}t^n=P_n(x)t^n, \quad \text{...(12.44)}$$

where $N=\left\lfloor\frac{n}{2}\right\rfloor$, the integral value less than or equal to $n/2$. Thus $(1-2xt+t^2)^{-1/2}$ is the generating function for the Legendre polynomials $P_n(x)$.

Example 12.8: Using the generating function of the Legendre polynomials prove that for each integral $n \geq 0$,

(a) $P_n(1)=1$, (b) $P_n(-1)=(-1)^n$, (c) $P_{2n}(0)=(-1)^n\dfrac{(2n+1)!}{2^{2n}(n!)^2}$, and $P_{2n+1}(0)=0$.

Solution:

(a) Set $x=1$ in (12.42), we have $L(1,t)=\dfrac{1}{\sqrt{1-2t+t^2}}=\dfrac{1}{1-t}=\sum_{n=0}^{\infty}P_n(1)t^n$.

Also for $-1<t<1$, $\dfrac{1}{1-t}=\sum_{n=0}^{\infty}t^n$, and thus, $P_n(1)=1$, for each integral $n\geq 0$.

(b) Similarly set $x=-1$ in (12.42), we have $L(-1,t)=\dfrac{1}{\sqrt{1+2t+t^2}}=\dfrac{1}{1+t}=\sum_{n=0}^{\infty}P_n(-1)t^n$.

Also for $-1<t<1$, $\dfrac{1}{1+t}=\sum_{n=0}^{\infty}(-1)^n t^n$, and thus $P_n(-1)=(-1)^n$, for each integral $n\geq 0$.

(c) Further, set $x = 0$ in (12.42), we obtain $L(0, t) = \dfrac{1}{\sqrt{1+t^2}} = (1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)t^n$.

Also for $t^2 < 1$, $\quad (1+t^2)^{-1/2} = \sum_{n=0}^{\infty} (-1)^n \dfrac{(2n+1)!}{2^{2n}(n!)^2} t^{2n}$

and thus $\quad P_{2n}(0) = (-1)^n \dfrac{(2n+1)!}{2^{2n}(n!)^2}$ and $P_{2n+1}(0) = 0$, $\quad$ for $n = 0, 1, 2, \ldots$

12.3.5 Recurrence Relations for Legendre Polynomials

The Legendre polynomials $P_n(x)$ satisfy the following recurrence relations

1. $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$ ...(12.45)
2. $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$ ...(12.46)
3. $xP_n'(x) = nP_n(x) + P_{n-1}'(x)$ ...(12.47)
4. $(1-x^2)P_{n-1}'(x) = n[xP_{n-1}(x) - P_n(x)]$ ...(12.48)
5. $(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$ ...(12.49)
6. $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$ ...(12.50)

Proof. We use Rodrigue's formula (12.38) to prove these relations.

1. We have, $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}[(x^2-1)^n]$. Differentiating w.r.t. x, to obtain

$$P_n'(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[n(x^2-1)^{n-1}2x] = \frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dx^n}[x(x^2-1)^{n-1}]$$

$$= \frac{1}{2^{n-1}(n-1)!}\left[x\frac{d^n}{dx^n}(x^2-1)^{n-1} + n\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right] \qquad \text{(using Leibnitz's rule)}$$

$$= x\frac{d}{dx}\left[\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}[(x^2-1)]^{n-1}\right] + n\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}[(x^2-1)]^{n-1}$$

$$= xP_{n-1}'(x) + nP_{n-1}(x).$$

2. We have, $\quad P_{n+1}'(x) = \dfrac{d}{dx}\left[\dfrac{1}{2^{n+1}(n+1)!}\dfrac{d^{n+1}}{dx^{n+1}}[(x^2-1)^{n+1}]\right]$

$$= \frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}[(n+1)(x^2-1)^n.2x] = \frac{1}{2^n n!}\frac{d^{n+1}}{dx^{n+1}}[x(x^2-1)^n]$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} [xn(x^2-1)^{n-1} 2x + (x^2-1)^n]$$

$$= \frac{1}{2^{n-1}(n-1)!} \frac{d^n}{dx^n} [x^2(x^2-1)^{n-1}] + P_n(x)$$

$$= \frac{1}{2^{n-1}(n-1)!} \frac{d^n}{dx^n} [(x^2-1)^n + (x^2-1)^{n-1}] + P_n(x)$$

$$= \frac{2n}{2^n n!} \frac{d^n}{dx^n} [(x^2-1)^n] + \frac{d}{dx}\left[\frac{1}{2^{n-1}(n-1)!} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^{n-1}\right] + P_n(x)$$

$$= 2nP_n(x) + P'_{n-1}(x) + P_n(x)$$

Thus, $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x).$

From this recurrence relation, there follows another important result concerning the integral of Legendre polynomials, as

$$\int P_n(x)dx = \frac{1}{2n+1}[P_{n+1}(x) - P_{n-1}(x)]. \quad ...(12.51)$$

3. Replacing n by $(n+1)$ in (12.45) to obtain

$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x),$$

or,
$$xP'_n(x) = P'_{n+1}(x) - (n+1)P_n(x) \quad ...(12.52)$$

Also from (12.46),
$$P'_{n+1}(x) = P'_{n-1}(x) + (2n+1)P_n(x) \quad ...(12.53)$$

Substituting for $P'_{n+1}(x)$ from (12.53) in (12.52) we get (12.47).

4. Subtracting x times multiple of (12.45) from (12.47) to obtain

$$0 = [nP_n(x) + P'_{n-1}(x)] - x[xP'_{n-1}(x) + nP_{n-1}(x)]$$

or, $(1-x^2)P'_{n-1}(x) = n[xP_{n-1}(x) - P_n(x)]$, which is (12.48)

5. Multiplying (12.47) by x and subtracting it from (12.45) to obtain

$$(1-x^2)P'_n(x) = [xP'_{n-1}(x) + nP_{n-1}(x)] - x[nP_n(x) + P'_{n-1}(x)]$$

or, $(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]$, which is (12.49)

6. Replacing n by $(n+1)$ in (12.48) to obtain

$$(1-x^2)P'_n(x) = (n+1)[xP_n(x) - P_{n+1}(x)]$$

or,
$$(n+1)P_{n+1}(x) = (n+1)xP_n(x) - (1-x^2)P'_n(x) \quad ...(12.54)$$

Substituting for $(1-x^2)P'_n(x)$ from (12.49) in (12.54), we obtain (12.50) as

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x),$$

These recurrence relations can also be proved using generating function for $P_n(x)$. For example, to prove (12.49), we can begin as follows.

Differentiating the generating function (12.42) with respect to t, we obtain

$\frac{\partial L(x,t)}{\partial t} = -\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \frac{(x-t)}{(1-2xt+t^2)^{3/2}}$. This gives

$$(1-2xt+t^2)\frac{\partial L(x,t)}{\partial t} - (x-t)\,L(x,t) = 0. \qquad ...(12.55)$$

Substituting $L(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n$ in (12.55) to obtain

$$(1-2xt+t^2)\sum_{n=1}^{\infty} n\,P_n(x)t^{n-1} - (x-t)\sum_{n=0}^{\infty} P_n(x)t^n = 0$$

or,
$$\sum_{n=1}^{\infty} nP_n(x)t^{n-1} - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1} - \sum_{n=0}^{\infty} xP_n(x)t^n + \sum_{n=0}^{\infty} P_n(x)t^{n+1} = 0.$$

Making the index of t same in every summation, we obtain

$$\sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n - \sum_{n=0}^{\infty} xP_n(x)t^n + \sum_{n=1}^{\infty} P_{n-1}(x)t^n = 0.$$

...(12.56)

Rewriting it as

$$[P_1(x) - xP_0(x)] + [2P_2(x) - 3xP_1(x) + P_0(x)]t$$
$$+ \sum_{n=2}^{\infty}[(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x)]t^n = 0. \qquad ...(12.57)$$

For (12.57) to be true the coefficient of t^n must be zero for $n = 0, 1, 2, ...$

Thus, $P_1(x) - xP_0(x) = 0$, $2P_2(x) - 3xP_1(x) + P_0(x) = 0$

and, for $n \geq 2$, $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$.

This is valid for $n = 0$ and 1 also; this establishes (12.49) for all integers $n \geq 0$.

Example 12.9: Using the recurrence relation $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - n\,P_{n-1}(x)$,

(a) Generate the Legendre polynomials P_2, P_3, P_4 given that $P_0(x) = 1$ and $P_1(x) = x$.

(b) Show that the coefficient of x^n in $P_n(x)$ is $a_n = \dfrac{(2^n)!}{2^n(n!)^2}$.

Solution: (a) The recurrence relation is $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.

For $n = 1$, it gives, $2P_2(x) = 3xP_1(x) - P_0(x) = 3x^2 - 1$, (using $P_0(x) = 1$ and $P_1(x) = x$)

or, $P_2(x) = (3x^2 - 1)/2$.

For $n = 2$, we have $3P_3(x) = 5xP_2(x) - 2P_1(x) = 5x(3x^2 - 1)/2 - 2x = 3(5x^3 - 3x)/2$

or, $P_3(x) = (5x^3 - 3x)/2$.

For $n = 3$, we have
$$4P_4(x) = 7xP_3(x) - 3P_2(x) = 7x(5x^3 - 3x)/2 - 3(3x^2 - 2)/2$$
$$= (35x^4 - 30x^2 + 3)/2$$

or, $P_4(x) = (35x^4 - 30x^2 + 3)/8$.

(b) We have, $P_1(x) = x$, thus $a_1 = 1$;

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad \text{thus } a_2 = \frac{3}{2} = \frac{4!}{2^2(2!)^2}; \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \quad \text{thus } a_3 = \frac{5}{2} = \frac{6!}{2^3(3!)^2}$$

To prove it in general, equate the coefficient of the highest power of x, that is, x^{n+1} on both sides of the recurrence relation, we obtain $(n+1)a_{n+1} = (2n+1)a_n$, $n \geq 0$.

It gives,
$$a_{n+1} = \frac{2n+1}{(n+1)}a_n = \frac{2n+1}{n+1}\cdot\frac{2n-1}{n}a_{n-1} = \frac{(2n+1)(2n-1)(2n-3)\ldots 3.1.}{(n+1)n(n-1)\ldots 2.1.}a_1$$
$$= \frac{(2n+1)(2n-1)(2n-3)\ldots 3.1.}{(n+1)n(n-1)\ldots 2.1.}, \quad \text{since } a_1 = 1.$$

Hence,
$$a_{n+1} = \frac{(2n+2)!}{[(n+1)!]^2 2^{n+1}}$$

Replacing $n+1$ by n, we get $a_n = \dfrac{(2n)!}{2^n(n!)^2}$, the desired result.

12.3.6 Orthogonality of the Legendre Polynomials on [–1, 1]

We have the following theorem.

Theorem 12.1 (Orthogonality of Legendre Polynomials): *If n and m are non-negative integers, then*

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases} \qquad \ldots(12.58)$$

This integral relationship is called *orthogonality* of the Legendre polynomials on [–1, 1].

Proof. We prove (12.58) using the fact that $P_n(x)$ is a solution of the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

for $\alpha = n$. Thus for some non-negative integers m and n, we have

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \qquad ...(12.59)$$

and,

$$(1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0. \qquad ...(12.60)$$

Multiplying (12.59) by $P_m(x)$ and (12.60) by $P_n(x)$ and subtracting the resultant equations, we obtain

$$(1-x^2)(P_n''P_m - P_m''P_n) - 2x(P_n'P_m - P_m'P_n) + [n(n+1) - m(m+1)]P_nP_m = 0$$

or,

$$\frac{d}{dx}[(1-x^2)(P_n'P_m - P_m'P_n)] + [n(n+1) - m(m+1)]\,P_nP_m = 0.$$

Integrating it over the interval $[-1, 1]$, we obtain

$$[(1-x^2)(P_n'P_m - P_m'P_n)]_{-1}^{+1} + [n(n+1) - m(m+1)]\int_{-1}^{1} P_nP_m\,dx = 0.$$

The first term vanishes at $x = \pm 1$, and thus for $m \neq n$, we get

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0, \quad m \neq n.$$

The case $m = n$ can be proved using the generating function (12.42) given by

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

Squaring both sides of it and integrating w.r.t. x over $[-1, 1]$, we obtain

$$\int_{-1}^{1} \frac{dx}{(1-2xt+t^2)} = \int_{-1}^{1}\left[\sum_{n=0}^{\infty} P_n(x)\,t^n\right]^2 dx. \qquad ...(12.61)$$

From left side of Eq. (12.61), we obtain

$$\int_{-1}^{1} \frac{dx}{(1-2xt+t^2)} = \left[\frac{\ln(1-2xt+t^2)}{-2t}\right]_{-1}^{1} = -\frac{1}{2t}[\ln(1-2t+t^2) - \ln(1+2t+t^2)]$$

$$= \frac{1}{t}[\ln(1+t) - \ln(1-t)] = 2\left[1 + \frac{t^2}{3} + \frac{t^4}{5} + \ldots \frac{t^{2n}}{2n+1} + \ldots\right] \qquad ...(12.62)$$

Also right side of Eq. (12.61), using the orthogonal property of the Legendre polynomials for $m \neq n$ gives

$$\int_{-1}^{1}\left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 dx = \sum_{n=0}^{\infty}\left(\int_{-1}^{1} P_n^2(x)\,dx\right)t^{2n}. \qquad ...(12.63)$$

From (12.62) and (12.63), equating the coefficient of t^{2n} gives

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}, \quad \text{for } n \ge 0.$$

This proves (12.58).

12.3.7 Fourier-Legendre Series

Suppose $f(x)$ is defined and have continuous derivatives over the interval $[-1, 1]$. We want to explore the possibility of expanding $f(x)$ in a series of Legendre polynomials, that is,

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \qquad \text{...(12.64)}$$

where the coefficients c_i's are constants.

To determine these coefficients, multiply the proposed expansion (12.64) by $P_m(x)$ and integrate the resulting equation over $[-1, 1]$, interchanging the summation and the integral, we get

$$\int_{-1}^{1} f(x)P_m(x)\,dx = \sum_{n=0}^{\infty} c_n\left(\int_{-1}^{1} P_n(x)P_m(x)\,dx\right). \qquad \text{...(12.65)}$$

Because of the orthogonal property, all terms in the summation on the right side of Eq. (12.65) are zeros, except when $n = m$, and thus

$$\int_{-1}^{1} f(x)P_m(x)\,dx = c_m \frac{2}{2m+1},$$

or,
$$c_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)\,dx, \quad m = 0, 1, 2, \ldots \qquad \text{...(12.66)}$$

The coefficients c_m's are called the *Fourier-Legendre coefficients* of f, and the resultant series is called the *Fourier-Legendre series*.

As a special case of Fourier-Legendre expansion, any polynomial $f(x)$ is a linear combination of Legendre polynomials. Thus in case of a polynomial, this series can be obtained by just solving for x^n in terms of $P_n(x)$ and writing each power of x in $f(x)$ in terms of Legendre polynomials.

For example, let $f(x) = 4x^3 - 2x^2 - 3x + 8$. Write

$$f(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x)$$

or,
$$4x^3 - 2x^2 - 3x + 8 = c_0(1) + c_1(x) + c_2\left(\frac{3x^2-1}{2}\right) + c_3\left(\frac{5x^3-3x}{2}\right).$$

Equating the coefficients of like powers of x on both sides, we get

$$c_0 = 22/3, \quad c_1 = -3/5, \quad c_2 = -4/3, \text{ and } \quad c_3 = 8/5$$

hence, $$f(x) = \frac{22}{3}P_0(x) - \frac{3}{5}P_1(x) - \frac{4}{3}P_2(x) + \frac{8}{5}P_3(x).$$

12.3.8 Zeros of Legendre Polynomials

The orthogonality property of Legendre polynomials is also applied to know the zeros of the Legendre polynomials. We have the following result.

Theorem 12.2 (Zeros of Legendre Polynomials): *For each non-negative integer n, $P_n(x)$ has n real and distinct roots all lying in $(-1, 1)$.*

Proof. The result is obviously true for $n = 0$, since $P_0(x) = 1$ has no zero. Also we observe that $P_1(x) = x$ has exactly one zero, which is $x = 0$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$ has two zeros, which are $x = \pm 1/\sqrt{3}$, lying in $(-1, 1)$. To prove it in general, we first show that *if $P_n(x)$ has a real root x_0 in $(-1, 1)$, then this root must be simple.*

Suppose that x_0 is a repeated root of $P_n(x)$, then $P_n(x_0) = P_n'(x_0) = 0$, and $P_n(x)$ is a solution of the initial value problem

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0; \quad y(x_0) = y'(x_0) = 0. \qquad \text{...(12.67)}$$

But we can check very easily, that this initial value problem has the unique solution and which is the trivial solution $y(x) = 0$, thus $P_n(x)$ is the zero function on $(-1, 1)$, which is a contradiction and hence $P_n(x)$ cannot have a repeated root in $(-1, 1)$.

Now suppose n is a positive integer, then since $P_n(x)$, $P_0(x)$ are orthogonal on $[-1, 1]$, thus

$$\int_{-1}^{1} P_n(x)P_0(x)dx = \int_{-1}^{1} P_n(x)dx = 0. \qquad \text{...(12.68)}$$

Equation (12.68) implies that $P_n(x)$ cannot be strictly positive or negative on $(-1, 1)$, and hence must change sign on $(-1, 1)$. Further $P_n(x)$ is continuous, there must exist some x_1 in $(-1, 1)$ with $P_n(x_1) = 0$.

Let $x_1, x_2, ..., x_m$ be all zeros of $P_n(x)$ in $(-1, 1)$, with $-1 < x_1 < ... < x_m < 1$, and

$q_m(x) = (x - x_1)(x - x_2) ... (x - x_m)$, be a polynomial of degree m.

If $m < n$, then $q_m(x)$ is orthogonal to $P_n(x)$ and so $\int_{-1}^{1} q_m(x)P_n(x)dx = 0$.

But $q_m(x)$ and $P_n(x)$ change sign at exactly the same points $x_1, x_2, ..., x_m$ in $(-1, 1)$, thus $q_m(x)$ and $P_n(x)$ are either of same sign or of the opposite sign on each interval $(-1, x_1)$, (x_1, x_2) ..., $(x_m, 1)$, and hence $q_m(x)P_n(x)$ is either strictly positive or negative on $(-1, 1)$ except at the finite number of points

$x_1, x_2, ..., x_m$ at which this is zero. Hence $\int_{-1}^{1} q_m(x)P_n(x)dx$ cannot be zero, which is a contradiction.

Thus, $m = n$ and hence $P_n(x)$ has exactly n simple zeros in $(-1, 1)$.

The result is evident from the graphs of $P_n(x)$, for $n = 0, 1, 2, 3, 4, 5$ as in Fig. 12.1a & b.

Example 12.10: Let $f(x)$ be a polynomial of degree m and let $n > m$, then $\int_{-1}^{1} f(x)P_n(x)dx = 0$.

Solution: Since $f(x)$ is a polynomial of degree m, thus $f(x) = c_0P_0(x) + c_1P_1(x) + ... + c_mP_m(x)$, for some c_i's.

Multiplying it with $P_n(x)$ and integrating w.r.t. x over the interval $[-1, 1]$, we get

$$\int_{-1}^{1} f(x)P_n(x)dx = \sum_{k=0}^{m} c_k \int_{-1}^{1} P_k(x)P_n(x)dx = 0,$$

using the orthogonality property of the Legendre polynomials for $m \neq n$.

Example 12.11: Show that $\int_{-1}^{1} x^2P_{n-1}(x)P_{n+1}(x)dx = \dfrac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$.

Solution: Consider the recurrence relation $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$.

Replacing n by $(n - 1)$ and $(n + 1)$, we get respectively

$$xP_{n-1} = \frac{1}{2n-1}[nP_n + (n-1)P_{n-2}], \text{ and } xP_{n+1} = \frac{1}{2n+3}[(n+2)P_{n+2} + (n+1)P_n].$$

Multiplying these two and integrating w.r.t. x over $[-1, 1]$, we get

$$\int_{-1}^{1} x^2P_{n-1}(x)P_{n+1}(x)dx = \frac{1}{(2n-1)(2n+3)} n(n+1) \int_{-1}^{1} P_n^2(x)dx,$$

since all other integrals on the r.h.s. are zeros by the orthogonal property of the Legendre polynomials. Thus,

$$\int_{-1}^{1} x^2P_{n-1}(x)P_{n+1}(x)dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}.$$

Exmaple 12.12: Show that $\int_{-1}^{1} f(x)P_n(x)dx = \dfrac{(-1)^n}{2^n.n!} \int_{-1}^{1} f^{(n)}(x).(x^2-1)^n dx$.

Hence, deduce that $\int_{-1}^{1} x^m P_n(x)dx = \begin{cases} 0, & \text{if } m<n \\ \dfrac{2^{n+1}(n!)^2}{(2n+1)!}, & \text{if } m=n \end{cases}$

Solution: Using Rodrigue's formula, we have

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} f(x)\,\frac{d^n(x^2-1)^n}{dx^n}\,dx$$

$$= \frac{1}{2^n n!}\left[\left\{f(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right\}_{-1}^{1} - \int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx\right]$$

$$= \frac{-1}{2^n.n!}\int_{-1}^{1} f'(x)\,\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\,dx,$$

since the first term is zero at $x = \pm 1$. Integrating further $(n-1)$ times, we obtain

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{(-1)^n}{2^n\,n!}\int_{-1}^{1} f^{(n)}(x)(x^2-1)^n dx.$$

For $f(x) = x^m$, $f^{(n)}(x) = 0$, when $m<n$ and, $f^{(n)}(x) = n!$, when $m = n$. Thus,

$$\int_{-1}^{1} x^n P_n(x)dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} n!(x^2-1)^n dx = \frac{2}{2^n}\int_{0}^{1}(1-x^2)^n dx$$

$$= \frac{2}{2^n}\int_{0}^{\pi/2} \cos^{(2n+1)}\theta\, d\theta, \quad \text{using } x = \sin\theta$$

$$= \frac{2}{2^n}\left[\frac{(2n)(2n-2)\ldots 4.2.}{(2n+1)(2n-1)\ldots 3.1}\right] = \frac{2}{2^n}\left[\frac{2^{2n}(n!)^2}{(2n+1)!}\right] = \frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

Example 12.13: Expand $f(x) = \begin{cases} 0, & -1<x\le 0 \\ x, & 0<x<1 \end{cases}$ as a Fourier-Legendre series.

Solution: Let $f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$, where c_i's are Fourier-Legendre coefficients given by

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\,P_n(x)dx, \quad n = 0, 1, 2, \ldots$$

For given $f(x)$, $c_n = \frac{2n+1}{2}\int_{0}^{1} x\,P_n(x)dx$. Thus,

$$c_0 = \frac{1}{2}\int_0^1 xP_0(x)dx = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{4},\ c_1 = \frac{3}{2}\int_0^1 xP_1(x)dx = \frac{3}{2}\int_0^1 x^2dx = \frac{1}{2},$$

$$c_2 = \frac{5}{2}\int_0^1 xP_2(x)dx = \frac{5}{2}\int_0^1 x\left(\frac{3x^2-1}{2}\right)dx = \frac{5}{16},\ c_3 = \frac{7}{2}\int_0^1 xP_3(x)dx = \frac{7}{2}\int_0^1 x\left(\frac{5x^3-3x}{2}\right)dx = 0$$

$$c_4 = \frac{9}{2}\int_0^1 xP_4(x)dx = \frac{9}{2}\int_0^1 x\left(\frac{35x^4-30x^2+3}{8}\right)dx = -\frac{3}{32}, \text{ and so on.}$$

Hence, $f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \ldots$

Example 12.14: Expand $f(x) = \cos\frac{\pi x}{2}$ as a Fourier-Legendre series.

Solution: Let $f(x) = \sum_{n=0}^{\infty} c_nP_n(x)$, where c_i's are Fourier-Legendre coefficients given as

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx, \quad n = 0, 1, 2, \ldots$$

For $f(x) = \cos\frac{\pi x}{2}$, $c_n = \frac{2n+1}{2}\int_{-1}^{1}\cos\frac{\pi x}{2}\,P_n(x)dx$

Because $\cos\frac{\pi x}{2}$ is an even function of x and $P_n(x)$ is an odd function for n odd. Thus, $\cos\frac{\pi x}{2}\,P_n(x)$ is odd for n odd. Thus $c_n = 0$, if n is odd. So we need to calculate only for $n = 0, 2, 4, \ldots$.

We have, $c_0 = \frac{1}{2}\int_{-1}^{1}\cos\frac{\pi x}{2}\,dx = \frac{1}{2}\left[\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_{-1}^{1} = \frac{2}{\pi} = 0.63662$

$$c_2 = \frac{5}{2}\int_{-1}^{1} \cos\frac{\pi x}{2} P_2(x)dx = \frac{5}{2}\int_{-1}^{1} \cos\frac{\pi x}{2}\left[\frac{1}{2}(3x^2-1)\right]dx$$

$$= \frac{15}{4}\int_{-1}^{1} x^2\cos\frac{\pi x}{2}\,dx - \frac{5}{4}\int_{-1}^{1}\cos\frac{\pi x}{2}\,dx$$

$$= \frac{15}{4}\left[x^2\frac{2}{\pi}\sin\frac{\pi x}{2} + 2x\frac{4}{\pi^2}\cos\frac{\pi x}{2} + 2\frac{8}{\pi^3}\sin\frac{\pi x}{2}\right]_{-1}^{1} - \frac{5}{4}\left[\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_{-1}^{1}$$

$$= 10\,\frac{\pi^2-12}{\pi^3} = 0.34355.$$

$$c_4 = \frac{9}{2}\int_{-1}^{1} \cos\frac{\pi x}{2} P_4(x)dx = \frac{9}{2}\int_{-1}^{1} \cos\frac{\pi x}{2}\left(\frac{35x^4-30x^2+3}{8}\right)dx$$

$$= 18\,\frac{(\pi^4-180\pi^2+1680)}{\pi^5} = 0.0064724, \text{ and so on.}$$

Thus, $\cos(\pi x/2) = 0.63662 - 0.34355 P_2(x) + 0.0064724 P_4(x) \ldots$

EXERCISE 12.2

1. For $n = 0, 1, 2, 3, 4$ verify by substitution that $P_n(x)$ is a solution of the Legendre equation
$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$
2. Show that for $n = 0, 1$ one of the solutions $y_1(x)$, $y_2(x)$ of the Legendre equation reduces to a polynomial and the other solution becomes a polynomial multiplied by $\ln[(1+x)/(1-x)]$.
3. Use Rodrigue's formula to reproduce the first five Legendre polynomials.
4. Express x^5 in terms of Legendre polynomials.
5. Express $x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre polynomials.
6. Show that

   (a) $P_n(-x) = (-1)^n P_n(x)$ (b) $P_n'(-x) = (-1)^{n+1} P_n'(x)$.
7. Using the Rodrigue's formula, show that

   (a) $\int_{-1}^{1} P_n(x)dx = 0.$ (b) $\int_{-1}^{1} x^m P_n(x)dx = 0, \quad m < n.$

8. Prove that $\int_{-1}^{1} (1-x^2)P'_m P'_n dx = \begin{cases} 0, & m \neq n \\ \dfrac{2n(n+1)}{2n+1}, & m = n \end{cases}$

for any non-negative integers m and n.

9. Prove that for any non-negative integer n

(a) $\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \dfrac{2n}{4n^2-1}$ (b) $\int_{-1}^{1} xP'_n(x)P_n(x)dx = \dfrac{2n}{2n+1}$

10. Using the generating function, $L(x;t) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$,

prove the recurrence relations

(a) $nP_n(x) - xP'_n(x) + P'_{n-1}(x) = 0$ (b) $(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x)$.

11. Find a solution of $(a^2-x^2)y'' - 2xy' + 12y = 0$, $a \neq 0$ by substituting $x = az$.

12. Expand the following in a series of Legendre polynomials

(a) $1 - 3x + 3x^2$. (b) $\sin \pi x$. (c) $f(x) = \begin{cases} -1, & -1 \le x \le 0 \\ 1, & 0 \le x \le 1 \end{cases}$

12.4 SINGULAR POINTS OF LINEAR DIFFERENTIAL EQUATIONS. METHOD OF FROBENIUS

So far we have derived the power series solution of a differential equation of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad \text{...(12.69)}$$

about a point x_0 at which the coefficients $a_1(x)$ and $a_2(x)$ are analytic and $a_0(x_0) \neq 0$. To put it in a different form, power series methods have been applied to differential equations of the form

$$y'' + p(x)y' + q(x)y = 0, \quad \text{...(12.70)}$$

with $p(x) = \dfrac{a_1(x)}{a_0(x)}, q(x) = \dfrac{a_2(x)}{a_0(x)}$ to find a solution about any point x_0 at which the coefficients $p(x)$ and $q(x)$ are analytic.

In this section we consider the problem of finding series solution about the points at which the coefficients $p(x)$ and $q(x)$ are not analytic.

12.4.1 Regular Points. Singular Points

The points at which $p(x)$ and $q(x)$ are analytic are called *regular points* or *ordinary points* of the differential equation. If x_0 is not an ordinary point of the differential equation, then it is called a *singular point* of the equation. Thus, x_0 is a singular point, if $a_0(x_0) = 0$ or if, any one of $a_1(x)/a_0(x)$, $a_2(x)/a_0(x)$ fails to be analytic at x_0.

A singular point x_0 of Eq. (12.70) is said to be *regular singular point*, if the functions $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0. A singular point, that is not regular, is said to be an *irregular singular point*.

For example, the differential equation $x^2y'' + xy' + (x^2 - 1)y = 0$ has singular point at zero. Every other real number is a regular point of this equation. However, since

$$xp(x) = x\frac{a_1(x)}{a_0(x)} = 1, \text{ and } \quad x^2q(x) = x^2\frac{a_2(x)}{a_0(x)} = x^2 - 1$$

both are analytic at $x = 0$, thus zero is a regular singular point of the differential equation.

As another example, the differential equation

$$x^3(x-2)y'' + 5(x+2)(x-2)y' + 3x^2y = 0 \qquad ...(12.71)$$

has singular points at $x = 0$ and 2. Consider

$$xp(x) = \frac{5x(x+2)(x-2)}{x^3(x-2)^2} = \frac{5}{x^2}\frac{x+2}{x-2}.$$

It is not analytic at $x = 0$, hence zero is an irregular singular point. But at $x = 2$,

$$(x-2)p(x) = \frac{5(x+2)(x-2)^2}{x^3(x-2)^2} = \frac{5(x+2)}{x^3} \text{ and, } (x-2)^2q(x) = \frac{3x^2(x-2)^2}{x^3(x-2)^2} = \frac{3}{x};$$

both of these are analytic at $x = 2$, therefore, 2 is a regular singular point of the differential Eq. (12.71).

Generally, it is more convenient to deal with regular singular point at 0 than at a point $x_0 \neq 0$. We observe that a regular singular point located at x_0 can always be shifted to the origin by setting $X = x - x_0$. For example, as we have noted above that 2 is a regular singular point of the differential Eq. (12.71). Setting $X = x - 2$, and $y(x) = Y(x-2) = Y(X)$, the Eq. (12.71) becomes

$$X(X+2)^3Y'' + 5X(X+4)Y' + 3(X+2)^2Y = 0, \qquad ...(12.72)$$

which has $x = 0$ as its regular singular point.

Further *an ordinary power series solution cannot be developed about a regular singular point*. We will attempt to find a series solution, a some sort of extension of an ordinary power series, about a regular singular point x_0. The solution may contain negative, or/and fractional powers of $(x - x_0)$. The series is called a *Frobenius series* and the method as *Frobenius method*. This method always generates two linearly independent solutions about a regular singular point.

12.4.2 THE FROBENKUS METHOD

Let x_0 be a regular singular point of the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \qquad ...(12.73)$$

The Frobenius method claims that at least one solution of the differential Eq. (12.73) can be represented in the form

$$y(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^{n+r} \qquad ...(12.74)$$

where the exponent r is any real number and is chosen so that $c_0 \neq 0$.

The equation also has a second solution, linearly independent of the first. This may be of the type (12.74) with different r and different coefficients, or may contain a logarithmic term. Taken together, these two solutions form a basis of solutions for the differential Eq. (12.73).

We will consider Frobenius method for equations of the form

$$x^2y'' + a_1(x)y' + a_2(x)y = 0 \qquad ...(12.75)$$

when $x = 0$ is a regular singular point.

Such equations arise frequently in applications and moreover a singular point located at x_0 can always be shifted to the origin, and it is convenient to compute while taking $x_0 = 0$.

Since zero is a regular singular point, we can rewrite Eq. (12.75) as

$$x^2y'' + xp(x)y' + q(x)y = 0 \qquad ...(12.76)$$

where $p(x)$ and $q(x)$ are analytic at $x = 0$.

Let the series solution be of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad c_0 \neq 0, \qquad ...(12.77)$$

where r is any real number.

Differentiating (12.77), we obtain

$$y'(x) = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}, \text{ and } \quad y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}.$$

Also since $p(x)$ and $q(x)$ are analytic at $x = 0$, thus

$$p(x) = p_0 + p_1x + p_2x^2 + ..., \text{ and } \quad q(x) = q_0 + q_1x + q_2x^2 + ...$$

Substituting for $y(x)$, $y'(x)$, $y''(x)$, $p(x)$, and $q(x)$ in Eq. (12.76), we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r} + [p_0 + p_1x + p_2x^2 + ...]\left(\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}\right)$$

$$+[q_0 + q_1x + q_2x^2 + ...]\left(\sum_{n=0}^{\infty} c_n x^{n+r}\right) = 0. \qquad ...(12.78)$$

For (12.77) to be a solution of (12.76), the coefficient of each power of x in (12.78) must be equal to zero to make it an identity.

Equating the coefficient of the lowest power of x, that is, x^r to zero, we obtain

$$[r(r-1)+rp_0+q_0]c_0=0$$

Since $c_0 \neq 0$, we obtain

$$r(r-1)+rp_0+q_0=0 \qquad \text{...(12.79)}$$

This equation is called *indicial equation* and the two roots $r=r_1$ and $r=r_2$ are called *indicial roots*.

Equating the coefficients of the remaining powers of x to zero, we obtain a *recurrence relation* relating the coefficients c_n. From it, for specific values of $r=r_1$ and $r=r_2$, we can find for the values of successive coefficients in terms of a few initial values.

Depending upon the nature of the indicial roots r_1 and r_2, the following cases arise:

Case I: r_1 and r_2 distinct and do not differ by an integer, (complex conjugate roots belong to this category). The two solutions, $y_1(x)=[y(x)]_{r=r_1}$, and $y_2(x)=[y(x)]_{r=r_2}$, obtained from (12.77) are linearly independent, thus, the complete solution of the differential equation is

$$y(x)=Ay_1(x)+By_2(x), \qquad \text{...(12.80)}$$

where A and B are two arbitrary constants.

Case II: *Repeated roots,* $r_1=r_2=p$. In this case one solution is, $y_1(x)=[y(x)]_{r=p}$ and it can be shown that the second linearly independent solution is of the form

$$y_2(x)=\left(\frac{\partial y}{\partial r}\right)_{r=p}$$

and hence the complete solution is of the form (12.80).

Case III: r_1-r_2 *equals to an integer.* In this case the smaller root $r=r_2$ either leads to both solutions $y_1(x)$ and $y_2(x)$, or to none, (the coefficients become infinite). In both cases the larger root gives one solution.

In case when the coefficients become infinite the solution is modified by substituting $c_0=a_0(r-r_2)$. Then one of the linearly independent solution is given by $y_1(x)=[y(x)]_{r=r_2}$, and the second linearly independent solution is $y_2(x)=\left[\frac{\partial y}{\partial r}\right]_{r=r_2}$. The solution corresponding to the larger root $r=r_1$ turns out to be linearly dependent on $y_1(x)$.

We illustrate these cases one by one by considering suitable examples.

Example 12.15: Solve the differential equation $4xy''+2y'+y=0$ using the Frobenius method.

Solution: It is easy to verify that $x=0$ is a regular singular point of the given equation. Let the series solution be

$$y(x)=\sum_{n=0}^{\infty} c_n x^{n+r}, \quad c_0\neq 0. \qquad \text{...(12.81)}$$

Differentiating, we obtain

$$y'(x)=\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}, \text{ and } \quad y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}.$$

Substituting in the given differential equation, we obtain

$$4\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}+2\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}+\sum_{n=0}^{\infty}c_n x^{n+r}=0,$$

or, $$\sum_{n=0}^{\infty}(n+r)(4n+4r-2)c_n x^{n+r-1}+\sum_{n=0}^{\infty}c_n x^{n+r}=0,$$

or, $$2r(2r-1)c_0x^{r-1}+\sum_{n=1}^{\infty}(n+r)(4n+4r-2)c_n x^{n+r-1}+\sum_{n=0}^{\infty}c_n x^{n+r}=0,$$

or, $$2r(2r-1)c_0x^{r-1}+\sum_{n=0}^{\infty}(n+r+1)(4n+4r+2)c_{n+1}x^{n+r}+\sum_{n=0}^{\infty}c_n x^{n+r}=0. \quad ...(12.82)$$

Equating to zero the coefficient of x^{r-1}, the lowest power of x, in (12.82), we obtain $2r(2r-1)c_0=0$. Since $c_0 \neq 0$, the indicial equation is $r(2r-1)=0$, which gives, $r=0, 1/2$.

Equating to zero the coefficient of x^{n+r} in (12.82), we obtain

$$2(n+r+1)(2n+2r+1)c_{n+1}+c_n=0,$$

which gives, $$c_{n+1}=\frac{-c_n}{2(n+r+1)(2n+2r+1)},\ n=0,1,2,... \quad ...(12.83)$$

as the recurrence relation. For $r=0$, (12.83) gives

$$c_{n+1}=\frac{-c_n}{2(n+1)(2n+1)},\quad n=0,1,2,...,$$

which gives, $c_1=-\frac{1}{2}c_0,\quad c_2=-\frac{1}{2.2.3}c_1=\frac{1}{4!}c_0\quad c_3=-\frac{1}{2.3.5}c_2=\frac{1}{6!}c_0$ etc.

Hence, one Frobenius solution corresponding to $r=0$, from (12.81), is

$$y_1(x)=c_0\left(1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+...\right)=c_0\cos\sqrt{x}=\cos\sqrt{x},$$ setting the constant $c_0=1$.

For $r=½$, (12.83) gives $$c_{n+1}=\frac{-c_n}{2\left(n+\frac{3}{2}\right)(2n+2)}=\frac{-c_n}{2(n+1)(2n+3)},\ n=0,1,2,...$$

which gives, $c_1=-\frac{1}{3!}c_0,\quad c_2=-\frac{1}{2.2.5}c_1=\frac{1}{5!}c_0\quad c_3=-\frac{1}{2.3.7}c_2=-\frac{1}{7!}c_0$

Hence, second Frobenius solution corresponding to $r=½$ from (12.81), is

$$y_2(x) = c_0\left[x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \frac{x^{7/2}}{7!} + \dots\right] = c_0 \sin\sqrt{x} = \sin\sqrt{x}\text{, setting the constant } c_0 = 1.$$

Hence, the complete solution of the given equation is $y(x) = A\cos\sqrt{x} + B\sin\sqrt{x}$, where A and B are two arbitrary constants.

Example 12.16: Solve the equation $x^2y''(x) - xy'(x) + 10y(x) = 0$ by the Frobenius method.

Solution: The equation has a regular singular point at $x = 0$. Let the series solution be

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad c_0 \neq 0. \qquad \text{...(12.84)}$$

Substituting in the given equation, we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + 10\sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Collecting the terms of the lowest power of x, that is, x^r and regrouping the terms of x^{n+r}, we obtain

$$(r^2 - 2r + 10)c_0 x^r + \sum_{n=1}^{\infty}[(n+r)(n+r-2) + 10]c_n x^{n+r} = 0. \qquad \text{...(12.85)}$$

Equating the coefficient of x^r on the left side Eq. (12.85) to zero and since $c_0 \neq 0$, we obtain $r^2 - 2r + 10 = 0$ as the indicial equation, which gives $r = 1 \pm 3i$ as the indicial roots.

Equating the coefficients of x^{n+r} in (12.85) to zero, the recurrence relation is

$$[(n+r)(n+r-2) + 10]c_n = 0, \quad n = 1, 2, \dots \qquad \text{...(12.86)}$$

For $r = 1 \pm 3i$, left side of Eq. (12.86) is non-zero for any value of n and hence for (12.86) to hold $c_n = 0$ for $n = 1, 2, \dots$. Thus, from (12. 84) we have $y(x) = c_0x^r$, $r = 1 \pm 3i$.

The two linearly independent solutions are obtained by taking the real and imaginary part of $y(x)$, as

$$y(x) = c_0\, x^{1\pm 3i} = c_0 x \exp[\pm i\,(3\ln x)] = c_0 x\,[\cos(3\ln x) \pm i\sin(3\ln x)].$$

Thus two linearly independent solutions are

$$y_1(x) = x\cos[3\ln x], \quad \text{and} \quad y_2(x) = x\sin[3\ln x].$$

Hence, the complete solution is $y(x) = x[A\cos(3\ln x) + B\sin(3\ln x)]$, where A and B are two arbitrary constants.

Example 12.17: Solve $x^2y'' - (x + x^2)y' + y = 0$ using the Frobenius method.

Solution: It is easy to verify that $x = 0$ is a regular singular point of the given equation $x^2y'' - (x + x^2)y' + y = 0$. Let the series solution be

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad c_0 \neq 0.$$

Substituting this in the given equation, we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)c_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)c_n x^{n+r+1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

To shift the indecis in the third summation on the left side of the above equation replace $n+1$ by n, we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)c_n x^{n+r} - \sum_{n=1}^{\infty} (n+r-1)c_{n-1} x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

or, $(r^2 - 2r + 1)\, c_0 + \sum_{n=1}^{\infty} [\{(n+r)(n+r-2)+1\}c_n - \{n+r-1\}\, c_{n-1}]x^{n+r} = 0.$ ...(12.87)

Equating the coefficient of x^r on left side of Eq. (12.87) to zero, we obtain $(r^2 - 2r + 1)c_0 = 0$, which gives $r^2 - 2r + 1 = 0$ as the indicial equation since $c_0 \neq 0$. The roots are repeated $r = 1, 1$.

Equating the coefficient of x^{n+r} to zero, we obtain the recurrence relation as

$$c_n = \frac{(n+r-1)}{(n+r)(n+r-2)+1}\, c_{n-1}, \quad n = 1, 2, \ldots$$

It gives $\quad c_1 = \frac{1}{r} c_0, \quad c_2 = \frac{1}{r(r+1)} c_0, \quad c_3 = \frac{1}{r(r+1)(r+2)} c_0$, etc.

Hence the solution taking $c_0 = 1$, is

$$y(x) = x^r\left[1 + \frac{x}{r} + \frac{x^2}{r(r+1)} + \frac{x^3}{r(r+1)(r+2)} + \ldots\right]. \qquad \text{...(12.88)}$$

One Frobenius solution corresponding to $r = 1$ is $y_1(x) = x\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right]$.

Since the indicial roots are equal the second linearly independent Frobenius solution is obtained from (12.88) as

$$y_2(x) = \left.\frac{\partial y}{\partial r}\right|_{r=1} = \left[x^r (\ln x)\left\{1 + \frac{x}{r} + \frac{x^2}{r(r+1)} + \frac{x^3}{r(r+1)(r+2)} + \ldots\right\}\right]_{r=1}$$

$$+\left[x^r\left\{-\frac{x}{r^2}-\frac{x^2}{r(r+1)}\left(\frac{1}{r}+\frac{1}{r+1}\right)-\frac{x^3}{r(r+1)(r+2)}\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}\right)-\ldots\right\}\right]_{r=1}$$

or, $$y_2(x)=(\ln x)y_1(x)-x^2\left[1+\frac{x}{2!}\left(1+\frac{1}{2}\right)+\frac{x^3}{3!}\left(1+\frac{1}{2}+\frac{1}{3}\right)+\ldots\right]$$

Hence the general solution is $y(x) = Ay_1(x) + By_2(x)$, where A and B are two arbitrary constants.

Example 12.18: Solve $4x^2y'' + 4xy' - y = 0$ using the Frobenius method.

Solution: The point $x = 0$ is a regular singular point of the equation. Let the series solution be

$$y(x)=\sum_{n=0}^{\infty} c_n x^{n+r},\quad c_0 \neq 0, \qquad \ldots(12.89)$$

where r is any real number.

Substituting for y'', y' and y in the given equation, we obtain

$$4\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r}+4\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}-\sum_{n=0}^{\infty}c_n x^{n+r}=0$$

or, $$\sum_{n=0}^{\infty}[4(n+r)^2-1]\,c_n x^{n+r}=0. \qquad \ldots(12.90)$$

Equating the coefficient of the least power of x, that is of x^r, on the left side of Eq. (12.90) to zero, we obtain $c_0(4r^2-1) = 0$. Since $c_0 \neq 0$, hence the indicial equation is $(4r^2 - 1) = 0$, which gives, $r = \pm 1/2$, as its roots. Next, equating the coefficient of x^{n+r} on the left side of Eq. (12.90) to zero, we obtain

$$[4(n+r)^2-1]c_n=0,\quad n=1,2,\ldots \qquad \ldots(12.91)$$

as the recurrence relation.

For $r = 1/2$, (12.91) becomes $4n(n + 1)c_n = 0$, which gives, $c_n = 0$, for all n except $n = 0$, or -1.

Since $n = -1$ is not admissible, therefore one Frobenius solution from (12.89) is

$$y_1(x)=c_0x^{1/2}. \qquad \ldots(12.92)$$

For $r = -1/2$, (12.91) becomes $4n(n - 1)c_n = 0$, which gives, $c_n = 0$, for all n, except $n = 0$, or 1, therefore, the second Frobenius solution from (12.89) is

$$y_2(x)=c_0x^{-\frac{1}{2}}+c_1x^{\frac{1}{2}}. \qquad \ldots(12.93)$$

We note that the solution (12.92) is contained in (12.93) as its part and hence the general solution is given by (12.93) itself, where c_0 and c_1 are two arbitrary constants.

Example 12.19: Solve $x^2y'' + x(x+2)y' - 2y = 0$ by the Frobenius method.

Solution: It is easy to check that $x = 0$ is a regular singular point of the given equation. Let the series solution be

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad c_0 \neq 0.$$

Substituting in the given equation, we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r+1} + 2\sum_{n=0}^{\infty} (n+r)c_n x^{n+r} - 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Replacing $(n+1)$ by n in the second summation on the left, we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1)c_{n-1}x^{n+r} + 2\sum_{n=0}^{\infty} (n+r)c_n x^{n+r} - 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0. \quad ...(12.94)$$

Comparing the coefficient of x^r on the left side of Eq. (12.94) to zero, we obtain $(r^2 + r - 2)c_0 = 0$.

Since $c_0 \neq 0$, it gives $r^2 + r - 2 = 0$ as the indicial equation with $r_1 = 1$ and $r_2 = -2$ as its two roots. Here $r_1 - r_2 = 3$, an integer.

Equating the coefficient of x^{n+r} on the left side of Eq. (12.94) to zero, the recurrence relation obtained is

$$c_n = \frac{(n+r-1)}{[2-(n+r)(n+r+1)]}c_{n-1}, \quad n = 1, 2, ... \quad ...(12.95)$$

For $r = -2$, (12.95) becomes $c_n = \dfrac{(n-3)}{2-(n-2)(n-1)}c_{n-1} = -\dfrac{1}{n}c_{n-1}$.

Hence, $c_1 = -c_0$, $c_2 = -\dfrac{1}{2}c_1 = \dfrac{1}{2!}c_0$, $c_3 = -\dfrac{1}{3}c_2 = -\dfrac{1}{3!}c_0$, $c_4 = -\dfrac{1}{4}c_3 = -\dfrac{1}{4!}c_0$ etc.

Thus one Frobenius solution is, $y_1(x) = c_0 x^{-2}\left[1 - x + \dfrac{1}{2!}x^2 - \dfrac{1}{3!}x^3 + \dfrac{1}{4!}x^4 - ...\right]$

or, $$y_1(x) = c_0\left[\frac{1}{x^2} - \frac{1}{x} + \frac{1}{2!}\right] - c_0\left[\frac{1}{3!}x - \frac{1}{4!}x^2 + \frac{1}{5!}x^3 + ...\right]. \quad ...(12.96)$$

For $r = 1$, (12.95) becomes $c_n = \dfrac{n}{[2-(n+1)(n+2)]} = -\dfrac{1}{n+3}c_{n-1}$. This gives

$$c_1 = -\frac{c_0}{4}, \quad c_2 = \frac{c_0}{4.5}, \quad c_3 = -\frac{c_0}{4.5.6}, \quad c_4 = \frac{c_0}{4.5.6.7}, \text{ etc.}$$

Thus the second solution is

$$y_2(x) = c_0 x\left[1 - \frac{1}{4}x + \frac{1}{4.5}x^2 - \frac{1}{4.5.6}x^3 + \frac{1}{4.5.6.7}x^4 - \ldots\right] = 3!c_0\left[\frac{1}{3!}x - \frac{1}{4!}x^2 + \frac{1}{5!}x^3 \ldots\right] \quad \ldots(12.97)$$

It is clear from (12.96) and (12.97) that the solution $y_1(x)$ includes the solution $y_2(x)$.

The general solution can be given as

$$y(x) = A\left[\frac{1}{x^2} - \frac{1}{x} + \frac{1}{2}\right] + Bx\left[\frac{1}{3!} - \frac{x}{4!} + \frac{x^2}{5!} - \ldots\right],$$

where A and B are two arbitrary constants.

Example 12.20: Solve the equation $x(1 + x)y'' + 3xy' + y = 0$ about $x = 0$ by Frobenius method.

Solution: Obviously $x = 0$ is a regular singular point of the given equation. Let the series solution be $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$, $c_0 \neq 0$.

Substituting in the given equation, we obtain

$$\sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r} + 3\sum_{n=0}^{\infty}(n + r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Collecting the terms of the least power of x, that is, x^{r-1}; replacing $n - 1$ with n in the first summation on left hand side and then regrouping the terms of x^{n+r}, we obtain

$$r(r - 1)c_0 x^{r-1} + \sum_{n=0}^{\infty}[(n + r + 1)(n + r)c_{n+1} + \{(n + r)(n + r + 2) + 1\}c_n]x^{n+r} = 0. \quad \ldots(12.98)$$

Equating the coefficient of x^{r-1} on the left hand side of Eq. (12.98) to zero, and since $c_0 \neq 0$, the indicial equation is $r(r - 1) = 0$, which gives, $r = 0, 1$. Here $r_1 - r_2$ equals to an integer.

Equating the coefficient of x^{n+r} on the left hand side of Eq. (12.98) to zero, the recurrence relation obtained is

$$c_{n+1} = -\frac{[(n + r)(n + r + 2) + 1]}{(n + r + 1)(n + r)}c_n = -\frac{(n + r + 1)^2}{(n + r + 1)(n + r)}c_n = -\frac{(n + r + 1)}{n + r}c_n.$$

This gives for $n = 0, 1, 2, \ldots$

$$c_1 = -\frac{r+1}{r}c_0, \quad c_2 = -\frac{r+2}{r+1}c_1 = \frac{r+2}{r}c_0, \quad c_3 = -\frac{r+3}{r+2}c_2 = -\frac{r+3}{r}c_0 \quad \text{etc.}$$

The series solution can be expressed as

$$y(x) = c_0 x^r \left[1 - \frac{r+1}{r}x + \frac{r+2}{r}x^2 - \frac{r+3}{r}x^3 + \ldots\right]. \quad \ldots(12.99)$$

We observe that the coefficients c_1 onward in (12.99) become infinite when $r = 0$. We modify this by putting $c_0 = a_0(r-0) = a_0 r$, $a_0 \neq 0$, so that (12.99) becomes

$$y(x) = a_0 x^r [r - (r+1)x + (r+2)x^2 - (r+3)x^3 + \ldots]. \quad \ldots(12.100)$$

For $r = 0$, it gives $y_1(x) = a_0\,[-x + 2x^2 - 3x^3 + \ldots] = -a_0\,x(1-x)^{-2}$, as one Frobenius solution.

The second Frobenius solution is obtained from (12.100) as

$$y_2(x) = \left[\frac{\partial y}{\partial r}\right]_{r=0} = [(\ln x)y(x) + a_0 x^r\{1 - x + x^2 - x^3 + \ldots\}]_{r=0}$$

$$= (\ln x)y_1(x) + \frac{a_0}{1+x} = -a_0 x(\ln x)(1-x)^{-2} + \frac{a_0}{1+x}.$$

The general solution is $y(x) = Ay_1(x) + By_2(x)$, where A and B are two arbitrary constants.

We can verify that solution corresponding to the larger root $r = 1$ is linearly dependent on $y_1(x)$.

We know that if a series solution is derived about an analytic point of a differential equation, then it is a power series. Now we attempt Frobenius method to find a series solution about an analytic point. We expect the roots of the indicial equation to be non-negative integer, since the Frobenius method should also lead to the same power series.

We illustrate this by the following example.

Example 12.21: Solve the differential equation $y'' + xy = 0$ in series about $x = 0$ by Frobenius method.

Solution: Here $x = 0$ is an analytic point for the given equation.

Let the solution be $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$.

Substituting in the given equation, we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0.$$

Rewriting this as

$$r(r-1)c_0 x^{r-2} + (r+1)rc_1 x^{r-1} + (r+2)(r+1)c_2 x^r + \sum_{n=3}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0.$$

Replacing $(n-2)$ by n in the first summation and replacing $(n+1)$ by n in the second summation on the left side and regrouping the terms of x^{n+r}, we obtain

$$r(r-1)c_0x^{r-2} + (r+1)rc_1x^{r-1} + (r+2)(r+1)c_2x^r$$
$$+ \sum_{n=1}^{\infty} [(n+r+2)(n+r+1)c_{n+2} + c_{n-1}]x^{n+r} = 0. \qquad ...(12.101)$$

Equating the coefficient of the least power of x, that is, x^{r-2} in (12.101) to zero gives $r(r-1) = 0$, since $c_0 \neq 0$. Thus indicial roots are $r = 0, 1$.

Next, equating the coefficient of x^{r-1} in (12.101) to zero gives $(r+1)rc_1 = 0$.

It gives $c_1 = 0$, for $r = 1$, and c_1 as arbitrary constant for $r = 0$.

Next, equating to zero the coefficient of x^r gives $(r+1)(r+2)c_2 = 0$, which implies $c_2 = 0$, for $r = 0$, 1 both.

Equating to zero the coefficient of x^{n+r} in (12.101) the recurrence relation is

$$c_{n+2} = -\frac{c_{n-1}}{(n+r+2)(n+r+1)}, \quad n = 1, 2, ...$$

For $r = 0$, it becomes, $c_{n+2} = -\dfrac{c_{n-1}}{(n+1)(n+2)}, \quad n = 1, 2, ...$ Thus

$$c_3 = -\frac{c_0}{2.3}, \qquad c_4 = -\frac{c_1}{3.4} \qquad c_5 = -\frac{c_2}{4.5} = 0,$$

$$c_6 = -\frac{c_3}{5.6} = \frac{c_0}{2.3.5.6} \qquad c_7 = -\frac{c_4}{6.7} = \frac{c_1}{3.4.6.7}, \qquad c_8 = 0, \quad \text{etc.}$$

Hence one Frobenius solution corresponding to $r = 0$ is

$$y_1(x) = \left[c_0 + c_1x - \frac{c_0}{2.3}x^3 - \frac{c_1}{3.4}x^4 + \frac{c_0}{2.3.5.6}x^6 + \frac{c_1}{3.4.6.7}x^7 + ...\right]$$
$$= c_0\left[1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - ...\right] + c_1x\left[1 - \frac{1}{12}x^3 + \frac{1}{504}x^6 - ...\right],$$

where c_0 and c_1 are two arbitrary constants.

Since it involves two arbitrary constants this can be considered as the complete solution of the given differential equation.

We can check very easily that the solution corresponding to $r = 1$ is linearly dependent on $y_1(x)$. Also we observe that the general solution is a power series as expected, since $x = 0$ is a regular point of the given differential equation.

EXERCISE 12.3

Solve the following differential equations in power series about $x = 0$ by Frobenius method.

1. $xy'' + y' + xy = 0$
2. $x^2y'' + x(½ + 2x)y' + (x - ½)y = 0$

3. $x^2y'' - 4xy' + 20y = 0$
4. $x(x+1)y'' + (3x+1)y' + y = 0$
5. $2x^2y'' + xy' - (x^2+1)y = 0$
6. $y'' - xy' + y = 0$
7. $x^2y'' + xy' + (x^2-1)y = 0$
8. $x^2y'' + x(x^2-1)y' + y = 0$
9. $y'' + x^2y = 0$
10. $(4+x^2)y'' - 6xy' + 8y = 0$

12.5 BESSEL EQUATION. BESSEL FUNCTION OF THE FIRST KIND

The differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad \text{...(12.102)}$$

is called Bessel equation of order ν, $\nu \geq 0$. It is one of the most important differential equations in applied mathematics and appears in connection with electric fields, vibrations, heat conduction, etc. when the problem shows cylindrical symmetry. We must note that though it is a second order differential equation, yet it is described of order ν, referring to its parameter. The particular solutions of this equation are called *Bessel functions of order* ν. The Bessel functions come under the class of *special functions*, since they are different from the standard functions like sine, cosine, logarithmic functions, etc.

Another form of Bessel equation that often arises in applications is

$$x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0. \quad \text{...(12.103)}$$

This equation transforms to (12.102) by substitution $u = \lambda x$ and then again replacing u with x.

12.5.1 Solution of the Bessel Equation

Obviously origin is a regular singular point of the Bessel Eq. (12.102), so we attempt a solution of the form $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$, $c_0 \neq 0$.

Substituting for $y(x)$, $y'(x)$, and $y''(x)$ in (12.102), we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r} + (x^2 - \nu^2) \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

or, $$[r(r-1) + r - \nu^2]c_0x^r + [(r+1)r + (r+1) - \nu^2]c_1x^{r+1} + \sum_{n=2}^{\infty} [\{(n+r)^2 - \nu^2\}c_n + c_{n-2}]x^{n+r} = 0 \quad \text{...(12.104)}$$

Equating the coefficient of the lowest power of x, that is, x^r in (12.104) to zero, the indicial equation, since $c_0 \neq 0$, is $r^2 - \nu^2 = 0$. This gives $r = \pm\nu$ as the indicial roots.

Equating the coefficient of the next higher power of x, that is, x^{r+1} in (12.104) to zero, we obtain

$$[(r+1)^2 - \nu^2]c_1 = 0. \quad \text{...(12.105)}$$

Setting $r = \nu$ in (12.105), we have $(2\nu+1)c_1 = 0$, which gives $c_1 = 0$, since $(2\nu+1) \neq 0$ for $\nu \geq 0$.

Equating the coefficient of x^{n+r} in (12.104) to zero, the recurrence relation is

$$c_n = -\frac{c_{n-2}}{[(n+r)^2 - \nu^2]}, \quad n = 2, 3, \dots \qquad \text{...(12.106)}$$

For $r = \nu$, (12.106) reduces to

$$c_n = -\frac{c_{n-2}}{n(n+2\nu)}, \quad n = 2, 3, \dots \qquad \text{...(12.107)}$$

Since $c_1 = 0$, (12.107) yields, $c_3 = c_5 = \dots 0$, that is, $c_n = 0$ for odd n.
For even n, (12.107) yields

$$\begin{aligned} c_{2n} &= \frac{-1}{2n(2n+2\nu)} c_{2n-2} = \frac{-1}{2^2 n(n+\nu)} c_{2n-2} \\ &= \frac{-1}{2^2 n(n+\nu)} \frac{-1}{2(n-1)[2(n-1)+2\nu]} c_{2n-4} \\ &= \frac{(-1)^2}{2^4 n(n-1)(n+\nu)(n+\nu-1)} c_{2n-4} \\ &\dots \\ &= \frac{(-1)^n}{2^{2n} n!(1+\nu)(2+\nu)\dots(n+\nu)} c_0. \end{aligned} \qquad \text{...(12.108)}$$

Thus one of the two linearly independent solutions of the Bessel Eq. (12.102) is

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!(1+\nu)(2+\nu)\dots(n+\nu)} x^{2n+\nu}. \qquad \text{...(12.109)}$$

12.5.2 Bessel Function of the First Kind

For a suitable choice of c_0, the expression on the right side of (12.109) defines *Bessel function of the first kind* denoted by $J_\nu(x)$. The value of c_0 is chosen as

$$c_0 = 1/[2^\nu \Gamma(\nu+1)], \qquad \text{...(12.110)}$$

where Γ is the gamma function defined by $\Gamma(\nu+1) = \int_0^\infty e^{-x} x^\nu dx$, refer to Section 6.9.

Substituting the value of c_0 from (12.110) in (12.108), and using the property of gamma function we obtain

$$c_{2n} = \frac{(-1)^n}{2^{2n+\nu}(n!)\,\Gamma(\nu+1)\,[(1+\nu)(2+\nu)\dots(n+\nu)]} = \frac{(-1)^n}{2^{2n+\nu}(n!)\,\Gamma(n+\nu+1)}. \qquad \text{...(12.111)}$$

Therefore, the *Bessel's function of the first kind of order* ν is

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu} \quad ...(12.112)$$

$$= \left(\frac{x}{2}\right)^{\nu}\left[\frac{1}{\Gamma(\nu+1)} - \frac{1}{1!\Gamma(\nu+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\Gamma(\nu+3)}\left(\frac{x}{2}\right)^4 - \dots\right]$$

The test for the convergence of infinite series shows that this series converges for all x.

Since, Bessel equation is of second order differential equation, thus we need another linearly independent solution to write the complete solution. The indicial roots of the Bessel's equation are $r = \pm\nu$.

We can check when $J_{-\nu}(x)$ gives us the second desired solution. We have

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)}\left(\frac{x}{2}\right)^{2n-\nu} \quad ...(12.113)$$

$$= \left(\frac{x}{2}\right)^{-\nu}\left[\frac{1}{\Gamma(1-\nu)} - \frac{1}{1!\Gamma(2-\nu)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(3-\nu)}\left(\frac{x}{2}\right)^3 \dots\right]$$

called the *Bessel function of the first kind of order* $-\nu$.

The leading terms in $J_\nu(x)$ and $J_{-\nu}(x)$ contain x^ν and $x^{-\nu}$ respectively, and hence, *if* ν *is not an integer, then* J_ν *and* $J_{-\nu}$ *are linearly independent*, (neither is a constant multiple of the other), and thus the general solution of the Bessel equation is

$$y(x) = A\,J_\nu(x) + B\,J_{-\nu}(x), \quad ...(12.114)$$

where A and B are arbitrary constants.

12.5.3 Bessel Function $J_\nu(x)$, ν Being an Integer

Let $\nu = k$, then $\Gamma(n+\nu+1) = \Gamma(n+k+1) = (n+k)!$, and thus, expression (12.111) for c_{2n} is simplified

to $c_{2n} = \dfrac{(-1)^n}{2^{2n+k}(n!)(n+k)!}$, and hence

$$J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)(n+k)!}\left(\frac{x}{2}\right)^{2n+k} \quad ...(12.115)$$

for integral value of k. The series (12.115) converges rapidly for all x.

Consider $$J_{-k}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)(n-k)!}\left(\frac{x}{2}\right)^{2n-k} = \sum_{n=k}^{\infty} \frac{(-1)^n}{(n!)(n-k)!}\left(\frac{x}{2}\right)^{2n-k}.$$

Replace n with $(n+k)$, we get

$$J_{-k}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+k}}{(n+k)!\,n!}\left(\frac{x}{2}\right)^{2n+k} = (-1)^k J_k(x) \qquad \text{...(12.115a)}$$

for integral value of k.

Thus for $v = k$, an integer, $J_v(x)$ and $J_{-v}(x)$ are linearly dependent and hence to find the general solution in this case we need another independent solution. We will come to this problem in Section 12.6.

Bessel functions of order zero and order one. For $k = 0$ and 1 we obtain from (12.115)

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \dots \qquad \text{...(12.116)}$$

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1} = \frac{x}{2} - \frac{x^3}{2^3(1!)(2!)} + \frac{x^5}{2^5(2!)(3!)} - \dots \qquad \text{...(12.117)}$$

We find that $J_0(0) = 1$ and $J_1(0) = 0$.

The presence of alternate positive and negative terms in $J_0(x)$ and $J_1(x)$ ensure that their graphs, as shown in Fig. 12.2, must oscillate and decay fast as $x \to \infty$.

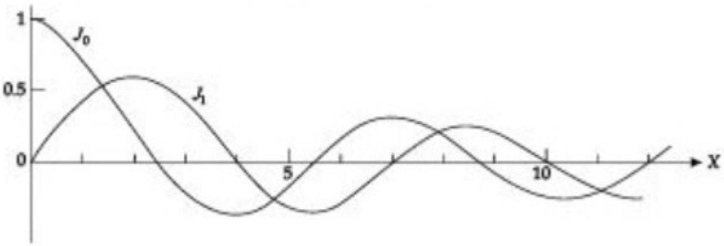


Fig. 12.2

The graph of $J_0(x)$ closely resembles to that of $\cos x$ and of $J_1(x)$ resembles to $\sin x$. The first five zeros of $J_0(x)$ and $J_1(x)$ are given as

$J_0(x)$: 2.405, 5.520, 8.653, 11.792, 14.931

$J_1(x)$: 3.832, 7.016, 10.174, 13.324, 16.471

Though the successive zeros of these functions are not completely regularly spaced, but approximately differ by $\pi = 3.141$. Also as indicated from the graphs, the height of the 'waves' decreases with increasing x.

12.5.4 Relationships Between Derivatives of $J_v(x)$

We have the following results:

(a) $$\frac{d}{dx}[x^v J_v(x)] = x^v J_{v-1}(x) \qquad \text{...(12.118)}$$

(b) $\dfrac{d}{dx}[x^{-\nu}J_\nu(x)] = -x^{-\nu}J_{\nu+1}(x)$ ...(12.119)

To prove (a), multiplying the Bessel function given by (12.112) by x^ν, we have

$$x^\nu J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2\nu}}{2^{2n+\nu} n!\,\Gamma(n+\nu+1)}$$

Differentiating it w.r.t. x and using $\Gamma(\nu + n + 1) = (\nu + n)\,\Gamma(\nu + n)$, we get

$$[x^\nu J_\nu(x)]' = \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+\nu)\, x^{2n+2\nu-1}}{2^{2n+\nu} n!\,(n+\nu)\Gamma(n+\nu)} = x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu)}\left(\frac{x}{2}\right)^{2n+\nu-1} = x^\nu J_{\nu-1}(x).$$

In particular for $\nu = 0$, we have

$$J'_0(x) = J_{-1}(x) = -J_1(x), \quad ...(12.120)$$

since $J_{-k}(x) = (-1)^k J_k(x)$, for integral k.

To prove (b), multiply the Bessel function given by (12.112) by $x^{-\nu}$ and differentiate, we get

$$[x^{-\nu} J_\nu(x)]' = \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{n!\,2^{2n+\nu}\,\Gamma(n+\nu+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(n-1)!\,2^{2n+\nu-1}\,\Gamma(n+\nu+1)}$$

Replacing $(n-1)$ by n and resetting the index, we obtain

$$[x^{-\nu} J_\nu(x)]' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{n!\,2^{2n+\nu+1}\,\Gamma(n+\nu+2)} = -x^{-\nu}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+2)}\left(\frac{x}{2}\right)^{2n+\nu+1} = -x^{-\nu}J_{\nu+1}(x).$$

The following two results involving *integrals of Bessel's functions* follow respectively from (12.118) and (12.119).

$$\int x^\nu J_{\nu-1}(x)\,dx = x^\nu J_\nu(x) + c_1 \quad ...(12.121)$$

and, $$\int x^{-\nu} J_{\nu+1}(x)\,dx = -x^{-\nu}J_\nu(x) + c_2 \quad ...(12.122)$$

where c_1 and c_2 are two arbitrary constants.

Recurrence relations involving $J_\nu(x)$. We prove the following relations

(a) $J_{\nu-1}(x) + J_{\nu+1}(x) = \dfrac{2\nu}{x} J_\nu(x)$ ...(12.123)

(b) $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)$ ...(12.124)

To prove (a), from (12.118), we have

$$\nu x^{\nu-1} J_\nu(x) + x^\nu J'_\nu(x) = x^\nu J_{\nu-1}(x) \quad ...(12.125)$$

Similarly from (12.119), we obtain

$$-\nu x^{-\nu-1} J_\nu(x) + x^{-\nu} J_\nu'(x) = -x^{-\nu} J_{\nu+1}(x)$$

Multiplying both sides of this by $x^{2\nu}$, we obtain

$$-\nu x^{\nu-1} J_\nu(x) + x^\nu J_\nu'(x) = -x^\nu J_{\nu+1}(x) \qquad ...(12.126)$$

Subtracting (12.126) from (12.125) and dividing by x^ν, we obtain

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x).$$

To prove **(b)**, add (12.125) and (12.126) and divide by x^ν, we obtain

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x).$$

12.5.5 Bessel Functions $J_\nu(x)$ of order $\nu = \pm 1/2, \pm 3/2$, etc.

The half-integer order values for $\nu = \pm 1/2, \pm 3/2, \ldots$ of the Bessel's function represent elementary functions. For example, for $\nu = 1/2$ form (12.112)

$$J_{1/2}(x) = x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+\frac{1}{2}}\, n!\, \Gamma\left(n+\frac{3}{2}\right)} = \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}\, n!\, \Gamma\left(n+\frac{3}{2}\right)}$$

Also, $$\Gamma\left(n+\frac{3}{2}\right) = \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\cdots\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{(2n+1)!}{2^{2n+1} n!}\sqrt{\pi}.$$

Thus, $$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sqrt{\frac{2}{\pi x}} \sin x \qquad ...(12.127)$$

is expressed in the form of simple sine function.

Also from (12.118), for $\nu = \frac{1}{2}$, we have $\frac{d}{dx}[\sqrt{x} J_{1/2}(x)] = \sqrt{x} J_{-1/2}(x)$. Hence,

$$J_{-1/2}(x) = \frac{1}{\sqrt{x}} \frac{d}{dx}[\sqrt{x} J_{1/2}(x)] = \sqrt{\frac{2}{\pi x}} \cos x, \text{ using (12.127)}. \qquad ...(12.128)$$

The graphs of $J_{1/2}(x)$ and $J_{-1/2}(x)$ are shown in Fig. 12.3.

Next, from (12.123) for $\nu = \frac{1}{2}$, we have

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) \qquad ...(12.129)$$

and for $\nu = -1/2$, (12.123) gives

$$J_{-3/2}(x) = -J_{1/2}(x) - \frac{1}{x} J_{-1/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right) \qquad ...(12.130)$$

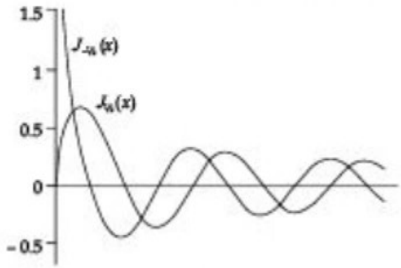


Fig. 12.3

Similarly for $v = 3/2$ from (12.123), we have

$$J_{5/2}(x) = -J_{½}(x) + \frac{3}{x} J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[-\sin x + \frac{3}{x^2}\sin x - \frac{3}{x}\cos x\right] \quad ...(12.131)$$

The process can be continued indefinitely and it is a better way to generate Bessel functions $J_v(x)$ for $v = \pm k/2$, than by referring back to $J_v(x)$ each time. These results are quite useful for investigating the various properties of Bessel functions.

12.5.6 Zeros of $J_v(x)$

In some of the applications, sometimes we need to know the zeros of $J_v(x)$, that is, solutions of the equation $J_v(x) = 0$. It can be shown that *$J_v(x)$ has infinitely many simple zeros, and in general, if α is any sufficiently large number and $k > 1$, then $J_v(x)$ has a simple zero between α and $\alpha + k\pi$. Thus positive zeros of $J_v(x)$ can be ordered in an increasing sequence.* Also we mention the following results concerning the zeros of $J_v(x)$.

1. Let v be a real number, then, except possibly for $x = 0$, J_v has no zero in common with either $J_{v-1}(x)$ and $J_{v+1}(x)$, that is, $J_v(x)$ cannot share a zero with $J_{v-1}(x)$, or $J_{v+1}(x)$.

2. If v be any real number and a and b are distinct positive zeros of $J_v(x)$, then $J_{v-1}(x)$ and $J_{v+1}(x)$ each have at least one zero between a and b. This property is called the interlacing property of successively indexed Bessel functions.

The interlacing property of functions $J_0(x)$, $J_1(x)$ and $J_2(x)$ is evident from the following table giving the first five positive zeros of $J_v(x)$ for $v = 0, 1, 2$ and their graphs in Fig. 12.4.

$J_0(x)$	:	2.405	5.520	8.654	11.792	14.931
$J_1(x)$	:	3.832	7.016	10.173	13.323	16.470
$J_2(x)$	:	5.135	8.417	11.620	14.796	17.960

It can be seen from Fig. 12.4 that Bessel functions $J_0(x)$, $J_1(x)$ and $J_2(x)$ are oscillatory in nature and resemble damped sinusoids. In fact, the oscillatory property is true for all $J_k(x)$, k being an integer, but Bessel functions are not strictly periodic.

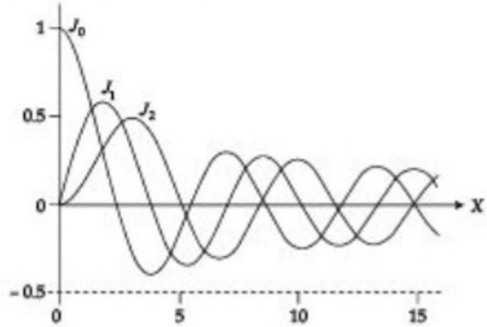


Fig. 12.4

12.5.7 Generating Function For $J_k(x)$, k Being an Integer

We prove that $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{k=-\infty}^{\infty} J_k(x)t^k$ ...(12.132)

the function on the left side is called the *generating function of the Bessel function* $J_k(x)$. Consider

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}}\, e^{-\frac{x}{2t}} = \left[1+\frac{xt}{2}+\frac{1}{2!}\left(\frac{xt}{2}\right)^2+\ldots\right]\left[1-\frac{x}{2t}+\frac{1}{2!}\left(\frac{x}{2t}\right)^2-\ldots\right] \quad \text{...(12.133)}$$

The coefficient of t^k on the right side of Eq. (12.133) is

$$\frac{1}{k!}\left(\frac{x}{2}\right)^k - \frac{1}{(k+1)!}\left(\frac{x}{2}\right)^{k+2} + \frac{1}{2!\,(k+2)!}\left(\frac{x}{2}\right)^{k+4} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(k+n)!}\left(\frac{x}{2}\right)^{2n+k} = J_k(x).$$

Also the coefficient of $(-t)^{-k}$ in (12.133) is the same as above, that is, $J_k(x)$, therefore, the coefficient of t^{-k} is $(-1)^k J_k = J_{-k}(x)$. Hence,

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + J_1(x)t + J_2(x)t^2 + \ldots + J_{-1}(x)t^{-1} + J_{-2}(x)t^{-2} + \ldots = \sum_{k=-\infty}^{\infty} J_k(x)t^k.$$

12.5.8 An Integral Formula for $J_k(x)$

Using the generating function we can derive an integral formula for $J_k(x)$ when k is a non-negative integer. We show that

$$J_k(x) = \frac{1}{\pi}\int_0^{\pi} \cos(k\theta - x\sin\theta)\,d\theta. \quad \text{...(12.134)}$$

To prove it, consider

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{k=-\infty}^{\infty} J_k(x)t^k = \sum_{k=-\infty}^{-1} J_k(x)t^k + J_0(x) + \sum_{k=1}^{\infty} J_k(x)t^k$$

$$= \sum_{k=1}^{\infty} (-1)^k J_k(x)t^{-k} + J_0(x) + \sum_{k=1}^{\infty} J_k(x)t^k = J_0(x) + \sum_{k=1}^{\infty} J_k(x)\left[t^k + \frac{(-1)^k}{t^k}\right]$$

$$= J_0(x) + \sum_{k=1}^{\infty} J_{2k}(x)\left(t^{2k} + \frac{1}{t^{2k}}\right) + \sum_{k=1}^{\infty} J_{2k-1}(x)\left(t^{2k-1} - \frac{1}{t^{2k-1}}\right) \qquad \text{...(12.135)}$$

Let $t = e^{i\theta} = \cos\theta + i\sin\theta$, then

$$t^{2k} + \frac{1}{t^{2k}} = 2\cos(2k\theta), \quad t^{2k-1} - \frac{1}{t^{2k-1}} = 2i\sin(2k-1)\theta, \quad \text{and} \quad \frac{1}{2}\left(t - \frac{1}{t}\right) = i\sin\theta.$$

Thus, (12.135) becomes

$$e^{i(x\sin\theta)} = \cos(x\sin\theta) + i\sin(x\sin\theta)$$

$$= J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos(2k\theta) + 2i\sum_{k=1}^{\infty} J_{2k-1}(x)\sin(2k-1)\theta.$$

Equating the real and imaginary parts, we obtain

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos(2k\theta) \qquad \text{...(12.136)}$$

and, $\sin(x\sin\theta) = 2\sum_{k=1}^{\infty} J_{2k-1}(x)\sin(2k-1)\theta.$ ...(12.137)

Multiplying both sides of (12.136) by $\cos k\theta$ and of (12.137) by $\sin k\theta$, integrating each of the resulting expressions from 0 to π, we obtain respectively

$$\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos k\theta\, d\theta = \begin{cases} J_k(x), & \text{for k even or zero} \\ 0, & \text{for } k \text{ odd} \end{cases}$$

and, $$\frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin k\theta\, d\theta = \begin{cases} 0, & \text{for } k \text{ even} \\ J_k(x), & \text{for } k \text{ odd.} \end{cases}$$

Hence, if k is a non-negative integer, then

$$J_k(x) = \frac{1}{\pi}\int_0^{\pi} [\cos(x\sin\theta)\cos k\theta + \sin(x\sin\theta)\sin k\theta]d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(k\theta - x\sin\theta)d\theta.$$

This proves (12.134).

Example 12.22: Express $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Solution: From (12.123), we have $J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x) - J_{\nu-1}(x)$. Setting $\nu = 3, 2, 1$ to obtain

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x),\quad J_3(x) = \frac{4}{x}J_2(x) - J_1(x),\quad J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$$

Eliminating $J_3(x)$ and $J_2(x)$ gives the required expression

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x).$$

Similarly we can express $J_k(x)$, $k > 1$, being positive integer, in terms of $J_0(x)$ and $J_1(x)$.

Example 12.23: Evaluate the following

(a) $\int\left(x^2 + \frac{1}{x}\right)J_1(x)dx$ (b) $\int x^3 J_0(x)\,dx$

Solution: (a) We have $\int\left(x^2 + \frac{1}{x}\right)J_1(x)dx = \int x^2 J_1(x)\,dx + \int \frac{1}{x}J_1(x)dx.$

For first integral on the right; set $\nu = 2$ in (12.121), we have

$\int x^2 J_1(x)\,dx = x^2 J_2(x) + c_1.$

The second integral on the right is

$$\int x^{-1}J_1(x)dx = \int x^{-2}[xJ_1(x)]dx = x[J_1(x)]\left(-\frac{1}{x}\right) - \int [xJ_1(x)]'\left(-\frac{1}{x}\right)dx + c_2$$

$$= -J_1(x) + \int J_0(x)dx + c_2,\quad \text{using (12.118) for } v = 1.$$

Hence, $\int\left(x^2 + \frac{1}{x}\right)J_1(x)dx = x^2 J_2(x) - J_1(x) + \int J_0(x)dx + c$, where c is an arbitrary constant.

(b)
$$\int x^3 J_0(x)dx = \int x^2[x\,J_0(x)]dx = \int x^2\frac{d}{dx}[xJ_1(x)]dx$$

$$= x^3 J_1(x) - 2\int x^2 J_1(x)dx = x^3 J_1(x) - 2x^2 J_2(x) + c.$$

Example 12.24: Prove that $J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + ...) = 1$.

Solution: We have, $[J_k^2 + J_{k+1}^2]' = 2J_kJ_k' + 2J_{k+1}J_{k+1}'$ ...(12.138)

For $\nu = k$, from (12.126) we obtain $J'_k = \dfrac{k}{x}J_k - J_{k+\nu}$

and for $\nu = (k+1)$ from (12.125), we have $J'_{k+1} = J_k - \dfrac{k+1}{x}J_{k+1}$.

Substituting for J_k' and J'_{k+1} in (12.138), we obtain

$$[J_k^2 + J_{k+1}^2]' = 2\left[\frac{k}{x}J_k^2 - \frac{k+1}{x}J_{k+1}^2\right] \qquad ...(12.139)$$

Setting $k = 0, 1, 2, ...$ successively, from (12.139) we have

$$[J_0^2 + J_1^2]' = 2\left[0 - \frac{1}{x}J_1^2\right]$$

$$[J_1^2 + J_2^2]' = 2\left[\frac{1}{x}J_1^2 - \frac{2}{x}J_2^2\right]$$

$$[J_2^2 + J_3^2]' = 2\left[\frac{2}{x}J_2^2 - \frac{3}{x}J_3^2\right], \text{ and so on.}$$

Adding we obtain

$$[J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + ... \infty)]' = 0, \qquad ...(12.140)$$

since $J_k \to 0$ as $k \to \infty$.

Integrating (12.140) both sides, we obtain $J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + ...) = c$.

Using the fact that for $x = 0$, $J_0(x) = 1$ and $J_k(x) = 0$, for $k > 1$, we obtain $c = 1$, and hence the desired result

EXERCISE 12.4

1. Show that $J_\nu(x)$ is an even function when ν is even and odd function when ν is odd.
2. Express $J_6(x)$ in terms of $J_0(x)$ and $J_1(x)$.
3. Express $J_{7/2}(x)$ in terms of sine and cosine functions.
4. Show that
   (a) $J_{-1/2}(x) = J_{1/2}(x)\cot x$ (b) $J'_{1/2}(x)J_{-1/2}(x) - J'_{-1/2}(x)J_{1/2}(x) = 2/\pi x$
5. Show by use of recurrence relation that
   (a) $J_0''(x) = [J_2(x) - J_0(x)]/2$ (b) $J_1''(x) = J_1(x) - (1/x)J_2(x)$
6. Show that
   (a) $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$ (b) $4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$
7. $\int x^\nu J_{\nu-1}(x)dx = x^\nu J_\nu(x) + c$
8. $\int x^{-\nu} J_{\nu+1}(x)dx = -x^{-\nu} J_\nu(x) + c$

9. $\int J_{\nu+1}(x)dx = \int J_{\nu-1}(x)dx - 2J_\nu(x)$

10. Evaluate

(a) $\int J_3(x)dx$ (b) $\int J_5(x)dx.$

11. Show that

(a) $\int J_0(x)\cos x\, dx = xJ_0(x)\cos x + xJ_1(x)\sin x + c$

(b) $\int J_1(x)\sin x\, dx = xJ_1(x)\sin x + J_0(x)(x\cos x - \sin x) + c.$

12. Let $u = J_0(\alpha x)$ and $v = J_0(\beta x)$. Then prove that $xu'' + u' + \alpha^2 xu = 0,\quad xv'' + v' + \beta^2 xv = 0.$

13. Show that $\int_0^\infty e^{-ax} J_0(bx)dx = \dfrac{1}{\sqrt{a^2+b^2}}$.

14. Establish the Jacobi series

$\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \ldots$

$\sin(x\cos\theta) = 2[J_1\cos\theta - J_3\cos 3\theta + J_5\cos 5\theta - \ldots]$

15. Using the series in Problem 14, above, prove that $J_0^2 + 2\sum_{n=1}^{\infty} J_n^2 = 1.$

12.6 BESSEL FUNCTIONS OF THE SECOND KIND

It has been shown in the preceding section, refer to Eq. (12.115a), that the two Bessel functions $J_\nu(x)$ and $J_{-\nu}(x)$ of the first kind are linearly dependent when ν is an integer. Therefore to find the general solution of the Bessel's equation with parameter $\nu = n$, an integer, we need to obtain the second solution linearly independent of $J_n(x)$.

12.6.1 Bessel Equation of Order Zero

We begin by considering the case $n = 0$, that is, Bessel equation of order zero given by

$$xy'' + y' + xy = 0. \quad ...(12.141)$$

Let the series solution be $y = \sum_{n=0}^{\infty} c_n x^{n+r},\quad c_0 \neq 0.$

Substituting in (12.141) gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}+\sum_{n=0}^{\infty}c_n x^{n+r+1}=0.$$

Collecting the terms of x^{r-1} and x^r, and then making the index of each summation x^{n+r}, we have

$$r^2c_0x^{r-1}+(1+r)^2c_1x^r+\sum_{n=1}^{\infty}(n+r+1)(n+r)c_{n+1}x^{n+r}+\sum_{n=1}^{\infty}(n+r+1)c_{n+1}x^{n+r}+\sum_{n=1}^{\infty}c_{n-1}x^{n+r}=0.$$

...(12.142)

Comparing the coefficient of x^{r-1} in (12.142) to zero, gives $r^2=0$, since $c_0\neq 0$. This gives two equal roots of $r=0, 0$.

Comparing the coefficient of x^r in (12.142) to zero gives $(1+r^2)\,c_1=0$, which is valid for $r=0$ only if, $c_1=0$.

Comparing the coefficient of x^{n+r} in (12.142) to zero gives the recurrence relation

$$c_{n+1}=\frac{-c_{n-1}}{(n+r+1)^2},\ \text{for}\quad n=1,2,\ldots \qquad ...(12.143)$$

Since $c_1=0$, thus (12.143) gives, $c_3=c_5=c_7=\ldots=0.$

Further $\quad c_2=-\dfrac{c_0}{(r+2)^2},\quad c_4=-\dfrac{c_2}{(r+4)^2}=\dfrac{c_0}{(r+2)^2(r+4)^2},\quad$ etc.

Hence, one Frobenius solution is

$$y(x)=c_0x^r\left[1-\frac{x^2}{(r+2)^2}+\frac{x^4}{(r+2)^2(r+4)^2}-\frac{x^6}{(r+2)^2(r+4)^2(r+6)^2}+\ldots\right],$$

where $r=0$. Taking $c_0=1$, it is

$$y_1(x)=\left[1-\frac{x^2}{2^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots\right]=\left[1-\frac{x^2}{2^2(1!)^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+\ldots\right]$$

$=J_0(x)$, refer to (12.116).

Since the two indicial roots are equal, the second linearly independent solution taking $c_0=1$, is given by

$$y_2(x)=\left.\frac{\partial y}{\partial r}\right|_{r=0}=J_0(x)\ln x+x^r\left[\frac{x^2}{(r+2)^2}\cdot\frac{2}{r+2}-\frac{x^4}{(r+2)^2(r+4)^2}\left(\frac{2}{r+2}+\frac{2}{r+4}\right)+\ldots\right]_{r=0}$$

$$=J_0(x)\ln x+\left[\frac{x^2}{2^2}-\frac{x^4}{2^2.4^2}\left(1+\frac{1}{2}\right)+\frac{x^6}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\ldots\right]$$

$$=J_0(x)\ln x+\left[\frac{x^2}{2^2(1!)^2}-\frac{x^4}{2^4(2!)^2}\left(1+\frac{1}{2}\right)+\frac{x^6}{2^6(3!)^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\ldots\right]$$

$$= J_0(x)\ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{2^{2n}(n!)^2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right). \quad \ldots(12.144)$$

The series in (12.144) can be shown to be convergent but the logarithmic term becomes infinite at the origin. Thus (12.144) is finite only for $x > 0$.

12.6.2 Bessel's Functions of Second Kind

Since J_0 and y_2 are linearly independent functions they form a basis of the Eq. (12.141). Another basis is obtained by replacing y_2 with an independent particular solution of the form $a(y_2 + bJ_0)$, where $a = 2/\pi$ and $b = \gamma - \ln 2$; the number γ is the Euler's constant defined as

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right) = 0.57721566490\ \ldots$$

The particular solution thus obtained, denoted by $Y_0(x)$, is given as

$$Y_0(x) = \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2)J_0\right]$$

$$= \frac{2}{\pi}\left[J_0(x)\ln x - \sum_{n=1}^{\infty} \frac{(-1)^n h_n}{2^{2n}(n!)^2} x^{2n} + (\gamma - \ln 2)J_0\right]$$

$$= \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma\right) - \sum_{n=1}^{\infty} \frac{(-1)^n h_n}{2^{2n}(n!)^2} x^{2n}\right], \quad \ldots(12.145)$$

where $h_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$.

The solution given by (12.145) is called the *Bessel's function of the second kind of order zero*. It is also called the *Neuman function of the second kind*.

The graphs of $J_0(x)$ and $Y_0(x)$ are shown in Fig. 12.5. The graphs look somewhat like damped cosine and sine functions except that $Y_0(x) \to -\infty$ as $x \to 0$, since $Y_0(x)$ contains the $\ln x$ term.

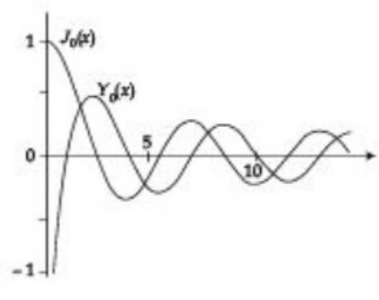


Fig. 12.5

Bessel functions of the second kind of integral order $\nu = n \neq 0$ can be defined in a similar fashion but to make them compatible with the functions $J_{-\nu}(x)$, defined in the preceding section, the following definition is considered:

$$\left.\begin{aligned} Y_\nu(x) &= \frac{1}{\sin \nu\pi}\left[J_\nu(x)\cos \nu\pi - J_{-\nu}(x)\right] \\ \text{and,}\quad Y_n(x) &= \lim_{\nu\to n} Y_\nu(x). \end{aligned}\right\} \quad \ldots(12.146a)$$

As $\nu \to n$, an integer on the right side of (12.146a) is an indeterminate from 0/0 and can be evaluated. It is possible to show that $Y_n(x)$ works out to be

$$Y_n(x) = \frac{2}{\pi}\left[J_n(x)\left(\ln\frac{x}{2}+\gamma\right) - \sum_{k=0}^{\infty}\frac{(-1)^k(h_k+h_{k+n})}{2^{2k+n+1}(k!)(k+n)!}x^{2k+n}\right] - \frac{2}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{2^{2k-n+1}k!}x^{2k-n}, \quad \text{...(12.146b)}$$

where by definition $h_0 = 1$ and $h_k = \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}\right)$.

The expression (12.146b) for Bessel's function of second kind of order $\nu = n$ contains a logarithmic term and hence $J_n(x)$ and $Y_n(x)$ are two linearly independent solutions of the Bessel's equation.

Thus the general solution of Bessel's equation of positive integral order $\nu = n$ is therefore

$$y(x) = AJ_n(x) + BY_n(x).$$

The graphs of some Bessel functions of the second kind are shown in Fig. 12.6.

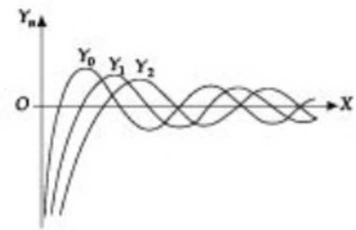


Fig. 12.6

12.6.3 A Closed Form Expression for the Bessel's Function of Second Kind of Order *n*.

A closed form expression for the Bessel's function of the second kind of order n can be obtained by solving the Bessel equation of order n by reduction of order method.

Let $y(x) = u(x)J_n(x)$ be another independent solution of the Bessel equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0, \quad \text{...(12.147)}$$

where n is an integer. Thus,

$$x^2(u''J_n + 2u'J_n' + uJ_n'') + x(u'J_n + uJ'_n) + (x^2 - n^2)uJ_n = 0,$$

or

$$[x^2J_n'' + xJ_n' + (x^2 - n^2)J_n]u + x^2J_nu'' + 2x^2J'_nu' + xJ_nu' = 0 \quad \text{...(12.148)}$$

Since J_n is a solution of the Bessel equation (12.147), thus $x^2J_n'' + xJ_n' + (x^2 - n^2)J_n = 0$, and hence Eq. (12.148) becomes $x^2J_nu'' + 2x^2J'_nu' + xJ_nu' = 0$.

Dividing it throughout by x^2J_nu', we obtain $\frac{u''}{u'} + 2\frac{J_n'}{J_n} + \frac{1}{x} = 0$, or, $\frac{d}{dx}(\ln u') + 2\frac{d}{dx}(\ln J_n) + \frac{d}{dx}(\ln x) = 0$, or $\frac{d}{dx}[\ln(u' J_n^2 x)] = 0$.

Integrating, it gives, $u' J_n^2 x = c_1$, a constant,

or, $u(x) = c_1 \int \frac{dx}{x[J_n(x)]^2} + c_2$.

Taking $c_1 = 1$ and $c_2 = 0$, we obtain

$$Y_n(x) = J_n(x) \int \frac{dx}{x[J_n(x)]^2} \qquad ...(12.149)$$

as a closed form expression for $Y_n(x)$.

12.7 MODIFIED BESSEL FUNCTIONS

In applications Bessel equation normally occurs in implicit form, requiring a suitable substitution to write the solution in terms of Bessel functions. This leads to *modified Bessel functions*. Consider the equation

$$y'' + \frac{1}{x}y' + k^2 y = 0, \; x > 0. \qquad ...(12.150)$$

This equation is not a Bessel equation because of the presence of the factor k^2. Choose $t = kx$, we have

$$\frac{d}{dx} = \frac{d}{dt}\cdot\frac{dt}{dx} = k\frac{d}{dt}, \text{ and } \frac{d^2}{dx^2} = k^2\frac{d^2}{dt^2}. \text{ Thus (12.150) becomes}$$

$$tk^2\frac{d^2y}{dt^2} + k^2\frac{dy}{dt} + k^2 ty = 0, \quad \text{or} \quad t\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty = 0$$

which is Bessel equation of order zero, with general solution

$$y(x) = AJ_0(t) + BY_0(t) = AJ_0(kx) + BY_0(kx).$$

Let $k = i$. Then $y(x) = AJ_0(ix) + BY_0(ix)$ is the general solution of the equation

$$y'' + \frac{1}{x}y' - y = 0, \quad x > 0. \qquad ...(12.151)$$

Equation (12.151) is a *modified Bessel equation of order zero and $J_0(ix)$ is a modified Bessel function of the first kind of order zero.* Usually it is denoted by $I_0(x)$. Thus using (12.116), we have

$$I_0(x) = J_0(ix) = 1 + \frac{1}{2^2}x^2 + \frac{1}{2^2 4^2}x^4 + \frac{1}{2^2 4^2 6^2}x^6 + ... \qquad ...(12.152)$$

Normally $Y_0(ix)$ is not used, however, the second solution in this case is chosen as

$$K_0(x) = -[\gamma + \ln(x/2)]\, I_0(x) + \frac{1}{2^2}x^2 + \frac{1}{2^2 4^2}x^4 + \ldots,\quad x > 0 \qquad \ldots(12.153)$$

where γ is Euler's constant, and hence the general solution of Eq. (12.151) is

$$y(x) = AI_0(x) + BK_0(x), \quad \text{for} \quad x > 0$$

Similarly, the general solution of the equation

$$y'' + \frac{1}{x}y' - b^2 y = 0 \qquad \ldots(12.154)$$

is
$$y(x) = AI_0(bx) + BK_0(bx), \quad x > 0.$$

12.7.1 Ber and Bei Functions

Next for $k^2 = -i$, Eq. (12.150) becomes

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - iy = 0. \qquad \ldots(12.155)$$

This equation occurs in certain problems of electrical engineering. A particular solution of Eq. (12.155) is

$$J_0(kx) = J_0[(-i)^{1/2}x] = J_0[i^{3/2}x] = 1 - \frac{i^3x^2}{2^2} + \frac{i^6x^4}{(2!)^2 2^4} - \frac{i^9x^6}{(3!)^2 2^6} + \frac{i^{12}x^8}{(4!)^2 2^8} \cdots \qquad \text{(using 12.116)}$$

$$= \left[1 - \frac{x^4}{2^2.4^2} + \frac{x^8}{2^2.4^2.6^2.8^2} - \ldots\right] + i\left[\frac{x^2}{2^2} - \frac{x^6}{2^2.4^2.6^2} + \frac{x^{10}}{2^2.4^2.6^2.8^2.10^2} - \ldots\right]$$

which is complex for real x.

The series in the real part is defined as *Bessel-real, or ber function* and the series in the imaginary part is defined as *Bessel-imaginary or bei* function. Thus,

$$\text{ber } x = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^2.4^2.6^2 \ldots (4n)^2}, \qquad \ldots(12.156)$$

and,
$$\text{bei } x = -\sum_{n=1}^{\infty} \frac{(-1)^n x^{4n-2}}{2^2.4^2.6^2 \ldots (4n-2)^2} \qquad \ldots(12.157)$$

so that
$$y = \text{ber } x + i \text{ bei } x \text{ is a solution of Eq. (12.155).}$$

Example 12.25: Solve the differential equation $x^2y'' + xy' + (k^2x^2 - \nu^2)y = 0$.

Solution: Put $t = kx$, we have $\frac{d}{dx} = \frac{d}{dt}.\frac{dt}{dx} = k\frac{d}{dt}$, and $\frac{d^2}{dx^2} = k^2\frac{d^2}{dt^2}$.

Substituting in the given equation, we obtain $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0$, which is Bessel equation of order v. Its solution is

$$y = \begin{cases} AJ_v(t) + BJ_{-v}(t), \text{ for non-integral } v \\ AJ_v(t) + BY_v(t), \text{ for integral } v. \end{cases}$$

Hence the solution of the given equation is

$$y = \begin{cases} AJ_v(kx) + BJ_{-v}(kx), \quad \text{for non-integral } v \\ AJ_v(kx) + BY_v(kx), \quad \text{for integral } v. \end{cases}$$

Example 12.26: Solve the differential equation $9x^2y'' - 27xy' + (9x^2 + 35)y = 0$.

Solution: Let $y = x^2u$. It gives $y' = 2xu + x^2u'$, and $y'' = 2u + 4xu' + x^2u''$

Substituting in the given differential equation and rearranging the terms, we obtain

$$9x^4u'' + 9x^3u' + (9x^4 - x^2)u = 0.$$

Dividing by $9x^2$, we get $x^2u'' + xu' + (x^2 - 1/9)u = 0$, which is Bessel equation of order $v = 1/3$. Its general solution is $u(x) = AJ_{1/3}(x) + BJ_{-1/3}(x)$.

Therefore the solution of the given differential equation is

$$y(x) = Ax^2J_{1/3}(x) + Bx^2J_{-1/3}(x), \quad x > 0.$$

Example 12.27: Solve the differential equation $xy'' + \alpha y' + k^2xy = 0$, and from the solution obtained find the general solution of $xy'' + 2y' + 3xy = 0$.

Solution: Put $y = x^v z$, so that $y' = x^v z' + vx^{v-1}z$, and $y'' = x^v z'' + 2vx^{v-1}z' + v(v-1)x^{v-2}z$. Substituting in the given equation we obtain

$$x^{v+1}z'' + (2v + \alpha)x^v z + [k^2x^2 + v^2 + (\alpha - 1)v]x^{v-1}z = 0.$$

Dividing by x^{v-1} and selecting v such that $2v + \alpha = 1$, it becomes

$$x^2z'' + xz' + (k^2x^2 - v^2)z = 0, \quad v = (1 - \alpha)/2.$$

which can be reduced to Bessel's differential equation of order v by substitution $t = kx$, refer to Example (12.25). So solution of the given equation using $y = x^v z$ is

$$y = \begin{cases} x^v[AJ_v(kx) + BJ_{-v}(kx)], \quad \text{for non-integral } v \\ x^v[AJ_v(kx) + BY_v(kx)], \quad \text{for integral } v \end{cases}$$

where $v = (1 - \alpha)/2$.

Next, in case of differential equation $xy'' + 2y' + 3xy = 0$, we have, $\alpha = 2$. Thus, $v = -1/2$, and $k^2 = 3$. Hence the general solution is

$$y = \frac{1}{\sqrt{x}}\left[AJ_{1/2}(\sqrt{3}x) + BJ_{-1/2}(\sqrt{3}x)\right],$$

A and B being two arbitrary constants.

12.8 ORTHOGONALITY OF BESSEL FUNCTIONS. FOURIER-BESSEL SERIES

In this section we prove the orthogonality of Bessel's function which is of somewhat different form when compared to Legendre polynomials. Using this orthogonality relationship we will go for Fourier-Bessel series.

12.8.1 Orthogonality Property of Bessel Functions

Let ν be any positive number. Then $J_\nu(x)$ has infinitely many positive zeros which can be arranged in an ascending sequence $j_1 < j_2 < j_3 < \ldots$. For each j_n, consider the function $\sqrt{x}\, J_\nu(j_n x)$ over $0 \le x \le 1$. These functions are orthogonal on $[0, 1]$ in the sense that the integral of the product of any two of these over the interval $[0, 1]$ is zero. In fact, we have the following result:

Theorem 12.3 (Orthogonality of Bessel Functions). *For $\nu \ge 0$ and $n = 1, 2, 3, \ldots$*

$$\int_0^1 x J_\nu(j_n x) J_\nu(j_m x)\,dx = \begin{cases} 0, \text{ if } n \ne m \\ \dfrac{1}{2}[J_{\nu+1}(j_n)]^2, n = m \end{cases} \qquad \ldots(12.158)$$

where j_n and j_m are the zeros of $J_\nu(x)$.

Proof. We know that $u(x) = J_\nu(j_n x)$ satisfies the Bessel equation

$$x^2 u'' + x u' + (j_n^2 x^2 - \nu^2) u = 0, \qquad \ldots(12.159)$$

and $v(x) = J_\nu(j_m x)$ satisfies the equation

$$x^2 v'' + x v' + (j_m^2 x^2 - \nu^2) v = 0. \qquad \ldots(12.160)$$

Multiply (12.159) by v and (12.160) by u and subtract the resulting equations to obtain

$$x^2(u''v - uv'') + x(u'v - uv') = (j_m^2 - j_n^2) x^2 uv.$$

Dividing by x, it gives $x(u''v - uv'') + (u'v - uv') = (j_m^2 - j_n^2) xuv$

or,
$$[x(u'v - uv')]' = (j_m^2 - j_n^2) xuv$$

Integrating it over $[0, 1]$, we have

$$\int_0^1 [x(u'v - uv')]'\,dx = \int_0^1 (j_m^2 - j_n^2) xuv\,dx \qquad \ldots(12.161)$$

$$\text{L.H.S of (12.161)} = \int_0^1 [x(u'v - uv')]'\,dx = \left[x(u'v - uv')\right]_0^1 = [u'v - uv']_{x=1}$$

Further, $u = J_\nu(j_n x)$ gives $u' = j_n J_\nu'(j_n x)$. Similarly, $v = J_\nu(j_m x)$ gives $v' = j_m J_\nu'(j_m x)$. Thus,

$$\begin{aligned} \text{L.H.S} = [u'v - uv']_{x=1} &= [j_n J_\nu'(j_n x) J_\nu(j_m x) - j_m J_\nu'(j_m x) J_\nu(j_n x)]_{x=1} \\ &= j_n J_\nu'(j_n) J_\nu(j_m) - j_m J_\nu'(j_m) J_\nu(j_n) \\ &= 0, \text{ since } j_m \text{ and } j_n \text{ are the zeros of } J_\nu(x). \end{aligned}$$

Next, the R.H.S of (12.161) $= (j_m^2 - j_n^2)\int_0^1 xuv\,dx = (j_m^2 - j_n^2)\int_0^1 xJ_\nu(j_n x)J_\nu(j_m x)\,dx.$

Comparing the two sides, we obtain $\int_0^1 xJ_\nu(j_n x)J_\nu(j_m x)\,dx = 0,\quad$ for $j_n \neq j_m$.

In case $j_n = j_m$, then considering j_n as a root of $J_\nu(x) = 0$ and j_m as a variable approaching j_n, the L.H.S. is equal to $j_n J_\nu'(j_n)J_\nu(j_m)$, and thus, we have

$$\lim_{j_m \to j_n}\int_0^1 xJ_\nu(j_n x)J_\nu(j_m x)\,dx = \lim_{j_m \to j_n}\frac{j_n J_\nu'(j_n)J_\nu(j_m)}{j_m^2 - j_n^2} = \lim_{j_m \to j_n}\frac{j_n J_\nu'(j_n)J_\nu'(j_m)}{2j_m}$$

$$= = \frac{1}{2}\{J_\nu'(j_n)\}^2 = \frac{1}{2}[J_{\nu+1}(j_n)]^2,\quad \text{using (12.119) for } x = j_n.$$

This proves the result.

This result also verifies the fact that J_ν and $J_{\nu+1}$ cannot have a positive zero in common, that is, $J_\nu(j_n) = 0$ implies that $J_{\nu+1}(j_n) \neq 0$.

12.8.2 Fourier-Bessel Series

Let $f(x)$ be defined on $[0, 1]$. We choose the coefficients a_n to have an expansion of the form

$$f(x) = \sum_{n=1}^{\infty} a_n J_\nu(j_n x), \qquad \text{...(12.162)}$$

where ν is fixed

To arrive at the requisite coefficients, multiply (12.162) by $xJ_\nu(j_m x)$ and integrate over $[0,1]$, we obtain

$$\int_0^1 xf(x)J_\nu(j_m x)dx = \sum_{n=1}^{\infty} a_n\int_0^1 xJ_\nu(j_n x)J_\nu(j_m x)\,dx = a_m\int_0^1 xJ_\nu^2(j_m x)\,dx = \frac{1}{2}a_m J_{\nu+1}^2(j_m),$$

using orthogonality of Bessel functions. This implies

$$a_m = \frac{2\int_0^1 xf(x)J_\nu(j_m x)dx}{J_{\nu+1}^2(j_m)} \qquad \text{...(12.163)}$$

These numbers are called the Fourier-Bessel coefficients of f, and the resultant series (12.162) is called the *Fourier-Besel series of f in terms of the functions* $J_\nu(j_n x)$. Fourier-Bessel series occurs in the solution of the heat equation for certain kind of regions. Generally the coefficients are difficult to compute, however, with the modern computational techniques, one can often make approximations to the requisite degree of accuracy.

Example 12.28: For the function $f(x) = x, 0 \le x \le 1$, develop a Fourier-Bessel series of the form

$$f(x) = \sum_{n=1}^{\infty} a_n J_1(j_n x),$$

where j_n are the zeros of $J_1(x) = 0$.

Solution: The proposed Fourier-Bessel series is $x = a_1 J_1(j_1 x) + a_2 J_1(j_2 x) + \ldots$

To obtain Fourier-Bessel coefficients, multiply both sides by $x\, J_1(j_n x)$ and integrate from 0 to 1, using orthogonality of Bessel function, we obtain

$$\int_0^1 x^2 J_1(j_n x)\, dx = a_n \int_0^1 x J_1^2(j_n x) dx = \frac{1}{2} a_n J_2^2(j_n).$$

$$\text{This gives } a_n = \frac{2}{J_2^2(j_n)} \int_0^1 x^2 J_1(j_n x) dx = \frac{2}{J_2^2(j_n)} \left[\frac{x^2}{j_n} J_2(j_n x) \right]_0^1, \quad \text{(refer to (12.121) for } \nu = 2\text{)}$$

$$= \frac{2}{j_n J_2(j_n)}.$$

$$\text{Thus,} \qquad x = 2 \sum_{n=1}^{\infty} \frac{J_1(j_n x)}{j_n J_2(j_n)}, \text{ where } j_n \text{ are the zeros of } J_1(x) = 0.$$

Example 12.29: Expand $f(x) = (1 - x^2)$, $0 \le x \le 1$, in Fourier-Bessel series in terms of Bessel function of order zero.

Solution: Let the required series be $f(x) = \sum_{n=1}^{\infty} a_n J_0(j_n x)$,

where j_n are the zeros of $J_0(x)$, and the coefficients a_n are given by

$$a_n = \frac{2}{J_1^2(j_n)} \int_0^1 x(1 - x^2) J_0(j_n x) dx, \quad \text{refer to (12.163).}$$

$$\text{Consider} \quad \int_0^1 x(1 - x^2) J_0(j_n x) dx = \int_0^1 (1 - x^2)\, x J_0(j_n x) dx$$

$$=\left[(1-x^2)\int xJ_0(j_nx)dx\right]_0^1-\int_0^1\left[(-2x)\int xJ_0(j_nx)dx\right]dx$$

$$=\left[(1-x^2)x\frac{J_1(j_nx)}{j_n}\right]_0^1+\frac{2}{j_n}\int_0^1 x^2J_1(j_nx)dx,\quad \int xJ_0(x)dx=xJ_1(x),\text{ refer to (12.121]}.$$

$$=\left[\frac{2}{j_n^2}x^2J_2(j_nx)\right]_0^1,\qquad \int x^2J_1(x)dx=x^2J_2(x)]$$

$$=\frac{2}{j_n^2}J_2(j_n).$$

Thus,
$$a_n=\frac{4}{j_n^2}\frac{J_2(j_n)}{J_1^2(j_n)}. \quad ...(12.164)$$

Also from the recurrence relation (12.123), for $v = 1$, we have $J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$ which for $x=j_n$ becomes $J_2(j_n)=\frac{2}{j_n}J_1(j_n)$, since $J_0(j_n)=0$. Thus, (12.164) becomes

$$a_n=\frac{8}{j_n^3}\frac{1}{J_1(j_n)}, \quad ...(12.165)$$

where j_n are the zeros of $J_0(x)$.

The values of coefficients as given by (12.165) can be calculated using the table for Bessel functions, refer to Appendix IV. The first three coefficients are $a_1 = 1.1056$, $a_2 = -0.1393$, $a_3 = 0.0452$.

Thus, the expansion is

$$(1-x^2)\approx 1.1056\,J_0(j_1x)-0.1393\,J_0(j_2x)+0.0452\,J_0(j_3x)$$

where $j_1 = 2.405$, $j_2 = 5.520$, $j_3 = 8.654$ are the values of the first three zeros of $J_0(x)$.

EXERCISE 12.5

1. Show that $x^aJ_v(bx^c)$ is a solution of

$$y''-\left(\frac{2a-1}{x}\right)y'+\left(b^2c^2x^{2c-2}+\frac{a^2-v^2c^2}{x^2}\right)y=0.$$

Using this write the general solution of the following differential equations in terms of function $x^a J_\nu(bx^c)$ and $x^a J_{-\nu}(bx^c)$

(a) $y'' + (1/x)y' + (4x^2 - 4/9x^2)y = 0$

(b) $y'' + (3/x)y' + (16x^2 - 5/4x^2)y = 0$

(c) $y'' + (5/x)y' + (81x^4 + 7/4x^2)y = 0$

2. Using the given substitution reduce the given differential equation to Bessel equation and hence find the general solution of the equations

(a) $y'' + (1/x)y' + (8 - 1/x^2)y = 0, \quad z = 2\sqrt{2}x$

(b) $x^2y'' + xy' + (x-1)y/4 = 0, \quad z = \sqrt{x}$

(c) $xy'' + 5y' + xy = 0, \quad y = z/x^2$

(d) $y'' + x^2y = 0, \quad y = u\sqrt{x},\ x^2/2 = z.$

3. Show that $y(x) = \sqrt{x}\, J_{1/3}[(2/3)kx^{3/2}]$ is a solution of $y'' + k^2xy = 0$.

4. Let $u(x) = J_0(\alpha x)$ and $v(x) = J_0(\beta x)$. Show that $xu'' + u' + \alpha^2 xu = 0$ and $xv'' + v' + \beta^2 xv = 0$. Also show that

$$[x(u'v - v'u)]' = (\beta^2 - \alpha^2)xuv, \text{ and } (\beta^2 - \alpha^2)\int xJ_0(\alpha x)J_0(\beta x)dx = x[\alpha J_0'(\alpha x)J_0(\beta x) - \beta J_0'(\beta x)J_0(\alpha x)].$$

5. Show that $Y_0'(x) = -Y_1(x)$.

6. The function $I_\nu(x) = (i)^{-\nu}J_\nu(ix)$, $i = \sqrt{-1}$ is called the *modified Bessel function of the first kind of order* ν. Show that $I_\nu(x)$ is a solution of the differential equation $x^2y'' + xy' - (x^2 + \nu^2)y = 0$ and has the representation $I_\nu(x) = \sum_{n=0}^{\infty} \dfrac{x^{2n+\nu}}{2^{2n+\nu}\, n!\, \Gamma(n+\nu+1)}$.

7. Show that

(a) $[xI_0'(x)]' = xI_0(x)$.

(b) $I_{-\nu}(x) = I_\nu(x)$, for $\nu = n$, an integer.

8. For the following functions develop a Fourier-Bessel series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n J_2(j_n x), \text{ where } j_n \text{ are the zeros of } J_2(x) = 0.$$

(a) $f(x) = x^2, \quad 0 < x < R$

(b) $f(x) = \begin{cases} x^2, & 0 < x < 1/2 \\ 0, & 1/2 < x < 1 \end{cases}$

9. Expand $f(x) = x(1-x)$, $0 \le x \le 1$ in Fourier-Bessel series in terms of Bessel function of order one.

12.9 STURM-LIOUVILLE PROBLEM. EIGENFUNCTIONS AND ORTHOGONALITY

The mathematical models of many physical situations in engineering and physics results in a general homogeneous linear differential equation of the form

$$y'' + R(x)y' + [Q(x) + \lambda P(x)]y = 0, \quad \text{...(12.166)}$$

where the function $y(x)$ is defined over some interval [a, b] and satisfies the boundary conditions at $x = a$ and b, and λ is a real parameter.

The Eq. (12.166) can be put in a compact form by multiplying it with a function $r(x)$ defined as $r(x) = e^{\int R(x)dx}$. We get

$$r(x)\,[y'' + R(x)\,y'] + r(x)[Q(x) + \lambda P(x)]y = 0$$

or,
$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [q(x) + \lambda p(x)]y = 0 \quad \text{...(12.167)}$$

where $q(x) = r(x)Q(x)$ and $p(x) = r(x)P(x)$.

The differential equation of the form (12.167) is called *Sturm-Liouville equation*.

In the examples to follow, we show that most of the equations of engineering applications can be studied as a unified approach by means of Sturm-Liouville equation. We shall consider $p(x)$, $q(x)$, $r(x)$ and $r'(x)$ to be continuous functions defined on some closed interval $[a, b]$.

1. *Simple harmonic equation:* $y'' + w^2y = 0$.

This follows from (12.167) by setting $r(x) = 1$, $q(x) = 0$, $p(x) = 1$, and $\lambda = w^2$.

2. *The Legendre equation:* $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$.

This follows from (12.167) by setting $r(x) = 1 - x^2$, $q(x) = 0$, $p(x) = 1$, and $\lambda = \alpha(\alpha + 1)$.

3. *The Bessel equation:* $x^2y'' + xy' + (k^2x^2 - \nu^2)\,y = 0$.

This follows from (12.167) by setting $r(x) = x$, $q(x) = -\nu^2/x$, $p(x) = x$, and $\lambda = k^2$.

4. *The Chebyshev equation:* $(1 - x^2)y'' - xy' + n^2y = 0$.

This follows by setting $r(x) = \sqrt{1-x^2}$, $q(x) = 0$, $p(x) = 1/\sqrt{1-x^2}$, and $\lambda = n^2$.

5. *The Laguerre equation:* $xy'' + (1 - x)\,y' + ny = 0$.

This follows by setting $r(x) = xe^{-x}$, $q(x) = 0$, $p(x) = e^{-x}$, and $\lambda = n$.

6. *The Hermite equation:* $y'' - 2xy' + 2ny = 0$.

It follows by setting $r(x) = e^{-x^2}$, $q(x) = 0$, $p(x) = e^{-x^2}$, and $\lambda = 2n$.

Thus all these equations, being particular cases of (12.167), can be subjected to a unified approach by means of Sturm-Liouville equation.

12.9.1 Sturm-Liouville Problem

The Sturm-Liouville Eq. (12.167) considered on some interval $a \le x \le b$ along with the boundary conditions at $x = a$ and $x = b$ given by

$$A_1 y(a) + A_2 y'(a) = 0 \quad \text{and} \quad B_1 y(b) + B_2 y'(b) = 0, \tag{12.168}$$

when A_1, A_2, B_1 and B_2 are real constants with $A_1^2 + A_2^2 \neq 0$ and $B_1^2 + B_2^2 \neq 0$, is called a Sturm-Liouville problem.

Obviously $y = 0$ is a trivial solution of the Sturm-Liouville problem for any value of the parameter λ but this solution is of no practical utility. The Sturm-Liouville problem has an important property, *the orthogonality of the eigenfunctions.* Before discussing that we introduce the the concept of eigenvalue and eigenvalue function, and orthogonal and onthonormal functions.

Eigenvalue and eigenfunction. Each value of λ for which a non-trivial solution of the problem under study can be found is called an *eigenvalue* of the problem and the corresponding solution $y(x)$ is called an *eigenfunction* of the problem. In practical situations an eigenvalue λ is associated with an important physical characteristic of the problem, such as the frequency of vibration of a string and the corresponding eigenfunction $y(x)$ can be considered to give description of the particular mode of the vibration of the string when it vibrates at the frequency determined by the associated eigenvalue.

Orthogonal and orthonormal functions. The functions $y_1, y_2, \ldots$ defined on some interval $a \le x \le b$ are called *orthogonal* on $a \le x \le b$ with respect to a *weight function* $p(x) > 0$, if

$$\int_a^b p(x) y_n(x) y_m(x)\, dx = 0, \quad \text{for } m \neq n.$$

The norm, $\|y_n\|$ of y_n, is defined by $\|y_n\| = \sqrt{\int_a^b p(x) y_n^2(x)\, dx}$.

The functions are called *orthonormal* on $a \le x \le b$, if they are orthogonal on $a \le x \le b$ and all have norm one. For example, $y_n(x) = \sin nx$, $n = 1, 2, \ldots$ form an orthogonal set on the interval $[-\pi, \pi]$, (here weight function $p(x) = 1$), since

$$\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = 0, \quad n \neq m.$$

Also since, $\int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi$, $n = 1, 2, \ldots$, thus the norm $\|y_n\| = \sqrt{\pi}$, $n = 1, 2, \ldots$ and hence $y_n(x) = \sin nx$, $n = 1, 2, \ldots$ are not orthonormal over $[-\pi, \pi]$, but $\frac{\sin nx}{\sqrt{\pi}}$, $n = 1, 2, \ldots$ form an orthonormal set over $[-\pi, \pi]$.

Now we consider the orthogonality of the eigenfunctions.

12.9.2 Orthogonality of Eigenfunctions

We give the following theorem.

Theorem 12.4 (Orthogonality of Eigenfunctions): *Let the functions $p(x)$, $q(x)$, $r(x)$ and $r'(x)$ in the Sturm-Liouville Eq. (12.167) be real valued continuous functions and let $p(x) > 0$, for $a \le x \le b$. If $y_m(x)$, $y_n(x)$ are the two eigenfunctions of the Sturm-Liouville problem corresponding to two distinct eigenvalues λ_m, λ_n, then $y_m(x)$, $y_n(x)$ are orthogonal to each other on the interval $[a, b]$ with respect to the weight function $p(x)$, that is,*

$$\int_a^b p(x)\, y_m(x)\, y_n(x)\, dx = 0, \quad m \neq n. \qquad \text{...(12.169)}$$

Proof: Since λ_m, λ_n are the distinct eigenvalues and $y_m(x)$, $y_n(x)$ are the corresponding eigenfunctions of the Eq(12.167), we have

$$(ry_m')' + (q + \lambda_m p)y_m = 0, \qquad \text{...(12.170)}$$

and,

$$(ry_n')' + (q + \lambda_n p)y_n = 0. \qquad \text{...(12.171)}$$

Multiplying (12.170) by y_n and (12.171) by y_m and subtracting we obtain

$$[y_m(ry_n')' - y_n(ry_m')'] + (\lambda_n - \lambda_m)py_m\, y_n = 0$$

or,

$$[y_m(ry_n') - y_n(ry_m')]' + (\lambda_n - \lambda_m)py_m y_n = 0$$

or,

$$(\lambda_n - \lambda_m)py_n y_m = [y_n(ry_m') - y_m(ry_n')]'.$$

Integrating both sides with respect to x over the interval $[a, b]$, we obtain

$$(\lambda_n - \lambda_m)\int_a^b py_n\, y_m dx = [y_n(ry_m') - y_m(ry_n')]_a^b$$

$$= r(b)[y_m'(b)\, y_n(b) - y_n'(b)\, y_m(b)] - r(a)[y_m'(a)\, y_n(a) - y_n'(a)\, y_m(a)]$$

$$= r(b)\Delta(b) - r(a)\Delta(a), \qquad \text{...(12.172)}$$

where

$$\Delta(x) = \begin{vmatrix} y_n(x) & y_n'(x) \\ y_m(x) & y_m'(x) \end{vmatrix}.$$

Since the eigenfunctions $y_n(x)$ and $y_m(x)$ satisfy the boundary conditions (12.168), thus at $x = a$ we have

$$A_1 y_n(a) + A_2 y_n'(a) = 0, \quad \text{and} \quad A_1 y_m(a) + A_2 y_m'(a) = 0.$$

Here both A_1 and A_n are not zero, so the non-trivial solution exists for this set of homogeneous equations, and thus, we must have

$$\begin{vmatrix} y_n(a) & y_n'(a) \\ y_m(a) & y_m'(a) \end{vmatrix} = 0, \text{ or } \quad \Delta(a) = 0.$$

Similarly at $x = b$, for the homogeneous set of equations

$$B_1 y_n(b) + B_2 y_n'(b) = 0, \quad \text{and} \quad B_1 y_m(b) + B_2 y_m'(b) = 0$$

since the non-trivial solution B_1, B_2 exists and thus $\begin{vmatrix} y_n(b) & y_n'(b) \\ y_m(b) & y_m'(b) \end{vmatrix} = 0$, or $\quad \Delta(b) = 0$.

Substituting for $\Delta(a)$ and $\Delta(b)$ in (12.172), we obtain $(\lambda_n - \lambda_m) \int_a^b p(x) y_n(x) y_m(x) dx = 0$

and, since $\lambda_n \neq \lambda_m$, it gives

$$\int_a^b p(x) y_n(x) y_m(x) dx = 0, \quad n \neq m$$

the orthogonality condition.

There are very important implications of this orthogonality property of the eigenfunctions. For example, the Legendre equation

$$[(1 - x^2) y']' + n(n+1) y = 0, \quad -1 < x < 1$$

is Sturm-Liouville equation with $r(x) = (1 - x^2)$, $q(x) = 0$, $p(x) = 1$ and parameter $\lambda = n(n + 1)$. Since $y(-1) = y(1) = 0$, and for $n = 0, 1, 2, \ldots; \lambda = 0, 2, 6, \ldots,$ the eigenfunctions are the Legendre polynomials and hence their orthogonality follows from the orthogonality of the eigenfunctions, a result proved in Section 12.3.6.

Example 12.30: Find the eigenvalues and eigenfunctions of Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(l) = 0$$

representing the vibrations of an elastic string of length l stretched between two fixed points $x = 0$ and $x = l$ and allowed to vibrate; here $y(x)$ represents the displacement.

Solution: The given b.v.p. is Sturm-Liouville problem with $r(x) = 1$, $q(x) = 0$, $p(x) = 1$ and λ as parameter. We consider the three cases, λ being negative, zero and positive.
When $\lambda = -k^2$, the general solution is $y(x) = Ae^{kx} + Be^{-kx}$.

Applying the boundary conditions $y(0) = 0$, $y(l) = 0$, we have

$$A + B = 0 \text{ and } \quad Ae^{kl} + Be^{-kl} = 0,$$

which give $A = B = 0$, and hence the solution is $y = 0$, a trivial one, which is not an eigenfuction.

When $\lambda = 0$, the general solution is $y = Ax + B$.

Applying the boundary conditions $y(0) = y(l) = 0$ give $A = B = 0$, and so the solution is again $y = 0$, trivial one, which is not an eigenfunction.

When $\lambda = k^2$, the general solution is $y(x) = A\cos kx + B\sin kx$.

The condition $y(0) = 0$ gives $A = 0$, and $y(l) = 0$ gives $B\sin kl = 0$.

For non-zero solution we must have $\sin kl = 0$, that is, $kl = \pm n\pi$, $n = 0, 1, \ldots$; which gives $k = 0$, $\pm\pi/l, \pm 2\pi/l, \ldots$

Thus the eigenvalues are $\lambda = 0,\ \ \pi^2/l^2,\ \ 4\pi^2/l^2, \ldots$

Since the eigenfunctions multiplied by non-zero constant remain eigenfunction; thus taking $B_n = 1$, the corresponding eigenfunctions are $y_n(x) = \sin(n\pi x/l)$, $\quad n = 0, 1, 2, \ldots$

From the orthogonality of eigenfunctions, $y_n(x)$ are orthogonal over the interval $[0, l]$ with weight function $p(x) = 1$, that is

$$\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = 0, \quad m \neq n.$$

Example 12.31: Find the eigenvalues and eigenfunctions of the Sturm-Liouville equations $y'' + \lambda y = 0$ subject to the boundary conditions $y(0) = y(l)$, and $y'(0) = y'(l)$.

Solution: We consider the three cases, λ being negative, zero, and positive.

When $\lambda = -k^2$, the general solution is $y(x) = Ae^{kx} + Be^{-kx}$

The boundary condition $y(0) = y(l)$ gives $A + B = Ae^{kl} + Be^{-kl}$

or,
$$A(1 - e^{kl}) = B(e^{-kl} - 1) \qquad \ldots(12.173)$$

Similarly, the condition $y'(0) = y'(l)$ gives

$$A(1 - e^{kl}) = -B(e^{-kl} - 1). \qquad \ldots(12.174)$$

The Eqs. (12.173) and (12.174) imply that $A = 0$, and hence, B is also zero. Thus, the solution is only trivial one. So $\lambda = -k^2$ cannot be eigenvalues.

When $\lambda = 0$, the general solution is $y(x) = Ax + B$.

The boundary conditions $y(0) = y(l)$ gives $B = Al + B$, which implies that $A = 0$, since $l \neq 0$.

Next, since $y'(x) = 0$, the boundary condition $y'(0) = y'(l)$ is satisfied for all values of B.

Thus, $\lambda = 0$ is an eigenvalue and $y(x) = B$, where B is an arbitrary constant not equal to zero, is the corresponding eigenfunction.

When $\lambda = k^2$, the general solution is $y(x) = A\cos kx + B\sin kx$.

The boundary condition $y(0) = y(l)$ gives

$$A(1 - \cos kl) = B\sin kl, \qquad \ldots(12.175)$$

and the boundary condition $y'(0) = y'(l)$ gives

$$B(1 - \cos kl) = -A\sin kl. \qquad \ldots(12.176)$$

Multiplying (12.175) by A and (12.176) by B and adding, we obtain

$$(A^2 + B^2)(1 - \cos kl) = 0.$$

This is satisfied if either $A^2 + B^2 = 0$, which gives, $A = 0$ and $B = 0$ leading to a trivial solution $y(x) = 0$, or $1 - \cos kl = 0$. This gives $kl = \pm 2n\pi$, or $k = \pm \dfrac{2n\pi}{l}$, for $n = 0, 1, 2, ...$

Hence, the eigenvalues are $\lambda_n = 4n^2\pi^2/l^2$, $n = 0, 1, 2, ...$

The corresponding eigenfunctions are

$$y_n(x) = A\cos(2n\pi x/l) + B\sin(2n\pi x/l), \quad n = 0, 1, 2, ...,$$

where A and B are two arbitrary constants not both zero simultaneously. Taking $A = 1, B = 0$, and then $A = 0, B = 1$, it is observed that in this case corresponding to the single eigenvalue $\lambda_n = 4n^2\pi^2/l^2$, there are two eigenfuctions $y_n^{(1)} = \cos(2n\pi x/l)$ and $y_n^{(2)} = \sin(2n\pi x/l)$.

Example 12.32: Find the eigenvalues and eigenfunctions of the Bessel equation $x^2y'' + xy' + (j^2x^2 - \nu^2)y = 0$ on the interval $0 \le x \le 1$ when the solution is bounded with $y(1) = 0$.

Solution: The given equation can be written in the form $(xy')' + \left(-\dfrac{\nu^2}{x} + j^2 x\right)y = 0$

with $r(x) = x$, $q(x) = -\nu^2/x$, $p(x) = x$ and $\lambda = j^2$. The general solution is

$$y(x) = AJ_\nu(jx) + BY_\nu(jx).$$

Since $Y_\nu(jx)$ is infinite at $x = 0$, thus for the solution to remain finite over the interval $0 \le x \le 1$, we must set $B = 0$. The solution becomes $y(x) = AJ_\nu(jx)$.

The boundary condition $y(1) = 0$ gives $AJ_\nu(j) = 0$. So for non-trivial solution j must be the zeros of $J_\nu(x)$ and let the zeros be denoted by $j_1, j_2, j_3, ...$.

Thus the eigenvalues are $\lambda_n = j_n^2$, $n = 1, 2, ...$ and the corresponding eigenfunctions are

$$y_n(x) = J_\nu(j_n x), \quad n = 1, 2, ...$$

where we have set $A = 1$. By the orthogonality property of the eigenfunctions $y_n(x) = J_\nu(j_n x)$ are orthogonal over the interval $[0, 1]$ w.r.t the weight function $p(x) = x$, that is,

$$\int_0^1 xJ_\nu(j_m x)J_\nu(j_n x)\,dx = 0, \quad m \ne n,$$

a result which confirms to the already established one, refer to (12.158).

EXERCISE 12.6

Find the eigenfunctions of the following Sturm-Liouville problems and verify their orthogonality.

1. $y'' + \lambda y = 0;\ y(0) = 0,\ y(\pi) = 0$
2. $y'' + \lambda y = 0;\ y'(0) = 0,\ y'(\pi) = 0$
3. $y'' + \lambda y = 0;\ y(0) = 0,\ y'(\pi) = 0$
4. $(xy')' + (\lambda/x)y = 0;\ y(1) = 0,\ y(l) = 0.$
5. $(xy')' + (\lambda/x)y = 0;\ y'(1) = 0,\ y(l) = 0.$

6. Verify the orthogonality of Bessel functions direct from the Bessel equation.
7. Verify the orthogonality of Legendre polynomials direct from the Legendre equation.

ANSWERS

Exercise 12.1 (p. 697)

1. $y = 4 - 4x + x^3 + \frac{1}{12}x^4 + \ldots$

2. $y = a - ax + (1 + a)\frac{x^2}{2!} - (1 + 3a)\frac{x^3}{3!} + \ldots$

3. $y = 1 - (x - 1) + \frac{3}{2}(x - 1)^2 - \frac{1}{6}(x - 1)^3 + \ldots$

4. $y = 1 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$

5. $y = 1 - \frac{1}{2}(x + 2)^2 - \frac{1}{3}(x + 2)^3$

6. $y = 1 + 4x + \frac{3}{2}x^2 - \frac{1}{6}x^3 + \ldots$

7. $y = 3 + \frac{5}{2}(x - 1)^2 + \frac{5}{6}(x - 1)^3 + \frac{5}{24}(x - 1)^4 + \ldots$

8. $y = 1 + x - \frac{1}{4}x^2 + \frac{1}{4}x^3 + \frac{1}{32}x^4 + \ldots$

9. $y = 3 + 3x + \frac{7}{4}x^2 + \frac{23}{24}x^3 + \ldots$

10. $y = 1 - (x - 1) - \frac{5}{3}(x - 1)^2 + \frac{1}{3}(x - 1)^3 \ldots$

11. $y = a + bx + \frac{1}{3}bx^3 - \frac{1}{12}ax^4 + \frac{1}{20}bx^5 + \ldots$

12. $y = a + bx - \frac{1}{2}ax^2 + \left(\frac{1 - 2b}{6}\right)x^3 + \frac{5a}{24}x^4 + \ldots$

13. $y = a + bx - ax^2 - \frac{b}{12}x^3 - \frac{a}{24}x^4 + \ldots$

14. $y = a + bx + \frac{1 - a}{2}x^2 + \frac{1}{6}(1 - 4b)x^3 - \frac{1}{24}(9 - 11a)x^4 + \ldots$

15. $y = a + bx - \frac{1}{4}ax^2 + \frac{2 - b}{12}x^3 + \frac{5a - 12b}{96}x^4 + \ldots$

16. $y = a_0\left(1 - \frac{w^2x^2}{2!} + \frac{w^4x^4}{4!} - \ldots\right) + \frac{a_1}{w}\left(wx - \frac{w^3x^3}{3!} + \ldots\right)$

17. $y = a_0\left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^6 + \ldots\right) + a_1\left(x - \frac{1}{20}x^5 + \frac{1}{1440}x^9 + \ldots\right)$

18. $y = a_0\left(1 - 2x^2 + \frac{2}{3}x^4 + \ldots\right) + a_1\left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5 + \ldots\right) + \frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{20}x^5 + \ldots$

19. $y(x) = -1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{8}x^4 + \ldots$ 20. $y(x) = x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{5}{24}x^4$.

Exercise 12.2 (p. 714)

4. $\frac{8}{63}P_5 + \frac{28}{63}P_3 + \frac{3}{7}P_1$ 5. $\frac{8}{35}P_4 + \frac{6}{5}P_3 - \frac{2}{21}P_2 + \frac{34}{5}P_2 - \frac{224}{105}P_0$ 11. $P_3(x/a)$

12. (a) $2P_0(x) - 3P_1(x) + 2P_2(x)$ (b) $0.9549P_1(x) - 1.1582P_3(x) + 0.2143P_5(x) - \ldots$
(c) $1.5P_1(x) - 1.7317P_3(x) + 5.2857P_5(x) - \ldots$.

Exercise 12.3 (p. 726)

1. $y = A\left[1 - \frac{1}{2^2}x^2 + \frac{1}{2^2.4^2}x^4 - \ldots\right] + B\left[y_1 \ln x + c_0\left\{\frac{1}{2^2}x^2 - \frac{1}{2^2.4^2}\left(1 + \frac{1}{2}\right)x^4 + \ldots\right\}\right]$

2. $y = Ax\left(1 - \frac{6}{5}x + \frac{6}{7}x^2 - \frac{4}{9}x^3 + \ldots\right) + Bx^{-1/2}$

3. $y(x) = x^2\,[A\cos(4\ln x) + B\sin(4\ln x)]$. 4. $y = \frac{A + B\ln x}{1 + x}$

5. $y = Ax\left[1 + \frac{x^2}{14} + \frac{x^4}{616} + \ldots\right] + Bx^{-\frac{1}{2}}\left[1 + \frac{x^2}{2} + \frac{x^4}{40} + \ldots\right]$

6. $y = A\left[1 - \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{3x^6}{6!} - \ldots\right] + Bx$

7. $y = \left(A - \frac{B}{2}\ln x\right)x\left[1 - \frac{x^2}{2.4} + \frac{x^4}{2.4^2.6} - \ldots\right] + \frac{B}{x}\left[1 + \frac{x^2}{2^2} - \frac{1 + \frac{1}{4}}{2^2.4}x^4 + \ldots\right]$

8. $y = (A + B\ln x)\left[1 - \frac{1}{2^2}x^2 + \frac{1.3}{2^2.4^2}x^4 - \ldots\right] - B\left(\frac{1}{128}x^5 - \frac{59}{65536}x^9 + \ldots\right)$

9. $y = A\left(1 - \frac{x^4}{3.4} + \frac{x^8}{3.4.7.8} - \ldots\right) + Bx\left(1 - \frac{x^4}{4.5} + \frac{x^8}{4.5.8.9} - \ldots\right)$

10. $y = A\left(1 - \frac{x^2}{4} - \frac{x^4}{24} - \ldots\right) + Bx\left(1 - \frac{x^2}{12} - \ldots\right).$

Exercise 12.4 (p. 737)

2. $J_6(x) = \left(\frac{3840}{x^4} - \frac{768}{x^3} - \frac{2}{x}\right)J_1(x) + \left(\frac{144}{x^2} - 1 - \frac{1920}{x^4}\right)J_0(x).$

3. $\sqrt{\frac{2}{\pi x}}\left[\left(\frac{15 - 6x^2}{x^3}\right)\sin x + \left(\frac{15}{x^2} - 1\right)\cos x\right]$

10. (a) $-2J_2(x) - J_0(x) + c$ (b) $-2J_4(x) - 2J_2(x) - J_0(x) + c.$

Exercise 12.5 (p. 749)

1. (a) $y = AJ_{1/3}(x^2) + BJ_{-1/3}(x^2)$ (b) $y = Ax^{-1}J_{3/4}(2x^2) + Bx^{-1}J_{-3/4}(2x^2)$
   (c) $y = Ax^{-2}J_{1/2}(3x^3) + Bx^{-2}J_{-1/2}(3x^3)$
2. (a) $y = AJ_1(2\sqrt{2}x) + BY_1(2\sqrt{2}x)$ (b) $y = AJ_1(\sqrt{x}) + BY_1(\sqrt{x})$
   (c) $y = A\left[\frac{1}{x^2}J_2(x) + \frac{1}{x^2}Y_2(x)\right]$ (d) $y = A\left[\sqrt{x}\,J_{1/4}\left(\frac{1}{2}x^2\right) + \sqrt{x}\,Y_{1/4}\left(\frac{1}{2}x^2\right)\right]$
8. (a) $c_n = \frac{2R}{j_n}J_3(j_nR)$ (b) $c_n = J_3(j_n/2)/[4j_n J_3^2(j_n)].$
9. $x(1 - x) = 0.4522\,J_1(3.8317\,x) - 0.03152\,J_1(7.0156\,x) + 0.0320\,J_1(10.1735\,x) - \ldots.$

Exercise 12.6 (p. 758)

1. $y_n(x) = \sin nx, \quad n = 1, 2, \ldots$
2. $y_n(x) = \cos nx, \quad n = 0, 1, 2, \ldots$
3. $y_n(x) = \sin\frac{(2n+1)x}{2}, \quad n = 0, 1, 2, \ldots$
4. $y_n(x) = \sin\left(\frac{n\pi \ln x}{\ln l}\right), \quad n = 1, 2, \ldots$
5. $y_n(x) = \cos\left(\frac{(2n+1)\pi \ln x}{\ln l}\right), \quad n = 1, 2, \ldots$

13 Laplace Transforms

CHAPTER

The Laplace transform converts a differential equation and corresponding initial value problem to an algebraic problem, comparatively easier to solve, and then transforms the solution back to the solution of the initial value problem. Practically Laplace transforms form the most important operational method to solve problems directly, initial value problems without first determining a general solution, and non-homogeneous differential equations without first solving the corresponding homogeneous equations.

13.1 DEFINITION AND EXISTENCE CONDITIONS. LINEARITY AND FIRST SHIFTING PROPERTIES

Let $f(t)$ be defined for $0 \le t < \infty$. Then the Laplace transform of $f(t)$, written symbolically as $F(s) = \mathcal{L}\{f(t)\}$, is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt, \qquad ...(13.1)$$

for all s, such that this integral converges. The parameter can be complex, but we limit it here to real values only.

We observe that the original function f depends on t and the new function F, the transform of f, depends on s. The original function $f(t)$ in (13.1) is called the *inverse Laplace transform* of $F(s)$, denoted by $\mathcal{L}^{-1}[F(s)]$, and we write

$$f(t) = \mathcal{L}^{-1}[F(s)]. \qquad ...(13.2)$$

Since the interval of integration in (13.1) is infinite, so it is an improper integral and is thus evaluated according to the rule $\int_0^\infty e^{-st} f(t)dt = \lim_{\tau \to \infty} \int_0^\tau e^{-st} f(t)dt.$

Example 13.1: Find the Laplace transform of $f(t) = 1$, when $t \ge 0$.

Solution: By definition, $\mathcal{L}[1] = \int_0^\infty e^{-st}\, 1\, dt = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$, provided $s > 0$.

Example 13.2: Find the Laplace transform of $f(t) = e^{at}$, a is real and $t \geq 0$.

Solution: By definition, $\mathcal{L}[e^{at}] = \int_0^\infty e^{-st}\, e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}$, provided $s > a$.

Example 13.3: Find the Laplace transform of $f(t) = t$, $t \geq 0$.

Solution: By definition, $\mathcal{L}[t] = \int_0^\infty t e^{-st}\, dt = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2}$, $s \geq 0$.

Example 13.4: Find the Laplace transform of $f(t) = \sin t$.

Solution: By definition, $\mathcal{L}[\sin t] = \int_0^\infty e^{-st} \sin t\, dt = -\left[\frac{e^{-st}\cos t + se^{-st}\sin t}{s^2+1}\right]_0^\infty = \frac{1}{s^2+1}$.

Practically, a Laplace transform is rarely computed directly from the definition. The Laplace transform has many general properties which are helpful in developing the Laplace transforms of some functions from the transforms of elementary functions. An important property satisfied by the Laplace transform is that of linearity given as follows.

13.1.1 Linearity of the Laplace Transform

Let $f(t)$ and $g(t)$ be any two functions whose Laplace transforms exist. For any two constants a and b,

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] \qquad ...(13.3)$$

The proof follows immidiately from definition, since

$$\mathcal{L}[af(t) + bg(t)] = \int_0^\infty e^{-st}[af(t) + bg(t)]dt = a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)dt = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)].$$

Example 13.5: Find the Laplace transform of cosh at and sinh at.

Solution: By definition,

$$\mathcal{L}[\cosh at] = \mathcal{L}\left[\frac{e^{at}+e^{-at}}{2}\right] = \frac{1}{2}\mathcal{L}[e^{at}] + \frac{1}{2}\mathcal{L}[e^{-at}] = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}, s > |a|.$$

Similarly, $\mathcal{L}[\sinh at] = \frac{a}{s^2-a^2}, s > |a|$.

Example 13.6: Find the Laplace transform of $\sin wt$ and $\cos wt$.

Solution: We have, $\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$. Set $a = iw$, it gives

$$\mathcal{L}[e^{iwt}] = \frac{1}{s-iw} = \frac{(s+iw)}{(s-iw)(s+iw)} = \frac{s+iw}{s^2+w^2},\ e^{iwt} = \cos wt + i\sin wt, \text{ since}$$

or, $$\mathcal{L}[\cos wt + i\sin wt] = \frac{s+iw}{s^2+w^2}$$

or, $$\mathcal{L}[\cos wt] + i\mathcal{L}[\sin wt] = \frac{s}{s^2+w^2} + i\frac{w}{s^2+w^2}.$$

Equating the real and imaginary parts on both sides, we obtain

$$\mathcal{L}[\cos wt] = \frac{s}{s^2+w^2}, \text{ and } \mathcal{L}[\sin wt] = \frac{w}{s^2+w^2}.$$

The table below gives some functions $f(t)$ and their Laplace transforms $\mathcal{L}[f(t)]$.

	$f(t)$	$F(s) = \mathcal{L}[f(t)]$		$f(t)$	$F(s) = \mathcal{L}[f(t)]$
1	1	$1/s$	7	$\dfrac{ae^{at} - be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
2	t	$1/s^2$	8	$\sin at$	$a/(s^2+a^2)$
3	$t^n, n = 1, 2, \ldots$	$n!/s^{n+1}$	9	$\cos at$	$s/(s^2+a^2)$
4	$t^a, \alpha > 0$	$\Gamma(\alpha+1)/s^{a+1}$	10	$\sinh at$	$a/(s^2-a^2)$
5	e^{at}	$1/(s-a)$	11	$\cosh at$	$s/(s^2-a^2)$
6	$\dfrac{e^{at}-e^{bt}}{(a-b)}$	$\dfrac{1}{(s-a)(s-b)}$	12	$1/\sqrt{t}$	$\sqrt{\dfrac{\pi}{s}}$

The two functions $f(t)$ and $F(s)$ are called a *Laplace transform pair.*

13.1.2 Sufficient Conditions for the Existence of Laplace Transform

Let $f(t)$ be a piecewise continuous on $0 \le t < \infty$. This means that f is continuous on any finite interval $[a, b]$ except perhaps at finitely many points in $[a, b]$ at which the function $f(t)$ has finite jumps. For example, let

$$f(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 2, & t = 2 \\ 1, & 2 < t \le 3 \\ -1, & 3 < t \le 4 \end{cases}$$

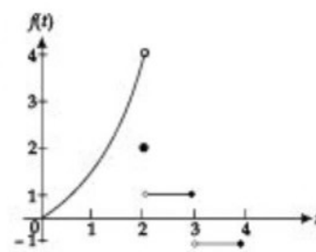


Fig. 13.1

Then f is continuous on the interval [0, 4] except at 2 and 3, where f has finite discontinuities. A graph of this function is shown in Fig. 13.1.

If function $f(t)$ is piecewise continuous on $[0, \tau]$, then so is $e^{-st} f(t)$, and hence the integral $\int_0^\tau e^{-st} f(t)\, dt$ exists. But the existence of the integral $\int_0^\tau e^{-st} f(t)\, dt$ for every $\tau > 0$ does not necessarily ensure the existence of the integral $\int_0^\tau e^{-st} f(t)\, dt$ as $\tau \to \infty$. For example, $f(t) = e^{t^2}$ is continuous on every interval $[0, k]$ for some finite k, but $\int_0^\infty e^{-st} e^{t^2}\, dt$ diverges for every real value of s. Thus for the convergence of the integral $\int_0^\infty e^{-st} f(t)\, dt$ we need some further condition on $f(t)$.

The form of the integral suggests that, if $f(t)$ is of exponential order α, that is, if there exist some constants α and $M > 0$ such that $|f(t)| \le Me^{\alpha t}$, which implies $e^{-st}\,|f(t)| \le Me^{(\alpha-s)t}$ for $t \ge 0$, and further, since the integral $\int_0^\infty Me^{(\alpha-s)t} dt$ converges to $\frac{M}{s-\alpha}$ for $s > \alpha$, then by comparison test, refer to Section 7.8.3, the integral $\int_0^\infty e^{-st}\,|f(t)|\, dt$, and hence, the integral $\int_0^\infty e^{-st} f(t)\, dt$ converges for $s > \alpha$. This ensures the existence of the Laplace transform for $f(x)$.

Thus, we have the following result:

Theorem 13.1: *If $f(t)$ is a piecewise continuous function on the interval $[0, \infty)$ and is of exponential order α for $t \ge 0$, then Laplace transform of $f(t)$ exists for $s > \alpha$.*

Geometrically, the condition of exponential order α means that the graph of $f(t)$, $t \ge 0$ does not grow faster than the graph of the exponential function $g(t) = Me^{\alpha t}$, $t \ge 0$. Many functions, e.g., sin at, cos at, e^{at}, etc satisfy this condition.

Also we note that *the conditions stated above are sufficient but not necessary for a function to have a Laplace transform.*

Consider, for example, $f(t) = 1/\sqrt{t}$. It is infinite at $t = 0$ and hence is not piecewise continuous on interval $[0, \infty)$, but its Laplace transform exists, since

$$\mathcal{L}\left[\frac{1}{\sqrt{t}}\right] = \int_0^\infty e^{-st} \frac{1}{\sqrt{t}}\, dt = 2\int_0^\infty e^{-sx^2}\, dx, \quad (t = x^2)$$

$$= \frac{2}{\sqrt{s}} \int_0^\infty e^{-u^2}\, du, \quad (u = x\sqrt{s})$$

$$= \frac{\Gamma(1/2)}{\sqrt{s}}, \quad \text{refer to (7.54)}$$

$$= \sqrt{\frac{\pi}{s}}, s > 0, \quad \text{since } \Gamma(1/2) = \sqrt{\pi}.$$

Remark: We observe that if the Laplace transform of a given function exists, then it is uniquely determined. Conversely, it can be shown that if two functions, both defined on the positive real axis, have the same transform, then these functions cannot differ over an interval of positive length, although they may differ at various isolated points. But two continuous functions having the same Laplace transforms must be equal and thus, in general, we may say that the inverse of a given transform is essentially unique and this is of wide practical importance since Laplace transforms are extensively used in solving the initial and boundary value problems.

Example 13.7: Find the Laplace tansform of $f(t) = \begin{cases} t/\tau, & \text{when } 0 \le t < \tau \\ 1, & \text{when } t \ge \tau \end{cases}$.

Solution: By definition, $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\tau e^{-st} \frac{t}{\tau} dt + \int_\tau^\infty e^{-st} dt$

$$= \frac{1}{\tau}\left[\left| t\,\frac{e^{-st}}{-s}\right|_0^\tau - \left|\frac{e^{-st}}{s^2}\right|_0^\tau\right] + \left|\frac{e^{-st}}{-s}\right|_\tau^\infty = \frac{1}{\tau}\left[\frac{\tau e^{-s\tau}}{-s} - \frac{e^{-st}-1}{s^2}\right] + \frac{e^{-st}}{s} = \frac{1-e^{-s\tau}}{\tau s^2}.$$

13.1.3 First Shifting Theorem (or the s-Shifting)

We have the following result:

Theorem 13.2 **(s–Shifting):** *If $f(t)$ has the transform $F(s)$, then $e^{at}f(t)$ has the transform $F(s-a)$, that is,*

$$\mathcal{L}[e^{at}f(t)] = F(s-a). \qquad ...(13.4)$$

Proof. By definition, $\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-(s-a)t} f(t) dt = F(s-a)$,

provided $F(s) = \mathcal{L}[f(t)]$ exists for s greater than some k, then $\mathcal{L}[e^{at}f(t)]$ exists for $(s-a) > k$.

Application of the first shifting property leads us to the following results:

1. $\mathcal{L}[e^{at}t^n] = \dfrac{n!}{(s-a)^{n+1}}$, for $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$

2. $\mathcal{L}[e^{at}\sin bt] = \dfrac{b}{(s-a)^2+b^2}$, for $\mathcal{L}[\sin bt] = \dfrac{b}{s^2+b^2}$

3. $\mathcal{L}[e^{at}\cos bt] = \dfrac{s-a}{(s-a)^2+b^2}$, for $\mathcal{L}[\cos bt] = \dfrac{s}{s^2+b^2}$

4. $\mathcal{L}[e^{at}\sinh bt] = \dfrac{b}{(s-a)^2-b^2}$, for $\mathcal{L}[\sinh bt] = \dfrac{b}{s^2-b^2}$

5. $\mathcal{L}[e^{at}\cosh bt] = \dfrac{s-a}{(s-a)^2-b^2}$, for $\mathcal{L}[\cosh bt] = \dfrac{s}{s^2-b^2}$

6. $\mathcal{L}\left[\dfrac{e^{at}}{\sqrt{t}}\right] = \sqrt{\dfrac{\pi}{s-a}}$, for $\mathcal{L}\left[\dfrac{1}{\sqrt{t}}\right] = \sqrt{\dfrac{\pi}{s}}$.

Example 13.8: Find the Laplace transform of (a) $\sinh t \cos t$, (b) $\sin^3 2t$, (c) $2e^{-t}\cos^2 t$.

Solution: (a) We have $\sinh t \cos t = \frac{1}{2}(e^t - e^{-t})\cos t = \frac{1}{2}[e^t \cos t - e^{-t}\cos t]$.

Therefore, $\mathcal{L}[\sinh t \cos t] = \frac{1}{2}[\mathcal{L}[e^t \cos t] - \mathcal{L}[e^{-t}\cos t]]$

$$= \frac{1}{2}\left[\frac{s-1}{(s-1)^2+1} - \frac{s+1}{(s+1)^2+1}\right] = \frac{(s^2-2)}{[(s-1)^2+1]\,[(s+1)^2+1]}.$$

(b) Since $\sin 3t = 3\sin t - 4\sin^3 t$, we have, $\sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$, therefore, $\sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$. Thus

$$\mathcal{L}[\sin^3 2t] = \frac{3}{4}\mathcal{L}[\sin 2t] - \frac{1}{4}\mathcal{L}[\sin 6t]$$

$$= \frac{3}{4}\frac{2}{s^2+4} - \frac{1}{4}\frac{6}{s^2+36} = \frac{3}{2}\left[\frac{32}{(s^2+4)(s^2+36)}\right] = \frac{48}{(s^2+4)(s^2+36)}.$$

(c) Since $2\cos^2 t = [1 + \cos 2t]$, therefore,

$$\mathscr{L}[2e^{-t}\cos^2 t] = \mathscr{L}[e^{-t}] + \mathscr{L}[e^{-t}\cos 2t] = \frac{1}{s+1} + \frac{s+1}{(s+1)^2+4} = \frac{2[(s+1)^2+2]}{(s+1)\{(s+1)^2+4\}}.$$

Example 13.9: Find the Laplace transforms of

(a) $\sin\sqrt{t}$ (b) $erf(\sqrt{t})$ (c) $J_0(t)$.

Solution: (a) We have $\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \dots$.

Therefore, $\mathscr{L}[\sin\sqrt{t}] = \mathscr{L}\left[t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \dots\right]$

$$= \mathscr{L}[t^{1/2}] - \frac{1}{3!}\mathscr{L}[t^{3/2}] + \frac{1}{5!}\mathscr{L}[t^{5/2}] - \frac{1}{7!}\mathscr{L}[t^{7/2}] + \dots$$

$$= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3!s^{5/2}} + \frac{\Gamma(7/2)}{5!s^{7/2}} - \frac{\Gamma(9/2)}{7!s^{9/2}} + \dots$$

$$= \frac{\Gamma(3/2)}{s^{3/2}}\left[1 - \frac{3}{2.3!s} + \frac{5.3}{2^2.5!s^2} - \frac{7.5.3}{2^3.7!s^3} + \dots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \frac{1}{3!(4s)^3} + \dots\right] = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-\frac{1}{4s}}.$$

(b) By definition,

$$erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left[1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots\right]dx$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{t^{1/2}}{1} - \frac{t^{3/2}}{3.1!} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \dots\right]$$

Therefore, $\mathscr{L}[erf(\sqrt{t})] = \frac{2}{\sqrt{\pi}}\mathscr{L}\left[\frac{t^{1/2}}{1} - \frac{t^{3/2}}{3.1!} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \dots\right]$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{s^{5/2}.3.1!} + \frac{\Gamma(7/2)}{s^{7/2}.5.2!} - \frac{\Gamma(9/2)}{s^{9/2}.7.3!} + \dots\right]$$

$$= \frac{2\Gamma(3/2)}{\sqrt{\pi}s^{3/2}}\left[1 - \frac{3}{2.3.s} + \frac{5.3}{2^2.5.s^2\,2!} - \frac{7.5.3}{2^3.7.3!s^3} + \dots\right]$$

$$= \frac{1}{s^{3/2}}\left[1 - \frac{1}{2s} + \frac{(-1/2)(-3/2)}{2!}\frac{1}{s^2} + \frac{(-1/2)(-3/2)(-5/2)}{3!}\frac{1}{s^3} + \dots\right]$$

$$= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-\frac{1}{2}} = \frac{1}{s^{3/2}}\frac{s^{1/2}}{(1+s)^{1/2}} = \frac{1}{s\sqrt{(1+s)}}.$$

(c) We know that, refer to (12.116),

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \dots$$

Therefore, $\mathcal{L}[J_0(t)] = \mathcal{L}(1) - \dfrac{\mathcal{L}(t^2)}{2^2} + \dfrac{\mathcal{L}(t^4)}{2^2.4^2} - \dfrac{\mathcal{L}(t^6)}{2^2.4^2.6^2} + \dots$

$$= \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2.4^2}\frac{4!}{s^5} - \frac{1}{2^2.4^2.6^2}\frac{6!}{s^7} + \dots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{2}\cdot\frac{3}{4}\left(\frac{1}{s^4}\right) - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\left(\frac{1}{s^6}\right)\dots\right]$$

$$= \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{-1/2} = \frac{1}{\sqrt{s^2+1}}.$$

EXERCISE 13.1

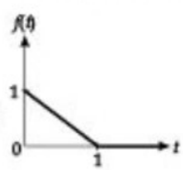
1. Find the Laplace transforms of the following functions.

(a) $\dfrac{(\sqrt{t}-1)^2}{\sqrt{t}}$ (b) $(\sin t - \cos t)^2$ (c) $\sin\sqrt{t}$

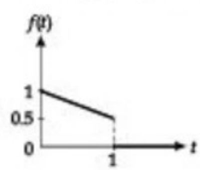
(d) $t^3 \cos 4t$ (e) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$ (f) $e^{-at}\sinh bt$.

2. Find the Laplace transforms of the functions $f(t)$ represented by

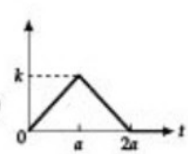
(a)

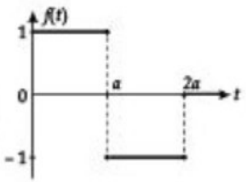


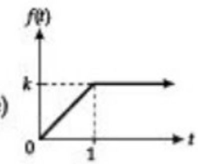
(b)



(c)



(d) 

(e) 

3. Find the Laplace transform of the following functions using the first shifting property:
   (a) $e^{4t}\sin 2t\cos t$ (b) $(1+te^{-t})^3$
   (c) $\sinh t\cos^2 t$ (d) $te^{2t}\cos 5t$.
4. If $\mathcal{L}[f(t)] = F(s)$ and c is any positive constant, then show that $\mathcal{L}[f(ct)] = F(s/c)/c$; use it to obtain Laplace transform of $\cos at$ from that of $\cos t$.
5. Find the Laplace transform of $f(t) = |t-1| + |t+1|$, $t \geq 0$.
6. Suppose f is piecewise continuous on $[0, k]$ for every $k > 0$, and M and b are such that $|f(t)| \leq Me^{bt}$ for $t \geq 0$. If $F(s) = \mathcal{L}[f(t)]$, then prove that $\lim_{s\to\infty} F(s) = 0$.
7. If $f(t)$ is a periodic function with period $\tau > 0$, then show that

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-s\tau}}\int_0^{\tau} e^{-st}f(t)\,dt.$$

8. Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} 5, \text{ for } & 0 < t \leq 3 \\ 0, \text{ for } & 3 < t \leq 6 \end{cases} \quad \text{with} \quad f(t+6) = f(t),\ t > 0.$$

9. Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} E\sin wt, & \text{for } 0 < t \leq \dfrac{\pi}{w} \\ 0, & \text{for } \dfrac{\pi}{w} \leq t \leq \dfrac{2\pi}{w} \end{cases} \quad \text{with} \quad f\left(t + \frac{2\pi}{w}\right) = f(t).$$

10. Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} h, & 0 < t \leq a \\ 0, & a < t \leq 2a \end{cases} \quad \text{with} \quad f(t+2a) = f(t).$$

13.2 TRANSFORMS OF DERIVATIVES AND INTEGRALS

The Laplace transform replaces operations of calculus by operations of algebra on transforms. In this section we shall observe that differentiation of $f(t)$ is replaced by multiplication of $\mathcal{L}\{f(t)\}$ by s and integration of $f(t)$ is replaced by division of $\mathcal{L}\{f(t)\}$ by s.

13.2.1 Laplace Transform of the Derivative of $f(t)$

We state the following theorem:

Theorem 13.3 (Transform of Derivative): *If $f(t)$ is continuous for all $t \geq 0$ and has a derivative $f'(t)$ which is piecewise continuous on every finite interval contained in $t \geq 0$, then*

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0). \qquad ...(13.5)$$

Proof. Assuming that $f(t)$ satisfies the sufficiency conditions for the existence of a Laplace transform, we have

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)dt = \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st}f(t)dt = -f(0) + s\int_0^\infty e^{-st} f(t)dt = sF(s) - f(0).$$

Thus Laplace transform of $f'(t)$ exists, provided $\mathcal{L}[f(t)]$ exists.

This result can be extended to piecewise continuous functions also. Suppose $f(t)$, $t \geq 0$ is continuous except for a finite jump at $t = a > 0$, then we can show that (13.5) is modified to

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) - [f(a+0) - f(a-0)]\, e^{-as},$$

provided $\mathcal{L}[f(t)]$ exists.

Transform of higher order derivatives

By applying (13.5) to the second order derivative $f''(t)$, we obtain

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0) = s[s\mathcal{L}[f(t)] - f(0)] - f'(0) = s^2\mathcal{L}[f(t)] - sf(0) - f'(0);$$

Thus, $$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0). \qquad ...(13.6)$$

Using induction, we arrive at the following result.

Theorem 13.4: *Let $f(t)$ and its derivative $f'(t), f''(t), \ldots, f^{(n-1)}(t)$ be continuous functions for all $t \geq 0$ and let $f^{(n)}(t)$ be piecewise continuous on every finite interval contained in $t \geq 0$, then*

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0). \qquad ...(13.7)$$

Example 13.10: Using the Laplace transform of derivatives, find the Laplace transform of the functions

(a) t^2 (b) $\sin wt$ (c) $\sin^2 t$.

Solution: The function $f(t) = t^2$ gives $f(0) = 0, f'(0) = 0, f''(t) = 2$.

We have $$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0).$$

Thus, $$\mathcal{L}[f(t)] = \frac{1}{s^2}\mathcal{L}[f''(t)] = \frac{1}{s^2}\mathcal{L}[2] = \frac{2}{s^3}.$$

(b) $f(t) = \sin wt$, gives, $f'(t) = w\cos wt, f''(t) = -w^2 \sin wt$.

We have $$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0).$$

Thus, $$\mathcal{L}[-w^2 \sin wt] = s^2\mathcal{L}[\sin wt] - 0 - w$$

or, $$-w^2\mathcal{L}[\sin wt] = s^2\mathcal{L}[\sin wt] - w, \quad \text{which gives } \mathcal{L}[\sin wt] = \frac{w}{s^2 + w^2}.$$

(c) $f(t) = \sin^2 t$, gives $f'(t) = 2\sin t\cos t = \sin 2t$.

We have, $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$.

Substituting for $f(t) = \sin^2 t$, we obtain, $\mathcal{L}[\sin 2t] = s\mathcal{L}[\sin^2 t]$, thus

$$\mathcal{L}[\sin^2 t] = \frac{2}{s(s^2+4)}.$$

13.2.2 Laplace Transform of Integral of $f(t)$

We have the following theorem.

Theorem 13.5 (Transform of Integral): *Let $f(t)$ be a piecewise continuous function such that $|f(t)| \le Me^{\alpha t}$, for $\alpha > 0$ and all $t \ge 0$. If $\mathcal{L}[f(t)] = F(s)$, then*

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}, \text{ for } s > \alpha. \qquad ...(13.8)$$

Proof. Let $g(t) = \int_0^t f(\tau)\,d\tau$. We have,

$$|g(t)| \le \int_0^t |f(\tau)|\,d\tau \le M\int_0^t e^{\alpha\tau}d\tau = M\frac{e^{\alpha t}}{\alpha}, \quad \text{for } t \ge 0.$$

This shows that $|g(t)|$ grows no faster than $|f(t)|$ as $t \to \infty$, so the existence of the Laplace transform of $f(t)$ ensures the existence of Laplace transform of $g(t)$. Also $g'(t) = f(t)$, except for points at which $f(t)$ is discontinuous. Hence, $g'(t)$ is piecewise continuous on each finite interval, and using the result for the Laplace transform of the derivative of a function, we have

$$\mathcal{L}[f(t)] = \mathcal{L}[g'(t)] = s\mathcal{L}[g(t)] - g(0) = s\mathcal{L}[g(t)], \text{ since } g(0) = 0.$$

Hence, $$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}.$$

Example 13.11: Find (a) $\mathcal{L}\left[\int_0^t \tau\cos a\tau\,d\tau\right]$ (b) $\mathcal{L}\left[\int_0^t \tau^2 e^{\tau}\,d\tau\right]$.

Solution: (a) Let $f(t) = t\cos at$. We have

$\mathcal{L}[f(t)] = \mathcal{L}[t\cos at] = Re.\ \mathcal{L}[te^{iat}]$

$$= Re.\frac{1}{(s-ia)^2} = Re\frac{(s+ia)^2}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

Thus, $$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{(s^2 - a^2)}{s(s^2 + a^2)^2}, \text{ using (13.8)}$$

(b) Let $f(t) = t^2e^t$. We have $\mathcal{L}[f(t)] = \mathcal{L}[t^2e^t] = \dfrac{2}{(s-1)^3}$.

Thus, $$\mathcal{L}\left[\int_0^t \tau^2 e^{\tau} d\tau\right] = \frac{2}{s(s-1)^3}, \text{ using (13.8)}$$

13.2.3 Differentiation and Integration of Transforms

The concept of differentiation and integration of Laplace transform helps us to apply Laplace transforms to some specific problems. We consider the corresponding operations for the transformed function $F(s)$.

Theorem 13.6 **(Differentiation of Transform):** *If $\mathcal{L}[f(t)] = F(s)$, then*

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}, \; n = 1, 2, \ldots \qquad \text{...(13.9)}$$

Proof. By definition $F(s) = \int_0^\infty e^{-st} f(t)dt$.

Differentiating $F(s)$ *w.r.t.* s under the integral sign, we obtain

$$\frac{dF(s)}{ds} = \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)dt = -\int_0^\infty e^{-st} t f(t)dt = -\mathcal{L}[tf(t)], \qquad \text{...(13.10)}$$

which is (13.9) for $n = 1$.

Differentiating (13.10) *w.r.t.* s again under the integral sign, we obtain

$$\frac{d^2 F(s)}{ds^2} = (-1)^2 \mathcal{L}[t^2 f(t)].$$

By induction we get the general result (13.9).

Theorem 13.7 (**Integration of Transform**): *If $f(t)/t$ be piecewise continuous defined for $t > 0$, and $|f(t)/t| \leq Me^{\alpha t}$, for some constants α, and $M > 0$, then*

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds, \quad s > \alpha, \qquad \text{...(13.11)}$$

where $F(s) = \mathcal{L}[f(t)]$.

Proof. Since $f(t)/t$ is piecewise continuous for $t > 0$ and is of exponential order, thus it satisfies the sufficient conditions for the existence of Laplace transform.

Let $$G(s) = \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt, \quad \text{for } s > \alpha. \qquad ...(13.12)$$

Differentiating (13.12) *w.r.t.* s under the integral sign, we have

$$G'(s) = \int_0^\infty e^{-st}(-t)\frac{f(t)}{t}\, dt = -\int_0^\infty e^{-st} f(t)\, dt = -F(s).$$

Integrating $F(s)$ *w.r.t.* s over the interval (s, ∞), we have

$$\int_s^\infty F(s)\, ds = -\int_s^\infty G'(s)\, ds = G(s) - \lim_{s\to\infty} G(s) = G(s),$$

since from (13.12) $\lim_{s\to\infty} G(s) = 0$, $f(t)/t$ being of exponential order α. Thus,

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds, \quad \text{for } s > \alpha,$$

this proves (13.11).

Example 13.12: Find the Laplace transforms of

(a) $t\cos at$ (b) $te^{at}\cos t$ (c) t^3e^{-3t}

Solution: (a) Since $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$, therefore, using (13.10),

$$\mathcal{L}[t\cos at] = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = -\frac{(s^2+a^2) - s.2s}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

(b) Since $\mathcal{L}[e^{at}\cos t] = \frac{(s-a)}{(s-a)^2+1}$, therefore, using (13.10)

$$\mathcal{L}[te^{at}\cos t] = -\frac{d}{ds}\left(\frac{(s-a)}{(s-a)^2+1}\right) = \frac{(s-a)^2-1}{[(s-a)^2+1]^2}, \text{ for } s > a$$

(c) Since $\mathcal{L}(e^{-3t}) = \frac{1}{s+3}$, therefore, using (13.9)

$$\mathcal{L}[t^3e^{-3t}] = (-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) = -\frac{(-1)^3\, 3!}{(s+3)^4} = \frac{6}{(s+3)^4}, \text{ for } s > -3.$$

Example 13.13: Find (a) $L\left[\frac{\sin at}{t}\right]$ (b) $L\left[\frac{\cos at - \cos bt}{t}\right]$

Solution: (a) Since $\mathscr{L}[\sin at] = \frac{a}{s^2+a^2}$, therefore,

$$\mathscr{L}\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{u^2+a^2}du = \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}.$$

(b) Since $\mathscr{L}[\cos at - \cos bt] = \mathscr{L}[\cos at] - \mathscr{L}[\cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$, therefore,

$$\mathscr{L}\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left(\frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\right)du = \left[\frac{1}{2}\ln(u^2+a^2) - \frac{1}{2}\ln(u^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln\frac{u^2+a^2}{u^2+b^2}\right]_s^\infty = -\frac{1}{2}\ln\left[\frac{s^2+a^2}{s^2+b^2}\right] = \ln\left(\frac{s^2+b^2}{s^2+a^2}\right)^{\frac{1}{2}}.$$

Example 13.14: Evaluate (a) $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$ (b) $\int_0^\infty te^{-2t}\sin t\, dt$

Solution: (a) Since $\mathscr{L}[e^{-t} - e^{-3t}] = \frac{1}{s+1} - \frac{1}{s+3}$, $s > -1$, therefore,

$$\mathscr{L}\left[\frac{e^{-t}-e^{-3t}}{t}\right] = \int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+3}\right)ds = -\left(\ln\frac{s+3}{s+1}\right)_s^\infty = \ln\frac{s+3}{s+1}.$$

Thus, $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt = \mathscr{L}\left[\frac{e^{-t}-e^{-3t}}{t}\right]_{s=0} = \ln 3.$

Alternatively, $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt = \int_0^\infty \frac{e^{-t}}{t}dt - \int_0^\infty \frac{e^{-3t}}{t}dt = \mathscr{L}\left[\frac{1}{t}\right]_{s=1} - \mathscr{L}\left[\frac{1}{t}\right]_{s=3}$

$$= \int_1^\infty \frac{1}{s}ds - \int_3^\infty \frac{1}{s}ds, \quad \text{using (13.11)}$$

$$= \int_1^3 \frac{1}{s} ds = [\ln s]_1^3 = \ln 3.$$

(b) $\int_0^\infty te^{-2t} \sin t\, dt = \int_0^\infty e^{-2t} (t \sin t) dt = \mathscr{L}[t \sin t]_{s=2} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\Big|_{s=2} = \frac{2s}{(s^2+1)^2}\Big|_{s=2} = \frac{4}{25}.$

EXERCISE 13.2

1. Given that $4f''(t) + f(t) = 0, f(0) = 0, f'(0) = 2$, show that $\mathscr{L}[f(t)] = \frac{8}{4s^2+1}$.
2. Given that $f(0) = 1, f'(0) = 0, f''(0) = 1$, find $\mathscr{L}[f'''(t)]$
3. Given that $f(t) = \begin{cases} \sin t, & 0 \le t < \pi/2 \\ 0, & t \ge \pi/2 \end{cases}$,

   find $\mathscr{L}[f(t)]$ using the transformation of derivatives.
4. Find the Laplace transform of $f(t)$, when $f''(t) + 4f(t) = 8 \sin t, f(0) = 0 f'(0) = 2$.
5. Using the transform of derivatives, find the Laplace transform of sin at and from it derive the transform of cos at.
6. Show that

   (i) If $\mathscr{L}\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$, then $\mathscr{L}\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

   (ii) If $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$, then $L\left\{\frac{d^2 f}{dt^2}\right\} = \frac{2(1-e^{-s})}{s}$

   (iii) If $\mathscr{L}\left[\frac{d^2 f}{dt^2}\right] = \tan^{-1}\frac{1}{s}$, $f(0) = 0$ and $\left(\frac{df}{dt}\right)_{t=0} = 1$, then $\mathscr{L}\{f(t)\} = \left[2s + 1 + \tan^{-1}\left(\frac{1}{s}\right)\right]\frac{1}{s^2}$

   (iv) If $\mathscr{L}\{tf(t)\} = \frac{1}{s(s^2+1)}$, then $\mathscr{L}\{e^{-t}f(2t)\} = \frac{1}{4}\ln\frac{(s+1)^2+4}{(s+1)^2}$
7. Find the Laplace transforms of the following

   (a) $\int_0^t \tau^2 \sin 2\tau\, d\tau$ (b) $\int_0^t \frac{e^t \sin t}{t} dt$
8. Find the Laplace transforms of the following

   (a) $te^{-t}\cosh t$ (b) $t^2 e^{-3t} \sin 2t$

9. Find the Laplace transforms of the following

(a) $\dfrac{e^{-at}-e^{-bt}}{t}$ (b) $\dfrac{\sin^2 t}{t}$ (c) $\dfrac{1-\cos t}{t^2}$

10. Evaluate the following integrals using the Laplace transforms

(a) $\displaystyle\int_0^\infty \frac{\sin at}{t}\,dt,\ a>0$ (b) $\displaystyle\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt$

(c) $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$ (d) $\displaystyle\int_0^\infty e^{-at^2}\,dt$

13.3 INVERSE LAPLACE TRANSFORM. CONVOLUTION THEOREM

In this section we turn our attention to the inverse Laplace transform of a function. As defined in Section 13.1; *given a function F(s), a function f(t) such that* $\mathscr{L}[f(t)] = F(s)$ *is called an inverse Laplace transform of F(s)*, and we write, $f(t) = \mathscr{L}^{-1}[F(s)]$. For example,

$$\mathscr{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}, \quad \text{since} \quad \mathscr{L}[e^{at}] = \frac{1}{s-a}.$$

The inverse process is ambiguous because given $F(s)$ there will be many functions whose Laplace transform is $F(s)$. For example, we know that the Laplace transform of $f(t) = e^{-t}$ is $1/(s+1)$ for $s > -1$. However, if we change $f(t)$ at just one point, say

$$g(t) = \begin{cases} e^{-t}, & \text{for } t \neq 3 \\ 0, & \text{for } t = 3 \end{cases}$$

then $\int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} g(t)dt$ and $g(t)$ has the same Laplace transform as $f(t)$. So should we conclude that the process of inverse Laplace is not unique. An answer is provided by *Lerch's Theorem*, which states that *two continuous functions having the same Laplace transform must be equal.*

Further we observe that in case $f(t)$ satisfies the sufficient conditions for the existence of Laplace transform and $\mathscr{L}[f(t)] = F(s)$, then $\lim_{s\to\infty} F(s) = 0$ and $\lim_{s\to\infty} [sF(s)]$ is bounded, thus these two results indicate that *it is not necessary that corresponding to any function F(s) of s there exist inverse Laplace transform f(t).* For example, corresponding to $F(s) = s$, there does not exist inverse transform $f(t)$.

Next, because of the linearity of the Laplace transform *inverse transforms is also linear*, that is

$$\mathscr{L}^{-1}[af(t) + bg(t)] = a\mathscr{L}^{-1}[f(t)] + b\mathscr{L}^{-1}[g(t)], \quad ...(13.13)$$

where $f(t)$, $g(t)$ are any two functions whose Laplace transforms exist and, a and b two arbitrary constants.

The table below gives some functions $F(s)$ and their inverse Laplace transforms $\mathcal{L}^{-1}[F(s)] = f(t)$.

	$F(s)$	$f(t) = \mathcal{L}^{-1}[F(s)]$		$F(s)$	$f(t) = \mathcal{L}^{-1}[F(s)]$
1	$1/s$	1	9	$a/(s^2 - a^2)$	$\sinh at$
2	$1/(s-a)$	e^{at}	10	$s/(s^2 - a^2)$	$\cosh at$
3	$1/s^{n+1}$	$\dfrac{t^n}{n!}, n = 1, 2, \ldots$	11	$1/\sqrt{s}$	$1/\sqrt{\pi t}$
4	$1/s^{\alpha+1}$	$t^{\alpha}/\Gamma(\alpha+1), \alpha > 0$	12	$1/s\sqrt{s+1}$	$erf(\sqrt{t})$
5	$\dfrac{1}{(s-a)(s-b)}$	$\dfrac{(e^{at} - e^{bt})}{a-b}$	13	$b/[(s-a)^2 + b^2]$	$e^{at} \sin bt$
6	$\dfrac{s}{(s-a)(s-b)}$	$\dfrac{(ae^{at} - be^{at})}{a-b}$	14	$(s-a)/[(s-a)^2 + b^2]$	$e^{at} \cos bt$
7	$a/(s^2 + a^2)$	$\sin at$	15	$2as/(s^2 + a^2)^2$	$t \sin at$
8	$s/(s^2 + a^2)$	$\cos at$	16	$(s^2 - a^2)/(s^2 + a^2)^2$	$t \cos at$

We note that $F(s)$ generally is a rational algebraic function of s. To find inverse transform, we simplify $F(s)$ by applying partial fractions, which then will be expressible as one of the standard forms given above.

Example 13.15: Find the inverse Laplace transforms of

(a) $\dfrac{2s+5}{s^2+25}$ (b) $\dfrac{3}{s^2+3s-10}$

(c) $\dfrac{s+2}{s^2-4s+13}$ (d) $\dfrac{s^2-3s+4}{s^3}$

Solution: (a) Using the linearity property of inverse transforms

$$\mathcal{L}^{-1}\left[\frac{2s+5}{s^2+25}\right] = 2\mathcal{L}^{-1}\left[\frac{s}{s^2+25}\right] + \mathcal{L}^{-1}\left[\frac{5}{s^2+25}\right] = 2\cos 5t + \sin 5t.$$

(b) Let $$\frac{3}{s^2+3s-10} = \frac{A}{(s+5)} + \frac{B}{(s-2)}$$

Multiplying both sides by $(s+5)(s-2)$, we get

$$3 = A(s-2) + B(s+5) = (A+B)s + (-2A + 5B).$$

Equating the coefficients of s and constant terms on both sides of it, we get $A + B = 0$, and $-2A + 5B = 3$, which give $A = -3/7$ and $B = 3/7$. Thus,

$$\mathcal{L}^{-1}\left[\frac{3}{s^2+3s-10}\right]=\mathcal{L}^{-1}\left[\frac{-3/7}{s+5}+\frac{3/7}{s-2}\right]$$

$$=-\frac{3}{7}\mathcal{L}^{-1}\left[\frac{1}{s+5}\right]+\frac{3}{7}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]=-\frac{3}{7}e^{-5t}+\frac{3}{7}e^{2t}.$$

(c) We have $\mathcal{L}^{-1}\left[\frac{s+2}{s^2-4s+13}\right]=\mathcal{L}^{-1}\left[\frac{(s-2)+4}{(s-2)^2+9}\right]$

$$=\mathcal{L}^{-1}\left[\frac{(s-2)}{(s-2)^2+3^2}\right]+\frac{4}{3}\mathcal{L}^{-1}\left[\frac{3}{(s-2)^2+3^2}\right]$$

$$=e^{2t}\cos 3t+\frac{4}{3}e^{2t}\sin 3t.$$

(d) We have $\mathcal{L}^{-1}\left[\frac{s^2-3s+4}{s^3}\right]=\mathcal{L}^{-1}\left[\frac{1}{s}-\frac{3}{s^2}+\frac{4}{s^3}\right]$

$$=\mathcal{L}^{-1}\left(\frac{1}{s}\right)-3\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)+4\mathcal{L}^{-1}\left(\frac{1}{s^3}\right)$$

$$=1-3t+4\frac{t^2}{2!}=1-3t+2t^2.$$

Example 13.16: Find the inverse Laplace transforms of

(a) $\frac{s}{s^4+s^2+1}$ (b) $\frac{s^3}{s^4-a^4}$

Solution: We have $s^4+s^2+1=(s^2+1)^2-s^2=(s^2+s+1)(s^2-s+1)$. Let

$$\frac{s}{s^4+s^2+1}=\frac{As+B}{s^2+s+1}+\frac{Cs+D}{s^2-s+1} \quad ...(13.14)$$

Multiplying both sides by s^4+s^2+1, we get

$$s=(As+B)(s^2-s+1)+(Cs+D)(s^2+s+1).$$

Equating coefficients of constant term: $B+D=0.$

Equating coefficients of s: $A-B+C+D=1$

Equating coefficients of s^2: $-A+B+C+D=0$

Equating coefficients of s^3: $A+C=0$

Solving for A, B, C, and D we get $A = 0, \quad B = -\frac{1}{2}, \quad C = 0, \quad D = \frac{1}{2}$.

Thus (13.14) becomes

$$\frac{s}{s^4+s^2+1} = \frac{1}{2}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right] = \frac{1}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right].$$

Therefore, $$\mathcal{L}^{-1}\left[\frac{s}{s^4+s^2+1}\right] = \frac{1}{2}\left[\frac{2}{\sqrt{3}}e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t - \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t\right]$$

$$= \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\left(\frac{e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}}{2}\right) = \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\sinh\frac{t}{2}.$$

(b) We have

$$\frac{s^3}{s^4-a^4} = s\,\frac{s^2}{(s^2-a^2)(s^2+a^2)} = \frac{1}{2}s\left[\frac{(s^2+a^2)+(s^2-a^2)}{(s^2-a^2)(s^2+a^2)}\right] = \frac{1}{2}\left[\frac{s}{s^2-a^2}+\frac{s}{s^2+a^2}\right].$$

Therefore, $$\mathcal{L}^{-1}\left[\frac{s^3}{s^4-a^4}\right] = \frac{1}{2}\left[\mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right)+\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right)\right] = \frac{1}{2}\,(\cosh at + \cos at).$$

A few other inversion results, respectively corresponding to the transforms of derivatives (13.7), transforms of integrals (13.8), differentiation of transforms (13.9) and integration of transforms (13.11) are as follows:

1. *If,* $\mathcal{L}^{-1}[F(s)] = f(t)$ *and* $f(0) = f'(0) = \ldots = f^{n-1}(0) = 0$, *then*

$$\mathcal{L}^{-1}[s^n F(s)] = \frac{d^n}{dt^n}[f(t)]. \qquad \ldots(13.15)$$

2. *If* $\mathcal{L}^{-1}[F(s)] = f(t)$, *then*

$$\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt; \qquad \ldots(13.16)$$

$$\mathcal{L}^{-1}\left[\frac{F(s)}{s^2}\right] = \int_0^t\left(\int_0^t f(t)\,dt\right)dt \quad \text{etc.}$$

3. *If* $\mathcal{L}^{-1}[F(s)] = f(t)$, *then*

$$\mathcal{L}^{-1}\left[(-1)^n \frac{d^n}{ds^n} F(s)\right] = t^n f(t), \quad n = 1, 2, \ldots \qquad \ldots(13.17)$$

4. *If* $\mathcal{L}^{-1}[F(s)] = f(t)$, *then*

$$\mathcal{L}^{-1}\left[\int_s^{\infty} F(s)ds\right] = \frac{f(t)}{t} \qquad \ldots(13.18)$$

provided $f(t)/t$ *satisfies the sufficient conditions for the existence of Laplace transform.*

Example 13.17: Find the inverse Laplace transforms of

(a) $\dfrac{1}{s^2(s^2+a^2)}$ (b) $\dfrac{s}{(s^2+a^2)^2}$

Solution: (a) Since, $\mathcal{L}^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{1}{a}\sin at$, therefore, by (13.16) we have

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t \frac{1}{a}\sin at\,dt = \frac{1}{a^2}(1-\cos at), \text{ and using this again,}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \int_0^t \frac{1}{a^2}(1-\cos at)dt = \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right].$$

(b) Let $f(t) = \mathcal{L}^{-1}\left(\dfrac{s}{(s^2+a^2)^2}\right)$. By (13.18), we have

$$\frac{f(t)}{t} = \mathcal{L}^{-1}\left[\int_s^{\infty} \frac{s}{(s^2+a^2)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{2}\int_s^{\infty} \frac{2s}{(s^2+a^2)^2}ds\right] = \mathcal{L}^{-1}\left[-\frac{1}{2}\left(\frac{1}{s^2+a^2}\right)_s^{\infty}\right]$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{2a}. \text{ Hence, } f(t) = \frac{1}{2a}t\sin at.$$

Example 13.18: Find the inverse Laplace transforms of

(a) $\ln\dfrac{s+1}{s-1}$ (b) $\dfrac{1}{\sqrt{s^2+1}}$

(c) $\tan^{-1}\left(\dfrac{2}{s^2}\right)$ (d) $s\ln\dfrac{1}{\sqrt{s^2+1}} + \cot^{-1}s$

Solution:

(a) Let, $f(t) = \mathscr{L}^{-1}\left[\ln\frac{s+1}{s-1}\right]$. By (13.17), we have

$$tf(t) = \mathscr{L}^{-1}\left[-\frac{d}{ds}\ln\left(\frac{s+1}{s-1}\right)\right] = -\mathscr{L}^{-1}\left[\frac{d}{ds}\{\ln(s+1) - \ln(s-1)\}\right]$$

$$= -\mathscr{L}^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right] = e^{t} - e^{-t} = 2\sinh t. \text{ Hence, } f(t) = \frac{2}{t}\sinh t.$$

(b) We have, $\dfrac{1}{\sqrt{s^2+1}} = (1+s^2)^{-1/2} = \dfrac{1}{s}\left[1+\dfrac{1}{s^2}\right]^{-1/2}$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{2}\cdot\frac{3}{4}\left(\frac{1}{s^2}\right)^2 - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\left(\frac{1}{s^2}\right)^3 + \ldots\right]$$

$$= \frac{1}{s} - \frac{1}{2^2}\cdot\frac{2!}{s^3} + \frac{1}{2^2.4^2}\cdot\frac{4!}{s^5} - \frac{1}{2^2.4^2.6^2}\cdot\frac{6!}{s^7} + \ldots$$

Thus, $\mathscr{L}^{-1}\left[\dfrac{1}{\sqrt{s^2+1}}\right] = \mathscr{L}^{-1}\left[\dfrac{1}{s}\right] - \dfrac{1}{2^2}\mathscr{L}^{-1}\left[\dfrac{2!}{s^3}\right] + \dfrac{1}{2^2.4^2}\mathscr{L}^{-1}\left(\dfrac{4!}{s^5}\right) - \dfrac{1}{2^2.4^2.6^2}L^{-1}\left(\dfrac{6!}{s^7}\right) + \ldots$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{1}{2^2.4^2.6^2}t^6 + \ldots$$

$= J_0(t)$, Bessel's function of the first kind of order zero, refer to (12.116).

(c) Let, $f(t) = \mathscr{L}^{-1}\left[\tan^{-1}\frac{2}{s^2}\right]$. By (13.17), we have

$$tf(t) = \mathscr{L}^{-1}\left[-\frac{d}{ds}\tan^{-1}\frac{2}{s^2}\right] = \mathscr{L}^{-1}\left[\frac{4s}{s^4+4}\right]$$

$$= \mathscr{L}^{-1}\left[\frac{4s}{(s^2+2)^2-(2s)^2}\right] = \mathscr{L}^{-1}\left[\frac{(s^2+2s+2)-(s^2-2s+2)}{(s^2+2s+2)(s^2-2s+2)}\right]$$

$$= \mathscr{L}^{-1}\left[\frac{1}{(s^2-2s+2)} - \frac{1}{(s^2+2s+2)}\right] = \mathscr{L}^{-1}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$= e^{t}\sin t - e^{-t}\sin t = 2\sinh t\sin t.$

(d) Let, $f(t) = \mathcal{L}^{-1}\left[s \ln \dfrac{s}{\sqrt{s^2+1}} + \cot^{-1} s\right]$. By (13.17), we have

$$tf(t) = \mathcal{L}^{-1}\left[-\frac{d}{ds}\left(s \ln \frac{s}{\sqrt{s^2+1}} + \cot^{-1} s\right)\right]$$

$$= -\mathcal{L}^{-1}\left[\frac{d}{ds}\left\{s \ln s - \frac{1}{2} s \ln (s^2+1) + \cot^{-1} s\right\}\right]$$

$$= -\mathcal{L}^{-1}\left[\ln s + 1 - \frac{1}{2}\ln (s^2+1) - \frac{s^2}{s^2+1} - \frac{1}{s^2+1}\right] = -\mathcal{L}^{-1}\left[\ln s - \frac{1}{2}\ln (s^2+1)\right]$$

Reapplying (13.17), we have

$$t^2 f(t) = (-1)^2 \mathcal{L}^{-1}\left[\frac{d}{ds}\left(\ln s - \frac{1}{2}\ln (s^2+1)\right)\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = 1 - \cos t. \text{ Hence, } f(t) = \frac{1-\cos t}{t^2}.$$

13.3.1 Laplace Convolution Theorem

In Example 13.17b, the function can be expressed as the product of two functions, as

$$\frac{s}{(s^2+a^2)^2} = \frac{s}{(s^2+a^2)}\frac{1}{(s^2+a^2)} = F(s)\,G(s), \text{ say.} \qquad \text{...(13.19)}$$

The inverse transformation was determined there using the property of integration of transforms. However in (13.19), we know the inverse transforms of $F(s)$ and $G(s)$, but this information could not be applied as the inverse Laplace transforms of the product of two functions is simply not the product of their inverse Laplace transforms. Next, we present *Laplace convolution theorem, a result concerning the inverse Laplace transform of the product function.*

Theorem 13.8 (Convolution Theorem): *If $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[g(t)] = G(s)$, then*

$$\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = F(s)G(s) \qquad \text{...(13.20)}$$

or, equivalently

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)d\tau. \qquad \text{...(13.21)}$$

(The expression $\int_0^t f(\tau)g(t-\tau)d\tau$ is called the *convolution* of $f(t)$ and $g(t)$ and is denoted by $f * g$. It is easy to prove that convolutions are commutative, distributive and associative.)

Proof. By definition,

$$\mathscr{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right]dt = \int_0^\infty \int_0^t e^{-st} f(\tau)g(t-\tau)d\tau\, dt. \qquad ...(13.22)$$

The region of integration $R(\tau, t) = \begin{cases} 0 \le \tau < t \\ 0 \le t < \infty \end{cases}$

is shown in Fig. 13.2. Changing the order of integration (13.22) becomes

$$\mathscr{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)g(t-\tau)dt\, d\tau$$

$$= \int_0^\infty e^{-s\tau} f(\tau)\left\{\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)dt\right\}d\tau$$

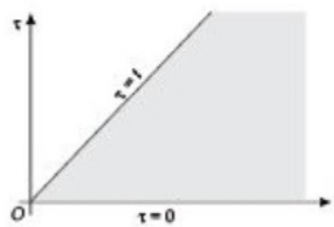


Fig. 13.2

$$= \int_0^\infty e^{-s\tau} f(\tau)\left\{\int_0^\infty e^{-s\theta} g(\theta)d\theta\right\}d\tau, \quad (t-\tau=\theta.)$$

$$= \left(\int_0^\infty e^{-s\tau} f(\tau)d\tau\right)G(s) = F(s)G(s), \text{ which is (13.20).}$$

Example 13.19: Apply convolution theorem to find the inverse Laplace transforms of

(a) $\dfrac{s}{(s^2+a^2)^2}$ (b) $\dfrac{1}{(s-2)(s+3)}$ (c) $\dfrac{1}{s\sqrt{s+4}}$

Solution: (a) Since $\mathscr{L}^{-1}\left(\dfrac{s}{(s^2+a^2)}\right) = \cos at$ and $\mathscr{L}^{-1}\left(\dfrac{1}{(s^2+a^2)}\right) = \dfrac{1}{a}\sin at$, therefore, by convolution theorem,

$$\mathscr{L}^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{a}\int_0^t \cos a\tau \sin a(t-\tau)\,d\tau = \frac{1}{2a}\int_0^t [\sin at + \sin a(2\tau - t)]d\tau$$

$$= \frac{1}{2a}\left[\tau \sin at - \frac{1}{2a}\cos a(2\tau - 1)\right]_0^t$$

$$= \frac{1}{2a}\left[t \sin at - \frac{1}{2a}\cos at + \frac{1}{2a}\cos at\right] = \frac{1}{2a} t \sin at$$

the same as obtained in Example (13.17b).

(b) Since $\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$, and $\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$, therefore, by convolution theorem

$$\mathcal{L}^{-1}\left(\frac{1}{(s-2)(s+3)}\right) = \int_0^t e^{2\tau} e^{-3(t-\tau)} d\tau = \int_0^t e^{5\tau - 3t} d\tau = \frac{1}{5}\left[e^{5\tau - 3t}\right]_0^t = \frac{1}{5}\left[e^{2t} - e^{-3t}\right].$$

(c) Since $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$, and $\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{t^{-1/2}}{\Gamma(1/2)} = \frac{1}{\sqrt{\pi t}}$ gives $\mathcal{L}^{-1}\frac{1}{\sqrt{s+4}} = \frac{e^{-4t}}{\sqrt{\pi t}}$, thus by convolution theorem

$$\mathcal{L}^{-1}\left(\frac{1}{s\sqrt{s+4}}\right) = \int_0^t \frac{e^{-4\tau}}{\sqrt{\pi\tau}} 1\, d\tau = \frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-4\tau}}{\sqrt{\tau}} d\tau$$

$$= \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-u^2}\, du \quad (4\tau = u^2.)$$

$$= \frac{1}{2}\,\text{erf}.(2\sqrt{t}).$$

EXERCISE 13.3

1. Find the inverse Laplace transforms of the functions

(a) $\frac{s^2 - 1}{s(s^2+4)}$ (b) $\frac{s^3 - 1}{(s+2)^2(s^2-9)}$ (c) $\frac{5s+3}{(s-1)(s^2-2s+5)}$

2. Find the inverse Laplace transforms of the functions

(a) $\frac{s}{s^4 + 4a^4}$ (b) $\frac{s+2}{(s^2+4s+5)^2}$ (c) $\frac{s^2+s}{(s^2+1)(s^2+2s+2)}$

3. Find the inverse Laplace transforms of the functions

(a) $\frac{s^2}{(s^2+a^2)^2}$ (b) $\frac{1}{(s^2+a^2)^2}$

4. Find the inverse Laplace transforms of the functions

(a) $\ln\left(1+\frac{1}{s^2}\right)$ (b) $\ln\left(\frac{1+s}{s}\right)$.

5. Find the inverse Laplace transforms of the function

(a) $\cot^{-1}\frac{s}{2}$ (b) $\tan^{-1}(s-1)$ (c) $\tan^{-1}\left(\frac{2}{s^2}\right)$

6. Apply Convolution theorem to find the inverse Laplace transforms of the functions

(a) $\frac{1}{(s^2+a^2)^2}$ (b) $\frac{s}{(s^2+1)(s^2+4)}$ (c) $\frac{1}{(s+1)(s+9)^2}$

7. Find (a) $\mathcal{L}^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right)$ (b) $\mathcal{L}^{-1}\left(\frac{1}{s^2+4}F(s)\right)$

8. Find (a) $\mathcal{L}^{-1}\left(\frac{1}{s}\sin\frac{1}{s}\right)$ (b) $\mathcal{L}^{-1}\left(\frac{1}{s}\cos\frac{1}{s}\right)$

9. Show that $\mathcal{L}^{-1}\left\{s\sqrt{s+a}\right\}^{-1} = \frac{erf\sqrt{at}}{\sqrt{a}}$

10. Evaluate the transform of the functions

(a) $\int_0^t e^{t-\tau}\sin 2\tau\, d\tau$ (b) $\int_0^t \cosh 3(t-\tau)\, d\tau$

13.4 TRANSFORM SOLUTION OF INITIAL VALUE PROBLEMS

The Laplace transform is a powerful tool for solving initial value problems. A particular feature of this technique is that the initial value problem is converted into an algebra problem incorporating the initial conditions into the algebraic manipulations. Once we obtain $Y(s)$, the Laplace transform of the dependent variable $y(t)$, our problem becomes that of the inversion transformation to get $y(t)$. The method is useful in solving linear differential equations with constant coefficients, linear differential equation with variable coefficients of the polynomial type, simultaneous differential equations, integral equations and integro-differential equations.

13.4.1 Solutions of Differential Equations

We consider a few examples which elaborate the applications of Laplace transform in solving linear differential equations with given initial conditions.

Example 13.20: Solve the initial value problem

$$y''(t) + 3y'(t) + 2y(t) = \sin 2t \qquad ...(13.23)$$

with $\qquad y(0) = 2 \text{ and } y'(0) = -1.$ ...(13.24)

Solution: Taking the Laplace transforms on both sides of (13.23) and applying the linearity property, we obtain

$$\mathcal{L}[y''(t)] + 3\mathcal{L}[y'(t)] + 2\mathcal{L}[y(t)] = \mathcal{L}[\sin 2t] \qquad ...(13.25)$$

Setting $\mathcal{L}[y(t)] = Y(s)$ and using the initial conditions (13.24), we have

$$\mathcal{L}[y''(t)] = s^2Y(s) - 2s + 1, \text{ and } \quad \mathcal{L}[y'(t)] = sY(s) - 2. \text{ Also } \mathcal{L}[\sin 2t] = \frac{2}{s^2+4}$$

Using all these, Eq. (13.25) becomes

$$s^2Y(s) - 2s + 1 + 3[sY(s) - 2] + 2Y(s) = \frac{2}{s^2+4}$$

or,

$$(s^2 + 3s + 2)Y(s) = \frac{2s^3 + 5s^2 + 8s + 22}{s^2+4}.$$

or,

$$Y(s) = \frac{2s^3 + 5s^2 + 8s + 22}{(s^2+4)(s+2)(s+1)} \qquad ...(13.26)$$

When resolved in partial fraction form (13.26) gives

$$Y(s) = -\frac{5}{4}\cdot\frac{1}{s+2} + \frac{17}{5}\,\frac{1}{s+1} - \frac{1}{20}\,\frac{2}{s^2+4} - \frac{3}{20}\,\frac{s}{s^2+4}.$$

Taking inverse Laplace transform and using linear property, we obtain

$$\mathcal{L}^{-1}[Y(s)] = -\frac{5}{4}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \frac{17}{5}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{20}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) - \frac{3}{20}\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right),$$

which gives

$$y(t) = -\frac{5}{4}e^{-2t} + \frac{17}{5}e^{-t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t, \quad t \geq 0$$

as the solution of the given initial value problem.

Example 13.21: Solve the initial value problem

$$y^{(iv)}(t) - k^4y(t) = 0 \qquad ...(13.27)$$

with $\qquad y(0) = 1, \quad y'(0) = y''(0) = y'''(0) = 0.$ ...(13.28)

Solution: Taking the Laplace transforms on both sides of (13.27) and using the linearity property, we obtain

$$\mathcal{L}[y^{(iv)}t] - k^4\mathcal{L}[y(t)] = \mathcal{L}[0] \qquad ...(13.29)$$

Setting $\mathcal{L}[y(t)] = Y(s)$, and using the initial conditions (13.28), we have

$$\mathcal{L}[y^{(iv)}t] = s^4Y(s) - s^3, \text{ and } \mathcal{L}(0) = 0.$$

Using these, Eq. (13.29) becomes $(s^4 - k^4)Y(s) = s^3$, which gives

$$Y(s) = \frac{s^3}{s^4 - k^4} = \frac{s^3}{(s^2+k^2)(s-k)(s+k)}.$$

When resolved in partial fractions, we obtain

$$Y(s) = \frac{1}{4}\frac{1}{s-k} + \frac{1}{4}\frac{1}{s+k} + \frac{1}{2}\frac{s}{s^2+k^2} \quad \text{...(13.30)}$$

Taking the inverse Laplace transforms and using the linearity of inverse transforms, (13.30) gives

$$y(t) = \frac{1}{4}e^{kt} + \frac{1}{4}e^{-kt} + \frac{1}{2}\cos kt = \frac{1}{2}\left[\frac{e^{kt}+e^{-kt}}{2}\right] + \frac{1}{2}\cos kt = \frac{1}{2}[\cosh kt + \cos kt], \quad t \ge 0.$$

Example 13.22: Solve the boundary value problem

$$y''(t) + 9y(t) = \cos 2t \quad \text{...(13.31)}$$

with
$$y(0) = 1, \quad y(\pi/2) = -1. \quad \text{...(13.32)}$$

Solution: Since $y'(0)$ is not given we assume $y'(0) = c$, say. Taking the Laplace transforms of both sides of (13.31) and using the linearity property, we obtain

$$\mathcal{L}[y''(t)] + 9\mathcal{L}[y(t)] = \mathcal{L}[\cos 2t]. \quad \text{...(13.33)}$$

Setting $\mathcal{L}[y(t)] = Y(s)$ and using the initial conditions $y(0) = 1$ and $y'(0) = c$, we have

$$\mathcal{L}[y''(t)] = s^2Y(s) - s - c. \text{ Also } \mathcal{L}[\cos 2t] = \frac{2}{s^2+4}.$$

Using these, Eq. (13.33) becomes

$$[s^2Y(s) - s - c] + 9Y(s) = \frac{2}{s^2+4} \text{ or, } (s^2+9)Y(s) = s + c + \frac{2}{s^2+4}, \text{ which gives}$$

$$Y(s) = \frac{c}{s^2+9} + \frac{s}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}. \quad \text{...(13.34)}$$

Taking the inverse Laplace transform and using the linearity of inverse transforms, (13.34) gives

$$y(t) = \frac{c}{3}\sin 3t + \cos 3t + \frac{1}{3}\int_0^t \cos 2\tau \sin 3(t-\tau)\,d\tau$$

$$= \frac{c}{3}\sin 3t + \cos 3t + \frac{1}{6}\int_0^t [\sin(3t-\tau) + \sin(3t-5\tau)]\,d\tau$$

$$= \frac{c}{3}\sin 3t + \cos 3t + \frac{1}{6}\left[\cos(3t-\tau) + \frac{1}{5}\cos(3t-5\tau)\right]_0^t$$

$$= \frac{c}{3} \sin 3t + \cos 3t + \frac{1}{5} (\cos 2t - \cos 3t)$$

or, $$y(t) = \frac{c}{3} \sin 3t + \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t. \quad ...(13.35)$$

To find c, use $y(\pi/2) = -1$, (13.35) gives $-1 = -\frac{c}{3} - \frac{1}{5}$, or $c = \frac{12}{5}$.

Hence, the solution is $y(t) = \frac{4}{5} \sin 3t + \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t, \quad t \geq 0.$

Example 13.23: Solve the differential equation

$$ty'' - (t+2)y' + 3y = t - 1 \quad ...(13.36)$$

with $$y(0) = 0, \quad y(2) = 9. \quad ...(13.37)$$

Solution: Since $y'(0)$ is not given, we assume $y'(0) = c$, say. Taking the Laplace transforms of both sides of (13.36) and using the linearity property, we obtain

$$\mathcal{L}[ty''] - \mathcal{L}[(t+2)y'] + 3\mathcal{L}[y] = \mathcal{L}[t-1] \quad ...(13.38)$$

Setting $\mathcal{L}[y(t)] = Y(s)$ and using

$$\mathcal{L}[ty''] = -\frac{d}{ds}\mathcal{L}[y''], \quad \mathcal{L}[ty'] = -\frac{d}{ds}\mathcal{L}[y'], \quad \mathcal{L}(t-1) = \frac{1}{s^2} - \frac{1}{s},$$ (13.38) gives

$$-\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] + \frac{d}{ds}[sY(s) - y(0)] - 2[sY(s) - y(0)] + 3Y(s) = \frac{1}{s^2} - \frac{1}{s}.$$

Using the initial conditions $y(0) = 0$ and $y'(0) = c$, it becomes

$$-\frac{d}{ds}[s^2Y(s) - c] + \frac{d}{ds}[sY(s)] - 2sY(s) + 3Y(s) = \frac{1}{s^2} - \frac{1}{s},$$

or, $$-s^2Y'(s) - 2sY(s) + sY'(s) + Y(s) - 2sY(s) + 3Y(s) = \frac{1}{s^2} - \frac{1}{s} \quad ...(13.39)$$

where the prime denotes the differentiation w.r.t. s. Simplifying, (13.39) gives

$$(-s^2 + s)Y'(s) + 4(1-s)Y(s) = \frac{s - s^2}{s^3} \text{ or, } Y'(s) + \frac{4}{s}Y(s) = \frac{1}{s^3} \quad ...(13.40)$$

Equation (13.40) is a first order linear differential equation in Y with integrating factor

$$\exp\left(\int \frac{4}{s}\, ds\right) = e^{4 \ln s} = s^4.$$

Multiplying (13.40) throughout by s^4 and then integrating, we get

$$Y(s) = \frac{1}{2s^2} + \frac{c_1}{s^4}, \quad ...(13.41)$$

where c_1 is an arbitrary constant.

Taking the inverse transform of (13.41), we obtain

$$y(t) = \frac{1}{2}t + \frac{c_1}{6}t^3. \qquad ...(13.42)$$

To evaluate c_1, use $y(2) = 9$, we obtain $9 = \frac{2}{2} + \frac{8c_1}{6}$, or $c_1 = 6$.

The solution is therefore $y(t) = \frac{1}{2}t + t^3, \quad t \geq 0$.

Example 13.24: Solve the differential equation

$$ty'' + y' + ty = 0 \qquad ...(13.43)$$

with $\quad y(0) = 2, \quad y'(0) = 0. \qquad ...(13.44)$

Solution: Taking the Laplace transforms of both sides of (13.43) and using the linearity property we have $\mathcal{L}[ty''(t)] + \mathcal{L}[y'(t)] + \mathcal{L}[ty(t)] = 0$.

Setting $\mathcal{L}[y(t)] = Y(s)$ and using $\mathcal{L}[ty(t)] = -\frac{d}{ds}\mathcal{L}[y(t)]$ etc., it becomes

$$-\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] - \frac{d}{ds}[Y(s)] = 0.$$

Using the initial conditions (13.44), it gives

$$-[s^2Y'(s) + 2sY(s) - 2] + [sY(s) - 2] - Y'(s) = 0$$

or, $$(s^2 + 1)Y'(s) + sY(s) = 0 \qquad ...(13.45)$$

where prime denotes derivative w.r.t.s. Eq. (13.45) is a first order linear differential equation in Y, separating the variables and integrating, we have

$$\int \frac{dY(s)}{Y(s)} + \int \frac{s\,ds}{s^2+1} = c', \text{ which gives, } \ln Y(s) + \frac{1}{2}\ln(s^2+1) = c',$$

or, $$Y(s) = \frac{c}{\sqrt{s^2+1}}, \qquad ...(13.46)$$

where c is an arbitrary constant.

Taking inverse transform of (13.46), we obtain

$$y(t) = c\mathcal{L}^{-1}\left[\frac{1}{\sqrt{1+s^2}}\right] = cJ_0(t), \qquad \text{[ref. to Example (13.18 b)]}$$

where $J_0(t)$ is Bessel function of the first kind of order zero.

To find c, we have $y(0) = cJ_0(0) = c$, since $J_0(0) = 1$, which gives $c = 2$, and hence, $y(t) = 2J_0(t)$.

Example 13.25: Solve the differential equation

$$y'' + 2ty' - 4y = 1, \qquad ...(13.47)$$

with $$y(0) = y'(0) = 0. \qquad ...(13.48)$$

Solution: Taking the Laplace transforms of both sides of (13.47) and using the linearity property, we obtain

$$\mathcal{L}[y''] + 2\mathcal{L}[ty'] - 4\mathcal{L}[y] = \mathcal{L}[1]. \qquad ...(13.49)$$

Setting $\mathcal{L}[y(t)] = Y(s)$ and using $\mathcal{L}[ty'(t)] = -\frac{d}{ds}\mathcal{L}[y'(t)]$, it gives

$$[s^2Y(s) - sy(0) - y'(0)] - 2\frac{d}{ds}[sY(s) - y(0)] - 4Y(s) = \frac{1}{s}.$$

Using the initial conditions (13.48), it becomes

$$s^2Y(s) - 2[Y(s) + sY'(s)] - 4Y(s) = \frac{1}{s}, \text{ or } Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right)Y(s) = -\frac{1}{2s^2}, \qquad ...(13.50)$$

where prime denotes the derivative w.r.t. s.

Eq. (13.50) is a linear first order differential equation in Y, with integrating factor

$$\exp\left\{\int\left(\frac{3}{s} - \frac{s}{2}\right)ds\right\} = \exp\left\{3\ln s - \frac{s^2}{4}\right\} = s^3e^{-s^2/4}.$$

Multiplying (13.50) throughout by $s^3e^{-s^2/4}$ and then integrating, we get

$$Y(s) = \frac{1}{s^3} + \frac{c}{s^3}e^{s^2/4}. \qquad ...(13.51)$$

There is no further initial condition to determine c, an arbitrary constant, in (13.51). However, in order to have $\lim_{s\to\infty} Y(s) = 0$, we must choose $c = 0$. Thus,

$$Y(s) = \frac{1}{s^3} \qquad ...(13.52)$$

Taking inverse transform of (13.52), we obtain $y(t) = \frac{1}{2}t^2, t \geq 0$

as the solution of the given initial value problem.

13.4.2 Solution of Simultaneous Differential Equations

Here, we apply the Laplace transform in solving system of simultaneous differential equations.

Example 13.26: Solve the system of simultaneous equations

$$x'(t) + y'(t) + x(t) = -e^{-t}, \quad x'(t) + 2y'(t) + 2x(t) + 2y(t) = 0; \qquad ...(13.53)$$

$$x(0) = -1, \quad y(0) = 1. \qquad ...(13.54)$$

Solution: Taking Laplace transform on both sides of Eqs. (13.53), we obtain

$$sX(s) - x(0) + sY(s) - y(0) + X(s) = -\frac{1}{s+1}$$

and,
$$sX(s) - x(0) + 2[sY(s) - y(0)] + 2X(s) + 2Y(s) = 0$$

Using the initial conditions (13.54), these equations become respectively

$$(s+1)X(s) + sY(s) = -\frac{1}{s+1} \qquad \text{...(13.55)}$$

and,
$$(s+2)X(s) + 2(s+1)Y(s) = 1. \qquad \text{...(13.56)}$$

Solving (13.55) and (13.56) simultaneously for $X(s)$ and $Y(s)$, we obtain

$$X(s) = -\frac{s+2}{s^2+2s+2}, \text{ and } Y(s) = \frac{s^2+3s+3}{(s^2+2s+2)(s+1)}$$

or,
$$X(s) = -\left[\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\right], \text{ and } Y(s) = \left[\frac{1}{s+1} + \frac{1}{(s+1)^2+1}\right]$$

Taking inverse Laplace transforms, we get $x(t) = -e^{-t}(\cos t + \sin t)$, and $y(t) = e^{-t}(1 + \sin t)$ as the solution of the given system of equations.

Example 13.27: An electron is projected into a uniform magnetic field which is perpendicular to its direction of motion. If (x, y) denotes its position at any time t, then equations of motion are

$$mx'' = ky', \text{and } \; my'' = -kx' \qquad \text{...(13.57)}$$

with
$$x(0) = y(0) = y'(0) = 0, \text{ and } \; x'(0) = c; \qquad \text{...(13.58)}$$

where m, k and c are constants.

Use Laplace transform to find x and y at time t.

Solution: Taking the Laplace transforms to both sides of the equations (13.57), we get

$$m\mathcal{L}[x''(t)] = k\mathcal{L}[y'(t)], \text{ and } \; m\mathcal{L}[y''(t)] = -k\mathcal{L}[x'(t)].$$

Setting $\mathcal{L}[x(t)] = X(s)$ and $\mathcal{L}[y(t)] = Y(s)$, and using the property of transform of derivatives, these become

$$m[s^2X(s) - sx(0) - x'(0)] = k[sY(s) - y(0)],$$

and,

$$m[s^2Y(s) - sy(0) - y'(0)] = -k[sX(s) - x(0)].$$

Using the initial conditions (13.58), we obtain respectively

$$ms^2X(s) - ksY(s) = mc \qquad \text{...(13.59)}$$

and,
$$ms^2Y(s) + ksX(s) = 0. \qquad \text{...(13.60)}$$

Solving Eqs. (13.59) and (13.60) for $X(s)$ and $Y(s)$, we have

$$X(s) = \frac{c}{s^2+k^2/m^2} \text{ and } Y(s) = \frac{c}{k/m}\left(\frac{s}{s^2+k^2/m^2} - \frac{1}{s}\right).$$

Taking inverse Laplace transforms, we obtain

$$x(t) = \frac{mc}{k}\sin\frac{kt}{m}, \quad \text{and} \quad y(t) = \frac{mc}{k}\left(\cos\frac{kt}{m} - 1\right), \quad t \geq 0$$

as the solution of the given system.

13.4.3 Solution of Integral and Integro-Differential Equations

An *integral equation* involves the integral of an unknown function and an *integro-differential equation* involves both the integral of an unknown function and its derivative. These equations occur in many applications of mathematics to various branches of engineering.

For example, in *R-C* circuit, refer to Fig. 13.3, with electromotive force $E(t)$, using the Kirchoff's law the flow of current satisfies the integral equation $Ri + \frac{1}{C}\int_0^t i\,dt = E(t)$.

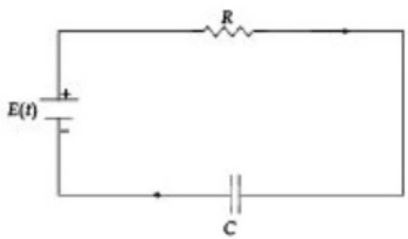


Fig. 13.3

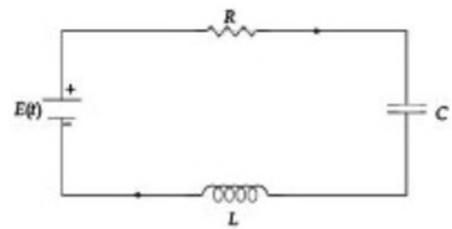


Fig. 13.4

An example of an integro-differential equation is obtained while considering the *R-L-C* circuit, refer to Fig. 13.4. Using the Kirchoff's law, the flow of current satisfies the integro-differential equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = E(t).$$

Example 13.28: Solve the integral equation $y(t) = 2e^{-t} + \int_0^t \sin(t-\tau)y(\tau)d\tau$.

Solution: Taking the Laplace transforms of both sides to the given integral equation and setting $\mathcal{L}[y(t)] = Y(s)$, we get

$$Y(s) = \frac{2}{s+1} + \mathcal{L}\left[\int_0^t \sin(t-\tau)\,y(\tau)d\tau\right]$$

Using the convolution theorem, refer to (13.20), it gives

$$Y(s) = \frac{2}{s+1} + \frac{Y(s)}{s^2+1}, \text{ or } Y(s) = \frac{2(s^2+1)}{s^2(s+1)}.$$

Resolving it in partial fractions, we obtain $Y(s) = \frac{2}{s^2} - \frac{2}{s} + \frac{4}{s+1}$.

Taking the inverse transform, we obtain $y(t) = 2t - 2 + 4e^{-t}, t \geq 0$ as the solution of the integral equation.

Example 13.29: Solve the integro-differential equation $y' + 3y + 2\int_0^t y\,dt = t, y(0) = 0$.

Solution: Taking the Laplace transform on both sides of the given equation and setting $\mathcal{L}[y(t)] = Y(s)$, we obtain $[sY(s) - y(0)] + 3Y(s) + \frac{2}{s}Y(s) = \frac{1}{s^2}$.

Using the initial condition $y(0) = 0$ and simplifying, we get $Y(s) = \frac{1}{s(s+1)(s+2)}$.

Resolving it in partial fractions, we obtain $Y(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}$.

Taking the inverse transform, we obtain $y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$

as the solution of the given integro-differential equation value problem.

EXERCISE 13.4

Solve the following initial value problem.

1. $y'(t) - 4y(t) = t, \quad y(0) = -1$
2. $y'(t) + 2y(t) = \sin t, \quad y(0) = -1$
3. $y''(t) + 4y(t) = 5e^{-t}, \quad y(0) = 2, \quad y'(0) = 3$
4. $y''(t) + 2y'(t) + y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1$
5. $y'''(t) - y'(t) = 2\cos t, \quad y(0) = 3, \quad y'(0) = 2, \quad y''(0) = 1$
6. $y''(t) + 4y'(t) + 13y = 2e^{-2t}\sin 3t, \quad y(0) = 1, \quad y'(0) = 0$
7. $y''(t) + 9y(t) = 18t, \quad y(0) = 0, \quad y(\pi/2) = 0$
8. $y'''(t) + 2y''(t) - y'(t) - 2y(t) = 0, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 2$
9. $y^{iv}(t) + 2y''(t) + y(t) = \sin t, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0$
10. $y''(t) + n^2y(t) = a\sin(nt + \alpha), \quad y(0) = y'(0) = 0$

Solve the following initial value problems with polynomial coefficients

11. $ty''(t) + 2y'(t) + ty(t) = \cos t, \quad y(0) = 1$

12. $ty''(t) + (1 - 2t)\, y'(t) - 2y(t) = 0, \quad y(0) = 1, \quad y'(0) = 2$

13. $y''(t) + 2ty'(t) - y(t) = t, \quad y(0) = 0, \quad y'(0) = 1$

14. $y''(t) + ty'(t) - 2y(t) = 6 - t, \quad y(0) = 0, \quad y'(0) = 1$

15. $ty''(t) + 2ty'(t) + 2y(t) = 2, \quad y(0) = 1, \quad y'(0)$ is arbitrary.

Solve the following system of initial value problems

16. $x'(t) - 2x(t) + y(t) = \sin t, \quad y'(t) + 2x(t) - y(t) = 1, \quad x(0) = 1, y(0) = -1$

17. $x'(t) + y'(t) + x(t) = -e^{-t}, \quad x'(t) + 2y'(t) + 2(x(t) + y(t)) = 0; \quad x(0) = -1, y(0) = 1$

18. $x'(t) + 2y''(t) = e^{-t}, \quad x'(t) + 2x(t) - y(t) = 1, \quad x(0) = y(0) = y'(0) = 0$

19. $x'(t) + x(t) - z(t) = 1, \; y'(t) - x(t) + y(t) = 1, \; z'(t) + y(t) - x(t) = 0, \; x(0) = 1, \; y(0) = 0, \; z(0) = 1$

20. $x'(t) - z(t) = e^t, \quad y'(t) - z(t) = 2, \quad z'(t) - x(t) = 1; \quad x(0) = 1, \quad y(0) = 0, z(0) = 1.$

Solve the following integral equations

21. $y(t) = \sin t + \int_0^t \sin(t - \tau) y(\tau) d\tau$

22. $y(t) = t^2 + \int_0^t \cos(t - \tau) y(\tau) d\tau$

23. $y(t) = 2t^2 + \int_0^t y(t - \tau) e^{-\tau} d\tau$

24. $y(t) = e^{-3t}\left[e^t - 3\int_0^t y(\tau) e^{3\tau} d\tau\right]$

25. $y(t) = 1 - \sinh t + \int_0^t (1 + \tau) y(t - \tau) d\tau$

Solve the following integral-differential equations

26. $y'(t) - 4y(t) + 3\int_0^t y(t) dt = t, \quad y(0) = 1.$

27. $y'(t) + 4y(t) = 4\int_0^t \sin \tau \, y(t - \tau) d\tau, \quad y(0) = 1$

28. $y'(t) - y(t) - 6\int_0^t y(t) dt = \sin t, \quad y(0) = 2$

29. $y''(t) + y(t) = \int_0^t \sin \tau \, y(t - \tau) d\tau, \quad y(0) = 1, \quad y'(0) = 0$

30. $y''(t) - y(t) = \int_0^t \sinh \tau \, y(t - \tau) d\tau, \quad y(0) = 1, \quad y'(0) = 0.$

13.5 THE HEAVISIDE FUNCTION. THE UNIT PULSE FUNCTION. SECOND SHIFTING THEOREM

So far we have considered the Laplace transform of a function if it is piecewise continuous on $0 \le t < a$, for every a and is of exponential order as $t \to \infty$. We have thus far avoided functions with discontinuities. In applications, however, a system is oftenly subjected to discontinuous forcing function. For example, an external force acting on a mechanical system or a voltage applied to an electrical circuit can be turned off after a certain period of time. Now we study systems with forcing functions that are discontinuous, although we still assume that they are piecewise continuous on $0 \le t < a$, for every a and are of exponential order as $t \to \infty$, so that their Laplace transforms exist. This will lay the foundation for solving certain initial value problems having discontinuous forcing functions. Before proceeding further, we remark that a function $f(t)$ has a *jump discontinuity* at $t = a$, if the left and right hand limits of $f(t)$ as $t \to a$ both exist but are not equal. The difference $\left|\lim\limits_{t \to a+} f(t) - \lim\limits_{t \to a-} f(t)\right|$ is termed as the *magnitude of the jump*, refer to Fig. 13.5. Functions with jump discontinuities can be effectively represented by the *Heaviside function* or, the *unit step function* as explained next.

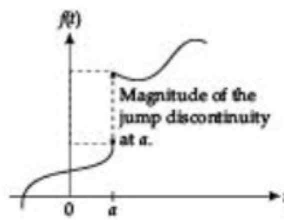


Fig. 13.5

13.5.1 Heaviside Function (or, Unit Step Function)

The *Heaviside function* H is defined by

$$H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \ge 0 \end{cases} \qquad ...(13.61)$$

A graph of $H(t)$ is shown in Fig. 13.6. It has a jump discontinuity of magnitude 1 at 0. The Heaviside function may be thought of a flat switching function, 'on' when $t \ge 0$, where $H(t) = 1$, and 'off' when $t < 0$, where $H(t) = 0$. Further, if a is any number, then $H(t - a)$ is the Heaviside function shifted a units to the right as shown in Fig. 13.7, since

$$H(t-a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t \ge a \end{cases} \qquad ...(13.62)$$

$H(t - a)$ models a flat signal of magnitude 1, turned off until time $t = a$ and then switched on.

The Heaviside function $H(t)$, also called the *unit step function*, is also denoted by $u(t)$, and $H(t - a)$ by $u(t - a)$, or $u_a(t)$.

The following are some examples of functions represented in terms of unit step functions

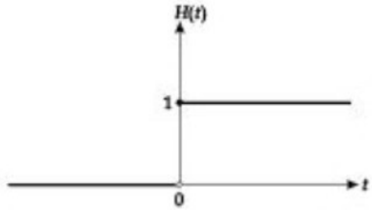


Fig. 13.6

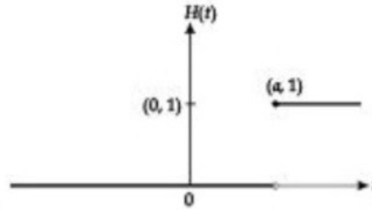


Fig. 13.7

1. $f(t) = u(t-\pi)\cos t = \begin{cases} 0 & \text{if } t < \pi \\ \cos t & \text{if } t \geq \pi \end{cases}$ as shown in Fig. 13.8.

2. $g(t) = u(t-2)t^2 = \begin{cases} 0 & \text{if } t < 2 \\ t^2 & \text{if } t \geq 2 \end{cases}$ as shown in Fig. 13.9.

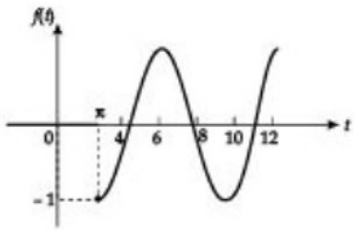


Fig. 13.8

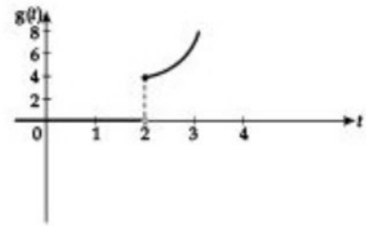


Fig. 13.9

The unit step function is frequently used in engineering applications which often involve functions that are either 'off' or 'on'. Fig. 13.10 shows the effect of three unit step functions

$$f(t) = k[u_1(t) - 2u_4(t) + u_6(t)]$$

and Fig. 13.11 shows the effect of infinitely many unit step functions,

$$g(t) = 4\sin\left(\frac{\pi}{2}t\right)[u_0(t) - u_2(t) - u_4(t) - \ldots].$$

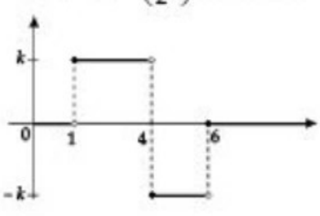


Fig. 13.10

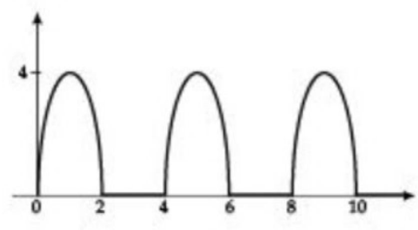


Fig. 13.11

13.5.2 The Unit Pulse Function

A *unit pulse function* is of the form $f(t) = u(t-a) - u(t-b)$, $a < b$, and is represented as shown in Fig. 13.12. It has value 0, if $t < a$, value 1, if $a \le t < b$ and value 0, if $t \ge b$. As another example, the function

$$f(t) = [u(t-1) - u(t-2)]e^t = \begin{cases} 0, & \text{if } t < 1 \\ e^t, & \text{if } 1 \le t < 2 \\ 0, & \text{if } t \ge 2 \end{cases}$$

is represented in Fig. 13.13.

Laplace transform of unit-step function. By definition

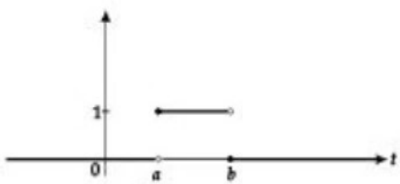


Fig. 13.12

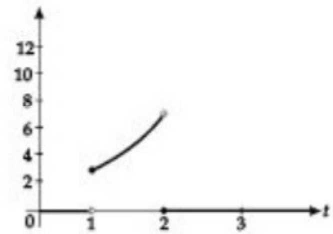


Fig. 13.13

$$\mathcal{L}[u(t-a)] = \int_0^\infty u(t-a)e^{-st}dt$$

$$= \int_a^\infty e^{-st}dt = \left[\frac{-e^{-st}}{s}\right]_a^\infty = \frac{e^{-as}}{s}, \text{ for } s \ge 0, \quad a \ge 0. \quad ...(13.63)$$

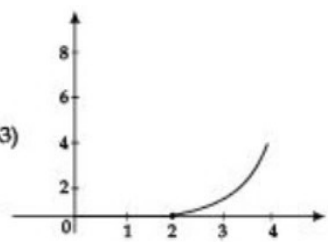


Fig. 13.14

Laplace transform of unit-pulse function. By definition

$$\mathcal{L}[u(t-a) - u(t-b)] = \int_0^\infty [u(t-a) - u(t-b)]e^{-st}dt$$

$$= \int_a^b e^{-st}dt = \int_a^\infty e^{-st}dt - \int_b^\infty e^{-st}dt = \frac{e^{-as} - e^{-bs}}{s}, \text{ for } s > 0, \quad b > a \ge 0. \quad ...(13.64)$$

Next, we consider *shifted function* of the form $u(t-a)f(t-a)$. It is given by

$$u(t-a)\,f(t-a) = \begin{cases} 0, & t<a \\ f(t-a), & t \geq a \end{cases}$$

which is same as $f(t)$ shifted a units to the right.

For example, the graph of the shifted function $u(t-2)(t-2)^2$ is shown in Fig. 13.14.

Comparing it with the graph in Fig. 13.9, we observe that the graph of $u(t-2)(t-2)^2$ is zero along the horizontal axis until $t=2$, and for $t \geq 2$, the graph is of t^2 for $t \geq 0$ shifted 2 units to the right to begin at 2 instead of zero.

Now we are in a position to give the *Second Shifting Theorem* which deals with the Laplace transform of such a function.

13.5.3 Second Shifting Theorem (or, the *t*-Shifting)

We have the following result:

Theorem 13.9 (t-Shifting): *If $\mathcal{L}\{f(t)\} = F(s)$, then*

$$\mathcal{L}[u(t-a)f(t-a)] = e^{-as}\,F(s) \qquad ...(13.65)$$

where $u(t-a)\,f(t-a) = \begin{cases} 0, & \text{if } t<a \\ f(t-a), & \text{if } t \geq a \end{cases}$ *is the shifted function.*

Proof. By definition $\mathcal{L}[u(t-a)\,f(t-a)] = \int_0^\infty u(t-a)\,f(t-a)e^{-st}\,dt$

$$= \int_a^\infty f(t-a)e^{-st}\,dt = \int_0^\infty f(\tau)e^{-s(\tau+a)}\,d\tau,\ (\tau = t-a)$$

$$= e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = e^{-as}\mathcal{L}[f(t)] = e^{-as}F(s).$$

Thus, the Laplace transform of $u(t-a)f(t-a)$ is obtained by multiplying the Laplace transform of $f(t)$ by e^{-as}.

For example, $\mathcal{L}[u(t-a)] = \mathcal{L}[u(t-a)f(t-a)]$, with $f(t) = 1$ for all t

$$= e^{-as}\mathcal{L}[f(t)] = e^{-as}\mathcal{L}[1] = \frac{e^{-as}}{s}, \text{ as obtained in (13.63).}$$

Also applying the inverse transform to both sides of (13.65), we obtain

$$\mathcal{L}^{-1}[e^{-as}\,F(s)] = u(t-a)f(t-a). \qquad ...(13.66)$$

Example 13.30: Find the Laplace transform of the function

$$f(t) = \begin{cases} 2, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \\ \sin t, & t \ge 2\pi \end{cases}$$

Solution: Writing $f(t)$ as $f(t) = 2u(t) - 2u(t-\pi) + u(t-2\pi)\sin t$

$$= 2u(t) - 2u(t-\pi) + u(t-2\pi)\sin(t-2\pi), \quad \text{...(13.67)}$$

since $\sin t = \sin(t-2\pi)$, because of the periodicity of $\sin t$.

Taking the Laplace transforms on both sides of (13.67) and using the second shifting theorem, we obtain $F(s) = \dfrac{2}{s} - \dfrac{2e^{-\pi s}}{s} + \dfrac{e^{-2\pi s}}{s^2+1}$.

Example 13.31: Find the Laplace transform of

(a) $f(t) = \begin{cases} (t-1), & 1 \le t < 2 \\ (3-t), & 2 \le t < 3 \end{cases}$ (b) $f(t) = \begin{cases} 2+t^2 & 0 \le t < 2 \\ 6, & 2 \le t < 3 \\ (2t-5), & 3 \le t < \infty \end{cases}$

Solution: (a) Writing $f(t)$ as $f(t) = (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$

$$= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3). \quad \text{...(13.68)}$$

Taking the Laplace transforms on both sides of (13.68) and using the second shifting theorem, we obtain

$$F(s) = \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2} = (e^{-s} - 2e^{-2s} + e^{-3s})/s^2.$$

(b) Writing $f(t)$ as

$$f(t) = (2+t^2)[u(t) - u(t-2)] + 6[u(t-2) - u(t-3)] + (2t-5)u(t-3)$$
$$= 2u(t) + t^2u(t) - (t^2-4)u(t-2) + (2t-11)u(t-3)$$
$$= 2u(t) + t^2u(t) - (t-2)^2u(t-2) - 4(t-2)u(t-2) + 2(t-3)u(t-3) - 5u(t-3). \quad \text{...(13.69)}$$

Taking the Laplace transforms on both sides of (13.69) and using the second shifting theorem, we obtain

$$F(s) = \frac{2}{s} + \frac{2!}{s^3} - \frac{2!e^{-2s}}{s^3} - \frac{4e^{-2s}}{s^2} + \frac{2e^{-3s}}{s^2} - \frac{5e^{-3s}}{s} = 2\left(\frac{1}{s} + \frac{1}{s^3}\right) - 2\left(\frac{1}{s^3} + \frac{2}{s^2}\right)e^{-2s} + \left(\frac{2}{s^2} - \frac{5}{s}\right)e^{-3s}.$$

Example 13.32: Find the inverse Laplace transforms of

(a) $\dfrac{3s+1}{s^2(s^2+4)}e^{-3s}$ (b) $\dfrac{\left(e^{-2s} + e^{-3s}\right)}{s(s+4)}$

Solution: Let $f(t) = \mathcal{L}^{-1}\left[\dfrac{3s+1}{s^2(s^2+4)}e^{-3s}\right]$.

Using partial fraction, $\dfrac{3s+1}{s^2(s^2+4)} = \dfrac{3}{4s} + \dfrac{1}{4s^2} - \dfrac{(3s+1)}{4(s^2+4)}$. Thus,

$$\mathcal{L}^{-1}\left[\frac{3s+1}{s^2(s^2+4)}e^{-3s}\right] = \frac{3}{4}\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right] + \frac{1}{4}\left[\frac{e^{-3s}}{s^2}\right] - \frac{3}{4}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}e^{-3s}\right] - \frac{1}{8}\left[\frac{2e^{-3s}}{s^2+4}\right] \quad \text{...(13.70)}$$

We have $\mathcal{L}^{-1}\left[\dfrac{1}{s}\right] = 1$, $\mathcal{L}^{-1}\left[\dfrac{1}{s^2}\right] = t$, $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+4}\right] = \cos 2t$, and $\mathcal{L}^{-1}\left[\dfrac{2}{s^2+4}\right] = \sin 2t$

Also from t-shifting, refer to (13.66), $\mathcal{L}^{-1}[e^{-as}F(s)] = u(t-a)f(t-a)$. Thus the inverse transform of (13.70) gives

$$f(t) = \frac{3}{4}u(t-3) + \frac{1}{4}(t-3)\,u(t-3) - \frac{3}{4}\cos 2(t-3)u(t-3) - \frac{1}{8}\sin 2(t-3)u(t-3).$$

or,
$$f(t) = \begin{cases} 0, & \text{for } t < 3 \\ \dfrac{3}{4} + \dfrac{1}{4}(t-3) - \dfrac{3}{4}\cos 2(t-3) - \dfrac{1}{8}\sin 2(t-3), & \text{for } t \ge 3 \end{cases}$$

(b) Let $f(t) = \mathcal{L}^{-1}\left[\dfrac{e^{-2s}+e^{-3s}}{s(s+4)}\right]$.

Using partial fraction, $\dfrac{1}{s(s+4)} = \dfrac{1}{4}\left(\dfrac{1}{s} - \dfrac{1}{s+4}\right)$ Thus,

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}+e^{-3s}}{s(s+4)}\right] = \frac{1}{4}\left[\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s}\right) - \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+4}\right) + \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s}\right) - \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s+4}\right)\right] \quad \text{...(13.71)}$$

We have $\mathcal{L}^{-1}\left(\dfrac{1}{s}\right) = 1$, $\mathcal{L}^{-1}\left(\dfrac{1}{s+4}\right) = e^{-4t}$, and $\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = u(t-a)f(t-a)$.

Thus the inverse transform of (13.71) gives

$$f(t) = \frac{1}{4}u(t-2) - \frac{1}{4}e^{-4(t-2)}u(t-2) + \frac{1}{4}u(t-3) - \frac{1}{4}e^{-4(t-3)}u(t-3)$$

$$f(t) = \begin{cases} 0, & 0 \le t < 2 \\ \dfrac{1}{4}(1 - e^{-4(t-2)}), & 2 \le t < 3 \\ \dfrac{1}{4}(1 - e^{-4(t-2)} - e^{-4(t-3)}), & t \ge 3 \end{cases}$$

Example 13.33: In an R-C circuit, the current $i(t)$ is given by $R\,i(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$.

Find the current i at any instant t, if a single square wave with voltage v_0 is connected at $t = a$ and disconnected at $t = b$.

Solution: We have $R\,i(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$, ...(13.72)

where
$$v(t) = \begin{cases} 0, & 0 \le t < a \\ v_0, & a \le t < b \\ 0, & t \ge b \end{cases}$$

Representing $v(t)$ in terms of the pulse function, $v(t) = v_0[u(t-a) - u(t-b)]$, (13.72) becomes

$$R\,i(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v_0[u(t-a) - u(t-b)].$$

Taking the Laplace transform on both sides of this equation and setting $\mathcal{L}[i(t)] = I(s)$, we obtain

$$RI(s) + \frac{I(s)}{sC} = \frac{v_0}{s}(e^{-as} - e^{-bs}), \text{ or } I(s) = \frac{v_0/R}{s + 1/(RC)}(e^{-as} - e^{-bs}). \quad ...(13.73)$$

Since $\mathcal{L}^{-1}\left(\frac{v_0/R}{s + 1/RC}\right) = \frac{v_0}{R}e^{-\frac{1}{RC}t}$, and $\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = u(t-a)f(t-a)$ thus the inverse transform of (13.73) gives

$$i(t) = \frac{v_0}{R}\left[e^{-\frac{1}{RC}(t-a)}u(t-a) - e^{-\frac{1}{RC}(t-b)}u(t-b)\right]$$

or,
$$i(t) = \begin{cases} 0, & 0 \le t < a \\ \frac{v_0}{R}e^{-\frac{1}{RC}(t-a)}, & a \le t < b \\ \frac{v_0}{R}\left(e^{-\frac{1}{RC}(t-a)} - e^{-\frac{1}{RC}(t-b)}\right), & t \ge b \end{cases}$$

Example 13.34: Solve the initial value problem

$$y''(t) + 3y'(t) + 2y(t) = \begin{cases} 0, & 0 \le t < \pi \\ \sin 2t & t \ge \pi \end{cases} \quad ...(13.74)$$

with $$y(0) = 1 \text{ and } y'(0) = 0 \qquad ...(13.75)$$

Solution: Let $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ \sin 2t & t \ge \pi \end{cases}$

Representing $f(t)$ in terms of unit step function, we have $f(t) = u(t-\pi)\sin 2t$.

The Eq. (13.74) becomes $y''(t) + 3y'(t) + 2y(t) = u(t-\pi)\sin 2t$.

Taking Laplace transforms on both sides of this and setting $\mathcal{L}[y(t)] = Y(s)$, we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = \frac{2e^{-\pi s}}{s^2+4}.$$

Using the initial conditions (13.75) and simplifying, it gives

$$(s^2+3s+2)\,Y(s) = s+3+\frac{2e^{-\pi s}}{s^2+4}, \text{ or } Y(s) = \frac{s+3}{(s^2+3s+2)} + \frac{2e^{-\pi s}}{(s^2+3s+2)(s^2+4)}.$$

Using partial fraction expansion, we have

$$Y(s) = \frac{2}{s+1} - \frac{1}{s+2} + e^{-\pi s}\left[\frac{2}{5}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s+2} - \frac{1}{20}\frac{2}{s^2+4} - \frac{3}{20}\frac{s}{s^2+4}\right].$$

Taking inverse transform and using that $\mathcal{L}^{-1}(e^{-as}F(s)) = u(t-a)f(t-a)$, we obtain for $t \ge 0$,

$$y(t) = 2e^{-t} - e^{-2t} + u(t-\pi)\left[\frac{2}{5}e^{-(t-\pi)} - \frac{1}{4}e^{-2(t-\pi)} - \frac{1}{20}\sin 2(t-\pi) - \frac{3}{20}\cos 2(t-\pi)\right],$$

or, $$y(t) = \begin{cases} 2e^{-t} - e^{-2t}, & 0 \le t < \pi \\ 2e^{-t} - e^{-2t} + \left[\frac{2}{5}e^{-(t-\pi)} - \frac{1}{4}e^{-2(t-\pi)} - \frac{1}{20}\sin 2(t-\pi) - \frac{3}{20}\cos 2(t-\pi)\right], & t \ge \pi. \end{cases}$$

13.6 DIRAC-DELTA FUNCTION. TRANSFORM AND FILTERING PROPERTY

Many-engineering applications involve the concept of an *impulse*, which may be interpreted *as a force of large magnitude applied over for an instant of time*, for example, the force due to hammer blow. We can model an impulse as follows.

For any positive number ϵ, consider the pulse δ_ϵ defined by

$$\delta_\epsilon(t) = \frac{1}{\epsilon}[u(t) - u(t-\epsilon)] \qquad ...(13.76)$$

As shown in Fig. 13.15, this is a pulse of magnitude $\frac{1}{\epsilon}$ and duration ϵ. By letting ϵ approach zero, we obtain pulse of increasing magnitude over shorter time intervals.

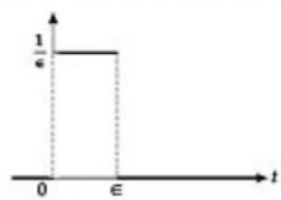


Fig. 13.15

13.6.1 The Dirac-Delta Function

Dirac delta function is actually a limiting mathematical operation and not a function as its name implies. *It can be considered as a pulse of infinite magnitude over an infinitely short duration and is defined to be*

$$\delta(t) = \lim_{\epsilon \to 0^+} \delta_\epsilon(t). \qquad ...(13.77)$$

The delta function can be considered to be the limit of a rectangular pulse of height $1/\epsilon$ and width ϵ in the limit as $\epsilon \to 0^+$. Thus the area of the graph representing the pulse remains constant at 1 as $\epsilon \to 0^+$, while its width decreases to zero and its height increases to infinity. Therefore, we may interpret $\delta(t)$ as

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \infty & \text{for } t = 0 \end{cases} \qquad ...(13.78)$$

Further taking the limit as $\epsilon \to 0$, from (13.76) and (13.77) *we can interpret $\delta(t)$ as the derivative of the unit step function $u(t)$, though the ordinary derivative of $u(t)$ does not exist since it is discontinuous at $t = 0$, thus, $\delta(t)$ is to be considered as a generalized function.*

The shifted delta function $\delta(t - a)$ is zero except for $t = a$, where it has an infinite spike.

13.6.2 Laplace Transform of the Delta Function

The delta function does not satisfy the existence conditions. However, we can derive the Laplace transform of the delta function using its definition. To begin with consider

$$\delta_\epsilon(t-a) = \frac{1}{\epsilon}[u(t-a) - u(t-a-\epsilon)].$$

Then, $$\mathcal{L}[\delta_\epsilon(t-a)] = \frac{1}{\epsilon}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+\epsilon)s}}{s}\right] = \frac{e^{-as}(1-e^{-\epsilon s})}{\epsilon s}.$$

Taking the limit as $\epsilon \to 0^+$, we obtain

$$\mathcal{L}[\delta(t-a)] = \lim_{\epsilon \to 0^+}\left[\frac{e^{-as}(1-e^{-\epsilon s})}{\epsilon s}\right] = e^{-as}. \qquad ...(13.79)$$

In particular, when $a = 0$, we have

$$\mathcal{L}[\delta(t)] = 1. \qquad ...(13.80)$$

Also applying the inverse transformation to (13.79) and (13.80) we obtain receptively

$$\mathcal{L}^{-1}[e^{-as}] = \delta(t-a), \qquad ...(13.81)$$

and $$\mathcal{L}^{-1}[1] = \delta(t). \qquad ...(13.82)$$

Next we prove an important property of delta function.

13.6.3 Filtering Property of the Dirac-Delta Function

Theorem 13.10 (Filtering Property): Let $f(t)$ be defined and integrable on $[0, \infty)$, and let it be continuous in a neighbourhood of a. Then for $a \geq 0$

$$\int_0^\infty f(t)\delta(t-a)dt = f(a). \qquad ...(13.83)$$

Proof. First consider

$$\int_0^\infty f(t)\delta_\epsilon(t-a)dt = \int_0^\infty \frac{1}{\epsilon}[u(t-a)-u(t-a-\epsilon)]f(t)dt = \frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)dt \qquad ...(13.84)$$

Using the mean value theorem for integrals, we have

$$\int_a^{a+\epsilon} f(t)dt = f(t_\epsilon)\int_a^{a+\epsilon} dt = \epsilon f(t_\epsilon),$$ for some t_ϵ: $a < t_\epsilon < a+\epsilon$. Thus, (13.84) becomes

$$\int_0^\infty f(t)\,\delta_\epsilon(t-a)dt = f(t_\epsilon). \qquad ...(13.85)$$

As $\epsilon \to 0^+$, $a+\epsilon \to a$, so $t_\epsilon \to a$ and by continuity, $f(t_\epsilon) \to f(a)$. Thus, taking limit $\epsilon \to 0^+$, (13.85), becomes

$$\int_0^\infty f(t)\delta(t-a)dt = f(a),$$

this proves the desired result.

The filtering property interprets that, *if at time a, a signal is hit with an impulse, by multiplying it by $\delta(t-a)$, and the resulting signal is summed over all possible time by integrating from zero to infinity, then the result is exactly the signal value $f(a)$.*

We note that the Laplace transform of $\delta(t-a)$ can also be obtained by using the filtering property. Set $f(t) = e^{-st}$ in (13.83), we have

$$\int_0^\infty e^{-st}\delta(t-a)dt = f(a) = e^{-as}.$$

The delta function has many engineering applications. In mechanical problems it is used to represent an impulse, defined as the integral of a large force applied locally for a very short time. In electrical systems it can be used to represent the brief application of a very large voltage, or the sudden discharge of energy contained in a capacitor. The function is applied to study the behaviour of circuits that have been subjected to transients, like high input voltage generated during switching. Occasionally the transient may result in break-downing the system. Because of this reason, before a circuit is built, transients are modelled by the delta function and their effects on the circuit are studied to set safety standards for the system to be fabricated.

Example 13.35: Determine the response of the damped mass-spring system governed by

$$y''(t) + 3y'(t) + 2y(t) = \delta(t-1); \quad \text{...(13.86)}$$

$$y(0) = 0, \quad y'(0) = 1. \quad \text{...(13.87)}$$

Solution: Taking Laplace transforms on both sides of (13.86), setting $\mathcal{L}[y(t)] = Y(s)$, we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = e^{-s}.$$

Using the initial conditions (13.87), it becomes $s^2Y(s) + 3sY(s) + 2Y(s) = e^{-s}$

Solving for $Y(s)$, we obtain $Y(s) = \dfrac{1}{(s^2+3s+2)}e^{-s} = \left[\dfrac{1}{s+1} - \dfrac{1}{s+2}\right]e^{-s}.$

Taking inverse transform on both sides of this and using $\mathcal{L}^{-1}[e^{-as}F(s)] = u(t-a)f(t-a)$, we obtain

$$y(t) = u(t-1)(e^{-(t-1)} - e^{-2(t-1)}), \quad \text{or} \quad y(t) = \begin{cases} 0, & 0 \le t < 1 \\ e^{-(t-1)} - e^{-2(t-1)}, & t \ge 1 \end{cases}$$

is the desired solution.

Example 13.36: Find the output voltage response $q(t)/C$ in an *RLC*- circuit with $R = 10$ ohms, $L = 1$ henry and $C = 0.01$ farad connected to a transient source of voltage modelled by $\delta(t)$, assuming that the current and the charge are zero on the capacitor at time zero.

Solution: By Kirchoff's voltage law $Li'(t) + Ri(t) + \dfrac{1}{C}q(t) = \delta(t)$.

Setting $i(t) = q'(t)$, and substituting for R, L and C, it becomes

$$q''(t) + 10q'(t) + 100q(t) = \delta(t). \quad \text{...(13.88)}$$

The initial conditions are

$$q(0) = q'(0) = 0. \quad \text{...(13.89)}$$

Taking Laplace transforms on both sides of (13.88) and setting $\mathcal{L}[q(t)] = Q(s)$, we have

$$[s^2Q(s) - sq(0) - q'(0)] + 10[sQ(s) - q(0)] + 100Q(s) = 1.$$

Using the initial conditions (13.89) in it and simplifying for $Q(s)$, we obtain

$$Q(s) = \frac{1}{s^2+10s+100} = \frac{1}{(s+5)^2+75}.$$

Taking inverse transform it gives $q(t) = \mathcal{L}^{-1}\left[\dfrac{1}{(s+5)^2+75}\right] = \dfrac{1}{5\sqrt{3}}e^{-5t}\sin\left(5\sqrt{3}\,t\right).$

Hence the output voltage response is $\dfrac{1}{C}q(t) = 100\,q(t) = \dfrac{20}{\sqrt{3}}e^{-5t}\sin\left(5\sqrt{3}\,t\right).$

EXERCISE 13.5

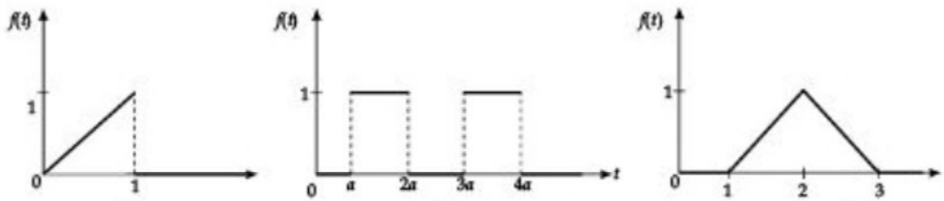
1. Find the Laplace transforms of the functions

   (a) $t^2u(t-1)$ (b) $4u(t-\pi)\cos t$ (c) $\frac{1}{2}(1+e^{-2(t-1)})u(t-1)$

2. Find the Laplace transforms of the following functions which are assumed to be zero outside the given interval.

   (a) e^t, $(0<t<1)$ (b) $10\cos \pi t$, $(1<t<2)$ (c) t^2, $(t>3)$

3. Write the following functions whose graphs are given below in terms of unit step functions and find their Laplace transforms



(a) (b) (c)

4. Find the inverse transform of the functions

   (a) $\dfrac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$ (b) $\dfrac{2}{s^2}-2e^{-2s}\left[\dfrac{1}{s^2}+\dfrac{2}{s}\right]+\dfrac{se^{-\pi s}}{s^2+1}$ (c) $\dfrac{3(1-e^{\pi s})}{s^2+9}$

5. Sketch the graph of the following functions and express them in terms of unit step function. Hence find their Laplace transforms

   (a) $f(t)=\begin{cases}k, & 0\le t<2\\ 0, & 2\le t<4\\ k, & t\ge 4\end{cases}$ (b) $f(t)=\begin{cases}\cos t, & 0\le t<\pi\\ \cos 2t, & \pi\le t<2\pi\\ \cos 3t, & t\ge 2\pi\end{cases}$ (c) $f(t)=\begin{cases}\cos(\omega t+\phi), & 0\le t<T\\ 0, & t\ge T\end{cases}$

Solve the following initial value problems

6. $y'(t)+y(t)=f(t)$, $f(0)=3$, $f(t)=\begin{cases}1, & 0\le t<1\\ -1, & t\ge 1\end{cases}$

7. $y''(t)+y(t)=f(t)$, $y(0)=1$, $y'(0)=0$, $f(t)=\begin{cases}3, & 0\le t<4\\ 2t-5, & t\ge 4\end{cases}$

8. $y'(t)+4\int_0^t y(t)\,dt=tu(t-1)$, $y(0)=2$.

9. $y''(t)+5y'(t)+6y(t)=u(t-\pi)\cos(t-\pi)$, $y(0)=1$, $y'(0)=0$

10. $y''(t) + 5y'(t) + 6y(t) = 1 - u(t-3) - u(t-5), \quad y(0) = 0, \quad y'(0) = 0$

Solve the following initial value problems

11. $y''(t) + 3y'(t) + 2y(t) = \delta(t-1) - \delta(t-2), \quad y(0) = 0, \quad y'(0) = 0$
12. $y''(t) + 4y = 1 - u(t-1) + \delta(t-2), \quad y(0) = 1, \quad y'(0) = 0$
13. $y''(t) + 4y'(t) + 8y(t) = 16\delta(t-1), + 8\delta(t-2) \quad y(0) = y'(0) = 0$
14. The integro-differential equation governing the flow of current $i(t)$ in an RC-circuit is given by $R\, i(t) + \dfrac{1}{C}\displaystyle\int_0^t i(\tau)\, d\tau = E(t)$. If initially at $t = 0$ there is no current and $E(t) = v_0[u(t-1) - u(t-2)]$, v_0 being constant, find the current $i(t)$ at instant t.
15. An impulse of I (kg-sec) is applied to a mass m attached to a spring having a spring constant k. The system is damped with damping constant μ. The motion is governed by

$$y''(t) = I\delta(t) - ky(t) - \mu y'(t).$$

Find an expression for displacement of mass, assuming initial conditions $y(0) = y'(0) = 0$.

ANSWERS

Exercise 13.1 (p. 766)

1. (a) $\dfrac{\sqrt{\pi}(1+2s) - 4s^{1/2}}{2s^{3/2}}$ (b) $\dfrac{s^2 - 2s + 4}{s(s^2+4)}$ (c) $\dfrac{\sqrt{\pi}}{2s^{2/3}} e^{-\frac{1}{4}s}$

   (d) $\dfrac{6s^4 - 576s^2 + 1536}{(s^2+16)^4}$ (e) $\dfrac{s + (s-1)e^{-\pi s}}{1+s^2}$ (f) $\dfrac{b}{(s+a)^2 - b^2}$

2. (a) $1/s + (e^{-s} - 1)/s^2$ (b) $(e^{-s} - 1)/2s^2 - (e^{-s}/2s) + (1/s)$

   (c) $k(1 + e^{-2as} - 2\,e^{-as})/as^2$ (d) $(1/s) + (e^{-2as} - 2e^{-as})/s$

   (e) $k/s^2 - k\,e^{-s}/s^2$

3. (a) $\dfrac{1}{2}\left[\dfrac{3}{(s-4)^2+9} + \dfrac{1}{(s-4)^2+1}\right]$ (b) $\dfrac{1}{s} + \dfrac{3}{(s+1)^2} + \dfrac{6}{(s+2)^3} + \dfrac{6}{(s+3)^4}$

   (c) $\dfrac{3}{2}\left[\dfrac{1}{s^2-9} + \dfrac{s^2-13}{s^4 - 10s^2 + 169}\right]$ (d) $\dfrac{(s-2)^2 - 25}{[(s-2)^2 + 25]^2}$

5. $\dfrac{2}{s}\left(1 + \dfrac{e^{-s}}{s}\right)$ 8. $\dfrac{5}{s(1+e^{-3s})}$ 9. $\dfrac{Ew}{s^2+w^2}\dfrac{1}{1-e^{-\pi s/w}}$ 10. $\dfrac{h}{s(1+e^{-as})}$

Exercise 13.2 (p. 773)

2. $s^3F(s) - s^2 - 1$ 3. $(1 - se^{-\pi s/2})/(s^2+1)$ 4. $\dfrac{8}{(s^2+1)(s^2+4)} + \dfrac{2}{s^2+4}$

7. (a) $(12s^2 - 16)/[s(s^2+4)^3]$ (b) $\dfrac{1}{s}\cot^{-1}(s-1)$

8. (a) $\dfrac{s^2+2s+2}{(s^2+2s)^2}$ (b) $\dfrac{12(s+3)^2 - 16}{[(s+3)^2+4]}$

9. (a) $\ln\dfrac{s+b}{s+a}$ (b) $\dfrac{1}{4}\ln\dfrac{s^2+4}{s^2}$ (c) $\cot^{-1}s - \dfrac{1}{2}s\ln\left(1+\dfrac{1}{s^2}\right)$

10. (a) $\dfrac{\pi}{2}$ (b) $\dfrac{1}{4}\ln 5$ (c) $\dfrac{\pi}{2}$ (d) $\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}$

Exercise 13.3 (p. 782)

1. (a) $-1/4 + (5/4)\cos 2t$ (b) $(9/5)te^{-2t} - (96/25)e^{-2t} + (13/75)e^{3t} + (14/3)e^{-3t}$
   (c) $e^t - e^{-t}\cos 2t + (3/2)e^{-t}\sin 2t$
2. (a) $(\sin at \sinh at)/2a^2$ (b) $(1/2)te^{-2t}\sin t$
   (c) $(1/5)(1+e^{-\pi})\sin t + (3/5)(1-e^{-\pi})\cos t$
3. (a) $(\sin at + at\cos at)/2a$ (b) $(\sin at - at\cos at)/2a^3$
4. (a) $2(1-\cos t)/t$ (b) $(1-e^{-t})/t$
5. (a) $\sin 2t/t$ (b) $-(e^t\sin t)/t$ (c) $2\sinh t\sin t$.
6. (a) $[\sin at - at\cos at]/2a^3$ (b) $[\cos t - \cos 2t]/3$
   (c) $[1 - e^{-8t}(1+8t)]\,e^{t}/64$
7. (a) $\dfrac{a\sin at - b\sin bt}{a^2-b^2}$ (b) $\dfrac{1}{2}\displaystyle\int_0^t \sin 2\tau\, f(t-\tau)d\tau$
8. (a) $t - \dfrac{t^3}{(3!)^2} + \dfrac{t^5}{(5!)^2} - \ldots$ (b) $1 - \dfrac{t^2}{(2!)^2} + \dfrac{t^4}{(4!)^2} - \ldots$
10. (a) $\dfrac{2}{(s-1)(s^2+4)}$ (b) $\dfrac{1}{s^2-1}$

Exercise 13.4 (p. 791)

1. $y(t) = -\dfrac{1}{16} - \dfrac{1}{4}t - \dfrac{15}{16}e^{4t}$ 2. $y(t) = -\dfrac{1}{5}\cos t + \dfrac{2}{5}\sin t + \dfrac{6}{5}e^{-2t}$

3. $y(t) = e^{-t} + 2\sin 2t + \cos 2t$

4. $y(t) = \frac{1}{2}(t^2 + 2t)e^{-t}$

5. $y(t) = 2 + 2e^{t} - e^{-t} - \sin t$

6. $y(t) = e^{-2t}\left(\cos 3t + \frac{7}{9}\sin 3t - \frac{1}{3}t\cos 3t\right)$

7. $y(t) = 2t + \pi\sin 3t$

8. $y(t) = [(5e^{t} + e^{-2t})/3] - e^{-t}$

9. $y(t) = [(3 - t^2)\sin t - 3t\cos t]/8$

10. $y(t) = (\alpha/2n^2)[\sin nt\cos\alpha - nt\cos(nt + \alpha)]$

11. $y(t) = \frac{1}{2}\left(1 + \frac{2}{t}\right)\sin t$

12. $y(t) = e^{2t}$

13. $y(t) = t$

14. $y(t) = 3t^2 + t$

15. $y(t) = 1 - (4 - c)te^{-2t}$

16. $x(t) = \frac{4}{9} + \frac{1}{3}t - \frac{1}{5}\sin t - \frac{2}{5}\cos t + \frac{43}{45}e^{3t}$, $y(t) = \frac{5}{9} + \frac{2}{3}t + \frac{1}{5}\sin t - \frac{3}{5}\cos t - \frac{43}{45}e^{3t}$

17. $x = -e^{-t}(\cos t + \sin t)$, $y = e^{-t}(1 + \sin t)$

18. $x(t) = 1 + e^{-t} - e^{-at} - e^{-bt}$, $y(t) = 1 + e^{-t} - be^{-at} - ae^{-bt}$; $a = (2 - \sqrt{2})/2$, $b = (2 + \sqrt{2})/2$

19. $x(t) = 7/8 + (5/4)t - (1/4)t^2 + (1/8)e^{-2t}$, $y(t) = 1/8 + (7/4)t - (1/4)t^2 - (1/8)e^{-2t}$
$z(t) = 9/8 + (3/4)t - (1/4)t^2 - (1/8)e^{-2t}$

20. $x(t) = -1 + (1/4)e^{-t} + (1/4)(3 + 2t)e^{t}$, $y(t) = 1 + 2t + (1/4)e^{-t} + (1/4)(2t - 1)e^{-t}$
$z(t) = -(1/4)e^{-t} + (1/4)(1 + 2t)e^{t}$

21. $y(t) = t$

22. $y(t) = t^2 + 2t + 2 - e^{t/2}\{(2/\sqrt{3})\sin(\sqrt{3}/2) + 2\cos(\sqrt{3}/2t)\}$

23. $y(t) = 2t^2 + \frac{2}{3}t^3$

24. $y(t) = \frac{1}{4}e^{-2t} + \frac{3}{4}e^{-6t}$

25. $y(t) = \cosh t$

26. $y(t) = \frac{1}{3} - e^{t} + \frac{5}{3}e^{3t}$

27. $y(t) = 1 - (4/\sqrt{3})e^{-2t}\sinh\sqrt{3}\,t$

28. $y(t) = -(7/50)\cos t - (1/50)\sin t + (63/50)e^{3t} + (22/50)e^{-2t}$

29. $y(t) = (1 + \cos\sqrt{2}\,t)/2$

30. $y(t) = (1 + \cosh\sqrt{2}\,t)/2$

Exercise 13.5 (p. 804)

1. (a) $e^{-s}(2s^{-3} + 2s^{-2} + s^{-1})$ (b) $-4e^{-\pi s}s/(s^2 + 1)$ (c) $\frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2}\right)e^{-s}$

2. (a) $\frac{(1 - e^{1-s})}{s - 1}$ (b) $\frac{-10s(e^{-s} + e^{-2s})}{s^2 + \pi^2}$ (c) $e^{-3s}\left[\frac{9}{s} + \frac{6}{s^2} + \frac{2}{s^3}\right]$

3. (a) $t[u(t)-u(t-1)]$, $[1-e^{-s}(s+1)]/s^2$
   (b) $[u(t-a)-u(t-2a)+u(t-3a)-\ldots]$, $e^{-as/2}[2s\cosh(as/2)]$
   (c) $[(t-1)u(t-1)-2(t-2)u(t-2)+(t-3)u(t-3)]$, $e^{-s}(1-e^{-s})^2/s^2$
4. (a) $[u(t-1/2)-u(t-1)]\sin \pi t$ (b) $2t-2tu(t-2)-\cos t.\, u(t-\pi)$
   (c) $[1+u(t-\pi)]\sin 3t$
5. (a) $\dfrac{k[1-e^{-2s}+e^{-4s}]}{s}$

   (b) $\dfrac{1}{s^2+1}+e^{-\pi s}\left(\dfrac{s}{s^2+4}-\dfrac{s}{s^2+1}\right)+e^{-2\pi s}\left(\dfrac{s}{s^2+9}-\dfrac{s}{s^2+4}\right)$

   (c) $[(s\cos\phi-w\sin\phi)-e^{-s\tau}\{s\cos(\phi+w\tau)-w\sin(\phi+w\tau)\}]/(s^2+w^2)$
6. $y(t)=(1+2e^{-t})u(t)-2[1-e^{-(t-1)}u(t-1)]$
7. $y(t)=3-2\cos t+2\{t-4-\sin(t-4)\}u(t-4)$
8. $y(t)=2\cos 2tu(t)+[1-\cos 2(t-1)+2\sin 2(t-1)]u(t-1)/4$
9. $y(t)=3e^{-2t}-2e^{-3t}+(1/10)(3e^{3(\pi-t)}-4e^{-2(\pi-1)}-\cos t-\sin t)u(t-\pi)$
10. $y(t)=(1/6)[1+2e^{-3t}-3e^{-2t}]\,u(t)-(1/6)\,[1+2e^{-3(t-3)}-3e^{-2(t-3)}]\,u(t-3)$
    $-(1/6)\,[1+2e^{-3(t-5)}-3e^{-2(t-5)}]u(t-5)$
11. $y(t)=[e^{1-t}-e^{2-2t}]u(t-1)-[e^{2-t}-e^{4-2t}]u(t-2)$
12. $y(t)=(1/2)+\cos 2t-(1/2)\cos^2 t-(1/2)\,\{1-\cos^2(t-1)\}u(t-1)+(1/2)\sin(2t-4)u(t-2)$
13. $y(t)=8e^{-2(t-1)}\sin 2(t-1)u(t-1)+4e^{2(t-2)}\sin 2(t-2)u(t-2)$
14. $i(t)=[t\sin t+\cos t-1]e^t$ 15. $y(t)=\dfrac{1}{mn}e^{-\mu t/2m}\sin nt.$

PART E

Fourier Analysis and Partial Differential Equations

14 Fourier Series

CHAPTER

Fourier series arise naturally while analyzing many physical phenomena like electrical oscillations, vibrating mechanical systems, longitudinal oscillations in crystals, etc. Many functions including some discontinuous periodic functions of practical interest, which do not find Taylor series representation, can be expanded in a Fourier series and, as such, Fourier series are more universal than Taylor series. They are very powerful tools in solving certain ordinary and partial differential equations. Modern-day applications of Fourier series include areas like data compression, filtering and signal analysis, CAT scans, sattelite communication, etc.

14.1 THE FOURIER SERIES OF A FUNCTION

A Fourier series is a representation of a periodic function as a series of cosine and/or sine terms. Before studying Fourier series we need to consider periodic functions.

14.1.1 Periodic Functions

A function f is called a periodic function, if there is some positive number T such that for every x in the domain of f

$$f(x+T)=f(x); \qquad \ldots(14.1)$$

and the number $T > 0$, with this property, is called a period of f.

For example, $\sin x$ is periodic with period 2π, since $\sin(x+2\pi)=\sin x$ for all x. The function $\tan x$ is periodic with period π, for $\tan(x+\pi)=\tan x$ for all x. Examples of *non-periodic* functions are x, e^x, etc. Further, we note that if f is periodic with period T, it is necessarily periodic with period $2T$, $3T$, $4T$ ... as well, since $f(x+2T)=f((x+T)+T)=f(x+T)=f(x)$. Of all these possible periods, the smallest one, if it exists, is called the *fundamental period* of f. For example, for $\cos x$ and $\sin x$ the fundamental period is 2π while for $\cos 2x$ and $\sin 2x$ it is π.

The function $f(x)$ = constant is periodic and every $T > 0$ is a period. Thus there is no smallest period so, $f(x)$ = constant does not have a fundamental period.

Further, if $f(x)$ and $g(x)$ are periodic with fundamental period T, then the function $h(x) = af(x) + bg(x)$, a, b being constants is also periodic with period T.

Now we define the Fourier series representation of a function $f(x)$.

14.1.2 Fourier Series

If a function f is periodic with period 2π and is integrable over $-\pi < x < \pi$, then the Fourier series representation of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \qquad ...(14.2)$$

where coefficients $a_0, a_1, a_2, ...; b_0, b_1, b_2, ...,$ called the Fourier coefficients, are determined by the function $f(x)$.

To determine the coefficients we need the following results which follow from the orthogonality property of the trigonometric functions $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$ ... $\cos nx$, $\sin nx$. However, the results can be proved otherwise also, for all integral values of m and n.

1. $$\int_{-\pi}^{\pi} \sin nx\,dx = \int_{-\pi}^{\pi} \cos nx\,dx = 0 \qquad ...(14.3)$$

2. $$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases} \qquad ...(14.4)$$

3. $$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases} \qquad ...(14.5)$$

4. $$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0, \text{ for all } m \text{ and } n. \qquad ...(14.6)$$

In fact, the results from (14.3) to (14.6) hold for the interval of integration $(\alpha, \alpha + 2\pi)$, for an arbitrary α. In particular, for $\alpha = -\pi$, the interval becomes $(-\pi, \pi)$ and for $\alpha = 0$ the interval becomes $(0, 2\pi)$. Next, we determine the coefficients a_0, a_n and b_n.

Determination of the constant term a_0. Integrating both sides of (14.2) from $-\pi$ to π and assuming that term by term integration is possible, we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx \right). \qquad ...(14.7)$$

Using (14.3), and integrating the first term on the right side of (14.7), we obtain

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Determination of the coefficients a_n. Multiplying (14.2) by $\cos mx$ for any fixed positive integer m and integrating on both sides from $-\pi$ to π; assuming that term by term integration is possible, we obtain

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx\right). \quad ...(14.8)$$

Using (14.3), (14.5) and (14.6) on the right side of (14.8), we obtain $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$.

Determination of the coefficients b_n. Similarly multiplying (14.2) by sin mx, for any fixed positive integer m, and integrating on both sides from $-\pi$ to π; assuming that term by term integration is possible, we obtain

$$\int_{-\pi}^{\pi} f(x)\sin mx = a_0\int_{-\pi}^{\pi}\sin mx\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\sin mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\sin mx\,dx\right). \quad ...(14.9)$$

Using (14.3), ((14.4) and (14.6) on the right side of (14.9), we obtain $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx$.

The results:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad ...(14.10)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad ...(14.11)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad ...(14.12)$$

for $n = 1, 2, \ldots$ are called the *Euler's formulae* for the Fourier coefficients a_n and b_n associated with the Fourier series representation of $f(x)$ given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx). \quad ...(14.13)$$

14.1.3 Convergence and Sum of a Fourier Series

Suppose that $f(x)$ is any given periodic function of period 2π which is continuous or merely piecewise continuous over the interval of integration. Then we can compute the Fourier coefficients a_0, a_n, b_n of $f(x)$ and use them to form the Fourier series (14.13) of $f(x)$. We would expect that the series thus obtained converges to $f(x)$ over the domain of definition of f.

Various results are available that give sufficient conditions on f for the Fourier series of $f(x)$ to represent $f(x)$. However, one such result which covers the majority of periodic functions appearing in practical applications, is stated as follows.

Theorem 14.1: *If $f(x)$ is a periodic function with period 2π and if $f(x)$ and $f'(x)$ both are piecewise continuous in the interval $-\pi \le x \le \pi$, then the Fourier series of $f(x)$ is convergent. It converges to $f(x)$ at every point x at which $f(x)$ is continuous, and to the mean value $[f(x+)+f(x-)]/2$ at every point x at which $f(x)$ is discontinuous, where $f(x+)$ and $f(x-)$ are the right and left hand limits respectively.*

But we must note that the jumps at the points of discontinuity must be finite. Figure 14.1 shows the graph of a typical piecewise continuous function.

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ x/2, & 1 < x \le 2. \\ 2x & 2 < x \le 3 \end{cases}$$

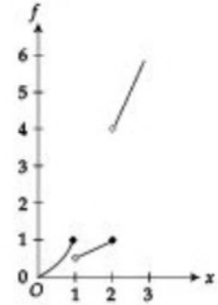


Fig. 14.1

At points of discontinuity (e.g. at $x = 1, 2$), the function $f(x)$ has finite jumps.

An example of a simple function which is not piecewise continuous is

$$f(x) = \begin{cases} 0, & x = 0 \\ 1/x, & 0 < x \le 1 \end{cases}$$

since $\lim_{x \to 0^+} f(x) = \infty$ and so the jump at the discontinuity $x = 0$ is not finite and thus $f(x)$ is not piecewise continuous on $[0,1]$.

Example 14.1 (Saw-tooth wave): Find the Fourier series for the function $f(x) = x$, $-\pi < x < \pi$, when $f(x) = f(x + 2\pi)$.

Solution: The graph of the function $f(x)$ is shown in Fig. 14.2. It is periodic with period 2π. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left[\left(\frac{x\sin nx}{n}\right)_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\sin nx\,dx\right] = \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{\pi} = 0.$$

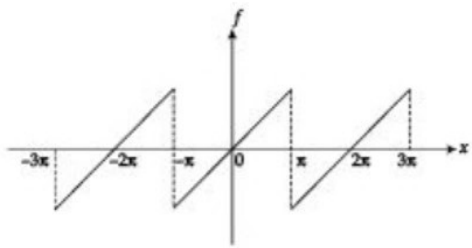


Fig. 14.2

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx = \frac{1}{\pi}\left[-\left(\frac{x\cos nx}{n}\right)_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} = -\frac{2}{n}\cos n\pi = (-1)^{n+1}\frac{2}{n}.$$

Hence Fourier series of $f(x)$ on $[-\pi, \pi]$ is

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \ldots$$

We note that the Fourier series converges to $f(x)$ at every point at which it is continuous. The function $f(x)$ has points of finite discontinuity at $x = -3\pi, -\pi, \pi, 3\pi$ ... The average of the extremes at each discontinuity is $\frac{1}{2}(\pi + (-\pi)) = 0$; and it can be verified by direct substitution that the above series converges to zero at $x = -3\pi, -\pi, \pi, 3\pi$, etc.

Example 14.2 (Rectangular wave): Find the Fourier series for the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ 4, & 0 < x < \pi \end{cases}$$

Solution: The graph of the periodic function $f(x)$ is shown in Fig. 14.3.

It is periodic with period 2π. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 4\,dx\right] = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nx\,dx + \int_{0}^{\pi} 4\cos nx\,dx\right] = \frac{4}{n\pi}[\sin nx]_0^{\pi} = 0$$

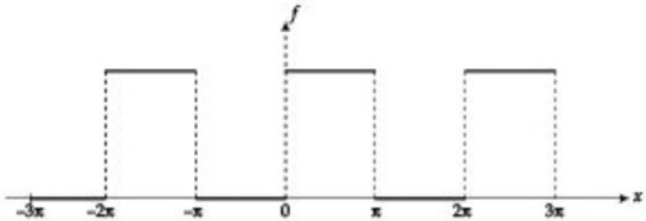


Fig. 14.3

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin nx\,dx + \int_{0}^{\pi} 4\sin nx\,dx\right] = \frac{-4}{n\pi}[\cos nx]_0^{\pi} = \frac{4}{n\pi}[1-(-1)^n].$$

Thus,
$$b_n = \begin{cases} \dfrac{8}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Hence the Fourier series of $f(x)$ on $(-\pi, \pi)$ is

$$f(x) = 2 + \frac{8}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right) = 2 + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}. \qquad \ldots(14.14)$$

We note that at points of finite discontinuity of $f(x)$, that is, at $x = -2\pi, -\pi, 0, \pi, 2\pi$, etc., the series converges to 2, the mean value $\frac{1}{2}(0+4)$, and for all others x in the domain of definition of f, the series converges to $f(x)$. To illustrate how this convergence to f is achieved, we plot some of the partial sums of the series (14.14). The partial sums are

$$S_1(x) = 2, \quad S_2(x) = 2 + \frac{8}{\pi}\sin x, \quad S_3(x) = 2 + \frac{8}{\pi}\sin x + \frac{8}{3\pi}\sin 3x,$$

$$S_4(x) = 2 + \frac{8}{\pi}\sin x + \frac{8}{3\pi}\sin 3x + \frac{8}{5\pi}\sin 5x, \text{ and so on.}$$

Their graphs in Fig. 14.4 show that the series is convergent and has the sum $f(x)$ Also we note that at $x = -\pi, 0, \pi$, the points of discontinuity of $f(x)$, all partial sums have the value '2', the average of the values 0 and 4 of $f(x)$.

The graph in Fig. 14.4 shows upto the first four partial sum approximations to the function $f(x)$ defined in Example (14.2). The Nth partial sum is given by

$$S_N(x) = a_0 + \sum_{n=1}^{N-1}(a_n\cos nx + b_n\sin nx) \text{ and, } f(x) = \lim_{N\to\infty} S_N(x).$$

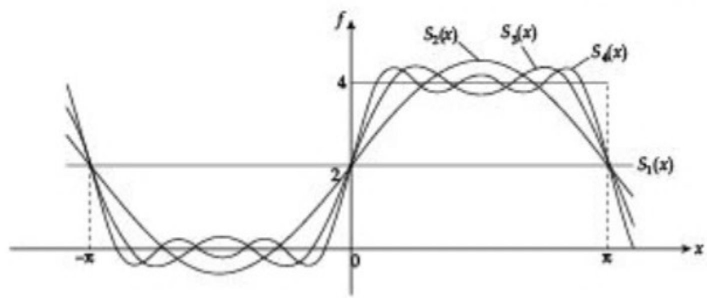


Fig. 14.4

Remark: It should be noted that not every function has a Fourier expansion involving an infinite number of terms. For example, $f(x) = 1 + 2\sin x \cos x$ when rewritten as $f(x) = 1 + \sin 2x$ is in fact its own Fourier series.

Example 14.3 (Rectangular pulse): Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 \le x \le \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

Solution: The graph of the function $f(x)$ on the interval $(-\pi, \pi)$ is shown in Fig. 14.5.

The function is periodic with period 2π. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dx = \frac{1}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx\, dx$$

$$= \frac{1}{\pi}\left(\frac{\sin nx}{n}\right)_{-\pi/2}^{\pi/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

which gives,

$$a_n = \begin{cases} \dfrac{2}{n\pi}(-1)^{\frac{n-1}{2}} & \text{for odd } n \\ 0, & \text{for even } n \end{cases}$$

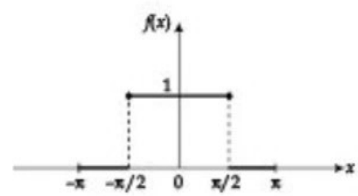


Fig. 14.5

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nx\,dx = \frac{1}{n\pi}[-\cos nx]_{-\pi/2}^{\pi/2} = 0.$$

Hence the Fourier series of $f(x)$ on $(-\pi, \pi)$ is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \ldots\right]. \quad \ldots(14.15)$$

Figs. 14.6a and 14.6b show respectively that graphs of the first five and the first ten terms of the Fourier series expansion (14.15) in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$. It can be observed that the graphs of $S_5(x)$ and $S_{10}(x)$ exhibit over and undershoots close to the discontinuities $x = -\pi/2, \pi/2$. This oscillatory behaviour of the partial sums $S_N(x)$ near a point of jump discontinuity continues even for large value N and is called the *Gibbs phenomenon*.

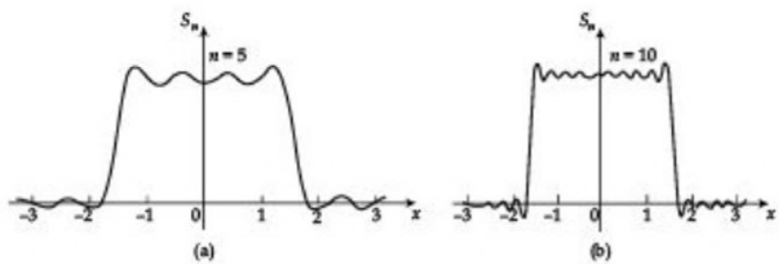


Fig. 14.6

Example 14.4: Find the Fourier series for the function $f(x) = e^{-x}$, $0 < x < 2\pi$ with $f(x + 2\pi) = f(x)$.

Solution: The function $f(x)$ is periodic with period 2π. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}dx = -\frac{1}{2\pi}(e^{-x})_0^{2\pi} = \frac{1-e^{-2\pi}}{2\pi},$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}[e^{-x}(-\cos nx + n\sin nx)]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi(n^2+1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-x}(-\sin nx - n\cos nx)\right]_0^{2\pi} = \frac{n(1-e^{-2\pi})}{\pi(n^2+1)}.$$

Hence the Fourier series of $f(x)$ on $(0, 2\pi)$ is

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2+1} + \frac{n\sin nx}{n^2+1}\right)\right\}.$$

14.2 FOURIER SERIES OF FUNCTIONS OF PERIOD $T = 2l$

So far we have considered the Fourier series expansion of functions with period 2π. In many applications, we need to find the Fourier series expansion of periodic functions with arbitrary period, say $2l$. The transition from period $T = 2l$ to period $T = 2\pi$ is quite simple and involves only a proportional change of scale.

Consider the periodic function $f(x)$ with period $2l$ defined in $(-l, l)$. To change the problem to period 2π, set $v = \frac{\pi x}{l}$, which gives, $x = \frac{lv}{\pi}$.

Thus $x = \pm l$ corresponds to $v = \pm\pi$ and the function $f(x)$ of period $2l$ in $(-l, l)$ may be regarded as function $g(v)$ of period 2π in $(-\pi, \pi)$. Hence,

$$g(v) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv) \quad \text{...(14.16)}$$

with coefficients

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)\,dv \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\,dv \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\sin nv\,dv \end{aligned}\right\} \quad \text{...(14.17)}$$

Making the inverse substitutions, $v = \frac{\pi x}{l}$ and $g(v) = f(x)$ in (14.16) and (14.17), we obtain the Fourier series expansion of $f(x)$ in the interval $(-l, l)$ given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

with coefficients

$$\left.\begin{aligned} a_0 &= \frac{1}{2l}\int_{-l}^{l} f(x)dx \\ a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \end{aligned}\right\} \quad \text{...(14.18)}$$

We may replace the interval of integration by any interval of length $T = 2l$, say by the interval $(0, 2l)$.

Example 14.5: Find the Fourier series for the function $f(x) = \begin{cases} x, & -1 < x \le 0 \\ x+2, & 0 < x < 1, \end{cases}$

where $f(x) = f(x+2)$. From the series obtained deduce the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$.

Solution: The graph of the periodic function $f(x)$ in the interval $(-1, 1)$ is shown in Fig. 14.7.

The function is periodic with period 2. The Fourier coefficients are

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\left[\int_{-1}^{0} x\,dx + \int_{0}^{1}(x+2)dx\right] = 1,$$

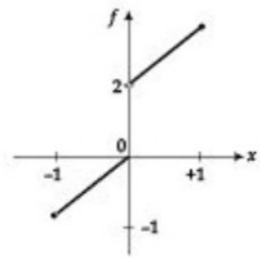


Fig. 14.7

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0} x\cos n\pi x\,dx + \int_{0}^{1}(x+2)\cos n\pi x\,dx$$

$$= 2\int_{0}^{1}\cos n\pi x\,dx = \frac{2}{n\pi}(\sin n\pi x)_0^1 = \frac{2}{n\pi}(\sin n\pi) = 0,$$

$$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0} x\sin n\pi x\,dx + \int_{0}^{1}(x+2)\sin n\pi x\,dx$$

$$= 2\int_{0}^{1} x\sin n\pi x\,dx + 2\int_{0}^{1}\sin n\pi x\,dx = 2\left[-\frac{x\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 - 2\left[\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$= -\frac{2\cos n\pi}{n\pi} - \frac{2\cos n\pi}{n\pi} + \frac{2}{n\pi} = \frac{2}{n\pi} - \frac{4}{n\pi}(-1)^n = \frac{2}{n\pi}[1-(-1)^n 2].$$

Thus $$b_n = \begin{cases} \dfrac{6}{n\pi}, & \text{for odd } n \\ \dfrac{-2}{n\pi}, & \text{for even } n \end{cases}$$

Hence the Fourier series of $f(x)$ on $(-1, 1)$ is

$$f(x) = 1 + \frac{2}{\pi}\left[3\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}(3\sin 3\pi x) - \frac{1}{4}\sin 4\pi x + \frac{1}{5}(3\sin 5\pi x) - \frac{1}{6}\sin 6\pi x + - \ldots\right].$$

Further, for $x = 1/2$, $f(x) = x + 2 = 1/2 + 2 = 5/2$.

Setting $x = 1/2$ on both sides of the series above, we obtain

$$\frac{5}{2} = 1 + \frac{2}{\pi}\left[3\sin\frac{\pi}{2} - \frac{1}{2}\sin\pi + \sin\frac{3\pi}{2} - \frac{1}{4}\sin 2\pi + \frac{3}{5}\sin\frac{5\pi}{2} - \frac{1}{6}\sin 3\pi + \frac{3}{7}\sin\frac{7\pi}{2} - \ldots\right]$$

$$= 1 + \frac{2}{\pi}\left[3 - 1 + \frac{3}{5} - \frac{3}{7} + \ldots\right]$$

or, $$\frac{3\pi}{4} = 3 - 1 + \frac{3}{5} - \frac{3}{7} + \ldots, \text{ This gives } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

Example 14.6 (Half-wave rectifiers): Find the Fourier series of the periodic function

$$f(x) = \begin{cases} 0, & -l < x \le 0 \\ E\sin wx, & 0 < x < l \end{cases}$$

with period $T = 2l = \dfrac{2\pi}{w}$.

Solution: The graph of the periodic function $f(x)$ with period $2\pi/w$ is shown in Fig. 14.8.

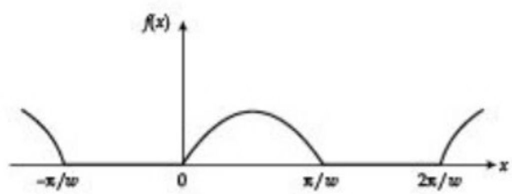


Fig. 14.8

The Fourier coefficients are

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx = \frac{w}{2\pi}\left[\int_{-\pi/w}^{0} 0dx + \int_{0}^{\pi/w} E\sin wx\,dx\right] = \frac{w}{2\pi}\left[-\frac{E\cos wx}{w}\right]_0^{\pi/w} = \frac{w}{2\pi}\left(\frac{2E}{w}\right) = \frac{E}{\pi}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{w}{\pi}\left[\int_{-\pi/w}^{0} 0\cos nwx\,dx + \int_{0}^{\pi/w} E\sin wx\cos nwx\,dx\right]$$

$$= \frac{wE}{2\pi}\int_{0}^{\pi/w}[\sin(1+n)wx + \sin(1-n)wx]dx = \frac{wE}{2\pi}\left[\frac{-\cos(1+n)wx}{(1+n)w} - \frac{\cos(1-n)wx}{(1-n)w}\right]_0^{\pi/w}, n \neq 1$$

$$= \frac{E}{2\pi}\left[\left(\frac{-\cos(1+n)\pi+1}{1+n}\right) + \left(\frac{-\cos(1-n)\pi+1}{1-n}\right)\right] = \frac{E}{2\pi}\left[\frac{-(-1)^{1+n}+1}{1+n} + \frac{-(-1)^{1-n}+1}{1-n}\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \quad (\text{except } n=1) \\ \dfrac{-2E}{(n-1)(n+1)\pi}, & \text{if } n \text{ is even} \end{cases}$$

For $n = 1$, $a_1 = \frac{w}{\pi}\int_{0}^{\pi/w} E\sin wx\cos wx\,dx = \frac{wE}{2\pi}\int_{0}^{\pi/w}\sin 2wx\,dx = \frac{-E}{2\pi}\left[\frac{\cos 2wx}{2}\right]_0^{\pi/w} = 0.$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{w}{\pi}\left[\int_{-\pi/w}^{0} 0.\sin nwx\,dx + \int_{0}^{\pi/w} E\sin wx\sin nwx\,dx\right]$$

$$= \frac{wE}{2\pi}\int_{0}^{\pi/w}[\cos(1-n)wx - \cos(1+n)wx]dx$$

$$= \frac{wE}{2\pi}\left[\frac{\sin(1-n)wx}{(1-n)w} - \frac{\sin(1+n)wx}{(1+n)w}\right]_0^{\pi/w}, \quad n \neq 1$$

$$= \frac{E}{2\pi}(0) = 0.$$

For $\quad n = 1, b_1 = \frac{wE}{\pi}\int_{0}^{\pi/w}\sin wx\sin wx\,dx = \frac{wE}{\pi}\int_{0}^{\pi/w}\sin^2 wx\,dx$

$$= \frac{wE}{2\pi}\int_0^{\pi/w}(1-\cos 2wx)\,dx = \frac{wE}{2\pi}\left[x - \frac{\sin 2wx}{2w}\right]_0^{\pi/w} = \frac{wE}{2\pi}\left(\frac{\pi}{w}\right) = \frac{E}{2}.$$

Hence Fourier series of $f(x)$ on $\left(\frac{-\pi}{w}, \frac{\pi}{w}\right)$ is

$$f(x) = \frac{E}{\pi} + \frac{E}{2}\sin wx - \frac{2E}{\pi}\left[\frac{1}{1.3}\cos 2wx + \frac{1}{3.5}\cos 4wx + \ldots\right].$$

Example 14.7: Obtain the Fourier series for the periodic function

$$f(x) = e^{-x}, \quad -l < x < l, \text{ where } f(x+2l) = f(x).$$

Solution: The function $f(x)$ is periodic with period $2l$. The Fourier coefficients are

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2l}\int_{-l}^{l} e^{-x}\,dx = \frac{1}{2l}[-e^{-x}]_{-l}^{l} = \frac{1}{2l}(e^{l} - e^{-l}) = \frac{\sinh l}{l},$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \frac{l}{l^2+n^2\pi^2}[-e^{-l}\cos n\pi + e^{l}\cos n\pi] = \frac{2l\cos n\pi}{l^2+n^2\pi^2}\left(\frac{e^{l}-e^{-l}}{2}\right) = \frac{2(-1)^n l}{l^2+n^2\pi^2}\sinh l.$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= -\frac{l}{l^2+n^2\pi^2}\left[\frac{n\pi}{l}(e^{-l} - e^{l})\cos n\pi\right] = \frac{2n\pi\cos n\pi}{l^2+n^2\pi^2}\left(\frac{e^{l}-e^{-l}}{2}\right) = \frac{2(-1)^n n\pi}{l^2+n^2\pi^2}\sinh l.$$

Hence the Fourier series of $f(x)$ on $(-l, l)$ is

$$e^{-x} = \frac{\sinh l}{l} + 2\sinh l\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{l^2+n^2\pi^2}\left(l\cos\frac{n\pi x}{l} + n\pi \sin\frac{n\pi x}{l}\right)\right)$$

$$= \sinh l\left[\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^3\pi^2}\cos\frac{3\pi x}{l} - \dots\right)\right.$$

$$\left. - 2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{\pi x}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \dots\right)\right]$$

EXERCISE 14.1

1. Obtain the Fourier series to represent $f(x) = \frac{1}{4}(\pi - x)^2, \quad 0 < x < 2\pi, \ f(x+2\pi) = f(x)$.
2. Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1, & -\pi < x \le 0 \\ -1, & 0 < x < \pi \end{cases}, \quad f(x+2\pi) = f(x).$$

3. Find the Fourier series expansion of $f(x) = x \sin x, \quad 0 < x < 2\pi, \ f(x+2\pi) = f(x)$.
4. Find the Fourier series to represent $f(x) = x - x^2, -\pi < x < \pi$. Hence show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}.$$

5. Obtain the Fourier series for the function $f(x) = x^2, -\pi < x < \pi$. Hence show that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$ (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$

6. Let f be the periodic function shown in the Fig. 14.9, each segment of which is a semicircle of radius π. Show that its Fourier series expansion is

$$f(x) = \frac{\pi^2}{4} + \frac{\pi^2}{2}\sum_{n=1}^{\infty}[J_0(n\pi) + J_2(n\pi)]\cos nx, \text{ where}$$

J_0 and J_2 have their usual meanings.

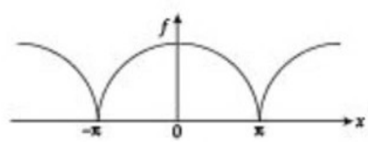


Fig. 14.9

7. Find the Fourier series expansion of $f(x)= e^{-4x}, \quad -2 \le x \le 2, \quad f(x+4) = f(x)$.
8. Find the Fourier series of the *periodic square wave* given by

$$f(x) = \begin{cases} 0, & -2 < x \le -1 \\ k, & -1 < x < 1 \\ 0, & 1 \le x < 2 \end{cases} \quad f(x+4) = f(x).$$

9. Find the Fourier series expansion of $f(x) = \pi \sin \pi x, \quad 0 < x < 1, f(x+1) = f(x)$.
10. Prove that in the range $-\pi < x < \pi$,

$$\cosh ax = \frac{2a^2}{\pi} \sinh a\pi \left(\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} \cos nx \right).$$

14.3 FOURIER SERIES EXPANSIONS OF EVEN AND ODD FUNCTIONS

We can save some work in computing the Fourier coefficients, if a function is even or odd. Before proceeding further, first we discuss the concept of even and odd functions and a few of their properties from calculus.

14.3.1 Even and Odd Functions

A function $f(x)$ is an even function on $[-l, l]$, if $f(-x) = f(x)$, for $-l \le x \le l$.

For example, $y = x^2, x^4, \cos x, e^{-|x|}$ are even functions of x on any interval $[-l, l]$. The graph of such a function is symmetric with respect to the y-axis as shown in Fig. 14.10 for $y = x^2$.

A function $g(x)$ is an odd function on $[-l, l]$, if $g(-x) = -g(x)$, for $-l \le x \le l$.

For example, $y = x, x^3, \sin x$ are odd functions of x on any interval $[-l, l]$. The graph of such a function is symmetric with respect to the orgin as shown in Fig. 14.11 for $y = x^3$.

A function may neither be an even function nor an odd function. For example, functions $y = x + x^2, e^x$ are not even nor odd on any interval $[-l, l]$.

Further, the product of two even functions or two odd functions is an even function and the product of an even function with an odd function is an odd function. Also we have following results from calculus

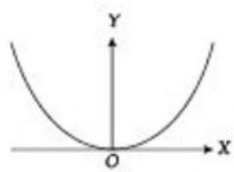


Fig. 14.10

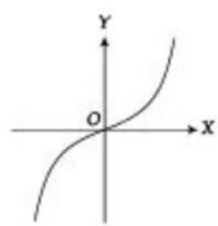


Fig. 14.11

1. $$\int_{-l}^{l} f(x)dx = 2\int_{0}^{l} f(x)dx, \qquad ...(14.19)$$

if f is an even function of x on $[-l, l]$, and

2. $$\int_{-l}^{l} f(x)dx = 0, \qquad ...(14.20)$$

if f is an odd function on $[-l, l]$.

14.3.2 Fourier Series Expansions of Even and Odd Functions

We have already observed in Example 14.1 with $f(x) = x$, which is an odd function of x on $[-\pi, \pi]$, that the cosine coefficients were all zeros, since $x \cos nx$ is an odd function of x and thus the Fourier expansion of $f(x) = x$ consists only sine terms. We have the following results for the Fourier series expansion of even and odd functions:

Theorem 14.2: *If $f(x)$ is an even function on $[-l, l]$ of period $2l$, then*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} dx \qquad ...(14.21)$$

with coefficients $a_0 = \frac{1}{l}\int_0^l f(x)dx$ and $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$. $\qquad ...(14.22)$

Theorem 14.3: *If $f(x)$ is an odd function on $[-l, l]$ of period $2l$, then*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} dx, \qquad ...(14.23)$$

with coefficients

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx. \qquad ...(14.24)$$

These results follow easily from the applications of (14.19) and (14.20) to the Euler's formulae for the Fourier coefficients given by (14.18).

The series in (14.21) is called the *Fourier cosine series* and the series in (14.23) is called the *Fourier sine series*.

When the period is 2π, then (14.21) reduces to

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \qquad ...(14.25)$$

with coefficients, $$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx, \text{ and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx; \quad ...(14.26)$$

and (14.23) reduces to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad ...(14.27)$$

with coefficients, $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx. \quad ...(14.28)$$

Example 14.8: Find the Fourier series of $f(x) = x^2$ on $(-1, 1)$, when $f(x+2) = f(x)$.

Solution: Since the function $f(x) = x^2$ is an even function of x on $[-1, 1]$, thus its Fourier series expansion consists only of constant term and cosine terms; also $f(x)$ is periodic with period 2. The Fourier coefficients are

$$a_0 = \frac{1}{l}\int_0^{l} f(x)dx = \int_0^1 x^2 dx = \left|\frac{x^3}{3}\right|_0^1 = \frac{1}{3},$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}dx = 2\int_0^1 x^2\cos n\pi x\,dx = 2\left[\left[x^2\frac{\sin n\pi x}{n\pi}\right]_0^1 - 2\int_0^1 x\frac{\sin n\pi x}{n\pi}dx\right]$$

$$= -\frac{4}{n\pi}\left[-x\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 = \frac{4}{n^2\pi^2}\cos n\pi = \frac{4(-1)^n}{n^2\pi^2}.$$

Hence the Fourier series for $f(x)$ on $(-1, 1)$ is, $x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$

Example 14.9: Find the Fourier series expansion of the function $f(x) = \sin ax$, $-\pi < x < \pi$, where a is not an integer.

Solution: Since $\sin ax$ is an odd function of x on $[-\pi, \pi]$, thus its Fourier series expansion consists of only sine terms. The Fourier coefficients are

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{\pi}\int_0^{\pi}\sin ax\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}[\cos(n-a)x - \cos(n+a)x]dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{(n+a)}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{(n-a)} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right] = (-1)^{n+1}\frac{2n\sin a\pi}{\pi(n^2-a^2)}.$$

Hence the Fourier series for $f(x)$ on $(-\pi, \pi)$ is $\sin ax = \dfrac{2\sin a\pi}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{n^2-a^2}\sin nx$.

Example 14.10: Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1+2x/\pi, & -\pi < x < 0 \\ 1-2x/\pi, & 0 \le x < \pi \end{cases} \quad f(x+2\pi) = f(x).$$

Also deduce that
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

Solution: The graph of the function $f(x)$ is shown in Fig. 14.12. The graph is symmetrical about y-axis. Hence, $f(x)$ is an even function of x over $(-\pi, \pi)$ with period 2π. Thus its Fourier expansion consists of only constant term and cosine terms. The Fourier coefficients are

$$a_0 = \frac{1}{l}\int_0^l f(x)dx = \frac{1}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx = \frac{1}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = 0,$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{2}{\pi}\frac{\sin nx}{n}dx\right]$$

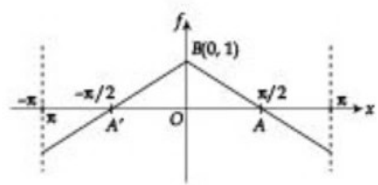


Fig. 14.12

$$= \frac{4}{\pi^2}\left[\frac{-\cos nx}{n^2}\right]_0^{\pi} = \frac{4}{n^2\pi^2}(1-\cos n\pi) = \frac{4}{n^2\pi^2}(1-(-1)^n).$$

Thus, $a_n = \begin{cases} \dfrac{8}{n^2\pi^2}, & \text{for odd } n \\ 0, & \text{for even } n \end{cases}$.

Hence the Fourier series of $f(x)$ on $(-\pi, \pi)$ is

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]. \quad \ldots(14.29)$$

At $x = 0$, $f(0) = 1$. Setting $x = 0$ in (14.29), we obtain

$$1 = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right), \quad \text{or} \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

14.4 FOURIER HALF-RANGE COSINE AND SINE SERIES

We have seen that in case $f(x)$ is defined on $-l \le x \le l$ we can write its Fourier series, and the coefficients of the series are determined by the function and the interval. Let us suppose that a function $f(x)$ of period $2l$ is specified only on a half-range interval $0 \le x \le l$. In such a case, we have a choice to extend the definition of the function to the interval $-l \le x \le l$ in a suitable manner, even or odd, to find its Fourier cosine or sine expansion respectively and then restricting the Fourier series representation of the extended function to the original half-range interval $0 \le x \le l$.

14.4.1 The Fourier Cosine Series on $0 \le x \le l$

Let a function $f(x)$ specified on the interval $0 \le x \le l$ is extended to the interval $-l \le x \le l$ as an even function $g(x)$ of x, given by $g(x) = \begin{cases} f(-x), & -l \le x \le 0 \\ f(x), & 0 \le x \le l \end{cases}$ which coincides with $f(x)$ on the interval $0 \le x \le l$, refer to Figs. 14.13a and 14.13b, then the Fourier series representation of $f(x)$ on the interval $0 \le x \le l$ is the cosine series given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad \ldots(14.30)$$

where $\quad a_0 = \frac{1}{l}\int_0^l f(x)dx$ and $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx.$

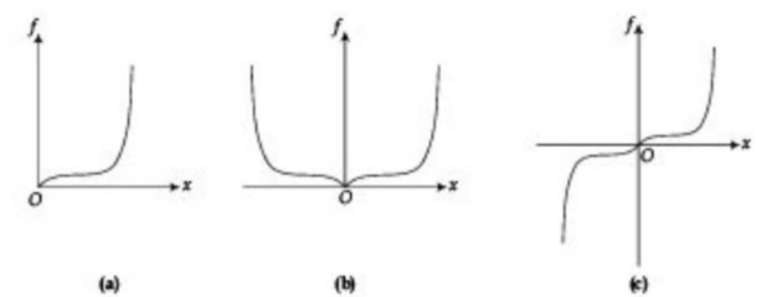


Fig. 14.13

14.4.2 The Fourier Sine Series on $0 \leq x \leq l$

If $f(x)$ specified on the interval $0 \le x \le l$, is extended to the interval $-l \le x \le l$ as an odd function $g(x)$ of x, given by $g(x) = \begin{cases} -f(-x), & -l \le x \le 0 \\ f(x), & 0 \le x \le l \end{cases}$ which coincides with $f(x)$ on the interval $0 \le x \le l$, refer to Figs. 14.13a and 14.13c, then the Fourier series representation of $f(x)$ on the interval $0 \le x \le l$ is the sine series given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad \text{...(14.31)}$$

where
$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$$

The expansions (14.30) and (14.31) are respectively referred to as *half-range cosine series* and *half-range sine series* expansions of $f(x)$.

Example 14.11: Find the Fourier sine and cosine series expansions of $f(x) = x$ for $0 \le x \le \pi$.

Solution: The sine series representation of

$$f(x) \text{ is } f(x) = \sum_{n=1}^{\infty} b_n \sin nx; \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx.$$

Substituting for $f(x)$, we have

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx = \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = -\frac{2}{n}\cos n\pi = (-1)^{n+1}\frac{2}{n}.$$

Hence, the required sine series expansion of $f(x) = x$ is

$$x = 2\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sin nx}{n}, \quad 0 \le x \le \pi.$$

Next, the cosine series representation of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx; \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx \text{ and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx.$$

Substituting for $f(x)$, we have $a_0 = \frac{1}{\pi}\int_0^{\pi} x\, dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi/2$, and

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi n^2}[(-1)^n - 1].$$

Thus, $$a_n = \begin{cases} -\dfrac{4}{\pi n^2}, & \text{when } n \text{ is odd.} \\ 0, & \text{when } n \text{ is even.} \end{cases}$$

Hence the cosine series expansion of $f(x) = x$ is $x = \dfrac{\pi}{2} - \dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{\cos(2n-1)x}{(2n-1)^2}, \quad 0 \le x \le \pi.$

Example 14.12: Write the sine series expansion of $f(x) = \begin{cases} 1, & 0 < x \le \pi/2 \\ 2, & \pi/2 < x < \pi \end{cases}$ on $[0, \pi]$ and also discuss its convergence.

Solution: The sine series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \text{ with } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Substituting for $f(x)$, we have

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2}\sin nx\,dx + \int_{\pi/2}^{\pi} 2\sin nx\,dx\right] = \frac{2}{\pi}\left[\left[-\frac{\cos nx}{n}\right]_0^{\pi/2} + 2\left[-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{n\pi}\left[\left(1 - \cos\frac{n\pi}{2}\right) - 2\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right] = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} + 1 - 2(-1)^n\right].$$

Hence, the sine series expansion of $f(x)$ is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos\frac{n\pi}{2} + 1 - 2(-1)^n\right)\sin nx.$$

The series converges to 0, for $x = 0$; to 1, for $0 < x < \pi/2$; to $\frac{1}{2}(1+2) = 3/2$, for $x = \pi/2$; and to 2, for $\pi/2 < x < \pi$, and again to 0 for $x = \pi$.

EXERCISE 14.2

1. Expand the function $f(x) = x^4$ on $[-1, 1]$ as a Fourier series.
2. Expand the function $f(x) = x\sin x$ as a Fourier series in the interval $[-\pi, \pi]$. Also deduce that

$$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\ldots=\frac{1}{4}(\pi-2).$$

3. Expand the function $f(x) = |x|, -\pi < x < \pi$ as Fourier series and hence deduce that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}.$$

4. Expand $f(x) = |\cos x|, -\pi < x < \pi$ as a Fourier series.

5. Expand $f(x) = \begin{cases} -1/2, & -\pi < x < 0 \\ 1/2, & 0 < x < \pi \end{cases}$ $f(x + 2\pi) = f(x)$, as a Fourier series.

6. Expand $f(x) = \begin{cases} -x+1, & -\pi \le x \le 0 \\ x+1, & 0 \le x \le \pi \end{cases}$ $f(x + 2\pi) = f(x)$

   as a Fourier series. Also deduce the value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

7. Obtain the Fourier series expansion of $f(x) = 4 - x^2, -2 \le x \le 2$. Also show that

$$\frac{\pi^2}{12}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

8. Find the Fourier cosine and sine series of $f(x) = 1, 0 \le x \le 2$.

9. Find the Fourier cosine series of the function $f(x) = \begin{cases} x^2, & 0 \le x < 2 \\ 4, & 2 \le x \le 4. \end{cases}$

10. Find the Fourier sine series of the function $f(x) = \begin{cases} x, & 0 \le x < \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi. \end{cases}$

11. Expand $\sin\left(\frac{\pi x}{l}\right)$ in half-range cosine series in the interval $[0, l]$.

12. Find the Fourier cosine series for the function $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi/2 \\ 0, & \pi/2 < x \le \pi. \end{cases}$

14.5 INTEGRATION AND DIFFERENTIATION OF FOURIER SERIES. THE PARSEVAL'S FORMULA

First we discuss the termwise integration and differentiation of the Fourier series of a function $f(x)$ and then using the concept of termwise integration, we derive the Parseval's formula.

14.5.1 Termwise Integration and Differentiation of Fourier Series

Let $f(x)$ be piecewise continuous on $[-l, l]$ with Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right). \text{ Then for any } x,\ -l \le x \le l,$$

$$\int_{-l}^{x} f(t)\,dt = a_0(x+l) + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[a_n \sin\frac{n\pi x}{l} - b_n\left(\cos\frac{n\pi x}{l} - (-1)^n\right)\right]. \quad ...(14.32)$$

Note that the expression on the right side of (14.32) is exactly what we get by integrating the Fourier series term by term from $-l$ to x. This holds even if the Fourier series does not converge to $f(x)$ at a particular value of x.

Example 14.13: Use the Fourier series representation of $f(x) = x, -\pi < x < \pi$ to find the Fourier series representation for x^2 over $-\pi < x < \pi$.

Solution: The Fourier series representation of the function $f(x) = x$ over $[-\pi, \pi]$, refer to Example 14.1 is

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx. \quad ...(14.33)$$

Integrating (14.33) term by term over the interval $(-\pi, x)$ for any x in $-\pi < x < \pi$, we obtain

$$\int_{-\pi}^{x} x\,dx = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\int_{-\pi}^{x}\sin nx\,dx, \text{ or, } \frac{1}{2}(x^2-\pi^2) = \sum_{n=1}^{\infty}\frac{2}{n^2}(-1)^n[\cos nx - \cos n\pi],$$

$$\text{or, } x^2-\pi^2 = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}[\cos nx - (-1)^n] = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 4\sum_{n=1}^{\infty}\frac{1}{n^2}. \quad ...(14.34)$$

Using the result $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, (refer to Problem 5 Exercise 14.1), (14.34) becomes

$$x^2-\pi^2 = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - \frac{2\pi^2}{3}, \text{ or, } x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx,$$

the Fourier series representation of x^2 over $-\pi \le x \le \pi$.

We have seen that termwise integration of Fourier series of a function $f(x)$ leads to some meaningful results. But same is not always true in case of termwise differentiation, Consider again the Fourier series expansion of x over the interval $-\pi \le x \le \pi$. It is

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx. \qquad ...(14.35)$$

The series converges to x for $-\pi < x < \pi$.

Differentiating w.r.t. x for $-\pi < x < \pi$, we obtain

$$1 = \sum_{n=1}^{\infty} 2(-1)^{n+1} \cos nx, \qquad ...(14.36)$$

which is absurd, since the right side of (14.36) does not even converge over $-\pi < x < \pi$.

Thus, in this case termwise derivative of Fourier series is not related to the derivative of $f(x)$. However, for the validity of termwise differentiation of Fourier series, the series should be uniformly convergent over the given interval which is not true in case of the right side of (14.35).

Example 14.14: Use the Fourier series representation of $f(x) = x^2, -\pi < x < \pi$ to find the Fourier series representation for x over $-\pi < x < \pi$.

Solution: The Fourier series representation of $f(x) = x^2$ over $[-\pi, \pi]$, refer to Example 14.13, is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \qquad ...(14.37)$$

The series on the right side of (14.37) is uniformly convergent over $-\pi < x < \pi$, thus termwise differentiation of (14.37) is admissible. Then for $-\pi < x < \pi$

$$f'(x) = 2x = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \text{ or, } x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx,$$

which is the Fourier series representation of x over $-\pi < x < \pi$.

14.5.2 The Parseval's Formula

We state the following result:

Theorem 14.4 (Parseval's formula): *If the Fourier series for $f(x)$ converges uniformly on $(-l, l)$, then*

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{2l}\int_{-l}^{l} [f(x)]^2 dx, \qquad ...(14.38)$$

where $a_0, a_n, b_n, n = 1, 2, ...$ are the Fourier coefficients of f on $(-l, l)$.

Proof. The Fourier series for $f(x)$ on $(-l, l)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right).$$

Since the series is uniformly convergent over $(-l, l)$, multiplying both sides of it by $f(x)$ and integrating termwise from $-l$ to l, we get

$$\int_{-l}^{l} [f(x)]^2 dx = a_0 \int_{-l}^{l} f(x) dx + \sum_{n=1}^{\infty} \left(a_n \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx + b_n \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \right)$$

$$= l \left[2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right], \quad \text{using (14.18)}$$

or,
$$a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{2l} \int_{-l}^{l} [f(x)]^2 dx, \text{ which is (14.38).}$$

In case the interval is $(0, 2l)$, then the Parseval formula corresponding to (14.38), is

$$a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{2l} \int_{0}^{2l} [f(x)]^2 dx. \qquad \text{...(14.39)}$$

On the similar lines we can prove the following two results corresponding to half-range expansion of $f(x)$ over the interval $[0, l]$.

Theorem 14.5: *If half-range cosine series of $f(x)$ converges uniformly to $f(x)$ over $(0, l)$, then*

$$a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 = \frac{1}{l} \int_{0}^{l} [f(x)]^2 dx, \qquad \text{...(14.40)}$$

where $\quad a_0 = \frac{1}{l} \int_{0}^{l} f(x) dx,$ *and* $a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n\pi x}{l} dx.$

Theorem 14.6: *If the half-range sine series of $f(x)$ converges uniformly to $f(x)$ over $(0, l)$, then*

$$\frac{1}{2} \sum_{n=1}^{\infty} b_n^2 = \frac{1}{l} \int_{0}^{l} [f(x)]^2 dx, \qquad \text{...(14.41)}$$

where
$$b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} dx.$$

The expression, $[f(x)]_{rms} = \sqrt{\frac{1}{2l} \int_{-l}^{l} [f(x)]^2 dx}$ $\qquad$ ...(14.42) is called the *root mean square (r.m.s.) value of the function f(x) over the interval* $(-l, l)$. The *r.m.s.* value of a periodic function finds applications in engineering physics.

Example 14.15: Find the Fourier series expansion of the function $f(x) = |x|$ defined over the interval $(-2, 2)$. Using Parseval equality, prove that $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{96}$.

Solution: Since $f(x)$ is an even function of $f(x)$ over the interval $(-2, 2)$, thus the Fourier series expansion of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}, \qquad \ldots(14.43)$$

We have
$$a_0 = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1$$

$$a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx = \int_0^2 x\cos\frac{n\pi x}{2}dx = \frac{4}{n^2\pi^2}(\cos n\pi - 1) = \frac{4}{n^2\pi^2}[(-1)^n - 1].$$

Hence (14.43) becomes

$$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\frac{n\pi x}{2}.$$

Applying the Parseval equality $a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 = \frac{1}{l}\int_0^l [f(x)]^2 dx$, we obtain

$$1^2 + \frac{1}{2}\cdot\frac{16}{\pi^4}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]^2 = \frac{1}{2}\int_0^2 x^2 dx$$

or, $1 + \frac{32}{\pi^4}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right) = 4/3$, which gives $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots = \frac{\pi^4}{96}$.

Example 14.16: Using the Fourier coefficients of $f(x) = \cos(x/2)$ on $(-\pi, \pi)$ prove that

$$\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2} = \frac{\pi^2 - 8}{16}.$$

Solution: The function $f(x) = \cos(x/2)$ is an even function of x on the interval $(-\pi, \pi)$. The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_0^\pi \cos\frac{x}{2}dx = \frac{2}{\pi}\left[\sin\frac{x}{2}\right]_0^\pi = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^\pi \cos\frac{x}{2}\cos nx\,dx = \frac{1}{\pi}\int_0^\pi \left[\cos\left(n+\frac{1}{2}\right)x + \cos\left(n-\frac{1}{2}\right)x\right]dx$$

$$= \frac{2}{\pi}\left[\frac{\sin\left(n+\frac{1}{2}\right)x}{2n+1} + \frac{\sin\left(n-\frac{1}{2}\right)x}{2n-1}\right]_0^\pi = \frac{2}{\pi}\left[\frac{\sin\left(n+\frac{1}{2}\right)\pi}{2n+1} + \frac{\sin\left(n-\frac{1}{2}\right)\pi}{2n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n-1}\right] = \frac{-4(-1)^n}{\pi(4n^2-1)}.$$

By Parseval formula $a_0^2 + \frac{1}{2}\sum a_n^2 = \frac{1}{l}\int_0^l [f(x)]^2 dx$. Substituting for a_0, a_n, l and $f(x)$, we get

$$\frac{4}{\pi^2} + \frac{16}{2\pi^2}\sum\frac{1}{(4n^2-1)^2} = \frac{1}{\pi}\int_0^\pi \cos^2(x/2)\,dx = \frac{1}{2\pi}\int_0^\pi (1+\cos x)dx = \frac{1}{2}$$

$$\text{or,}\quad \sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2} = \frac{\pi^2-8}{16}.$$

EXERCISE 14.3

1. If $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$ has the Fourier series representation, $f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$,

   then find the Fourier series representation of $g(x) = \begin{cases} -x-\pi, & -\pi < x < 0 \\ x-\pi, & 0 < x < \pi. \end{cases}$

2. Find the Fourier series of $f(x) = \pi^2 - x^2$ for $-\pi < x < \pi$ and use it to find the Fourier series of x and $x(\pi^2 - x^2)$.

3. Let $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x, & 0 < x \le \pi \end{cases}$. Write the Fourier series of $f(x)$ on $[-\pi, \pi]$ and show that this series converges to $f(x)$ on $(-\pi, \pi)$, and can be integrated term by term, and thus obtain a