trigonometric series expansion for $g(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x^2/2, & 0 < x \le \pi. \end{cases}$

4. If $f(x) = x \sin x, -\pi < x < \pi$ with the Fourier series representation

$$f(x) = \pi - \frac{1}{2}\pi \cos x + 2\pi \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1} \cos nx,$$

show that the series can be differentiated term by term and hence find the Fourier series expansion for $g(x) = x \cos x + \sin x$, for $-\pi < x < \pi$.

5. Given $f(x) = \begin{cases} \sin 2x, & -\pi \le x < -\pi/2 \\ 0, & -\pi/2 \le x \le \pi/2 \\ \sin 2x, & \pi/2 < x \le \pi \end{cases}$

Find the Fourier series expansion for $f'(x)$ by differentiating the Fourier expansion for $f(x)$.

6. Using the Fourier coefficients for the function $f(x) = \begin{cases} -1, & -\pi < x \le 0 \\ 1, & 0 < x < \pi \end{cases}$, show that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

7. Using the Fourier coefficients for the function $y = x^2$ on $[-\pi, \pi]$, show that $\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}$.

8. From the coefficients of the half-range cosine series of the function

$$f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}, \quad \text{find } \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}.$$

14.6 COMPLEX FORM OF THE FOURIER SERIES

To simplify the calculations it is sometimes convenient to work in terms of complex numbers even when the parameters under reference are reals. In this context we study the complex form of the Fourier series of a real function $f(x)$. This form is of special interest in the study of electrical circuits.

14.6.1 Complex Fourier Series

Let $f(x)$ be a real periodic function of period $2l$ over the interval $(-l, l)$. Then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right).$$

Since, $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, therefore, this series can be expressed as

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left\{\frac{a_n}{2}\left(e^{\frac{in\pi x}{l}} + e^{\frac{-in\pi x}{l}}\right) + \frac{b_n}{2i}\left(e^{\frac{in\pi x}{l}} - e^{\frac{-in\pi x}{l}}\right)\right\}$$

and after regrouping the terms, we have

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{\frac{-in\pi x}{l}}. \qquad ...(14.44)$$

We define $c_0 = a_0$, $c_n = \frac{a_n - ib_n}{2}$ and $c_{-n} = \frac{a_n + ib_n}{2}$, for $n = 1, 2, \ldots$.

Clearly c_n and c_{-n} are complex conjugates. Using these, (14.44) becomes

$$f(x) = \lim_{k\to\infty}\sum_{n=-k}^{k} c_n e^{\frac{in\pi x}{l}}, \quad \text{for } -l < x < l, \qquad ...(14.45)$$

where
$$c_n = \frac{1}{2}\left[\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx - \frac{i}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{-in\pi x}{l}}dx$$

and,
$$c_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{in\pi x}{l}}dx.$$

Combining these two, we have $c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{\frac{-in\pi x}{l}}dx, \quad n = 0, \pm 1, \pm 2$ ...(14.46)

Thus (14.45) is the complex form of the Fourier series representation of the periodic function $f(x)$ defined over the interval $-l < x < l$, with Fourier coefficients given by (14.46).

Since, the complex form of the Fourier series representation of a function is derived from its real variable definition, the convergence properties of the complex Fourier series are the same as those for the real variable case. Thus, at points of continuity of $f(x)$ the series converges to $f(x)$, while at points of discontinuity it converges to the mid-point.

Example 14.17: Find the complex Fourier series representation of the function

$$f(x) = \begin{cases} 0, & 0 < x \le 1 \\ 1, & 1 < x < 4 \end{cases} \text{ when } f(x) = f(x+4).$$

Solution: The function $f(x)$ is periodic with period 4 defined on the interval (0, 4), with $2l = 4$. Thus the complex Fourier coefficients c_n are given by $c_n = \frac{1}{4}\int_0^4 f(x)e^{\frac{-in\pi x}{2}}\,dx = \frac{1}{4}\int_1^4 e^{\frac{-in\pi x}{2}}\,dx$. For $n = 0$, we get $c_0 = \frac{1}{4}\int_1^4 dx = 3/4$.

For all c_n, except $n = 0$ $c_n = \frac{1}{4}\left[\frac{-2}{in\pi}e^{\frac{-in\pi x}{2}}\right]_1^4 = \frac{i}{2\pi n}[1 - e^{-in\pi/2}]$.

Hence, the complex Fourier series representation of $f(x)$ is

$$f(x) = \frac{3}{4} + \lim_{k\to\infty}\sum_{n=-k}^{k}\frac{i}{2\pi n}(1 - e^{-in\pi/2})e^{\frac{in\pi x}{2}},\quad (n \neq 0).$$

Example 14.18: Find the complex Fourier series representation of the function

$$f(x) = e^{-x},\quad -\pi < x < \pi;\quad f(x) = f(x + 2\pi)$$

Solution: The function $f(x)$ is periodic with period 2π, defined on the interval $(-\pi, \pi)$. Here $l = \pi$, thus the complex Fourier coefficients are

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(1+in)x}\,dx$$

$$= \frac{-1}{2\pi(1+in)}\left[e^{-(1+in)x}\right]_{-\pi}^{\pi} = \frac{-1}{2\pi(1+in)}\left[e^{-(1+in)\pi} - e^{(1+in)\pi}\right]$$

$$= \frac{-1}{2\pi(1+in)}\left[e^{-\pi}(\cos n\pi - i\sin n\pi) - e^{\pi}(\cos n\pi + i\sin n\pi)\right]$$

$$= \frac{(1-in)}{2\pi(1+n^2)}\left[(e^{\pi} - e^{-\pi})\cos n\pi\right] = (-1)^n\frac{(1-in)\sinh\pi}{\pi(1+n^2)}.$$

Hence, the complex Fourier series is $f(x) = \frac{\sinh\pi}{\pi}\lim_{k\to\infty}\sum_{n=-k}^{k}(-1)^n\left(\frac{1-in}{1+n^2}\right)e^{inx}$.

14.6.2 Frequency Spectra of a Function $f(x)$

In applications of Fourier series to periodic physical phenomena with fundamental period T, it is sometimes more convenient to work in terms of the angular frequency w, defined as $w = 2\pi/T = 2\pi/2l = \pi/l$, called the *fundamental angular frequency*. In terms of w, the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right),$$ can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nwx + b_n \sin nwx), \quad ...(14.47)$$

where $$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx,\quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos nwx\,dx \text{ and } b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin nwx\,dx;\quad l = \pi/w.$$

In terms of the fundamental angular frequency, the complex Fourier series form (14.45) can be expressed as

$$f(x) = \lim_{k\to\infty}\sum_{n=-k}^{k} c_n e^{inwx}, \quad ...(14.48)$$

where $$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-inwx}dx,\quad l = \pi/w,\ n = 0,\ \pm1,\ \pm2,\ ... \quad ...(14.49)$$

The plot of the points $(nw, |c_n|)$, where w is the fundamental angular frequency and c_n are the Fourier coefficients as defined in (14.49) is called the *frequency spectrum* or *amplitude spectrum* of the function $f(x)$ and the number nw is called the *nth harmonic frequency* of the function $f(x)$.

Example 14.19: Find the frequency spectrum of the periodic pulse defined by

$f(x) = 3x/4,\quad 0 \le x \le 8$ and $f(x+8) = f(x)$.

Solution: The function $f(x)$ is periodic with period $T = 2l = 8$ defined on $[0, 8]$. Thus the fundamental angular frequency w is $w = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$. The complex Fourier coefficients are

$$c_n = \frac{1}{8}\int_0^8 \frac{3}{4}xe^{-\frac{in\pi}{4}x}dx = \frac{3}{32}\left[x\left(\frac{4}{-in\pi}\right)e^{-\frac{in\pi}{4}x} - \left(\frac{4}{-in\pi}\right)^2 e^{-\frac{in\pi}{4}x}\right]_0^8 = \frac{3i}{n\pi},\quad n \ne 0.$$

For $n = 0$, $$c_0 = \frac{3}{32}\int_0^8 x\,dx = \frac{3}{32}\left(\frac{x^2}{2}\right)_0^8 = 3.$$

The frequency spectrum of $f(x)$ is a plot of points $(nw, |c_n|)$, where

$$nw = \frac{n\pi}{4},\quad |c_0| = 3 \text{ and } |c_n| = \frac{3}{|n|\pi}, \text{ for } n = \pm1,\ \pm2,\ \pm3, \dots .$$

The plot is shown in Fig. 14.14.

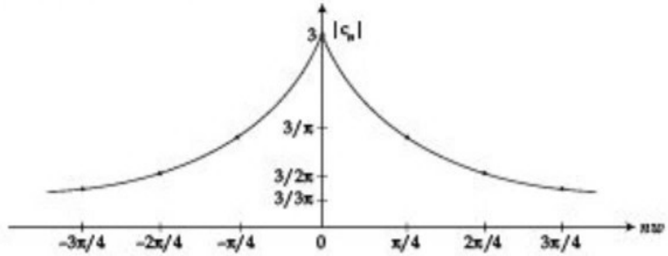


Fig. 14.14

Example 14.20: Find the frequency spectrum of the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 \leq x \leq \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

when $f(x + 2\pi) = f(x)$

Solution: The function $f(x)$ is periodic with period $T = 2\pi$ defined over the interval $(-\pi, \pi)$. The fundamental angular frequency w is

$$w = 2\pi/T = 2\pi/2\pi = 1.$$

The complex Fourier coefficients are

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-inx}dx.$$

For $n = 0$, $c_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi/2}^{\pi/2} dx = \dfrac{1}{2}$; and for all other $n \neq 0$

$$c_n = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-inx}dx = \frac{1}{n\pi}\left[\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}\right] = \frac{1}{n\pi}\sin\frac{n\pi}{2}, \quad n = \pm 1, \pm 2, \ldots$$

The frequency spectrum of $f(x)$ is a plot of points $(nw, |c_n|)$. Here

$$nw = n,\ |c_0| = \frac{1}{2} \text{ and } |c_n| = \frac{1}{n\pi}\left|\sin\frac{n\pi}{2}\right|, \text{ for } n = \pm 1, \pm 2, \ldots .$$

The plot is as shown in Fig. 14.15.

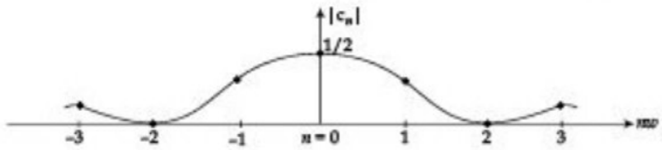


Fig. 14.15

EXERCISE 14.4

Find the complex Fourier series representation of $f(x)$ on the given interval

1. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \end{cases}, \quad f(x+2) = f(x)$
2. $f(x) = e^x, \quad 0 < x < 1, \quad f(x+1) = f(x).$
3. $f(x) = |E \sin \lambda x|, \quad 0 < x < \pi/\lambda, \quad f(x + \pi/\lambda) = f(x)$
4. $f(x) = e^{-|x|}, \quad -2 < x < 2, \quad f(x+4) = f(x)$

Find the frequency spectrum of $f(x)$ for the following problems

5. $f(x) = \begin{cases} 0, & -\pi/2 < x < 0 \\ \sin x, & 0 \le x < \pi/2 \end{cases}, \quad f(x+\pi) = f(x)$
6. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \end{cases}, \quad f(x+2) = f(x)$
7. $f(x) = |E \sin \lambda x|, \quad 0 < x < \pi/\lambda, \quad f(x + \pi/\lambda) = f(x)$
8. Plot some points of the frequency spectrum of the function defined by

$$f(x) = 4 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{26}{n(12-5i)} e^{2nix}.$$

14.7 NUMERICAL HARMONIC ANALYSIS

So far we have derived the Fourier series expansion of a function $f(x)$ when it was known analytically. However, in many practical problems the analytic nature of the periodic function $f(x)$ is not known but one may be in a position to observe only a set of values of x and y; y being dependent on x, say $y = f(x)$. In such a case, to evaluate the Fourier coefficients, the Euler's formulae studied previously need some modifications given as below.

Let (x_i, y_i), $i = 1, 2, \ldots k$ be the given set of values, where the x_i's are equally spaced. The Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} ydx$$

$$= [\text{Mean value of } y \text{ over the one period } T = 2\pi] = \frac{1}{k}\sum_{i=1}^{k} y_i, \quad \text{...(14.50)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 2.\frac{1}{2\pi}\int_{-\pi}^{\pi} y\cos nx\,dx$$

$$= 2[\text{Mean value of } y\cos nx \text{ over the one period } T = 2\pi] = \frac{2}{k}\sum_{i=1}^{k} y_i\cos nx_i \quad \text{...(14.51)}$$

and, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 2.\frac{1}{2\pi}\int_{-\pi}^{\pi} y\sin nx\,dx$$

$$= 2[\text{Mean value of } y\sin nx \text{ over the one period } T = 2\pi] = \frac{2}{k}\sum_{i=1}^{k} y_i\sin nx_i \quad \text{...(14.52)}$$

Then the Fourier series for $y = f(x)$ is

$$y = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx), \quad \text{...(14.53)}$$

where the Fourier coefficients are given by (14.50), (14.51) and (14.52).

The process of finding the Fourier series for a function given by numerical values is known as *numerical harmonic analysis*. The term $(a_1\cos x + b_1\sin x)$ is called *the fundamental* or *first harmonic*, the term $(a_2\cos 2x + b_2\sin 2x)$ is called *the second harmonic* and so on.

Example 14.21: Given that x is a function of θ over the interval $0 \le \theta \le 2\pi$. Find the Fourier series expansion of x upto the second harmonic on the basis of the following data

θ :	0,	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$
x :	298	356	373	337	254	155	80	51	60	93	147	221

Solution: The Fourier series for $x = f(\theta)$ upto the second harmonic is

$$x = a_0 + \sum_{n=1}^{2}(a_n\cos n\theta + b_n\sin n\theta),$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{12}\sum_{i=1}^{12} x_i, \quad a_n = \frac{1}{6}\sum_{i=1}^{12} x_i \cos n\theta_i \text{ and } b_n = \frac{1}{6}\sum_{i=1}^{12} x_i \sin n\theta_i.$$

To evaluate the coefficients we form the following table:

θ	$\sin\theta$	$\cos\theta$	$\sin 2\theta$	$\cos 2\theta$	x	$x\sin\theta$	$x\cos\theta$	$x\sin 2\theta$	$x\cos 2\theta$
0	0.00	1.00	0.00	1.00	298	0.00	298.00	0.00	298.00
$\pi/6$	0.50	0.87	0.87	0.50	356	178.00	309.72	309.72	178.00
$\pi/3$	0.87	0.50	0.87	−0.50	373	324.51	186.50	324.51	−186.50
$\pi/2$	1.00	0.00	0.00	−1.00	337	337.00	0.00	0.00	−337.00
$2\pi/3$	0.87	−0.50	−0.87	−0.50	254	220.98	−127.00	−220.98	−127.00
$5\pi/6$	0.50	−0.87	−0.87	−0.50	155	77.50	−134.85	−134.85	−77.50
π	0.00	−1.00	0.00	1.00	80	0.00	−80.00	0.00	80.00
$7\pi/6$	−0.50	−0.87	0.87	0.50	51	−25.50	−44.37	44.37	25.50
$4\pi/3$	−0.87	−0.50	0.87	−0.50	60	−52.20	−30.00	52.20	−30.00
$3\pi/2$	−1.00	0.00	0.00	−1.00	93	−93.00	0.00	0.00	−93.00
$5\pi/3$	−0.87	0.50	−0.87	−0.50	147	−102.90	73.50	−102.90	−73.50
$11\pi/6$	−0.50	0.87	−0.87	0.50	221	−110.50	192.27	−192.27	110.50
			Total:	2425	753.89	643.77	54.18	−77.50	

We have,

$$\Sigma x = 2425, \quad \Sigma x \sin\theta = 753.89, \quad \Sigma x \cos\theta = 643.77, \quad \Sigma x \sin 2\theta = 54.18, \quad \Sigma x \cos 2\theta = -77.50$$

Thus, $a_0 = \dfrac{2425}{12} = 202.09 \approx 202, \quad a_1 = \dfrac{643.77}{6} = 107.30 \approx 107, \quad a_2 = \dfrac{-77.50}{6} = -12.92 \approx -13.$

$$b_1 = \frac{753.89}{6} = 125.65 \approx 126, \text{ and } b_2 = \frac{54.18}{6} = 9.03 \approx 9.$$

Hence the Fourier series expansion upto two harmonics is given by

$$x \approx 202 + 107\cos\theta - 13\cos 2\theta + 126\sin\theta + 9\sin 2\theta.$$

Example 14.22: The turning moment T units of the crank shaft of a steam engine is given for a series of values of the crank-angle θ in degrees;

$\theta°$:	0	30	60	90	120	150	180
T :	0	5224	8097	7850	5499	2626	0

Find the first four terms in a series of sines to represent T and calculate T for $\theta = 75°$.

Solution: The half-range sine series to represent T is

$$T \approx b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta,$$

where the coefficients b_i's are given by $b_n = \dfrac{2}{6}\Sigma T \sin n\theta = \dfrac{1}{3}\Sigma T \sin n\theta.$

To calculate b_i's we form the following table:

θ	T	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$	$T\sin\theta$	$T\sin 2\theta$	$T\sin 3\theta$	$T\sin 4\theta$
0	0	0.00	0.00	0.00	0.00	0.00	0.00	0.0	0.00
30	5224	0.50	0.87	1.00	0.87	2612.00	4544.88	5224.00	4544.88
60	8097	0.87	0.87	0.00	−0.87	7044.39	7044.39	0	−7044.39
90	7850	1.00	0.00	−1.00	0.00	7850.00	0.00	−7850.00	0.00
120	5499	0.87	−0.87	0.00	0.87	4784.13	−4784.13	0.00	4784.13
150	2626	0.50	−0.87	1.00	−0.87	1313.00	−2284.62	2626.00	−2284.62
					Total:	23603.52	4520.52	0.00	0.00

Thus, $b_1 = \dfrac{23603.52}{3} = 7867.67 \approx 7868$, $b_2 = \dfrac{4520.52}{3} = 1506.84 \approx 1507$, $b_3 = 0$, $b_4 = 0$.

Hence, the Fourier series is given by $T \approx 7868 \sin\theta + 1507 \sin 2\theta$. When $\theta = 75°$, then

$T = 7868 \sin 75° + 1507 \sin 150° = 7868 \times 0.9659 + 1507 \times 0.50 \approx 8353.20.$

Example 14.23: The following table gives the variations of a periodic current over a fundamental period of T second

t(sec) :	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A(amp) :	1.98	1.30	1.05	1.30	−0.88	−0.25	1.98

Show that there is direct current part of 0.75 amp. in the variable current and obtain the amplitude of the first harmonic.

Solution: The series is periodic over the interval $(0, T)$ hence the period $2l = T$, that is, $l = T/2$. Thus the current A is given as

$$A = a_0 + a_1 \cos\frac{2\pi t}{T} + b_1 \sin\frac{2\pi t}{T} + a_2 \cos\frac{4\pi t}{T} + b_2 \sin\frac{4\pi t}{T}.$$

Here a_0 represents the direct current part and $\sqrt{a_1^2 + b_1^2}$ gives the amplitude of the first harmonic. To calculate the coefficients, we form the following table:

t	$2\pi t/T$	$\cos(2\pi t/T)$	$\sin(2\pi t/T)$	A	$A\cos(2\pi t/T)$	$A\sin(2\pi t/T)$
0	0	1.00	0.000	1.98	1.98	0.00
$T/6$	$\pi/3$	0.50	0.87	1.30	0.65	1.13
$T/3$	$2\pi/3$	−0.50	0.87	1.05	−0.53	0.91
$T/2$	π	−1.0	0.00	1.30	−1.30	0.00
$2T/3$	$4\pi/3$	−0.5	0.87	−0.88	0.44	0.76
$5T/6$	$5\pi/3$	0.5	0.87	−0.25	−.13	0.22
			Total:	4.5	1.11	3.02

Here, $\Sigma A = 4.5$, $\Sigma A \cos(2\pi t/T) = 1.11$, $\Sigma A \sin(2\pi t/T) = 3.02$. Hence

$$a_0 = \frac{4.5}{6} = 0.75, \quad a_1 = \frac{1.11}{3} = 0.37, \quad b_1 = \frac{3.02}{3} = 1.01.$$

Thus, the direct current part is 0.75 amp. and amplitude of the first harmonic is $\sqrt{(0.37)^2 + (1.01)^2} = 1.07$ amp.

EXERCISE 14.5

1. The following values of y give the displacement of a certain machine part for the rotation x of the flywheel

x :	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y :	1.98	2.15	2.77	−0.22	−0.31	1.43	1.98

Express y in Fourier series upto the third harmonic.

2. The following values of y give the displacement in inches of a certain machine part for the rotation x of the flywheel. Expand y in the form of a Fourier series upto fourth hormonic

x :	0	30°	60°	90°	120°	150°	180°
y :	0	9.2	14.4	17.8	17.3	11.7	0

3. Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table:

x :	0	1	2	3	4	5
y :	4	8	15	7	6	2

4. The turning moment T on the crank-shaft of a steam engine for the crank angle θ in degrees is recorded as follow. Express T in a series of sines upto the fourth harmonic

θ :	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
T :	0	2.7	5.2	7	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2	0

5. A part of a machine has an oscillatory motion. The displacement y at a time t is given below:

t :	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
y :	0	0.64	1.13	1.34	0.95	0.00	−0.92	−1.33	−1.17	−0.66	0.0

Find constants in the equation $y = A\sin(10\pi t + \alpha_1) + B\sin(20\pi t + \alpha_2)$.

ANSWERS

Exercise 14.1 (p. 824)

1. $\frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots$ 2. $-\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$

3. $-1+\pi\sin x-\frac{1}{2}\cos x+\frac{2}{2^2-1}\cos 2x+\frac{2}{3^2-1}\cos 3x+\ldots$

4. $-\frac{\pi^2}{3}+4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\ldots\right]+2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\ldots\right]$

5. $\frac{\pi^2}{3}-4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+\ldots\right)$

7. $\frac{\sinh 8}{8}+\sinh 8\sum_{n=1}^{\infty}\left[\frac{16(-1)^n}{64+\pi^2n^2}\cos\left(\frac{n\pi x}{2}\right)+\frac{2n\pi(-1)^n}{64+\pi^2n^2}\sin\left(\frac{n\pi x}{2}\right)\right]$

8. $\frac{k}{2}+\frac{2k}{\pi}\left(\cos\frac{\pi x}{2}-\frac{1}{3}\cos\frac{\pi x}{3}+\frac{1}{5}\cos\frac{\pi x}{5}-\ldots\right)$

9. $4\left(\frac{1}{2}-\frac{1}{1.3}\cos 2\pi x-\frac{1}{3.5}\cos 4\pi x-\frac{1}{5.7}\cos 6\pi x\ldots\right)$

Exercise 14.2 (p. 831)

1. $\frac{1}{5}-\frac{8}{\pi^4}\left[\frac{\pi^2-6}{1^4}\cos\pi x-\frac{2^2\pi^2-6}{2^4}\cos 2\pi x+\frac{3^2\pi^2-6}{3^4}\cos 3\pi x+\ldots\right]$

2. $1-\frac{1}{2}\cos x-\frac{2}{1.3}\cos 2x+\frac{2}{2.4}\cos 3x-\frac{2}{3.5}\cos 4x+\ldots$

3. $\frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right)$

4. $\frac{2}{\pi}+\frac{4}{\pi}\left(\frac{1}{3}\cos 2x-\frac{1}{15}\cos 4x+\ldots\right)$

5. $\frac{2}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\ldots\right)$

6. $\frac{\pi}{2}+1-\frac{4}{\pi}\left(\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\ldots\right)$

7. $\frac{8}{3}+\frac{16}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2}-\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}-\ldots\right]$

8. $1, \frac{4}{\pi}\left[\sin\frac{\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}+\frac{1}{5}\sin\frac{5\pi x}{2}+\ldots\right]$

9. $\frac{8}{3}+\frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\frac{n\pi}{2}-\frac{2}{n\pi}\sin\frac{n\pi}{2}\right]\cos\frac{n\pi x}{4}$

10. $\frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right]$

11. $\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{1.3}\cos\frac{2\pi x}{l} + \frac{1}{3.5}\cos\frac{4\pi x}{l} + \frac{1}{5.7}\cos\frac{6\pi x}{l} + \ldots\right]$

12. $\frac{1}{\pi} + \frac{1}{\pi}\left[\cos x - \frac{2}{3}\cos 2x - \cos 3x - \frac{2}{15}\cos 4x + \frac{1}{3}\cos 5x - \ldots\right]$

Exercise 14.3 (p. 837)

1. $\frac{-\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$

2. $\pi^2 - x^2 = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2}, \quad x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$

   $x(\pi^2 - x^2) = 12\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n^3}$

3. $f(x) = \frac{1}{4}\pi + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$

   $g(x) = \frac{1}{4}x\pi + \frac{1}{4}\pi^2\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^3}\sin nx + \frac{(-1)^{n+1}}{n^2}(-\cos nx + (-1)^n\right]$

4. $x\cos x + \sin x = \frac{1}{2}\pi\sin x + 2\pi\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}n\sin(nx)$

5. $f'(x) = \frac{1}{\pi}\left(\frac{-2}{3}\sin x + \frac{6}{5}\sin 3x + \ldots\right) + \frac{1}{\pi}\left(\frac{-2}{3}\cos x + \frac{3\pi}{3}\cos 2x - \ldots\right)$

8. $\frac{\pi^4}{96}$

Exercise 14.4 (p. 843)

1. $f(x) = 1 + \lim_{k\to\infty}\sum_{n=-k}^{k}\frac{i}{n\pi}(1-(-1)^n)e^{-in\pi x}$ 2. $f(x) = e - \lim_{k\to\infty}\sum_{n=-k}^{k}\frac{e^{-in\pi x}}{1-2n\pi i}, n = 0, \pm 1, \pm 2, \ldots$

3. $f(x) = -2\frac{E}{\pi} \lim_{k \to \infty} \sum_{n=-k}^{k} \frac{1}{(4n^2-1)} e^{2n\lambda i x}$.

4. $f(x) = \sum_{n=-\infty}^{\infty} \frac{2}{4-n^2\pi^2} [\{1-(-1)^n e^{-2}\}] e^{\frac{-in\pi x}{2}}$

5. $\left(2n, \frac{4|n|}{\pi(4n^2-1)}\right)$

6. $\left(n\pi, \frac{[1-(-1)^n]}{|n|\pi}\right)$

7. $\left(2n\lambda, \left|\frac{2E}{(4n^2-1)\pi}\right|\right)$

8. The frequency spectrum consists of the point (0, 4) and the points $(2n, 2/|n|)$.

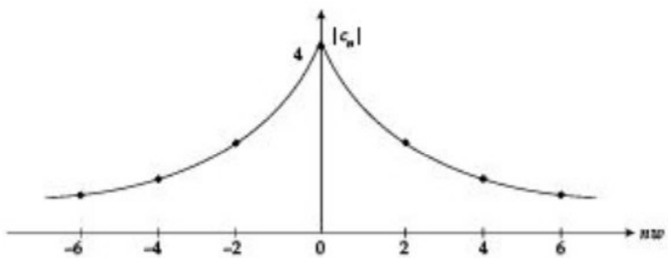


Fig. 14.16

Exercise 14.5 (p. 847)

1. $y = 1.3 + 0.92 \cos x + 1.097 \sin x - 0.42 \cos 2x - 0.681 \sin 2x + 0.36 \cos 3x$
2. $y = 11.73 - (7.73 \cos 2x + 1.57 \sin 2x) + (-2.83 \cos 4x + .116 \sin 4x)$
3. $y = 7 - 2.8 \cos x - 1.5 \cos 2x + 2.7 \cos 3x$
4. $T = 7.8 \sin \theta + 1.5 \sin 2\theta - 9.2 \sin 3\theta + 11.6 \sin 4\theta$
5. $A = 1.317, \quad B = -0.1524, \quad \alpha_1 = -0.0083, \quad \alpha_2 = -0.315.$

15 Fourier Integrals and Fourier Transforms

CHAPTER

Fourier integrals and Fourier transforms extend the concept of Fourier series to non-periodic functions defined for all x. A non-periodic function which cannot be represented as Fourier series over the entire real line may be represented in an integral form. Fourier transforms are integral transforms similar to Laplace transforms. In fact, 'Fourier analysis', the term including various kinds of Fourier series, integrals and transforms find variety of applications in science and engineering.

15.1 FOURIER INTEGRAL

In the preceding chapter we have seen that if a function $f(x)$ is defined on $-\infty < x < \infty$ and is periodic over an interval $-l < x < l$ (and satisfies the other conditions), then it can be represented by a Fourier series. In many practical problems we come across functions defined on $-\infty < x < \infty$ that are not periodic, e.g. $f(x) = e^{-x^2}$, the graph of which is shown in Fig. 15.1.

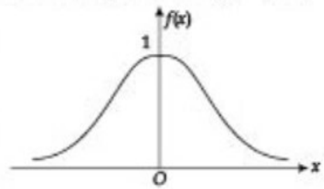


Fig. 15.1

We cannot expand such functions in Fourier series since they are not periodic, however we can consider such functions to be periodic but with an infinite period. *The Fourier integral can be regarded as an extension of the concept of Fourier series to non-periodic (or aperiodic) functions by letting $l \to \infty$.*

Consider any periodic function $f(x)$ of period $2l$ that can be represented by a Fourier series, then

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right), \quad ...(15.1)$$

where $a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx$, $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$ and $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$.

Substituting the values for a_0, a_n and b_n, (15.1) becomes

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(u)du + \frac{1}{l}\sum_{n=1}^{\infty}\left[\cos\frac{n\pi x}{l}\int_{-l}^{l} f(u)\cos\frac{n\pi u}{l}du + \sin\frac{n\pi x}{l}\int_{-l}^{l} f(u)\sin\frac{n\pi u}{l}du\right]. \quad ...(15.2)$$

Set $w_n = \frac{n\pi}{l}$ and $\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{l} - \frac{n\pi}{l} = \frac{\pi}{l}$, (15.2) becomes

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(u)du + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\Delta w\int_{-l}^{l} f(u)\cos w_n u\,du + (\sin w_n x)\,\Delta w\int_{-l}^{l} f(u)\sin w_n u\,du\right]. \quad ...(15.3)$$

The Eq. (15.3) is valid for any fixed finite l, arbitrary large.

We now let $l \to \infty$, and assume $f(x)$ to be absolutely integrable over the interval $(-\infty, \infty)$, that is, $\int_{-\infty}^{\infty} |f(x)|dx$ converges, then the value of the integral $\frac{1}{2l}\int_{-l}^{l} f(u)du$ tends to zero as $l \to \infty$; also $\Delta w = \pi/l \to 0$ and the infinite series in (15.3) becomes an integral from 0 to ∞, which represents $f(x)$ as

$$f(x) = \int_{0}^{\infty} [A(w)\cos wx + B(w)\sin wx]dw, \quad ...(15.4)$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos wu\,du$ and $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin wu\,du$, ...(15.5)

are called the *Fourier coefficients* and (15.4) is called the *Fourier integral representation of* $f(x)$.

The sufficient conditions for the validity of (15.4) are

1. *$f(x)$ is piecewise continuous on every interval $[-l, l]$.*
2. *$f(x)$ is absolutely integrable on the real axis, that is, $\int_{-\infty}^{\infty} |f(x)|dx$ converges.*
3. *At every x on the real line $f(x)$ has left and right hand derivatives.*

We state without proof the following convergence theorem for the Fourier integral, called the *Fourier Integral Theorem.*

Theorem 15.1 **(Fourier integral theorem):** *If $f(x)$ satisfies the conditions 1 to 3 stated above, then the Fourier integral of f converges to $f(x)$ at every point x at which f is continuous, and to the mean value $[f(x+0) + f(x-0)]/2$ at every point x at which f is discontinuous, where $f(x+)$ and $f(x-)$ are the right and left hand limits respectively.*

Example 15.1: Find the fourier integral representation of $f(x) = \begin{cases} 1, & \text{for } -1 \le x \le 1 \\ 0, & \text{for } |x| > 1 \end{cases}$ and hence prove that $\int_0^\infty \frac{\sin w}{w} dw = \frac{\pi}{2}$.

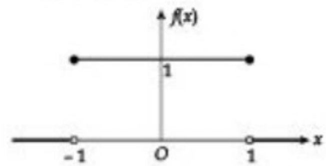


Fig. 15.2

Solution: The graph of $f(x)$ is shown in Fig. 15.2. Clearly $f(x)$ is piecewise smooth and is absolutely integrable over $(-\infty, \infty)$. Thus $f(x)$ has a Fourier integral representation. The Fourier coefficients of $f(x)$ are

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos wu\, du = \frac{1}{\pi}\int_{-1}^{1}\cos wu\, du = \left[\frac{\sin wu}{\pi w}\right]_{-1}^{1} = \frac{2\sin w}{\pi w}, \text{ and,}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin wu du = \frac{1}{\pi}\int_{-1}^{1}\sin wu\, du = 0.$$

Hence the Fourier integral of $f(x)$ is $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w} dw$.

Since $x = \pm 1$ are the points of discontinuity of $f(x)$, thus at $x = \pm 1$

$$\frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w} dw = \frac{1}{2}[f(x+0) + f(x-0)] = 1/2, \text{ for } \quad x = \pm 1.$$

$$\text{Thus, } \int_0^\infty \frac{\cos wx \sin w}{w} dw = \begin{cases} \frac{\pi}{2}, & \text{for } -1 < x < 1 \\ \frac{\pi}{4}, & \text{for } x = \pm 1 \\ 0, & \text{for } |x| > 1 \end{cases} \quad \text{...(15.6)}$$

Set $x = 0$ in (15.6), we have $\int_0^\infty \frac{\sin w}{w} dw = \pi/2$.

Example 15.2: Find the Fourier integral representation of $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases}$

and find the value of the resulting integral when, (a) $x < 0$, (b) $x = 0$, (c) $x > 0$. Also derive that

$$\int_0^\infty \frac{dw}{1+w^2} = \pi/2.$$

Solution: The given function $f(x)$ is piecewise smooth and is absolutely integrable over $(-\infty, \infty)$, since $\int_{-\infty}^{\infty} |f(x)|\,dx = \int_0^{\infty} e^{-x}dx = 1$. Thus $f(x)$ has a Fourier integral representation. The Fourier coefficients of $f(x)$ are $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos wu\,du = \frac{1}{\pi}\int_0^{\infty} e^{-u}\cos wu\,du = \frac{1}{\pi(1+w^2)}$, and,

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin wu\,du = \frac{1}{\pi}\int_0^{\infty} e^{-u}\sin wu\,du = \frac{w}{\pi(1+w^2)}.$$

Thus the Fourier integral repesentation of $f(x)$ is $f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\cos wx + w\sin wx}{1+w^2}dw.$

At the point of discontinuity, $x = 0$,

$$\frac{1}{\pi}\int_0^{\infty} \frac{\cos wx + w\sin wx}{1+w^2}dw = \frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2}[1+0] = \frac{1}{2}.$$

Thus,

$$\int_0^{\infty} \frac{\cos wx + w\sin wx}{1+w^2}dw = \begin{cases} 0, & x < 0 \\ \pi/2, & x = 0 \\ \pi e^{-x}, & x > 0 \end{cases} \qquad \text{...(15.7)}$$

Set $x = 0$ in (15.7), we have $\int_0^{\infty} \frac{dw}{1+w^2} = \pi/2.$

15.2 FOURIER COSINE AND FOURIER SINE INTEGRALS

For an even or odd function the Fourier integral becomes simpler, analogous to the Fourier series expansion for the even or odd function. When $f(x)$ is an even function, then $f(u)\sin wu$ is an odd function of u, so from (15.5) we have $B(w) = 0$ and

$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(u)\cos wu\,du. \qquad \text{...(15.8)}$$

Thus (15.4) simplifies to, $f(x) = \int_0^{\infty} A(w)\cos wx\,dw,$ ...(15.9)

called the *Fourier cosine integral representation* of $f(x)$.

Similarly, when $f(x)$ is an odd function, then $f(u)\cos wu$ is an odd function of u, so from (15.5), we have $A(w) = 0$ and

$$B(w) = \frac{2}{\pi}\int_0^\infty f(u)\sin wu\,du. \quad ...(15.10)$$

Thus (15.4) simplifies to, $f(x) = \int_0^\infty B(w)\sin wx\,dw,$ ...(15.11)

called the *Fourier sine integral representation* of $f(x)$.

The convergence result for the integral representations of even and odd functions is as follows.

Theorem 15.2: *If $f(x)$ is (i) piecewise continuous on each interval $[0, b]$, (ii) absolutely integrable on the real axis, and (iii) at every $x \in [0, \infty]$, $f(x)$ has left and right hand derivatives then the Fourier cosine and sine integrals of f converge to $f(x)$ at every point x at which f is continuous, and to the mean value $[f(x+0) + f(x-0)]/2$ at every point x at which f is discontinuous.*

Also similar to Fourier cosine and sine series defined on half period $[0, l]$, we can define Fourier cosine and Fourier sine integral representations of functions defined on the real half line $[0, \infty]$ by using respectively even or odd expansion of $f(x)$ to the whole real line.

Example 15.3: Find the Fourier cosine and sine integrals of $f(x) = e^{-kx}$, $x > 0$, $k > 0$.

Solution: Clearly $f(x)$ is differentiable and is absolutely integrable over $(0, \infty)$.
To obtain Fourier cosine representation, we have

$$A(w) = \frac{2}{\pi}\int_0^\infty f(u)\cos wu\,du = \frac{2}{\pi}\int_0^\infty e^{-ku}\cos wu\,du.$$

Consider $I = \int_0^\infty e^{-ku}\cos wu\,du = \left[\frac{e^{-ku}}{w^2+k^2}(w\sin wu - k\cos wu)\right]_0^\infty.$

When u tends to infinity, it becomes zero, and when u tends to zero, it becomes $-k/(w^2+k^2)$, since $k > 0$. Thus $I = k/(w^2+k^2)$ and hence

$$A(w) = \frac{2}{\pi}\int_0^\infty e^{-ku}\cos wu\,du = \frac{2k}{\pi(w^2+k^2)}. \quad ...(15.12)$$

Thus the Fourier cosine integral $f(x) = \int_0^\infty A(w)\cos wx\,dx$ becomes

$$e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}\,dx,\ k > 0. \quad ...(15.13)$$

On the similar lines, to obtain the Fourier sine integral representation of $f(x)$, we have

$$B(w) = \frac{2}{\pi}\int_0^{\infty} e^{-ku}\sin wu\,du = \frac{2w}{\pi(k^2+w^2)}, \qquad ...(15.14)$$

and thus $$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^{\infty} \frac{w\sin wx}{k^2+w^2}\,dw,\ w>0. \qquad ...(15.15)$$

The integral representations (15.12) and (15.14) are called *Laplace integrals* because $A(w)$ is $2/\pi$ times the Laplace transform of $\cos wx$ and $B(w)$ is $2/\pi$ times the Laplace transform of $\sin wx$.

Example 15.4: Let $f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & x\geq 1 \end{cases}$. Using the Fourier cosine integral representation of f, show that $\int_0^{\infty} \frac{\sin t}{t}\,dt = \pi/2$.

Solution: The function $f(x)$ is piecewise smooth and is also absolutely integrable over $(0, \infty)$. To obtain Fourier cosine representation, we have

$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(u)\cos wu\,du = \frac{2}{\pi}\int_0^{1}\cos wu\,du = \frac{2}{\pi}\left[\frac{\sin wu}{w}\right]_0^1 = \frac{2\sin w}{\pi w}.$$

Thus the Fourier cosine integral representation of $f(x)$ is

$$f(x) = \int_0^{\infty} A(w)\cos wx\,dw = \frac{2}{\pi}\int_0^{\infty}\frac{\sin w}{w}\cos wx\,dw.$$

The representation converges to $f(x)$ for every x in $(0, \infty)$ except at the point $x = 1$ which is a point of discontinuity of $f(x)$. At $x = 1$, the representation converges to

$$\frac{f(x+0)+f(x-0)}{2} = \frac{f(1+0)+f(1-0)}{2} = \frac{1}{2}.$$

Therefore $$\int_0^{\infty}\frac{\sin w}{w}\cos wx\,dw = \begin{cases} \pi/2, & 0<x<1 \\ \pi/4, & x=1 \\ 0, & x>1 \end{cases} \qquad ...(15.16)$$

and, hence for $x = 1$, (15.16) gives $\int_0^{\infty}\frac{\sin w\cos w}{w}\,dw = \frac{\pi}{4}$, or $\int_0^{\infty}\frac{\sin 2w}{2w}\,dw = \frac{\pi}{4}$.

Setting $2w = t$ in it, we obtain $\int_0^{\infty}\frac{\sin t}{t}\,dt = \frac{\pi}{2}$.

Example 15.5: Solve the integral equation $\int_0^\infty f(x)\sin ax\,dx = e^{-a}$, where a is constant.

Solution: The given integral is Fourier sine integral represention.

Let $$f(x) = \int_0^\infty A(w)\sin wx\,dw, \quad ...(15.17)$$

where $$A(w) = \frac{2}{\pi}\int_0^\infty f(u)\sin wu\,du. \quad ...(15.18)$$

Comparing (15.18) with the given equation, we get

$$w = a \quad \text{and} \quad \frac{\pi A(w)}{2} = e^{-a}, \text{thus} \quad A(w) = \frac{2}{\pi}e^{-w},$$

and hence from (15.17) we have

$$f(x) = \frac{2}{\pi}\int_0^\infty e^{-w}\sin wx\,dw. \quad ...(15.19)$$

Consider $$I = \int_0^\infty e^{-w}\sin wx\,dw = \left[-\frac{e^{-w}}{1+x^2}(x\cos wx + \sin wx)\right]_0^\infty$$

When w tends to infinity, this becomes zero and at $w = 0$, it is $-x/(1+x^2)$ and thus $I = x/(1+x^2)$.

Using in (15.19), we get $f(x) = \dfrac{2x}{\pi(1+x^2)}, \quad x > 0.$

15.3 THE COMPLEX FOURIER INTEGRAL REPRESENTATION

Analogous to the complex form of the Fourier series discussed in Section 14.6, the Fourier integral can also be expressed in the equivalent complex form. This complex form provides the necessary plateform to develop the Fourier transform, (refer to Section 15.5), which are highly developed as a methodology like the Laplace transform.

Substituting the expressions for $A(w)$ and $B(w)$ from (15.5) into (15.4), we obtain

$$f(x) = \frac{1}{\pi}\int_0^\infty\left[\int_{-\infty}^\infty f(u)\{\cos wu\cos wx + \sin wu\sin wx\}du\right]dw$$

$$= \frac{1}{\pi}\int_0^\infty\left[\int_{-\infty}^\infty f(u)\cos w(u-x)du\right]dw. \quad ...(15.20)$$

Inserting, $\cos w(u-x) = \frac{1}{2}\left(e^{iw(u-x)} + e^{-iw(u-x)}\right)$, it becomes

$$f(x) = \frac{1}{2\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(u)\left(e^{iw(u-x)} + e^{-iw(u-x)}\right)du\right]dw$$

$$= \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)e^{iw(u-x)}\,du\,dw + \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)e^{-iw(u-x)}du\,dw. \qquad \text{...(15.21)}$$

In the first integral on the right side of (15.21), replace w by $-w$, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(u)e^{-iw(u-x)}\,du\,dw + \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)e^{-iw(u-x)}du\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{-iw(u-x)}du\,dw. \qquad \text{...(15.22)}$$

This is the *complex Fourier integral representation* of f on the real line. If we put

$$c(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-iwu}du, \qquad \text{...(15.23)}$$

then the integral (15.22) becomes

$$f(x) = \int_{-\infty}^{\infty} c(w)e^{iwx}dw. \qquad \text{...(15.24)}$$

The $c(w)$ as given in (15.23) is called the *complex Fourier integral coefficient of* f.

Example 15.6: If $f(x) = e^{-a|x|}$ for all real x and with $a > 0$, a positive constant, then find the complex Fourier integral representation of f.

Solution: The function is $f(x) = \begin{cases} e^{-ax}, & \text{for } x \geq 0 \\ e^{ax}, & \text{for } x < 0 \end{cases}$ $a > 0$ being a constant.

Obvious $f(x)$ is piecewise smooth and is absolutely integrable over the interval $(-\infty, \infty)$. The complex Fourier integral coefficient of f is given by

$$c(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-iwu}du = \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{au}e^{-iwu}du + \int_0^{\infty} e^{-au}e^{-iwu}du\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{(a-iw)u}du + \int_{0}^{\infty} e^{-(a+iw)u}du\right] = \frac{1}{2\pi}\left[\left[\frac{e^{(a-iw)u}}{a-iw}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+iw)u}}{-(a+iw)}\right]_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{a+iw}+\frac{1}{a-iw}\right) = \frac{a}{\pi(a^2+w^2)}.$$

Thus, the complex Fourier integral representation, $f(x) = \int_{-\infty}^{\infty} c(w)e^{iwx}dw$, becomes

$$e^{-a|x|} = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{1}{a^2+w^2}e^{iwx}dw.$$

EXERCISE 15.1

1. Show that $f(x) = 1$, $(0 < x < \infty)$, cannot be represented by a Fourier integral.
   Derive the Fourier integral representations of the following functions (Problems 2-5). At which points, if any, does the Fourier integral fail to converge to $f(x)$? To what value does the integral converge at those points?

2. $f(x) = \begin{cases} 100, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} bx/a, & |x| \le a \quad a, b > 0 \\ 0, & |x| > a \end{cases}$

4. $f(x) = \begin{cases} (\pi/2)\cos x, & |x| \le \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$

5. $f(x) = \begin{cases} 0, & x < 0 \\ \cos x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

In the following problems, find the integral representation as mentioned

6. $f(x) = e^{-2x} + e^{-3x}$, $x > 0$; cosine representation.

7. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$; cosine representation.

8. $f(x) = \begin{cases} \sinh x, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$; sine representation.

In the following problems, find the complex Fourier integral of the function and determine what this integral converges to

9. $f(x) = xe^{|x|}$, for all real x.

10. $f(x)=\begin{cases}\sin \pi x, & |x|\le 5\\ 0, & |x|>5\end{cases}$ **11.** $f(x)=\begin{cases}\cos x, & 0\le x\le \pi/2\\ \sin x, & -\pi/2<x<0\\ 0, & |x|>\pi/2\end{cases}$

12. Define a suitable function $f(x)$ and use the Fourier integral representation to show that

$$\int_0^{\infty}\frac{\sin ax}{x}dx=\pi/2,\ (a>0).$$

13. If $\int_0^{\infty} f(x)\sin ax\,dx=\begin{cases}1, & 0<a<1\\ 0, & a>1\end{cases}$, then find $f(x)$.

14. Using the Fourier integral representation, show that

$$\int_0^{\infty}\frac{1-\cos \pi w}{w}\sin(xw)dw=\begin{cases}\pi/2, & 0<x\le\pi\\ 0, & x>\pi\end{cases}$$

15. Show that $\int_0^{\infty}\frac{\sin \pi w\sin wx}{1-w^2}dw=\begin{cases}\frac{1}{2}\pi\sin x, & 0\le x\le\pi\\ 0, & x>\pi\end{cases}$

15.4 FOURIER TRANSFORM AND ITS PROPERTIES

An *integral transform* is a transformation that produces from a given function, a new function which depends on a different variable and appears in the form of an integral. These transformations are mainly employed as a tool to solve certain initial and boundary value ordinary and partial differential equations arising in many areas of science and engineering. The Laplace transform as discussed in Chapter 13, is one such transform which has wide applications. Fourier transforms are the next other integral transforms which are of vital importance from the applications viewpoint in solving initial and boundary value problems.

We will discuss three transforms: *The Fourier transform, the Fourier cosine transform* and *the Fourier sine transform;* the first being complex and the latter two real. These transforms are obtained from the corresponding Fourier integral representations.

15.4.1 The Fourier Transform

The complex Fourier integral representation of function $f(x)$ on real line, refer to Eq. (15.22) is

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u)e^{-iw(u-x)}du\,dw=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)e^{-iwu}du\right]e^{iwx}dw. \quad ...(15.25)$$

The expression in bracket, a function of w denoted by $F(w)$, is called the *Fourier transform of f;* and since u is a dummy variable, we replace u by x and have

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx, \qquad \text{...(15.26)}$$

so that (15.25) becomes $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w) e^{iwx} dw, \qquad \text{...(15.27)}$$

and is called the *inverse Fourier transform of F(w).*

The function $f(x)$ and the associated Fourier transform $F(w)$ are called a *Fourier transform pair.*

Other common notations used for the Fourier transform of $f(x)$ are $\hat{f}(w)$ or, $\mathcal{F}(f(x))$ and the inverse Fourier transform is denoted by $\mathcal{F}^{-1}(f(x))$. Further, the choice of the normalizing factors $1/\sqrt{2\pi}$ in integrals (15.26) and (15.27) is optional and it is chosen here so, to make the two integrals as symmetric as possible. All that is required for the normalizing factors is that their product be $1/2\pi$. In fact we can write the normalizing factors in the general form as $k/2\pi$ and $1/k$, where k is an arbitrary scale factor.

The sufficient conditions for the existence of the Fourier transform are:

1. *f(x) is piecewise continuous on every finite interval; and*
2. *f(x) is absolutely integrable on the real axis.*

Similarly, for the existence of inverse Fourier transform of F(w), F(w) must be absolutely integrable over $(-\infty, \infty)$, *and thus* $\lim_{|w| \to \infty} F(w) = 0$.

Example 15.7: Find the Fourier transforms of

(a) $f(x) = \begin{cases} k, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$ (b) $f(x) = \begin{cases} a, & -l < x < 0 \\ b, & 0 < x < l \quad a, b > 0 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(x) = u(x+1) - u(x-1)$, where $u(x)$ is the unit-step function.

(d) $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$

Solution: (a) By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{a} k e^{-iwx} dx = \frac{k}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{-iw} \right]_0^a = \frac{k}{iw\sqrt{2\pi}} (1 - e^{-iwa}).$$

(b) $$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-l}^{0} a e^{-iwx} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{l} b e^{-iwx} dx$$

$$= \frac{a}{\sqrt{2\pi}}\left[\frac{e^{-iwx}}{-iw}\right]_{-l}^{0} + \frac{b}{\sqrt{2\pi}}\left[\frac{e^{-iwx}}{-iw}\right]_{0}^{l} = \frac{1}{iw\sqrt{2\pi}}\left[(b-a) + ae^{-iwl} - be^{-iwl}\right].$$

(c) The graph of $f(x) = u(x+1) - u(x-1) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

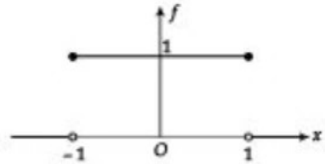


Fig. 15.3

is shown in Fig. 15.3.

By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-iwx}}{-iw}\right)_{-1}^{1}$$

$$= \frac{e^{iw} - e^{-iw}}{\sqrt{2\pi}\, iw} = \sqrt{\frac{2}{\pi}}\,\frac{\sin w}{w}.$$

(d) By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-iwx}}{-iw}\right]_{-a}^{a}$$

$$= \frac{1}{w\sqrt{2\pi}}\left[\frac{e^{iwa} - e^{-iwa}}{i}\right] = \sqrt{\frac{2}{\pi}}\,\frac{\sin wa}{w}.$$

Example 15.8: Find the Fourier transform of $f(x) = e^{-ax^2}, a > 0$.

Solution: By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2} e^{-iwx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(ax^2+iwx)}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\left[-\left(\sqrt{a}x + \frac{iw}{2\sqrt{a}}\right)^2 + \left(\frac{iw}{2\sqrt{a}}\right)^2\right]}dx$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty} e^{-\left[\sqrt{a}x + \frac{iw}{2\sqrt{a}}\right]^2}dx = \frac{1}{\sqrt{2\pi a}}e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty} e^{-t^2}dt, \text{ setting } \sqrt{a}x + \frac{iw}{2\sqrt{a}} = t$$

$$= \frac{1}{\sqrt{2\pi a}}e^{-\frac{w^2}{4a}}.\sqrt{\pi}, \text{ since } \int_{-\infty}^{\infty} e^{-t^2}dt = 2\int_{0}^{\infty} e^{-t^2}dt = \Gamma(1/2) = \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}.$$

Example 15.9: Find the Fourier transform of the following functions

(a) $f(x) = e^{-|x|}$ (b) $\delta(x) = \lim_{k \to 0} \frac{1}{k}[u(x) - u(u-k)]$; $u(x)$ being the unit-step function.

Solution: (a) The function is $f(x) = \begin{cases} e^{x} & -\infty < x \le 0 \\ e^{-x} & 0 < x < \infty \end{cases}$

By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(1-iw)x} dx + \int_{0}^{\infty} e^{-(1+iw)x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[\frac{e^{(1-iw)x}}{(1-iw)} \right]_{-\infty}^{0} - \left[\frac{e^{-(1+iw)x}}{(1+iw)} \right]_{0}^{\infty} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(1-iw)} + \frac{1}{(1+iw)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2}{1+w^2} = \sqrt{\frac{2}{\pi}} \frac{1}{1+w^2}.$$

(b) The function is $u(x) - u(x-k) = \begin{cases} 0, & x < 0 \\ 1, & 0 \le x < k \\ 0, & x \ge k \end{cases}$

Thus $\mathcal{F}(\delta(x)) = \lim_{k \to 0} \left[\frac{1}{k} \mathcal{F}[u(x) - u(x-k)] \right] = \lim_{k \to 0} \left[\frac{1}{k\sqrt{2\pi}} \int_{0}^{k} e^{-iwx} dx \right]$

$$= \frac{1}{\sqrt{2\pi}} \lim_{k \to 0} \left[\frac{1 - e^{-iwk}}{iwk} \right] = \frac{1}{\sqrt{2\pi}}.$$

Remark. A graph of $|F(w)|$ versus w is called the *amplitute spectrum* of $f(x)$.

For example, if $f(x) = u(x+1) - u(x-1)$, then $F(w) = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$, refer to Example 15.7c. The graph of $\left(w, \sqrt{\frac{2}{\pi}} \left| \frac{\sin w}{w} \right| \right)$ is as shown in Fig. 15.4 for $w \ge 0$.

Fig. 15.4

15.4.2 Properties of Fourier Transform

The properties of Fourier transform help to simplify the calculations involving Fourier transform and to obtain some results which are otherwise difficult to obtain.

1. ***Linearity:*** We state the following result.

Theorem 15.3 **(Linearity theorem):** *For any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and for any constants a, b*

$$\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}(f(x)) + b\mathcal{F}(g(x)), \qquad \text{...(15.28)}$$

where $\mathcal{F}(f(x))$ denotes the Fourier transform of $f(x)$.

The proof follows directly from the definition of Fourier transform.

2. ***Fourier transform of derivatives:*** It is stated as follows:

Theorem 15.4 **(Transform of derivatives):** *If $f(x)$ is a continuous function of x with $f(x) \to 0$ as $|x| \to \infty$, and $f'(x)$ is absolutely integrable over $(-\infty, \infty)$, then*

(a) $$\mathcal{F}[f'(x)] = iw\mathcal{F}[f(x)], \qquad \text{...(15.29)}$$

(b) $$\mathcal{F}[f^{(n)}(x)] = (iw)^n \mathcal{F}[f(x)], \qquad \text{...(15.30)}$$

and holds for all n such that the derivatives $f^{(r)}(x)$, $r = 1, 2, \ldots, n$ satisfy the sufficient conditions for the existence of the Fourier transforms.

Proof. (a) By definition, $\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-iwx}dx$. Integrating by parts, we obtain

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}}\left[\left(f(x)e^{-iwx}\right)_{-\infty}^{\infty} - (-iw)\int_{-\infty}^{\infty} f(x)e^{-iwx}dx\right]$$

Since $f(x) \to 0$ as $|x| \to \infty$, therefore, $\mathcal{F}[f'(x)] = (iw)\mathcal{F}[f(x)]$

(b) The repeated application of result (a) gives result (b) provided the desired conditions are satisfied at each step.

Example 15.10: Find the Fourier transform of $f(x) = x e^{-ax^2}$, $a > 0$.

Solution: We have $\mathcal{F}[f(x)] = \mathcal{F}[xe^{-ax^2}] = \mathcal{F}\left[-\frac{1}{2a}(e^{-ax^2})'\right] = -\frac{1}{2a}\mathcal{F}[(e^{-ax^2})']$

$$= -\frac{1}{2a}(iw)\mathcal{F}[e^{-ax^2}], \quad \text{using differentiability}$$

$$= -\frac{iw}{2a}\left(\frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}\right), \quad \text{refer to Example 15.8}$$

$$= \frac{-iw}{2a\sqrt{2a}} e^{-\frac{w^2}{4a}}.$$

Example 15.11: Show that

(a) $\mathcal{F}[x^n f(x)] = i^n \frac{d^n}{dw^n}[F(w)]$ ...(15.31)

(b) $\mathcal{F}[x^m f^{(n)}(x)] = i^{m+n} \frac{d^m}{dw^m}[w^n F(w)]$, ...(15.32)

where $F(w) = \mathcal{F}[f(x)]$ is the Fourier transform of $f(x)$.

Solution: (a) By definition of Fourier transform $F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$.

Differentiating w.r.t. w and using Leibnitz rule to differentiate under integral sign, we have

$$\frac{d}{dw}[F(w)] = \frac{1}{\sqrt{2\pi}} \frac{d}{dw} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{-iwx} dx$$

or,

$$i \frac{d}{dw}[F(w)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{-iwx} dx = \mathcal{F}[xf(x)].$$

The repeated applications of the differentiation w.r.t. w leads to the desired result

$$\mathcal{F}[x^n f(x)] = i^n \frac{d^n}{dw^n}[F(w)].$$

(b) Consider $\mathcal{F}[x^m f^{(n)}(x)] = i^m \frac{d^m}{dw^m} \mathcal{F}[f^{(n)}(x)]$, using (15.31)

$$= i^m \frac{d^m}{dw^m}[(iw)^n F(w)], \quad \text{using (15.30)}$$

$$= i^{m+n} \frac{d^m}{dw^m}[w^n F(w)],$$

provided $f(x)$ and its successive derivatives satisfy the requisite conditions.

Example 15.12: Using the property of the Fourier transform of derivatives, find the Fourier transform of $f(x) = e^{-ax^2}$, $a > 0$.

Solution: Clearly $f(x)$ satisfies the requisite conditions of continuity and absolute integrability over the real axis for the existence of Fourier transform.

It is easy to see that $f(x)$ satisfies the differential equation

$$f'(x) + 2axf(x) = 0. \quad ...(15.33)$$

Taking the Fourier transform of (15.33), we have

$$\mathcal{F}[f'(x)] + 2a\,\mathcal{F}[xf(x)] = 0. \text{ It gives } iwF(w) + 2a\,(i\,F'(w)) = 0$$

or,
$$2a\,F'(w) + wF(w) = 0, \quad ...(15.34)$$

where $F(w)$ is the Fourier transform of $f(x)$. Rewriting (15.34) as $\frac{F'(w)}{F(w)} = -\frac{1}{2a}w$.

Integrating it w.r.t. w, we get $F(w) = A\exp\left[-\frac{w^2}{4a}\right]$, where A is an arbitrary constant.

To determine A we have $F(0) = A$ and also at $w = 0$,

$$F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{\sqrt{a}} = \frac{1}{\sqrt{2a}}.$$

Thus,
$$F(w) = \frac{1}{\sqrt{2a}}e^{-w^2/4a}, \quad a > 0.$$

3. Shifting x by x_0 (the time-shifting); Scaling x by a; and Shifting w by w_0 (the frequency-shifting): The result is stated as follows.

Theorem 15.5 **(Shifting and scaling):** *If $f(x)$ has Fourier transform $F(w)$, then*

(a) $\mathcal{F}[f(x - x_0)] = e^{-iwx_0}F(w)$; *shifting on x-axis by* x_0.

(b) $\mathcal{F}[f(ax)] = \frac{1}{a}F(w/a)$, $a > 0$; *scalling x by a,*

(c) $\mathcal{F}[e^{iw_0x}f(x)] = F(w - w_0)$; *shifting w by* w_0.

The results follows immediately from the definition of the Fourier transform of $f(x)$.

Example 15.13: Find the Fourier transform of $f(x) = e^{-a(x-5)^2}$, $a > 0$, using shifting property.

Solution: By shifting property, we have

$$\mathcal{F}[e^{-a(x-5)^2}] = e^{-i5w}\mathcal{F}[e^{-ax^2}] = e^{-i5w}.\frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}, \quad \text{refer to Example 15.8}$$

$$= \frac{1}{\sqrt{2a}}e^{-\left(\frac{w^2}{4a} + i5w\right)}.$$

Example 15.14: Find the Fourier transform of $f(x) = 4e^{-|x|} - 5e^{-3|x+2|}$.

Solution: Using the linearity property

$$\mathcal{F}[f(x)] = 4\mathcal{F}(e^{-|x|}) - 5\mathcal{F}(e^{-3|x+2|}) = 4\mathcal{F}(e^{-|x|}) - 5e^{2iw}\mathcal{F}(e^{-|3x|}), \text{ using } x\text{-shifting}$$

$$= 4\mathcal{F}(e^{-|x|}) - \frac{5}{3}e^{2iw}\mathcal{F}(e^{-|x|})_{w \to w/3}, \text{ using scaling}$$

$$= 4.\frac{1}{\sqrt{2\pi}}\frac{2}{1+w^2} - \frac{5}{3}e^{2iw}.\frac{1}{\sqrt{2\pi}}\frac{2}{1+\left(\frac{w}{3}\right)^2}, \text{ refer to Example 15.9a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{8}{1+w^2} - \frac{30e^{2iw}}{9+w^2}\right].$$

4. Fourier transform of integrals: It is stated as follows.

Theorem 15.6 **(Transforms of integrals):** *If $\mathcal{F}[f(x)] = F(w)$, then*

$$\mathcal{F}\left[\int_{-\infty}^{x} f(t)\,dt\right] = \frac{1}{iw}F(w), \qquad ...(15.35)$$

provided $F(w)$ satisfies $F(0) = 0$.

Proof. Let $g(x) = \int_{-\infty}^{x} f(t)dt$, then $g'(x) = f(x)$, since $\lim_{x\to-\infty} f(x) = 0$.

Also, $\mathcal{F}[g'(x)] = iw\,\mathcal{F}[g(x)]$. Substituting for $g(x)$ and $g'(x)$, it becomes

$$\mathcal{F}[f(x)] = iw\mathcal{F}\left[\int_{-\infty}^{x} f(t)\,dt\right],$$

which gives (15.35).

Example 15.15: Using the transform of integrals, find the Fourier transform of $f(x) = e^{-ax^2}$.

Solution: We have,

$$\mathcal{F}(e^{-ax^2}) = \mathcal{F}\left[-2a\int_{-\infty}^{\infty}\right] = -2a\mathcal{F}\left[\int_{-\infty}^{x} xe^{-ax^2}\,dx\right]$$

$$= -2a\frac{1}{iw}\mathcal{F}(xe^{-ax^2}) = -2a.\frac{1}{iw}\left(\frac{-iw}{2a\sqrt{2a}}e^{-\frac{w^2}{4a}}\right), \text{ refer to Example 15.10}$$

$$= \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}.$$

5. Fourier transform of convolutions: The *convolution* $f*g$ of two functions f and g is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$$

We have the following result.

Theorem 15.7 (Convolution theorem): *Let $f(x)$ and $g(x)$ be two piecewise continuous, bounded and absolutely integrable functions on the x-axis, then the Fourier transform of $f*g$, the convolution of f and g, is*

$$\mathcal{F}(f*g) = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g). \qquad ...(15.36)$$

Proof. By definition of Fourier transform

$$\mathcal{F}[(f*g)(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{-iwx} \right) dx\, dt$$

Put $x - t = s$, then $x = (t + s)$, this becomes

$$\mathcal{F}[(f*g)(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(s) e^{-iw(t+s)}\, ds\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt} dt \int_{-\infty}^{\infty} g(s) e^{-iws} ds = \sqrt{2\pi}\, \mathcal{F}(f(x))\, \mathcal{F}(g(x)).$$

We observe that result in case of Fourier transform of convolution is the same as that of Laplace transform of convolution except for the factor $\sqrt{2\pi}$.

Taking inverse Fourier transform of (15.36), we obtain

$$(f*g)(x) = \sqrt{2\pi} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(f(x))\, \mathcal{F}(g(x)) e^{iwx} dw$$

or,

$$(f*g)(x) = \int_{-\infty}^{\infty} F(w)\, G(w) e^{iwx} dw, \qquad ...(15.37)$$

where $F(w) = \mathcal{F}(f(x))$ and $G(w) = \mathcal{F}(g(x))$.

The result (15.37) is particularly useful while solving partial differential equations using Fourier transforms.

Example 15.16: Find the inverse Fourier transform of $F(w) = \dfrac{1}{(4+w^2)(9+w^2)}$

Solution: Lel $h(x)$ be the inverse Fourier transform, then

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{(4+w^2)(9+w^2)} e^{jwx}\, dw = \frac{1}{\sqrt{2\pi}} (f * g)(x), \quad \text{using (15.37)}$$

where $f(x) = \mathcal{F}^{-1}\left(\dfrac{1}{4+w^2}\right)$, $g(x) = \mathcal{F}^{-1}\left(\dfrac{1}{9+w^2}\right)$ and $(f*g)$ is the convolution of f and g.

Using Example 15.9a and scaling property, we have

$$f(x) = \mathcal{F}^{-1}\left(\frac{1}{4+w^2}\right) = \frac{1}{4}\sqrt{2\pi}\, e^{-2|x|},$$

and,

$$g(x) = \mathcal{F}^{-1}\left(\frac{1}{9+w^2}\right) = \frac{1}{6}\sqrt{2\pi}\, e^{-3|x|}.$$

Hence,

$$h(x) = \frac{1}{\sqrt{2\pi}}(f * g)(x) = \frac{\sqrt{2\pi}}{24} e^{-2|x|} * e^{-3|x|} = \frac{\sqrt{2\pi}}{24} \int_{-\infty}^{\infty} e^{-2|x-t|}\, e^{-3|t|}\, dt$$

For $x > 0$, $\displaystyle\int_{-\infty}^{\infty} e^{-2|x-t|}\, e^{-3|t|}\, dt = \int_{-\infty}^{0} e^{-2|x-t|} e^{-3|t|}\, dt + \int_{0}^{x} e^{-2|x-t|} e^{-3|t|}\, dt + \int_{x}^{\infty} e^{-2|x-t|} e^{-3|t|}\, dt$

$$= \int_{-\infty}^{0} e^{-2(x-t)} e^{3t}\, dt + \int_{0}^{x} e^{-2(x-t)}\, e^{-3t}\, dt + \int_{x}^{\infty} e^{-2(t-x)} e^{-3t}\, dt = \frac{6e^{-2x}}{5} - \frac{4e^{-3x}}{5}.$$

Similarly for $x < 0$, $\displaystyle\int_{-\infty}^{\infty} e^{-2(x-t)} e^{-3|t|}\, dt = \frac{6e^{2x}}{5} - \frac{4e^{3x}}{5}.$

and, for $x = 0$ $\displaystyle\int_{-\infty}^{\infty} e^{-2|x-t|} e^{-3|t|}\, dt = \frac{2}{5}.$

Therefore,

$$h(x) = \begin{cases} \sqrt{2\pi}\left(\dfrac{1}{20}e^{2x} - \dfrac{1}{30}e^{3x}\right) & x<0 \\ \sqrt{2\pi}/60 & x=0 \\ \sqrt{2\pi}\left(\dfrac{1}{20}e^{-2x} - \dfrac{1}{30}e^{-3x}\right) & x>\infty \end{cases}$$

$$= \sqrt{2\pi}\left(\frac{1}{20}e^{-2|x|} - \frac{1}{30}e^{-3|x|}\right), -\infty < x < \infty.$$

The table below gives some functions $f(x)$ and their Fourier transforms $F(\omega)$.

$f(x)$	$F(\omega) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx$
1. $\begin{cases}1 & \lvert x\rvert < a\\ 0 & \lvert x\rvert > a\end{cases}$, $(a > 0)$, or	$\sqrt{\frac{2}{\pi}}\left(\frac{\sin a\omega}{w}\right)$
2. $\frac{1}{x}$	$\begin{cases}\frac{i}{\sqrt{2\pi}}, & \omega > 0\\ 0 & \omega = 0\\ \frac{-i}{\sqrt{2\pi}}, & \omega < 0\end{cases}$
3. $\begin{cases}1, & a < x < b\\ 0, & \text{otherwise}\end{cases}$ $(0 < a < b)$	$\frac{1}{\sqrt{2\pi}}\left(\frac{e^{-ia\omega} - e^{-ib\omega}}{i\omega}\right)$
4. $\begin{cases}a - \lvert x\rvert, & \lvert x\rvert < a\\ 0, & \lvert x\rvert > a\end{cases}$	$\sqrt{\frac{2}{\sqrt{\pi}}}\left(\frac{1-\cos aw}{\omega^2}\right)$
5. $\frac{\sin ax}{x}, a > 0$	$\begin{cases}\sqrt{\pi/2} & \lvert\omega\rvert < a\\ 0, & \lvert\omega\rvert > a\end{cases}$
6. $\begin{cases}e^{-ax}, & x > 0\\ 0, & x < 0\end{cases}$ $(a > 0)$	$\frac{1}{\sqrt{2\pi}}\frac{1}{a+i\omega}$
7. $\begin{cases}e^{ax}, & b < x > c\\ 0, & \text{otherwise}\end{cases}$ $(a > 0)$	$\frac{1}{\sqrt{2\pi}}\left[\frac{e^{(a-i\omega)c} - e^{(a-i\omega)b}}{a - i\omega}\right]$
8. $e^{-a\lvert x\rvert}, (a > 0)$	$\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+\omega^2}\right)$
9. $xe^{-a\lvert x\rvert}(a > 0)$	$-\sqrt{\frac{2}{\pi}}\frac{2ia\omega}{(a^2+\omega^2)^2}$
10. $\lvert x\rvert e^{-a\lvert x\rvert}, (a > 0)$	$\sqrt{\frac{2}{\pi}}\frac{(a^2-\omega^2)}{(a^2+\omega^2)}$
11. $e^{-ax^2}, (a > 0)$	$\frac{1}{\sqrt{2a}}e^{-\omega^2/4a}$

Contd.

12.	$\frac{1}{a^2+x^2}, (a>0)$	$\frac{1}{a}\sqrt{\frac{\pi}{2}}\, e^{-a\lvert w\rvert}$
13.	$\frac{x}{a^2+x^2}, (a>0)$	$\frac{-i}{2a}\sqrt{\frac{\pi}{2}}\, we^{-a\lvert w\rvert}$
14.	$\begin{cases} e^{-x}x^{a} & x>0 \\ 0, & x\le 0 \end{cases}$	$\frac{\Gamma(a)}{\sqrt{2\pi}(1+iw)^{a}}$
15.	$\delta(x)$	$\frac{1}{\sqrt{2\pi}}$
16.	$J_0(ax), (a>0)$	$\sqrt{\frac{2}{\pi}}\frac{u(a-\lvert w\rvert)}{(a^2-w^2)^{1/2}}$

EXERCISE 15.2

1. Find the Fourier transforms of

(a) $f(x)=\begin{cases}1, & 0<x<a \\ 0, & \text{otherwise}\end{cases}$ (b) $f(x)=\begin{cases}e^{iw}, & 0<x<1 \\ 0, & \text{otherwise}\end{cases}$

(c) $f(x)=\begin{cases}e^{x}, & |x|<a \\ 0, & \text{otherwise}\end{cases}$ (d) $f(x)=\begin{cases}x, & 0<x<a \\ 0, & \text{otherwise}\end{cases}$

2. Find the Fourier transform of $f(x)=\begin{cases}1-x^2, & |x|<1 \\ 0, & |x|>1\end{cases}$

Hence show that $\int_0^\infty \frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}dx=-\frac{3\pi}{16}$

3. Find the Fourier transform of

(a) $f(x)=e^{-x^2/2}$ (b) $f(x)=\frac{\sin ax}{x}, a>0$

(c) $f(x)=\begin{cases}x^{a}e^{-x}, & x>0 \\ 0, & x\le 0\end{cases}$ (d) $f(x)=\begin{cases}x^2, & |x|<x_0 \\ 0, & \text{otherwise}\end{cases}$

4. Show that if $f(x)$ has a finite jump discontinuity at $x=a$, then

$$\mathcal{F}[f'(x)] = iwF(w)-\frac{1}{2\pi}[f(a+)-f(a-)]e^{-iaw}$$

and hence find the Fourier transform of $f'(x)$ when $f(x) = \begin{cases} x, & 0 \le x < a \\ 0, & \text{otherwise.} \end{cases}$

5. Using Fourier transform solve $y'(x) - 4y(x) = u(x)e^{-4x}$, where $u(x)$ is the unit step function.
6. Using the fact that the Bessel function $J_0(x)$ satisfies the differential equation $xf''(x) + f'(x) + xf(x) = 0$, find the Fourier transform of $J_0(x)$, use $\int_0^\infty J_0(x)dx = 1$.
7. Use convolution to find the inverse Fourier transform of the functions

   (a) $\dfrac{1}{(1+iw)^2}$ (b) $\dfrac{\sin 3w}{w(2+iw)}$

8. Using convolution find the Fourier transform of $f(x) = \begin{cases} xe^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$
9. Show that Fourier transform of $f(x) = \begin{cases} 1, & b > x c \\ 0, & \text{otherwise} \end{cases}$ is $F(w) = \dfrac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$.

   Using this result find the inverse Fourier transform of $\dfrac{i}{\sqrt{2\pi}} \dfrac{e^{-ib(a-w)} - e^{-ic(a-w)}}{(a-w)}$.

10. Evaluate $\int_{-\infty}^{\infty} \delta(x-3)\, u(x-3) e^{-5x} dx$, where $\delta(x)$ is the Dirac-Delta function and $u(x)$ is the unit-step function.

15.5 FOURIER COSINE AND FOURIER SINE TRANSFORMS AND THEIR PROPERTIES

The Fourier cosine and sine transforms can be considered as special cases of the Fourier transform of $f(x)$ when $f(x)$ is even or odd function of x over the real axis.

15.5.1 The Fourier Cosine Transform

Consider $f(x)$ to be a piecewise continuous and absolutely integrable function of x over the real axis and so its Fourier transform $F(w)$ exists, refer to 15.26, and is given by

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)[\cos wx - i \sin wx] dx \quad ...(15.38)$$

Further if $f(x)$ is an even function of x, then $f(x)\cos wx$ is even function of x and $f(x)\sin wx$ is odd function of x and so the right side of (15.38) simplifies to

$$F_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx \qquad ...(15.39)$$

is called the *Fourier cosine transform of $f(x)$*, (also denoted by $\mathcal{F}_c$ or $\hat{f}_c$).

The *inverse Fourier cosine transform of $F_c(w)$* corresponding to the inverse Fourier transform (15.27) can be obtained as follows. Consider

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(w) e^{iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(w)[\cos wx + i \sin wx] dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(w) \cos wx \, dx,$$

for we note from (15.39) that $F_c(w)$ and, hence $\mathcal{F}_c(w) \cos wx$ is an even function of w, and $\mathcal{F}_c(w) \sin wx$ is an odd function of w.

The integral denoted by $f(x) = \mathcal{F}_c^{-1}(w)$ and defined as

$$f(x) = \mathcal{F}_c^{-1}(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(w) \cos wx \, dx \qquad ...(15.40)$$

is called *the inverse Fourier cosine transform* of $\mathcal{F}_c(w)$.

15.5.2 The Fourier Sine Transform

Similarly, considering $f(x)$ to be an odd function of x, piecewise continuous and absolutely integrable over the real axis, we arrive at integrals

$$F_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx \, dx \qquad ...(15.41)$$

(also denoted by $\mathcal{F}_s$, or $\hat{f}_s$), defined as the *Fourier sine transform of $f(x)$*, and its inverse

$$f(x) = \mathcal{F}_s^{-1}(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(w) \sin wx \, dw \qquad ...(15.42)$$

defined as *the inverse Fourier sine transform of $F_s(\omega)$*.

The sufficient conditions for the existence of Fourier cosine and sine transforms are:

1. *$f(x)$ is piecewise continuous on each finite interval [0, l]; and*
2. *$f(x)$ is absolutely integrable on the positive real axis.*

Similarly, as in case of inverse Fourier transform, *for the existence of inverse Fourier cosine and sine transforms, $F_c(w)$ and $F_s(w)$ must be absolute integrable over $(0, \infty)$.*

Remarks: 1. Whenever $f(x)$ is discontinuous, then expression on the left of (15.40) and (15.42) is replaced by $[f(x+0)+f(x-0)]/2$ because the Fourier cosine and sine transforms satisfy the same convergence properties as the Fourier transform.

2. We have derived Fourier cosine and sine transforms as special cases of the Fourier transform, when $f(x)$ being even or odd, however, as in case of Fourier cosine and Fourier sine integrals, these two transfoms respectively can be defined when $f(x)$ is given on semi-infinite interval say, $0 < x < \infty$, and is extended to the domain $-\infty < x < \infty$ as even or odd function.

Example 15.17: Find Fourier cosine and sine transforms of $f(x) = \begin{cases} 1, & 0 \le x \le a \\ 0, & x > a \end{cases}$

Solution: By definition

$$F_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos wx \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{\sin wx}{w} \right)_0^a = \sqrt{\frac{2}{\pi}} \frac{\sin aw}{w},$$

$$F_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx = \sqrt{\frac{2}{\pi}} \int_0^a \sin wx \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{-\cos wx}{w} \right)_0^a = \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos aw}{w} \right)$$

15.5.3 Properties of Fourier Cosine and Sine Transforms

Like Fourier transform the Fourier cosine and sine transforms also satisfy certain properties which are useful from applications point of view.

1. Linearity: For any two functions $f(x)$ and $g(x)$ whose Fourier cosine and sine transforms exist and for any constants a and b

$$\hat{f}_c\,[af(x) + bg(x)] = a\,\hat{f}_c\,[f(x)] + b\,\hat{f}_c\,[g(x)] \text{ and, } \hat{f}_s\,[af(x) + bg(x)] = a\,\hat{f}_s\,[f(x)] + b\,\hat{f}_s\,[g(x)]$$

The proofs follow directly from the definition of Fourier cosine and sine transforms.

2. Shifting w by w_0 and scaling x by a: If $F_c(w)$ and $F_s(w)$ are the Fourier cosine and sine transforms of $f(x)$, then

(a) $\hat{f}_c\,[\cos(w_0 x) f(x)] = \frac{1}{2}[F_c(w + w_0) + F_c(w - w_0)]$

(b) $\hat{f}_c\,[\sin(w_0 x) f(x)] = \frac{1}{2}[F_s(w + w_0) - F_s(w - w_0)]$

(c) $\hat{f}_s\,[\cos(w_0 x) f(x)] = \frac{1}{2}[F_s(w + w_0) - F_s(w - w_0)]$

(d) $\hat{f}_s\,[\sin(w_0 x) f(x)] = \frac{1}{2}[F_c(w - w_0) - F_c(w + w_0)]$

(e) $\hat{f}_c[f(ax)] = \frac{1}{a} F_c(w/a),\ a > 0$

(f) $\hat{f}_s[f(ax)] = \frac{1}{a} F_s(w/a),\ a > 0.$

These results follow directly from the definitions of the Fourier cosine and sine transforms. For example, to prove (b) we have

$$\sin w_0 x \cos wx = \frac{1}{2}[\sin (w_0 + w)x + \sin (w_0 - w)x] = \frac{1}{2}[\sin (w + w_0)x - \sin(w - w_0)x].$$

Thus, $\hat{f}_c[\sin (w_0 x) f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(w_0 x)\cos(wx) f(x) dx$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty \sin(w + w_0)x\, f(x)\, dx - \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(w - w_0)x\, f(x)\, dx\right]$$

$$= \frac{1}{2}[F_s(w + w_0) - F_s(w - w_0)].$$

To prove (e), we have $\hat{f}_c[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(ax) \cos wx\, dx = \frac{1}{a}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \frac{w}{a} x\, dx = \frac{1}{a} F_c(w/a)$

3. Fourier cosine and sine transforms of derivatives

Let $f(x)$ and $f'(x)$ be continuous and absolutely integrable on the interval $[0, \infty)$ and $f''(x)$ be piecewise continuous on every subinterval $[0, l]$. Then

(a) $\hat{f}_c[f'(x)] = w F_s(w) - \sqrt{\frac{2}{\pi}} f(0)$ (b) $\hat{f}_s[f'(x)] = -wF_c(w)$

(c) $\hat{f}_c[f''(x)] = -w^2 F_c(w) - \sqrt{\frac{2}{\pi}} f'(0)$ (d) $\hat{f}_s[f''(x)] = -w^2 F_s(w) + w\sqrt{\frac{2}{\pi}} f(0)$

Proof. (a) By definition

$$\hat{f}_c[f'(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx\, dx = \sqrt{\frac{2}{\pi}}\left[[f(x) \cos wx]_0^\infty + w\int_0^\infty f(x) \sin wx\, dx\right]$$

$$= wF_s(w) - \sqrt{\frac{2}{\pi}} f(0), \text{ assuming that } f(x) \to 0 \text{ as } x \to \infty.$$

The result (b) can be proved on the similar lines as in (a).

To prove (c), by definition

$$\hat{f}_c[f''(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\cos wx\, dx = \sqrt{\frac{2}{\pi}}\left[[f'(x)\cos wx + wf(x)\sin wx]_0^\infty + w^2\int_0^\infty f(x)\cos wx\, dx\right]$$

$$= -w^2 F_c(w) - \sqrt{\frac{2}{\pi}}\, f'(0), \text{ assuming } f(x), f'(x) \to 0 \text{ as } x \to \infty.$$

The result (d) can be proved on the similar lines as in (c).

Example 15.18: Find the Fourier cosine and sine transforms of $f(x) = e^{-ax},\ x \geq 0, a > 0$, by using the Fourier cosine and sine transforms of derivatives.

Solution: Here $f(x) = e^{-ax}$, this gives $f'(x) = -ae^{-ax}$ and $f''(x) = a^2 e^{-ax}$.

Thus, $$\hat{f}_c[f''(x)] = \hat{f}_c[a^2 e^{-ax}] = a^2 \hat{f}_c[e^{-ax}] = a^2 F_c(w), \quad ...(15.43)$$

where $F_c(w)$ denotes the Fourier cosine transform of $f(x) = e^{-ax}$.

Also, $$\hat{f}_c[f''(x)] = -w^2 F_c(w) - \sqrt{\frac{2}{\pi}}\, f'(0) = -w^2 F_c(w) + a\sqrt{\frac{2}{\pi}}, \quad ...(15.44)$$

since $f'(0) = -a$

From (15.43) and (15.44), we have

$$a^2 F_c(w) = -w^2 F_c(w) + a\sqrt{\frac{2}{\pi}} \text{ or, } F_c(w) = \sqrt{\frac{2}{\pi}}\,\frac{a}{w^2 + a^2}$$

To find Fourier sine transform, consider

$$\hat{f}_s[f''(x)] = a^2\, \hat{f}_s[e^{-ax}] = a^2 F_s(w), \quad ...(15.45)$$

where $F_s(w)$ denotes the Fourier sine transform of $f(x) = e^{-ax}$.

Also, $$\hat{f}_s[f''(x)] = -w^2 F_s(w) + w\sqrt{\frac{2}{\pi}}\, f(0) = -w^2 F_s(w) + w\sqrt{\frac{2}{\pi}} \quad ...(15.46)$$

From (15.45) and (15.46), we have

$$a^2 F_s(w) = -w^2 F_s(w) + w\sqrt{\frac{2}{\pi}} \text{ or, } F_s(w) = \sqrt{\frac{2}{\pi}}\,\frac{w}{w^2 + a^2}$$

Remark: While solving second order differential equations using integral transforms when the domain is semi-infinite real line, $(0 < x < \infty)$, we need to choose among the Laplace, Fourier cosine and Fourier sine transforms. The Laplace transform will possibly be the best in case of the initial value problems. In case of boundary-value type, the choice will be between Fourier cosine and sine transforms. To use Fourier cosine transform we need to know $f'(0)$; and to use sine transform we require $f(0)$. Thus, the choice may be made accordingly between the two transforms on the basis of the conditions prescribed. We illustrate this in the example to follow next. However, a few specific applications of Fourier transforms to the solutions of partial differential equations are considered in Section 17.15.

Example 15.19: By applying an integral transform, solve the boundary value problem

$$f''(x) - f(x) = 3e^{-2x}, \quad (0 < x < \infty), \quad f(0) = x_0, \quad f(\infty) \text{ bounded} \qquad ...(15.47)$$

Solution: The domain of definition $0 < x < \infty$ is semi-infinite and the problem is boundary-value problem with $f(0) = x_0$; so the clear choice is to apply Fourier sine transform. Applying Fourier sine transform to (15.47) and using the linearity property, we have

$$\hat{f}_s[f''(x)] - \hat{f}_s[f(x)] = 3\hat{f}_s[e^{-2x}]$$

or,

$$-w^2F_s(w) + w\sqrt{\frac{2}{\pi}}f(0) - F_s(w) = 3\sqrt{\frac{2}{\pi}}\frac{w}{w^2+4} \qquad ...(15.48)$$

(refer to Example 15.18 for $a = 2$)

Using $f(0) = x_0$ and solving (15.48) for $F_s(w)$, we have

$$F_s(w) = \sqrt{\frac{2}{\pi}}\frac{wx_0}{w^2+1} - 3\sqrt{\frac{2}{\pi}}\frac{w}{(w^2+4)(w^2+1)} = \sqrt{\frac{2}{\pi}}\left[(x_0-1)\frac{w}{w^2+1} + \frac{w}{w^2+4}\right] \qquad ...(15.49)$$

Taking inverse Fourier sine transform of (15.49) and using the linearity property of the inverse transform we have, refer to Example 15.18 for $a = 1, 2$,

$$f(x) = (x_0 - 1)e^{-x} + e^{-2x}, \qquad ...(15.50)$$

as the solution of (15.47).

Remark: While using the result of $\hat{f}_s[f''(x)]$ to obtain the solution of Eq. (15.47), we have implicitly used that $f(x)$ and $f'(x)$ both tends to zero as x tends to infinity and in fact we can verify that (15.50) satisfies these conditions.

15.6 PARSEVAL IDENTITIES FOR FOURIER TRANSFORMS

The Parseval identities for Fourier transform and Fourier cosine and sine transforms are given by

(a) $\int_{-\infty}^{\infty} f(x)\bar{g}(x)dx = \int_{-\infty}^{\infty} F(w)\bar{G}(w)dw$ (b) $\int_{-\infty}^{\infty} |f(x)|^2dx = \int_{-\infty}^{\infty} |F(w)|^2dw$

(c) $\int_0^{\infty} f(x)\,g(x)dx = \int_0^{\infty} F_c(w)G_c(w)dw$ (d) $\int_0^{\infty} |f(x)|^2dx = \int_0^{\infty} |F_c(w)|^2dw$

(e) $\int_0^{\infty} f(x)g(x)dx = \int_0^{\infty} F_s(w)G_s(w)dw$ (f) $\int_0^{\infty} |f(x)|^2dx = \int_0^{\infty} |F_s(w)|^2dw$,

where $F(w)$, $F_c(w)$ and $F_s(w)$ are respectively the Fourier transform, Fourier sine and Fourier cosine transforms of $f(x)$ respectively and 'bar' denotes the complex conjugate.

We note that Fourier transform is defined for real and complex functions both, thus to prove (a) consider

$$\int_{-\infty}^{\infty} f(x)\ \bar{g}(x)dx = \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{G}(w)\, e^{-iwx}\, dw\right\}dx = \int_{-\infty}^{\infty} \bar{G}(w)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx\right\}dw$$

$$= \int_{-\infty}^{\infty} \bar{G}(w)\, F(w)dw = \int_{-\infty}^{\infty} F(w)\ \bar{G}(w)dw, \text{ which is (a).}$$

Put $g(x) = f(x)$ in (a), we obtain

$$\int_{-\infty}^{\infty} f(x)\ \bar{f}(x)dx = \int_{-\infty}^{\infty} F(w)\ \bar{F}(w)dw, \text{ or, } \int_{-\infty}^{\infty} |f(x)|^2dx = \int_{-\infty}^{\infty} |F(w)|^2dw, \text{ which is (b).}$$

Similarly to prove (c), consider

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} f(x)\left\{\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} G_c(w)\cos wx\, dw\right\}dx = \int_{-\infty}^{\infty} G_c(w)\left\{\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos wx\, dx\right\}dw$$

$$= \int_{0}^{\infty} G_c(w)F_c(w)\,dw = \int_{0}^{\infty} F_c(w)G_c(w)dw, \text{ which is (c).}$$

The result (d) follows from (c). Similarly we can prove (e) and (f).

These results are useful in solving certain improper integrals.

Example 15.20: Find the Fourier cosine transform of $f(x) = xe^{-ax}$, $x > 0$, $a > 0$ and then evaluate

$$\int_{0}^{\infty} \frac{(a^2 - x^2)(b^2 - x^2)}{(a^2 + x^2)^2(b^2 + x^2)^2}dx$$

using the Parseval identity for the cosine transforms.

Solution: By definition of the Fourier cosine transform

$$F_c(w) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos wx\, dx = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} xe^{-ax}\cos wx\, dx = \text{Re}\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} xe^{-ax}\, e^{iwx}dx$$

$$= \text{Re}\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} xe^{-(a-iw)x}dx = \text{Re}\sqrt{\frac{2}{\pi}}\left[\left(\frac{xe^{-(a-iw)x}}{-(a-iw)}\right)_0^{\infty} + \int_{0}^{\infty}\frac{e^{-(a-iw)x}}{(a-iw)}\,dx\right]$$

$$= \text{Re}\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{e^{-(a-iw)x}}{(a-iw)}dx = \text{Re}\sqrt{\frac{2}{\pi}}\left(\frac{e^{-(a-iw)x}}{-(a-iw)^2}\right)_0^{\infty}$$

$$= \text{Re}\sqrt{\frac{2}{\pi}}\frac{1}{(a-iw)^2} = \text{Re}\sqrt{\frac{2}{\pi}}\frac{(a+iw)^2}{(a^2+w^2)^2} = \sqrt{\frac{2}{\pi}}\frac{a^2-w^2}{(a^2+w^2)^2}.$$

Thus, $F_c(w) = \sqrt{\frac{2}{\pi}}\frac{a^2-w^2}{(a^2+w^2)^2}$.

For the Fourier cosine transform the Parseval identity is

$$\int_0^{\infty} F_c(w)G_c(w)\,dw = \int_0^{\infty} f(x)g(x)\,dx \qquad ...(15.51)$$

Set $f(x) = xe^{-ax}$, $a > 0$ and $g(x) = xe^{-bx}$, $b > 0$ and correspondingly

$F_c(w) = \sqrt{\frac{2}{\pi}}\frac{a^2-w^2}{(a^2+w^2)^2}$ and $G_c(w) = \sqrt{\frac{2}{\pi}}\frac{b^2-w^2}{(b^2+w^2)^2}$, (15.51) becomes

$$\frac{2}{\pi}\int_0^{\infty}\frac{(a^2-w^2)(b^2-w^2)}{(a^2+w^2)^2(b^2+w^2)^2}\,dw = \int_0^{\infty} x^2e^{-(a+b)x}\,dx$$

$$= \left[x^2\frac{e^{-(a+b)x}}{-(a+b)} - (2x)\frac{e^{-(a+b)x}}{(a+b)^2} + 2\frac{e^{-(a+b)x}}{-(a+b)^3}\right]_0^{\infty} = \frac{2}{(a+b)^3}$$

or, $$\int_0^{\infty}\frac{(a^2-w^2)(b^2-w^2)}{(a^2+w^2)^2(b^2+w^2)^2}\,dw = \frac{\pi}{(a+b)^3}$$

Changing w to x, we get the desired integral.

Example 15.21: Using Parseval identities show that

(a) $\int_0^{\infty}\frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$ (b) $\int_0^{\infty}\frac{x^2}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a}$

Solution: Consider $f(x) = e^{-ax}$, $a > 0$.

The Fourier cosine and sine transform of $f(x)$, respectively are

$F_c(w) = \sqrt{\frac{2}{\pi}}\frac{a}{w^2+a^2}$ and $F_s(w) = \sqrt{\frac{2}{\pi}}\frac{a}{w^2+a^2}$, (refer to Example 15.18)

To prove (a) consider the Parseval identity for the Fourier cosine transform

$\int_0^{\infty}[F_c(w)]^2 = \int_0^{\infty}[f(x)]^2dx$. Set $f(x) = e^{-ax}$ and $F_c(w) = \sqrt{\frac{2}{\pi}}\frac{a}{w^2+a^2}$, we have

$$\frac{2}{\pi}\int_0^\infty \frac{a^2}{(w^2+a^2)^2}dw = \int_0^\infty e^{-2ax}dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{1}{2a}, \text{ or } \int_0^\infty \frac{1}{(w^2+a^2)^2}dw = \frac{\pi}{4a^3}.$$

Changing w to x we get the desired result.

To prove (b), consider the Parseval's identity for the Fourier sine transform,

$\int_0^\infty [F_s(w)]^2 = \int_0^\infty [f(x)]^2 dx$. Set $f(x) = e^{-ax}$ and $F_s(w) = \sqrt{\frac{2}{\pi}}\frac{w}{(w^2+a^2)}$, we have

$$\frac{2}{\pi}\int_0^\infty \frac{w^2}{(w^2+a^2)^2}dw = \int_0^\infty e^{-2ax}dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{1}{2a}, \text{ or } \int_0^\infty \frac{w^2}{(w^2+a^2)^2} = \frac{\pi}{4a}.$$

Changing w to x, we get the desired result.

Example 15.22: Using the Parseval identity prove that $\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$.

Solution: Consider $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

The Fourier transform of $f(x)$, refer to Example 15.7d for $a = 1$, is $F(w) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin w}{w}\right)$.

The Parseval identity for the Fourier transform is $\int_{-\infty}^{\infty} |F(w)|^2 dw = \int_{-\infty}^{\infty} |f(x)|^2 dx$.

Substituting for $F(w)$ and $f(x)$, it gives

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 w}{w^2}dw = \int_{-1}^{1} dx = 2, \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{\sin^2 w}{w^2}dw = \pi$$

Changing w to x, we get, $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}dx = \pi$, or $\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$.

15.7 THE FINITE FOURIER COSINE AND SINE TRANSFORMS

The Fourier cosine and sine transforms defined on $[0, \infty)$ are motivated by the respective integral representations of a function. In many applications we are to deal with problems defined on finite intervals, and hence we define *the finite Fourier cosine and sine transforms* using Fourier cosine and sine series instead of integrals.

Finite Fourier cosine transform: Suppose f is piecewise continuous on $[0, \pi]$, then *the finite Fourier cosine transform of* f denoted by $F_c(n)$ is defined as

$$F_c(n) = \int_0^\pi f(x) \cos nx \, dx \quad \text{...(15.52)}$$

for $n = 0, 1, 2, \ldots$

Also the Fourier cosine series representation of $f(x)$ on the interval $[0, \pi]$, refer to Eq. (14.30)

for $l = \pi$, is $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, where

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)dx = \frac{1}{\pi}F_c(0), \text{ and } a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx \, dx = \frac{2}{\pi}F_c(n). \text{ Thus,}$$

$$f(x) = \frac{1}{\pi}F_c(0) + \frac{2}{\pi}\sum_{n=1}^{\infty} F_c(n) \cos nx \quad \text{...(15.53)}$$

can be interpreted as the *inverse finite Fourier cosine transform.*

Finite Fourier sine transform: Suppose f is piecewise continuous on $[0, \pi]$, then *the finite Fourier sine transform* of $f(x)$ denoted by $F_s(n)$ is defined as

$$F_s(n) = \int_0^\pi f(x) \sin nx \, dx \quad \text{...(15.54)}$$

for $n = 1, 2, \ldots$

Also the Fourier sine representation series of $f(x)$ on the interval $[0, \pi]$, refer to Eq. (14.31)

for $l = \pi$, is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx \, dx = \frac{2}{\pi}F_s(n)$. Thus,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n) \sin nx \quad \text{...(15.55)}$$

can be interpreted as the *inverse finite Fourier sine transform.*

In case the domain of definition for $f(x)$ is $[0, l]$, then (15.52), (15.53), (15.54) and (15.55) respectively become

$$F_c(n) = \int_0^l f(x) \cos \frac{n\pi x}{l} dx, \quad \text{...(15.56)}$$

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{l}, \quad \text{...(15.57)}$$

$$F_s(n) = \int_0^l f(x) \sin \frac{n\pi x}{l} dx, \qquad \text{...(15.58)}$$

and,

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{l}. \qquad \text{...(15.59)}$$

Example 15.23: If $f(x) = x^2$, $0 \le x \le 4$, find finite Fourier cosine and sine transform of $f(x)$.

Solution: By definition

$$F_c(n) = \int_0^4 f(x) \cos \frac{n\pi x}{l} dx = \int_0^4 x^2 \cos \frac{n\pi x}{4} dx$$

$$= \left[\frac{4x^2}{n\pi} \sin \frac{n\pi x}{4}\right]_0^4 - \int_0^4 \frac{4}{n\pi} 2x \sin \frac{n\pi x}{4} dx = -\frac{8}{n\pi} \int_0^4 x \sin \frac{n\pi x}{4} dx$$

$$= -\frac{8}{n\pi}\left[\frac{-4x}{n\pi} \cos \frac{n\pi x}{4} + \frac{16}{n^2\pi^2} \sin \frac{n\pi x}{4}\right]_0^4 = \frac{128}{n^2\pi^2}(-1)^n.$$

Similarly,

$$F_s(n) = \int_0^l f(x) \sin \frac{n\pi x}{l} dx = \int_0^4 x^2 \sin \frac{n\pi x}{4} dx$$

$$= \left[\frac{-4}{n\pi} x^2 \cos \frac{n\pi x}{4}\right]_0^4 + \int_0^4 2x \frac{4}{n\pi} \cos \frac{n\pi x}{4} dx$$

$$= -\frac{64}{n\pi}(-1)^n + \frac{8}{n\pi}\left[\frac{4x}{n\pi} \sin \frac{n\pi x}{4} + \frac{16}{n^2\pi^2} \cos \frac{n\pi x}{4}\right]_0^4$$

$$= \frac{-(-1)^n 64}{n\pi} + \frac{128}{n^3\pi^3}((-1)^n - 1).$$

Example 15.24: Find $f(x)$, $0 < x < \pi$, if its finite Fourier sine transform is

$$F_s(n) = \frac{1 - \cos n\pi}{n^2\pi^2}, n = 1, 2, ...$$

Solution: By definition

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{l} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{n^2\pi^2} \sin nx$$

$$= \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} \sin nx = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin (2n-1)x}{(2n-1)^2}.$$

EXERCISE 15.3

1. Find the Fourier cosine and Fourier sine transforms of
   (a) $f(x) = e^{-x}, x > 0$ (b) $f(x) = xe^{-ax}$
   (c) $f(x) = x^{\alpha-1}, 0 < \alpha < 1$ (d) $h(x) = \int_0^\infty f(x)\, g(x)\, dx$
2. Find the Fourier cosine and Fourier sine transforms of
   (a) $f(x) = \begin{cases} \cos x, & 0 \le x \le a \\ 0, & x > a \end{cases}$ (b) $f(x) = e^{-x} \cos x, x > 0$
3. Explain why the following functions have neither Fourier cosine transform nor Fourier sine transform
   (a) $f(x) = 1$ (b) $f(x) = e^x$
4. Find the Fourier sine transform of $e^{-ax}, a > 0$ and prove that
   $$\int_0^\infty \frac{x \sin \alpha x}{a^2 + x^2} dx = \frac{\pi}{2} e^{-a\alpha}, \quad \alpha > 0.$$
   Hence, obtain the Fourier sine transform of $x/(a^2 + x^2)$.
5. Find the Fourier cosine transform of e^{-ax} and hence evaluate $\int_0^\infty \frac{\cos \alpha x}{x^2 + a^2} dx$
6. Find the finite Fourier cosine and sine transforms of the following functions defined on $[0, \pi]$
   (a) $f(x) = \sin ax, a > 0$ (b) $f(x) = \sinh ax, a > 0$
7. Solve the integral equation $\int_0^\infty f(x) \cos \alpha x\, dx = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

   Hence show that $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \pi/2.$
8. Solve the integral equation $\int_0^\infty f(x) \sin \alpha x\, dx = \begin{cases} 1, & 0 \le \alpha < 1 \\ 2, & 1 \le \alpha < 2 \\ 0, & \alpha \ge 2 \end{cases}$
9. Using Parseval identities for sine and cosine transforms of $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$'

evaluate (a) $\int_0^\infty \frac{(1-\cos x)^2}{x^2} dx$ (b) $\int_0^\infty \frac{\sin^2 x}{x^2} dx.$

10. Solve using a cosine or sine transform the boundary value problem

$$f''(x) - 9f(x) = 50e^{-3x}, \quad 0 < x < \infty, \quad f(0) = 0, \quad f(\infty) \text{ bounded.}$$

ANSWERS

Exercise 15.1 (p. 859)

2. $\frac{100}{\pi}\int_0^\infty \frac{1}{w}[\sin 2w \cos wx + (1 - \cos 2w)\sin wx]dw$ for all x, except at $x = 0$ and $x = 2$; and $f(0) = f(2) = 100$ but Fourier integral converges to the average value 50.

3. $\frac{2b}{aw}\int_0^\infty \frac{\sin wx(\sin wa - wa\cos wa)}{w^2} dw$, for all x except at $x = \pm a$; and $f(a) = b$, $f(-a) = -b$ but Fourier integral converges to the average value $b/2$ and $-b/2$.

4. $\int_0^\infty \frac{\cos\frac{\pi w}{2}\cos wx}{1-w^2} dw$, for all x.

5. $\frac{1}{\pi}\int_0^\infty \frac{w[\sin wx - \sin w(\pi + x)]}{w^2-1} dw$, for all x, except at $x = 0$ and $x = \pi$. At $x = 0$ Fourier integral converges to the average value $1/2$ and at $x = \pi$, converges to $-1/2$.

6. $\frac{2}{\pi}\int_0^\infty \left(\frac{2}{4+w^2} + \frac{3}{9+w^2}\right)\cos wx\, dw$

7. $\frac{-2}{\pi}\int_0^\infty \frac{[1+\cos w\pi]}{w^2-1}\cos wx\, dw$

8. $\frac{2}{\pi}\int_0^\infty \frac{[\sin 3w\cosh 3 - w\cos 3w\sinh 3]}{1+w^2}\sin wx\, dw$, except at $x = 3$. At $x = 3$, converges to $\frac{1}{2}\sin h\, 3$, the average of $f(3+0)$ and $f(3-0)$.

9. $\frac{i}{\pi}\int_{-\infty}^\infty \left[\frac{-2w(1-w^2)^2}{((1-w^2)^2+4w^2)^2} - \frac{8w^3}{((1-w^2)^2+4w^2)^2}\right] e^{-iwx} dw$; converges to $xe^{|x|}$ for all real x.

10. $i\int_{-\infty}^{\infty}\left(\dfrac{\sin 5w}{w^2-\pi^2}\right)e^{iwx}\,dw$; converges to $\sin \pi x$ for $|x| < 5$ and to zero for $|x| \geq 5$.

11. $\dfrac{1}{2\pi}\int_{-\infty}^{\infty}\left[-\dfrac{\cos(\pi w/2)}{w^2-1}+\dfrac{i\sin(\pi w/2)-w}{w^2-1}+\dfrac{1-w\sin(\pi w/2)}{w^2-1}+i\dfrac{w}{w^2-1}\cos(\pi w/2)\right]e^{iwx}\,dw$

converges to $\cos x$, for $0 < x < \pi/2$, to $\sin x$ for $-\pi/2 < x < 0$, to 0 for $|x| > \pi/2$, to $1/2$ at $x = 0$, to $-1/2$ at $x = -\pi/2$ and to 0 at $x = \pi/2$.

13. $f(x) = 2(1 - \cos x)/\pi$, $x > 0$.

Exercise 15.2 (p. 871)

1. (a) $\dfrac{1}{\sqrt{2\pi}}\left(\dfrac{1-e^{iwa}}{iw}\right)$ (b) $\dfrac{i}{\sqrt{2\pi}}\left(\dfrac{1-e^{-i(w-a)}}{a-w}\right)$

(c) $\dfrac{i}{\sqrt{2\pi}}\dfrac{(e^{a-iaw}-e^{-a+iaw})}{(1-iw)}$ (d) $\dfrac{[(1+iaw)e^{-iaw}-1]}{\sqrt{2\pi}\,w^2}$

2. $-\dfrac{4}{\sqrt{2\pi}\,w^3}(w\cos w - \sin w)$

3. (a) $e^{-w^2/2}$ (b) $\sqrt{\pi/2}$, $|w| < a$; 0, otherwise

(c) $\dfrac{\sqrt{a}}{\sqrt{2\pi}(1+iw)^2}$ (d) $\sqrt{\dfrac{2}{\pi}}\dfrac{[(x_0^2w^2-2)\sin x_0w + 2x_0w\cos x_0\,w]}{w^3}$

5. $-\dfrac{1}{8}e^{-4|x|}$.

6. $\sqrt{\dfrac{2}{\pi}}\dfrac{1}{(1-w^2)^{1/2}}$, $0 < w^2 < 1$.

7. (a) $u(x)\cdot xe^{-x}$ (b) $\dfrac{1}{4}[1-e^{-2(x+3)}]u(x+3)-\dfrac{1}{4}[1-e^{-2(x-3)}]u(x-3)$

8. $\dfrac{1}{\sqrt{2\pi}}(1+iw)^2$

10. $4e^{-15}$.

Exercise 15.3 (p. 883)

1. (a) $F_c(w) = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{1+w^2}$, $F_s(w) = \sqrt{\dfrac{2}{\pi}}\dfrac{w}{1+w^2}$

(b) $F_c(w) = \sqrt{\frac{2}{\pi}} \frac{a^2 - w^2}{(a^2 + w^2)^2}$ $F_s(w) = \sqrt{\frac{2}{\pi}} \frac{2aw}{(w^2 + a^2)^2}$

(c) $F_c(w) = \sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{w\alpha} \cos \frac{\alpha\pi}{2}$, $F_s(w) = \sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{w\alpha} \sin \frac{\alpha\pi}{2}$

(d) $H_c(w) = \int_0^\infty F_c(w)\, G_c(w)\, dw$, $H_s(w) = \int_0^\infty F_s(w) G_s(w)\, dw$

2. (a) $F_c(w) = \frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1-w)}{1-w} + \frac{\sin a(1+w)}{1+w} \right]$,

$F_s(w) = \sqrt{\frac{2}{\pi}} \frac{w}{w^2 - 1} - \frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1+w)}{1+w} - \frac{\cos a(1-w)}{1-w} \right]$

(c) $F_c(w) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1+(1+w)^2} + \frac{1}{1+(1-w)^2} \right]$,

$F_s(w) = \frac{1}{\sqrt{2\pi}} \left[\frac{1+w}{1+(1+w^2)} - \frac{1-w}{1+(1-w)^2} \right]$

4. $\sqrt{\frac{2}{\pi}} \frac{w}{w^2 + a^2}$, $\sqrt{\frac{\pi}{2}} e^{-aw}$

5. $\sqrt{\frac{2}{\pi}} \frac{a}{(w^2 + a^2)}$, $\frac{\pi}{2a} e^{-\alpha a}$

6. (a) $c_o = (1 - \cos a\pi)/a$,

$c_n = \frac{1}{2(n-a)} [\cos \{(n-a)\pi\} - 1] - \frac{1}{2(n+a)} [\{(n+a)\pi - 1\}]$

$c_n = 0$, if $n = a$ an integer.

$s_n = \left[\frac{\sin(n-a)\pi}{2(n-a)} - \frac{\sin(n+a)\pi}{2(n+a)} \right]$, $s_n = \pi/2$, if $n = a$, an integer.

7. $f(x) = \frac{2(1 - \cos x)}{\pi x^2}$

8. $f(x) = (2 + 2\cos x - 4\cos 2x)/\pi x$.

9. (a) $\pi/2$ (b) $\pi/2$

10. $f(x) = -\frac{25}{3} xe^{-3x}$

16 Partial Differential Equations

CHAPTER

Modelling of a process that is distributed in space and time generally leads to a partial differential equation (PDE). The issue of existence and uniqueness of solution in case of a PDE is not straight one as like that an ODE. The solution of a PDE involves arbitrary functions and is generally not unique. Some additional conditions are needed to be specified on the boundary of the region where the solution is defined. Further there are no generally applicable methods to solve non-linear PDEs.

16.1 BASIC CONCEPTS

An equation which involves more than one independepent variable and one or more partial derivatives of the dependent variable with respect to them is called a *partial differential equation* (*PDE*). Mathematical models of physical situations involving two or more independent variables often lead to partial differential equations. A few important partial differential equations are:

1. *One-dimensional heat flow equation:* $c^2u_{xx} = u_t$.
2. *Laplace equation in two dimensions:* $u_{xx} + u_{yy} = 0$.
3. *Laplace equation in three dimensions:* $u_{xx} + u_{yy} + u_{zz} = 0$.
4. *One-dimensional wave equation:* $u_{tt} = c^2u_{xx}$.
5. *Two-dimensional wave equation:* $u_{tt} = c^2(u_{xx} + u_{yy})$.

The *order* of a *PDE* is the order of the highest order partial derivative of the dependent variable u that occurs in the equation. For example, all the equations given above are of order two. Next, as in ordinary differential equation (*ODE*) the *degree* of a *PDE* is defined as the power of the highest order derivative, occurring in the equation after the equation has been made free of radicals and fractions in its derivatives. The equations given above are of degree one.

A general first order PDE for the function $u(x, y)$ is of the form

$$F(x, y, u, u_x, u_y) = 0, \quad ...(16.1)$$

where F is an arbitrary function.

In general, a first order PDE for a function $u(x_1, x_2, ..., x_n)$ of the n independent variables $x_1, x_2, ..., x_n$ is of the form

$$F(x_1, x_2, ..., x_n, u, u_{x_1}, u_{x_2}, ..., u_{x_n}) = 0, \qquad ...(16.2)$$

where F is an arbitrary function and $u_{x_i} = \dfrac{\partial u}{\partial x_i}, i = 1, 2, ..., n.$

Similarly, *a general second order PDE for the function $u(x,y)$* is of the form

$$G(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \qquad ...(16.3)$$

where G is an arbitrary function and $u_x = \dfrac{\partial u}{\partial x}$ and $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$, etc.

The *PDE* of first and second order are of special importance because they occur frequently in practical problems. Further, most of the second order equations that occur in physical applications are *linear*, that is, in which *the unknown function $u(x, y)$ and its partial derivatives appear linearly.*

The general second order linear *PDE* in unknown variable $z(x, y)$ with constant coefficients may be expressed as

$$a\frac{\partial^2 z}{\partial x^2} + 2h\frac{\partial^2 z}{\partial x \partial y} + b\frac{\partial^2 z}{\partial y^2} + 2f\frac{\partial z}{\partial x} + 2g\frac{\partial z}{\partial y} + cz = f(x, y). \qquad ...(16.4)$$

Using the notations $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$, $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$, and $t = \dfrac{\partial^2 z}{\partial y^2}$, it becomes

$$ar + 2hs + bt + 2fp + 2gq + cz = f(x, y).$$

Comparing it with the general equation of the conic $ax^2 + 2hxy + by^2 + 2fx + 2gy + c = 0$, Eq. (16.4) is classified as

(i) *parabolic*, if $h^2 - ab = 0$,

(ii) *hyperbolic*, if $h^2 - ab > 0$, and

(iii) *elliptic*, if $h^2 - ab < 0$

over the region under consideration.

For example, the one-dimensional heat equation $u_t = c^2 u_{xx}$ (with $u \to z, t \to y$), is *parabolic*; the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$ (with $u \to z, t \to y$), is *hyperbolic*; while the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$, (with $u \to z$), is *elliptic*. The form of the general solution depends on the type of the equation being solved.

Remark. Since the constant $a, h, b, ...$ may be functions of x and y, thus the discriminant $h^2 - ac$ may be a function of x and y and hence it is also possible that it is zero, positive or negative in different parts of the x, y-plane. Consider the *Tricomi equation* $u_{xx} + xu_{yy} = 0$, which arises in the study of the two-dimensional steady transonic flow past a body such as wing of an aeroplane, here we can check that it is elliptic in the right half plane $x > 0$ and hyperbolic in the left plane $x < 0$. Such an equation is called a *change-of-type equation* with solutions that are qualitatively different in the two half planes.

16.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

In the chapter on *ODE*'s we have seen how ordinary differential equations are formed by the elimination of arbitrary constants. Partial differential equations can be formed by the elimination of arbitrary functions or by the elimination of the arbitrary constants from the relation involving three or more variables.

Elimination of arbitrary function(s): Suppose two arbitrary expressions $u = u(x, y, z)$ and $v = v(x, y, z)$ are connected by the relation

$$F(u, v) = 0, \quad \text{...(16.5)}$$

where F is an arbitrary function which we need to eliminate.

Assuming z to be dependent variable on x and y. Differentiating (16.5) partially with respect to x we obtain

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0. \quad \text{...(16.6)}$$

Similarly, differentiating (16.5) partially with respect to y, we obtain

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0. \quad \text{...(16.7)}$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (16.6) and (16.7), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z} \end{vmatrix}=0,$$

which gives

$$Pp + Qq = R, \quad \text{...(16.8)}$$

a first order partial differential equation linear in p and q, where

$$P=\frac{\partial(u,v)}{\partial(y,z)},\quad Q=\frac{\partial(u,v)}{\partial(z,x)} \text{ and } R=\frac{\partial(u,v)}{\partial(x,y)}$$

are functions of x, y and z.

Thus the elimination of an arbitrary function F given by (16.5) leads to a partial differential equation linear in p and q.

Elimination of arbitrary constants: Consider a relation of the type

$$f(x, y, z, a, b) = 0, \quad \text{...(16.9)}$$

where a and b are two arbitrary constants which we need to eliminate.

Considering z to be dependent on the two independent variables x and y, differentiate (16.9) partially w. r. t. x to obtain

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p=0. \qquad ...(16.10)$$

Similarly differentiating (16.9) w. r. t. y to obtain

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q=0. \qquad ...(16.11)$$

The three equations (16.9) to (16.11) consist of two arbitrary constants a and b, and, in general, it will be possible to eliminate a and b from these equations to obtain a relation of the form

$$g(x, y, z, p, q)=0. \qquad ...(16.12)$$

Thus, the elimination of two arbitrary constants from the relation (16.9) leads to a first order *PDE* (16.12). However, the equation obtained by eliminating the arbitrary constants need not necessarily be linear in p and q, refer to Example 16.2. But the *PDE* obtained by eliminating an arbitrary function F from (16.5) is always linear. Even sometimes it may not be possible to eliminate the arbitrary constants using Eqs. (16.9), (16.10) and (16.11). Then we go for the second order partial derivatives and eliminate the arbitrary constants but, the partial differential equation may not be unique.

Further every *ODE* of the nth order may be regarded as derived from a solution containing n arbitrary constants. It might be supposed that every *PDE* of the nth order was similarly derivable from a solution containing n arbitrary functions. However, this is not true in general. The elimination of n arbitrary functions sometimes leads to a *PDE* of still higher order and that too may not be unique.

Example 16.1: Derive a *PDE* by eliminating the arbitrary constants a and c from the equation

$$x^2+y^2+(z-c)^2=a^2 \qquad ...(16.13)$$

which represents the set of all spheres with center along the z-axis.

Solution: Differentiating Eq. (16.13) partially with respect to x and y to obtain respectively

$$x+p(z-c)=0, \text{ and } y+q(z-c)=0.$$

Eliminating c from these two equations we obtain $yp - xq = 0$, a partial differential equation linear in p and q.

Example 16.2: Eliminate the arbitrary constants a and b from the equation

$$(x-a)^2+(y-b)^2+z^2=1 \qquad ...(16.14)$$

which represents a family of spheres with unit radius and center in the *XOY* plane.

Solution: Differentiate Eq. (16.14) partially with respect to x and y to obtain respectively

$$(x-a)+zp=0, \text{ and } (y-b)+zq=0.$$

Substituting for $(x - a)$ and $(y - b)$ from these in (16.14), we obtain $(1 + p^2 + q^2)z^2 = 1$, a non-linear partial differential equation in p and q.

Example 16.3: Eliminate the arbitrary constants a and b from the equation

$$z = ae^{by}\cos bx. \quad ...(16.15)$$

Solution: Differentiating Eq. (16.15) partially with respect to x and y to obtain respectively

$$z_x = -abe^{by}\sin bx \quad ...(16.16)$$

and,
$$z_y = abe^{by}\cos bx. \quad ...(16.17)$$

It is not easy to eliminate a and b from the equations obtained.

Differentiating Eq. (16.16) again w.r.t. x and Eq. (16.17) w.r.t. y to obtain respectively

$$z_{xx} = -ab^2e^{by}\cos bx, \text{ and } \quad z_{yy} = ab^2e^{by}\cos bx, \quad \text{which give}$$

$$z_{xx} + z_{yy} = 0 \quad ...(16.18)$$

a second order partial differential equation.

Remark. We note that Eq. (16.18) obtained above is not necessarily unique. Since, from (16.16) we obtain,

$$z_{xy} = -ab^2e^{by}\sin bx = bz_x$$

and,
$$z_{xx} = -ab^2e^{by}\cos bx = -bz_y.$$

Eliminating b from these equations we obtain, $z_x z_{xx} + z_y z_{xy} = 0$, another partial differential equation.

Example 16.4: Form the *PDE* by eliminating the arbitrary function from the relation

$$f(x^2 + y^2,\ z - xy) = 0. \quad ...(16.19)$$

Solution: Differentiating Eq. (16.19) partially w.r.t. x and y to obtain respectively

$$2xf_u + (p - y)f_v = 0, \text{ and } \quad 2yf_u + (q - x)f_v = 0,$$

where $u = x^2 + y^2, v = z - xy$ and $f_u = \partial f/\partial u$, etc.

Eliminating f_u and f_v from these equations we obtain $\begin{vmatrix} 2x & p-y \\ 2y & q-x \end{vmatrix} = 0,$ or $yp - xq = y^2 - x^2,$

a partial differential equation linear in p and q.

Example 16.5: Form the *PDE* by eliminating the arbitrary functions from

$$z = yf(x) + xg(y). \quad ...(16.20)$$

Solution: Differentiating (16.20) partially with respect to x and y to obtain respectively

$$z_x = yf'(x) + g(y), \text{ and } z_y = f(x) + xg'(y).$$

Differentiating partially again to obtain $z_{xy} = f'(x) + g'(y)$.

Consider $\quad xz_x + yz_y = xy[f'(x) + g'(y)] + [xg(y) + yf(x)]$

or, $\quad xz_x + yz_y = xyz_{xy} + z$, a partial differential equation of second order.

Example 16.6: Form the *PDE* by eliminating the arbitrary functions from

$$z = \frac{1}{y}[f(y - ax) + F(y + ax)]. \quad ...(16.21)$$

Solution: Rewriting (16.21) as $zy = f(y - ax) + F(y + ax)$.

Differentiating it partially with respect to x and y to obtain respectively

$$yz_x = -af' + aF', \text{ and } z + yz_y = f' + F'.$$

Differentiating these again to obtain

$$yz_{xx} = a^2f'' + a^2F'' \text{ and } z_y + z_y + yz_{yy} = f'' + F''.$$

Eliminating f'' and F'' from these two we obtain $a^2(2z_y + yz_{yy}) = yz_{xx}$, a second order partial differential equation.

EXERCISE 16.1

Form the *PDE* by eliminating the arbitrary constants from

1. $z = axy + b$
2. $z = (x + a)(y + b)$
3. $ax^2 + by^2 + z^2 = 1$
4. $z = (x - a)^2 + (y - b)^2$
5. $2z = (ax + y)^2 + b$
6. $z = ae^{-p^2t} \cos qx \sin ry, \quad p^2 + q^2 = r^2$
7. Find the partial differential equation of all planes which are at a constant distance d from the origin.
8. Find the partial differential equation by eliminating the arbitrary constants from the equation $x^2 + y^2 = (z - c)^2 \tan^2 \alpha$ which represents the set of all right circular cones whose axes coincide with the z-axis.

   Form the partial differential equation by eliminating the arbitrary function(s) from the following equations, (9 -15)

9. $z = f(x^2 + y^2)$
10. $f(x^2 + y^2, x^2 - z^2) = 0$
11. $xyz = f(x + y + z)$
12. $z = f(x + ct) + g(x - ct)$
13. $z = f(x) + e^y g(x)$
14. $z = f\left(\frac{y}{x}\right) + \phi(xy)$
15. $z = f(x \cos \alpha + y \sin \alpha - at) + F(x \cos \alpha + y \sin \alpha + at)$
16. Verify that $f(x^2 - z^2, x^3 - y^3) = 0$ is a solution of the partial differential equation $z(y^2z_x + x^2z_y) = xy^2$.

16.3 TYPES OF SOLUTION OF A PDE

***A solution** of a PDE in some domain D of the independent variables x and y, is a function that has all partial derivatives appearing in the equation and satisfies the equation everywhere in D.*

In the preceding section we have seen that a relation of the type

$$f(x, y, z, a, b) = 0 \qquad ...(16.22)$$

leads to, in general, a partial differential equation

$$g(x, y, z, p, q) = 0 \qquad ...(16.23)$$

of the first order.

Any such relation of the form (16.22) which contains two arbitrary constants and is a solution of a partial differential equation (16.23), of the first order is said to be a ***complete solution*** *or, a* ***complete integral*** *of that partial differential equation.*

Such a solution is also called *integral surface* of the Eq. (16.23).

Next, a relation of the type

$$F(u, v) = 0 \qquad ...(16.24)$$

involving an arbitrary function F connecting two known functions $u = u(x, y, z)$, $v = v(x, y, z)$ and satisfying a first-order partial differential equation of the type (16.23) is called a ***general integral*** *or,* ***general surface*** *of that partial differential equation.*

It appears that in some sense, a general integral should provide a much broader set of solutions as compared to a complete integral, but, we shall see later that it is possible to find a general integral of a partial differential equation once its complete integral is known.

The solution obtained by determining the arbitrary constants in the complete integral or the arbitrary function in the general integral by using the given conditions, is called a ***particular integral*** *or a* ***particular solution*** *of the partial differential equations.*

The envelope of the family of surfaces $f(x, y, z, a, b) = 0$ with parameters a and b, if it exists, is called a ***singular integral*** *or,* ***singular solution*** *of the partial differential equation.*

Here we would like to recall that the equation of the envelope of the two-parameter family $f(x, y, z, a, b) = 0$ is obtained by eliminating a and b from the equations $f = 0$, $\partial f/\partial a = 0$ and $\partial f/\partial b = 0$.

The singular integral is different from the particular integral in the sense that it cannot be obtained from the complete integral by assigning some particular values to the arbitrary constants a and b.

We must note that every first order partial differential equation does not possess solution. For example, the non-linear equation $p^2 + q^2 = -1$ is not satisfied by any real function $z = z(x, y)$.

16.4 THE LAGRANGE'S EQUATION: LINEAR PDE OF THE FIRST ORDER

The linear first order partial differential equation of the form

$$Pp + Qq = R, \qquad ...(16.25)$$

where P, Q and R are given functions of x, y and z only, is called the Lagrange's equation in two independent variables x and y.

The generalization of Eq. (16.25) to n independent variables is the equation of the form

$$X_1p_1 + X_2p_2 + ... + X_np_n = X, \qquad ...(16.26)$$

where $X_1, X_2, ..., X_n$ and X are given functions of n independent variables $x_1, x_2, ..., x_n$ and a dependent variable z and $p_i = \dfrac{\partial z}{\partial x_i}$, $i = 1, 2, ..., n$.

Remark. It should be clearly observed in this connection that the term '*linear*' means that p and q appear in Eq. (16.25) in the linear form, but P, Q, R may be any functions of x, y and z, which is in contrast to the situation in case of ordinary differential equation where the dependent variable must also appear linearly. For example the equation $xp - yq = z^2$ is linear, while the equation $x\,(dz/dx) = z^2$ is not a linear one.

16.4.1 General Solution of the Lagrange's Equation

We have the following result:

Theorem 16.1: *The general solution of the linear partial differential equation*

$$Pp + Qq = R \qquad ...(16.27)$$

is

$$F(u, v) = 0, \qquad ...(16.28)$$

where F is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two linearly independent solutions of the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \qquad ...(16.29)$$

Proof: Since $u(x, y, z) = c_1$ is a solution of Eqs. (16.29), we have

$$u_x dx + u_y dy + u_z dz = 0, \qquad ...(16.30)$$

and also from (16.29), we have

$$dx = kP, \quad dy = kQ, \quad dz = kR, \qquad ...(16.31)$$

where $k \neq 0$ is an arbitrary constant. Thus, from (16.30) and (16.31), we obtain

$$Pu_x + Qu_y + Ru_z = 0. \qquad ...(16.32)$$

Similarly, since $v(x, y, z) = c_2$ is also a solution of Eqs. (16.29), we obtain

$$Pv_x + Qv_y + Rv_z = 0. \qquad ...(16.33)$$

Solving Eqs. (16.32) and (16.33) simultaneously for P, Q and R, we obtain

$$\frac{P}{\partial(u,v)/\partial(y,z)} = \frac{Q}{\partial(u,v)/\partial(z,x)} = \frac{R}{\partial(u,v)/\partial(x,y)}. \qquad ...(16.34)$$

Also elimination of arbitrary function F from the relation $F(u, v) = 0$ leads to the linear partial differential equation, refer to Section 16.2,

$$p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}. \qquad ...(16.35)$$

Setting each term in (16.34) equal to $\frac{1}{c} \neq 0$, an arbitrary constant, and substituting

$$\frac{\partial(u,v)}{\partial(y,z)} = cP, \quad \frac{\partial(u,v)}{\partial(z,x)} = cQ \quad \text{and} \quad \frac{\partial(u,v)}{\partial(x,y)} = cR$$

in (16.35) and cancelling c from both sides, we obtain $Pp + Qq = R$.

Thus $F(u, v) = 0$, where $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two linearly independent solutions of the auxiliary Eq. (16.29), is the general solution of the Lagrange's Eq. (16.27).

This result can be easily extended to the case of n independent variables stated as follow:

Theorem 16.2: *If $u(x_1, x_2, \ldots, x_n, z) = c_i$, $i = 1, 2, \ldots, n$ are n independent solutions of the auxiliary equations* $\frac{dx_1}{P_1}=\frac{dx_2}{P_2}=\ldots=\frac{dx_n}{P_n}=\frac{dz}{R}$, *then the relation* $\phi(u_1, u_2, \ldots, u_n) = 0$,

where ϕ is an arbitrary function, is a general solution of the linear partial differential equation

$$P_1p_1 + P_2p_2 + \ldots + P_np_n = R, \text{ where } p_i = \frac{\partial z}{\partial x_i}, \quad i = 1, 2, \ldots n.$$

To solve the linear partial differential equation $Pp + Qq = R$, we need to take the following steps:

1. Write the auxiliary equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

2. Find two independent solution of the auxiliary equations. To find these if it is possible to take any pair, say $\frac{dx}{P}=\frac{dy}{Q}$, when the third variable, in this case z, is absent, then we solve it by applying the method of ordinary differential equation and obtain an integral say $u(x, y) = c$. The second integral can be obtained by selecting another pair, if possible. Even we can use the first integral $u(x, y) = c$ while obtaining the second integral.

Sometimes we can find the multipliers l, m, n which are not necessarily constants such that $lP + mQ + nR = 0$, which implies $ldx + mdy + ndz = 0$. This can be integrated if the expression $ldx + mdy + ndz$ is an exact differential of some function. Also we can try for another set of multipliers λ, μ, ν, with this property to find the second integral.

In case $lP + mQ + nR \neq 0$, but $ldx + mdy + ndz = d(lP + mQ + nR)$, then integrating this we can find an integral surface to the auxiliary equations.

3. After obtaining the two independent solutions $u = c_1$ and $v = c_2$ of the auxiliary equations, the general solution of the Lagrange's equation is then of the form $F(u, v) = 0$, or $u = \phi(v)$, or $v = \psi(u)$, where F, ϕ or ψ are arbitrary functions.

Example 16.7: Find the general solution of the partial differential equation

$$x^2p + y^2q = (x + y)z.$$

Solution: The auxiliary equations are

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z}. \qquad \ldots(16.36)$$

From the first pair of terms in (16.36), $\frac{dx}{x^2}=\frac{dy}{y^2}$ we get the integral

$$x^{-1} - y^{-1} = c_1 \text{ or, } \frac{y-x}{xy} = c_1. \qquad \ldots(16.37)$$

Also from Eq. (16.36) we have $\frac{dx-dy}{x^2-y^2}=\frac{dz}{(x+y)z}$, which gives

$$\frac{dx - dy}{x - y} = \frac{dz}{z}. \quad ...(16.38)$$

Integrating (16.38), we obtain the integral

$$\frac{x - y}{z} = c_2. \quad ...(16.39)$$

Therefore, the general solution of the given equation from (16.37) and (16.39) is

$$F\left(\frac{y-x}{xy}, \frac{x-y}{z}\right) = 0,$$

where F is an arbitrary function.

Example 16.8: Find the general solution of the partial differential equation

$$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy.$$

Solution: The auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}. \quad ...(16.40)$$

From (16.40), we have

$$\frac{dx - dy}{(x^2 - y^2) + z(x - y)} = \frac{dy - dz}{(y^2 - z^2) + x(y - z)}$$

or,

$$\frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)}. \quad ...(16.41)$$

Cancelling the factor $(x + y + z)$ from the denominator in (16.41) to obtain

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}. \quad ...(16.42)$$

Integrating (16.42), we get the integral

$$\frac{x - y}{y - z} = c_1. \quad ...(16.43)$$

Similarly, from Eq. (16.40), we obtain $\frac{dx - dz}{x - z} = \frac{dy - dx}{y - x}$, which gives the integral

$$\frac{x - z}{y - x} = c_2. \quad ...(16.44)$$

Hence the general solution of the given equation is

$$F\left(\frac{x-y}{y-z}, \frac{x-z}{y-x}\right) = 0, \text{ where } F \text{ is an arbitrary function.}$$

Remark. Another independent solution from the auxiliary equation (16.40) can be obtained as follows.

Each of the equation in (16.40) is equal to $\dfrac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz}$.

Also the equations are equal to $\dfrac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy}$.

Equating the above two expressions and cancelling the common factor $x^2 + y^2 + z^2 - yz - zx - xy$ on both sides in denominator, we obtain

$$\frac{x\,dx + y\,dy + z\,dz}{(x + y + z)} = dx + dy + dz \quad \text{or,} \quad x\,dx + y\,dy + z\,dz = (x + y + z)d(x + y + z).$$

Integrating, we obtain $x^2 + y^2 + z^2 = (x + y + z)^2 + c_2'$

or, $$xy + yz + zx = c_2'. \qquad ...(16.45)$$

Hence the general solution using (16.43) and (16.45) can also be expressed as

$\dfrac{x - y}{y - z} = f(xy + yz + zx)$, where f is an arbitrary function.

Example 16.9: Find the general solution of the partial differential equation

$$p - q = \ln(x + y).$$

Solution: The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\ln(x + y)}. \qquad ...(16.46)$$

Taking the first pair of terms in (16.46), $dx = -dy$, the integral is

$$x + y = c_1. \qquad ...(16.47)$$

Next, consider $dx = \dfrac{dz}{\ln(x + y)}$. Using (16.47), we obtain $(\ln c_1)dx = dz$, which gives the integral

$$z - (\ln c_1)x = c_2, \quad \text{or,} \quad z - x\ln(x + y) = c_2. \qquad ...(16.48)$$

Hence, the general solution of the given equation using (16.47) and (16.48) is

$z = x\ln(x + y) + \phi(x + y)$, where ϕ is an arbitrary function.

Example 16.10: Find the general solution of the partial differential equation

$$px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3).$$

Solution: Rewriting the equation in the standard form, we get

$$x(z - 2y^2)p + y(z - y^2 - 2x^3)q = z(z - y^2 - 2x^3). \qquad ...(16.49)$$

The auxiliary equations are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}. \qquad ...(16.50)$$

From the last pair of terms in (16.50) we obtain $dy/y = dz/z$, which gives the integral

$$y/z = c_1. \qquad ...(16.51)$$

Next, consider the first and the last terms in (16.50), we have

$$\frac{dx}{x(z-2y^2)} = \frac{dz}{z(z-y^2-2x^3)}. \qquad ...(16.52)$$

Using (16.51) in (16.52), we obtain $\dfrac{dx}{x(z-2c_1^2z^2)} = \dfrac{dz}{z(z-c_1^2z^2-2x^3)}$

or,

$$\frac{dx}{x} = \frac{(1-2c_1^2z)dz}{(z-c_1^2z^2-2x^3)}. \qquad ...(16.53)$$

Set $z - c_1^2z^2 = t$, which gives $(1-2c_1^2z)dz = dt$ in (16.53), to obtain

$$\frac{dx}{x} = \frac{dt}{t-2x^3}, \quad \text{or,} \quad \frac{dt}{dx} - \frac{1}{x}t = -2x^2, \qquad ...(16.54)$$

a linear differential equation in t of order one and degree one. The solution of Eq. (16.54) is

$$\frac{t}{x} + x^2 = c_2. \qquad ...(16.55)$$

Substituting for t and then setting $c_1 = \dfrac{y}{z}$, (16.55) gives

$$\frac{z-y^2}{x} + x^2 = c_2. \qquad ...(16.56)$$

Hence the general solution of the given equation is $F\left(\dfrac{y}{z}, \dfrac{z-y^2}{x} + x^2\right) = 0$, where F is an arbitrary function.

Example 16.11: Find the general solution of the partial differential equation

$$p\cos(x+y) + q\sin(x+y) = z.$$

Solution: The auxiliary equations are

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}. \qquad ...(16.57)$$

These imply

$$\frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{dz}{z}. \qquad ...(16.58)$$

Setting $(x+y) = t$ in (16.58), we obtain

$$\frac{dt}{\cos t + \sin t} = \frac{dz}{z}$$

or,
$$\frac{1}{\sqrt{2}} \sec\left(t - \frac{\pi}{4}\right) dt = \frac{dz}{z}. \qquad ...(16.59)$$

Integrating (16.59), $\left|\sec\left(t - \frac{\pi}{4}\right) + \tan\left(t - \frac{\pi}{4}\right)\right| = c_1 z^{\sqrt{2}}$

or, $\tan\left(\frac{t - \frac{\pi}{4}}{2} + \frac{\pi}{4}\right) = c_1 z^{\sqrt{2}}$ or, $\tan\left(\frac{x+y}{2} + \frac{\pi}{8}\right) = c_1 z^{\sqrt{2}}$. ...(16.60)

Again from the first pair of terms in (16.57), we have $\frac{dy}{dx} = \frac{\sin(x+y)}{\cos(x+y)}$.

Setting $(x + y) = t$ in it gives $\frac{dt}{dx} - 1 = \frac{\sin t}{\cos t}$

or,
$$dx = \frac{\cos t}{\cos t + \sin t} dt = \frac{1}{2} \frac{[(\cos t + \sin t) + (\cos t - \sin t)]}{\cos t + \sin t} dt$$

or,
$$2dx = \left[1 + \frac{\cos t - \sin t}{\sin t + \cos t}\right] dt. \qquad ...(16.61)$$

Integrating (16.61) gives $\sin t + \cos t = c_2 e^{2x-t}$. Substituting for $t = x + y$ in it we obtain

$$[\sin(x+y) + \cos(x+y)]e^{y-x} = c_2. \qquad ...(16.62)$$

Hence the general solution of the given equation is $\phi(c_1, c_2) = 0$, where c_1 and c_2 are given by (16.60) and (16.62) respectively.

Example 16.12: Solve the partial differential equation $(y - z)\frac{\partial u}{\partial x} + (z - x)\frac{\partial u}{\partial y} + (x - y)\frac{\partial u}{\partial z} = 0$, where u is a function of the independent variables x, y and z.

Solution: The auxiliary equations are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{du}{0}. \qquad ...(16.63)$$

From Eqs. (16.63), we obtain $du = 0$, $dx + dy + dz = 0$, $x\,dx + y\,dy + z\,dz = 0$.

Integrating we obtain $u = c_1$, $x + y + z = c_2$, $x^2 + y^2 + z^2 = c_3$.

Hence the general solution of the given partial differential equation is

$u = f(x + y + z,\ x^2 + y^2 + z^2)$, where f is an arbitrary function.

16.4.2 Particular Integral Passing Through a Given Curve

So far we have found that the general solution of the linear partial differential equation $Pp + Qq = R$ is $F(u, v) = 0$, where $u\ (x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two linearly independent solutions of the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Now we explore how such a general solution may be used to find a particular integral surface which passes through a curve whose parametric equations are $x = x(t),\ y = y(t),\ z = z(t)$, where t is a parameter.

Since the curve lies on the integral surface so we must have

$$u[x(t), y(t), z(t)] = c_1, \quad v[x(t), y(t), z(t)] = c_2.$$

We eliminate t from these two equations to obtain a relation between c_1 and c_2. Substituting $c_1 = u(x, y, z),\ c_2 = v(x, y, z)$ in the relation obtained we arrive at the requisite particular integral.

Example 16.13: Find the integral surface of the linear partial differential equation $yp + xq + 1 = z$ which passes through the curve $z = x^2 + y + 1,\quad y = 2x$.

Solution: First we find the general solution of the equation $yp + xq + 1 = z$.

The auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z-1}. \qquad \text{...(16.64)}$$

From the first pair of terms in (16.64) we obtain

$$y^2 - x^2 = c_1. \qquad \text{...(16.65)}$$

Also from (16.64), $\frac{dx + dy}{x + y} = \frac{dz}{z-1}$, which gives the integral surface

$$\frac{z-1}{x+y} = c_2. \qquad \text{...(16.66)}$$

The parametric equations of the given curve $z = x^2 + y + 1,\quad y = 2x$ are, $x = t\ y = 2t$, and $z = (t + 1)^2$.

Substituting for x, y and z in (16.65) and (16.66), we obtain $3t^2 = c_1$ and $t + 2 = 3c_2$.

Eliminating t from these we obtain $3(3c_2 - 2)^2 = c_1$

Substituting for c_1 and c_2 respectively from (16.65) and (16.66) we obtain

$$3\left[3\left(\frac{z-1}{x+y}\right) - 2\right]^2 = (y^2 - x^2),$$

as the desired particular integral of the given equation.

Example 16.14: Find the integral surface of the linear partial differential equation $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line $x + y = 0,\ z = 1$.

Solution: First we find the general solution of the partial differential equation

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z.$$

The auxiliary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}. \quad ...(16.67)$$

Each term in (16.67) is equal to $(dx/x + dy/y + dz/z)/0$, which implies

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0. \quad ...(16.68)$$

Integrating (16.68), we obtain

$$xyz = c_1. \quad ...(16.69)$$

Also each term in (16.67) is equal to $\dfrac{(xdx + ydy - dz)}{0}$, which gives

$$xdx + ydy - dz = 0. \quad ...(16.70)$$

Integrating (16.70), we obtain

$$x^2 + y^2 - 2z = c_2. \quad ...(16.71)$$

The parametric equation of the given straight line $x + y = 0, z = 1$ are, $x = t$, $y = -t$ and $z = 1$. Substituting these values in (16.69) and (16.71), we obtain $-t^2 = c_1$ and $2(t^2 - 1) = c_2$.

Eliminating t from these equations, we obtain

$$2c_1 + c_2 + 2 = 0. \quad ...(16.72)$$

Substituting for $c_1 = xyz$ and $c_2 = x^2 + y^2 - 2z$ in (16.72), we obtain

$$x^2 + y^2 + 2xyz - 2z + 2 = 0$$

as the desired integral surface of the given equation.

EXERCISE 16.2

Find the general solution of the following partial differential equations:

1. $zp + yq = x$
2. $(y+z)p + (x+z)q = x + y$
3. $2yzp + zxq = 3xy$
4. $xy^2p + y^3q = (zxy^2 - 4x^3)$
5. $2xzp + 2yzq + x^2 + y^2 = z^2$
6. $px(x+y) = qy(x+y) - (x-y)(2x+2y+z)$
7. $(y+zx)p - (x+yz)q = x^2 - y^2$
8. $p + 3q = 5z + \tan(y - 3x)$
9. $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$
10. $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

11. $x(z-3y^3)p+y(3x^3-z)q=3(y^3-x^3)z$ **12.** $(y^2-x)p+yq=z+xy$

13. $xu_x+yu_y+zu_z=xyz$ **14.** $xu_x+2yu_y+3zu_z+4tu_t=0$

15. Find the integral surface of the partial differential equation

$$(3-2yz)p+x(2z-1)q=2x(y-3)$$

which passes through the circle $z=0, x^2+y^2=4$.

16. Find the integral surface of the partial differential equation

$$2y(z-3)p+(2x-z)q=y(2x-3)$$

which passes through the circle $z=0, x^2+y^2=2x$.

17. Find the integral surface of the partial differential equation

$$(x-y)y^2p+(y-x)x^2q=(x^2+y^2)z$$

which passes through the curve $xz=a^3, y=0$.

18. Find the integral surface of the partial differential equation

$$(2y^2+z)p+(y+2x)q=4xy-z$$

which passes through the straight line $z=1, y=x$.

19. Find the integral surface of the partial differential equation

$$y(x-z)p+(z^2-xz-x^2)q=y(2x-z)$$

which passes through the ellipse $z=0, 2x^2+4y^2=1$.

20. Find the integral surface of the partial differential equation

$$(x-y)p+(y-x-z)q=z$$

which passes through the circle $z=1, x^2+y^2=1$.

16.5 NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER. CHARPIT'S METHOD

The general non-linear partial differential equation of the first order is of the type

$$F(x, y, z, p, q)=0, \qquad \text{...(16.73)}$$

where the function F is not necessarily linear in p and q. Before introducing a general method of solving Eq. (16.73), we will discuss the different forms of the solutions of this equation. The various types of solutions are discusses below.

1. Complete integral, or Complete solution: We have seen in Section 16.2 that elimination of the arbitrary constants a and b from the equation

$$f(x, y, z, a, b)=0 \qquad \text{...(16.74)}$$

results in a partial differential equation of the form (16.73). It can be shown that the converse also holds, that is, any partial differential equation of the form (16.73) has solution of the form (16.74).

Any two parameters system of surfaces of the form (16.74) which satisfies the partial differential equation (16.73) is called a complete integral, or complete solution of this equation.

2. ***General solution, or General integral:*** If we obtain a one parameter family of surfaces

$$f(x, y, z, a, \psi(a)) = 0, \quad \text{...(16.75)}$$

a subsystem of surfaces (16.74) by choosing $b = \psi(a)$ and form its envelope by eliminating the arbitrary constant a from (16.75) and the equation $\partial f/\partial a = 0$, then we obtain a solution of the partial differential equation (16.73). When the function $\psi(a)$ is arbitrary this solution obtained is called the *general solution* or *general integral* of the Eq. (16.73). When a definite function $\psi(a)$ is used then the solution obtained is called a *particular integral* of the Eq. (16.73).

3. ***Singular integral, or Singular solution:*** The envelope of the two-parameter systems of surfaces (16.74) obtained by eliminating a and b from the equations $f(x, y, z, a, b) = 0$, $\partial f/\partial a = 0$, $\partial f/\partial b = 0$, if it exists, is also a solution of Eq. (16.73). It is called the *singular integral*, or *singular solution* of the Eq. (16.73).

Next, we discuss a general method, called *Charpit's method*, to find a complete integral of the partial differential equation (16.73).

16.5.1 Charpit's Method

Consider the first order partial differential equation

$$f(x, y, z, p, q) = 0. \quad \text{...(16.76)}$$

The Charpit's method consists of finding another partial differential equation

$$g(x, y, z, p, q, a) = 0 \quad \text{...(16.77)}$$

which is compatible with the given Eq. (16.76), that is, the Jacobian

$$J = \frac{\partial(f, g)}{\partial(p, q)} = f_q g_p - g_q f_p \neq 0.$$

Thus the Eqs. (16.76) and (16.77) are solvable for p and q as

$$p = p(x, y, z, a) \text{ and } q = q(x, y, z, a),$$

where a is an arbitrary constant.

Since $z = z(x, y)$, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p dx + q dy. \quad \text{...(16.78)}$$

Substituting for $p = p(x, y, z, a)$ and $q = q(x, y, z, a)$ in (16.78) and then integrating the resultant equation, we obtain the solution of Eq. (16.76) consisting of two arbitrary constants a and b.

To determine the Eq. (16.77) we proceed as follows.

Differentiating Eq. (16.76) with respect to x and y to obtain respectively

$$f_x + pf_z + p_x f_p + q_x f_q = 0 \quad \text{...(16.79)}$$

$$f_y + qf_z + p_y f_p + q_y f_q = 0. \quad \text{...(16.80)}$$

Similarly from Eq. (16.77), we obtain

$$g_x + pg_z + p_x g_p + q_x g_q = 0 \quad \text{...(16.81)}$$

$$g_y + qg_z + p_y g_p + q_y g_q = 0. \quad \text{...(16.82)}$$

Eliminating p_x between Eq. (16.79) and (16.81), and q_y between (16.80) and (16.82) we obtain respectively

$$(f_x g_p - g_x f_p) + (f_z g_p - g_z f_p)p + (f_q g_p - g_q f_p)q_x = 0 \qquad ...(16.83)$$

and,
$$(f_y g_q - g_y f_q) + (f_z g_q - g_z f_q)q + (f_p g_q - g_p f_q)p_y = 0. \qquad ...(16.84)$$

Adding (16.83) and (16.84), using

$$q_x = \frac{\partial q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = p_y$$

and, $f_q g_p - g_q f_p \neq 0$ and simplifying, we obtain

$$f_p g_x + f_q g_y + (pf_p + qf_q)g_z - (f_x + pf_z)g_p - (f_y + qf_z)g_q = 0, \qquad ...(16.85)$$

a linear partial differential equation of first order in g with x, y, z, p and q as independent variables.

The corresponding auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}. \qquad ...(16.86)$$

These equations are called *Charpit's equations.*

An integral of these equations, as simple as possible, involving p, or q, or both and an arbitrary constant a, is taken as the required second equation $g(x, y, z, p, q, a) = 0$. We solve $f = 0$ and $g = 0$ for p and q and obtain the complete solution of (16.76) from (16.78), involving two arbitrary constants a and b.

Example 16.15: Find a complete integral of the equation $z^2 = pqxy$.

Solution: The equation is

$$z^2 - pqxy = 0 \qquad ...(16.87)$$

The Charpit's equations are

$$\frac{dx}{-qxy} = \frac{dy}{-pxy} = \frac{dz}{-2pqxy} = \frac{dp}{pqy - 2pz} = \frac{dq}{pqx - 2qz}. \qquad ...(16.88)$$

From (16.88), we have $\dfrac{dx}{qxy} = \dfrac{dz}{2pqxy} = \dfrac{dp}{2pz - pqy}$,

Using (16.87), it becomes

$$\frac{dx}{qxy} = \frac{dz}{2z^2} = \frac{dp}{2pz - pqy}. \qquad ...(16.89)$$

Each term in (16.89) is equal to $\dfrac{(dp/p - dz/z + dx/x)}{0}$, and thus

$$\frac{dp}{p} = \frac{dz}{z} - \frac{dx}{x}. \qquad ...(16.90)$$

Integrating (16.90), we obtain

$$p = \frac{az}{x}, \qquad \text{...(16.91)}$$

where a is an arbitrary constant.

Substituting (16.91) in (16.87), we obtain

$$q = \frac{z}{ay}. \qquad \text{...(16.92)}$$

Thus, $dz = p\,dx + p\,dy$ becomes $dz = \frac{az}{x}dx + \frac{z}{ay}dy$, which gives

$$\frac{dz}{z} = a\frac{dx}{x} + \frac{1}{a}\frac{dy}{y}. \qquad \text{...(16.93)}$$

Integrating (16.93), we obtain $z = bx^a y^{1/a}$, as the complete integral of Eq. (16.87), where a and b are arbitrary constants.

Example 16.16: Find a complete integral of the equation $z = p^2 + qy$. Also find the singular solution if it exists.

Solution: The equation is

$$p^2 + qy - z = 0. \qquad \text{...(16.94)}$$

The Charpit's equations are $\frac{dx}{2p} = \frac{dy}{y} = \frac{dz}{2p^2 + qy} = \frac{dp}{p} = \frac{dq}{0}$.

From the last term of these equations, we obtain $q = a$.

Using it in (16.94), we obtain $p = \sqrt{z - qy} = \sqrt{z - ay}$. Thus $dz = p\,dx + q\,dy$ becomes

$$dz = \sqrt{z - ay}\,dx + a\,dy, \quad \text{or} \quad \frac{dz - a\,dy}{\sqrt{z - ay}} = dx.$$

Integrating it, we obtain $2\sqrt{z - ay} = x + b$, or $z = ay + \frac{1}{4}(x + b)^2$, as the complete integral of (16.94), where a and b are arbitrary constants.

To find singular solution we obtain the envelope of the two parameters family

$$\phi(a, b) = z - ay - \frac{1}{4}(x + b)^2 = 0. \qquad \text{...(16.95)}$$

Here, $\frac{\partial \phi}{\partial a} = 0$, gives $y = 0$, and $\frac{\partial \phi}{\partial b} = 0$, gives $x = -b$.

Substituting these in (16.95), we obtain $z = 0$ as a singular solution for the given partial differential equation.

Example 16.17: Find a complete integral of the equation $p^2x + q^2y = z$.

Solution: The equation is

$$p^2x + q^2y - z = 0. \qquad \text{...(16.96)}$$

The Charpit's equations are

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2(p^2x + q^2y)} = \frac{dp}{p - p^2} = \frac{dq}{q - q^2}. \qquad \text{...(16.97)}$$

From Eqs. (16.97), we have

$$\frac{p^2dx + 2pxdp}{p^2x} = \frac{q^2dy + 2qydy}{q^2y}, \text{ or } \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}, \text{ which gives}$$

$$p^2x = aq^2y, \qquad \text{...(16.98)}$$

where a is an arbitrary constant. Solving Eq. (16.96) and (16.98) for p, q, we have

$$p = \left[\frac{az}{(1+a)x}\right]^{1/2} \text{ and } q = \left[\frac{z}{(1+a)y}\right]^{1/2}. \text{ Thus } dz = p\,dx + q\,dy \text{ becomes}$$

$$dz = \left[\frac{az}{(1+a)x}\right]^{1/2} dx + \left[\frac{z}{(1+a)y}\right]^{1/2} dy \text{ or, } \left(\frac{1+a}{z}\right)^{1/2} dz = \left(\frac{a}{x}\right)^{1/2} dx + \left(\frac{1}{y}\right)^{1/2} dy.$$

Integrating it, we obtain $[(1 + a)z]^{1/2} = (ax)^{1/2} + y^{1/2} + b$ as the complete integral of Eq. (16.96), where a and b are arbitrary constants.

Example 16.18: Find a singular solution, if it exists, of the equation

$$6yz - 6pxy - 3qy^2 + pq = 0. \qquad \text{...(16.99)}$$

Solution: The Charpit's equations are

$$\frac{dx}{-(6xy - q)} = \frac{dy}{-(3y^2 - p)} = \frac{dz}{-[p(6xy - q) + q(3y^2 - p]} = \frac{dp}{-(-6py + 6py)} = \frac{dq}{-(6z - 6px - 6qy + 6qy)} \qquad \text{...(16.100)}$$

The fourth term in (16.100) is $\frac{dp}{0}$, which implies $dp = 0$, that is $p = a$, where a is an arbitrary constant. Using $p = a$ in Eq. (16.99) gives $q = \frac{6y(z - ax)}{3y^2 - a}$. Thus $dz = p\,dx + q\,dy$ becomes

$$dz = a\,dx + \frac{6y(z - ax)}{3y^2 - a} dy, \quad \text{or,} \quad \frac{d(z - ax)}{z - ax} = \frac{6y\,dy}{3y^2 - a}.$$

Integrating it, we obtain $z - ax = b(3y^2 - a)$ as the complete integral of (16.99), where a and b are arbitrary constants.

To find singular solution we are to obtain the envelope of the two parameter family

$$\phi(a, b) = z - ax - b(3y^2 - a) = 0. \quad ...(16.101)$$

Here, $\dfrac{\partial\phi}{\partial a} = 0$, gives $-x + b = 0$, that is, $x = b$

and, $\dfrac{\partial\phi}{\partial b} = 0$, gives $-(3y^2 - a) = 0$, that is, $y^2 = a/3$.

Substituting these in (16.101), we obtain $z = ab = 3xy^2$ as the singular solution of the given equation.

Example 16.19: Find a particular integral of the partial differential equation $ypq + xp^2 = 1$ which passes through the curve $x = 0$, $y = z$.

Solution: The equation is

$$ypq + xp^2 - 1 = 0. \quad ...(16.102)$$

The Charpit's equations are

$$\frac{dx}{yq + 2px} = \frac{dy}{yp} = \frac{dz}{p(yq + 2px) + ypq} = \frac{dp}{-p^2} = \frac{dq}{-pq}. \quad ...(16.103)$$

The last two terms in (16.103) are, $dp/p = dq/q$. Integrating it we obtain

$$p = aq \quad ...(16.104)$$

where a is an arbitrary constant.

Solving (16.102) and (16.104) for p and q, we obtain $p = \dfrac{\sqrt{a}}{\sqrt{y + ax}}$, and $q = \dfrac{1}{\sqrt{a}\sqrt{y + ax}}$.

Thus $dz = p\,dx + q\,dy$ becomes

$$dz = \frac{\sqrt{a}\,dx}{\sqrt{y + ax}} + \frac{dy}{\sqrt{a}\sqrt{y + ax}} \quad \text{or,} \quad \sqrt{a}\,dz = \frac{d(y + ax)}{\sqrt{y + ax}}.$$

Integrating, we obtain

$$\sqrt{a}z + b = 2\sqrt{y + ax}, \quad ...(16.105)$$

as the complete integral of the given Eq. (16.102), where a and b are arbitrary constants.

The parametric equation of the curve $x = 0$, $y = z$ are, $x = 0$, $y = z = t$. Hence (16.105) becomes

$$(\sqrt{a}\,t + b) = 2t^{1/2}, \text{ or } (\sqrt{a}\,t + b)^2 = 4t \text{ or, } at^2 + 2(\sqrt{a}\,b - 2)t + b^2 = 0.$$

The roots of this equation are equal if

$$4(\sqrt{a}\,b - 2)^2 - 4ab^2 = 0 \quad \text{or,} \quad -4\sqrt{a}\,b + 4 = 0, \quad \text{or} \quad \sqrt{a}\,b = 1.$$

Substituting $\sqrt{a} = 1/b$ in (16.105), we get

$$\frac{1}{b}z+b=2\left(y+\frac{1}{b^2}x\right)^{1/2} \text{ or, } \left(\frac{1}{b}z+b\right)^2=4\left(y+\frac{1}{b^2}x\right)$$

or, $$(z+b^2)^2-4(b^2y+x)=0. \quad \text{...(16.106)}$$

The singular solution is obtained by eliminating b from

$$\phi(b)=(z+b^2)^2-4(b^2y+x)=0 \text{ and } \frac{\partial\phi}{\partial b}=(z+b^2)-2y=0, \text{ which gives, } b^2=(2y-z).$$

Substituting $b^2=(2y-z)$ in (16.106) and simplifying, we obtain $y^2-yz+x=0$. as the particular integral of the given equation

EXERCISE 16.3

Find the complete integral of the partial differential equations

1. $z=p^2x+qy$
2. $p^2+q^2=x-y$
3. $(p^2+q^2)y=qz$
4. $z=p^2x+q^2y$
5. $4x^2p^2+9y^2q^2=z^2$
6. $2(z+xp+yq)=yp^2$
7. $px^5-4q^3x^2+6x^2z-2=0$
8. Verify that $(x-a)^2+(y-b)^2+z^2=1$ is the complete integral of the partial differential equation $z^2(p^2+q^2+1)=1$. Find its singular integral, if it exists.
9. Find the singular solution of the following differential equations, if exist
   (i) $px+qy+z=xq^2$ (ii) $z=px+qy+p^2+q^2$
10. Find the particular integral of the differential equation $pq=z$ which passes through the parabola $x=0, y^2=z$.
11. Find the particular integral of the differential equation $px+q^2y=z$ which passes through the curve $x=1, y+z=0$.
12. Find the particular integral of the differential equation $z=p^2-q^2$ which passes through the parabola $4z+x^2=0, y=0$.

16.6 SOME SPECIAL FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

We consider a few special types of first-order partial differential equations which can be solved very easily by Charpit's method.

I. Equations containing p and q only, $f(p, q)=0$: The equation is

$$f(p,q)=0. \quad \text{...(16.107)}$$

Charpit's equations are $\dfrac{dx}{f_p}=\dfrac{dy}{f_q}=\dfrac{dz}{pf_p+qf_q}=\dfrac{dp}{0}=\dfrac{dq}{0}$.

From the last two terms we can choose $p=a$, or $q=a$, the choice depends on the problem given. Say we choose $p=a$, then from (16.107) $f(a,q)=0$, or $q=\phi(a)$, again a constant.

Thus $dz = pdx + qdy$ becomes, $dz = adx + \phi(a)dy$, which gives $z = ax + \phi(a)y + b$, as the complete integral of Eq. (16.107), where a and b are two arbitrary constants.

Example 16.20: Find a complete integral of the partial differential equation $\sqrt{p} + \sqrt{q} = 1$.

Solutions: The equation involves only p and q so choose $p = a$. This gives

$$q = (1 - \sqrt{p}\,)^2 = (1 - \sqrt{a})^2.$$

Hence the complete integral of the given equation is $z = ax + (1 - \sqrt{a})^2 y + b$, where a and b are two arbitrary constants.

II. Equations not containing the independent variable x and y; f(z, p, q) = 0: The equation is

$$f(z, p, q) = 0. \qquad ...(16.108)$$

The Charpit's equations are $\dfrac{dx}{f_p} = \dfrac{dy}{f_q} = \dfrac{dz}{pf_p + qf_q} = \dfrac{dp}{-pf_z} = \dfrac{dq}{-qf_z}$.

The last two terms are, $dp/p = dq/q$, which give

$$p = aq, \qquad ...(16.109)$$

where a is an arbitrary constant.

We solve (16.108) and (16.109) to obtain expressions for p and q, using these in $dz = pdx + qdy$ we find the complete integral.

Example 16.21: Find a complete integral of the partial differential equation

$$p^2 z + q^2 = 1. \qquad ...(16.110)$$

Solution: The Eq. (16.110) is independent of variables x and y thus

$$p = aq, \qquad ...(16.111)$$

where a is an arbitrary constant.

Solving (16.110) and (16.111) simultaneously for p and q, we obtain

$$p = a(1 + a^2 z)^{-1/2} \text{ and } q = (1 + a^2 z)^{-1/2}.$$

Hence $dz = p\,dx + q\,dy$, becomes $(1 + a^2 z)^{1/2} dz = a\,dx + dy$. Integrating, we obtain

$\dfrac{2}{3}(1 + a^2 z)^{3/2} = a^3 x + a^2 y + a^2 b$ as the complete integral, where a and b are two arbitrary constants.

Example 16.22: Find a complete integral of the equation $z^2 = 1 + p^2 + q^2$.

Solution: Equation is of the form $f(z, p, q) = 0$, which is independent of x and y and thus $p = aq$, where a is an arbitrary constant. Substituting this in the given equation, we obtain

$z^2 - 1 = q^2(1 + a^2)$, which gives $q = \left[\dfrac{z^2 - 1}{1 + a^2}\right]^{1/2}$, and hence $p = a\left[\dfrac{z^2 - 1}{1 + a^2}\right]^{1/2}$.

Thus $dz = p\,dx + q\,dy$ becomes $(1+a^2)^{1/2} \dfrac{dz}{(z^2-1)^{1/2}} = a\,dx + dy.$

Integrating we obtain $(1 + a^2)^{1/2} \cosh^{-1} z = ax + y + b$, as a complete integral of the given equation, where a and b are arbitrary constants.

III. Separable form, $f(x, p) = g(y, q)$: The equation is

$$f(x, p) - g(y, q) = 0. \qquad ...(16.112)$$

The Charpit's equations are $\dfrac{dx}{f_p} = \dfrac{dy}{-g_q} = \dfrac{dz}{pf_p - qg_q} = \dfrac{dp}{-f_x} = \dfrac{dq}{g_y}$.

Consider the pair $dx/f_p = -dp/f_x$. It gives $f_x dx + f_p dp = 0$. Integrating we obtain

$$f(x, p) = a, \qquad ...(16.113)$$

where a is an arbitrary constant.

Using (16.113) in (16.112) gives

$$g(y, q) = a. \qquad ...(16.114)$$

We solve for p and q from (16.113) and (16.114) respectively and use in $dz = p\,dx + q\,dy$ to find the complete integral.

Example 16.23: Find a complete integral of the partial differential equation $p^2 + q^2 = x + y$.

Solution: Equation is of separable form $p^2 - x = y - q^2$. Hence p and q are given by $p^2 - x = a$ and $y - q^2 = a$, which give $p = (a + x)^{1/2}$ and $q = (y - a)^{1/2}$; a being an arbitrary constant.

Thus, $dz = p\,dx + q\,dy$ becomes $dz = (a + x)^{1/2}dx + (y - a)^{1/2}dy$, which gives

$z = (a+x)^{3/2} + (y-a)^{3/2} + b$ as the complete integral of the given equation.

IV. Clairaut equation: It is of the form

$$z = px + qy + f(p, q). \qquad ...(16.115)$$

The Charpit's equations are $\dfrac{dx}{x + f_p} = \dfrac{dy}{y + f_q} = \dfrac{dz}{px + qy + pf_p + qf_q} = \dfrac{dp}{0} = \dfrac{dq}{0}$.

From the last two terms we have $p = a$ and $q = b$, where a and b are two arbitrary constants.

Substituting these values in (16.115) we obtain $z = ax + by + f(a, b)$, as the complete integral. The same can be verified by substitution.

Example 16.24: Find a complete integral of the equation $pqz = p^2(xq + 1) + q^2(yp + 1)$.

Solution: Rewriting the given equation as $z = px + qy + \dfrac{p}{q} + \dfrac{q}{p}$, which is of the Clairaut form.

Hence the complete integral is $z = ax + by + \dfrac{a}{b} + \dfrac{b}{a}$, where a and b are two arbitrary constants.

Certain partial differential equation can be reduced to one of the special types of partial differential equations just discussed, after some simple substitution. In the examples to follow we consider a few such equations.

Example 16.25: Find a complete integral of the equation $z^2(p^2x^2+q^2)=1$.

Solution: Equation is

$$z^2\left[\left\{\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1. \qquad ...(16.116)$$

Set $X=\ln x$, it gives $\frac{\partial z}{\partial X}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial X}=x\frac{\partial z}{\partial x}$, thus Eq. (16.116) becomes

$$z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1, \qquad ...(16.117)$$

which is of the type $f(z, p, q)=0$, taking $p=\partial z/\partial X$ and $z=z(X, y)$.

Hence $p=aq$, where a is an arbitrary constant. Using this in (16.117) gives

$$q=\frac{1}{(\sqrt{1+a^2})z} \text{ and } p=\frac{a}{(\sqrt{1+a^2})z}.$$

Thus $dz=p\,dX+q\,dy$, becomes $\sqrt{1+a^2}\;z\,dz=a\,dX+dy$, which gives

$$\sqrt{(1+a^2)}\,z^2=2(aX+y)+b, \quad \text{or,} \quad \sqrt{(1+a^2)}\,z^2=2(a\ln x+y)+b$$

as a complete integral of the given equation.

Example 16.26: Find complete integral of the equation

$$(p^2-q^2)z=x-y. \qquad ...(16.118)$$

Solution: Substitute $u=z^{3/2}$, we have $u_x=\frac{3}{2}\sqrt{z}\,p$ and $u_y=\frac{3}{2}\sqrt{z}\,q$

which give respectively $zp^2=\frac{4}{9}(u_x)^2$ and $zq^2=\frac{4}{9}(u_y)^2$.

Thus the given equation becomes

$$(u_x)^2-(u_y)^2=\frac{9}{4}(x-y) \quad \text{or,} \quad (u_x)^2-\frac{9}{4}x=(u_y)^2-\frac{9}{4}y,$$

which is of separable form.

Hence, $\quad (u_x)^2-\frac{9}{4}x=(u_y)^2-\frac{9}{4}y=a$, where a is an arbitrary constant.

From this we obtain $u_x = \left(a + \frac{9}{4}x\right)^{1/2}$, and $u_y = \left(a + \frac{9}{4}y\right)^{1/2}$.

Hence $du = u_x dx + u_y dy$ becomes $du = \left(a + \frac{9}{4}x\right)^{1/2} dx + \left(a + \frac{9}{4}y\right)^{1/2} dy$.

Integrating we obtain,

$$\frac{8}{27}u = \left(a + \frac{9}{4}x\right)^{3/2} + \left(a + \frac{9}{4}y\right)^{3/2} + b \text{ or, } \frac{8}{27}z^{3/2} = \left(a + \frac{9}{4}x\right)^{3/2} + \left(a + \frac{9}{4}y\right)^{3/2} + b$$

as a complete integral, where a and b are two arbitrary constants.

Example 16.27: Find a complete integral of the differential equation

$$(x - y)(px - qy) = (p - q)^2. \quad ...(16.119)$$

Solution: Apply the transformations $u = x + y$ and $v = xy$.

Thus, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 1$ and $\frac{\partial v}{\partial x} = y$, $\frac{\partial v}{\partial y} = x$.

Hence, $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$, become respectively

$$p = z_u + yz_v \text{ and } q = z_u + xz_v.$$

Substituting for p and q in Eq. (16.119) and simplifying, we obtain

$$z_u = z_v^2 \quad ...(16.120)$$

which is of the form $f(p, q) = 0$.

Taking $z_v = a$ in (16.120) gives $z_u = a^2$, and hence the solution of (16.120) is $z = a^2u + av + b$, and, therefore, a complete integral of the given equation is $z = a^2(x + y) + axy + b$, where a and b are two arbitrary constants.

EXERCISE 16.4

Find complete integrals of the equations

1. $\sqrt{p} + \sqrt{q} = 1$
2. $p + q = pq$
3. $p^2 - 3q^2 = 5$
4. $zpq = p + q$
5. $p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$
6. $2\sqrt{p} + 3\sqrt{q} = 6x + 2y$
7. $(p + q)(z - px - qy) = 1$
8. $p^3 + q^3 = 216z$
9. $z^2(p^2 + q^2) = x^2 + y^2$
10. $2xyz = px^2y + qxy^2 + 4pq$
11. $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1.$

16.7 FINDING SURFACES ORTHOGONAL TO A GIVEN FAMILY OF SURFACES

An important application of the theory of linear partial differential equations of the first order is to find surfaces orthogonal to a given family of surfaces. Let

$$f(x, y, z) = c \qquad ...(16.121)$$

be the given one-parameter family of surfaces to which we find a system of surfaces which cut these surfaces at right angles and let the requisite surface be

$$z = g(x, y). \qquad ...(16.122)$$

The direction ratios of the normals to (16.121) and (16.122) at the point of intersection $(x, y, z,)$ are respectively $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$, and $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$.

Since the surfaces are orthogonal, thus

$$\frac{\partial f}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial z}{\partial y} - \frac{\partial f}{\partial z} = 0 \text{ or } p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}. \qquad ...(16.123)$$

The linear partial differential equation (16.123) is the general equation determining the surfaces orthogonal to the given family (16.121). The requisite surfaces are given by the integral curve of the equations

$$\frac{dx}{\partial f/\partial x} = \frac{dy}{\partial f/\partial y} = \frac{dz}{\partial f/\partial z}. \qquad ...(16.124)$$

Example 16.28: Find the surface which intersects the surfaces of the family $z(x + y) = c(3z + 1)$ orthogonally and passes through the curve $(x + y) = 0$ and $z = 0$.

Solution: The given family is

$$f(x, y, z) = \frac{z(x + y)}{3z + 1} = c. \qquad ...(16.125)$$

Thus equations giving the family orthogonal to surfaces (16.125) are

$$\frac{dx}{z/(3z + 1)} = \frac{dy}{z/(3z + 1)} = \frac{dz}{(x + y)/(1 + 3z)^2}$$

or,

$$\frac{dx}{z(3z + 1)} = \frac{dy}{z(3z + 1)} = \frac{dz}{(x + y)}. \qquad ...(16.126)$$

From the first two terms in (16.126), we obtain

$$x - y = a. \qquad ...(16.127)$$

Also, we have $\frac{dx + dy}{2z(3z + 1)} = \frac{dz}{(x + y)}$, which gives $(x + y)d(x + y) = 2z(3z + 1)dz$.

Integrating we have

$$(x+y)^2 = 2z^2(2z+1) + b. \quad ...(16.128)$$

The orthogonal surfaces are given by $b = \phi(a)$, which gives

$$(x+y)^2 - 2z^2(2z+1) = \phi(x-y), \quad ...(16.129)$$

where ϕ is an arbitrary function.

Since, the required surface passes through the curve $x+y=0$, $z=0$, thus from (16.129), we have $\phi = 0$. Therefore the requisite orthogonal surface is $(x+y)^2 = 2z^2(2z+1)$.

EXERCISE 16.5

1. Find the general equation of surfaces orthogonal to the family given by $x(x^2+y^2+z^2) = cy^2$, where c is an arbitrary constant.
2. Find the surface which cuts orthogonally the family of surfaces $(2x+3y)z = c(z+2)$, where c is an arbitrary constant and which passes through the circle $3(x^2+y^2) = 4$, $z = 1$.
3. Find the surface which is orthogonal to the family of surfaces $z = cxy(x^2+y^2)$, where c is an arbitrary constant and which passes through the hyperbola $x^2 - y^2 = a^2$, $z = 0$.
4. Find the surface which is orthogonal to the family of surfaces $x^2+y^2+z^2 = cy$, $c \neq 0$ is arbitrary constant and which passes through the circle $x^2+y^2 = 4$, $z = 1$.

16.8 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

A *homogeneous linear partial differential equation of the nth order with constant coefficients is of the form*

$$(D^n + a_1 D^{n-1}D' + a_2 D^{n-2}D'^2 + \dots + a_n D'^n)z = f(x, y), \quad ...(16.130)$$

where a_i's are constants. It is homogeneous in the sense that all terms contain derivatives of the same order.

Here, $D = \dfrac{\partial}{\partial x}$ and $D' = \dfrac{\partial}{\partial y}$. The equation (16.130) can be written as

$$F(D, D')z = f(x, y). \quad ...(16.131)$$

As in case of ordinary differential equation, the complete solution of the Eq. (16.131) is the sum of the *complementary function*, a general solution of the equation

$$F(D, D')z = 0 \quad ...(16.132)$$

and the *particular integral*, a solution of the Eq. (16.131) not containing any arbitrary constant.

Further, *if z_i, $i = 1, 2, \dots, n$ are n solutions of the Eq. (16.132), then $\sum_{r=1}^{n} c_r z_r$ where c_r's are arbitrary constants, is also a solution of Eq. (16.132).* The proof of this is on the lines as in case of ordinary differential equations.

We will assume the homogeneous differential operator $F(D, D')$ to be reducible, that is, it can be resolved into n linear factors of the form $(D - mD')$.

16.8.1 The Complementary Function

We explain the method by considering a homogeneous equation of order two given by

$$(D^2 + a_1 DD' + a_2 D'^2)z = 0. \quad ...(16.133)$$

however, the results are applicable to equations of higher orders also.

The auxiliary equation is

$$D^2 + a_1 DD' + a_2 D'^2 = 0. \quad ...(16.134)$$

Let its roots be $D/D' = m_1, m_2$.

Case I: When the roots are real and distinct

Then (16.133) can be written as

$$(D - m_1 D')(D - m_2 D')z = 0. \quad ...(16.135)$$

The Eq. (16.135) will be satisfied by the solution of the equation

$$(D - m_2 D')z = 0, \text{ or } p - m_2 q = 0, \quad ...(16.136)$$

which is a linear partial differential equation of order one. The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}$$

with two independent solutions as $y + m_2 x = k_1$, and $z = k_2$, where k_1, k_2 are two arbitrary constants.

Thus a solution of Eq. (16.135) is given by $k_2 = \phi(k_1)$, or $z = \phi(y + m_2 x)$.

Similarly another solution of Eq. (16.135) corresponding to $(D - m_1 D')z = 0$ is given by

$$z = \psi(y + m_1 x).$$

Hence, the complete solution of Eq. (16.135) is

$$z = \psi(y + m_1 x) + \phi(y + m_2 x), \quad ...(16.137)$$

where ψ and ϕ are two arbitrary functions.

In case the auxiliary equation is of degree three with three distinct roots m_1, m_2 and m_3, then the complementary function will be $z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x)$ and so on.

Case II: When the roots are equal

In case the roots are equal, say $m_1 = m_2 = m$, then the Eq. (16.135) becomes

$$(D - mD')^2 z = 0. \quad ...(16.138)$$

Let $u = (D - mD')z$, then (16.138) can be written as $(D - mD')u = 0$, which as in Case I, gives

$$u = \phi(y + mx), \quad ...(16.139)$$

where ϕ is an arbitrary function, and thus

$$(D - mD')z = \phi(y + mx). \quad ...(16.140)$$

Eq. (16.140) gives $p - mq = \phi(y + mx)$.

The corresponding subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{\phi(y+mx)},$$

which give two independent solutions as $y+mx=k_1$ and $z=\phi(k_1)x+k_2$ or, $z-x\phi(y+mx)=k_2$.

Thus the complete solution is $k_2=\psi(k_1)$, that is,

$$z-x\phi(y+mx)=\psi(y+mx) \text{ or, } z=\psi(y+mx)+x\phi(y+mx), \quad ...(16.141)$$

where ψ and ϕ are two arbitrary functions.

In case the auxiliary equation is of degree three and a root m is repeated thrice, then the solution corresponding to this is $z=\phi_1(y+mx)+x\phi_2(y+mx)+x^2\phi_3(y+mx)$, and so on.

Example 16.29: Solve the equation $(D^2+DD'-2D'^2)z=0$.

Solution: The auxiliary equation is $m^2+m-2=0$ with roots $m=1,-2$.

Hence, the general solution is $y=\phi_1(y+x)+\phi_2(y-2x)$, where ϕ_1, ϕ_2 are two arbitrary functions.

Example 16.30: Solve the equation $(D^3-6D^2D'+11DD'^2-6D'^2)z=0$.

Solution: The auxiliary equation is $m^3-6m^2+11m-6=0$, with roots $m=1,2,3$.

Hence, the general solution is $z=\phi_1(y+x)+\phi_2(y+2x)+\phi_3(y+3x)$, where ϕ_1, ϕ_2, ϕ_3 are three arbitrary functions.

Example 16.31: Solve the equation $(D^3-5D^2D'+8DD'^2-4D'^3)z=0$.

Solution: The auxiliary equation is $m^3-5m^2+8m-4=0$, with roots $m=1,2,2$.

Hence, the general solution is $z=\phi_1(y+x)+\phi_2(y+2x)+x\phi_3(y+2x)$, where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Example 16.32: Solve the biharmonic equation $(D^4-2D^2D'^2+D'^4)z=0$.

Solution: The auxiliary equation is $m^4-2m^2+1=0$, or $(m-1)^2(m+1)^2=0$, with roots $m=1,1,-1$ and -1.

Hence the general solution is $z=\phi_1(y+x)+x\phi_2(y+x)+\phi_3(y-x)+x\,\phi_4(y-x)$, where ϕ_i's are arbitrary functions.

16.8.2 The Particular Integral

Consider the equation $F(D,D')\,z=f(x,y)$, the particular integral is given as $z=\dfrac{1}{F(D,D')}f(x,y)$.

We find particular integrals for some specific forms of $f(x,y)$.

Case I: When $f(x,y)=e^{ax+by}$

Since $De^{ax+by}=ae^{ax+by}$ and $D'e^{ax+by}=be^{ax+by}$, thus we can easily find that

$F(D,D')e^{ax+by}=F(a,b)e^{ax+by}$. Hence we write

$$e^{ax+by}=[F(D,D')]^{-1}F(a,b)e^{ax+by}$$

or, $$[F(D, D')]^{-1}e^{ax+by} = \frac{1}{F(a,b)}e^{ax+by}, \quad ...(16.142)$$

provided $F(a, b) \neq 0$. In case $F(a, b) = 0$, we may follow the general method to be discussed as **Case IV**.

Case II: When $f(x) = \sin(ax + by)$ or $\cos(ax + by)$

Since, $$D^2 \sin(ax + by) = -a^2 \sin(ax + by)$$
$$DD' \sin(ax + by) = -ab \sin(ax + by)$$
$$D'^2 \sin(ax + by) = -b^2 \sin(ax + by)$$

and so on. Thus, we can easily find that

$$F(D^2, DD', D'^2) \sin(ax + by) = F(-a^2, -ab, -b^2) \sin(ax + by),$$

Hence, $$\sin(ax + by) = [F(D^2, DD', D'^2)]^{-1}F(-a^2, -ab, -b^2)\sin(ax + by)$$

or, $$[F(D^2, DD', D'^2)]^{-1} \sin(ax + b) = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax + by), \quad ...(16.143)$$

provided $F(-a^2, -ab, -b^2) \neq 0$. In case $F(-a^2, -ab, -b^2) = 0$, we may follow the general method to be discussed as **Case IV**.

Case III: When $f(x, y) = x^m y^n$, or a polynomial in x, y.

The particular integral is

$$z = [F(D, D')]^{-1}x^m y^n. \quad ...(16.144)$$

To evaluate it we expand $[F(D, D')]^{-1}$ as an infinite series in ascending powers of D or D', depending upon $m < n$ or $m > n$, and then operate on $x^m y^n$ term by term.

Case IV: The general method

When $f(x, y)$ is any function of x and y, under the assumption that $F(D, D')$ can be factorized linearly, consider $(D - mD')z = f(x, y)$ or, $p - mq = f(x, y)$.

The subsidiary equations are $\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(x, y)}$,

with one solution as $y + mx = c$.

We use this to find the second solution. Consider $dz = f(x, c - mx)dx$.

Integrating, we have $z = \int f(x, c - mx)dx + c_1$, where c is to be replaced by $y + mx$ after integration.

Here c_1 is an arbitrary constant which can be set as zero, since we are finding particular integral. Hence, we may take the particular integral as

$$\frac{1}{(D - mD')}f(x, y) = \int f(x, c - mx)dx, \quad ...(16.145)$$

where c is to be replaced by $y + mx$ after integration.

Example 16.33: Solve the equation $(D^2 + 2DD' + D'^2)z = e^{2x+3y}$.

Solution: The auxiliary equation is $(m^2 + 2m + 1) = 0$, or $(m+1)^2 = 0$, with roots $m = -1, -1$.

The complementary function is

$$z = \phi_1(y - x) + x\phi_2(y - x),$$

where ϕ_1 and ϕ_2 are two arbitrary functions.

The particular integral is

$$z = \frac{1}{(D^2 + 2DD' + D'^2)} e^{2x+3y} = \frac{1}{4 + 2(2)(3) + 9} e^{2x+3y} = \frac{1}{25} e^{2x+3y}.$$

Hence the complete solution is

$$z = \phi_1(y - x) + x\,\phi_2(y - x) + \frac{1}{25} e^{2x+3y}.$$

Example 16.34: Solve $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$.

Solution: The auxiliary equation is $(m^2 - 4m + 4) = 0$, with roots $m = 2, 2$.

The complementary function is $y = \phi_1(y + 2x) + x\phi_2(y + 2x)$, where ϕ_1 and ϕ_2 are two arbitrary functions.

The particular integral is $z = \dfrac{1}{(D - 2D')^2} e^{2x+y}$.

Here $F(D, D') = (D - 2D')^2 = 0$, for $D = 2$ and $D' = 1$. Thus, the usual procedure fails. We adopt the general method. The equation is

$$(D - 2D')(D - 2D')z = e^{2x+y}. \quad \text{...(16.146)}$$

Set $u = (D - 2D')z$, Eq. (16.146) gives

$$u = \frac{1}{D - 2D'} e^{2x+y} = \int e^{2x+(c-2x)}\,dx, \text{ replacing } y \text{ by } c - 2x, \text{ since } m = 2.$$

$$= xe^c = xe^{2x+y}.$$

Thus $(D - 2D')z = u$, gives

$$z = \frac{1}{D - 2D'} \cdot xe^{2x+y} = \int xe^{2x+(c-2x)}dx = \frac{1}{2}x^2 e^c = \frac{1}{2}x^2 e^{2x+y}.$$

Hence the complete solution is $z = \phi_1(y + 2x) + x\phi_2(y + 2x) + \dfrac{1}{2}x^2 e^{2x+y}$.

Example 16.35: Solve $(D^3 - 4D^2D' + 4DD'^2)z = \cos(2x + 3y)$.

Solution: The auxiliary equation is $m^3 - 4m^2 + 4m = 0$, with roots $m = 0, 2, 2$.

The complementary function is $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$, where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

The particular integral is

$$z = \frac{1}{D^3 - 4D^2D' + 4DD'^2}\cos(2x+3y) = \frac{1}{(-4)D - 4(-6)D + 4(-9)D}\cos(2x+3y)$$

$$= -\frac{1}{16D}\cos(2x+3y) = -\frac{1}{32}\sin(2x+3y).$$

Hence, the general solution is $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) - \dfrac{1}{32}\sin(2x+3y)$.

Example 16.36: Solve $(D^2 - 5DD' + 4D'^2)z = \sin(4x+y)$.

Solution: The auxiliary equation is $m^2 - 5m + 4 = 0$, with roots $m = 1, 4$.

The complementary function is $z = \phi_1(y+x) + \phi_2(y+4x)$, where ϕ_1 and ϕ_2 are two arbitrary functions.

The particular integral is $z = \dfrac{1}{D^2 - 5DD' + 4D'^2}\sin(4x+y)$.

Here $F(D, D') = 0$, for $D^2 = -16$, $DD' = -4$, $D'^2 = -1$. Thus the usual method fails. We adopt the general method. The equation is

$$(D - D')(D - 4D')z = \sin(4x+y) \qquad ...(16.147)$$

Set $u = (D - 4D')z$, the Eq. (16.147) becomes $(D - D')u = \sin(4x+y)$, which gives

$$u = \frac{1}{D - D'}\sin(4x+y) = \int \sin(3x+c)\,dx, \text{ replacing } y \text{ by } c - x, \text{ since } m = 1$$

$$= -\frac{1}{3}\cos(3x+c) = -\frac{1}{3}\cos(4x+y).$$

Thus $(D - 4D')z = u$ gives $z = \dfrac{1}{D - 4D'}\left[-\dfrac{1}{3}\cos(4x+y)\right]$

$$= -\frac{1}{3}\int \cos(4x + c - 4x)\,dx, \text{ replacing } y \text{ by } c - 4x, \text{ since } m = 4$$

$$= -\frac{1}{3}\int \cos c\,dx = -\frac{1}{3}x\cos c = -\frac{1}{3}x\cos(4x+y).$$

Hence the complete solution is $z = \phi_1(y+x) + \phi_2(y+4x) - \dfrac{1}{3}x\cos(4x+y)$.

Example 16.37: Solve $(D^2 + DD' - 6D'^2)z = y\cos x$.

Solution: The auxiliary equation is $m^2 + m - 6 = 0$, with roots $m = 2, -3$.

The complementary function is $z = \phi_1(y+2x) + \phi_2(y-3x)$, where ϕ_1 and ϕ_2 are two arbitrary functions.

The particular integral is

$$z = \frac{1}{(D-2D')(D+3D')} y\cos x = \frac{1}{(D-2D')}\frac{1}{D}\left[1+3\frac{D'}{D}\right]^{-1} y\cos x$$

$$= \frac{1}{D-2D'}\frac{1}{D}\left[1-3\frac{D'}{D}+9\frac{D'^2}{D^2}-\ldots\right] y\cos x$$

$$= \frac{1}{D-2D'}\frac{1}{D}[y\cos x - 3\sin x] = \frac{1}{D-2D'}[y\sin x + 3\cos x]$$

$$= \frac{1}{D}\left(1-\frac{2D'}{D}\right)^{-1}(y\sin x + 3\cos x) = \frac{1}{D}\left[1+\frac{2D'}{D}+\frac{4D'^2}{D^2}+\ldots\right](y\sin x + 3\cos x)$$

$$= \frac{1}{D}[y\sin x + 3\cos x - 2\cos x] = -y\cos x + \sin x.$$

Hence the complete solution is $z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x.$

Example 16.38: Solve $(D^2 + 3DD' + 2D'^2)z = x + y.$

Solution: The auxiliary equation is $m^2 + 3m + 2 = 0$, which gives $m = -1, -2.$

The complementary function is $z = \phi_1(y-x) + \phi_2(y-2x)$, where ϕ_1 and ϕ_2 are two arbitrary functions.

The particular integral is

$$z = \frac{1}{(D+D')(D+2D')}(x+y) = \frac{1}{D^2}\left[1+\frac{D'}{D}\right]^{-1}\left[1+2\frac{D'}{D}\right]^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1+\frac{D'}{D}\right]^{-1}\left[1-2\frac{D'}{D}\right](x+y) = \frac{1}{D^2}\left[1+\frac{D'}{D}\right]^{-1}(x+y-2x)$$

$$= \frac{1}{D^2}\left[1-\frac{D'}{D}\right](y-x) = \frac{1}{D^2}[y-x-x] = \frac{1}{D^2}[y-2x] = y\frac{x^2}{2} - \frac{x^3}{3}.$$

Hence the complete solution is $z = \phi_1(y-x) + \phi_2(y-2x) + \frac{1}{2}yx^2 - \frac{1}{3}x^3.$

EXERCISE 16.6

Solve the following homogeneous partial differential equations

1. $(D^3 - 3D^2D' + 2DD'^2)z = 0$
2. $(2D^2 + 5DD' + 2D'^2)z = 0$
3. $(D^3 - 4D^2D' + 4DD'^2)z = 0$
4. $(4D^4 - 4D^3D' - 3D^2D'^2 + 2DD'^3 + D'^4)z = 0$

Find the complete solution of the following partial differential equations

5. $(D^2+5DD'+6D'^2)z=e^{x-y}$
6. $(4D^2+12DD'+9D'^2)z=e^{3x-2y}$
7. $(2D^2+5DD'-3D'^2)z=\sin(2x-y)$
8. $(D^2-D'^2)z=\cos(x+y)$
9. $(D^2-DD'-2D'^2)z=(y-1)e^x$
10. $(D^2-2DD'+D'^2)z=\tan(y+x)$
11. $(D^2+DD'-2D'^2)z=8\ln(x+5y)$
12. $(D^2+2DD'+D'^2)z=2\cos y-x\sin y$
13. $(D^2-2DD'-15D'^2)z=12xy$
14. $(D^2-D'^2)z=\tan^3 x\tan y-\tan x\tan^3 y$
15. $(D^3+D^2D'-DD'^2-D'^3)z=e^x\cos 2y.$

16.9 NON-HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

By a non-homogeneous linear partial differential equation we mean an equation of the form

$$F(D,D')z=f(x,y), \qquad \text{...(16.148)}$$

where the polynomial expression on the right side of (16.148) is not homogeneous one. As in case of homogeneous linear partial differential equations, the complete solution is the sum of the complementary function and the particular integer.

To find complementary function we shall assume that $F(D,D')$ is reducible into linear factors of the form $(D-mD'-a)$.

To solve $(D-mD'-a)z=0$, we write it as

$$p-mq=az.$$

The subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{az}.$$

The integrals are $y+mx=c_1$ and $z=c_2e^{ax}$.

Writing the solution as $c_2=\phi(c_1)$, we get

$$z=e^{ax}\phi(y+mx) \qquad \text{...(16.149)}$$

as a solution .

In case the factor is repeated twice, then we can show that the integral corresponding to, (by substituting $(D-mD'-a)z=u$), $(D-mD'-a)^2z=0$ is

$$z=e^{ax}\phi_1(y+mx)+xe^{ax}\phi_2(y+mx), \qquad \text{...(16.150)}$$

where ϕ_1 and ϕ_2 are two arbitrary functions.

In case the linear factor is $(D-a)$, then the solution (16.149) simplifies to

$$z=e^{ax}\phi(y)$$

and if the linear factor is $(D-mD')$, then (16.149) becomes

$$z=\phi(y+mx)$$

as seen in case of homogeneous equations.

But if the linear factor is of the form $(D' - b)$, then to find the solution consider the differential equation

$$(D' - b)z = 0, \text{ or } q - bz = 0.$$

The auxiliary equations are

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dz}{bz}$$

with solutions $x = k_1, z = k_2 e^{by}$, and hence, the solution of $(D' - b)z = 0$ is

$$z = e^{by}\phi(x). \quad ...(16.151)$$

The complementary function of $F(D, D')z = f(x, y)$ is obtained as the sum of the corresponding solution to each linear factor as obtained above.

The method of obtaining particular integrals of non-homogeneous equations are similar to as in case of homogeneous one discussed in the preceding section and will be best explained in the examples to follow.

Example 16.39: Solve $(D^2 - DD' + D' - 1)z = \cos(x + 2y)$.

Solution: The auxiliary equation is $(D^2 - DD' + D' - 1) = 0$, or $(D - 1)(D - D' + 1) = 0$.

Hence complementary function is $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x)$, where ϕ_1, ϕ_2 are two arbitrary functions.

The particular integral is

$$z = \frac{1}{D^2 - DD' + D' - 1}\cos(x + 2y)$$

$$= \frac{1}{-1 + 2 + D' - 1}\cos(x + 2y), \quad (D^2 = -1,\ DD' = -2)$$

$$= \frac{1}{D'}\cos(x + 2y) = \int \cos(x + 2y)\,dy = \frac{1}{2}\sin(x + 2y).$$

Hence the complete solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + \frac{1}{2}\sin(x + 2y)$.

Example 16.40: Solve $(D - 3D' - 2)^3 z = 6e^{2x}\sin(3x + y)$.

Solution: The auxiliary equation is $(D - 3D' - 2)^3 = 0$.

Hence the complementary function is $z = e^{2x}\phi_1(y + 3x) + xe^{2x}\phi_2(y + 3x) + x^2e^{2x}\phi_3(y + 3x)$, where ϕ_1, ϕ_2 and ϕ_3 are three arbitrary functions.

The particular integral is

$$z = \frac{1}{(D - 3D' - 2)^3}6e^{2x}\sin(3x + y) = 6e^{2x}\frac{1}{[(D + 2) - 3D' - 2]^3}\sin(3x + y)$$

$$= 6e^{2x}\frac{1}{(D-3D')^3}\sin(3x+y) = 6e^{2x}\int\left(\int\left(\int \sin(3x+y)dx\right)dx\right)dx,$$

where y is to be replaced by $y-3x$, since $m=4$.

Thus, $$z = 6e^{2x}\int\left(\int\left(\int \sin c\,dx\right)dx\right)dx = e^{2x}x^3\sin c = x^3e^{2x}\sin(y+3x),$$

taking the arbitrary constants zeros since we are finding particular integral.

Hence, the complete solution is

$$z = e^{2x}\phi_1(y+3x) + xe^{2x}\phi_2(y+3x) + x^2e^{2x}\phi_3(y+3x) + x^3e^{2x}\sin(y+3x).$$

Example 16.41: Solve $(D+D'-1)(D+2D'-3)z = 4+3x+6y$.

Solution: The auxiliary equation is $(D+D'-1)(D+2D'-3)=0$.

Hence the complementary function is $z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-2x)$, where ϕ_1 and ϕ_2 are two arbitrary functions.

The particular integral is $z = \dfrac{1}{(D+D'-1)(D+2D'-3)}(4+3x+6y)$. ...(16.152)

Consider the operator

$$\frac{1}{(D+D'-1)(D+2D'-3)} = \frac{1}{3}[1-(D+D')]^{-1}\left[1-\frac{D+2D'}{3}\right]^{-1}$$

$$= \frac{1}{3}[1+(D+D') + \text{terms of higher orders}]$$

$$\times\left[1+\frac{D+2D'}{3} + \text{terms of higher orders}\right]$$

$$= \frac{1}{3}\left[1+(D+D') + \frac{D+2D'}{3} + \text{terms of higher orders}\right]$$

$$= \frac{1}{3}\left[1+\frac{4D+5D'}{3} + \text{terms of higher orders}\right].$$

Thus, (16.152) becomes $z = \dfrac{1}{3}\left[(4+3x+6y) + \dfrac{1}{3}(4D+5D')(4+3x+6y)\right]$

$$= \frac{1}{3}\left[4+3x+6y+\frac{1}{3}(12+30)\right] = x+2y+6.$$

Hence the complete solution is $z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-2x) + x + 2y + 6$.

Example 16.42: Solve $(D+D'+3)^2(D'-2)z = e^{2x+y}$.

Solution: The auxiliary equation is $(D+D'+3)^2(D'-2)=0$.

Hence the complementary function is $z = e^{-3x}\phi_1(y-x) + xe^{-3x}\phi_2(y-x) + e^{2y}\phi_3(x)$, where ϕ_1, ϕ_2, and ϕ_3 are arbitrary functions.

The particular integral is $z = \dfrac{1}{(D+D'+3)^2(D'-2)}e^{2x+y} = \dfrac{1}{(2+1+3)^2(-1)}e^{2x+y} = -\dfrac{1}{36}e^{2x+y}$.

Hence the complete solution is $z = e^{-3x}\phi_1(y-x) + xe^{-3x}\phi_2(y-x) + e^{2y}\phi_3(x) - \dfrac{1}{36}e^{2x+y}$.

Example 16.43: Solve $(x^2D^2 - 4xyDD' + 4y^2D'^2 + 6yD')z = x^3y^4$.

Solution: The equation can be reduced to partial differential equation with constant coefficients by substitution $x = e^u$, and $y = e^v$, so that $u = \ln x$, and $v = \ln y$.

The dependent variable $z(x, y)$ becomes function of u and v and for simplicity we again denote it by $z(u, v)$. It is easy to verify that

$$yD'z = D_v z,\ x^2D^2z = (D_u - 1)\,D_u z,\ xyDD'z = D_uD_v z \text{ and } y^2D'^2z = (D_v - 1)D_v z,$$

where $D_u = \dfrac{\partial}{\partial u}$ and $D_v = \dfrac{\partial}{\partial v}$. Thus the given equation becomes

$$[(D_u - 1)D_u - 4D_uD_v + 4(D_v - 1)D_v + 6D_v]z = e^{3u+4v}$$

or, $(D_u^2 - 4D_uD_v + 4D_v^2 - D_u + 2D_v)z = e^{3u+4v}$ or, $(D_u - 2D_v)(D_u - 2D_v - 1)z = e^{3u+4v}$.

The auxiliary equation is $(D_u - 2D_v)(D_u - 2D_v - 1) = 0$.

Hence the complementary function is

$$z = \phi_1(v + 2u) + e^u\phi_2(v + 2u) = \phi_1(\ln y + 2\ln x) + x\phi_2(\ln y + 2\ln x) = f_1(yx^2) + xf_2(yx^2),$$

where f_1, f_2 are two arbitrary functions.

The particular integral is

$$z = \frac{1}{(D_u - 2D_v)(D_u - 2D_v - 1)}e^{3u+4v} = \frac{1}{(3-8)(3-8-1)}e^{3u+4v} = \frac{1}{30}e^{3u+4v} = \frac{1}{30}x^3y^4.$$

Hence the complete solution is $z = f_1(yx^2) + xf_2(yx^2) + \dfrac{1}{30}x^3y^4$.

Incase the operator $F(D, D')$ is *irreducible*, then it may not always be possible to find a solution with the required number of arbitrary functions but we can construct solution which contains the required number of arbitrary constants. The method follows from the result

$$F(D, D')e^{ax+by} = F(a, b)e^{ax+by} \qquad \text{...(16.153)}$$

which is valid in case of reducible as well as irreducible $F(D, D')$.

To determine the complementary functions of an equation $F(D, D')z = f(x, y)$, we split $F(D, D')$ into factors. The reducible factors are treated by the method discussed already. To treat an irreducible factor, we observe from (16.153) that e^{ax+by} is a solution of the equation $F(D, D')z = 0$ provided, $F(a, b) = 0$, so that

$$z_k = \sum_k c_k e^{a_k x + b_k y} \qquad \text{... (16.154)}$$

is solution corresponding to the irreducible factors, where a_k, b_k and c_k are constants with $F(a_k, b_k) = 0$ and k takes as many value as are required.

The method is best illustrated in the examples to follow.

Example 16.44: Solve $(D^2 - D')z = e^{2x+y}$

Solution: The equation is irreducible, hence the solution is $z = e^{ax+by}$, provided $a^2 - b = 0$, or $a = \pm\sqrt{b}$. Thus the complementary function is $z = c_1 e^{\sqrt{b}x+by} + c_2 e^{-\sqrt{b}x+by}$, where c_1, c_2 and b are arbitrary constants.

The particular integral is, $z = \dfrac{1}{D^2 - D'} e^{2x+y} = \dfrac{1}{4-1} e^{2x+y} = \dfrac{1}{3} e^{2x+y}$

Hence the complete solution is $z = c_1 e^{\sqrt{b}x+by} + c_2 e^{-\sqrt{b}x+by} + \dfrac{1}{3} e^{2x+y}$.

Example 16.45: Solve $(D^2 - DD' - 2)z = \sin(3x + 4y)$.

Solution: The equation is irreducible. Hence the complementary function is

$z = e^{ax+by}$, provided $a^2 - ab - 2 = 0$.

Next, the particular integral is

$$z = \frac{1}{D^2 - DD' - 2} \sin(3x + 4y) = \frac{1}{-9+12-2} \sin(3x + 4y) = \sin(3x + 4y).$$

Hence the complete solution is $z = \sum_{k=1}^{2} c_k e^{a_k x + b_k y} + \sin(3x + 4y)$, where a_k, b_k and c_k are arbitrary constants with $a_k^2 - a_k b_k - 2 = 0$.

EXERCISE 16.7

Solve the following equations

1. $DD'(D - 2D' - 3)z = 0$
2. $(D^2 - D'^2 + D - D')z = 0$
3. $(D^2 - 4D'^2 + 2D + 1)z = 0$
4. $r + 2s + t + 2p + 2q + z = 0$
5. $(4D^3 - 8D^2D' + 4D^2 - 8DD' + D - 2D')z = 0$

Find the complete solution of the equations

6. $(D^2 + D'^2 + 2DD' + 2D + 2D' + 1)z = e^{2x+y}$
7. $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x + 2y)$
8. $(D^2 - D'^2 + 3D' - 3D)z = e^{x+2y} + xy$
9. $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y$
10. $(D - 3D' - 2)^2 z = 2e^{2x} \tan(y + 3x)$
11. $(D - D'^2)z = \cos(x - 3y)$
12. $(D^2 - D')z = A\cos(lx + my)$; A, l, m are constants.
13. $(D^2 - D')z = 2y - x^2$

14. Show that a *l.p.d.e.* of the type $\sum_{r,s} c_{rs} x^r y^s \frac{\partial^{r+s} z}{\partial x^r \partial y^s} = f(x, y)$ reduces to a *l. p. d. e.* with constant coefficients by substitution $x = e^u$, $y = e^v$. Hence solve the equation $x^2 r - y^2 t + xp - yq = \ln x$.
15. Solve the equation $x^2 r - 3xys + 2y^2 t + xp + 2yq = x + 2y$.

16.10 MONGE'S METHOD OF SOLVING QUASI-LINEAR SECOND ORDER EQUATIONS

Most of the non-linear second order equations of the general form

$$f(x, y, z, p, q, r, s, t) = 0$$

cannot be solved exactly. Monge's method is employed to solve quasi-linear partial differential equations of the form

$$Rr + Ss + Tt = V, \quad \text{...(16.155)}$$

where r, s, t, appear in the first degree and the coefficients R, S, T, V are functions of x, y, z, p and q only.

The method consists of reducing the Eq. (16.155) into an equivalent system of two equations from which we determine p, or q, or both p and q. If both p and q are determined, then we use $dz = pdx + qdy$ to find the solution, otherwise solution is obtained on the lines of solving a Lagrange's equation.

We have, $dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = rdx + sdy$, and $dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = sdx + tdy$

which give respectively $r = \frac{dp - sdy}{dx}$, and $t = \frac{dq - sdx}{dy}$.

Substituting for r and t in (16.155) and rearranging the terms we obtain

$$(Rdpdy + Tdqdx - Vdxdy) + s(Sdxdy - R(dy)^2 - T(dx)^2) = 0. \quad \text{...(16.156)}$$

Since the Eq. (16.156) holds for arbitrary value of r, thus to hold this we must have

$$Rdpdy + Tdqdx - Vdxdy = 0 \quad \text{...(16.157)}$$

and,

$$R(dy)^2 - Sdydx + T(dx)^2 = 0. \quad \text{...(16.158)}$$

These equations are called *Monge's subsidiary equations*.

We can solve Eq. (16.158) as quadratic in dy/dx. We may get two distinct values for dy/dx. Both of these values may be used along with Eq. (16.157) to get two expressions involving p and q. These are solved for p and q and then substituting for p and q in $dz = pdx + qdy$, we solve for the general integral. Alternatively, we may use one of dy/dx along with Eq. (16.157) to get a first order linear partial differential equation, which is solved by Lagrange's method leading to the general solution.

In case both the values of dy/dx are equal then evidently we are left with the alternate approach only as discussed above.

Example 16.46: Solve $r + (a+b)s + abt = xy$ using Monge's method.

Solution: Here we have $R = 1$, $S = (a+b)$, $T = ab$, $V = xy$.

The Monge's subsidiary equations

$$R(dy)^2 - Sdx\,dy + T(dx)^2 = 0, \text{ and } R\,dp\,dy + Tdq\,dx - V\,dx\,dy = 0,$$

become respectively,

$$(dy)^2 - (a+b)dx\,dy + ab(dx)^2 = 0 \qquad ...(16.159)$$

and,

$$dp\,dy + (ab)dq\,dx - xy\,dx\,dy = 0. \qquad ...(16.160)$$

Factorizing (16.159), we get $(dy - adx)(dy - bdx) = 0$, which give

$$dy - adx = 0 \text{ and } dy - bdx = 0.$$

Integrating, we get respectively

$$y - ax = \text{const. and } y - bx = \text{const.}$$

Substituting $dy = adx$ in (16.160), we obtain

$$dp(adx) + ab\,dq\,dx - xy\,dx(adx) = 0$$

or,

$$dp + bdq - xydx = 0.$$

Integrating, we obtain

$$p + bq - y\frac{x^2}{2} = \text{const.} = \phi_1(y - ax), \qquad ...(16.161)$$

where ϕ_1 is an arbitrary function.

Similarly using $dy = bdx$ in (16.160) and integrating, we obtain

$$p + aq - y\frac{x^2}{2} = \text{const.} = \phi_2(y - bx), \qquad ...(16.162)$$

where ϕ_2 is an arbitrary function.

Solving (16.161) and (16.162) for p and q, we get

$$p = \frac{b\phi_2 - a\phi_1}{b-a} + \frac{x^2y}{2} \text{ and } q = \frac{\phi_2 - \phi_1}{a-b}.$$

Substituting for p and q in $dz = pdx + qdy$, we have

$$dz = \left(\frac{b\phi_2 - a\phi_1}{b-a} + \frac{x^2y}{2}\right)dx + \left(\frac{\phi_2 - \phi_1}{a-b}\right)dy$$

or,

$$(a-b)dz = -\phi_1(dy - adx) + \phi_2(dy - bdx) + (a-b)\frac{x^2y}{2}dx.$$

Integrating, we get

$$(a-b)z = -\int \phi_1(y-ax)d(y-ax) + \int \phi_2(y-bx)d(y-bx) + (a-b)\frac{y}{2}\int x^2dx$$

or, $$z = f_1(y - ax) + f_2(y - bx) + \frac{1}{6}x^3y,$$

where f_1 and f_2 are two arbitrary functions.

Remark. We can verify that the same solution is obtained by solving the equation $r + (a+b)s + abt = xy$ using the method of solving homogeneous equations as discussed in Section 16.8.

Example 16.47: Solve $x^2r + 2xys + y^2t = 0$ using Monge's method.

Solution: Here we have

$$R = x^2,\ S = 2xy,\ T = y^2,\ V = 0.$$

The Monge' subsidiary equations

$$R(dy)^2 - Sdy\,dx + T(dx)^2 = 0 \quad \text{and} \quad Rdp\,dy + Tdq\,dx - Vdy\,dx = 0,$$

become respectively

$$x^2(dy)^2 - 2xy\,dy\,dx + y^2(dx)^2 = 0 \qquad ...(16.163)$$

and, $$x^2dp\,dy + y^2dq\,dx = 0. \qquad ...(16.164)$$

From Eq. (16.163), we have $(xdy - ydx)^2 = 0$, or $xdy - ydx = 0$ with solution $y = ax$.

Substituting $y = ax$ in (16.164) to obtain $ax^2dp\,dx + a^2x^2dq\,dx = 0$, or $dp + adq = 0$, which gives $p + aq = \text{const.}$ or, $p + (y/x)q = \phi_1(y/x)$, or $xp + yq = x\phi_1(y/x)$, which is Lagrange's equation.

The subsidiary equations are $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{x\phi_1(y/x)}$.

From the first two terms, we get $y = k_1x$, and from the first and the last term, we have

$$z = x\phi_1(k_1) + k_2 = x\phi_1(y/x) + k_2.$$

Hence the solution can be obtained as $z = x\phi_1(y/x) + \phi_2(y/x)$, where ϕ_1 and ϕ_2 are two arbitrary functions.

Remark. We note that the equation $x^2r + 2xys + y^2t = 0$ can be reduced to *l.p.d.e.* with constant coefficients by using the transformation $x = e^u$ and $y = e^v$ and the same solution can be obtained by solving the equation using the method as discussed in Section 16.8.

Example 16.48: Solve $(x-y)(xr - xs - ys + yt) = (x+y)(p-q)$

Solution: Here $R = x(x-y)$, $S = -(x^2 - y^2)$, $T = y(x-y)$, $V = (x+y)(p-q)$.

The Monge's equations $R(dy)^2 - Sdy\,dx + T(dx)^2 = 0$ and $Rdp\,dy + Tdq\,dx - Vdxdy = 0$,

become respectively, $x(x-y)(dy)^2 + (x^2 - y^2)dy\,dx + y(x-y)(dx)^2 = 0$

and, $$x(x-y)dp\,dy + y(x-y)dq\,dx - (x+y)(p-q)dxdy = 0$$

Simplifying these we get respectively

$$x(dy)^2 + (x+y)dy\,dx + y(dx)^2 = 0 \qquad ...(16.165)$$

and, $$xdp\,dy + ydq\,dx - \frac{x+y}{x-y}(p-q)dx\,dy = 0 \qquad ...(16.166)$$

Eq. (16.165) may be factorized as $(x\,dy + y\,dx)(dx + dy) = 0$, which gives

$$xy = \text{const. or, } \quad x + y = \text{const.}$$

Taking $xy =$ constant and dividing each term of (16.166) by $x\,dy$, or its equivalent $-y\,dx$, we get

$$dp - dq - \frac{p-q}{x-y}(dx - dy) = 0$$

or,

$$\frac{dp - dq}{p - q} = \frac{dx - dy}{x - y}.$$

This gives

$$\frac{(p-q)}{(x-y)} = \text{const.} = \phi_1(xy), \quad \text{say}$$

or,

$$p - q = (x - y)\phi_1(xy),$$

a Lagrange's linear equation.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{(x-y)\phi_1(xy)}.$$

The first two terms give $x + y = c$, c being a constant.

The first and the third terms give

$$\begin{aligned} dz &= (x - y)\,\phi_1(xy)dx \\ &= -(c - 2x)\phi_1(xc - x^2)dx \\ &= -\phi_1(cx - x^2)d(cx - x^2). \end{aligned}$$

Integrating, we have

$$z = f_1(cx - x^2) + c'.$$

Hence the solution can be given as

$$\begin{aligned} z &= f_1(cx - x^2) + f_2(c) \\ &= f_1(xy) + f_2(x + y), \end{aligned}$$

where f_1 and f_2 are two arbitrary functions.

Example 16.49: Solve the equation $q^2 r - 2pqs + p^2 t = 0.$

Solution: Here, $R = q^2$, $S = -2pq$, $T = p^2$ and $V = 0.$

The Monge's equations

$$R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0 \text{ and } R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0,$$

become respectively,

$$q^2(dy)^2 + 2pq\,dx\,dy + p^2(dx)^2 = 0 \qquad \text{...(16.167)}$$

and,

$$q^2\,dp\,dy + p^2\,dq\,dx = 0. \qquad \text{...(16.168)}$$

From (16.167), we have $(p\,dx + q\,dy)^2 = 0$, or $dz = 0$, which gives, $z = k_1$.

Dividing Eq. (16.168) by $q\,dy$, or its equivalent $-p\,dx$, we obtain

$$q\,dp = p\,dq, \text{ which gives } \quad p = k_2 q, \text{ or } p = \phi_1(k_1)q$$

Thus $p = \phi_1(z)q$, where ϕ_1 is an arbitrary function.

The equation $p - q\ \phi_1(z) = 0$ is Lagrange's equation with subsidiary equations

$$\frac{dx}{1} = \frac{dy}{-\phi_1(z)} = \frac{dz}{0},$$

with integrals $z = k_3$ and $y + x\phi_1(k_3) = k_4 = \phi_2(k_3)$, say.

Thus the general solution is $y + x\phi_1(z) = \phi_2(z)$, where ϕ_1 and ϕ_2 are two arbitrary functions.

Example 16.50: Solve the equation $r - t\cos^2 x + p\tan x = 0$.

Solution: Here, $R = 1$, $S = 0$, $T = -\cos^2 x$ and $V = -p\tan x$

The Monge's equations $R(dy)^2 - Sdx\,dy + T(dx)^2 = 0$ and $Rdp\,dy + Tdq\,dx - Vdx\,dy = 0$,

become respectively, $$(dy)^2 - \cos^2 x(dx)^2 = 0, \quad \text{...(16.169)}$$

and $$dp\,dy - \cos^2 x\,dq\,dx + p\tan x\,dx\,dy = 0. \quad \text{...(16.170)}$$

From (16.169), we have $dy = \cos x\,dx$ and $dy = -\cos x\,dx$

The equation $dy = \cos x\,dx$ gives $y - \sin x = k_1$.

Using $dy = \cos x\,dx$ in (16.170) and simplifying, we obtain

$dp - \cos x\,dq + p\tan x\,dx = 0$ or, $\sec x\,dp - dq + p\tan x\sec x\,dx = 0$ or, $d(p\sec x) - dq = 0$.

Integrating it we obtain $p\sec x - q = k_2 = \phi(k_1)$, where ϕ is arbitrary.

This gives $p\sec x - q = \phi(y - \sin x)$ a Lagrange's equation.

The subsidiary equations are $\dfrac{dx}{\sec x} = \dfrac{dy}{-1} = \dfrac{dz}{\phi(y - \sin x)}$.

The first two terms give, $y + \sin x = k_3$. Also from the last two terms, we obtain

$dz = -\phi(y - \sin x)dy = -\phi(2y - k_3)dy = -\dfrac{1}{2}\phi(2y - k_3)d(2y)$, which gives

$$z = -\frac{1}{2}\int \phi(2y - k_3)d(2y) + k_4 = \phi_1(2y - k_3) + \phi_2(k_3), \quad \text{say,}$$

$$= \phi_1(y - \sin x) + \phi_2(y + \sin x),$$

where ϕ_1 and ϕ_2 are two arbitrary functions.

EXERCISE 16.8

Solve the following equations by Monge's method

1. $r = a^2 t$
2. $x^2 - 2xs + t + q = 0$
3. $y^2 r - 2ys + t = p + 6y$
4. $2x^2 r - 5xys + 2y^2 t + 2(px + qy) = 0$
5. $pt - qs = q^3$
6. $z(qs - pt) = pq^2$

7. $rq^2 - 2pqs + tp^2 = pt - qs$
8. $t - r\sec^4 y = 2q\tan y$
9. $(1-q)^2 r - 2(2-p-2q+pq)s + (2-p^2)t = 0$
10. $(1+q)^2 r - 2(1+p)(1+q)s + (1+p)^2 t = 0.$

ANSWERS

Exercise 16.1 (p. 892)

1. $px - qy = 0$
2. $pq = z$
3. $z(px + qy) = z^2 - 1$
4. $p^2 + q^2 = 4z$
5. $px + qy = q^2$
6. $z_{xx} + z_{yy} + z_{tt} = 0$
7. $z = px + qy + d\sqrt{1 + p^2 + q^2}$
8. $yp - xq = 0$
9. $yp - xq = 0$
10. $zyp - xzq = xy$
11. $x(y-z)p + y(z-x)q = z(x-y)$
12. $z_{tt} = c^2 z_{xx}$
13. $z_{yy} = z_y$
14. $x^2 z_{xx} - y^2 z_{yy} = yz_y - xz_x$
15. $a^2(z_{xx} + z_{yy}) = z_{tt}$.

Exercise 16.2 (p. 901)

1. $x^2 - z^2 = f\left(\dfrac{x+z}{y}\right)$
2. $f\left(\dfrac{x-y}{y-z}, \dfrac{y-z}{z-x}\right) = 0$
3. $3y^2 - z^2 = f(x^2 - 2y^2)$
4. $f\left[\dfrac{y}{x}, x - \ln\left(z - \dfrac{4x^2}{y^2}\right)\right] = 0$
5. $x^2 + y^2 + z^2 = xf\left(\dfrac{y}{x}\right)$
6. $(x+y)(x+y+z) = f(xy)$.
7. $x^2 + y^2 - z^2 = f(xy + z)$
8. $f[y - 3x,\ e^{-5x}\{5z + \tan(y - 3x)\}] = 0$
9. $f\left(xyz, \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right) = 0$
10. $f(x^2 + y^2 + z^2, xyz) = 0$
11. $f(x^3 + y^3 + z, xyz) = 0$
12. $f\left[3xy - y^3, \dfrac{1}{y}(2z - y^3 + 2xy)\right] = 0$
13. $u = \dfrac{1}{3}xyz + \dfrac{1}{3}f\left(\dfrac{x}{y}, \dfrac{y}{z}\right)$
14. $u = f\left(\dfrac{x^2}{y}, \dfrac{x^3}{z}, \dfrac{x^4}{t}\right)$
15. $x^2 + y^2 + z = 4$
16. $x^2 + y^2 - 2x = z^2 - 4z$
17. $z^3(x^3 + y^3)^2 = a^9(x-y)^3$
18. $y^2 - x^2 + yz - x - z + 1 = 0$

19. $2x^2 + 4y^2 + 3z^2 + 2xz = 1$
20. $(x - y + z)^2 + z^4(x + y + z)^2 - 2z^2(x - y + z) - 2z^4(x + y + z) = 0.$

Exercise 16.3 (p. 908)

1. $z = ay + (\sqrt{x} + b)^2$
2. $\dfrac{3z}{2} = (x - a)^{3/2} - (a - y)^{3/2} + b$
3. $az^2 = (x + b)^2 + y^2$
4. $(1 + a)z = \left[\sqrt{ax} + \sqrt{b + y}\right]^2$
5. $z = bx^{a/2} y^{\sqrt{1-a^2}/3}$
6. $z = \dfrac{ax}{y^2} + \dfrac{b}{y} - \dfrac{a^2}{4y^2}$
7. $z = \dfrac{2}{3}(y + a)^{3/2} + b\, e^{3/x^2} + \dfrac{1}{9}\left(\dfrac{3}{x^2} + 1\right)$
8. $z = \pm 1$
9. (i) no singular solution (ii) $4z + (x^2 + y^2) = 0$
10. $16z = (4y + x)^2$
11. $xy = z(x - 2)$
12. $4z + (x + \sqrt{2}y)^2 = 0.$

Exercise 16.4 (p. 912)

1. $z = ax + (1 - \sqrt{a})^2 y + b$
2. $z = ax + \dfrac{ay}{a - 1} + b$
3. $z = \sqrt{5 + 3a^2}\, x + ay + b$
4. $z^2 = 2(a + 1)\left(x + \dfrac{y}{a}\right) + b$
5. $z = \dfrac{1}{3}(x^2 + a^2)^{3/2} + (y^2 - a^2)^{1/2} + b$
6. $z = \dfrac{1}{72}(6x + a)^3 + \dfrac{1}{54}(2y - a)^3 + b$
7. $z = ax + by + \dfrac{1}{a + b}$
8. $(1 + a^3)z^2 = 64(ax + y + b)^3$
9. $z^2 = x\sqrt{x^2 + a} + y\sqrt{y^2 - a} + a \ln \dfrac{x + \sqrt{x^2 + a}}{y + \sqrt{y^2 - a}} + 2b$
10. $z = \sqrt{a(x + y)} + \sqrt{(1 - a)(x - y)} + b.$
11. $z = ax^2 + by^2 + 8ab$

Exercise 16.5 (p. 914)

1. $x^2 + y^2 + z^2 = zf\left(\dfrac{2x^2 + y^2}{z^2}\right)$
2. $3(x^2 + y^2) = z^2(z + 3)$

3. $(x^2 + y^2 + 4z^2)(x^2 - y^2)^2 = a^4(x^2 + y^2)$ 4. $x^2 + y^2 + z^2 = 5z.$

Exercise 16.6 (p. 920)

1. $z = \phi_1(y) + \phi_2(y + x) + \phi_3(y + 2x)$ 2. $z = \phi_1(y - 2x) + \phi_2(2y - x)$
3. $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$
4. $z = \phi_1(x + y) + x\phi_2(y + x) + \phi_3(x - 2y) + x\phi_4(x - 2y)$
5. $z = \phi_1(y - 2x) + \phi_2(y - 3x) + e^{x-y}/2$
6. $z = \phi_1(2y - 3x) + x\phi_2(2y - 3x) + x^2e^{3x-2y}/8$
7. $z = \phi_1(y - 3x) + \phi_2(2y + x) + \sin(2x - y)/5$
8. $z = \phi_1(y + x) + \phi_2(y - x) + [2x\sin(x + y) + \cos(x + y)]/4$
9. $z = \phi_1(y - x) + \phi_2(y + 2x) + ye^x$
10. $z = \phi_1(y + x) + x\phi_2(y + x) + \dfrac{x^2}{2}\tan(y + x)$
11. $z = \phi_1(y - 2x) + \phi_2(y + x) - \left[4x^2 + \dfrac{1}{22}(x + 5y)^2\{2\ln(x + 5y) - 1\}\right]$
12. $z = \phi_1(y - x) + x\phi_2(y - x) + x\sin y$
13. $z = \phi_1(y + 5x) + \phi_2(y - 3x) + x^4 + 2x^3y$
14. $z = \phi_1(y + x) + \phi_2(y - x) + (\tan x\tan y)/2$
15. $z = \phi_1(y + x) + \phi_2(y - x) + x\phi_3(y - x) + e^x(\cos 2y + 2\sin 2y)/25.$

Exercise 16.7 (p. 925)

1. $z = \phi_1(x) + \phi_2(y) + e^{3x}\phi_3(y + 2x)$
2. $z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$
3. $z = \sum c_k e^{a_k x + b_k y}$, provided $a_k^2 - 4b_k^2 + 2a_k + 1 = 0$
4. $z = e^{-x}\phi_1(y - x) + xe^{-x}\phi_2(y - x)$
5. $z = \phi_1(y + 2x) + e^{-x/2}\phi_2(y) + xe^{-x/2}\phi_3(y)$
6. $z = e^{-x}\phi_1(y - x) + xe^{-x}\phi_2(y - x) + e^{2x+y}/16$
7. $z = \phi_1(y - x) + e^{2x}\phi_2(y - x) + [2\cos(x + 2y) - 3\sin(x + 2y)]/39$
8. $z = \phi_1(y + x) + e^{3x}\phi_2(y - x) - ye^{x+2y} - \dfrac{1}{3}\left(\dfrac{x^2y}{2} + \dfrac{xy}{3} + \dfrac{x^2}{2} + \dfrac{2x}{9}\right)$
9. $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + (\sin(x + 2y))/2 - xe^y$
10. $z = e^{2x}[x^2\tan(y + 2x) + x\phi_1(y + 3x) + \phi_2(y + 3x)]$
11. $z = \dfrac{1}{82}[\sin(x - 3y) + 9(x - 3y)] + \sum c_k e^{a_k x + b_k y}; a_k - b_k^2 = 2$

12. $z = \frac{A}{m^2 - l^2}\{m \sin(lx + my) + l^2 \cos(lx + my)\} + \sum_k c_k e^{a_k x + b_k y}; \quad a_k^2 - b_k = 0$

13. $z = \sum c_k e^{a_k x + b_k y} - \left(y^2 + \frac{x^4}{12}\right); \qquad a_k^2 - b_k = 0$

14. $z = \phi_1\left(\frac{x}{y}\right) + \phi_2(xy) + \frac{1}{6}(\ln x)^2$

15. $z = \phi_1(xy) + \phi_2(x^2 y) + x + y.$

Exercise 16.8 (p. 930)

1. $z = \phi_1(y + ax) + \phi_2(y - ax)$
2. $z = \phi_1(y + \ln x) + x\phi_2(y + \ln x)$
3. $z = y^3 + y\phi_1(y^2 + 2x) + \phi_2(y^2 + 2x)$
4. $z = \phi_1(x^2 y) + \phi_2(xy^2)$
5. $y = xz + \phi_1(z) + \phi_2(x)$
6. $y = \phi_1(z) + z\phi_2(x)$
7. $y = \phi_1(x + z) + \phi_2(z)$
8. $z = \phi_1(x + \tan y) + \phi_2(x - \tan y)$
9. $x = \phi_1(y + 2x - z) + y\phi_2(y + 2x - z)$
10. $y = \phi_1(x + y + z) + x\phi_2(x + y + z).$

17 Applications of Partial Differential Equations

CHAPTER

The scope of applications of partial differential equations is much wider as compared to ordinary differential equations. They arise in many diversified areas like epidemiology, traffic flow studies and economic analysis. Models of three major kinds of physical phenomena: wave motion, heat conduction and potential theory lead respectively to three important partial differential equations, the wave equation, the heat equation and the Laplace's equation. Various mathematical tools including Fourier series, integrals, transforms and special functions are employed to solve these equations.

17.1 METHOD OF SEPARATION OF VARIABLES

To find the solution of a particular problem involving a partial differential equation, depending upon the nature of the physical phenomena, it is necessary to specify that the solution satisfies some specific conditions. In case these conditions are imposed on spatial boundaries belonging to the region D, where the solution is required, then such conditions are called *boundary conditions* and the problem is called *boundary value problem* (BVP). However, in case one of the independent variable is time, then it becomes necessary to specify how the solution starts and a condition of this type is called an *initial condition* and the problem is called *initial value problem* (IVP). Problems, with both initial and boundary conditions specified, are called *initial boundary value problems* (IBVP).

Method of separating variables, or product method is a powerful technique to solve linear partial differential equations with specified conditions. For a *PDE* in the unknown function u of two independent variables x and y, we assume that the desired solution is separable, that is,

$$u(x, y) = X(x)Y(y),$$

where X is a function of x alone and Y that of y alone. The substitution of u and its partial derivatives in terms of X and Y and their derivatives reduce the given *PDE* to the form

$$f(X, X', X'', ...) = g(Y, Y', Y'', ...)$$

which is separable in X and Y. Here primes denote the derivatives w.r.t. the corresponding independent variables.

Since f is a function of x alone and g is a function of y alone, x and y being independent, thus it follows that each must be equal to a common constant and hence the problem of solving the *PDE* reduces to finding the solutions to two *ODE's* given by

$$f(X, X', X'' \ldots) = 0, \quad g(Y, Y', Y'' \ldots) = 0$$

under specified conditions. From the expressions of $X(x)$ and $Y(y)$ so obtained we find expression for $u(x, y)$, the desired solution.

Example 17.1: Solve by the method of separation of variables the initial value problem (IVP)

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0; \quad u(0, y) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(0, y) = e^{-3y}, \text{ for all } y.$$

Solution: Let the solution be $u(x, y) = X(x)Y(y)$. ...(17.1)

It gives
$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY' \text{ and } \frac{\partial^2 u}{\partial x^2} = X''Y,$$

where the primes denote the derivatives w.r.t. the corresponding independent variable.

Substituting these in the given equation, we obtain $X''Y - 2X'Y + XY' = 0$, which gives

$$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y}. \qquad \text{...(17.2)}$$

Since x and y are independent variables, thus Eq. (17.2) can hold only if each side is equal to some constant, say k. Hence we get

$$X'' - 2X' - kX = 0 \qquad \text{...(17.3)}$$

and,
$$Y' + kY = 0 \qquad \text{...(17.4)}$$

two ordinary differential equations.

The solution of Eq. (17.3) is $X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$
and of Eq. (17.4) is $Y = c_3\, e^{-ky}$, where c_1, c_2 and c_3 are arbitrary constants.

Substituting for X and Y in (17.1), the solution of the given equation is

$$u(x, y) = \left[Ae^{(1+\sqrt{1+k})x} + Be^{(1-\sqrt{1+k})x}\right]e^{-ky}, \qquad \text{...(17.5)}$$

where $A = c_1c_3$ and $B = c_2c_3$.

The three arbitrary constants A, B and k can be determined from the conditions specified.

Using $u(0, y) = 0$ in (17.5) gives, $(A + B)e^{-ky} = 0$ for all y, which implies $A + B = 0$.

Similarly, using $\frac{\partial u}{\partial x}(0, y) = e^{-3y}$ in (17.5) gives, $[(1 + \sqrt{1+k})A + (1 - \sqrt{1+k})B]e^{-ky} = e^{-3y}$, for all y. This implies $(1 + \sqrt{1+k})A + (1 - \sqrt{1+k})B = 1$ and $k = 3$, and hence $3A - B = 1$.

Solving for A and B, we obtain $A = -B = \frac{1}{4}$. Thus (17.5) becomes $u(x, y) = \frac{1}{4}(e^{3x} - e^{-x})e^{-3y}$, the desired solution of the given *IVP*.

EXERCISE 17.1

Solve the following equations by the method of separation of variables:

1. $x^2 \dfrac{\partial u}{\partial x} + y^2 \dfrac{\partial u}{\partial y} = 0$

2. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y}$

3. $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial y} + u;\quad u(x, 0) = 6e^{-3x}$

4. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u;\quad u(0, y) = 0,\quad \dfrac{\partial u(0, y)}{\partial x} = 1 + e^{-3y}$

5. $\dfrac{\partial^2 u}{\partial x \partial y} = e^{-y} \cos x;\quad u(x, 0) = 0$ and $\dfrac{\partial u}{\partial y}(0, y) = 0.$

17.2 VIBRATING STRING: ONE-DIMENSIONAL WAVE EQUATION

The transversal oscillations induced in a guitar or violin string are governed by a partial differential equation called the *one-dimensional wave equation*.

Consider an elastic string placed along the x-axis from 0 to l, fixed at the ends $x = 0$ and $x = l$. We distort it at some instant $t = 0$ and then release to allow it to vibrate in the xy-plane. Let the function $u(x, t)$ denotes the displacement of the string at any point x and at any instant $t > 0$, as shown in Fig. 17.1.

To determine the mathematical model, the *p.d.e* in the displacement function $u(x, t)$, of the physical system described we make the following assumptions:

(i) The mass of the string per unit length ρ is constant and the string is perfectly elastic.

(ii) The tension caused by stretching the string is so large that the gravitational force on the string can be neglected, and

(iii) The string performs small transverse motion in a vertical plane e.g., xy-plane in Fig. 17.1 and thus the displacement $u(x, t)$ and the slope at every point of the string always remains small in magnitude.

Consider a segment PQ of the string between x and $x + \Delta x$ and let T_1 and T_2 be the tensions at the end-points P and Q of the segment acting along the tangents to the curve of the string at that points.

Since the motion of the string is only in the vertical plane, thus the horizontal components of the tension must be constant. Hence with the notations as in Fig. 17.1, we have

$$T_1 \cos \alpha = T_2 \cos \beta = \text{const. say, } T. \qquad ...(17.6)$$

In the vertical direction we have two forces $-T_1 \sin \alpha$ at P and $T_2 \sin \beta$ at Q with upward direction taken as positive. By Newton's second law of motion, the resultant of the two forces must be equal to mass $\rho \Delta x$

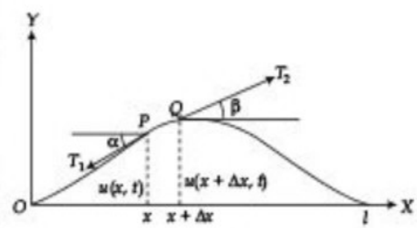


Fig. 17.1

of the segment PQ multiplied with the acceleration $\frac{\partial^2 u}{\partial t^2}$ evaluated at some point between the segment PQ, that is

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\Delta x \frac{\partial^2 u}{\partial t^2}. \quad ...(17.7)$$

Dividing (17.7) by (17.6), we obtain

$$\tan\beta - \tan\alpha = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}. \quad ...(17.8)$$

Since $\tan\alpha$ and $\tan\beta$ are the slopes to the curve of the string at $P(x)$ and $Q(x+\Delta x)$, thus

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)\bigg|_x, \text{ and } \tan\beta = \left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x}.$$

Using these in (17.8), we obtain $\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$.

Taking the limit as $\Delta x \to 0$, we obtain

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad ...(17.9)$$

where $c^2 = \rho/T$, since $\rho/T > 0$.

Eq. (17.9) is called the *one-dimensional wave equation*. It is called one-dimensional, since it involves only one space variable. The equation is second order homogeneous and of hyperbolic type, refer to Section 16.1 .

In order to solve the wave equation (17.9) we must incorporate the conditions imposed by the physical model under study.

If the ends of the string are fixed, then $u(0, t) = u(l, t) = 0$, for $t \geq 0$. These are the *boundary conditions*.

The *initial conditions* specifies the initial position $u(x, 0) = f(x)$, for $0 \leq x \leq l$, and the initial velocity $\frac{\partial u}{\partial t}\Big|_{t=0} = g(x)$, for $0 < x < l$, where f and g are given functions satisfying certain compatibility conditions like $f(0) = f(l) = 0$, if the string is fixed at ends; $g(x) = 0$, if the string is released from rest.

The wave equation together with the boundary and initial conditions constitute the *inital boundary value problem* for the position function $u(x, t)$. We will solve boundary value problems involving one-dimensional wave motion under variety of constraints.

17.3 SOLUTION OF THE WAVE EQUATION BY SEPARATION OF VARIABLES AND USE OF FOURIER SERIES

Let us consider an elastic string placed along the x-axis stretched to length l between two fixed points $x = 0$ and $x = l$. Let $f(x)$ and $g(x)$ respectively be the initial displacement and the initial velocity of the string. If $u(x, t)$ is the displacement of the string at any point x and at any instant $t > 0$, then the complete initial boundary value problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad 0 < t < \infty, \qquad ...(17.10)$$

with boundary conditions

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t \geq 0, \qquad ...(17.11)$$

and, initial conditions

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 \leq x \leq l, \qquad ...(17.12)$$

Using the method of separation of variables let the solution of Eq. (17.10) be of the form

$$u(x, t) = X(x)T(t), \qquad ...(17.13)$$

where X is a function of x only and T is a function of t only.

Substituting (17.13) in Eq. (17.10), we obtain $XT'' = c^2X''T$,

$$\frac{X''}{X} = \frac{T''}{c^2T}, \qquad ...(17.14)$$

where the primes denote the derivatives w.r.t. the corresponding independent variables.

The left side of Eq. (17.14) depends only on x and right side only on t, so both sides of Eq. (17.14) must be equal to a common constant, say k. This results in two differential equations

$$X'' - kX = 0, \qquad ...(17.15)$$

and,

$$T'' - kc^2T = 0. \qquad ...(17.16)$$

To solve (17.15) and (17.16), we consider the following three cases.

(i) When k is positive, say $k = \lambda^2$, then $X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$; $T = c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}$.

(ii) When k is negative, say $k = -\lambda^2$, then $X = c_5 \cos \lambda x + c_6 \sin \lambda x$; $T = c_7 \cos c\lambda t + c_8 \sin c\lambda t$.

(iii) When k is zero, then $X = c_9 x + c_{10}$; $T = c_{11} t + c_{12}$.

Thus, the various possible solutions of the wave equation are:

$$u = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}) \qquad ...(17.17)$$

$$u = (c_5 \cos \lambda x + c_6 \sin \lambda x)(c_7 \cos c\lambda t + c_8 \sin c\lambda t) \qquad ...(17.18)$$

$$u = (c_9 x + c_{10})(c_{11} t + c_{12}). \qquad ...(17.19)$$

In case of vibrating string problem u is a periodic funtion of x and t. Hence the solution (17.18) is the proper one, since it involves trigonometric expressions in x and t. We write

$$u(x, t) = (A \cos \lambda x + B \sin \lambda x)(C \cos c\lambda t + D \sin c\lambda t) \qquad ...(17.20)$$

as the suitable solution of the wave equation (17.10), where A, B, C and D are arbitrary constants to be determined using the initial and boundary conditions.

Using the boundary condition $u(0, t) = 0$ in (17.20), gives $A[C\cos c\lambda t + D\sin c\lambda t] = 0$, for all t, which implies that $A = 0$.

Hence, (17.20) becomes

$$u(x, t) = B\sin\lambda x[C\cos c\lambda t + D\sin c\lambda t]. \qquad \text{...(17.21)}$$

Using the condition $u(l, t) = 0$ in (17.21), gives $B\sin\lambda l[C\cos c\lambda t + D\sin c\lambda t] = 0$, for all t.

Now B can't be zero, since the solution will become zero one, thus $\sin\lambda l = 0 = \sin n\pi$, and hence, $\lambda = \frac{n\pi}{l}$, n being an integer. Therefore, the solution (17.21) can be expressed as

$$u_n(x, t) = \left[C_n\cos\frac{n\pi ct}{l} + D_n\sin\frac{n\pi ct}{l}\right]\sin\left(\frac{n\pi x}{l}\right).$$

Using the principle of superposition, we obtain

$$u(x, t) = \sum_{n=1}^{\infty}\left[C_n\cos\frac{n\pi ct}{l} + D_n\sin\frac{n\pi ct}{l}\right]\sin\left(\frac{n\pi x}{l}\right), \qquad \text{...(17.21a)}$$

where the constants are to be evaluated using the initial conditions (17.12).

Using the initial condition $u(x, 0) = f(x)$ in (17.21a), we obtain

$$f(x) = \sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{l}. \qquad \text{...(17.22)}$$

The series (17.22) is a Fourier half-range sine series in the interval $[0, l]$ and hence the coefficients C_n are given by

$$C_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \quad n = 1, 2, \ldots \qquad \text{... (17.23)}$$

Next, differentiating (17.21a) with respect to t, we obtain

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[-C_n\frac{n\pi c}{l}\sin\frac{n\pi ct}{l} + D_n\frac{n\pi c}{l} + \cos\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l}. \qquad \text{...(17.24)}$$

Using the initial condition $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$ in (17.24), we obtain

$$g(x) = \sum_{n=1}^{\infty}\left(D_n\frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l}. \qquad \text{...(17.25)}$$

The series (17.25) is also a Fourier half-range sine series in the interval [0, l] and hence the coefficients

$$D_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx,\ n = 1, 2, \ldots \qquad \ldots(17.26)$$

Hence, the general solution of the one-dimensional wave equation (17.10) with boundary conditions (17.11) and initial conditions (17.12) is given by (17.21a), where the constants C_n and D_n are respectively given by (17.23) and (17.26).

In case the string is released from rest, then

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = 0, \text{ for } 0 \le x \le l,$$

and thus from (17.26), $D_n = 0$ for $n = 1, 2, \ldots$ and hence the solution (17.21a) becomes

$$u(x, t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right), \qquad \ldots(17.27)$$

where the constants C_n are given by (17.23).

In case the string is initially in its equilibrium position, then $u(x, 0) = f(x) = 0$, for $0 \le x \le l$, thus from (17.23), $C_n = 0$ for $n = 1, 2, \ldots$ and hence the solution (17.21a) becomes

$$u(x, t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right), \qquad \ldots(17.28)$$

where the constants D_n are given by (17.26).

Example 17.2: A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $u(x) = u_0 \sin^3(\pi x/l)$. If it is released from rest from this position, find the displacement $u(x, t)$.

Solution: The displacement function $u(x, t)$ is given by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},\quad 0 < x < l,\quad t > 0 \qquad \ldots(17.29)$$

with boundary conditions

$$u(0, t) = 0,\quad u(l, t) = 0,\quad t \ge 0 \qquad \ldots(17.30)$$

and initial conditions

$$u(x, 0) = u_0 \sin^3(\pi x/l),\quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,\quad 0 \le x \le l. \qquad \ldots(17.31)$$

The solution of the one-dimensional wave equation (17.29) subject to the boundary conditions (17.30) and the initial condtion $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$, refer to Eq. (17.27), is given by

$$u(x,t)=\sum_{n=1}^{\infty} C_n \cos\frac{n\pi ct}{l}\sin\left(\frac{n\pi x}{l}\right), \qquad ...(17.32)$$

where C_n are constants to be determined using the second initial condition $u(x, 0) = u_0 \sin^3(\pi x/l)$ given in (17.31). Using this (17.32) becomes

$$u_0 \sin^3(\pi x/l)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \text{ or, } \frac{u_0}{4}\left[3\sin\frac{\pi x}{l}-\sin\frac{3\pi x}{l}\right]=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}. \qquad ...(17.33)$$

Comparing the coefficients of the like terms on both sides of (17.33) gives

$$C_1=\frac{3u_0}{4},\quad C_2=0,\quad C_3=-\frac{u_0}{4},\quad C_4=C_5=...=0$$

and hence (17.32) gives $u(x,t)=\dfrac{u_0}{4}\left[3\cos\dfrac{\pi ct}{l}\sin\dfrac{\pi x}{l}-\cos\dfrac{3\pi ct}{l}\sin\dfrac{3\pi x}{l}\right]$, the desired solution.

Example 17.3: A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $v_0 x(l - x)$, find the displacement of the string at any distance x from one end at any time t.

Solution: The displacement function $u(x, t)$ is given by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2},\quad 0<x<l,\quad t>0 \qquad ...(17.34)$$

with boundary conditions

$$u(0,t)=0,\quad u(l,t)=0,\quad t\ge 0 \qquad ...(17.35)$$

and initial conditions

$$u(x,0)=0,\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=u_0 x(l-x),\quad 0\le x\le l. \qquad ...(17.36)$$

The solution of the wave equation (17.34) subject to the boundary conditions (17.35) and the initial condition $u(x, 0) = 0$, refer to Eq. (17.28) is given by

$$u(x,t)=\sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right), \qquad ...(17.37)$$

where the constants D_n, $n = 1, 2, ...$ are to be determined using the second initial condition $\left.\dfrac{\partial u}{\partial t}\right|_{t=0}=u_0 x(l-x)$.

Differentiating (17.37) w.r.t. t and substituting $t = 0$ and using $\left.\dfrac{\partial u}{\partial t}\right|_{t=0}=u_0 x(l-x)$, it becomes

$$u_0x(l-x) = \sum_{n=1}^{\infty} D_n \frac{n\pi c}{l} \sin \frac{n\pi x}{l}. \qquad ...(17.38)$$

Next, expanding $u_0x(l-x)$ in a half-range sine series in $[0, l]$, we have

$$u_0x(l-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \qquad ...(17.39)$$

where $b_n = \frac{2}{l}\int_0^l u_0x(l-x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2u_0}{l}\left[-\frac{l}{n\pi}x(l-x)\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}(l-2x)\sin\frac{n\pi x}{l} - 2\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$= \frac{4u_0l^2}{n^3\pi^3}[1-\cos n\pi] = \frac{4u_0l^2}{\pi^3n^3}[1-(-1)^n]$, and hence

$$b_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{8u_0l^2}{\pi^3n^3}, & \text{when } n \text{ is odd} \end{cases} \qquad ...(17.40)$$

From (17.38), (17.39) and (17.40), we have $D_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{8u_0l^3}{\pi^4n^4c}, & \text{when } n \text{ is odd} \end{cases}$

Substituting for D_n in (17.37) the required solution is given by

$$u(x,t) = \sum_{n=1,3,5}^{\infty} \frac{8u_0l^3}{\pi^4n^4c}\sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right) = \frac{8u_0l^3}{\pi^4c}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi ct}{l}\sin\frac{(2n-1)\pi x}{l}.$$

Example 17.4: The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.

Solution: Let P and Q be the points of trisection of the string OA of length say l, fixed at the points O and A as shown in Fig. 17.2. Initially, it is taken to the form $OP'Q'A$, where $PP' = QQ' = a$, say and is then released.

The displacement $u(x, t)$ of string at a distance x from the fixed point O at $t > 0$ is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad ...(17.41)$$

with boundary conditions

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t \geq 0 \qquad ...(17.42)$$

and initial conditions

$$u(x, 0) = \begin{cases} \dfrac{3a}{l}x, & 0 \leq x \leq \dfrac{l}{3} \\ \dfrac{3a}{l}(l-2x), & \dfrac{l}{3} \leq x \leq \dfrac{2l}{3} \\ \dfrac{3a}{l}(x-l) & \dfrac{2l}{3} \leq x \leq l \end{cases} \qquad ...(17.43)$$

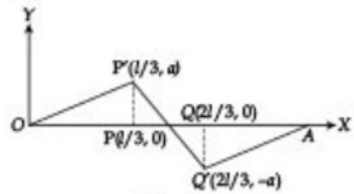


Fig. 17.2

and,
$$\left.\frac{\partial u}{\partial x}\right|_{t=0} = 0. \qquad ...(17.43a)$$

The solution of the one-dimensional wave equation (17.41) subject to the boundary conditions (17.42) and the initial condition (17.43a), refer to Eq. (17.27) is given by

$$u(x, t) = \sum_{n=1}^{\infty} C_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}, \qquad ...(17.44)$$

where C_n are constants to be determined using the initial condition at (17.43).

Substituting $t = 0$ in (17.44), we obtain

$$u(x, 0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}, \quad 0 \leq x \leq l. \qquad ...(17.45)$$

The series (17.45) is a Fourier half-range sine series expansion of $u(x, 0)$ in $(0, l)$, and hence the coefficients C_n's are given by $C_n = \dfrac{2}{l}\displaystyle\int_0^l u(x, 0)\sin\frac{n\pi x}{l}\,dx.$

In order for solution to satisfy the initial condition (17.43), we must have

$$C_n = \frac{2}{l}\left[\int_0^{l/3}\frac{3ax}{l}\sin\frac{n\pi x}{l}dx + \int_{l/3}^{2l/3}\frac{3a}{l}(l-2x)\sin\frac{n\pi x}{l}dx + \int_{2l/3}^{l}\frac{3a}{l}(x-l)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{6a}{l^2}\left[\left[x\left(\frac{-\cos(n\pi x/l)}{n\pi/l}\right) - 1\left(\frac{-\sin(n\pi x/l)}{(n\pi/l)^2}\right)\right]_0^{l/3} + \left[(l-2x)\left(-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right) - (-2)\left(-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right)\right]_{l/3}^{2l/3}\right.$$

$$\left. + \left[(x-l)\left(-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right) - (1)\left(-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right)\right]_{2l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right), = \frac{18a}{n^2\pi^2}\sin\frac{n\pi}{3}[1+(-1)^n], \text{since } \sin\frac{2n\pi}{3} = -(-1)^n \sin\frac{n\pi}{3}.$$

This gives $C_n = \begin{cases} 0, & \text{when } n \text{ is odd.} \\ \dfrac{36a}{n^2\pi^2}\sin\dfrac{n\pi}{3}, & \text{when } n \text{ is even.} \end{cases}$

Substituting the value for C_n in (17.44), we obtain the desired solution as

$$u(x,t) = \sum_{n=2,4,6}^{\infty} \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l},$$

or,
$$u(x,t) = \sum_{n=1}^{\infty} \frac{9a}{n^2\pi^2}\sin\frac{2n\pi}{3}\cos\frac{2n\pi ct}{l}\sin\frac{2n\pi x}{l}. \quad ...(17.46)$$

The displacement of the mid-point is obtained from (17.46) by substituting $x = l/2$, which gives $u(l/2, t) = 0$, since $\sin\dfrac{2n\pi x}{l} = 0$ at $x = l/2$ for all n. Hence the mid-point of the string is always at rest.

17.4 D' ALEMBERT'S SOLUTION OF THE WAVE EQUATION

The D' Alembert's method consists of finding the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad ...(17.47)$$

directly by change of variables.

Consider the transformation

$$\xi = x + ct \text{ and } \eta = x - ct. \quad ...(17.48)$$

It gives
$$\xi_x = \eta_x = 1 \text{ and } \xi_t = -\eta_t = c.$$

Also $u(x, t)$ becomes a function of ξ and η. By chain rule, we have

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta,$$
$$u_{xx} = (u_\xi + u_\eta)_\xi \xi_x + (u_\xi + u_\eta)_\eta \eta_x$$
$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}. \quad ...(17.49)$$

Similarly,
$$u_t = c(u_\xi - u_\eta) \text{ and,}$$
$$u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}). \quad ...(17.50)$$

Substituting from (17.49) and (17.50) in Eq. (17.47), we obtain

$$u_{\xi\eta} = \frac{\partial^2 u}{\partial\xi\partial\eta} = 0, \quad ...(17.51)$$

which can be solved directly by two successive integrations.

Integrating (17.51) with respect to ξ, we obtain

$$\frac{\partial u}{\partial \eta} = h(\eta),$$

where $h(\eta)$ is an arbitrary function of η. Integrating next with respect to η, we obtain

$$u = \int h(\eta)d\eta + \phi(\xi), \quad \text{...(17.52)}$$

where $\phi(\xi)$ is an arbitrary function of ξ.

Taking $\psi(\eta) = \int h(\eta)d\eta$, (17.52) becomes

$$u = \phi(\xi) + \psi(\eta), \text{ or, } u = \phi(x + ct) + \psi(x - ct), \quad \text{...(17.53)}$$

using (17.48).

The solution (17.53) is known as *D'Alembert's solution of the one-dimensional wave equation.*

In case the initial conditions are $u(x, 0) = f(x)$, and $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$,

then differentiating (17.53) w.r.t. t we obtain $\frac{\partial u}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$

and hence we have $\phi(x) + \psi(x) = f(x)$ and $c\phi'(x) - c\psi'(x) = g(x)$

or, $\phi'(x) + \psi'(x) = f'(x)$ and $\phi'(x) - \psi'(x) = \frac{1}{c}g(x)$.

Solving these for $\phi'(x)$ and $\psi'(x)$, we obtain

$$\phi'(x) = \frac{1}{2}f'(x) + \frac{1}{2c}g(x), \text{ and, } \psi'(x) = \frac{1}{2}f'(x) - \frac{1}{2c}g(x).$$

Integrating, we obtain

$$\left.\begin{aligned} \phi(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)ds + k(x_0) \\ \psi(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)ds - k(x_0) \end{aligned}\right\} \quad \text{...(17.54)}$$

where x_0 is arbitrary and k is a constant chosen such that $\phi(x) + \psi(x) = f(x)$.

Substituting (17.54) in (17.53), we obtain

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \quad \text{...(17.55)}$$

In case the initial velocity is zero, that is, $g(x) = 0$, then (17.55) reduces to

$$u(x, t) = \frac{1}{2}[f(x+ct) + f(x-ct)]. \qquad ...(17.56)$$

This result shows that even the two initial conditions are sufficient to determine the solution uniquely.

Also, as expected, the solution (17.27) of the wave equation obtained earlier by applying method of separation of variables is in confirmation with (17.56), for (17.27) can be expressed as

$$u(x, t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right) = \frac{1}{2}\sum_{n=1}^{\infty} C_n\left[\sin\frac{n\pi}{l}(x-ct) + \sin\frac{n\pi}{l}(x+ct)\right]$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{l}(x-ct) + \frac{1}{2}\sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{l}(x+ct). \qquad ...(17.57)$$

The two series on the right side of (17.57) can be obtained by substituting $x - ct$ and $x + ct$ respectively for the variable x in the Fourier sine series (17.22) for $f(x)$, and thus

$$u(x, t) = \frac{1}{2}[f^*(x-ct) + f^*(x+ct)],$$

f^* being the odd periodic extension of f over $[-l, l]$ with period $2l$.

Further (17.56) can be considered the solution for the *wave motion along an infinite string*. If long distances are involved, such as with sound waves in the ocean used to monitor temperature changes, then sometimes it can be modelled by an infinite string with no boundary conditions. It can be given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \qquad ...(17.58)$$

with initial conditions $u(x, 0) = f(x)$, and $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \ -\infty < x < \infty.$

However, the problem can be solved by the application of variable separation method also. Though there are no boundary conditions but to make the solution a bounded function, generally we need to impose certain boundary conditions.

Example 17.5: Use D'Alembert's method to find the displacement of a vibrating string with initial velocity zero and initial displacement $f(x) = k(\sin x - \sin 2x)$.

Solution: If $u(x, t)$ is the displacement, then $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ with initial conditions

$$u(x, 0) = f(x) = k(\sin x - \sin 2x) \text{ and } \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = 0.$$

By D' Alembert's method, refer to (17.26), the solution is

$$u(x,t) = \frac{1}{2}[f(x+ct)+f(x-ct)] = \frac{1}{2}[k\{\sin(x+ct) - \sin 2(x+ct) + \sin(x-ct) - \sin 2(x-ct)\}]$$

$$= \frac{k}{2}[\{\sin(x+ct) + \sin(x-ct)\} - \{\sin 2(x+ct) + \sin 2(x-ct)\}]$$

$$= k[\sin x \cos ct - \sin 2x \cos 2ct].$$

Obviously the given conditions are satisfied by the solution obtained.

Example 17.6: Use D' Alembert's method to find displacement of a vibrating string with initial displacement $f(x) = \sin x$ and initial velocity $g(x) = a$, where a is a constant.

Solution: If $u(x, t)$ is the displacement, then $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with initial conditions

$$u(x,0) = f(x) = \sin x \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = a.$$

By D' Alembert's method, refer to (17.55), the solution is

$$u(x,t) = \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x)dx = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} a\,dx$$

$$= \sin x \cos ct + at.$$

EXERCISE 17.2

1. In the vibrating string problem an elastic string of length l is fixed at $x = 0$ and at $x = l$. It is taken to the position $f(x) = A \sin \frac{2\pi x}{l}$ at $t = 0$ and then released. Find the displacement function of the string motion.
2. An elastic string of length l is fixed at both ends at $x = 0$ and $x = l$. At a distance 'd' units from the end $x = 0$, the string is transversely displaced to a distance 'd' and is released from rest when it is in that position. Find the expression for the displacement function $u(x, t)$.
3. An elastic string of length l which is fastened at its ends $x = 0$ and $x = l$ is released from its equilibrium position with initial velocity $g(x)$ given as

$$g(x) = \begin{cases} x, & 0 \le x \le l/3 \\ 0, & l/3 < x \le l. \end{cases}$$

Find the displacement of the string at any instant of time t.

4. Solve, $\quad \frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad t > 0;$

given that $\quad y(0,t) = y(2,t) = 0, \quad t \ge 0$

and, $\quad y(x, 0) = x(x-2), \quad \left.\dfrac{\partial y}{\partial t}\right|_{t=0} = g(x), \quad 0 \le x \le 2,$

where $\quad g(x) = \begin{cases} 0, & 0 \le x < 1/2, \ 1 < x \le 2 \\ 3, & 1/2 \le x \le 1 \end{cases}.$

5. Solve $\quad \dfrac{\partial^2 u}{\partial t^2} = 8\dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < 4, \quad t > 0;$

given that, $\quad y(0,t) = y(4,t) = 0, \quad t \ge 0$

and $\quad y(x, 0) = x^2(x-4), \quad \left.\dfrac{\partial y}{\partial t}\right|_{t=0} = 1, \quad 0 \le x \le 4.$

6. An elastic string of length 20 cm fixed at both ends is displaced from its position of equilibrium by imparting to each of its points an initial velocity given by

$$g(x) = \begin{cases} x, & 0 \le x \le 10 \\ 20 - x, & 10 < x \le 20 \end{cases}$$

x being the distance from one end. Determine the displacement function at any time t.

7. Show that the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0$$

subject to the conditions $u(0, t) = u(l, t) = 0, \quad t \ge 0, \quad u(x, 0) = f(x), \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = g(x)$

has the solution

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \cos \lambda_n t + B_n \sin \lambda_n t) \sin \frac{n\pi x}{l}\,; \lambda_n = \frac{n\pi x}{l},$$

where $\quad A_n = \dfrac{2}{l}\displaystyle\int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad$ and $\quad B_n = \dfrac{2}{cn\pi}\displaystyle\int_0^l g(x) \sin \frac{n\pi x}{l} dx.$

8. Use D' Alembert's method to find the displacement function of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection

(i) $f(x) = a(x - x^3)$ (ii) $f(x) = a \sin^2 \pi x.$

9. Use D' Alembert method to find the solution of the initial value problem defining the vibrations of an infinitely long elastic string when

(i) $f(x) = 0, \quad g(x) = \sin 3x,$

(ii) $f(x) = \sin 2x, \quad g(x) = \cos 2x,$

(iii) $f(x) = e^{-|x|}, \quad g(x) = \cos 4x.$

10. Solve the vibrating string problem when there is a resistance in the medium which is proportional to the velocity. If the initial displacement is $f(x)$ and the string starts from rest, then the bvp is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0, \quad 0 < a < 1$$

$$u(0, t) = u(\pi, t) = 0, \quad t > 0;$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi.$$

17.5 ONE-DIMENSIONAL HEAT FLOW EQUATION

Consider a straight thin bar or wire of constant cross-sectional area 'A' and of homogeneous material placed along the x-axis from 0 to l. Assume that the bar is insulated laterally and there is no heat flow along the surface. Thus, the heat flow is only along x-axis perpendicular to the cross-section of the bar. Let the density $\rho(\text{gm/cm}^3)$, specific heat $s(\text{cal/gm. deg})$ and the thermal conductivity $k(\text{cal/cm deg.sec})$ are constants and let $u(x, t)$ be the temperature at a distance x from O at time t.

Consider a typical segment of the bar of thickness δx between x and $x + \delta x$, as shown in Fig. 17.3, and let δu be the temperature change in this segment.

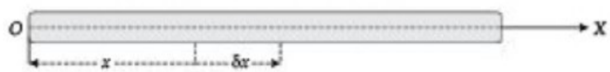


Fig. 17.3

If R_1 and R_2 respectively are the rates (cal/sec) of inflow and outflow of heat respectively at x and $x + \delta x$, then

$$R_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x \text{ and } R_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

and hence the net rate at which the heat enters this segment of the bar at time t is

$$R_1 - R_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right].$$

Under the assumption that there is no source or sink of heat in this segment, this must be equal to the rate at which the heat energy accumulates in this segment of width δx which is $s\rho A\delta x\,\frac{\partial u}{\partial t}$.

Hence, we have $\frac{\partial u}{\partial t} = \frac{k}{s\rho}\left[\frac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x}\right]$. Taking the limit as $\delta x \to 0$, we get

10. Solve the vibrating string problem when there is a resistance in the medium which is proportional to the velocity. If the initial displacement is $f(x)$ and the string starts from rest, then the bvp is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0, \quad 0 < a < 1$$

$$u(0, t) = u(\pi, t) = 0, \quad t > 0;$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi.$$

17.5 ONE-DIMENSIONAL HEAT FLOW EQUATION

Consider a straight thin bar or wire of constant cross-sectional area 'A' and of homogeneous material placed along the x-axis from 0 to l. Assume that the bar is insulated laterally and there is no heat flow along the surface. Thus, the heat flow is only along x-axis perpendicular to the cross-section of the bar. Let the density ρ(gm/cm^3), specific heat s(cal/gm. deg) and the thermal conductivity k(cal/cm deg.sec) are constants and let $u(x, t)$ be the temperature at a distance x from O at time t.

Consider a typical segment of the bar of thickness δx between x and $x + \delta x$, as shown in Fig. 17.3, and let δu be the temperature change in this segment.

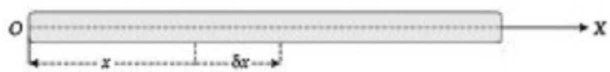


Fig. 17.3

If R_1 and R_2 respectively are the rates (cal/sec) of inflow and outflow of heat respectively at x and $x + \delta x$, then

$$R_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x \quad \text{and} \quad R_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

and hence the net rate at which the heat enters this segment of the bar at time t is

$$R_1 - R_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right].$$

Under the assumption that there is no source or sink of heat in this segment, this must be equal to the rate at which the heat energy accumulates in this segment of width δx which is $s\rho A\delta x \frac{\partial u}{\partial t}$.

Hence, we have $\frac{\partial u}{\partial t} = \frac{k}{s\rho}\left[\frac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x}\right]$. Taking the limit as $\delta x \to 0$, we get

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{...(17.59)}$$

where $c^2 = \frac{k}{s\rho}$ (cm²/sec) is called the *diffusivity* of the bar depending on its material.

This is *one-dimensional heat equation*.

The heat equation (17.59) together with certain initial and boundary conditions, which are associated with the physical model, uniquely determines the temperature distribution throughout the bar at any time $t > 0$.

For example, we may have the following initial boundary value problem:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0$$

with boundary conditions $u(0, t) = T_1, \quad u(l, t) = T_2, \quad t \geq 0,$

and, initial conditions $u(x, 0) = f(x), \quad 0 \leq x \leq l.$

17.6 SOLUTION OF THE HEAT EQUATION BY SEPARATION OF VARIABLES AND USE OF FOURIER SERIES

We solve the heat equation (17.59) for some specific boundary and initial conditions, using separation of variables and Fourier series. We consider the following cases.

Case I: *Ends of the bar kept at temperature zero*

If $u(x, t)$ is the temperature distribution in a thin, homogeneous bar of length l with both ends kept at zero temperature and $f(x)$ is the initial temperature distribution in the bar, then the boundary value problem modelling $u(x, t)$ is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0 \quad \text{...(17.60)}$$

$$u(0, t) = u(l, t) = 0, \; t \geq 0 \quad \text{...(17.61)}$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq l. \quad \text{...(17.62)}$$

Substituting $u(x, t) = X(x)T(t)$ in Eq. (17.60), we get

$$XT' = c^2 X'' T$$

or,

$$\frac{T'}{c^2 T} = \frac{X''}{X}, \quad \text{...(17.63)}$$

where the primes denote the derivatives w.r.t. the corresponding independent variables.

The left side of Eq. (17.63) depends only on t and the right side only on x, and since x and t are independent, so the both sides must be equal to a constant k. Thus (17.63) leads to two ordinary differential equations

$$X'' - kX = 0, \quad \text{...(17.64)}$$

and,

$$T' - kc^2 T = 0. \quad \text{...(17.65)}$$

To solve (17.64) and (17.65), we consider the following three cases.

(i) When k is positive, say $k = p^2$, then $X = c_1 e^{px} + c_2 e^{-px}$; $T = c_3 e^{c^2 p^2 t}$.

(ii) When k is negative, say $k = -p^2$, then $X = c_4 \cos px + c_5 \sin px$, $T = c_6 e^{-c^2 p^2 t}$.

(iii) When k is zero, then $X = c_7 x + c_8$, $T = c_9$.

Thus the various possible solutions of the heat equation are:

$$u(x, t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{c^2 p^2 t}) \quad ...(17.66)$$

$$u(x, t) = (c_4 \cos px + c_5 \sin px)(c_6 e^{-c^2 p^2 t}) \quad ...(17.67)$$

$$u(x, t) = (c_7 x + c_8) c_9 \quad ...(17.68)$$

Out of these we have to choose the solution consistent with the physical constraints of the model under study. Since we are dealing with the problem on heat condition the solution must be transient one, that is, u decreases with time t; and hence the solution (17.67) is the suitable one. We write

$$u(x, t) = (A \cos px + B \sin px)\, e^{-c^2 p^2 t} \quad ...(17.69)$$

as the solution of the heat equation, where A, B and p are constants to be determined by applying boundary and initial conditions.

Applying the boundary condition $u(0, t) = 0$ to (17.69) we obtain $A\, e^{-c^2 p^2 t} = 0$ for all t, which is true only if $A = 0$.

Further, using $u(l, t) = 0$ in (17.69) gives $(B \sin pl)\, e^{-c^2 p^2 t} = 0$, for all t. To obtain non-trivial solution B can't be 0 hence $\sin pl = 0$, which gives $p = \frac{n\pi}{l}$, n being an integer. We thus obtain

$$u_n(x, t) = B_n \sin \frac{n\pi x}{l}\, e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}.$$ By superposition principle, we obtain

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}, \quad 0 \le x \le l, \quad t > 0. \quad ...(17.70)$$

as the solution of the heat equation (17.60) satisfying the boundary conditions (17.61).

Applying the initial conditions (17.62) in (17.70), we obtain

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}, \quad 0 \le x \le l,$$

which is Fourier half-range sine series of $f(x)$ in $[0, l]$. Hence B_n are the Fourier coefficients given by

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx. \quad ...(17.71)$$

Thus, the solution of the heat equation (17.60) subject to the conditions (17.61) and (17.62) is given by (17.70), where the coefficients B_n are given by (17.71).

We observe that because of the exponential factor, all the terms in the series (17.70) approach zero as $t \to \infty$ and the rate of decay increases with n.

Case II: *Ends of the bar kept insulated*

When the ends of the bar are kept insulated then there is no heat loss across the ends. If the initial temperature is $f(x)$, then the boundary value problem modelling temperature distribution $u(x, t)$ is given as

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < l, \quad t > 0 \qquad \text{...(17.72)}$$

with boundary conditions

$$\frac{\partial u(0, t)}{\partial x} = \frac{\partial u(l, t)}{\partial x} = 0, \quad t > 0 \qquad \text{...(17.73)}$$

and initial condition

$$u(x, 0) = f(x), \quad 0 \le x \le l. \qquad \text{...(17.74)}$$

The appropriate solution of the heat equation (17.72), refer to (17.69), is

$$u(x, t) = (A \cos px + B \sin px) e^{-c^2 p^2 t}. \qquad \text{...(17.75)}$$

Differentiating (17.75) w.r.t. x, we obtain

$$\frac{\partial u}{\partial x} = (-A \sin px + B \cos px) p e^{-c^2 p^2 t}. \qquad \text{...(17.76)}$$

Using the condition $\frac{\partial u(0, t)}{\partial x} = 0$ in (17.76), we obtain $Bpe^{-c^2 p^2 t} = 0$, for all t. Since $p \neq 0$, thus $B = 0$, hence Eq. (17.76) becomes

$$\frac{\partial u}{\partial x} = -Ap \sin px \, e^{-c^2 p^2 t}. \qquad \text{...(17.77)}$$

Next using $\frac{\partial u(l, t)}{\partial x} = 0$ in (17.77), we obtain $-Ap \sin pl \, e^{-c^2 p^2 t} = 0$, for $t > 0$.

Since for non-zero solution of $u(x, t)$, A can't be zero and also $p \neq 0$ hence $\sin pl = 0$, which gives, $p = \frac{n\pi}{l}$, n being an integer. Hence the solution of the heat equation subject to the boundary conditions (17.73) is $u_n(x, t) = A_n \cos \frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$, n being an integer.

By the superposition principle the solution is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} \, e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}, \quad 0 < x < l, \quad t > 0. \qquad \text{...(17.78)}$$

Applying the initial condition (17.74), we obtain

$$u(x, 0) = f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l}, \quad 0 < x < l,$$

which is the Fourier half-range cosine series of $f(x)$ in $(0, l)$. Hence the coefficients A_n are the Fourier coefficients given by

$$A_0 = \frac{1}{l}\int_0^l f(x)dx, \quad A_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx, \ n = 1, 2, \ldots \qquad \ldots(17.79)$$

Thus, the solution of the heat equation (17.72) subject to the boundary conditions (17.73) and the initial condition (17.74) is given by (17.78), where the coefficients A_n are given by (17.79).

Example 17.7: Find the temperature distribution $u(x, t)$ in a laterally insulated bar 80 cm long if the initial temperature is $100 \sin (\pi x/80)$°C and the ends are kept at 0°C. How long will it take for the maximum temperature in the bar to drop to 50°C, given that $\rho = 8.92$ gm/cm^3, specific heat $s = 0.092$ cal/gm°C, and thermal conductivity $k = 0.95$ cal/cm sec°C?

Solution: The temperature distribution $u(x, t)$ is modelled as the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 80, \qquad \ldots(17.80)$$

subject to the boundary conditions

$$u(0, t) = 0, \quad u(80, t) = 0, \quad t \ge 0 \qquad \ldots(17.81)$$

and the initial condition

$$u(x, 0) = 100 \sin (\pi x/80), \quad 0 \le x \le 80. \qquad \ldots(17.82)$$

Here $c^2 = k/\rho s = 0.95/(8.92)(0.092) = 1.158$ cm^2/sec.

It is easy to see that solution of the heat equation (17.80) subject to the conditions (17.81), refer to Eq. (17.70), is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{80} e^{-\frac{1.158 n^2 \pi^2 t}{(80)^2}}, \quad 0 \le x \le 80, \quad t \ge 0 \qquad \ldots(17.83)$$

Using the condition (17.82), (17.83) gives $100 \sin (\pi x/80) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{80}$.

Comparing the coefficients of the like terms on both sides of this equation, we obtain $B_1 = 100, \quad B_2 = B_3 = \ldots = 0$, and thus (17.83) becomes

$$u(x, t) = 100 \sin \frac{\pi x}{80} e^{-\frac{1.158}{(80)^2}\pi^2 t} = 100 \sin \frac{\pi x}{80} e^{-.001782t}.$$

Further, $\quad u_{max} = 100 e^{-.001782t} = 50$, gives $t = 389$ sec $= 6.5$ minutes.

Example 17.8: A homogeneous laterally insulated bar of length 100 cm has its ends kept at zero temperature. Find the temperature distribution $u(x, t)$, if the initial temperature is

$$f(x) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 \le x \le 100. \end{cases}$$

Solution: The temperature distribution $u(x, t)$ is modelled as the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 100, \quad t > 0 \qquad ...(17.84)$$

subject to the boundary conditions $u(0, t) = 0, \quad u(100, t) = 0, \quad t \ge 0$ ...(17.85)

and the initial condition $f(x) = u(x,0) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 \le x \le 100. \end{cases}$ ...(17.86)

The solution of the heat equation (17.84) subject to the boundary conditions (17.85), refer to (17.70), is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{100} e^{-\frac{c^2 n^2 \pi^2 t}{(100)^2}}, \quad 0 \le x \le 100, \quad t \ge 0. \qquad ...(17.87)$$

Using the initial condition (17.86), (17.87) gives $f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{100}, \quad 0 \le x \le 100,$

which is Fourier half-range sine series expansion of $f(x)$ in the interval [0, 100]. Hence using (17.86) the coefficients B_n are given by

$$B_n = \frac{2}{100} \int_0^{100} f(x) \sin \frac{n\pi x}{100} dx = \frac{1}{50} \left[\int_0^{50} x \sin \frac{n\pi x}{100} dx + \int_{50}^{100} (100 - x) \sin \frac{n\pi x}{100} dx \right].$$

Integrating and simplifying, we obtain $B_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} \dfrac{400}{n^2 \pi^2}, & \text{if } n \text{ is odd} \end{cases}$

Substituting for B_n in (17.87), we get

$$u(x, t) = \frac{400}{\pi^2} \left[\sin \frac{\pi x}{100} e^{-\frac{c^2 \pi^2}{(100)^2} t} - \frac{1}{9} \sin \frac{3\pi x}{100} e^{-\frac{9c^2 \pi^2}{(100)^2}} + \frac{1}{25} \sin \frac{5\pi x}{100} e^{\frac{-25c^2 \pi^2 t}{(100)^2}} - \dots \right],$$

as the desired solution.

Example 17.9: Find the temperature distribution in a laterally insulated bar of length l with both the ends insulated and initial temperature in the rod being $\sin(\pi x/l)$.

Solution: If $u(x, t)$ is the temperature distribution in the bar, then it is modelled by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(17.88)$$

subject to the boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0, \qquad ...(17.89)$$

and initial condition

$$u(x, 0) = \sin\left(\frac{\pi x}{l}\right). \qquad ...(17.90)$$

The solution of the heat equation (17.88) subject to the boundary conditions (17.89), refer Eq. (17.78), is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2}{l^2} t}. \qquad ...(17.91)$$

Using the initial condition (17.90), (17.91) becomes $\sin\left(\frac{\pi x}{l}\right) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{l}$,

which is Fourier half-range cosine series of $\sin\frac{\pi x}{l}$ in the interval $[0, l]$. Hence

$$A_0 = \frac{1}{l}\int_0^l \sin\frac{\pi x}{l} dx \quad \text{and} \quad A_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l} dx. \text{ Thus}$$

$$A_0 = \frac{1}{l}\left[-\frac{l}{\pi}\cos\frac{\pi x}{l}\right]_0^l = -\frac{1}{\pi}\left[\cos\frac{\pi x}{l}\right]_0^l = \frac{2}{\pi}, A_1 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{\pi x}{l} dx = 0,$$

$$A_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l} dx = \frac{1}{l}\int_0^l \left[\sin\frac{(1+n)\pi x}{l} + \sin\frac{(1-n)\pi x}{l}\right]$$

$$= -\frac{1}{l}\left[\frac{l}{(1+n)\pi}\cos\frac{(1+n)\pi x}{l} + \frac{l}{(1-n)\pi}\cos\frac{(1-n)\pi x}{l}\right]_0^l, \quad n \neq 1$$

$$= -\frac{[(-1)^{n+1} - 1]}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{2[(-1)^{n+1} - 1]}{\pi(n^2 - 1)}, \quad n \neq 1$$

Thus,
$$A_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{-4}{\pi(n^2-1)}, & \text{if } n \text{ is even.} \end{cases}$$

Hence (17.91) becomes

$$u(x, t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)} \cos\frac{2n\pi x}{l} e^{-\frac{4c^2n^2\pi^2}{l^2}t},$$

the required temperature distribution.

Example 17.10: (a) A laterally insulated bar of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to and maintained at 0°C, find the temperature distribution in the rod at a distance x from A at time t.

(b) In case the temperature at A is raised to 20°C and that at B is reduced to 80°C, then what will be the temperature distribution.

Solution: (a) If $u(x, t)$ is the temperature distribution at a distance x from the fixed end A at any time $t > 0$, then it is modelled as the one-dimensional heat equation given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0. \qquad ...(17.92)$$

Now prior to the temperature change at the end B, the heat flow was independent of time, that is, the system was in steady state, and hence the Eq. (17.92) was reduced to $\frac{\partial^2 u}{\partial x^2} = 0$, which gives

$$u = ax + b, \qquad ...(17.93)$$

where a and b are arbitrary constants. Using the conditions $u(0) = 0$ and $u(l) = 100$, we obtain from (17.93), $a = \frac{100}{l}$ and $b = 0$, and hence the initial condition is given by

$$f(x) = u(x, 0) = \frac{100}{l}x. \qquad ...(17.94)$$

Also the boundary conditions for the subsequent flow are

$$u(0, t) = 0 \text{ and } u(l, t) = 0, \text{ for } t > 0. \qquad ...(17.95)$$

Thus, the temperature distribution will be the solution of Eq. (17.92) subject to initial condition (17.94) and boundary conditions (17.95). The solution of the one-dimensional heat equation (17.92) subject to the boundary conditions (17.95), refer to Eq. (17.70), is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{c^2n^2\pi^2}{l^2}t}. \qquad ...(17.96)$$

Using the initial condition (17.94) in (17.96), we obtain $\frac{100x}{l} = u(x, 0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$, which is the half-range sine series of $f(x) = \frac{100x}{l}$ in the interval $(0, l)$. Hence the coefficients B_n are given by

$$B_n = \frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}dx = \frac{200}{l^2}\left[-x\frac{l}{n\pi}\cos\frac{n\pi x}{l}+\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l}\right]_0^l$$

$$= -\frac{200}{n\pi}\cos n\pi = \frac{200}{n\pi}(-1)^{n+1}.$$

Substituting for B_n in (17.96), the required solution is

$$u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}e^{-\frac{c^2n^2\pi^2}{l^2}t}, \quad 0<x<l, \quad t \geq 0.$$

(b) Next, if the temperature at the end A is raised to 20°C and at the end B is lowered to 80°C, then the boundary conditions are

$$u(0, t) = 20, \quad u(l, t) = 80, \quad \text{for} \quad t>0 \qquad ...(17.97)$$

but the initial condition remains the same as (17.94).

Since the boundary values given by (17.97) are non-zero thus we need to modify the procedure. We consider the temperature function $u(x, t)$ as the sum of two parts given as

$$u(x, t) = u_s(x) + u_t(x, t) \qquad ...(17.98)$$

where $u_s(x)$ is the steady state solution of the heat equation (17.92) of the form (17.93), satisfying the boundary conditions (17.97) and $u_t(x, t)$ is the transient part of the temperature function defined by

$$u_t(x, t) = u(x, t) - u_s(x). \qquad ...(17.99)$$

We have, $u_s(0) = 20$ and $u_s(l) = 80$ and hence

$$u_s(x) = 20 + \frac{60}{l}x. \qquad ...(17.100)$$

Substituting $x = 0$ and $x = l$ in (17.99), we obtain respectively

and,
$$\left.\begin{aligned} u_t(0, x) &= u(0, t) - u_s(0) = 20 - 20 = 0 \\ u_t(l, x) &= u(l, t) - u_s(l) = 80 - 80 = 0. \end{aligned}\right\} \qquad ...(17.101)$$

Also, $u_t(x, 0) = u(x, 0) - u_s(x) = \dfrac{100x}{l} - \left(\dfrac{60x}{l}+20\right) = \dfrac{40x}{l} - 20.$ ...(17.102)

Now, we solve for the transient temperature function $u_t(x, t)$ as the solution of the heat equation (17.92) subject to the boundary conditions (17.101) and the initial condition (17.102).

As in part (a), the solution satisfying (17.101) is

$$u_t(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}e^{-\frac{c^2n^2\pi^2}{l^2}t} \qquad ...(17.103)$$

using the initial condition (17.102) in (17.103), we obtain

$$\left(\frac{40x}{l}-20\right)=u_t(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l},\quad 0<x<l$$

which is the half-range Fourier sine series expansion of the function $f(x)=20\left(\frac{2x}{l}-1\right)$ in the interval $(0, l)$ and hence the coefficients B_n are given by

$$B_n=\frac{2}{l}\int_0^l 20\left(\frac{2x}{l}-1\right)\sin\frac{n\pi x}{l}dx$$

$$=\frac{40}{l}\left[\left(-\frac{l}{n\pi}\left(\frac{2x}{l}-1\right)\cos\frac{n\pi x}{l}\right)+\left(\frac{l}{n\pi}\right)^2\frac{2}{l}\sin\frac{n\pi x}{l}\right]_0^l=-\frac{40}{n\pi}[1+\cos n\pi]=-\frac{40}{n\pi}[1+(-1)^n].$$

Thus, $B_n=\begin{cases}0, & \text{when } n \text{ is odd}\\ -\dfrac{80}{n\pi}, & \text{when } n \text{ is even}\end{cases}$ and hence (17.103) becomes

$$u_t(x,t)=-\frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}e^{-\frac{4c^2n^2\pi^2t}{l^2}}. \quad ...(17.104)$$

Thus the desired solution is obtained by combining (17.100) and (17.104) as

$$u(x,t)=u_s(x)+u_t(x,t)=20+\frac{60}{l}x-\frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}e^{-\frac{4c^2n^2\pi^2t}{l^2}}.$$

EXERCISE 17.3

1. Find the temperature distribution $u(x, t)$ in a laterally insulated bar 80 cm long if the initial temperature is $100\sin\left(\frac{3\pi x}{80}\right)$°C and the ends are kept at 0°C. How long will it take for the maximum temperature in the ball to drop to 50°C if the density, specific heat and thermal conductivity for the bar material are 8.92 gm/cm^3, 0.092 cal/gm°C and 0.5 cal/cm sec °C, respectively?
2. A rod of length l laterally insulated is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature distribution in the rod at any time t.
3. Find the temperature distribution in a laterally insulated bar of length l whose ends are kept at temperature 0°C assuming that the initial temperature is

$$f(x) = \begin{cases} x, & \text{if } 0 \le x \le l/2 \\ l - x, & \text{if } l/2 \le x \le l \end{cases}$$

4. Find the temperature distribution $u(x, t)$ in a laterally insulated bar of length 10 cm whose ends are kept at temperature 0°C when the initial temperature is $f(x) = x(10 - x)$ in °C. It is given that density is 10.6 gm/cm^3, specific heat is 0.056 cal/gm deg and thermal conductivity is 1.04 cal/cm deg.
5. A bar of length l laterally insulated has its ends A and B kept at 0° and u_0° respectively until steady-state conditions prevail. If the temperature at B is then suddenly reduced to 0° and kept so while that of B is maintained at 0°, find the temperature distribution in the bar at any subsequent time.
6. A bar 100 cm long, laterally insulated, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution at any subsequent time. Also show that the sum of the temperatures at any two points equidistant from the centre is always 100°C, a constant.
7. Solve $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, such that
   (i) u is not infinite, when $t \to \infty$,
   (ii) $\partial u / \partial x = 0$, when $x = 0$ and $u(x, t) = 0$, when $x = l$, for all t,
   (iii) $u(x, 0) = u_0$, for all t.
8. A bar AB of length 10 cm laterally insulated has its ends A and B kept at 30° and 100° temperatures respectively until steady-state prevails. Then the temperature at A is lowered to 20° and that of B to 40° and the ends are maintained at these temperatures. Find the temperature distribution in the bar at the subsequent time.

17.7 TWO-DIMENSIONAL HEAT FLOW EQUATION: THE LAPLACE'S EQUATION

Consider the flow of heat in a plate of heat conducting material of uniform thickness α(cm) with both faces insulated, so the flow is only along the plate. Let XOY plane be taken in one of face of the plate. Since the plate has been insulated on both of its surfaces, we may consider that temperature at any point is independent of the z-coordinate and depends only on x, y and time t and let it be denoted by $u(x, y, t)$. Such a flow is called *two-dimensional heat flow*. The flow is in the xy-plane only and is zero along the normal to the xy-plane. Further, let ρ(gm/cm^3), s(cal/gm °c) and k(cal/cm sec °c) be the density, specific heat and thermal conductivity of the material of the plate.

Consider a rectangular element $PQRS$ of the plate with sides δx and δy as shown in Fig. 17.4. Considering that the amount of heat that enters across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area, thus the amount of heat entering the element $\delta x\, \delta y$ per sec from the side PQ

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y.$$

Similarly, the amount of heat entering the element $\delta x \delta y$ per sec from the side PS

$$= -k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x.$$

Also the amount of heat flowing out through SR per sec

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y},$$

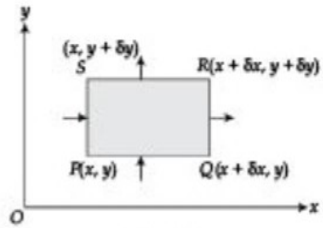


Fig. 17.4

and, the amount of heat flowing out through QR per sec $= -k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$.

Hence, the total gain of heat by the element $\delta x \delta y$ per sec

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right]. \qquad ...(17.105)$$

Also the gain of heat by the element $\delta x \delta y$ per sec

$$= \rho\alpha s\,\delta x\,\delta y\,\frac{\partial u}{\partial t}. \qquad ...(17.106)$$

Assuming that there are no source or sink of heat energy in the elements $\delta x \delta y$, equating (17.105) with (17.106), dividing both sides by $\alpha\delta y\,\delta y$ and taking the limit as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho s\frac{\partial u}{\partial t} \quad \text{or,} \quad c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t}, \qquad ...(17.107)$$

where $c^2 = \frac{k}{\rho s}$ (cm^2/sec) is called the *diffusivity* of the meterial. The Eq. (17.107) is called the *two-dimensional heat flow equation* and gives the temperature distribution in the xy-plane in the transient state.

17.7.1 The Steady State Heat Flow. The Laplace Equation

In the steady state when the temperature at a point is independent of time, so that $\frac{\partial u}{\partial t} = 0$, Eq. (17.107), reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(17.108)$$

which gives the temperature distribution in the xy-plane in the steady state. This is known as *Laplace equation in two dimensions.*

In case the heat flow is three-dimensional the equation corresponding to Eq. (17.107) is

$$c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial u}{\partial t}. \qquad ...(17.109)$$

This gives the temperature distribution in the xyz-space in the transient state, and is called the *three-dimensional heat flow equation.* In the steady state, Eq. (17.109) reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \qquad ...(17.110)$$

called the *Laplace equation in three dimensions.*

In addition to the heat flow in the steady state, the Laplace equation arises in problems involving potentials such as potentials for force fields in mechanics or electromagnetic or gravitational fields.

A function satisfying the Laplace equation in a certain region R is said to be *harmonic* on that region R. For example, $f(x, y) = x^2 - y^2$ satisfies the Eq. (17.108) over the entire plane. The theory of solution of Laplace's equation is called *potential theory.*

A steady state two-dimensional heat flow problem consists of Laplace Eq. (17.108) to be considered in some region R of the xy-plane along with the boundary conditions, say $u(x, y) = f(x, y)$ on the boundary curve of R. This forms the boundary value problem called the *Dirichlet problem.* The difficulty of a Dirichlet problem depends on the form of region R. We will consider this for rectangular and circular regions only.

17.8 SOLUTION OF LAPLACE'S EQUATION IN CARTESIAN COORDINATES

The equation modelling the steady state temperature distribution $u(x, y)$ in the rectangular region is given by the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b. \qquad ...(17.111)$$

We solve this equation by applying the method of separation of variables.

Let the solution be of the form

$$u(x, y) = X(x)Y(y). \qquad ...(17.112)$$

Substituting (17.112) in Eq. (17.111), we get $X''Y + XY'' = 0$, which gives

$$\frac{X''}{X} = -\frac{Y''}{Y}, \qquad ...(17.113)$$

where primes denote the derivatives w.r.t. the corresponding independent variables.

Since X''/X is a function of x only and Y''/Y is a function of y only and, x, y being independent, thus (17.113) holds when each side of (17.113) is equal to a constant, say k. Hence we have two differential equations $X'' - kX = 0$ and $Y'' + kY = 0$.

To solve these equations, we consider the following three cases.

(i) When k is positive, say $k = p^2$, then $X = c_1e^{px} + c_2e^{-px}$, $Y = c_3 \cos py + c_4 \sin py$

(ii) When k is negative, say $k = -p^2$, then $X = c_5 \cos px + c_6 \sin px$, $Y = c_7e^{py} + c_8e^{-py}$

(iii) When k is zero, then $X = c_9 x + c_{10}$, $Y = c_{11}y + c_{12}$.

Thus the three possible solutions of the Laplace equation (17.111) are

$$u(x, y) = (c_1e^{px} + c_2e^{-px})(c_3 \cos py + c_4 \sin py) \qquad ...(17.114)$$

$$u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7e^{py} + c_8e^{-py}) \qquad ...(17.115)$$

$$u(x, y) = (c_9x + c_{10})(c_{11}y + c_{12}). \qquad ...(17.116)$$

Of these we select the solution compatible with the boundary conditions given.

In the examples to follow we shall find the solution of the Laplace equation subject to different boundary conditions.

Example 17.11: Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b,$ ...(17.117)

subject to the boundary conditions $u(0, y) = u(a, y) = 0$ and, $u(x, 0) = 0$, $u(x, b) = u_0 \sin \dfrac{\pi x}{a}$.

Solution: The problem is represented as shown in Fig. 17.5.

The three possible solutions of the Laplace equation (17.117) are:

I. $u(x, y) = (c_1e^{px} + c_2e^{-px})(c_3 \cos py + c_4 \sin py)$

II. $u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7e^{py} + c_8e^{-py})$

III. $u(x, y) = (c_9x + c_{10})(c_{11}y + c_{12})$.

The solutions I and III don't comply with the boundary conditions. The solution II is the suitable one which is of the form

$$u(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py}) \qquad ...(17.118)$$

where A, B, C, D and p are constants to be determined.

Using condition $u(0, y) = 0$ in (17.118) gives

$$A(Ce^{py} + De^{-py}) = 0, \text{ for all } y \in (0, b)$$

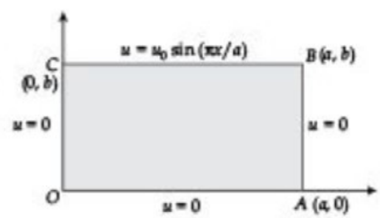


Fig. 17.5

which is satisfied only when $A = 0$, (since C and D both can't be zero). Hence (17.118) reduces to

$$u(x, y) = B \sin px(Ce^{py} + De^{-py}) \qquad \text{...(17.119)}$$

Using $u(a, y) = 0$, gives $0 = B \sin pa(Ce^{py} + De^{-py})$ for all $y \in (0, b)$, which gives $\sin pa = 0$, that is, $pa = n\pi$, or $p = \frac{n\pi}{a}$, n being an integer.

Further $u(x, 0) = 0$ gives, $0 = B \sin px(C + D) = 0$, for all $x \in (0, a)$ which gives, $C + D = 0$, or $D = -C$. Hence (17.119) becomes

$$u_n(x, y) = B_n \sin \frac{n\pi x}{a}\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right) = 2B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}, \quad n = 1, 2, \ldots$$

By superposition principle the solution is

$$u(x, y) = \sum_{n=1}^{\infty} 2B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}, \qquad \text{...(17.120)}$$

the coefficients B_n are to be determined using the boundary condition $u(x, b) = u_0 \sin \frac{\pi x}{a}$.

Using this in (17.120), we obtain $u_0 \sin \frac{\pi x}{a} = \sum_{n=1}^{\infty} 2B_n \sin \frac{\pi x}{a} \sinh \frac{n\pi b}{a}$

Comparing coefficients on both sides we obtain $B_1 = \frac{u_0}{2 \sinh (\pi b/a)}$, and $B_2 = B_3 = \ldots = 0$

and hence (17.120) becomes $u(x, y) = \frac{u_0}{\sinh (\pi b/a)} \sin \frac{\pi x}{b} \sinh \frac{\pi y}{a}$, the desired solution.

Example 17.12: Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$ ...(17.121)

subject to the boundary conditions $u(x, 0) = u(x, b) = 0$ and $u(0, y) = 0$, $u(a, y) = \pi y(b - y)$.

Solution: The problem is represented as shown in Fig. 17.6.

The three possible solutions of the Laplace equation (17.121) are:

I. $u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

II. $u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

III. $u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$.

It is easy to see that subject to the given boundary conditions, the solutions II and III lead to trivial solutions. Thus the suitable solution is I which is of the form

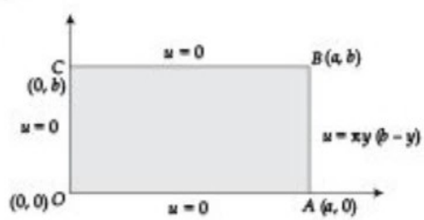


Fig. 17.6

$$u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py), \quad ...(17.122)$$

where A, B, C, D and p are constants to be determined.

Using $u(x, 0) = 0$ in (17.122) gives $0 = (Ae^{px} + Be^{-px})C$, for all $x \in (0, a)$ which implies $C = 0$, and hence (17.122) becomes

$$u(x, y) = D(Ae^{px} + Be^{-px})\sin py. \quad ...(17.123)$$

Using $u(x, b) = 0$ in (17.123), gives $0 = D(Ae^{px} + Be^{-px})\sin pb$ for all $x \in (0, a)$, which implies $\sin pb = 0$ (since D can't be zero), that is, $pb = n\pi$, or $p = \frac{n\pi}{b}$, where n is any integer

Further using $u(0, y)$ in (17.123), we obtain $0 = D(A + B)\sin py$, for all $y \in (0, b)$ which gives $A + B = 0$, that is $A = B$; and hence (17.123) becomes $u_n(x, y) = D_n \sin \frac{n\pi y}{b}\left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right)$. By superposition principle, the solution is

$$u(x, y) = \sum_{n=1}^{\infty} 2D_n \sin\frac{n\pi y}{b}\sinh\frac{n\pi x}{b}. \quad ...(17.124)$$

Using the boundary condition $u(a, y) = \pi y(b - y)$ in (17.124), gives

$$\pi y(b - y) = \sum_{n=1}^{\infty}\left(2D_n \sinh\frac{n\pi a}{b}\right)\sin\frac{n\pi y}{b},$$

which is the half-range sine series of $f(y) = \pi y(b - y)$ in the interval $(0, b)$; hence the coefficients $2D_n \sinh\frac{n\pi a}{b}$ are given by $2D_n \sinh\frac{n\pi a}{b} = \frac{2\pi}{b}\int_0^b y(b - y)\sin\frac{n\pi y}{b}\,dy$. Consider

$$I = \int_0^b y(b - y)\sin\frac{n\pi y}{b}\,dy$$

$$= \left[y(b - y)\left(\frac{-b}{n\pi}\right)\cos\frac{n\pi y}{b} - (b - 2y)\left(\frac{-b^2}{n^2\pi^2}\right)\sin\frac{n\pi y}{b} + (-2)\left(\frac{b^3}{n^3\pi^3}\right)\cos\frac{n\pi y}{b}\right]_0^b$$

$$= \frac{2b^3}{n^3\pi^3}[1 - (-1)^n].$$

Hence, $2D_n \sinh\frac{n\pi a}{b} = \begin{cases} \frac{8b^2}{n^3\pi^2}, & \text{when } n \text{ is odd.} \\ 0, & \text{when } n \text{ is even.} \end{cases}$

Substituting in (17.124), we get

$$u(x, y) = \frac{8b^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3 \sinh \frac{(2n-1)\pi a}{b}} \sin \frac{(2n-1)\pi y}{b} \sinh \frac{(2n-1)\pi x}{b}$$

as the desired solution.

Example 17.13: A long rectangular plate of width π cm with insulated surfaces has its temperature equal to zero on both the long sides and one of the short sides so that $u(0, y) = 0$, $u(\pi, y) = 0$, $u(x, \infty) = 0$, $u(x, 0) = kx$. Find the steady state temperature within the plate.

Solution: The steady state temperature $u(x, y)$ at any point is given by the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad ...(17.125)$$

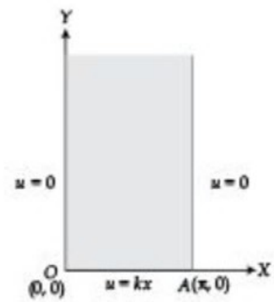


Fig. 17.7

The boundary conditions, as shown in Fig. 17.7 are

$$u(0, y) = 0, \quad \text{for} \quad 0 < y < \infty$$
$$u(\pi, y) = 0, \quad \text{for} \quad 0 < y < \infty$$
$$u(x, \infty) = 0, \quad \text{for} \quad 0 < x < \pi$$
$$u(x, 0) = kx, \quad \text{for} \quad 0 < x < \pi.$$

The three possible solutions of the Laplace equation are

I. $u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

II. $u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

III. $u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$.

From the condition that $u = 0$ as y tends to infinity for all values x, we observe that solutions I and III lead to trivial solution and hence solution II is the only suitable one, which is of the form

$$u(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py}) \qquad ...(17.126)$$

Using the boundary condition $u(0, y) = 0$, (17.126) gives $0 = A\,(Ce^{py} + De^{-py})$

which implies $A = 0$, and hence (17.126) reduces to the form

$$u(x, y) = \sin px(C'e^{py} + D'e^{-py}). \qquad ...(17.127)$$

Next using the condition $u(\pi, y) = 0$, (17.127) gives $\sin p\pi = 0$, or $p = n$, n being an integer.

Also using $u(x, \infty) = 0$ in (17.127) implies $C' = 0$.

Substituting $p = n$ and $C' = 0$ in (17.127), we obtain $u_n(x, y) = D \sin nxe^{-ny}$, n being integer and hence the general solution is of the form

$$u(x, y) = \sum_{n=1}^{\infty} D_n \sin nxe^{-ny}. \qquad ...(17.128)$$

Using the boundary condition $u(x, 0) = kx$ in (17.128), we obtain $kx = \sum_{n=1}^{\infty} D_n \sin nx, \quad 0 < x < a,$

which is half-range sine series expansion of $f(x) = kx$ in the interval $[0, \pi]$ hence the coefficients D_n are given by

$$D_n = \frac{2}{\pi}\int_0^{\pi} kx \sin nx\, dx = \frac{2k}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{2k}{n}(-1)^{n+1}.$$

Substituting for D_n in (17.128), we get $u(x, y) = 2k\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx e^{-ny}$, as the desired solution.

EXERCISE 17.4

1. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\quad 0 < x < a,\quad 0 < y < b$, subject to the boundary conditions

$$u(0, y) = u(a, y) = 0,\quad u(x, b) = 0 \quad\text{and}\quad u(x, 0) = f(x).$$

2. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\quad 0 < x < \pi,\quad y > 0$, subject to the boundary conditions

$$u(0, y) = u(\pi, y) = 0,\quad u(x, 0) = 1 \text{ and } \quad u(x, y) \to 0 \text{ as } y \to \infty.$$

3. A rectangular plate with insulated surface is 8 cm wide and is infinitely long. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 100 \sin(\pi x/8)$, $0 < x < 8$, while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plate is given by

$$u(x, y) = 100\, e^{-\pi y/8} \sin(\pi x/8).$$

4. Find the steady state temperature in a rectangular plate when the sides $x = 0$, $x = a$, $y = b$ are insulated while the edge $y = 0$ is kept at temperature $k \cos(\pi x/a)$.

5. A rectangular plate with insulated surface is 10 cm wide and is much long as compared to its width. If the temperature at the short edge $y = 0$ is given by

$$f(x) = \begin{cases} 20x, & \text{for } 0 \le x \le 5 \\ 20(10 - x), & \text{for } 5 \le x \le 10 \end{cases}$$

and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C. Show that steady state temperature distribution $u(x, y)$ at any point (x, y) in the plate is given by

$$u(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{10}e^{-(2n-1)\pi y/10}.$$

6. Find the steady-state temperature distribution in a thin sheet of metal plate which occupies the semi-infinite strip, $0 \le x \le a$ and $0 \le y < \infty$, when the edge $y = 0$ is kept at temperature $f(x) = kx(a - x)$, $0 < x < a$, while
   (i) the edges $x = 0$ and $x = a$ are kept at zero temperature,
   (ii) the edges $x = 0$ and $x = a$ are insulated,

   assuming in both the cases that $u(x, \infty) = 0$.
7. The temperature u is maintained at $0°$ along three edges of a square plate of length 100 cm and the fourth edge is maintained at $100°$ until steady-state conditions prevail. Find an expression for the temperature $u(x, y)$ at any point (x, y) of the plate. Hence, show that the temperature at the centre of the plate is
$$\frac{200}{\pi}\left[\frac{1}{\cosh(\pi/2)} - \frac{1}{3\cosh(3\pi/2)} + \frac{1}{5\cosh(5\pi/2)} - \dots\right].$$
8. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, $0 < x < a$, $0 < y < b$, subject to the boundary conditions
$$u(0, y) = u(a, y) = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=b} = 0 \quad \text{and} \quad u(x, 0) = f(x).$$
9. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, $0 < x < a$, $0 < y < b$, subject to the boundary conditions

   $u(x, 0) = 0$, $u(0, y) = u(a, y) = 0$ and $u(x, b) = (a - x)\sin x$.
10. Solve for the steady-state temperature distribution in a thin, flat plate covering the rectangle $0 \le x \le 4$, $0 \le y \le 1$, if the temperature on the horizontal sides is zero, while on the left side it is $f(y) = \sin \pi y$ and on the right side it is $f(y) = y(1 - y)$.

17.9 SOLUTION OF LAPLACE'S EQUATION IN POLAR COORDINATES

In case we need to solve the steady-state temperature distribution problem for a disc of radius R, then, in general, it is convenient to employ polar coordinates (r, θ) related to (x, y) by $x = r\cos\theta$, $y = r\sin\theta$. The Laplace equation in the cartesian form $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ is then replaced by its polar form given by
$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0, \qquad \text{...(17.129)}$$
see Appendix III.

Using the method of separation of variables, let the solution of Eq. (17.129) be of the form

$$u(r, \theta) = R(r)\Theta(\theta), \qquad ...(17.130)$$

where R is a function of r only and Θ that of θ only. Substituting (17.130) in the Eq. (17.129), we get

$$r^2R''\Theta + rR'\Theta + R\Theta'' = 0,$$

where primes denote the derivatives w.r.t. the corresponding independent variables.

Separating the variables, we obtain

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta}. \qquad ...(17.131)$$

Since the left side of Eq. (17.131) is a function of r only and right a function of θ only and, r and θ being independent, thus (17.131) holds when each side is equal to a constant, say k. This leads to two differential equations

$$r^2R'' + rR' - kR = 0 \qquad ...(17.132)$$

and,

$$\Theta'' + k\Theta = 0. \qquad ...(17.133)$$

Substituting $r = e^z$ in Eq. (17.132), it reduces to

$$\frac{d^2R}{dz^2} - kR = 0, \qquad ...(17.134)$$

a differential equation with constant coefficients.

To solve Eqs. (17.133) and (17.134), we consider the following cases

(a) When k is positive, say $k = p^2$, then

$$R = c_1e^{pz} + c_2e^{-pz} = c_1r^p + c_2r^{-p},\ \Theta = c_3\cos p\theta + c_4\sin p\theta.$$

(b) When k is negative, say $k = -p^2$, then

$$R = c_5\cos pz + c_6\sin pz = c_5\cos(p\ln r) + c_6\sin(p\ln r),\ \Theta = c_7e^{p\theta} + c_8e^{-p\theta}.$$

(c) When k is zero, then

$$R = c_9z + c_{10} = c_9\ln r + c_{10}\ \Theta = c_{11}\theta + c_{12}.$$

Thus the three possible solutions of the Laplace Eq. (17.129) are:

I. $u(r, \theta) = (c_1r^p + c_2r^{-p})(c_3\cos p\theta + c_4\sin p\theta)$

II. $u(r, \theta) = [c_5\cos(p\ln r) + c_6\sin(p\ln r)](c_7e^{p\theta} + c_8e^{-p\theta})$

III. $u(r, \theta) = (c_9\ln r + c_{10})(c_{11}\theta + c_{12})$.

Of these we select the solution(*s*) compatible with the boundary conditions given. In the examples to follow we shall solve the Laplace equation (17.129) subject to the various boundary conditions.

Example 17.14: Find the strady-state temperature distribution $u(r, \theta)$ of a semicircular plate of radius a which is such that its circumference is kept at temperature $k\theta(\pi - \theta)$ and the boundary diameter at zero temperature. Assuming that the lateral surface of the plate are insulated.

Solution: The steady-state temperature distribution $u(r, \theta)$ of a plate insulated laterally at any point (r, θ) is given by the Laplace equation

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0,\quad 0\le r\le a,\quad 0\le\theta\le\pi. \qquad ...(17.135)$$

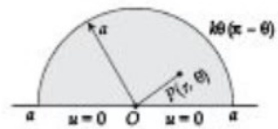


Fig. 17.8

The boundary conditions, as shown in Fig. 17.8, are

$$u(r, 0) = 0,\quad 0\le r\le a \qquad ...(17.136)$$

$$u(r, \pi) = 0,\quad 0\le r\le a \qquad ...(17.137)$$

$$u(a, \theta) = k\theta(\pi-\theta),\quad 0\le\theta\le\pi. \qquad ...(17.138)$$

The three possible solutions of the Laplace equation (17.135) are:

I. $u(r, \theta) = (c_1 r^p + c_2 r^{-p})(c_3\cos p\theta + c_4\sin p\theta)$

II. $u(r, \theta) = [c_5\cos(p\ln r) + c_6\sin(p\ln r)][c_7 e^{p\theta} + c_8 e^{-p\theta}]$

III. $u(r, \theta) = (c_9\ln r + c_{10})(c_{11}\theta + c_{12})$.

The boundary conditions (17.136) and (17.137) implies that u tends to zero as $r\to 0$ and hence the solutions II and III are not suitable. We consider the solution I which is of the form

$$u(r, \theta) = (Ar^p + Br^{-p})(C\cos p\theta + D\sin p\theta), \qquad ...(17.139)$$

where A, B, C, D and p are constants to be determined.

Using (17.136), (17.139) implies $C = 0$ for a non-zero solution, and hence (17.139) becomes

$$u(r, \theta) = (A'r^p + B'r^{-p})\sin p\theta. \qquad ...(17.140)$$

Next using (17.137) in (17.140) implies $\sin p\pi = 0$, which gives $p = n$, where n is any integer. Also since, $u = 0$ when $r\to 0$, hence from (17.140), we must have $B' = 0$. Thus (17.140) becomes

$$u_n(r, \theta) = A_n r^n \sin n\theta,$$

n being an integer. Hence the general solution is

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta, \qquad ...(17.141)$$

where the coefficients A_n's are to be determined using the condition (17.138). This gives

$$k\theta(\pi-\theta) = \sum_{n=1}^{\infty} A_n a^n \sin n\theta,\quad 0\le\theta\le\pi,$$

a half-range sine series expansion of the function $f(\theta) = k\theta(\pi-\theta)$ in the interval $[0, \pi]$. Thus the coefficients $A_n a^n$ are given by

$$A_n a^n = \frac{2}{\pi}\int_0^{\pi} k\theta(\pi-\theta)\sin n\theta\, d\theta$$

$$= \frac{2k}{\pi}\left[\theta(\pi-\theta)\left(-\frac{\cos n\theta}{n}\right)-(\pi-2\theta)\left(-\frac{\sin n\theta}{n^2}\right)+(-2)\left(\frac{\cos n\theta}{n^3}\right)\right]_0^{\pi} = \frac{4k}{\pi n^3}[1-(-1)^n]$$

or, $$A_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8k}{\pi a^n n^3}, & \text{when } n \text{ is odd} \end{cases} \qquad ...(17.142)$$

Using (17.142) in (17.141), we get

$$u(r,\theta) = \frac{8k}{\pi}\sum_{n=1,3,5}^{\infty}\left(\frac{r}{a}\right)^n \frac{1}{n^3}\sin n\theta = \frac{8k}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{2n-1}\frac{1}{(2n-1)^3}\sin(2n-1)\theta,$$

as the desired solution.

Example 17.15: Solve the Laplace equation

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0, \quad 0\le r\le a, \quad 0\le\theta\le 2\pi. \qquad ...(17.143)$$

subject to the condition $u(a,\theta)=f(\theta)$, $-\infty<\theta<\infty$. ...(17.144)

Solution: The solutions of the Laplace equation (17.143) are

I. $u(r,\theta)=(c_1r^p+c_2r^{-p})(c_3\cos p\theta+c_4\sin p\theta)$

II. $u(r,\theta)=[c_5\cos(p\ln r)+c_6\sin(p\ln r)][c_7e^{p\theta}+c_8e^{-p\theta})$

III. $u(r,\theta)=(c_9\ln r+c_{10})(c_{11}\theta+c_{12})$.

We need to find the solution of Eq. (17.143) subject to the condition (17.144). The *p.d.e.* (17.143) is of order two so we need additional boundary conditions to find the solution.

First reasonable condition we impose is that

$$u(r,\theta) \text{ is finite as } r\to 0, \qquad ...(17.145)$$

and, another that $u(r,\theta)$ is the single valued function of θ, and hence

$$u(r,\theta+2\pi)=u(r,\theta). \qquad ...(17.146)$$

Thus the possible solution is of the form I. Consider the solution as

$$u(r,\theta)=(Ar^p+Br^{-p})(C\cos p\theta+D\sin p\theta), \qquad ...(17.147)$$

where A, B, C, D and p are constants.

Using condition (17.145) in (17.147), we obtain $B=0$, thus (17.147) becomes

$$u(r,\theta)=r^p(C'\cos p\theta+D'\sin p\theta). \qquad ...(17.148)$$

Next, we subject (17.148) to the condition (17.146). Obviously $\cos p\theta$ and $\sin p\theta$ are periodic but we are to determine p such that these are 2π-periodic functions.

First we find p such that $\cos p(\theta+2\pi)=\cos p\theta$, for all θ, which implies

$$\cos p\theta\cos 2\pi p-\sin p\theta\sin 2\pi p=\cos p\theta.$$

Equating the coefficients of $\cos p\theta$ and $\sin p\theta$, we have

$$\left.\begin{aligned} &\cos 2p\pi = 1, \text{ which gives } p = 0, 1, 2, \ldots \\ \text{and,}\quad &\sin 2p\pi = 0, \text{ which gives } p = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots \end{aligned}\right\} \quad \text{...(17.149)}$$

Since both conditions at (17.149) need to be held simultaneously, we select the common values of p and hence $p = 0, 1, 2, \ldots$

Similarly, considering $\sin p(\theta + 2\pi) = \sin p\theta$, for all θ leads to $p = 0, 1, 2, \ldots$

Thus by superposition principle the solution (17.148) can be expressed as

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n (C_n \cos n\theta + D_n \sin n\theta) = C_0 + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + D_n \sin n\theta). \quad \text{...(17.150)}$$

Using condition (17.144) in (17.150), we obtain

$$f(\theta) = C_0 + \sum_{n=1}^{\infty} a^n (C_n \cos n\theta + D_n \sin n\theta), -\infty < \theta < \infty. \quad \text{...(17.151)}$$

We observe that (17.151) is the Fourier series expansion of the function $f(\theta)$ with period 2π, and hence

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta,\; C_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta, \text{ and } D_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta \quad \text{...(17.152)}$$

Thus, the solution of the given boundary value problem (17.143) is (17.150), where C_0, C_n and D_n are given by (17.152). The value of $u(r, \theta)$ as $r \to 0$ is given by C_0.

Example 17.16: Solve the Laplace equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 \le r \le 4, \quad -\pi \le \theta \le \pi, \quad \text{...(17.153)}$$

subject to the conditions

(i) $u(r, \theta)$ is finite as $r \to 0$,

(ii) $u(r, \theta) = u(r, \theta + 2\pi)$,

(iii) $u(4, \theta) = \theta^2$ for $-\pi \le \theta \le \pi$.

Solution: Proceeding as in Example 17.15, the solution of the Laplace Equation (17.153) subject to the given boundary conditions is

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta), \quad \text{...(17.154)}$$

where $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta,\ A_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta,$ and $B_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta.$

Here, $f(\theta) = \theta^2$ and $a = 4$. $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta^2 d\theta = \frac{\pi^2}{3}$,

$$A_n = \frac{1}{\pi 4^n}\int_{-\pi}^{\pi}\theta^2\cos n\theta d\theta = \frac{4(-1)^n}{n^2 4^n},\ B_n = \frac{1}{\pi 4^n}\int_{-\pi}^{\pi}\theta^2\sin n\theta\, d\theta = 0.$$

Substituting for A_0, A_n and B_n in (17.154), the required solution becomes

$$u(r,\theta) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{r}{4}\right)^n \cos n\theta.$$

Example 17.17: Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ x^2 + y^2 < 9$, subject to the condition

$$u(x, y) = x^2y^2, \text{ for } x^2 + y^2 = 9.$$

Solution: Using the transformation $x = r\cos\theta, y = r\sin\theta$ the boundary value problem becomes

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0,\quad 0 < r < 3, \qquad ...(17.155)$$

subject to $u(r,\theta) = r^4\cos^2\theta\sin^2\theta = 81\cos^2\theta\sin^2\theta = \frac{81}{4}\sin^2 2\theta = f(\theta)$, say.

The solution of Eq. (17.155) subject to the conditions that $u(r, \theta)$ is finite as $r \to 0$ and $u(r,\theta) = u(r, \theta + 2\pi)$, refer to Example 17.15, is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}(A_n\cos n\theta + B_n\sin n\theta), \qquad ...(17.156)$$

where $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta,\ A_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta$ and $B_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta.$

Here $f(\theta) = \frac{81}{4}\sin^2 2\theta$ and $a = 3$. Thus,

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{81}{4}\sin^2 2\theta d\theta = \frac{81}{8\pi}\left[\int_0^{\pi}(1-\cos 4\theta)\right] = \frac{81}{8}.$$

$$A_n = \frac{1}{\pi 3^n}\int_{-\pi}^{\pi}\frac{81}{4}\sin^2 2\theta\cos n\theta\, d\theta = \frac{81}{4\pi 3^n}\int_{-\pi}^{\pi}\sin^2 2\theta\cos n\theta\, d\theta.$$

$$= \frac{81}{8\pi 3^n}\left[\int_{-\pi}^{\pi}[\cos n\theta - \cos 4\theta\cos n\theta]\, d\theta\right.$$

Since, $\int_{-\pi}^{\pi}\cos n\theta = 0$ and $\int_{-\pi}^{\pi}\cos 4\theta\cos n\theta\, d\theta = \begin{cases}\pi, & \text{if } n = 4\\ 0, & \text{otherwise}\end{cases}$

thus, $\quad A_n = -\frac{81}{8.3^n}$, if $n = 4$, otherwise, zero,

Proceeding on similar lines, we obtain $B_n = \frac{1}{\pi 3^n}\int_{-\pi}^{\pi}\frac{81}{4}\sin^2 2\theta\sin n\theta\, d\theta = 0$, for all n.

Substituting for A_0, A_n and B_n in (17.156), we obtain

$$u(r, \theta) = \frac{81}{8} - \frac{1}{8}r^4\cos 4\theta, \qquad ...(17.157)$$

as the solution of the boundary value problem in polar coordinates.

To convert (17.157) in terms of cartesian coordinates, we use $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

Thus, $u(r, \theta) = \frac{81}{8} - r^4\cos^4\theta + r^4\cos^2\theta - \frac{r^4}{8}$. Substituting, $x = r\cos\theta$ and $r^2 = x^2 + y^2$, it gives

$$u(x, y) = \frac{81}{8} - x^4 + (x^2 + y^2)x^2 - \frac{(x^2 + y^2)^2}{8}$$

as the desired solution.

EXERCISE 17.5

1. In a semicircular plate with insulated faces of radius a, bounding diameter at 0°C and the semicircular boundary at 100°C, show that the steady state temperature distribution is given by $u(r, \theta) = \frac{400}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{2n-1}\frac{\sin(2n-1)\theta}{(2n-1)}$.
2. A semicircular plate of radius 10 cm has insulated faces with bounding diameter at 0°C and circumference temperature distribution is $u(10, \theta) = (400/\pi)(\pi\theta - \theta^2)$, $0 \le \theta \le \pi$. Determine the steady state temperature distribution of the plate at any point (r, θ) of the plate.
3. The bounding diameter of a semicircular plate of radius a is kept at 0°C and the temperature along the semicircular boundary is given by

$$u(a,\theta)=\begin{cases}50\theta, & \text{when } 0\le\theta<\pi/2\\ 50(\pi-\theta), & \text{when } \pi/2\le\theta\le\pi.\end{cases}$$

Show that the steady state temperature distribution is given by

$$u(r,\theta)=\frac{200}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{r}{a}\right)^{2n-1}\frac{\sin(2n-1)\theta}{(2n-1)^2},$$

assuming the lateral surfaces of the plate to be insulated.

4. A plate laterally insulated in the shape of truncated quadrant of a circle is bounded by $r=a$, $r=b$ and $\theta=0$, $\theta=\pi/2$. The plate is kept at temperature 0°C along three of the edges while along the edge $r=a$, it is kept at temperature $\theta(\pi/2-\theta)$. Determine the steady-state temperature distribution.

5. Solve the Laplace equation $r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial\theta^2}=0,\quad 0\le r\le b,\ -\infty<\theta<\infty,$

   subject to the boundary conditions

   (i) $u(r,\theta)$ is finite as $r\to 0$,

   (ii) $u(r,\theta+2\pi)=u(r,\theta)$,

   (iii) $u(a,\theta)=f(\theta)=\begin{cases}100, & 0<\theta<\pi\\ 0, & \pi<\theta<2\pi.\end{cases}$

6. Solve for $u(r,\theta)$ when

   $$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial\theta^2}=0,\quad 0<r<1,\ -\infty<\theta<\infty,$$

   and $u(r,\theta)$ is bounded and 2π periodic with $u(1,\theta)=50+20\cos\theta$.

7. Determine the steady-state temperature $u(r,\theta)$ in the annulus region $2\le r\le 4$ with the temperature along the boundaries given by $u(2,\theta)=6\cos\theta+10\sin\theta$ and $u(4,\theta)=15\cos\theta+17\sin\theta$.

8. Solve for $u(r,\theta)$ when

   $$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial\theta^2}=0,\quad 1\le r\le 2,\quad 0\le\theta\le\pi, \text{ with}$$

   $$u(1,\theta)=\left.\frac{\partial u}{\partial\theta}\right|_{(r,0)}=\left.\frac{\partial u}{\partial\theta}\right|_{(r,\pi)}=0 \quad\text{and}\quad u(2,\theta)=100.$$

17.10 VIBRATING MEMBRANE: TWO-DIMENSIONAL WAVE EQUATION

Consider the small transverse vibrations of an elastic stretched membrane over a flat frame in the *xy*-plane such as that of a drumhead. This may be considered the two-dimensional analogous of the vibrating string problem. We shall assume the membrane to be homogeneous, perfectly elastic which offers no resistance to bending. Also the tension per unit length T caused by stretching the membrane is the same at all points and in all directions and remains unchanged during the motion. Further, we assume that the deflections of the membrane during the motion are small resulting in small angles of inclination as compared to the size of the membrane.

Let $u(x, y, t)$ denote the transverse displacement of the membrane from its equilibrium position at time t. To derive the equation of motion we consider the forces acting on a small portion of the membrane. Since the deflections of the membrane and the angles of inclinations are small, the sides of the portion can be approximately taken equal to Δx and Δy, refer Figs. 17.9a & b. Thus, the forces acting on the sides of the portion are equal to $T\Delta x$ and $T\Delta y$, where the tension T is the force per unit length. Since the membrane is perfectly elastic, therefore, the direction of these forces is tangential to the membrane, refer Fig. 17.9b.

Considering the horizontal components of the forces, the components along x direction, refer to Fig. 17.9c, in the xu plane are:

$$T\Delta y \cos\beta \quad \text{and} \quad -T\Delta y \cos\alpha \qquad ...(17.158)$$

and in the y-direction, refer Fig. 17.9d, in the yu-plane, are:

$$T\Delta x \cos\delta \quad \text{and} \quad -T\Delta x \cos\gamma. \qquad ...(17.159)$$

Since α, β, γ and δ are small so their cosines are approximately equal to one. Hence the resultant horizontal component along the axes individually become zero. Thus we may regard the motion of the membrane as transversal, that is, each particle moves only vertically.

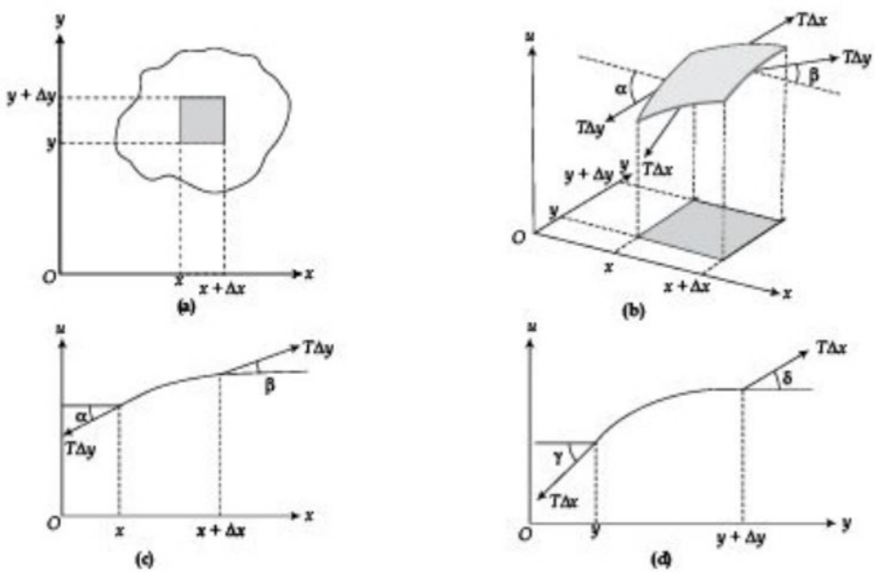


Fig. 17.9

Similarly, considering the vertical components of the forces acting on the small portion of the membrane, the resultants is:

$$T\Delta y(\sin\beta - \sin\alpha) + T\Delta x(\sin\delta - \sin\gamma). \qquad ...(17.160)$$

Since α, β, γ and δ are small so their sines can be replaced by their tangents and further using $\tan\alpha = u_x(x, y, t)$ and $\tan\beta = u_x(x+\Delta x, y, t)$ etc., the Eq. (17.160) becomes

$$T\Delta y[u_x(x+\Delta x, y, t) - u_x(x, y, t)] + T\Delta x[u_y(x, y+\Delta y, t) - u_y(x, y, t)], \qquad ...(17.161)$$

where the subscript x denotes partial derivatives w.r.t. x and the subscript y denotes partial derivatives w.r.t. y.

By Newton's second law of motion the resultant force must be equal to $\rho\Delta x\Delta y \dfrac{\partial^2 u}{\partial t^2}$, where ρ is the density (mass per unit area) of the undeflected membrane. Thus,

$$\rho\,\Delta x\Delta y\frac{\partial^2 u}{\partial t^2} = T\Delta y[u_x(x+\Delta x, y, t) - u_x(x, y, t)] + T\Delta x[u_y(x, y+\Delta y, t) - u_y(x, y, t)]. \qquad ...(17.162)$$

Dividing (17.162) throughout by $\Delta x\,\Delta y$ and taking the limits as $\Delta x \to 0$ and $\Delta y \to 0$, we get

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

when the derivatives on the right are evaluated at some suitable point in the portion $\Delta x\Delta y$.

Writing $\dfrac{T}{P} = c^2$, this equation becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad ...(17.163)$$

called the *two-dimensional wave equation*. In the form of Laplacian it can be expressed as

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u.$$

17.11 SOLUTION OF A VIBRATING RECTANGULAR MEMBRANE: WAVE EQUATION IN CARTESIAN COORDINATES

Consider an elastic membrane stretched and fixed along a rectangular membrane, $0 \le x \le a$, $0 \le y \le b$ in the xy-plane. To find the solution of this vibrating rectangular membrane we solve the two dimensional wave equation in cartesian coordinates given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 < x < a, \quad 0 < y < b, \; t > 0, \qquad ...(17.164)$$

subject to the boundary conditions $u(x, y, t) = 0$ on the boundary of the membrane for all t, that is,

$$u(x, 0, t) = u(x, b, t) = 0, \text{ for } 0 < x < a, t > 0, \quad ...(17.165)$$
$$u(0, y, t) = u(a, y, t) = 0, \text{ for } 0 < y < b, t > 0, \quad ...(17.166)$$

and the initial conditions

$$u(x, y, 0) = f(x, y) \quad ...(17.167)$$

and,
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x, y). \quad ...(17.168)$$

Here $f(x,y)$ and $g(x,y)$ are respectively the initial displacement and the initial velocity given to the membrane.

Let $u(x, y, t)$ be the displacement of the point (x, y) of the membrane (from its initial position $u = 0$) at time t and let it be of the separable form

$$u(x, y, t) = X(x)Y(y)T(t). \quad ...(17.169)$$

Substituting this in (17.164) and dividing by XYT, we obtain

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}, \quad ...(17.170)$$

where primes denote the derivatives w.r.t. the corresponding independent variables.

Since x, y and t are independent variables, thus (17.170) holds good with the selection of two suitable separation constants k and l resulting in three ordinary differential equations

$$X'' + k^2X = 0, \quad Y'' + l^2Y = 0 \text{ and } \quad T'' + c^2(k^2 + l^2)T = 0.$$

The respective solutions of these equations are

$$X = c_1 \cos kx + c_2 \sin kx \quad ...(17.171)$$
$$Y = c_3 \cos ly + c_4 \sin ly \quad ...(17.172)$$

and,
$$T = c_5 \cos \lambda ct + c_6 \sin \lambda ct; \quad \lambda = \sqrt{k^2 + l^2}. \quad ...(17.173)$$

Hence solution of the equation (17.164) using (17.169) is given by

$$u(x, y, t) = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)(c_5 \cos \lambda ct + c_6 \sin \lambda ct), \quad ...(17.174)$$

where $\lambda = \sqrt{k^2 + l^2}$ and c_1, c_2, etc. are constants to be obtained by using the boundary and initial conditions.

Using the boundary conditions (17.166) in (17.169) give $X(0) = X(a) = 0$;
and the conditions (17.165) give $Y(0) = Y(b) = 0$.

Using $X(0) = 0$ and $X(a) = 0$ in (17.171) give $c_1 = 0$ and $\sin ka = 0$, that is, $k = \frac{n\pi}{a}$, where n is an integer. Hence, $X_n(x) = c_2 \sin \frac{n\pi x}{a}$, $n = 1, 2, ...$

Similarly, $\quad Y_m(y) = c_4 \sin \frac{m\pi y}{b}$, $m = 1, 2, ...$

We note that it is sufficient to consider only positive integral values, since corresponding negative values don't lead to any independent solution.

Thus (17.174) becomes of the form

$$u_{nm}(x, y, t) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}[P_{nm}\cos\lambda_{nm}t + Q_{nm}\sin\lambda_{nm}t],$$

where $\lambda_{nm} = c\pi\sqrt{n^2/a^2 + m^2/b^2}$, $n, m = 1, 2, \ldots$; and P_{nm}, Q_{nm} are constants to be determined.

Using the principle of superposition, the general solution of Eq. (17.164) can be given as

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}[P_{nm}\cos\lambda_{nm}t + Q_{nm}\sin\lambda_{nm}t], \quad \text{...(17.175)}$$

where $\lambda_{nm} = c\pi\sqrt{n^2/a^2 + m^2/b^2}$.

Next using the boundary condition (17.167) in (17.175), we obtain

$$u(x, y, 0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}P_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} = f(x, y), \quad \text{...(17.176)}$$

which is the *double Fourier series.*

Assuming that f, $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x\partial y$ are continuous in the rectangular region $R = \{(x, y);\ 0 \le x \le a,\ 0 \le y \le b\}$, then $f(x, y)$ can be expressed as double Fourier series (17.176). To determine P_{nm}, multiplying both sides by $\sin(n\pi x/a)\sin(m\pi y/a)$ and integrating over the region R we obtain the coefficients P_{nm} as

$$P_{nm} = \frac{4}{ab}\int_0^a\int_0^b f(x, y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy, \quad n, m = 1, 2, \ldots \quad \text{...(17.177)}$$

called the *generalized Euler's formula.*

To obtain the coefficients Q_{nm}, we use the initial condition (17.168) in (17.175), we have

$$g(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}Q_{nm}\lambda_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}.$$

Assuming that $g(x, y)$ is expandable as a double Fourier series in R, then coefficients Q_{nm} are given by

$$Q_{nm} = \frac{4}{ab\lambda_{nm}}\int_0^a\int_0^b g(x, y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy, \quad n, m = 1, 2, \ldots \quad \text{...(17.178)}$$

Hence, the solution of the two-dimensional wave equation (17.164) is (17.175), where the coefficients P_{nm} and Q_{nm} are given by (17.177) and (17.178) respectively.

Example 17.18: Find the vibrations of a square membrane of side one *ft.* each, if the tension is 10.0*lb*/*ft*, the density is 2.5 slugs/ft^2, the initial velocity is zero and the initial displacement is $f(x, y) = a \sin \pi x \sin 2\pi y$ *ft.*

Solution: If $u(x, y, t)$ denotes the deflection of the point (x, y) at time t from its position of rest, then it is given by the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 \le x, y \le 1, \qquad \text{...(17.179)}$$

subject to the boundary conditions

$$u(0, y, t) = 0 = u(1, y, t), \text{ for } \quad 0 \le y \le 1 \text{ and } t > 0, \qquad \text{...(17.180)}$$

$$u(x, 0, t) = 0 = u(x, 1, t), \text{ for } \quad 0 \le x \le 1 \text{ and } t > 0, \qquad \text{...(17.181)}$$

and initial conditions

$$u(x, y, 0) = f(x, y) = a \sin \pi x \sin 2\pi y, \quad 0 \le x, y \le 1 \qquad \text{...(17.182)}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x, y) = 0. \qquad \text{...(17.183)}$$

Also here $\quad c^2 = \dfrac{T}{\rho} = \dfrac{10}{2.5} = 4 \text{ ft}^2/\text{sec}^2.$

The solution of Eq. (17.179) subject to boundary conditions (17.180) and (17.181), refer to Eq. (17.175), is

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin n\pi x \sin m\pi y [P_{nm} \cos \lambda_{nm} t + Q_{nm} \sin \lambda_{nm} t], \qquad \text{...(17.184)}$$

where $\lambda_{nm} = c\pi\sqrt{n^2 + m^2} = 2\pi\sqrt{n^2 + m^2}$.

Using the initial condition (17.183) in (17.184) we obtain $Q_{nm} = 0$, hence

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P_{nm} \sin n\pi x \sin m\pi y \cos \lambda_{nm} t. \qquad \text{...(17.185)}$$

Using the initial condition (17.182) gives

$$u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P_{nm} \sin n\pi x \sin m\pi y = f(x, y) = a \sin \pi x \sin 2\pi y.$$

Comparing we obtain that all $P_{nm} = 0$, except for $n = 1$, $m = 2$, which is, $P_{12} = a$. Substituting in (17.185), gives $u(x, y, t) = a \sin \pi x \sin 2\pi y \cos 2\sqrt{5}\, \pi t$, as the desired solution.

Example 17.19: Find the vibrations of a rectangular membrane of sides $a = 4$ ft and $b = 2$ ft, if the tension is 40.0 lb./ft., the density is 2.5 slugs/ ft^2, the initial velocity is zero and the initial displacement is $f(x, y) = 0.1(4x - x^2)(2y - y^2)$ ft.

Solution: The deflection $u(x, y, t)$ of the elastic membrane at the point (x, y) at time t is given by the two-dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad 0 \le x \le 4, \quad 0 \le y \le 2 \qquad ...(17.186)$$

subject to the boundary conditions

$$u(0, y, t) = u(4, y, t) = 0, \text{ for } \quad 0 \le y \le 2, \quad t > 0, \qquad ...(17.187)$$

$$u(x, 0, t) = u(x, 2, t) = 0, \text{ for } \quad 0 \le x \le 4, \quad t > 0, \qquad ...(17.188)$$

and the initial conditions

$$u(x, y, 0) = f(x, y) = 0.1(4x - x^2)(2y - y^2), \qquad ...(17.189)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 = g(x, y) = 0. \qquad ...(17.190)$$

Here, $c^2 = \dfrac{T}{\rho} = \dfrac{40.0}{2.5} = 16.0 \text{ ft}^2/\text{sec}^2.$

The solution of the equation (17.186) subject to the boundary conditions (17.187) and (17.188); and initial condition (17.190), as in Example 17.18, is

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} P_{nm} \sin\frac{n\pi x}{4} \sin\frac{m\pi y}{2} \cos\lambda_{nm} t, \qquad ...(17.191)$$

where $\lambda_{nm} = 4\pi\sqrt{n^2/16 + m^2/4} = \pi\sqrt{n^2 + 4m^2}$.

To evaluate the coefficients P_{nm} we apply the initial condition (17.189) in (17.191), we obtain

$$u(x, y, 0) = 0.1xy(4 - x)(2 - y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} P_{nm} \sin\frac{n\pi x}{4} \sin\frac{m\pi y}{2},$$

which is double Fourier series expansion of $f(x, y) = 0.1xy(4 - x)(2 - y)$ in the rectangular region $R = \{(x, y): 0 \le x \le 4, \; 0 \le y \le 2\}$. Hence, the coefficients P_{nm} are given by

$$P_{nm} = \frac{4}{4.2}\int_0^4\int_0^2 0.1xy(4 - x)(2 - y) \sin\frac{n\pi x}{4} \sin\frac{m\pi y}{2}\, dx\, dy$$

$$= \frac{1}{20}\left(\int_0^4 x(4 - x)\sin\frac{n\pi x}{4}\, dx\right)\left(\int_0^2 y(2 - y)\sin\frac{m\pi y}{2}\, dy\right).$$

Consider $\displaystyle\int_0^4 x(4 - x)\sin\frac{n\pi x}{4}\, dx$

$$= \left[x(4-x)\left(-\frac{4}{n\pi}\cos\frac{n\pi x}{4}\right) - (4-2x)\left\{-\left(\frac{4}{n\pi}\right)^2 \sin\frac{n\pi x}{4}\right\} + (-2)\left\{\left(\frac{4}{n\pi}\right)^3 \cos\frac{n\pi x}{4}\right\}\right]_0^4$$

$$= \frac{128}{n^3\pi^3}[1-(-1)^n] = \frac{256}{n^3\pi^3},\ n = 1, 3, 5, \ldots; \text{ and, zero otherwise.}$$

Similarly, $\int_0^2 y(2-y)\sin\frac{m\pi y}{2}\,dy = \frac{32}{m^3\pi^3};\quad m = 1, 3, 5, \ldots;$ and, zero otherwise.

Thus, $P_{nm} = \frac{2048}{5}\cdot\frac{1}{n^3m^3\pi^6},\quad n, m = 1, 3, 5, \ldots;$ and zero, otherwise.

Substituting for P_{nm} in (17.191), we obtain

$$u(x, y, t) = \frac{2048}{5\pi^6}\sum_n \sum_m \frac{1}{n^3m^3}\sin\frac{n\pi x}{4}\sin\frac{m\pi y}{2}\cos(\pi\sqrt{n^2+4m^2})t;\ n, m = 1, 3, 5, \ldots$$

as the desired solution.

Example 17.20: Find the vibrations of a rectangular membrane of sides a and b units with the entire boundary fixed and, with initial displacement and initial velocity, zero and one unit respectively.

Solution: The displacement $u(x, y, t)$ at the point (x, y) of the membrane at time t is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\quad 0 < x < a,\quad 0 < y < b, \qquad \ldots(17.192)$$

subject to the boundary conditions

$$u(0, y, t) = u(a, y, t) = 0, \text{ for } 0 \le y \le b,\quad t > 0 \qquad \ldots(17.193)$$

$$u(x, 0, t) = u(x, b, t) = 0, \text{ for } 0 \le x \le a,\quad t > 0 \qquad \ldots(17.194)$$

and the initial conditions

$$u(x, y, 0) = f(x, y) = 0, \text{ for } 0 \le x \le a,\quad 0 \le y \le b \qquad \ldots(17.195)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x, y) = 1, \text{ for } 0 \le x \le a,\quad 0 \le y \le b. \qquad \ldots(17.196)$$

The solution of Eq. (17.192) subject to the boundary conditions (17.193) and (17.194), refer to Eq. (17.175), is

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}[P_{nm}\cos\lambda_{nm}t + Q_{nm}\sin\lambda_{nm}t], \qquad \ldots(17.197)$$

where $\lambda_{nm} = c\pi\sqrt{n^2/a^2 + m^2/b^2}$.

Applying the initial condition (17.195), (17.197) gives $P_{nm} = 0$, and hence

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \lambda_{nm} t. \qquad \text{...(17.198)}$$

Next applying the initial condition (17.196), (17.198) gives

$$1 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{nm}\lambda_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b},$$

which is the double Fourier series expansion of the function $g(x, y) = 1$ in the region $R = \{(x, y) : 0 \le x \le a,\ 0 \le y \le b\}$. Thus, the coefficients Q_{nm} are given by

$$Q_{nm} = \frac{4}{ab\lambda_{nm}} \int_0^a \int_0^b \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}\, dx\, dy$$

$$= \frac{4}{ab\lambda_{nm}} \left(\int_0^a \sin \frac{n\pi x}{a} dx \right) \left(\int_0^b \sin \frac{m\pi y}{b} dy \right) = \frac{4}{mn\pi^2 \lambda_{nm}} [1 - (-1)^n][1 - (-1)^m]$$

or,

$$Q_{nm} = \begin{cases} \dfrac{16}{mn\pi^2 \lambda_{nm}}, & \text{if both } m, n \text{ are odd} \\ \text{zero}, & \text{otherwise} \end{cases}.$$

Hence the desired solution is $u(x, y, t) = \dfrac{16}{\pi^2} \sum_n \sum_m \dfrac{1}{nm\lambda_{nm}} \sin \dfrac{n\pi x}{a} \sin \dfrac{m\pi y}{b} \sin \lambda_{nm} t$, where summation is to be taken only for odd $n, m = 1, 3, 5, ...$; and $\lambda_{nm} = c\pi\sqrt{n^2/a^2 + m^2/b^2}$.

17.12 SOLUTION OF A VIBRATING CIRCULAR MEMBRANE: WAVE EQUATION IN POLAR COORDINATES

We now analyse the motion of a membrane fastened onto a circular frame and set in motion with given initial displacement and velocity. The circular membrane are of great importance in engineering; the drumhead, telephones, microphones, etc. provide a few useful examples of circular membrane. Let the rest position of the membrane be in the xy-plane with the origin as the centre and R_1 as its radius. The two-dimensional wave equation (17.163), in terms of polar-coordinates (r, θ), defined by $x = r\cos\theta$, $y = r\sin\theta$, is of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right), \quad 0 \le r \le R_1, \quad 0 \le \theta \le 2\pi. \qquad \text{...(17.199)}$$

We shall solve (17.199) for a radially symmetric membrane in which the motion of the membrane is independent of θ; the Eq. (17.199) then reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \quad 0 \le r \le R_1. \qquad ...(17.200)$$

Since the membrane is fixed along its frame, thus the boundary condition is

$$u(R_1, t) = 0, \text{ for all } t \ge 0. \qquad ...(17.201)$$

Since the solutions are independent of θ, the initial conditions are of the form

$$u(r, 0) = f(r), \text{ for } \quad 0 \le r \le R_1 \qquad ...(17.202)$$

and,

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r), \text{ for } \quad 0 \le r \le R_1; \qquad ...(17.203)$$

here $f(r)$ and $g(r)$ denote the initial displacement and velocity respectively.

We find the solution of (17.200) of the form

$$u(r, t) = R(r)T(t), \qquad ...(17.204)$$

where R is function of r only and T of t only. Substituting (17.204) in (17.200), we get

$$\frac{T''}{c^2 T} = \frac{1}{R}\left(R'' + \frac{1}{r}R'\right), \qquad ...(17.205)$$

where primes denote the ordinary derivatives w.r.t. the corresponding dependent variables. Since the expression on left side (17.205) is a function of t and on right side is a function of r only; t and r being independent so both must be equal to a common constant, say $-k^2$, since only negative constant leads to non-zero real solution under the given boundary condition (17.201).

Thus,

$$\frac{T''}{c^2 T} = \frac{1}{R}\left(R'' + \frac{R'}{r}\right) = -k^2.$$

It leads to two ordinary differential equations

$$T'' + k^2c^2T = 0 \qquad ...(17.206)$$

and,

$$R'' + \frac{1}{r}R' + k^2R = 0. \qquad ...(17.207)$$

Set $s = kr$, the Eq. (17.207) transforms to $\dfrac{d^2R}{ds^2} + \dfrac{1}{s}\dfrac{dR}{ds} + R = 0,$

which is Bessel's equation of order zero, refer to Eq. (12.141).

Its general solution is $R = AJ_0(s) + BY_0(s)$, where $J_0(s)$ and $Y_0(s)$ respectively are the Bessel's functions of the first kind and second kind of order zero.

Since the deflection of the membrane is always to remain finite but Y_0 becomes infinite at $s = 0$, (corresponding to $r = 0$), and hence taking $B = 0$; and then taking $A = 1$, we get

$$R(r) = J_0(s) = J_0(kr). \qquad ...(17.208)$$

Also on the boundary $r = R_1$ of the circular membrane, we must have $J_0(kR_1) = R(R_1) = 0$, which is satisfied for $kR_1 = \alpha_n, \quad n = 1, 2, ...,$ the zeros of the Bessel's function of order zero.

Thus from (17.208) the solution of the equation (17.207) are

$$R(r) = J_0(k_n r), \qquad \text{...(17.209)}$$

where $k_n = \alpha_n / R_1$, $n = 1, 2, \dots$.

The corresponding solutions of the Eq. (17.206) are $T(t) = (a_n \cos p_n t + b_n \sin p_n t)$, where $p_n = ck_n = c\alpha_n / R_1$. Hence the solution (17.204) of the Eq. (17.200) subject to the boundary condition (17.201) becomes

$$u_n(r, t) = (a_n \cos p_n t + b_n \sin p_n t) J_0(\alpha_n r / R_1), \quad n = 1, 2, \dots \qquad \text{...(17.210)}$$

The vibration of the membrane corresponding to u_n given by (17.210) is called the *nth normal mode* and has the frequency $p_n / 2\pi$ cycles per unit time.

Next to obtain the solution satisfying the initial conditions (17.202) and (17.203), we consider the general solution in the series form as

$$u(r, t) = \sum_{n=1}^{\infty} (a_n \cos p_n t + b_n \sin p_n t) J_0(\alpha_n r / R_1). \qquad \text{...(17.211)}$$

Setting $t = 0$ and using the condition (17.202), we obtain $\sum_{n=1}^{\infty} a_n J_0(\alpha_n r / R_1) = f(r)$.

Thus the coefficients a_n, the coefficients of the Fourier-Bessel series which represents $f(r)$ in the form $J_0(\alpha_n r / R_1)$, refer to Section 12.8.2, are given by

$$a_n = \frac{2}{R_1^2 J_1^2(\alpha_n)} \int_0^{R_1} r f(r) J_0(\alpha_n r / R_1)\, dr, \quad n = 1, 2, \dots \qquad \text{...(17.212)}$$

To obtain the coefficients b_n, we apply the initial condition (17.203) in (17.210) and proceed on similar lines, we obtain

$$b_n = \frac{2}{R_1^2 J_1^2(\alpha_n)\, p_n} \int_0^{R_1} r g(r) J_0(\alpha_n r / R_1)\, dr, \quad n = 1, 2, \dots \qquad \text{...(17.213)}$$

where $p_n = c\alpha_n / R_1$; α_n being the zeros of J_0, the Bessel's function of order zero.

Example 17.21: Determine the displacement for a radially symmetric circular membrane of unit radius fixed at its boundary with initial displacement $f(r) = 3J_0(\alpha_1 r) + J_0(\alpha_3 r)$ and initial velocity $g(r) = J_0(\alpha_2 r)$.

Solution: The displacement $u(r, t)$ for a radially symmetric circular membrane are given by the

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right], \quad 0 \le r \le 1, \quad t > 0, \qquad \text{...(17.214)}$$

subject to the boundary condition $u(1, t) = 0$, for $t \ge 0$, ...(17.215)

the initial conditions $u(r, 0) = f(r) = 3J_0(\alpha_1 r) + J_0(\alpha_3 r)$, for $0 \le r \le 1$ ...(17.216)

and, $\left. \frac{\partial u}{\partial t} \right|_{t=0} = g(r) = J_0(\alpha_2 r)$, for $0 \le r \le 1$. ...(17.217)

The solution of the Eq. (17.214) subject to (17.215), refer to Eq. (17.211) for $R_1 = 1$, is

$$u(r, t) = \sum_{n=1}^{\infty} (a_n \cos c\alpha_n t + b_n \sin c\alpha_n t) J_0(\alpha_n r), \quad ...(17.218)$$

where α_n's are the zeros of J_0, the Bessel's function of order zero.

Substituting $t = 0$ in (17.218) and using the initial condition (17.216), we obtain

$$3J_0(\alpha_1 r) + J_0(\alpha_3 r) = \sum_{n=1}^{\infty} a_n J_0(\alpha_n r).$$

Comparing both sides, we obtain $a_1 = 3$, $a_3 = 1$ and all others $a_n = 0$.

Similarly, using the initial condition (17.217) in (17.218), we obtain

$$J_0(\alpha_2 r) = \sum_{n=1}^{\infty} cb_n\alpha_n J_0(\alpha_n r),$$

which gives $cb_2\alpha_2 = 1$, that is, $b_2 = 1/c\alpha_2$ and all others $b_n = 0$.

Substituting in (17.218), the radial displacement $u(r, t)$ becomes

$$u(r, t) = 3J_0(\alpha_1 r)\cos(c\alpha_1 t) + \frac{1}{c\alpha_2} J_0(\alpha_2 r)\sin(c\alpha_2 t) + J_0(\alpha_3 r)\cos(c\alpha_3 t).$$

Example 17.22: Find the vibrations of a circular drumhead of radius 1.0 ft and density 2.0 slugs/ft^2, if the tension is 2.0 lb/ft, the initial velocity is zero and the initial displacement is $f(r) = 0.25\,(1 - r^2)$ ft.

Solution: The vibrations $u(r, t)$ of a circular drumhead are given by the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right], \quad 0 \le r \le 1, \quad t \ge 0, \quad ...(17.219)$$

subject to the boundary condition $u(1, t) = 0$, ...(17.220)

initial conditions $u(r, 0) = f(r) = 0.25(1 - r^2)$, ...(17.221)

and,

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0. \quad ...(17.222)$$

Here $c^2 = \frac{2}{2} = 1$, which gives $c = 1$.

The solution of Eq. (17.219) subject to the boundary condition (17.220) and the initial condition (17.222), refer to Eq. (17.218), for $g(r) = 0$, $R_1 = 1$ and $c = 1$ is given by

$$u(r, t) = \sum_{n=1}^{\infty} a_n \cos(\alpha_n t) J_0(\alpha_n r), \quad ...(17.223)$$

where α_n are the zeros of Bessel function J_0.

Next applying the initial condition (17.221) in (17.223), we get $0.25(1-r^2) = \sum_{n=1}^{\infty} a_n J_0(\alpha_n r)$.

Thus a_n are the coefficients of Fourier-Bessel series expansion of $f(x) = 0.25(1-r^2)$ in $J_0(\alpha_n r)$ hence a_n, refer to Eq. (17.212), are given by

$$a_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 0.25r(1-r^2)J_0(a_n r)dr \ldots \quad = \frac{1}{2J_1^2(\alpha_n)} \int_0^1 (1-r^2)[rJ_0(\alpha_n r)]dr$$

$$= \frac{2}{R_1^2 J_1^2(\alpha_n)} \left[\left| \frac{1}{\alpha_n}(1-r^2) r J_1(\alpha_n r) \right|_0^1 - \frac{1}{\alpha_n} \int_0^1 (-2r)\, r J_1(\alpha_n r)\, dr \right];$$

using $[rJ_1(\lambda r)]' = \lambda r J_0(\lambda r)$, refer to (12.121)

$$= \frac{1}{J_1^2(\alpha_n)} \left[\frac{1}{\alpha_n} \int_0^1 r^2 J_1(\alpha_n r)dr \right] = \frac{1}{a_n^2 J_1^2(\alpha_n)} [r^2 J_2(\alpha_n r)]_0^1;$$

using $[r^2 J_2(\lambda r)]' = \lambda r^2 J_1(\lambda r)$, refer to (12.121)

$$= \frac{J_2(\alpha_n)}{\alpha_n^2 J_1^2(\alpha_n)} = \frac{2}{\alpha_n^3 J_1(\alpha_n)}; \text{ using } J_2(\alpha_n) = \frac{2}{\alpha_n} J_1(\alpha_n) - J_0(\alpha_n) = \frac{2}{\alpha_n} J_1(\alpha_n), \text{ refer to (12.123).}$$

Substituting for a_n in (17.223) gives $u(r, t) = 2\sum_{n=1}^{\infty} \frac{1}{\alpha_n^3 J_1(\alpha_n)} \cos(\alpha_n t) J_0(\alpha_n r)$, as the desired solution.

EXERCISE 17.6

1. Find the vibrations of a square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is

   (i) $0.1 \sin 3\pi x \sin 4\pi y$ (ii) $k \sin 2\pi x \sin \pi y$ (iii) $kxy(1-x)(1-y)$.

2. Find the vibrations of a rectangular membrane with sides a and b units and fixed boundary, when it is given that it starts from rest and initial displacement is $kxy(a-x)(b-y)$.

3. Solve $$\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad 0 \le x \le a, \quad 0 \le y \le b$$

   subject to the boundary conditions

   $u(0, y, t) = 0 = u(a, y, t)$, for $0 \le y \le b$, $t > 0$,

   $u(x, 0, t) = 0 = u(x, b, t)$, for $0 \le x \le a$, $t > 0$.

and initial conditions

$$u(x, y, 0) = \sin\frac{2\pi x}{a}\sin\frac{3\pi y}{b} \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

4. Solve $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, $0 \le x, y \le 2\pi$, $t > 0$,

subject to boundary conditions

$$u(0, y, t) = u(2\pi, y, t) = 0, \quad 0 \le y \le 2\pi, \quad t \ge 0,$$
$$u(x, 0, t) = u(x, 2\pi, t) = 0, \quad 0 \le x \le 2\pi, \quad t \ge 0$$

and initial conditions

$$u(x, y, 0) = x^2 \sin y, \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

5. Show that the vibrations $u(r, t)$ of a circular membrane of radius one unit with initial velocity zero and initial displacement $f(r) = (1 - r^2)$, assuming $c = 1$ are given by

$$\sum_{n=1}^{\infty} \frac{4}{\alpha_n^3 J_1(\alpha_n)} \cos(\alpha_n t) . J_0(\alpha_n r).$$

6. Show that the vibrations $u(r, t)$ of a circular membrane of unit radius with initial displacement zero and initial velocity $g(r) = (1 - r^2)$, assuming $c = 1$ are given by

$$\sum_{n=1}^{\infty} \frac{4}{\alpha_n^4 J_1(\alpha_n)} \sin(\alpha_n t)\, J_0(\alpha_n r).$$

17.13 TRANSMISSION LINE AND RELATED EQUATIONS

The transmission lines network transfers energy from the generating units to the distribution system which supplies the load to the destinations. Consider a long cable which is imperfectly insulated so that leakage occurs along the entire length of the cable. Let R, L, C and G respectively be the resistance, inductance, capacitance to the ground and conductance to the ground of the cable per unit length. Further, let the source S of the current I in the cable be at $x = 0$, and the receiving end T be at $x = l$, as shown in Fig. 17.10.

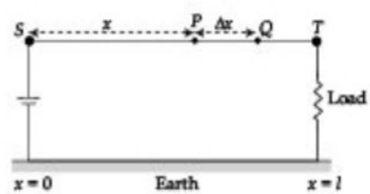


Fig. 17.10

Let $I(x, t)$ and $V(x, t)$ be the instantaneous current and voltage respectively at any point P at time t which is at a distance of x units from S. Consider a small segment PQ of the cable between x and $x + \Delta x$. Applying Kirchhoff's voltage law, that is, *the voltage drop across the segment Δx is the sum of the voltage drops due to resistance and due to inductance*, we obtain

$$-\Delta V = IR\Delta x + L\frac{\Delta I}{\Delta t}\Delta x.$$

Dividing by Δx and taking limit as $\Delta x \to 0$, we obtain the equation

$$-\frac{\partial V}{\partial x} = RI + L\frac{\partial I}{\partial t} \qquad ...(17.224)$$

called the *first transmission line equation.*

Next, applying the Kirchhoff's current law, that is, *the loss of current across the segment PQ is the sum of the losses of currents due to capacitance and due to leakages to the ground,* we obtain

$$-\Delta I = C\frac{\Delta V}{\Delta t}\Delta x + GV\Delta x.$$

Dividing by Δx and taking the limit $\Delta x \to 0$, we obtain the equation

$$-\frac{\partial I}{\partial x} = C\frac{\partial V}{\partial t} + GV, \qquad ...(17.225)$$

called the *second transmission line equation.*

Rewriting (17.224) and (17.225) respectively as

$$\left(R + L\frac{\partial V}{\partial t}\right)I + \frac{\partial V}{\partial x} = 0 \qquad ...(17.226)$$

and,
$$\frac{\partial I}{\partial x} + \left(C\frac{\partial}{\partial t} + G\right)V = 0. \qquad ...(17.227)$$

Eliminating I and V from the transmission Eqs. (17.226) and (17.227), we obtain respectively

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (RC + GL)\frac{\partial V}{\partial t} + RGV \qquad ...(17.228)$$

and,
$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (RC + GL)\frac{\partial I}{\partial t} + RGI \qquad ...(17.229)$$

called *the telephone equations.*

For a submarine cable, leakages are low (that is, $G = 0$) and the frequencies are low (that is $L = 0$) hence the equations (17.228) and (17.229) become respectively

$$\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t} \text{ and } \frac{\partial^2 I}{\partial x^2} = RC\frac{\partial I}{\partial t} \qquad ...(17.230)$$

called the *telegraph equations* or the *submarine cable equations*, similar to one-dimensional heat equation.

In case of transmission of alternating current of high frequencies, leakages and resistances are negligible. Taking $G = 0$ and $R = 0$, the equations (17.228) and (17.229) become respectively

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} \text{ and } \frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} \qquad ...(17.231)$$

called the *high-frequency line equations* or *radio equations*, similar to one-dimensional wave equation.

In the examples to follow we solve these equations subject to certain initial conditions.

Example 17.23: Find the current I and voltage V in a cable line of length l, t seconds after the ends are suddenly grounded, when $I(x, 0) = I_0$ and $V(x, 0) = V_0 \sin L(\pi x/b)$ and, R and G are negligible.

Solution: The transmission line equations are

$$-\frac{\partial V}{\partial x} = RI + L\frac{\partial I}{\partial t} \text{ and } -\frac{\partial I}{\partial x} = GV + C\frac{\partial V}{\partial t}.$$

Since R and G are negligible, the equations reduce respectively to

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \quad \text{...(17.232)}$$

and,

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}. \quad \text{...(17.233)}$$

To eliminate I between these two equations, differentiate (17.232) w.r.t. x and (17.233) w.r.t. t, we obtain respectively

$$\frac{\partial^2 V}{\partial x^2} = -L\frac{\partial^2 I}{\partial x \partial t} \text{ and } \frac{\partial^2 I}{\partial x \partial t} = -C\frac{\partial^2 V}{\partial t^2},$$

which give

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}. \quad \text{...(17.234)}$$

The initial conditions are

$$I(x, 0) = I_0, \quad V(x, 0) = V_0 \sin(\pi x/l); \quad 0 \le x \le l \quad \text{...(17.235)}$$

and since the ends are grounded the boundary conditions are

$$V(0, t) = V(l, t) = 0, \text{ for } t > 0. \quad \text{...(17.236)}$$

Also, $I = I_0$ at $t = 0$, therefore,

$$\left.\frac{\partial I}{\partial x}\right|_{t=0} = 0 \text{ and, hence } \left.\frac{\partial V}{\partial t}\right|_{t=0} = 0. \quad \text{...(17.237)}$$

Let a solution of Eq. (17.234) be

$$V = XT, \quad \text{...(17.238)}$$

where X is a function of x only and T is a function of t only.

Substituting (17.238) in (17.234), we obtain $X''T = LC\,XT''$, which gives

$$\frac{X''}{X} = LC\frac{T''}{T}. \quad \text{...(17.239)}$$

where primes denote the derivatives w.r.t. the corresponding independent variable.

Since left side of (17.239) is a function of x only and right side is a function of t only, and x and t are independent, which is possible when each side is equal to a constant, say $-k^2$. This leads to two second order ordinary differential equations $X'' + k^2X = 0$ and $T'' + \frac{k^2}{LC}T = 0$.

The solutions are respectively

$$X = c_1 \cos kx + c_2 \sin kx \quad \text{and} \quad T = c_3 \cos\frac{k}{\sqrt{LC}}t + c_4 \sin\frac{k}{\sqrt{LC}}t.$$

Hence, from (17.238) the solution is

$$V(x, t) = (c_1 \cos kx + c_2 \sin kx)\left(c_3 \cos \frac{kt}{\sqrt{LC}} + c_4 \sin \frac{kt}{\sqrt{LC}}\right). \quad \text{...(17.240)}$$

Applying the boundary conditions (17.236), we obtain $c_1 = 0$ and $k = \frac{n\pi}{l}$, n being integer. Thus (17.240) can be expressed as

$$V_n(x, t) = \sin \frac{n\pi x}{l}\left[A \cos \frac{n\pi t}{l\sqrt{LC}} + B \sin \frac{n\pi t}{l\sqrt{LC}}\right],\ n = 1, 2, \ldots \quad \text{...(17.241)}$$

Applying the initial condition (17.237), we obtain $B = 0$, and hence (17.241) becomes

$$V_n(x, t) = A \sin \frac{n\pi x}{l} \cos \frac{n\pi}{l\sqrt{LC}} t,\ n = 1, 2, \ldots$$

Applying the principle of superposition, the solution is

$$V(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi}{l\sqrt{LC}} t \quad \text{...(17.242)}$$

To determine coefficients A_n, we apply the initial condition $V = V_0 \sin(\pi x/l)$, (17.242) gives

$$V_0 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

This implies $A_1 = V_0$ and rest A_i's are zeros and hence (17.242) becomes

$$V(x, t) = V_0 \sin \frac{\pi x}{l} \cos \frac{\pi}{l\sqrt{LC}} t, \quad \text{...(17.243)}$$

the desired expression for voltage.

To find $I(x, t)$, from (17.232) we have

$$\frac{\partial I}{\partial t} = -\frac{1}{L}\frac{\partial V}{\partial x} = -\frac{1}{L}\frac{V_0\pi}{l} \cos \frac{\pi x}{l} \cos \frac{\pi}{l\sqrt{LC}} t.$$

Integrating w.r.t. t, treating x as constant, we get

$$I = -V_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi}{l\sqrt{LC}} t + c(x). \quad \text{...(17.244)}$$

To evaluate $c(x)$, we apply the initial condition $I = I_0$ at $t = 0$, (17.244) gives $c(x) = I_0$, and hence

$$I = I_0 - V_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi}{l\sqrt{LC}} t$$

the desired expression for the current.

Example 17.24: A transmission line of 1000 km long is initially under steady-state conditions with potential 1300 volts at the source end, $(x = 0)$ and 1200 volt at the terminal end, $(x = 1000)$. The terminal end is suddenly grounded while the potential at the source end is maintained at 1300 volts. Find potential $V(x, t)$, assuming the inductance and leakage to be negligible.

Solution: The equation of the transmission lines are

$$-\frac{\partial V}{\partial x} = RI + L\frac{\partial I}{\partial t} \quad \text{and} \quad -\frac{\partial I}{\partial x} = GV + C\frac{\partial V}{\partial t}.$$

When G and L are negligible, then these equations become

$$-\frac{\partial V}{\partial x} = RI \quad \text{and} \quad -\frac{\partial I}{\partial x} = C\frac{\partial V}{\partial t}.$$

To eliminate I, differentiate first equation w.r.t. x and substitute for $\frac{\partial I}{\partial x}$ from the second, we obtain

$$-\frac{\partial^2 V}{\partial x^2} = R\frac{\partial I}{\partial x} = R\left(-C\frac{\partial V}{\partial t}\right) \text{ or, } \frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t}, \text{ or } \frac{\partial V}{\partial t} = \frac{1}{RC}\frac{\partial^2 V}{\partial x^2}, \quad \text{...(17.245)}$$

the equation of telegraph line.

The steady-state voltage distribution, say $V = V_s$ is independent of time and thus from Eq. (17.245), we obtain $\frac{\partial^2 V_s}{\partial x^2} = 0$, which gives $V_s = c_1 + c_2 x$.

Applying the initial conditions $V = 1300$ at $x = 0$ and $V = 1200$ at $x = 1000$, we obtain $c_1 = 1300$ and $c_2 = -0.1$ and thus the initial steady-state voltage is given by

$$V_s(x) = V(x, 0) = 1300 - (0.1)x. \quad \text{...(17.246)}$$

At time $t = 0$, the terminal end is grounded so the boundary conditions are $V = 1300$ at $x = 0$, and $V = 0$ at $x = 1000$, and let $V_s^*(x)$ be the steady state voltage after grounding the cable at the terminal end, then $V_s^* = c_1^* + c_2^* x$, gives $c_1^* = 1300$, $c_2^* = -1.3$ thus

$$V_s^*(x) = 1300 - (1.3)x. \quad \text{...(17.247)}$$

Therefore,

$$V(x, t) = V_s^*(x) + V^*(x, t),$$

where $V^*(x, t)$ is the transient part, the transient solution of the telegraph equation ...(17.248)

$$\frac{\partial V}{\partial t} = a^2\frac{\partial^2 V}{\partial x^2}, \quad a^2 = \frac{1}{RC}$$

with boundary conditions $V^*(0, t) = 0$, and $V^*(1000, t) = 0$,

obtained using $V^*(x, t) = V(x, t) - V_s^*(x)$.

It is given by, refer to Example 17.10b,

$$V^*(x,t)=\sum_{n=1}^{\infty} b_n e^{-\frac{a^2n^2\pi^2}{l^2}t}\sin\frac{n\pi x}{l}; \qquad ...(17.249)$$

here $l=1000$ k.m. and $a^2=1/RC$ and hence from (17.247), (17.248) and (17.249), we obtain

$$V(x,t)=1300-(1.3)x+\sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2 t)/l^2RC}\sin\frac{n\pi x}{l}. \qquad ...(17.250)$$

Setting $t=0$ and using the initial condition (17.246), we have

$$1300-(0.1)x=1300-(1.3)x+\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l},\quad \text{or,}\quad 1.2x=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l},$$

the Fourier sine series expansion of $f(x)=1.2x$ in the interval $0\le x\le l$, where $l=1000$. Thus,

$$b_n=\frac{2}{1000}\int_0^{1000} 1.2x\sin\frac{n\pi x}{1000}dx$$

$$=\frac{2.4}{1000}\left[x\left(-\frac{1000}{n\pi}\right)\cos\frac{n\pi x}{1000}+1\left(\frac{1000}{n\pi}\right)^2\sin\frac{n\pi x}{1000}\right]_0^{1000}=\frac{2400}{\pi}\frac{(-1)^{n+1}}{n}.$$

Substituting in (17.250), we obtain

$$V(x,t)=1300-(1.3)x+\frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{n^2\pi^2 t}{10^6 RC}}\sin\frac{n\pi x}{1000}, \text{ as the required potential.}$$

Example 17.25: A submarine cable of length l km has a resistance of R ohms/km and capacitance of C farad/km. Initially both the ends are grounded so that the cable line is uncharged. At time $t=0$, constant e.m.f. E_0 is applied at the one end while the other is kept grounded. Find the expressions for the transient and steady state currents at the grounded end, assuming the inductance and leakage to the ground negligible.

Solution: Since inductance and leakage are negligible thus taking $L=0$ and $G=0$, we obtain the submarine equation as, refer to Example 17.24,

$$\frac{\partial^2 V}{\partial x^2}=RC\frac{\partial V}{\partial t}. \qquad ...(17.251),$$

subject to the initial conditon

$$V(x,0)=0,\text{ for } 0\le x\le l. \qquad ...(17.252)$$

Let $V(x,t)=X(x)T(t)$ be a solution of (17.251). Substituting it we obtain

$$X''T = RCXT', \text{ or } \quad \frac{X''}{X} = RC\frac{T'}{T} = -k^2, \text{ say.}$$

This gives two differential equations $X'' + k^2 X = 0$, and $T' + (k^2/RC)T = 0$.

Solving we obtain $X = c_1 \cos kx + c_2 \sin kx$ and $T = c_3 e^{-k^2 t/RC}$.

Thus the solution $V(x, t)$ is of the form

$$V(x, t) = (A \cos kx + B \sin kx)\, e^{-k^2 t/RC}. \qquad ...(17.253)$$

Applying the initial condition (17.252) at $x = 0$ and at $x = l$, we obtain from (17.253)

$$A = 0, \text{ and } kl = n\pi, \text{ or } \quad k = \frac{n\pi}{l}, \quad n \text{ being an integer.}$$

Substituting in (17.253), we obtain

$$V(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2 RC}}. \qquad ...(17.254)$$

Also the end conditions of the system at $t = 0$, are $V = 0$ at $x = 0$ and $V = E_0$ at $x = l$.

Thus taking $V(x, 0) = a + bx$, gives $a = 0$ and $b = \frac{E_0}{l}$, which gives $V(x, 0) = \frac{E_0}{l}x$.

Adding this factor to (17.254) we obtain the complete solution as

$$V(x, t) = \frac{E_0}{l}x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2 RC}} \qquad ...(17.255)$$

Set $t = 0$ and apply the initial condition $V(x, 0) = 0$, we obtain from (17.255)

$$-\frac{E_0 x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l},$$

which is the Fourier sine-series expansion of the function $f(x) = -\frac{E_0 x}{l}$ in $0 < x < l$, hence

$$b_n = \frac{2}{l}\int_0^l \left(-\frac{E_0 x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{-2E_0}{l^2}\left[x\left(\frac{-l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) + \left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l}\right]_0^l = \frac{2E_0}{n\pi}(-1)^n.$$

Substituting in (17.255), we obtain

$$V(x, t) = \frac{E_0 x}{l} + \frac{2E_0}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2 RC}}. \qquad ...(17.256)$$

Next for $L = 0$, the first transmission equation, refer to Eq. 17.224, becomes

$$I = -\frac{1}{R}\frac{\partial V}{\partial x} = -\frac{E_0}{lR} - \frac{2E_0}{lR}\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2 RC}}.$$

Thus at the grounded end $x = 0$, the current is

$$I(t) = -\frac{E_0}{lR} - \frac{2E_0}{lR}\sum_{n=1}^{\infty}(-1)^n e^{-\frac{n^2\pi^2 t}{l^2 RC}}.$$

When $t \to \infty$, the steady state current is $-E_0/lR$.

EXERCISE 17.7

1. Obtain potential $V(x, t)$ in a cable l km long immediate after t seconds the ends are suddenly grounded, assuming the resistance and leakage to be negligible, when it is given that initially $I(x, 0) = I_0$ and $V(x, 0) = E_0 \sin(\pi x/l)$.
2. Solve the radio equation $\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}$ when a periodic e.m.f. E_0 cos pt is applied at the end $x = 0$ of the cable.
3. Neglecting R and G, find the potential $V(x, t)$ in a line of length l, t seconds after the ends were suddenly grounded, given that $I(x, 0) = I_0$ and $V(x, 0) = a \sin(\pi x/l) + b \sin(5\pi x/l)$.
4. A steady voltage distribution of 20 volts at the source end and of 12 volts at the terminal end, is maintained in a telephone wire of length l. At time $t = 0$, the terminal end is grounded, find the voltage t sec later, neglecting leakage and inductance.
5. A telephone line 3000 km long has a resistance of 4 ohms/km and a capacitance of 5×10^{-7} farad/km. Initially both the ends are grounded so that the line is uncharged. At time $t = 0$, a constant e.m.f. of 2×10^4 volts is applied to one end, while the other end is left grounded. Find the current at the grounded end after 1.0 sec. assuming the inductance and leakage to be negligible.

17.14 LAPLACE'S EQUATION IN THREE DIMENSIONS. SOLUTIONS OF CARTESIAN, CYLINDRICAL AND SPHERICAL POLAR FORMS

The Laplace's equation in three dimensions is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \quad ...(17.257)$$

It is one of the most important partial differential equations in engineering applications. The expression $\nabla^2 u$ is called the Laplacian of u. The theory of the solution of Eq. (17.257) is called *potential theory* and the solutions which have continuous second order derivatives are called *harmonic functions*. The equation occurs in three-dimensional problem involving potentials, such as potential for force fields in mechanics or electromagnetic or gravitational field, steady-state heat flow and fluid flow. For example, the gravitational potential $\phi(x, y, z)$ at a point (x, y, z) resulting from a particle of mass m situated at (x_0, y_0, z_0) is

$$\phi(x, y, z) = \frac{Gm}{r}, \text{ where } \quad r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}.$$

Here Gm = constant. We can easily verify that the gravitational potential function $\phi(x, y, z)$ satisfies the Laplace's equation, that is, $\nabla^2\phi = 0$.

If the mass at the point (x_0, y_0, z_0) is replaced by the charged particle, then the gravitational potential is replaced by the electrostatic potential at (x, y, z) which again satisfies the Laplace's equation (17.257).

In most of the practical problems involving Laplace's equation we need to solve the boundary value problem in a region R with boundary surface S. Such a problem is called *Dirichlet problem* if the function $u(x, y, z)$ is prescribed on S. It is called *Neumann problem* if the normal derivative $\frac{\partial u}{\partial n}$ is prescribed on S. In case u is prescribed on a portion of S and $\frac{\partial u}{\partial n}$ on the remaining portion of S, the problem is called *mixed boundary value problem*. We will consider only the Dirichlet problems.

Further, a proper choice of the coordinate system, depending on the region R, makes the solution of the problem simpler.

In case the region is a cuboid we consider the Laplace's equation in cartesian coordinates. In case the problem involves spherical boundary we prefer spherical coordinates and in case of cylindrical boundaries the equation in cylindrical coordinates is considered. Next we consider the solution of the Laplace's equation in different coordinate systems.

17.14.1 Solution of Laplace's Equation in Three-Dimensional Cartesian Form

The Laplace's equation in cartesian form is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0. \qquad \text{...(17.258)}$$

Let

$$u(x, y, z) = X(x)Y(y)Z(z) \qquad \text{...(17.259)}$$

be a solution of Eq. (17.258) in separable form. Substituting and dividing by XYZ, we obtain

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z} = 0, \qquad \text{...(17.260)}$$

where the primes denote the derivatives w.r.t. the corresponding independent variables.

As x, y, z are independent, Eq. (17. 260) will hold if each term on the left side of it is a constant. Introducing two separable constants, say k^2 and l^2, we obtain three second order ordinary differential equations $X'' - k^2X = 0$, $Y'' - l^2Y = 0$ and $Z'' + (k^2 + l^2)Z = 0$.

Their solutions respectively are

$$X = c_1e^{kx} + c_2e^{-kx}, \quad Y = c_3e^{ly} + c_4e^{-ly}, \quad \text{and} \quad Z = c_5\cos\left(\sqrt{k^2+l^2}\right)z + c_6\sin\left(\sqrt{k^2+l^2}\right)z$$

and hence the solution (17.259) becomes

$$u(x, y, z) = (c_1 e^{kx} + c_2 e^{-kx})(c_3 e^{ly} + c_4 e^{-ly})\left(c_5 \cos\left(\sqrt{k^2 + l^2}\right)z + c_6 \sin\left(\sqrt{k^2 + l^2}\right)z\right). \quad ...(17.261)$$

With the selection of constants as $-k^2$ and $-l^2$, an alternate form of solution is

$$u(x, y, z) = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)\left(c_5 e^{(\sqrt{k^2+l^2})z} + c_6 e^{-(\sqrt{k^2+l^2})z}\right). \quad ...(17.261a)$$

The form to be selected and the values of the constants depend upon the boundary conditions prescribed on the surface.

17.14.2 Solution of Laplace Equation in Cylindrical Coordinates Form

The Laplace's equation in cylindrical coordinates (r, θ, z), related to (x, y, z) by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

is,
$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0. \quad ...(17.262)$$

The Eq. (17.262) is simply obtained from Laplace's equation in polar form, refer Eq. (17.129), by adding the term $\frac{\partial^2 u}{\partial z^2}$. Let

$$u(r, \theta, z) = R(r)\Theta(\theta)Z(z) \quad ...(17.263)$$

be a solution of Eq. (17.262) in separable form. Substituting and dividing by $R\Theta Z$, we obtain

$$\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{\Theta''}{r^2\Theta} + \frac{Z''}{Z} = 0, \text{ or, } \frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{\Theta''}{r^2\Theta} = -\frac{Z''}{Z}, \quad ...(17.264)$$

where the primes denote the derivatives w.r.t. the corresponding independent variables.

The left side of Eq. (17.264) depends on r and θ and right on z only. Thus (17.264) holds if each side is equal to a constant, say $-k^2$, which gives

$$Z'' - k^2 Z = 0, \quad ...(17.265)$$

and,
$$\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -k^2. \quad ...(17.266)$$

The Eq. (17.266) can be rewritten as

$$r^2\frac{R''}{R} + r\frac{R'}{R} + k^2 r^2 = -\frac{\Theta''}{\Theta}. \quad ...(17.267)$$

Applying the same argument, let each side be equal to a constant, say l^2. This gives the equations

$$\Theta'' + l^2\Theta = 0 \quad ...(17.268)$$

and, $r^2\frac{R''}{R} + r\frac{R'}{R} + k^2r^2 - l^2 = 0$, or, $r^2R'' + rR' + (k^2r^2 - l^2)R = 0.$ ...(17.269)

Eq. (17.269) is reducible to Bessel's equation of order l, refer to Example 12.25, and its solution is

$$R = c_1 J_l(kr) + c_2 Y_l(kr),$$

where J_l and Y_l respectively are the Bessel's functions of first kind and second kind of order l.

Also the solutions of Eqs. (17.268) and (17.265) are respectively

$\Theta = (c_3 \cos l\theta + c_4 \sin l\theta)$ and, $Z = c_5 e^{kz} + c_6 e^{-kz}$.

Thus the solution (17.263) becomes

$u(r, \theta, z) = [c_1 J_l(kr) + c_2 Y_l(kr)][c_3 \cos l\theta + c_4 \sin l\theta][c_5 e^{kz} + c_6 e^{-kz}]$,

which is known as a *cylindrical harmonic.*

17.14.3 Solution of Laplace Equation in Spherical Polar Coordinates Form

The Laplace's equation in spherical polar co-ordinate form (r, θ, ϕ), related to (x, y, z) by

$$x = r \cos\theta \sin\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\phi$$

is,
$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} = 0. \quad ...(17.270)$$

Let
$$u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi) \quad ...(17.271)$$

be a solution of (17.270) in separable form.

Substituting this in (17.270) and dividing by $R\Theta\Phi$, we obtain

$$\frac{r^2 R'' + 2rR'}{R} + \frac{\Phi'' + (\cot\phi).\Phi'}{\Phi} + \frac{\Theta''}{\Theta \sin^2\phi} = 0$$

or,
$$\frac{\Phi'' + (\cot\phi)\Phi'}{\Phi} + \frac{\Theta''}{\Theta\sin^2\phi} = -\frac{r^2R'' + 2rR'}{R}, \quad ...(17.272)$$

where the primes denote the derivatives w.r.t. the corresponding independent variables.

The left side of (17.272) is a function of θ and ϕ only while right side is a function of r only, and since r, θ and ϕ are independent, so each side must be equal to a constant, say $-n(n + 1)$, thus

$$\frac{\Phi'' + (\cot\phi)\Phi'}{\Phi} + \frac{\Theta''}{\Theta\sin^2\phi} = -\frac{r^2R'' + 2rR'}{R} = -n(n+1)$$

This gives two differential equations

$$r^2R'' + 2rR' - n(n+1)R = 0, \quad ...(17.273)$$

and,
$$\frac{[\Phi'' + (\cot\phi)\Phi']}{\Phi}\sin^2\phi + n(n+1)\sin^2\phi = -\frac{\Theta''}{\Theta}. \quad ...(17.274)$$

Applying the same argument to Eq. (17.274) and setting each side equal to a constant say m^2, we obtain the differential equations

$$\Theta'' + m^2\Theta = 0 \quad ...(17.275)$$

and,
$$\Phi'' + (\cot\phi)\Phi' + [n(n+1) - m^2 \operatorname{cosec}^2\phi]\Phi = 0 \quad ...(17.276)$$

To solve Eq. (17.273), set $R = r^k$, this gives

$$k(k-1) + 2k = n(n+1), \text{ that is, } k = n, \text{ or } -(n+1), \text{ and hence}$$

$$R = c_1 r^n + c_2 r^{-(n+1)}. \qquad ...(17.277)$$

Eq. (17.275) gives

$$\Theta = (c_3 \cos m\theta + c_4 \sin m\theta). \qquad ...(17.278)$$

To solve Eq. (17.276) set, $\alpha = \cos\phi$, we obtain the equation

$$(1-\alpha^2)\frac{d^2\Phi}{d\alpha^2} - 2\alpha\frac{d\Phi}{d\alpha} + \left(n(n+1) - \frac{m^2}{1-\alpha^2}\right)\Phi = 0,$$

with solution $\Phi = c_5 P_n^m(\alpha) + c_6 Q_n^m(\alpha)$, where

$$P_n^m(\alpha) = (1-\alpha^2)^{m/2}\frac{d^m P_n}{d\alpha^m} \quad \text{and} \quad Q_n^m(\alpha) = (1-\alpha^2)^{m/2}\frac{d^m Q_n}{d\alpha^m}$$

are *associated Legendre functions*, see Appendix III.

Substituting for $\alpha = \cos\phi$ it becomes

$$\Phi = c_5 P_n^m(\cos\phi) + c_6 Q_m^n(\cos\phi) \qquad ...(17.279)$$

Hence the solution (17.271) becomes

$$u_{nm}(r, \theta, \phi) = [c_1 r^n + c_2 r^{-(n+1)}][c_3 \cos m\theta + c_4 \sin m\theta][c_5 P_n^m(\cos\phi) + c_6 Q_n^m(\cos\phi)]$$

which is known as a *spherical harmonic*.

The general solution by superposition is

$$u(r, \theta, \phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[c_1 r^n + c_2 r^{-(n+1)}][c_3 \cos m\theta + c_4 \sin m\theta]\,[c_5 P_n^m(\cos\phi) + c_6 Q_n^m(\cos\phi)].$$

The choice of the constants depends upon the boundary conditions specified.

Example 17.26: Solve the three-dimensional problem $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$ in the unit prism $0 < x < 1,\ 0 < y < 1,\ 0 < z < 1$, where $u = 0$ on each of face except for the face $z = 1$ on which it is prescribed as $f(x, y) = xy$.

Solution: We are to solve the Laplace's equation

$$u_{xx} + u_{yy} + u_{zz} = 0, \qquad ...(17.280)$$

subject to the boundary conditions

$$u(0, y, z) = u(1, y, z) = u(x, 0, z) = u(x, 1, z) = u(x, y, 0) = 0 \text{ and } u(x, y, 1) = xy.$$

The solution of the Eq. (17.280) in separable form, refer to 17.261a, is

$$u(x, y, z) = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)\left(c_5 e^{(\sqrt{k^2+l^2})z} + c_6 e^{-(\sqrt{k^2+l^2})z}\right). \qquad ...(17.281)$$

Applying the boundary conditions $u(0, y, z) = 0$ and $u(1, y, z) = 0$ to (17.281), we obtain $c_1 = 0$ and $\sin k = 0$, which gives $k = n\pi$, n being an integer. Similarly, applying boundary conditions $u(x, 0, z) = u(x, 1, z) = 0$ implies $c_3 = 0$ and $\sin l = 0$, which gives $l = m\pi$, m being an integer.

Next, applying the boundary condition $u(x, y, 0) = 0$, (17.281) gives, $c_5 = -c_6 = c$, say. Thus the solution (17.281) becomes

$$u_{nm}(x, y, z) = cc_2c_4 \sin n\pi x \sin m\pi y\,(e^{(\sqrt{n^2+m^2})\pi z} - e^{-(\sqrt{n^2+m^2})\pi z}),\ n, m \text{ being integer.}$$

By superposition principle the solution is given as

$$u(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm} \sin n\pi x \sin m\pi y \sinh\left(\pi\sqrt{n^2+m^2}\right)z \quad \text{...(17.282)}$$

Next applying the boundary condition $u(x, y, 1) = xy$ in (17.282), we obtain

$$xy = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm} \sinh\left(\pi\sqrt{n^2+m^2}\right) \sin n\pi x \sin m\pi y.$$

which is double Fourier series expansion of $f(x, y) = xy$ in the region $0 \le x \le 1$ and $0 \le y \le 1$, and hence

$$D_{nm} = \frac{4}{\sinh\left(\pi\sqrt{n^2+m^2}\right)} \int_0^1\int_0^1 xy \sin n\pi x \sin m\pi y\, dx\, dy$$

$$= \frac{4}{\sinh\left(\pi\sqrt{n^2+m^2}\right)} \left(\int_0^1 x \sin n\pi x\, dx\right)\left(\int_0^1 y \sin m\pi y\, dy\right) = \frac{4(-1)^{n+m}}{\sinh\left(\pi\sqrt{n^2+m^2}\right) nm\pi^2}.$$

Substituting for D_{nm} in (17.282), we obtain

$$u(x, y, z) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{nm \sinh\left(\pi\sqrt{n^2+m^2}\right)} \sin n\pi x \sin m\pi y \sinh\left(\pi\sqrt{n^2+m^2}\right)z$$

as the desired solution.

Example 17.27: Solve the Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$ with boundary conditions $u = 0$ when $x \to \infty$, also when $y \to \infty$ and, also when $z = 0$.

Solution: The solution of the Laplace's equation in the separable form, refer to 17.261, is

$$u(x, y, z) = (c_1e^{kx} + c_2e^{-kx})(c_3e^{ly} + c_4e^{-ly})\left[c_5 \cos\left(\sqrt{k^2+l^2}\right)z + c_6 \sin\left(\sqrt{k^2+l^2}\right)z\right] \quad \text{...(17.283)}$$

Applying $u = 0$ when $x \to \infty$, we have $c_1 = 0$. Next, applying $u = 0$ when $y \to \infty$, we have $c_3 = 0$ and, applying $u = 0$ when $z = 0$, we have $c_5 = 0$. Thus (17.283) reduces to

$$u(x, y, z) = Ae^{-(kx + ly)} \sin\left(\sqrt{k^2+l^2}\right)z,$$

where A, k and l are constants. In case k and l are integers, the general solution by the principle of superposition is, $u(x, y, z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A_{kl}e^{-(kx+ly)} \sin\left(\sqrt{k^2+l^2}\right)z.$

Example 17.28: Find the steady state temperature distribution in radially symmetric circular cylinder of radius a and height h, given that $u = 0$ on the surface $z = 0$, the curved surface and $u = f(r)$, $0 \le r \le a$ on the surface $z = h$.

Solution: The steady-state temperature distribution is given by Laplace's equation in cylindrical coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0. \qquad ...(17.284)$$

Since the cylinder is radially symmetric the temperature distribution is independent of θ, say $u(r, z)$, and is the solution of the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \qquad ...(17.285)$$

subject to the boundary conditions

$$u(r, 0) = 0,\quad 0 \le r \le a;\quad u(a, z) = 0,\quad 0 \le z \le h;\quad \text{and}\quad u(r, h) = f(r),\quad 0 \le r \le a.$$

Let the solution be $u(r, z) = R(r)\, Z(z)$. Substituting in Eq. (17.285), we obtain

$$\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) = -\frac{Z''}{Z} = -k^2, \quad \text{(say)} \qquad ...(17.286)$$

where a negative sign of the constant is chosen in compatible with the boundary conditions given.

From (17.286), we have $R'' + \frac{1}{r}R' + k^2R = 0$, and $Z'' - k^2Z = 0$.

Solving these we obtain respectively

$$R(r) = c_1J_0(kr) + c_2Y_0(kr) \quad \text{and} \quad Z(z) = c_3e^{kz} + c_4e^{-kz},$$

where J_0 and Y_0 are the Bessel functions of order zero of first kind and second kind respectively.

Hence the solution is

$$u(r, z) = [c_1Jo(kr) + c_2Y_0(kr)][\,c_3e^{kz} + c_4e^{-kz}]. \qquad ...(17.287)$$

Since the temperature should be finite at $r = 0$, thus $c_2 = 0$, for $Y_0(kr)$ is infinite at $r = 0$. Hence

$$u(r, z) = J_0(kr)[c_3'e^{kz} + c_4'e^{-kz}]. \qquad ...(17.288)$$

Applying the boundary condition $u(r, 0) = 0$, (17.288) gives $c_4' = -c_3'$ and hence (17.288) becomes

$$u(r, z) = c_3'J_0(kr)[e^{kz} - e^{-kz}]. \qquad ...(17.289)$$

Next using the boundary condition $u(a, z) = 0$, gives $J_0(ka) = 0$, that is $ka = \alpha_n$, $n = 1, 2, ...$ where α_n's are the zeros of $J_0(x)$. Hence the general solution is

$$u(r,z) = \sum_{n=1}^{\infty} A_nJ_0\left(\frac{\alpha_n r}{a}\right)\left(e^{\frac{\alpha_n z}{a}} - e^{-\frac{\alpha_n z}{a}}\right). \qquad ...(17.290)$$

To evaluate A_n, we apply the boundary condition $u(r, h) = f(r)$ to (17.290). It gives

$$f(r) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_n r}{a}\right)\left(e^{\frac{\alpha_n h}{a}} - e^{-\frac{\alpha_n h}{a}}\right).$$

Thus $A_n\left(e^{\frac{\alpha_n h}{a}} - e^{-\frac{\alpha_n h}{a}}\right)$ are the coefficients of the Fourier-Bessel series expansion of $f(r)$, refer to Section 12.8, and are given by

$$\left(e^{\frac{\alpha_n h}{a}} - e^{-\frac{\alpha_n h}{a}}\right)A_n = \frac{2}{a^2 J_1^2\left(\frac{\alpha_n h}{a}\right)} \int_0^a r f(r)\, J_0\left(\frac{\alpha_n r}{a}\right) dr,$$

where α_n are the zeros of $J_0(x)$. If $f(r)$ is given, then coefficients A_n can be evaluated.

Example 17.29: Find the potential inside and outside a radially symmetric spherical capacitor of two metallic hemispheres of radius 1 ft separated by a small slit for reason of insulation, if the upper hemisphere is kept at 110 volts and the lower is grounded.

Solution: Since the spheres are radially symmetric the solution u is independent of θ. Thus $u_{\theta\theta} = 0$, so the Laplace's equation (17.270) becomes

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \phi^2} + \cot \phi \frac{\partial u}{\partial \phi} = 0, \quad \text{...(17.291)}$$

subject to the boundary conditions that $u(r, \phi)$ is finite, continuous and $u(r, \phi) \to 0$ as $r \to 0$; also

$$u(1, \phi) = f(\phi) = \begin{cases} 110, & \text{if } 0 \le \phi \le \pi/2 \\ 0, & \text{if } \pi/2 < \phi \le \pi \end{cases} \quad \text{...(17.292)}$$

To solve (17.291) substitute $u(r, \phi) = R(r)\,\Phi(\phi)$, and then divide by $R\Phi$, to obtain

$$\frac{r^2 R'' + 2rR'}{R} + \frac{\Phi'' + \cot \phi \Phi'}{\Phi} = 0 \text{ or, } \frac{1}{R}(r^2 R'' + 2rR') = -\frac{1}{\Phi}(\Phi'' + \cot \phi\, \Phi'). \quad \text{...(17.293)}$$

The independent variables in (17.293) are separated, the two sides must be equal to a constant, say $k = n(n + 1)$, this gives the two differential equations as

$$r^2 R'' + 2rR' - n(n+1)R = 0 \quad \text{...(17.294)}$$

and,
$$\Phi'' + (\cot \phi)\Phi' + n(n+1)\Phi = 0. \quad \text{...(17.295)}$$

To solve (17.294), substitute $R = r^k$, we obtain $k(k-1) + 2k - n(n+1) = 0$, which gives $k = n$ and $-(n+1)$, where n is arbitrary. Thus the solutions are

$$R_n(r) = r^n, \text{ and } R_n^*(r) = \frac{1}{r^{n+1}}.$$

To solve Eq. (17.295), set $\alpha = \cos \phi$, we have

$$\frac{d}{d\phi} = \frac{d}{d\alpha}\frac{d\alpha}{d\phi} = -\sin \phi \frac{d}{d\alpha} \text{ and, } \frac{d^2}{d\phi^2} = \frac{d}{d\phi}\left(-\sin \phi \frac{d}{d\alpha}\right) = -\left[\cos \phi \frac{d}{d\alpha} - \sin^2 \phi \frac{d^2}{d\alpha^2}\right].$$

Hence Eq. (17.295) becomes

$$\sin^2\phi\frac{d^2\Phi}{d\alpha^2}-2\cos\phi\frac{d\Phi}{d\alpha}+n(n+1)\Phi=0$$

or,
$$(1-\alpha^2)\frac{d^2\Phi}{d\alpha^2}-2\alpha\frac{d\Phi}{d\alpha}+n(n+1)\Phi=0, \quad ...(17.296)$$

which is Legendre's equation.

For the solutions of the system under consideration to be continuous, we need to take n as non-negative integer. For integer $n=0,1,2,...,$ the solutions of Eq. (17.296) are the Legendre polynomials

$$\Phi=P_n(\alpha)=P_n(\cos\phi),\ n=0,1,2,...$$

Hence,
$$u_n(r,\phi)=\left(c_1r^n+c_2\frac{1}{r^{n+1}}\right)P_n(\cos\phi),\ n=0,1,2,... \quad ...(17.297)$$

give the sequence of solution.

Next for inside the spherical capacitor $u(r,\phi)$ must be finite at $r=0$ and hence from (17.297), $c_2=0$. Thus inside the capacitor potential is

$$u(r,\phi)=\sum_{n=0}^{\infty}a_nr^nP_n(\cos\phi),\quad 0\le r\le 1. \quad ...(17.298)$$

For outside the spherical capacitor, since $u(r,\phi)$ is to be zero as $r\to\infty$, thus we should have from (17.297), $c_1=0$, and hence outside the capacitor the potential is

$$u(r,\phi)=\sum_{n=0}^{\infty}\frac{b_n}{r^{n+1}}P_n(\cos\phi),\quad r\ge 1. \quad ...(17.299)$$

To find a_n we apply the initial condition (17.292), thus from (17.298)

$$u(1,\phi)=f(\phi)=\sum_{n=0}^{\infty}a_nP_n(\cos\phi) \quad ...(17.300)$$

which is Fourier-Legendre series of $f(\phi)$, where $f(\phi)=\begin{cases}110, & 0\le\phi\le\pi/2\\ 0, & \pi/2<\phi\le\pi.\end{cases}$

Rewriting (17.300) as $f(\phi)=\sum_{n=0}^{\infty}a_nP_n(\alpha)$, where $\alpha=\cos\phi$.

Multiplying both sides by $P_n(\alpha)$, integrating from -1 to 1 and then using the orthogonality of the Lagendre polynomials on $[-1,1]$, refer to Section 12.3, we have

$$a_n\int_{-1}^{1}P_n^2(\alpha)d\alpha=\int_{-1}^{1}f(\phi)P_n(\alpha)d\alpha$$

or,
$$a_n \cdot \frac{2}{2n+1} = \int_{-1}^{1} f(\phi) P_n(\alpha)\, d\alpha = \int_0^{\pi} f(\phi) P_n(\cos\phi) \sin\phi\, d\phi$$

or,
$$a_n = 55(2n+1)\int_0^{\pi/2} P_n(\cos\phi)\sin\phi\, d\phi, \text{ (subsitituting for } f(\phi))$$

$$= 55(2n+1)\int_0^1 P_n(w)dw, \quad (w = \cos\phi)$$

$$= 55(2n+1)\int_0^1 \left[\sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} w^{n-2r}\right] dw, \text{ refer to Eq. (12.41)}$$

where $N = n/2$ for even n and equal to $(n-1)/2$ for odd n. Thus,

$$a_n = \frac{55(2n+1)}{2^n} \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{r!(n-r)!(n-2r+1)!}, \text{ since } \int_0^1 w^{n-2r} dw = \frac{1}{(n-2r+1)}.$$

For $n = 0$, $a_0 = 55$; $n = 1$, $a_1 = 165/2$; $n = 2$, $a_2 = 0$, $n = 3$, $a_3 = -385/8$ etc.

Hence the potential inside the sphere, from (17.298), is

$$u(r, \phi) = 55 + \frac{165}{2} r P_1(\cos\phi) - \frac{385}{8} r^3 P_3(\cos\phi) + \ldots, \ 0 \le r \le 1,$$

since, $P_0 = 1$. Here P_1, P_3 are the Legendre polynomials.

To find b_n we again apply the initial condition (17.292). Thus, from (17.299)

$$u(1, \phi) = \sum_{n=0}^{\infty} b_n P_n(\cos\phi),$$

which is same as (17.300), since $r = 1$; and hence $b_n = a_n$. Thus the potential outside the sphere, from (17.299), is

$$u(r, \phi) = \frac{55}{r} + \frac{165}{2r^2} P_1(\cos\phi) - \frac{385}{8r^4} P_3(\cos\phi) + \ldots, r \ge 1$$

From these series potentials can be computed upto the desired degree of accuracy.

EXERCISE 17.8

1. Solve $\nabla^2\phi\,(x, y, z) = 0$ in the region $0 \le x, y, z \le a$, subject to the conditions
   (i) $\phi = 0$ on $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, (ii) $\phi = f(x, y)$ on $z = a$.
2. Solve $\nabla^2 u(x, y, z) = 0$, for $0 \le x \le 1$, $0 \le y \le 2\pi$, $0 \le z \le \pi$ subject to the conditions $u(0, y, z) = u(1, y, z) = u(x, 0, z) = u(x, y, 0) = 0$, $u(x, 2\pi, z) = 2$ and $u(x, y, \pi) = 1$.

3. Find an expression for the steady state temperature distribution in the sphere of radius R with radial symmetry, if the initial data is given by $f(\phi) = A\phi^2$, where $A > 0$ is a constant.
4. Solve for the steady-state temperature in a solid, closed hemisphere, which in spherical coordinates is given by $0 \le r \le R, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi/2$. The disk base is kept at temperature zero and the hemispherical surface is kept at constant temperature u_0. Assume that the temperature distribution is independent of θ.
5. The potential on the surface of a unit sphere is $f(\phi) = \cos 2\phi$. Show that the potential at all points of the space is given by

$$u(r, \phi) = \begin{cases} 2r^2\left(\cos^2\phi - \dfrac{1}{3}\right) - \dfrac{1}{3}, & 0 \le r \le 1 \\ 2r^{-3}\left(\cos^2\phi - \dfrac{1}{3}\right) - \dfrac{r^{-1}}{3}, & r \ge 1 \end{cases}.$$

6. Show that the solution of a Laplace equation in cylindrical coordinates which remains finite at $r = 0$ may be expressed in the form

$$u(r, \theta, z) = \sum_{n=0}^{\infty} J_n(kr)\{e^{kz}(A_n \cos n\theta + B_n \sin n\theta) + e^{-kz}(C_n \cos n\theta + D_n \sin n\theta)\}.$$

7. Show that the steady-state temperature distribution in the material between two coaxial cylinders of radii r_1 and r_2 kept at temperatures t_1 and t_2 is given by

$$u(r) = (t_2 - t_1)\frac{\ln r}{\ln (r_2/r_1)} + \frac{t_1 \ln r_2 - t_2 \ln r_1}{\ln (r_2/r_1)}, \quad r_1 \le r \le r_2.$$

8. Find the temperature in a homogeneous ball of radius 1 unit if the lower boundary hemisphere is kept at 0°C and upper at 20°C.

17.15 SOLUTIONS OF HEAT, WAVE AND LAPLACE'S EQUATIONS BY FOURIER TRANSFORMS

In this section we apply Fourier transforms to solve heat equation, wave equation and Laplace's equation. The Fourier transform is applied to problems concerning the entire axis, while the Fourier cosine and sine transforms to problems involving the positive half-axis. Before applying the transforms to partial differential equations, we find the Fourier transforms of partial derivatives of a function $f(x, t)$ of two independent variables x and t.

The *Fourier transform with respect to x of a function f(x, t) of two independent variables x* and *t*,

denoted by $F(w, t)$, is defined as $F(w, t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x, t) e^{-iwx}\, dx$;

and the *inversion integral*, is given by

$$f(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w, t) e^{iwx} dw. \quad ...(17.301)$$

The Fourier transform of $f(x, t)$ is also denoted by ${}_x\mathcal{F}[f(x, t)]$ or, ${}_x\hat{f}[f(x, t)]$, where the pre-suffix x denotes the variables being transformed.

Next, the *Fourier transform of the nth partial derivative of $f(x, t)$ with respect to x* is given by

$${}_x\hat{f}\left\{\frac{\partial^n}{\partial x^n}[f(x,t)]\right\} = (iw)^n F(w, t), \quad ...(17.302)$$

Also we have, $${}_x\hat{f}\{x^n f(x, t)\} = i^n \frac{\partial^n}{\partial w^n}[F(w, t)], \quad ...(17.303)$$

and $${}_x\hat{f}\left\{x^m \frac{\partial^n}{\partial x^n}[f(x,t)]\right\} = i^{m+n} \frac{\partial^m}{\partial w^m}[w^n F(w, t)]. \quad ...(17.304)$$

The *Fourier cosine transform of a function $f(x, t)$ with respect to x* is defined by

$${}_x\hat{f}_c[f(x, t)] = F_c(w, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x, t) \cos wx \, dx, \quad ...(17.305)$$

and, the *inversion integral* is given by

$$f(x, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(w, t) \cos wx \, dw. \quad ...(17.306)$$

Similarly, the *Fourier sine transform* is defined by

$${}_x\hat{f}_s[f(x, t)] = F_s(w, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x, t) \sin wx \, dx, \quad ...(17.307)$$

and, the *inversion integral* is given by

$$f(x, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(w, t) \sin wx \, dw. \quad ...(17.308)$$

The *Fourier cosine and sine transforms of partial derivatives of the function $f(x, t)$* with respect to x are given by

$${}_x\hat{f}_c[f'(x, t)] = wF_s(w, t) - \sqrt{\frac{2}{\pi}} f(0, t) \quad ...(17.309)$$

$${}_x\hat{f}_s[f'(x, t)] = -wF_c(w, t) \quad ...(17.310)$$

$${}_x\hat{f}_c[f''(x, t)] = -w^2 F_c(w, t) - \sqrt{\frac{2}{\pi}} f'(0, t) \quad ...(17.311)$$

$$_x\hat{f}_s[f''(x, t)] = -w^2F_s(w, t) + \sqrt{\frac{2}{\pi}}\, wf(0, t). \quad ...(17.312)$$

Remarks 1. All the results stated above follow from the corresponding results derived in Sections 15.4 and 15.5.

2. In case the Fourier transform is applied with respect to one of the two independent variables in the partial differential equation, then the equation becomes an ordinary differential equation in terms of the other variables. We solve the differential equation obtained and the solution of the given boundary value problem is then obtained by taking the inverse Fourier transform.

Example 17.30: (Temperature in the infinite bar). The temperature distribution $u(x, t)$ in a thin laterally insulated homogeneous infinite bar is given by the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0. \quad ...(17.313)$$

Find $u(x, t)$ if the initial temperature is

$$u(x, 0) = f(x). \quad ...(17.314)$$

and $u(x, t)$ is finite as $x \to \pm\infty$.

Solution: Since the domain of definition is $(-\infty, \infty)$, we use Fourier transform with respect to x Let $_x\mathcal{F}[u(x, t)] = U(w, t)$ denote the Fourier transform of $u(x, t)$ w.r.t. x. Taking the Fourier transform w.r.t. x on both sides of the Eq. (17.313), we obtain

$$_x\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = {_x\mathcal{F}}\left[c^2 \frac{\partial^2 u}{\partial x^2}\right] = -c^2w^2\, {_x\mathcal{F}}[u(x, t)],$$

or,

$$_x\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = -c^2w^2U(w, t). \quad ...(17.315)$$

Also, $_x\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} ue^{-iwx}dx = \frac{\partial U(w, t)}{\partial t}$. Thus (17.315) becomes

$$\frac{\partial U(w, t)}{\partial t} = -c^2w^2U(w, t). \quad ...(17.316)$$

Since Eq. (17.316) involves only a derivative w.r.t. t, it can be considered as a first order ordinary differential equation with t as independent variable and w as a parameter. The general solution of Eq. (17.316) is

$$U(w, t) = ke^{-c^2w^2t}, \quad ...(17.317)$$

k being an arbitrary constant independent of t may depend on parameter w to be determined using the initial condition (17.314). The Fourier transform of the initial condition (17.314), is

$$\mathcal{F}_x[u(x, 0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx = U(w, 0) = a(w), \text{ say.} \qquad ...(17.318)$$

From (17.317) and (17.318), we have $k = U(w, 0) = a(w)$, thus, (17.317) becomes

$$U(w, t) = a(w)e^{-c^2w^2t}. \qquad ...(17.319)$$

The solution of given initial value problem is the inverse transform of $U(w, t)$. Using the definition of the inverse Fourier transform, (17.319) becomes

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(w)e^{-c^2w^2t} e^{iwx}dw \qquad ...(17.320)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v)e^{-iwv}\, dv \right] e^{-c^2w^2t^2} e^{iwx}dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v)e^{-iw(v-x)} e^{-c^2w^2t^2} dv\, dw$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) \cos w(v - x)e^{-c^2w^2t^2} dv\, dw - i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) \sin w(v - x)e^{-c^2w^2t^2} dv\, dw \right]. \qquad ...(17.321)$$

Since the real part on the right side of (17.321) is an even function of w and the imaginary part is an odd function of w, thus we have

$$U(x, t) = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(v) \cos w(v - x)e^{-c^2w^2t^2} dv \right) dw \qquad ...(17.322)$$

In case of specific initial conditions, like $u(x, 0) = f(x) = \begin{cases} u_0 = \text{constant}, & \text{if } |x| \le 1 \\ 0, & \text{if } |x| > 1 \end{cases}$, we have

$$U(w, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} u_0 e^{-iwx}dx = \frac{u_0}{\sqrt{2\pi}\, iw} [e^{iw} - e^{-iw}] = \frac{2u_0 \sin w}{w\sqrt{2\pi}} = a(w),$$

and thus (17.320) becomes

$$u(x, t) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin w}{w} e^{iwx} e^{-c^2w^2t} dw = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin w}{w} [\cos wx + i \sin wx] e^{-c^2w^2t} dw$$

$$= \frac{u_0}{\pi} \left[\int_{-\infty}^{\infty} \frac{1}{w} \sin w \cos wx e^{-c^2w^2t}dw + i \int_{-\infty}^{\infty} \frac{1}{w} \sin w \sin wx e^{-c^2w^2t^2} dw \right]$$

$$= \frac{2u_0}{\pi}\int_0^{\infty} \frac{1}{w} \sin w \cos wx e^{-c^2 w^2 t} dw,$$

since the real part on the right side is even function of w and the imaginary is an odd function of w.

Example 17.31: Solve the heat-flow equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0 \qquad ...(17.323)$$

subject to the conditions

$$u(x, 0) = f(x) \quad \text{and} \quad u(0, t) = 0. \qquad ...(17.324)$$

Solution: Since the domain of definition is $(0, \infty)$ we use Fourier sine transform, (application of Fourier cosine transfrom requires the knowledge of initial condition in terms of derivative). Taking the Fourier sine transform of the p.d.e. (17.323) w.r. t. x, we obtain

$${}_x\mathcal{F}_s\left[\frac{\partial u}{\partial t}\right] = {}_x\mathcal{F}_s\left[c^2 \frac{\partial^2 u}{\partial x^2}\right] = c^2\left[-w^2\, {}_x\mathcal{F}_s[u(x, t)] + \sqrt{\frac{2}{\pi}} w u(0, t)\right] = -c^2 w^2 U_s(w, t), \qquad ...(17.325)$$

since $u(0, t) = 0$.

$$\text{Also, } {}_x\mathcal{F}_s\left[\frac{\partial u}{\partial t}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{\partial u}{\partial t} \sin wx\, dx = \frac{\partial}{\partial t}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} u(x, t) \sin wx\, dx\right] = \frac{\partial U_s}{\partial t}(w, t),$$

thus, (17.325) becomes

$$\frac{\partial U_s}{\partial t}(w, t) = -c^2 w^2 U_s(w, t), \qquad ...(17.326)$$

an ordinary differential equation in U_s, with t as independent variable and w as a parameter. The general solution of Eq. (17.326) is

$$U_s(w, t) = k e^{-c^2 w^2 t}, \qquad ...(17.327)$$

k being an arbitrary constant depending on w, to be determined using the initial condition $u(x, 0) = f(x)$.

If ${}_x\mathcal{F}_s[f(x)] = {}_x\mathcal{F}_s[u(x, 0)] = U_s(w, 0) = a(w)$, then from (17.327), $k = a(w)$ and thus (17.327) becomes

$$U_s(w, t) = a(w)\, e^{-c^2 w^2 t}. \qquad ...(17.328)$$

The solution of the given heat flow problem is the inverse transform of $U_s(w, t)$. Using the definition of the inverse Fourier sine transform, (17.328) becomes

$$u(x, t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} a(w) e^{-c^2 w^2 t} \sin wx\, dw. \qquad ...(17.329)$$

Substituting $a(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \sin wv \, dv$ in (17.329), we obtain

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(v) \sin wv \sin wx \, e^{-c^2 w^2 t^2} dv \, dw,$$

as the solution.

In case of specific initial condition $u(x, 0) = f(x) = \begin{cases} u_0 = \text{cosntant}, & \text{if } 0 < x < l \\ 0, & \text{if } x \geq l \end{cases}$, we have

$$U_s(w, 0) = \sqrt{\frac{2}{\pi}} \int_0^l u_0 \sin wx \, dx = \sqrt{\frac{2}{\pi}} \frac{u_0(1 - \cos wl)}{w} = a(w).$$

Substituting in (17.329), we obtain the solution as

$$u(x, t) = \frac{2u_0}{\pi} \int_0^\infty \left(\frac{1 - \cos wl}{w} \right) e^{-c^2 w^2 t} \sin wx \, dw.$$

Example 17.32: Solve the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0. \qquad ...(17.330)$$

subject to the initial conditions $\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0$ and $y(x, 0) = f(x)$. ...(17.331)

Solution: Since the domain of definition is $(-\infty, \infty)$ we use Fourier transform w.r.t. x. Let ${}_x\mathcal{F}[y(x, t)] = Y(w, t)$ denote the Fourier transform of $y(x, t)$ w.r.t. x. Applying the Fourier transform w.r.t. x to the p.d.e. (17.330) we obtain

$$ {}_x\mathcal{F}\left[\frac{\partial^2 y}{\partial t^2} \right] = {}_x\mathcal{F}\left[c^2 \frac{\partial^2 y}{\partial x^2} \right] = -c^2 w^2 {}_x\mathcal{F}[y(x, t)] = -c^2 w^2 Y(w, t). \qquad ...(17.332)$$

Also, $$ {}_x\mathcal{F}\left[\frac{\partial^2 y}{\partial t^2} \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2} e^{-iwx} dx = \frac{\partial^2}{\partial t^2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x, t) e^{-iwx} dx \right] = \frac{\partial^2}{\partial t^2} Y(w, t).$$

Thus (17.332) becomes

$$\frac{\partial^2}{\partial t^2} Y(w, t) = -c^2 w^2 Y(w, t). \qquad ...(17.333)$$

Equation (17.333) can be considered as a second order ordinary differential equation in Y with t as independent variables and w as a parameter. The general solution of Eq. (17.333) is

$$Y(w, t) = k_1 \cos cwt + k_2 \sin cwt; \qquad ...(17.334)$$

k_1, k_2 being arbitrary constants independent of t, depending on the parameter w, to be determined from the initial conditions (17.331).

The Fourier transform w.r.t. x of the initial condition $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ gives

$$_x\mathcal{F}\left[\frac{\partial y}{\partial t}\right]_{t=0} = \frac{\partial}{\partial t}[_x\mathcal{F}\{y(x, t)\}]_{t=0} = \left(\frac{\partial Y}{\partial t}\right)_{t=0} = 0. \qquad ...(17.335)$$

Also, $$_x\mathcal{F}[y(x, 0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx = Y(w, 0) = a(w), \text{ say.} \qquad ...(17.336)$$

Using (17.335) and (17.336) in (17.334), we obtain $k_1 = a(w)$ and $k_2 = 0$, thus (17.334) becomes

$$Y(w, t) = a(w) \cos cwt. \qquad ...(17.337)$$

Solution $y(x, t)$ of the given initial value problem is the inverse transform of (17.337), given by

$$y(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(w) \cos cwt\, e^{iwx} dw. \qquad ...(17.338)$$

Substituting $a(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)e^{-iwv}\,dv$, (17.338) becomes

$$y(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v)\, e^{-iw(v-x)} \cos cwt\, dv\, dw$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) \cos w(v - x) \cos cwt\, dv\, dw - i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) \sin w(v - x) \cos cwt\, dv\, dw\right]$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(v) \cos w\,(v - x) \cos cwt\, dv\right)dw,$$

since the real part on the right is an even function of w and the imaginary is an odd function of w.

We can write the solution in the simpler form by rewriting (17.338) as

$$y(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(w)\, \frac{e^{icwt} + e^{-icwt}}{2} e^{iwx} dw = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} [a(w)e^{iw(x+ct)} + a(w)e^{-iw(x-ct)}]dw$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(w)e^{iw(x+ct)}dw + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(w)e^{iw(x-ct)}dw\right] = \frac{1}{2}[f(x+ct)+f(x-ct)],$$

a result already obtained in case of D'Alembert's solution of the wave equation, refer to Eq. (17.56).

Example 17.33: Solve the two-dimensional Laplace's equation $u_{xx} + u_{yy} = 0$ in the infinite strip $0 \le y \le a$, given that $u(x, 0) = 0$ and $u(x, a) = f(x)$, using an appropriate Fourier transform.

Solution: The Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad 0 < y < a, \qquad ...(17.339)$$

subject to the conditions $u(x, 0) = 0, \quad u(x, a) = f(x)$. ...(17.340)

Since, $-\infty < x < \infty$, thus we use Fourier transform with respect to x to solve this equation. Applying Fourier transform w.r.t. x, the Eq. (17.339) becomes

$$(iw)^2 U(w, y) + \frac{\partial^2 U(w, y)}{\partial y^2} = 0, \text{ or, } \frac{\partial^2 U(w, y)}{\partial y^2} - w^2 U(w, y) = 0, \qquad ...(17.341)$$

where $U(w, y) = {}_x\mathcal{F}[u(x, y)]$.

Eq. (17.341) is second order ordinary differential equation with U as dependent variable and y as independent variable and w as a parameter. The solution is

$$U(w, y) = k_1 e^{wy} + k_2 e^{-wy}, \qquad ...(17.342)$$

where k_1, k_2 are constants w.r.t. y, but may depend upon w. To determine k_1, k_2 apply Fourier transform w.r.t. x to the boundary conditions (17.340), we obtain

$$U(w, 0) = {}_x\mathcal{F}[u(x, 0)] = {}_x\mathcal{F}[0] = 0, \qquad ...(17.343)$$

and, $U(w, a) = {}_x\mathcal{F}[u(x, a)] = {}_x\mathcal{F}[f(x)] = F(w)$, say ...(17.344)

Using (17.343) and (17.344) in (17.342), we obtain

$$k_1 = -k_2 = \frac{F(w)}{e^{aw} - e^{-aw}} = \frac{F(w)}{2\sinh wa}, \text{ and hence (17.342) becomes}$$

$$U(w, y) = F(w)\frac{\sinh wy}{\sinh wa}. \qquad ...(17.345)$$

If $G(w, y)$ is defined as $G(w, y) = \dfrac{\sinh wy}{\sinh wa}$, then (17.345) becomes

$$U(w, y) = F(w)G(w, y). \qquad ...(17.346)$$

Applying inverse Fourier transform, (17.346) gives

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w)G(w, y)e^{iwx}\, dw. \qquad ...(17.347)$$

If $g(x, y) = {}_x\mathcal{F}^{-1}[G(w, y)]$ is the inverse Fourier transform of $G(w, y)$ w.r.t. x, then using the result $f * g = \int_{-\infty}^{\infty} F(w)G(w)e^{iwx}\, dw$, refer to (18.37), (17.347) becomes

$$u(x, y) = \frac{1}{\sqrt{2\pi}}(f * g) \qquad ...(17.348)$$

where $f * g$ denotes the convolution of f and g. Also by definition of inverse transform

$$g(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(w, y)e^{iwx}\, dw = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sinh wy}{\sinh wa} [\cos wx + i \sin wx]\, dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sinh wy}{\sinh wa} \cos wx\, dw + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sinh wy}{\sinh wa} \sin wx\, dw \qquad ...(17.349)$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{\sinh wy}{\sinh wa} \cos wx\, dw, \qquad ...(17.350)$$

since the first integral on the right side of (17.349) is an even function of w and second integral is an odd function of w.

Using (17.350) in the convolution (17.348) of f and g, we obtain

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)\, g(x - p, y)\, dp = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{w=0}^{\infty} \int_{-\infty}^{\infty} f(p) \frac{\sinh wy}{\sinh wa} \cos [w(x - p)]\, dp\, dw$$

$$= \frac{1}{\pi} \int_{w=0}^{\infty} \int_{-\infty}^{\infty} f(p) \frac{\sinh wy}{\sinh wa} \cos [w(x - p)]\, dp\, dw,$$

as the solution of the Laplace equation (17.339).

Remark: From the example discussed above we have observed that the type of Fourier transform to be used and the independent variable in $u(x, t)$ that is to be transformed depends on the region in which solution is required and also on the initial and boundary conditions given. When the variable to be transformed is defined over the interval $[0, \infty]$, then Fourier sine or cosine transform is used, selecting on the basis of nature of the initial boundary values given. In case the variable to be transformed is defined over the interval $(-\infty, \infty)$, then Fourier transform is applied.

EXERCISE 17.9

Solve the following initial/boundary value problems using Fourier transforms.

1. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$, $u(x, 0) = \begin{cases} 1, & -1 < x < 0 \\ -1, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$

2. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$, $u(x, 0) = u_0$ for $x > 0$, $u(0, t) = 0$, for $t > 0$.

3. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$, $u(x, 0) = e^{-2|x|}$, $-\infty < x < \infty$.

4. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$, $u(x, 0) = u_0$, for $x > 0$, $u(0, t) = u_0 \cos at$, for $t > 0$.

5. The steady state temperature distribution $u(x, y)$ in a thin, homogeneous, semi-infinite plate is given by the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < l, \quad 0 < y < \infty$$

with $\quad u(0, y) = e^{-2y}, \quad u(l, y) = 0, \quad y > 0$, and $\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$, $0 < x < l$.

Find the temperature distribution $u(x, y)$, $0 < x < l$, $y > 0$.

6. Use an integral transform to find the solution of the two-dimensional Laplace's equation $u_{xx} + u_{yy} = 0$ in the infinite strip $0 \le y \le \pi$, given that $u(x, 0) = e^{-2x} H(x)$, $u(x, \pi) = 0$, $-\infty < x < \infty$ where $H(x)$ is the Heaviside function.

ANSWERS

Exercise 17.1 (p. 937)

1. $u = ce^{k(1/y - 1/x)}$
2. $u = (Ae^{\sqrt{k}\,x} + Be^{-\sqrt{k}\,x})e^{2ky}$
3. $u = 6e^{-(3x + 2y)}$
4. $u = \frac{1}{\sqrt{2}} \sinh \sqrt{2}x + e^{-3y} \sin x$
5. $u = (1 - e^{-y}) \sin x$.

Exercise 17.2 (p. 948)

1. $u(x, t) = A \sin \frac{2\pi x}{l} \cos \frac{2\pi ct}{l}$

2. $u(x, t) = \dfrac{2dl^2}{a(l-a)\pi^2} \sum_{n=1}^{\infty} \dfrac{1}{n^2} \sin \dfrac{n\pi a}{l} \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$.

3. $u(x, t) = \dfrac{2l^2}{\pi^2 c} \sum_{n=1}^{\infty} \left[\dfrac{1}{\pi n^3} \sin\left(\dfrac{n\pi}{3}\right) - \dfrac{1}{3n^2} \cos\left(\dfrac{n\pi}{3}\right) \right] \sin\left(\dfrac{n\pi ct}{l}\right) \sin\left(\dfrac{n\pi x}{l}\right)$

4. $u(x, t) = \dfrac{-32}{\pi^3} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \sin \dfrac{(2n-1)\pi x}{2} \cos \dfrac{3(2n-1)\pi t}{2} \Bigg]$

$$+ \frac{4}{\pi^2} \sum \frac{1}{n^2} \sin\left(\frac{n\pi x}{2}\right)\left[\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{n\pi}{2}\right)\right] \sin \frac{3n\pi t}{2}$$

5. $u(x, t) = \dfrac{256}{\pi^3} \sum_{n=1}^{\infty} \dfrac{1}{n^3} [2(-1)^n + 1] \sin\left(\dfrac{n\pi x}{4}\right) \cos\left(\dfrac{n\pi t}{\sqrt{2}}\right)$

$$+ \frac{8}{\sqrt{2}\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin\left(\frac{(2n-1)\pi x}{4}\right) \sin\left(\frac{(2n-1)\pi t}{\sqrt{2}}\right)$$

6. $u(x, t) = \dfrac{1600}{c\pi^3} \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{(2n-1)^3} \sin \dfrac{(2n-1)\pi x}{20} \sin \dfrac{(2n-1)\pi ct}{20}$

8. (i) $u(x, t) = ax(1 - x^2 - 3c^2t^2)$; (ii) $u(x, t) = \dfrac{a}{2}(1 - \cos 2\pi x \cos 2\pi ct)$

9. (i) $u(x, t) = \dfrac{1}{3c} \sin(3x) \sin(3ct)$ (ii) $u(x, t) = \sin 2x \cos 2ct + \dfrac{1}{2c} [\cos 2x \sin 2ct]$

(iii) $u(x, t) = \dfrac{1}{2}(e^{-|x-ct|} + e^{-|x+ct|}) + \dfrac{1}{8} \cos 4x \sin 4ct$

10. $e^{-at/2} \sum \left\{ A_n \left[\cos \lambda_n t + \dfrac{a}{2\lambda_n} \sin \lambda_n t \right] \sin nx \right.$; $\lambda_n = n^2 - \dfrac{a^2}{4}$, $A_n = \dfrac{2}{\pi} \int_0^{\pi} f(x) \sin(\pi x) dx$.

Exercise 17.3 (p. 959)

1. $u(x, t) = 100 \sin \dfrac{3\pi x}{80} e^{-0.01607t}$, 43 seconds.

2. $u(x, t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi x}{l}e^{-\frac{c^2(2n-1)^2\pi^2 t}{l^2}}$

3. $u(x, t) = \frac{4l}{\pi^2}\left[\sin\frac{\pi x}{l}e^{-\frac{c^2\pi^2 t}{l^2}} - \frac{1}{9}\sin\frac{3\pi x}{l}e^{-\frac{9c^2\pi^2 t}{l^2}} + \ldots\right]$

4. $u(x, t) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{10}e^{-.0175(2n-1)^2\pi^2 t}$

5. $u(x, t) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}e^{-(n^2\pi^2c^2t/l^2)}$

6. $u(x, t) = \frac{l}{2} - \frac{4l}{\pi^2}\sum\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}e^{-c^2(2n-1)^2\pi^2 t/l}$

7. $u(x, t) = \frac{4u_0}{\pi}\left[\cos\frac{\pi x}{2l}e^{-\frac{\pi^2c^2}{4l^2}t} - \frac{1}{3}\cos\frac{3\pi x}{2l}e^{-\frac{9\pi^2c^2}{4l^2}t}\right]$

8. $u(x, t) = (2x + 20) + \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1-6(-1)^n}{n}\sin\frac{n\pi x}{10}e^{-\frac{c^2\pi^2n^2t}{100}}$.

Exercise 17.4 (p. 967)

1. $u(x, y) = \sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{a}\sinh\left(\frac{n\pi(b-y)}{a}\right); B_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$

2. $u(x, y) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)xe^{-(2n-1)y}$

4. $u(x, y) = k\,\mathrm{sech}\left(\frac{\pi b}{a}\right)\cos\left(\frac{\pi x}{a}\right)\cosh\left(\frac{\pi(b-y)}{a}\right)$

6. (i) $u(x, y) = -\frac{8ka^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{-(2n-1)\pi y/a}\sin\frac{(2n-1)\pi x}{a}$

(ii) $u(x, y) = \dfrac{ka^2}{6} + \dfrac{4ka^2}{\pi^2} \sum \dfrac{1}{4n^2} e^{-(2n\pi y/a)} \cos\left(\dfrac{2n\pi x}{a}\right)$

8. $u(x, y) = \sum B_n \sin\left(\dfrac{n\pi x}{a}\right) \cosh\left(\dfrac{n\pi(b-y)}{a}\right) \operatorname{sech}\left(\dfrac{n\pi b}{a}\right),\quad B_n = \dfrac{2}{a}\int_0^a f(x) \sin \dfrac{n\pi x}{a} dx.$

9. $u(x, y) = 4\pi a^2 \sum_{n=1}^{\infty} \dfrac{n[1-(-1)^n \cos a]}{\sinh(n\pi b/a)(a^2 - n^2\pi^2)^2} \sin\left(\dfrac{n\pi x}{a}\right) \sinh\left(\dfrac{n\pi y}{a}\right)$

10. $u(x, y) = \dfrac{-1}{\sinh(4\pi)} \sin(\pi y) \sinh(\pi(x-4)) + \sum_{n=1}^{\infty} \dfrac{2}{\sinh(4n\pi)}\left(2.\dfrac{1-(-1)^n}{\pi^3 n^3}\right) \sin(n\pi y) \sinh(n\pi x).$

Exercise 17.5 (p. 974)

2. $u(r, \theta) = \dfrac{3200}{\pi^2} \sum_{n=1}^{\infty} \left(\dfrac{r}{10}\right)^{2n-1} \dfrac{\sin(2n-1)\theta}{(2n-1)^3}$

4. $u(r, \theta) = \dfrac{2}{\pi} \sum_{n=1,3,5}^{\infty} \left(\dfrac{r}{a}\right)^{2n} \dfrac{r^{4n} - b^{4n}}{a^{4n} - b^{4n}} \dfrac{\sin 2n\theta}{n^3}$

5. $u(r, \theta) = 50 + \dfrac{200}{\pi} \sum_{n=1}^{\infty} \left(\dfrac{r}{a}\right)^{2n-1} \dfrac{\sin(2n-1)\theta}{2n-1}$

6. $u(r, \theta) = 50 + 20r\cos\theta$

7. $u(r, \theta) = 4\left(r - \dfrac{1}{r}\right)\cos\theta + 4\left(r + \dfrac{1}{r}\right)\sin\theta$

8. $u(r, \theta) = 100 \log_2 r.$

Exercise 17.6 (p. 987)

1. (i) $0.1 \sin 3\pi x \sin 4\pi y \cos 5\pi t$ (ii) $k \sin 2\pi x \sin \pi y \cos(\sqrt{5}\,\pi c t)$

(iii) $\dfrac{64k}{\pi^6} \sum_n \sum_m \dfrac{1}{n^3 m^3} \sin n\pi x \sin m\pi y \cos\left(\pi t\sqrt{m^2+n^2}\right); m, n = 1, 3, 5, \ldots$

2. $\dfrac{64a^2b^2k}{\pi^6} \sum_n \sum_m \dfrac{1}{n^3 m^3} \sin \dfrac{n\pi x}{a} \sin \dfrac{m\pi y}{b} \cos\left(\pi c t\sqrt{\dfrac{n^2}{a^2} + \dfrac{m^2}{b^2}}\right); m, n = 1, 3, 5, \ldots$

3. $\sin \dfrac{2\pi x}{a} \sin \dfrac{3\pi y}{b} \cos\left(\pi\sqrt{\left(\dfrac{4}{a^2}+\dfrac{9}{b^2}\right)}t\right)$

4. $\dfrac{1}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{8(-1)^{n+1}\pi^2}{n}+\dfrac{16}{n^3}[(-1)^n-1]\right]\sin\dfrac{nx}{2}\sin y\cos\sqrt{\left(\dfrac{n^2}{4}+1\right)}t.$

Exercise 17.7 (p. 995)

1. $E_0 \sin \dfrac{\pi x}{l} \cos \dfrac{\pi t}{l\sqrt{LC}}$

2. $E_0 \cos(pt - px\sqrt{LC})$

3. $a \sin \dfrac{\pi x}{l} \cos \dfrac{\pi t}{l\sqrt{LC}} + b \sin \dfrac{5\pi x}{l} \cos \dfrac{5\pi t}{l\sqrt{LC}}$

4. $\dfrac{20(l-x)}{l}+\dfrac{24}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sin\dfrac{n\pi x}{l}e^{-\frac{n^2\pi^2 t}{RCl^2}}$

5. 0.88 amp.

Exercise 17.8 (p. 1004)

1. $\phi(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\dfrac{n\pi x}{a}\sin\dfrac{m\pi y}{a}\dfrac{\sinh lz}{\sinh la};$

$$a_{nm} = \frac{4}{a^2}\int_0^a\int_0^a \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a} f(x, y)\,dx\,dy, \quad l = \frac{\pi}{a}\sqrt{m^2+n^2}$$

2. $u(x, y, z) = \dfrac{16}{\pi^2}\sum_{n}\sum_{m=1,3,5,}\dfrac{1}{nm\sinh\left(2\pi\sqrt{\pi^2 n^2+m^2}\right)}\sin(n\pi x)\sin\dfrac{my}{2}\sinh\left(\sqrt{n^2\pi^2+m^2}\right)z$

$$+\frac{16}{\pi^2}\sum_{n}\sum_{m=1,3,5}\frac{1}{nm\sinh\left(\pi\sqrt{\dfrac{m^2}{4}+\pi n^2}\right)}\sin(n\pi x)\sin\frac{my}{2}\sinh\left(\sqrt{\pi^2 n^2+\frac{m^2}{4}}\right)z$$

3. $u(r, \phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi), \quad a_n = \dfrac{(2n+1)A}{2}\int_{-1}^{1}[\cos^{-1}(w)]^2 P_n(w)\,dw$

4. $u(r, \phi) = \sum_{n=1}^{\infty} a_{2n-1} r^{2n-1} P_{2n-1}(\cos\phi),\quad a_{2n-1} = \frac{(4n-1)u_0}{R^{2n-1}} \int_0^1 P_{2n-1}(x)dx$

8. $u(r, \phi) = 10 + 15rP_1(\cos\phi) - \frac{35}{4} r^3 P_3(\cos\phi) + \ldots,\ 0 \le r \le 1.$

Exercise 17.9 (p. 1014)

1. $u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} (\cos w - 1) \sin(wx) e^{-c^2 w^2 t} dw$

2. $u(x, t) = u_0 \operatorname{erf.}\left[\frac{x}{2c\sqrt{t}}\right]$

3. $u(x, t) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{(4 + w^2)} \cos(wx) e^{-c^2 w^2 t} dw$

4. $u(x, t) = \frac{u_0}{2c\sqrt{\pi}} \int_0^t \frac{\cos at}{(t-p)^{3/2}} \exp\left(\frac{-x^2}{4c^2(t-p)}\right) dp \;\; - \frac{u_0 x}{2c\sqrt{\pi}} \int_0^t \frac{1}{p^3} \exp\left(\frac{-x^2}{4c^2 p}\right) dp + u_0$

5. $u(x, y) = \frac{4}{\pi} \int_0^{\infty} \frac{\sinh[w(l-x)] \cos(wy)}{(4 + w^2)\sinh(wl)} dw$

6. $u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh[w(\pi - y)]}{(4 + w^2)\sinh(w\pi)} [2\cos(wx) + w\sin(wx)] dw.$

PART F

Complex Analysis

18 CHAPTER Functions of a Complex Variable. Analytic Functions

The development of an analytic function is the extension of differential calculus to functions of a complex variable. The real and imaginary parts of an analytic function are solutions of Laplace equation in two dimensions and this characteristic makes them to find direct applications in two dimensional problems in elasticity, fluid mechanics and electrostatics. The conformal mapping concerns with the geometrical properties of analytic functions. Its practical utility lies in its characteristic to map region with a complicated boundary shape onto region with a simple boundary shape, facilitating in solving two dimensional boundary value problems for the Laplace equation.

18.1 COMPLEX NUMBERS

In the system of real number $\mathbb{R}$ we can solve all quadratic equations of the form $ax^2 + bx + c = 0$ with $a \neq 0$ and discriminant $b^2 - 4ac \geq 0$. But when the discriminant $b^2 - 4ac < 0$, the solutions of this quadratic equation do not belong to the system of reals. In fact, a simple quadratic equation of the form $x^2 + 1 = 0$ does not possess solution in reals. This difficulty was overcomed by introducing the imaginary unit i defined as $i^2 = -1$, or $i = \sqrt{-1}$ and expression like $\sqrt{-k^2}$, when k is a positive real number, can be written as $\sqrt{-k^2} = \sqrt{-1}.\sqrt{k^2} = ik$. Thus the system of real number is extended to the system of *complex number*, denoted by $\mathbf{C}$, in which a general complex number, denoted by z, is of the form $z = x + iy$, where x and y are reals and $i = \sqrt{-1}$. Here the plus sign in $x + iy$ does not denote addition, rather $x + iy$ is a single number. In fact, sometime a notation like (x, y) is used for a complex number $z = x + iy$, and then the complex number system can be considered to be a number system consisting of the set of all ordered pairs (x, y): $x, y \in \mathbb{R}$.

The real number x is called the *real part* and the real number y is called the *imaginary part* of the complex number z. We write $\text{Re } z = x$, and $\text{Im } z = y$.

18.1.1 Algebraic Rules for Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, we define the following:

(a) Equality: $z_1 = z_2$, that is, $x_1 + iy_1 = x_2 + iy_2$ if, and only if $x_1 = x_2$ and $y_1 = y_2$.

(b) Addition: $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$, that is, the sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is the complex number $(x_1 + x_2) + i(y_1 + y_2)$.

(c) Multiplication: $z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$, that is, the product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is the complex number $(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.

(d) Complex conjugate number: The complex number $\bar{z} = x - iy$ is called the *conjugate* of the complex number $z = x + iy$.

(e) Negative complex number: The complex number $-z = -x - iy$ is called the *negative* of the complex number $z = x + iy$.

(f) Modulus of a complex number: The real positive number, denoted by $|z|$, and defined by

$$|z| = |x + iy| = \sqrt{x^2 + y^2}$$

is called the *modulus* or the *absolute value* of the complex number $z = x + iy$.

(g) Division: Let $z_2 = (x_2 + iy_2) \neq 0$, then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right).$$

Thus division of two complex numbers is again a complex number.

We observe that the set of all complex numbers $z = x + iy$ in which the imaginary part y is zero, has all the properties of the set of real numbers x. An important departure of the set of complex numbers from that of the set of real numbers is the *order property*. For any two complex numbers we can't write $z_1 < z_2$, or $z_1 > z_2$. *Thus, the set of complex numbers is not an ordered set*. We simply compare the magnitude of two complex numbers, and only statements like Re $z > 5$, Im $z_1 <$ Im z_2, $|z| > 4$ are meaningful, since Re z, Im z, $|z|$, etc. are real numbers.

However, if z_1, z_2, z_3 are three complex numbers then the following properties of reals hold in case of complex numbers also.

1. ***Commutative laws:*** $z_1 + z_2 = z_2 + z_1$; $z_1z_2 = z_2z_1$.
2. ***Associative laws:*** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$; $(z_1z_2)z_3 = z_1(z_2z_3)$.
3. ***Distributive laws:*** $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$; $(z_1 + z_2)z_3 = z_1z_3 + z_2z_3$.
4. ***Additive and multiplicative identities:*** $z + 0 = 0 + z = z$; $1.z = z.1 = z$
5. ***Additive and multiplicative inverses:*** For every complex number z_1, there exists a unique complex number z_2 such that $z_1 + z_2 = 0$; z_2 is given by $-z_1$, the negative of z_1.

For every *non-zero* complex number z_1, there exists a unique complex number z_2 such that $z_1z_2 = 1$; z_2 is given by $1/z_1$, the inverse of z_1.

18.1.2 The Complex Plane

The form $z = (x, y)$ suggests a graphical representation of the complex number $z = x + iy$ as a point $P(x, y)$ as shown in Fig. 18.1 in the cartesian plane, called the *complex plane* or the *z-plane*. The *x-axis* is called the *real axis* and *y-axis* the *imaginary axis*.

The distance OP of P from the origin O gives the modulus of z. The point $P'(\bar{z})$, the conjugate of $P(z)$, is the reflection of z about the real axis and the point $P''(-z)$, the negative of $P(z)$, is the reflection of the point $P(z)$ about the origin O.

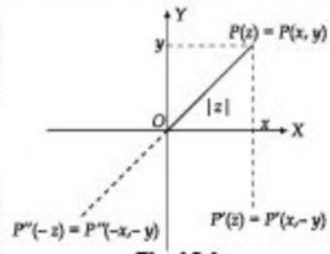


Fig. 18.1

This representation is also called the *Argand diagram*.

The addition $z_1 + z_2$ of two complex numbers z_1 and z_2 satisfies the parallelogram law for the addition of vectors as shown in Fig. 18.2a. So sometimes it is convenient to consider complex numbers as *vectors*. That is, a z vector is the vector from the origin to the point $z = (x, y)$. The subtraction $z_1 - z_2$ is represented as addition of z_1 with $(-z_2)$ as shown in Fig. 18.2b. Vectorially, we have

$$\overrightarrow{OP_1} + \overrightarrow{OP_2} = \overrightarrow{OP_3}, \text{ and } \overrightarrow{OP_1} - \overrightarrow{OP_2} = \overrightarrow{OP_3'} = \overrightarrow{P_2P_1}$$

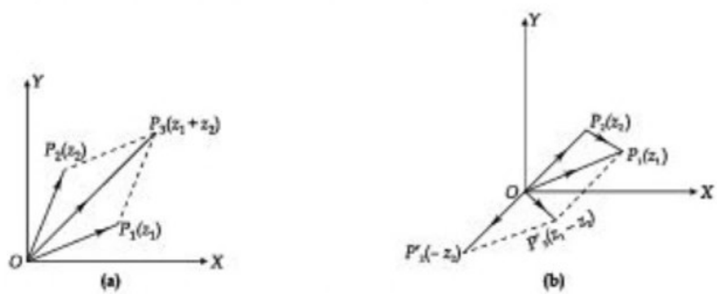


Fig. 18.2

Also we can easily verify the following results:

(i) $z\bar{z} = x^2 + y^2 = |z|^2$ and $|z| = \sqrt{z\bar{z}}$

(ii) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$

(iii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(iv) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, \quad z_2 \neq 0$

(v) $|z| \geq |\text{Re}\, z|, \ |z| \geq |\text{Im}\, z|$

(vi) $|z_1 z_2| = |z_1|\,|z_2|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, \quad z_2 \neq 0.$

The distance 'd' between the two complex numbers z_1 and z_2 is given by

$$d = |z_2 - z_1| = |(x_2 - x_1) + i(y_2 - y_1)| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 18.1: Let $z_1 = 2 + 3i$ and $z_2 = -6 + 4i$, then find $(z_1 + z_2)$, $z_1 z_2$, z_1/z_2 and the distance between z_1 and z_2.

Solution:

$$z_1 + z_2 = (2 + 3i) + (-6 + 4i) = (-4 + 7i)$$

$$z_1 z_2 = (2 + 3i)(-6 + 4i) = (-24 - 10i)$$

$$\frac{z_1}{z_2} = \frac{2 + 3i}{-6 + 4i} = \frac{(2 + 3i)(-6 - 4i)}{(-6 + 4i)(-6 - 4i)} = \frac{0 - 26i}{52} = -\frac{1}{2}i.$$

The distance between z_1 and z_2 is

$$d = |z_2 - z_1| = |(-6 - 2) + (4 - 3)i| = |-8 + i| = \sqrt{64 + 1} = \sqrt{65}.$$

18.1.3 Polar Form of Complex Numbers

Let $z = x + iy$ be any complex number represented by the point $P(x, y)$ as shown in Fig. 18.3. If (r, θ) are the polar coordinates of the point P, then we have

$$x = r\cos\theta, \quad y = r\sin\theta \qquad ...(18.1)$$

and,

$$z = x + iy = r(\cos\theta + i\sin\theta). \qquad ...(18.2)$$

The form (18.2) is called the *polar form* of the complex number z.

From (18.1), we have

$$r = \sqrt{x^2 + y^2} = |z| \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right). \qquad ...(18.3)$$

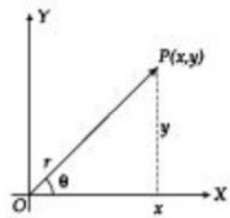


Fig. 18.3

Here r is called the *radius vector*, or the *modulus* of the complex number z, and θ is called the *argument* or the *amplitude* of z and is denoted by arg (z). Obviously θ is defined only when $z \neq 0$ and also θ in (18.1) can be replaced by the general value $\theta + 2n\pi$, where n is any integer. Hence, arg (z) has infinitely many values.

The value of θ which satisfies $-\pi < \theta \leq \pi$ is called the *principal value* of θ, and is denoted by Arg (z). Thus, we have arg (z) = Arg $(z) + 2n\pi$, $\quad n = 0, \pm 1, \pm 2, ...$

The identity $e^{i\theta} = \cos\theta + i\sin\theta$, (in short written as cis θ), called the *Euler's formula*, enables us to rewrite representation (18.2), as $z = re^{i\theta}$ and hence $\bar{z} = re^{-i\theta}$.

This representation, in turn, leads us to the following rules for calculating product, quotient, powers and roots of complex numbers.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ be two complex numbers, so that $|z_1| = r_1$, arg $(z_1) = \theta_1$; $|z_2| = r_2$, arg $(z_2) = \theta_2$.

Then, $\quad z_1 z_2 = r_1 e^{i\theta_1} . r_2 e^{i\theta_2} = r_1 r_2 \, e^{i(\theta_1 + \theta_2)}$, and hence

$$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|, \ \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2).$$

Thus, the product of two complex numbers is represented by a vector whose length is the product of the lengths of the two vectors and whose argument is the sum of their arguments.

Similarly, $\quad \dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$, and hence

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}, \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2).$$

In general, we note that $\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$.

Equality holds when both z_1 and z_2 lie in the right half plane or on the imaginary axis, but not both on the negative imaginary axis.

Similarly, in general, $\text{Arg}\left(\frac{z_1}{z_2}\right) \neq \text{Arg}(z_1) - \text{Arg}(z_2)$.

Also we observe that a vector may be rotated counter-clockwise through an angle θ by multiplying it by $e^{i\theta}$. Multiplication by i rotates a vector through 90°, by −1 through 180°, by $-i$ through 270°, etc.

Triangle Inequality: For any two complex numbers z_1 and z_2 we have the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2| \qquad \text{...(18.4)}$$

which is of vital importance. The result (18.4) follows from the fact that the points 0, z_1 and $z_1 + z_2$ are the vertices of a triangle with sides $|z_1|$, $|z_2|$ and $|z_1 + z_2|$ as shown in Fig. 18.4 and the sum of the two sides of a triangle is always greater than or equal to the third side.

By induction, the triangle inequality (18.4) can be extended to the sum of n complex numbers and is given by

$$|z_1 + z_2 + ... + z_n| \leq |z_1| + |z_2| + ... + |z_n|. \qquad \text{...(18.5)}$$

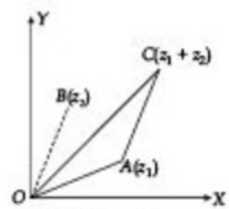


Fig. 18.4

Next we state an important result called *De Moivre's Theorem.*

Theorem 18.1: (De Moivre's Theorem) *If n is*

(i) an integer, positive, negative or zero, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \qquad \text{...(18.6)}$$

(ii) a fraction, positive or negative, then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Proof. The result is obviously true for $n = 0$.

When n is positive integer. Using the polar form of complex numbers, we have

$$z_1 z_2 \ldots z_n = r_1 e^{i\theta_1} r_2 e^{i\theta_2} \ldots r_n e^{i\theta_n},$$

or,
$$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n e^{i(\theta_1 + \theta_2 + \ldots + \theta_n)}. \qquad \text{...(18.7)}$$

Set $z_1 = z_2 = ... = z_n = z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ and then $r = 1$ in (18.7), we obtain

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

When n is negative integer. Let $n = -m$, where m is a positive integer, then

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i \sin m\theta}, \quad \text{since } m \text{ is a positive integer}$$

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos(-n)\theta - i \sin(-n)\theta = \cos n\theta + i \sin n\theta.$$

When n is fraction, positive or negative.

Let $n = p/q$, where p and q are integers in their reduced form, we have

$$(\cos\theta + i \sin\theta)^p = (\cos p\theta + i \sin p\theta) = \left(\cos\frac{p\theta}{q} + i \sin\frac{p\theta}{q}\right)^q, \quad \text{since } q \text{ is an integer.}$$

Taking qth root on both sides then one of the values of $[(\cos\theta + i \sin\theta)^p]^{1/q}$, that is, of $[\cos\theta + i \sin\theta]^{p/q}$ is $\left(\cos\frac{p\theta}{q} + i \sin\frac{p\theta}{q}\right)$. Hence, for $n = p/q$, one of the value of

$$(\cos\theta + i \sin\theta)^n = (\cos n\theta + i \sin n\theta).$$

This proves De Moivre's theorem for all rationals n.

18.1.4 Powers and Roots of a Complex Number

If n is a positive integer, we may apply the product formula to find *the nth power* of z as

$$z^n = z.z. \ldots z, \ n \text{ times.}$$

Let $\quad z = r(\cos\theta + i \sin\theta)$, then

$$z^n = r^n(\cos\theta + i \sin\theta)^n = r^n(\cos n\theta + i \sin n\theta), \quad \text{using De Moivre's theorem.}$$

Also, $\quad z^{-n} = r^{-n}(\cos n\theta - i \sin n\theta) = \dfrac{1}{r^n[\cos n\theta + i \sin n\theta]} = \dfrac{1}{z^n}.$

Further $|z^n| = r^n = |z|^n$, and $\arg z^n = n\theta = n \arg z$.

Next, if $z = r(\cos\theta + i \sin\theta)$ is a complex number different from zero and n is a positive integer, then there are precisely n different complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, that are the *nth roots of z*. To find these n different roots, let $\alpha = \rho[\cos\phi + i \sin\phi]$ be an nth root of a complex number $z = r(\cos\theta + i \sin\theta)$, so that

$$\alpha^n = z, \quad \text{or} \quad \rho^n[\cos\phi + i \sin\phi]^n = r(\cos\theta + i \sin\theta)$$

Using DeMoivre's theorem, we get

$$\rho^n[\cos n\phi + i \sin n\phi] = r[\cos\theta + i \sin\theta] = r[\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)],$$

where k is any integer.

Comparing, we obtain $\rho^n = r$, and $n\phi = \theta + 2k\pi$.

Thus, $\rho = |\alpha| = r^{1/n}$, is the nth root of a positive real number $r = |z|$, and

$$\phi = \arg(\alpha) = \frac{\theta + 2k\pi}{n}, k \text{ is any integer.}$$

Hence, $$z^{1/n} = \alpha = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right], k = 0 \pm 1, \pm 2 \ldots \quad \ldots(18.8)$$

There might appear to be infinitely many different answers corresponding to the infinitely many possible values of k. But we observe that $k = n + m$ gives the same answer as $k = m$.

In fact, we need to take n consecutive values for k to obtain all the different nth roots of z. For convenience, we take $k = 0, 1, 2, \ldots, n - 1$. All *the nth roots* of $r(\cos\theta + i\sin\theta)$ lie on a circle with origin as the centre and radius equal to the positive nth root of r. One of these has argument $\phi = \theta/n$, and the others are uniformly spaced around this circle, each being separated from its neighbour by an angle equal to $2\pi/n$. For example, the cube roots of unity are given by

$$\alpha = (1)^{1/3} = \left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right), k = 0, 1, 2.$$

These are $1, w, w^2$, where $w = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3} = \dfrac{-1+\sqrt{3}i}{2}$,

and lie on the unit circle and divide the circumference of the unit circle into three equal parts. Also we can easily verify that $1 + w + w^2 = 0$ and $w^3 = 1$.

18.1.5 Expansion of sin $n\theta$, cos $n\theta$ in Powers of sin θ and cos θ

By De Moivre's theorem, for positive integer n,

$$\begin{aligned}(\cos n\theta + i\sin n\theta) &= (\cos\theta + i\sin\theta)^n \\ &= \cos^n\theta + C_1^n\cos^{n-1}\theta\,(i\sin\theta) + C_2^n\cos^{n-2}\theta(i\sin\theta)^2 + C_3^n\cos^{n-3}\theta\,(i\sin\theta)^3 + \ldots \\ &= (\cos^n\theta - C_2^n\cos^{n-2}\theta\sin^2\theta + \ldots) + i(C_1^n\cos^{n-1}\theta\sin\theta - C_3^n\cos^{n-3}\theta\sin^3\theta + \ldots).\end{aligned}$$

Equating real and imaginary parts, we obtain

$$\cos n\theta = \cos^n\theta - C_2^n\cos^{n-2}\theta\sin^2\theta + C_4^n\cos^{n-4}\theta\sin^4\theta - \ldots \quad \ldots(18.9)$$

and, $$\sin n\theta = C_1^n\cos^{n-1}\theta\sin\theta - C_3^n\cos^{n-3}\theta\sin^3\theta + C_5^n\cos^{n-5}\theta\sin^5\theta - \ldots \quad \ldots(18.10)$$

Replacing $\sin^2\theta$ by $(1 - \cos^2\theta)$ in (18.9), we get the results in term of powers of $\cos\theta$. Similarly, the result can be obtained in powers of $\sin\theta$ from (18.10).

Further dividing (18.10) by (18.9), we obtain

$$\tan n\theta = \frac{C_1^n\cos^{n-1}\theta\sin\theta - C_3^n\cos^{n-3}\theta\sin^3\theta + C_5^n\cos^{n-5}\theta\sin^5\theta - \ldots}{\cos^n\theta - C_2^n\cos^{n-2}\theta\sin^2\theta + C_4^n\cos^{n-4}\theta\sin^4\theta - \ldots}$$

or, $$\tan n\theta = \frac{C_1^n\tan\theta - C_3^n\tan^3\theta + C_5^n\tan^5\theta - \ldots}{1 - C_2^n\tan^2\theta + C_4^n\tan^4\theta - \ldots}$$

obtained by dividing the numerator and denominator by $\cos^n \theta$.

Also if, $z = \cos\theta + i\sin\theta$, then $\dfrac{1}{z} = \cos\theta - i\sin\theta$. It gives

$$z^p = \cos p\theta + i\sin p\theta \text{ and } 1/z^p = \cos p\theta - i\sin p\theta.$$

Hence, $z + 1/z = 2\cos\theta$, $z - 1/z = 2i\sin\theta$; and $z^p + 1/z^p = 2\cos p\theta$, $z^p - 1/z^p = 2i\sin p\theta$.

These results can be used to *expand the powers of $\sin\theta$, or $\cos\theta$ in a series of sines or cosines multiples of θ*. For example,

$$\sin^7\theta = \frac{-1}{128i}(2i\sin\theta)^7 = \frac{-1}{128i}\left(z - \frac{1}{z}\right)^7$$

$$= \frac{-1}{128i}\left[z^7 - C_1^7 z^5 + C_2^7 z^3 - C_3^7 z + C_4^7\frac{1}{z} - C_5^7\frac{1}{z^3} + C_6^7\frac{1}{z^5} - C_7^7\frac{1}{z^7}\right]$$

$$= \frac{-1}{128i}\left[\left(z^7 - \frac{1}{z^7}\right) - C_1^7\left(z^5 - \frac{1}{z^5}\right) + C_2^7\left(z^3 - \frac{1}{z^3}\right) - C_3^7\left(z - \frac{1}{z}\right)\right]$$

$$= \frac{-1}{128i}\left[(2i\sin 7\theta) - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)\right]$$

$$= \frac{-1}{64}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right].$$

Example 18.2: Find $|z|$ and Arg $|z|$, when z is

(i) $1 + i$, (ii) $(1 + 2i)/[1 - (1 - i)^2]$

(iii) $(3 + 4i)(2 - i)/(2 + 3i)^2$.

Solution: (i) We have, $z = 1 + i = r(\cos\theta + i\sin\theta)$.

Equating the real and the imaginary parts, we get $1 = r\cos\theta$ and $1 = r\sin\theta$.

Therefore, $r^2 = 2 \Rightarrow r = \sqrt{2}$, and $\cos\theta = \dfrac{1}{\sqrt{2}}$, $\sin\theta = \dfrac{1}{\sqrt{2}}$, which gives $\theta = \pi/4$; the value of θ lying between $(-\pi, \pi)$. Therefore, $|z| = \sqrt{2}$ and $\text{Arg}(z) = \pi/4$.

(ii) We have $z = \dfrac{1+2i}{1-(1-i)^2} = \dfrac{1+2i}{1+2i} = 1$. Consider $1 = r(\cos\theta + i\sin\theta)$.

Equating the real and imaginary parts, we get $1 = r\cos\theta$ and $0 = r\sin\theta$
which gives $r = 1$ and $\theta = 0$.

Thus, $|z| = 1$ and $\text{Arg}(z) = 0$.

(iii) We have $z = \dfrac{(3+4i)(2-i)}{(2+3i)^2} = \dfrac{10+5i}{-5+12i} = \dfrac{10-145i}{169}$. Consider $\dfrac{10}{169} - i\dfrac{145}{169} = r(\cos\theta + i\sin\theta)$.

Equating the real and imaginary parts, we get $\dfrac{10}{169} = r\cos\theta$, $\dfrac{-145}{169} = r\sin\theta$.

This gives $|z| = r = \sqrt{\left(\frac{10}{169}\right)^2 + \left(\frac{145}{169}\right)^2} = \frac{5\sqrt{845}}{169}$; $\cos\theta = \frac{10}{5\sqrt{845}}$ and $\sin\theta = \frac{-145}{5\sqrt{845}}$,

which gives $\tan\theta = -\frac{29}{2}$.

Thus, $\text{Arg}\, z = -\tan^{-1}\left(\frac{29}{2}\right)$, since the number $z = \left(\frac{10}{169}, \frac{-145}{169}\right)$ lies in the fourth quadrant.

Example 18.3: Prove the inequalities

(i) $|z_1 + z_2| \le |z_1| + |z_2|$ ...(18.11)

(ii) $|z_1 - z_2| \ge \big|\, |z_1| - |z_2| \,\big|$ ...(18.12)

Solution: (i) We know that $|z|^2 = z\bar{z}$.

Therefore,
$$|z_1 + z_2|^2 = (z_1 + z_2)\,(\overline{z_1 + z_2}) = (z_1 + z_2)\,(\bar{z}_1 + \bar{z}_2)$$
$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 = |z_1|^2 + |z_2|^2 + 2R\,(z_1\bar{z}_2)$$
$$\le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|$$

or, $|z_1 + z_2|^2 \le [\,|z_1| + |z_2|\,]^2$, which implies $|z_1 + z_2| \le |z_1| + |z_2|$.

(ii)
$$|z_1 - z_2|^2 = (z_1 - z_2)\,(\bar{z}_1 - \bar{z}_2) = |z_1|^2 + |z_2|^2 - 2R(z_1\bar{z}_2)$$
$$\ge |z_1|^2 + |z_2|^2 - 2|z_1\bar{z}_2| = |z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2|$$

or, $|z_1 - z_2|^2 \ge \{|z_1| - |z_2|\}^2$, which implies $|z_1 - z_2| \ge \big|\,|z_1| - |z_2|\,\big|$.

We observe that the equality in (18.11) and (18.12) holds when $\text{Re}\,(z_1\bar{z}_2) = |z_1|\,|z_2|$, that is, when $z_1\bar{z}_2$ is real and positive, and since $z_1\bar{z}_2 = \frac{z_1}{z_2}|z_2|^2$, thus when the ratio z_1/z_2 is real and positive, that is, when $\text{Arg}\left(\frac{z_1}{z_2}\right) = 0$. or, $\text{Arg}\,(z_1) = \text{Arg}\,(z_2)$.

Example 18.4: Show that the equation of the ellipse having foci at z_1, z_2 and major axis $2a$ is $|z - z_1| + |z - z_2| = 2a$. Also find its eccentricity.

Solution: Let $P(z)$ be any point on the given ellipse with foci at $S_1(z_1)$ and $S_2(z_2)$ as shown in Fig. 18.5.

Then, $S_1P = |z - z_1|$ and $S_2P = |z - z_2|$.

In case of ellipse, we have $S_1P + S_2P = A_1A_2 = 2a$. This gives
$$|z - z_1| + |z - z_2| = 2a$$
as the equation of the ellipse.

If e is the eccentricity of this ellipse then $S_1S_2 = 2ae$, which gives
$$|z_2 - z_1| = 2ae, \text{ or } e = |z_2 - z_1|/2a.$$

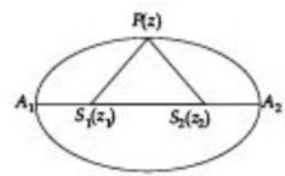


Fig. 18.5

Example 18.5: If z_1, z_2, z_3 be the vertices of an equilateral triangle, prove that

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

Solution: Let PQR be the equilateral triangle, with vertices at $P(z_1)$, $Q(z_2)$ and $R(z_3)$, as shown in Fig. 18.6. Clearly when PQ is rotated through $\pi/3$, it concides with PR.

Thus, $\overrightarrow{PQ}\, e^{i\pi/3} = \overrightarrow{PR}$. It gives

$$(z_2 - z_1)(\cos \pi/3 + i \sin \pi/3) = z_3 - z_1$$

or, $$(z_2 - z_1)(1 + i\sqrt{3}) = 2(z_3 - z_1)$$

or, $$i\sqrt{3}\,(z_2 - z_1) = 2(z_3 - z_1) - (z_2 - z_1) = 2z_3 - z_2 - z_1.$$

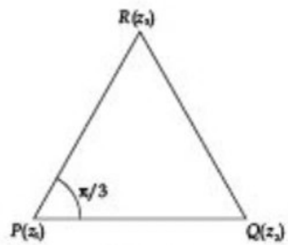


Fig. 18.6

Squaring both sides, we have

$$-3(z_2 - z_1)^2 = (2z_3 - z_2 - z_1)^2$$

or, $$4(z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1) = 0$$

or, $$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

Example 18.6: Show that the equation of a circle on a line segment joining z_1 and z_2 as diameter is $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$.

Solution: Let z, z_1, z_2 represent the points P, A, B in the complex plane, as shown in Fig. 18.7.

It is given that $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

or, $$PA^2 + PB^2 = AB^2.$$

Thus, APB is a right angle triangle, right angled at P.

Hence, the locus of P is a circle with AB as its diameter.

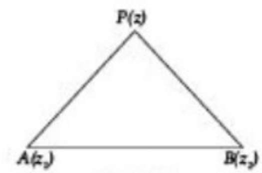


Fig. 18.7

Example 18.7: Find the fourth roots of -16.

Solution: We have $-16 = r(\cos\theta + i\sin\theta)$.

This implies $r\cos\theta = -16$, and $r\sin\theta = 0$.

Thus, $r = 16$, $\cos\theta = -1$, $\sin\theta = 0$.

This gives $\tan\theta = 0$, since the number $z(-16, 0)$ lies in the second quadrant. Therefore, $\theta = \pi$.

Thus, $-16 = 16[\cos\pi + i\sin\pi]$.

Hence, the four fourth roots, using De Moivre's theorem, are given by

$$\alpha_k = 2\left[\cos\frac{(\pi + 2k\pi)}{4} + i\sin\frac{(\pi + 2k\pi)}{4}\right], k = 0, 1, 2, 3. \text{ These are}$$

$$\alpha_0 = 2\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \sqrt{2}\,(1 + i), \alpha_1 = 2\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = \sqrt{2}\,(-1 + i),$$

$$\alpha_2 = 2\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = \sqrt{2}\,(-1 - i), \alpha_3 = 2\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = \sqrt{2}\,(1 - i).$$

Example 18.8(a): Use De Moivre's theorem to solve the equation $x^4 - x^3 + x^2 - x + 1 = 0$.

(b): Find the equation whose roots are $2\cos(\pi/7)$, $2\cos(3\pi/7)$ and $2\cos(5\pi/7)$

Solution(a): The equation is $x^4 - x^3 + x^2 - x + 1 = 0$. Multiplying by $(x+1)$, we have

$$(x+1)(x^4 - x^3 + x^2 - x + 1) = 0 \text{ or, } x^5 + 1 = 0, \quad \text{that is, } x^5 = -1 = \cos\pi + i\sin\pi.$$

Therefore, $$x = \cos\frac{(2k+1)\pi}{5} + i\sin\frac{(2k+1)\pi}{5}, k = 0, 1, 2, 3, 4.$$

Hence, the values of x are

$$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \quad \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}, \quad -1, \ \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}, \text{ and } \quad \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}.$$

The value -1 corresponds to the factor $(x+1) = 0$, thus the roots of the given equation are

$$\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5} \text{ and } \cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}.$$

Solution 18.8(b): Let $y = \cos\theta + i\sin\theta$, where

$$\theta = \pi/7, 3\pi/7, \ldots, 13\pi/7.$$

Consider, $y^7 = (\cos\theta + i\sin\theta)^7 = (\cos 7\theta + i\sin 7\theta) = -1$

Thus $y^7 + 1 = 0$ or, $(y+1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1) = 0$

The factor $y + 1$ corresponds to $\theta = \pi$. Thus the roots of the equation

$$y^6 - y^5 + y^4 - y^3 + y^2 - y + 1 = 0 \text{ are}$$

$$y = (\cos\theta + i\sin\theta), \text{ where } \pi/7, 3\pi/7, 5\pi/7, 9\pi/7, 11\pi/7, 13\pi/7.$$

Dividing this equation by y^3, we have $\left(y^3 + \frac{1}{y^3}\right)^3 - \left(y^2 + \frac{1}{y^2}\right) + \left(y + \frac{1}{y}\right) - 1 = 0$

or, $$\left[\left(y + \frac{1}{y}\right)^3 - 3\left(y + \frac{1}{y}\right)\right] - \left[\left(y + \frac{1}{y}\right)^2 - 2\right] + \left(y + \frac{1}{y}\right) - 1 = 0$$

or, $$x^3 - x^2 + 2x + 1 = 0, \text{ where } x = y + \frac{1}{y} = (\cos\theta + i\sin\theta) + (\cos\theta + i\sin\theta) = 2\cos\theta$$

Further, $(13\pi/7) = \cos(\pi/7)$, $\cos(11\pi/7) = \cos(3\pi/7)$, $\cos(9\pi/7) = \cos(5\pi/7)$.

Hence the equation with the desired roots

$$2\cos(\pi/7), 2\cos(3\pi/7) \text{ and } 2\cos(5\pi/7) \text{ is } x^3 - x^2 + 2x + 1 = 0$$

EXERCISE 18.1

1. Solve the following equations for the real numbers x and y

   (a) $(x+iy)(2-3i)=4+i$ (b) $\dfrac{x-1}{3+i}+\dfrac{y-1}{3-i}=i$ (c) $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2+3i)y+i}{3-i}=i.$

2. Express the following complex numbers in the form $re^{i\theta}$, with $r \geq 0$ and $-\pi < \theta \leq \pi$. Also draw an Argand diagram for each calculation

   (a) $\left(1+\sqrt{-3}\right)^2$ (b) $\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}.$

3. Find the modulus and arguments of the following complex numbers

   (a) $\tan\alpha - i$ (b) $1-\cos\alpha + i\sin\alpha$ (c) $1+\sin\alpha - i\cos\alpha.$

4. If $|z_1| = |z_2| = \ldots = |z_n| = 1$, then show that

$$|z_1+z_2+\ldots+z_n| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\ldots+\frac{1}{z_n}\right|.$$

5. Prove the parallelogram equality $|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$.
   Why is it named so?

6. Find the complex number z which simultaneously satisfies the equations

$$\left|\frac{z-12}{z-8i}\right| = \frac{5}{3} \text{ and } \left|\frac{z-4}{z-8}\right| = 1.$$

7. If $2\cos\theta = x+1/x$ and $2\cos\phi = y+1/y$, show that one of the values of

   (i) $x^m y^n + \dfrac{1}{x^m y^n}$ is $2\cos(m\theta+n\phi)$, (ii) $\dfrac{x^m}{y^n}+\dfrac{y^n}{x^m}$ is $2\cos(m\theta-n\phi)$.

8. If $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$, $z=\cos\gamma+i\sin\gamma$ and $x+y+z=0$, then prove that $x^{-1}+y^{-1}+z^{-1}=0.$

9. Prove that the points $x+iy$ and $\dfrac{1}{-x+iy}$ lie on a straight line through the origin.

10. Find all the values of $\left(\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)^{3/4}$; show that the continued product of these values is 1.

11. Solve $x^7=1$ and prove that the sum of the nth powers of the roots is 7 or zero, according as n is, or is not, a multiple of 7.

12. Solve the equation $x^7+x^4+x^3+1=0.$

13. Prove the n, nth roots of unity form a series in G.P. Also show that the sum of these roots is zero and their product is $(-1)^{n-1}$.

14. Using De Moivre's theorem show that

(a) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ (b) $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.

15. If $z_k = \cos\frac{\pi}{2^k} + i\sin\frac{\pi}{2^k}, k = 1, 2, \ldots,$ then prove that $z_1 z_2 \ldots \infty = -1$.
16. Find the roots common to the equations $x^4 + 1 = 0$ and $x^6 - i = 0$.
17. Show that $\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$.
18. Expand $\cos^5\theta\sin^7\theta$ in a series of sines of multiples of θ.
19. Show that the equation with roots $2\cos(2\pi/7)$, $2\cos(4\pi/7)$ and $2\cos(6\pi/7)$ is $x^3 + x^2 - 2x - 1 = 0$.
20. Show that the roots of the equation $(x-1)^n = x^n$, n being a positive integer, are $\frac{1}{2}[1 + i\cot(r\pi/n)], r = 1, 2, \ldots, (n+1)$.

18.2 SETS IN THE COMPLEX PLANE

Functions of a complex variable are defined on sets of complex number thus first we discuss various features of the sets in the complex plane.

A *point set* in the complex plane is a well-defined collection of finitely many or infinitely many points. The solutions of a quadratic equation, the points in the interior of a circle, etc. are examples of sets. Following are some most important sets which occur frequently in the study of functions of complex variables.

18.2.1 Circles, Disks and Half-plane

The equation $|z - z_0| = \delta$ defines a circle with center z_0 and radius δ. It is the set of all z whose distance from the center z_0 equals δ, refer to Fig. 18.8. Any point z on this circle has the polar form $z = z_0 + \delta e^{i\theta}, 0 \le \theta \le 2\pi$.

If $z_0 = 0$ and $\delta = 1$, then the equation $|z| = 1$ defines the *unit circle* $|z| = 1$.

The *open circular disk* $|z - z_0| < \delta$ gives the interior and $|z - z_0| > \delta$ gives the exterior of the circle $|z - z_0| = \delta$.

The *closed circular disk* $|z - z_0| \le \delta$ gives the set of all the points in the interior and the points on the circle itself.

The set of points $|z - z_0| < \delta$ is also called a *neighbourhood* of z_0, or in particular, an *open circular neighbourhood* of z_0. If we exclude the point z_0 from the open disk $|z - z_0| < \delta$, then it is called the *deleted neighbourhood* of the point z_0, and is written as $0 < |z - z_0| < \delta$.

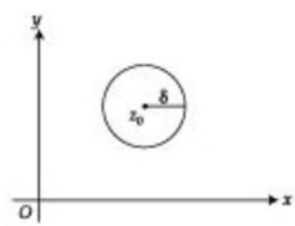


Fig. 18.8

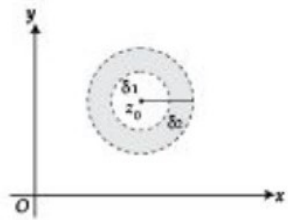


Fig. 18.9

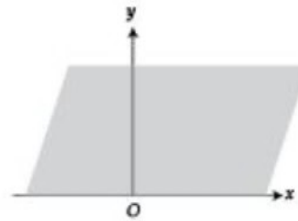


Fig. 18.10

The set of points $\delta_1 < |z - z_0| < \delta_2$, which lies between two concentric circles $|z - z_0| = \delta_1$ and $|z - z_0| = \delta_2$, defines an *open annulus*, or *circular ring*, refer to Fig. 18.9, while $\delta_1 \le |z - z_0| \le \delta_2$ defines the *closed annulus*.

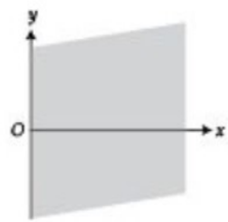


Fig. 18.11

The set of all points $z = x + iy$, $y > 0$ defines the *open upper half-plane*, refer to Fig. 18.10 and the set of all points $z = x + iy$, $y < 0$ defines the *open lower half-plane*. Similarly, the condition $x > 0$ defines the *open right half-plane*, refer to Fig. 18.11, and $x < 0$ defines the *open left half-plane*.

18.2.2 Open and Closed Sets. Domain and Region

A set S is called an open set, if every point of S has a neighbourhood consisting entirely of points that are in S. For example, the set $S = \{z: |z - z_0| < \delta\}$, the points in the interior of a circle is an open set.

A set S is called 'closed' if its complement is open. For example, the set $S = \{z: |z - z_0| \ge \delta\}$, the points in the exterior and the points on the circle $|z - z_0| = \delta$ is a closed set.

A set S is called 'connected', if any two of its points can be joined by a path consisting of finitely many straight line segments completely contained in S. For example, the set $\{z: Re(z) \ne 0\}$ is not connected but the set $\{z: Re(z) > 0\}$ is connected.

An open connected set is called a 'domain' and a 'region' is a set consisting of a domain together with all, one, or more, of its boundary points.

We note that *a domain is necessarily a region but converse may not be true.* For example, the open disk $|z - z_0| < \delta$ is domain as well as region, but the closed disk $|z - z_0| \le \delta$ is a region and not a domain.

Finally, the complex plane to which the point at $z = \infty$ has been included is called the *extended complex plane* and the complex plane without the point at $z = \infty$ is called the *finite complex plane*.

18.3 COMPLEX FUNCTION, LIMIT, CONTINUITY AND DIFFERENTIABILITY

The complex analysis mainly deals with the complex functions that are differentiable in some domain. In this section we introduce first, complex function and then define the concepts of limit, continuity and differentiability in the complex. The discussion, in general, is analogous to that in case of calculus of a real variable. However, it will be of great practical importance to know the basic differences between the behaviours of real and complex valued functions.

18.3.1 Complex Function

In case of calculus of a real variable, we recall that a function f defined on a set S of reals is a rule which assigns a unique value $f(x)$ to every $x \in S$.

Analogously, in complex domain is a set of complex numbers, and we define a *function* f on D as *a rule that assigns to every* $z \in D$ *a complex number* w, *called the value of* f *at* z, *and we write* $w = f(z)$.

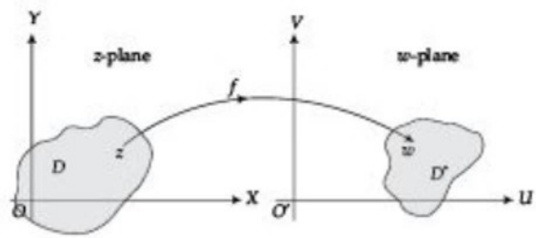


Fig. 18.12

The set D on which f is defined is called the *domain of definition* of f and the set D^* of the corresponding values of $w(z)$, is called the *range* of f, refer to Fig. 18.12. Nevertheless, the set D and D^* must be non-empty.

In general, w is complex so the mapping can be depicted from the $z = (x, y)$ plane to the $w = (u, v)$ plane. Further u and v, respectively the real and imaginary parts of w, must be functions of x and y. Thus we may write $w = f(z) = u(x, y) + iv(x, y)$.

Hence, a complex function $f(z)$ is equivalent to a pair of real functions $u(x, y)$ and $v(x, y)$, each depending on the two real variables x and y.

If $z = r\cos\theta + ir\sin\theta = re^{i\theta}$ is taken in polar form, then the real and imaginary parts of $f(z)$ can be expressed as real valued functions of the real variables r and θ and thus we may write

$$f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta).$$

If to each value of z there corresponds one and only one value of w, then the function $f(z)$ is called a *single-valued* function of z. For example, $w = 1/z$ is a single-valued function of z, defined for all values of z in the z-plane except at $z = 0$. However, in the theory of complex variable, we come across functions which take more than one value for every z belonging to the domain of definition. For example, the function $f(z) = \sqrt{z}$ is a *multi-valued* function of z, since it assumes two values for each non-zero value of z. In such cases, we restrict the discussion to those parts of the domain in which multiple-valued function behaves like a single-valued function. Each one of these single-valued functions is called a *branch* of the multiple-valued function.

Next, consider the function $f(z) = z^2$, where $z = x + iy$; $0 \le x < \infty$, $0 \le y < \infty$.

Here $u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$, so that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

Since $0 \le x, y < \infty$, it follows from the form of u and v that $-\infty < u < \infty$ and $0 \le v < \infty$. Thus, the range of f is the entire upper half-plane $v \ge 0$, while the domain was the first quadrant. In particular, if $z = 2 + i$, then $f(z) = 3 + 4i$.

To get some insight about the graphical display of f, we consider the images of some representative curves. For example, the image of the straight line $x = 1$, $(0 \le y < \infty)$ under $f(z) = z^2$ is given parametrically by $u = 1 - y^2$, $v = 2y$. Eliminating y, we get the parabola, $v^2 = -4(u - 1)$ and similarly, the image of $y = 1$, $(0 < x < \infty)$ is given by the parabola $v^2 = 4(u + 1)$.

Similarly, we can argue that the image of the hyperbola $x^2 - y^2 = a$ is the line $v = a$ and the image of the rectangular hyperbola $2xy = b$ is the line $v = b$, where a and b are real constants.

We shall return to this aspect in detail in Section 18.7.

18.3.2 Limit and Continuity

Let $f(z)$ be a function with D as its domain of definition in the z-plane. *The function $f(z)$ is said to have limit l as z approaches a point $z_0 \in D$, if given $\varepsilon > 0$, no matter how small, we can find a $\delta(\varepsilon) > 0$, such that*

$$|f(z) - l| < \varepsilon, \text{ whenever } |z - z_0| < \delta,$$

that is, we say that $f(z)$ has the limit l as z approaches z_0, if f is defined in a neighbourhood of z_0, possibly a deleted one, and that the values of f are 'close' to l for all z 'close' to z_0, refer to Fig. 18.13.

One can note that, whereas in the real case, x can approach an x_0 only along the real line thus there are only two possible paths, here in case of complex plane, z may approach z_0 from any direction. Thus in that sense the definition of limit is more strict in case of complex variable as compared to the real variable.

To define the limit of a function at $z = \infty$, we say that *the function $f(z)$ has a limit l as $z \to \infty$ if given $\epsilon > 0$, no matter how small, we can find a number $\delta(\epsilon) > 0$, such that*

$$|f(z) - l| < \epsilon, \quad \text{whenever } |z| > 1/\delta.$$

or alternatively, we substitute $z = 1/\xi$ since as $z \to \infty$, $\xi \to 0$, and obtain $\lim_{z\to\infty} f(z) = \lim_{\xi\to 0} f(1/\xi)$

A function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ is defined and $\lim_{z\to z_0} f(z) = f(z_0)$.

A function $f(z)$ which is not continuous at z_0 is said to be discontinuous at z_0.

Function $f(z)$ is said to be continuous in a domain D if it is continuous at each point of this domain D.

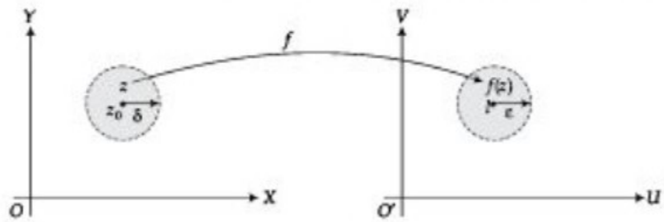


Fig. 18.13

If both $f(z_0)$ and $\lim_{z \to z_0} f(z) = l$ exist, but $f(z_0) \neq l$, then the point z_0 is called a point of *removable discontinuity*. In this case to make the function $f(z)$ continuous at $z = z_0$, we redefine it as $f(z_0) = l$.

Further, if the function $f(z) = u(x, y) + iv(x, y)$ is continuous at $z_0 = (x_0, y_0)$, then its real and imaginary parts are also continuous at z_0. Therefore, we can discuss the continuity of $f(z)$ by studying the continuity of $u(x, y)$ and $v(x, y)$.

We say that *a function $f(z)$ is continuous at $z = \infty$, if $f(1/\xi)$ is continuous at $\xi = 0$.*

Remark. A function of the complex variable $z = x + iy$, continuous in a domain D, is necessarily a continuous function of x when y is constant and also a continuous function of y when x is constant. But continuity with respect to x and y separately does not imply continuity with respect to z, refer to Example 18.13.

18.3.3 Derivative of a Complex Function

The derivative of a complex function f at a point $z = z_0$, denoted by $f'(z_0)$, is defined as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}, \text{ provided this limit exists.}$$

Substituting $\Delta z = z - z_0$, we have $f'(z_0) = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$.

More precisely, we require that there should exist a number l with the property that given any positive number ϵ, no matter how small, there must exist a positive number δ, depending on ϵ and possibly on z_0, such that the inequality

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - l \right| < \epsilon \text{ holds whenever } z \text{ is a point in the open disk } |z - z_0| < \delta.$$

We should remember that by the definition of limit $f(z)$ is defined in a neighbourhood of z_0 and z may approach z_0 from any direction in the complex plane. Hence, differentiability at z_0 means that along whatever path z approaches z_0 the $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ always approaches a certain fixed value and all these values are equal.

The various familiar rules of differentiation of functions of real variables, hold over to the complex case also. For example, we have

(a) $[f(z)+g(z)]' = f'(z)+g'(z)$ (b) $[f(z)\,g(z)]' = f'(z)\,g(z)+f(z)\,g'(z)$

(c) $\left[\dfrac{f(z)}{g(z)}\right]' = \dfrac{f'(z)\,g(z)-f(z)\,g'(z)}{[g(z)]^2},\ g(z)\neq 0$ (d) $[f(g(z))]' = f'(g(z))\,g'(z)$

Further, as in the real case, here also differentiability implies continuity but converse is not true. L'Hopital's rule also holds in case of functions of complex variables.

Example 18.9: Using the definition of limit, show that $\lim\limits_{z\to i}(z^2+iz) = -2$.

Solution: We show that given $\epsilon > 0$, no matter how small, we can find a $\delta(\epsilon) > 0$ such that $|(z^2+iz)-(-2)| < \epsilon$, for all z in $0 < |z-i| < \delta$. Consider

$$|z^2+iz+2| = |(z-i)(z+2i)| = |z-i|\ |z+2i|$$
$$= |z-i|\ |z-i+3i| \le |z-i|\ (|z-i|+3) \qquad \text{(Using triangular inequality)}$$

Thus, $|z^2+iz+2| < \epsilon$, if $|z-i|\ (|z-i|+3) < \epsilon$ or if $|z-i| < \dfrac{-3+\sqrt{9+4\epsilon}}{2}$.

Choosing $\delta(\epsilon) \le \dfrac{-3+\sqrt{9+4\epsilon}}{2}$ and with this choice of δ, we find that $|z^2+iz-(-2)| < \epsilon$, whenever $0 < |z-i| < \delta$, and thus, $\lim\limits_{z\to i}(z^2+iz) = -2$.

In fact at $z = i$, $f(z) = f(i) = -1-1 = -2$, so the function $f(z) = z^2+iz$ is continuous at $z = i$.

Example 18.10: Using the definition of limit show that $\lim\limits_{z\to\infty}(1/z^2) = 0$

Solution: We show that given $\epsilon > 0$, no matter how small, we can find a $\delta(\epsilon) > 0$, such that $\left|\dfrac{1}{z^2}\right| < \epsilon$, whenever $|z| > \dfrac{1}{\delta}$. Now $\left|\dfrac{1}{z^2}\right| < \epsilon$ implies $|z| > \dfrac{1}{\sqrt{\epsilon}}$. Thus choosing $\delta < \sqrt{\epsilon}$ and with this choice of δ we find that $\left|\dfrac{1}{z^2}\right| < \epsilon$, whenever $|z| > \dfrac{1}{\delta}$ and thus, $\lim\limits_{z\to\infty}\left|\dfrac{1}{z^2}\right| = 0$.

Example 18.11: Evaluate

(a) $\lim\limits_{z\to i}\dfrac{z^2+1}{z-1}$ (b) $\lim\limits_{z\to\infty}\left[\sqrt{z-2i}-\sqrt{z-i}\right]$

Solution: (a) $\lim\limits_{z\to i}\dfrac{z^2+1}{z-1}=\lim\limits_{z\to i}\dfrac{(z+i)(z-i)}{z-i}=\lim\limits_{z\to i}(z+i)=2i.$

(b) $\lim\limits_{z\to i}\left[\sqrt{z-2i}-\sqrt{z-i}\right]=\lim\limits_{z\to\infty}\dfrac{\left(\sqrt{z-2i}-\sqrt{z-i}\right)\left(\sqrt{z-2i}+\sqrt{z-i}\right)}{\sqrt{z-2i}+\sqrt{z-i}}$

$$=\lim_{z\to\infty}\left[\frac{-i}{\sqrt{z-2i}+\sqrt{z-i}}\right]=\lim_{\xi\to 0}\left[\frac{-i\sqrt{\xi}}{\sqrt{1-2i\xi}+\sqrt{1-i\xi}}\right]=\frac{0}{2}=0.$$

Example 18.12: Show that the following limits do not exist

(a) $\lim\limits_{z\to 0}f(z)$, where $f(z)=\begin{cases}\dfrac{\text{Re}(z)}{|z|}, & z\neq 0\\ 0 & z=0\end{cases}$ (b) $\lim\limits_{z\to 0}\left[\dfrac{1}{1-e^{1/z}}+iy^2\right]$

Solution: (a) First let z move along x-axis, that is, $y=0$ and then $x\to 0$; and second let z move along y-axis, that is, $x=0$ and then $y\to 0$.

In the first case, $\lim\limits_{z\to 0}f(z)=\lim\limits_{z\to 0}\dfrac{\text{Re}(z)}{|z|}=\lim\limits_{x\to 0}\dfrac{x}{|x|}=\pm 1.$

In the second case, $\lim\limits_{z\to 0}f(z)=\lim\limits_{z\to 0}\dfrac{\text{Re}(z)}{|z|}=\lim\limits_{y\to 0}\dfrac{0}{|y|}$; does not exist.

Since the limits along two different paths are not equal. Hence $\lim\limits_{z\to 0}f(z)$ does not exist.

(b) Consider the path, z moves along x-axis, that is, $y=0$ and then $x\to 0$. In this case,

$$\lim_{z\to 0}\left[\frac{1}{1-e^{1/z}}+iy^2\right]=\lim_{x\to 0}\left[\frac{1}{1-e^{1/x}}\right]=\begin{cases}0, & x>0\\ 1, & x<0\end{cases}$$

Next, consider the path z moves along y-axis, that is, $x=0$ and then $y\to 0$. In this case,

$$\lim_{z\to 0}\left[\frac{1}{1-e^{1/z}}+iy^2\right]=\lim_{y\to 0}[iy^2]=0$$

Since the limit is not unique along the two different paths, thus it does not exist.

Example 18.13: Show that the function $f(z)=\begin{cases}xy/(x^2+y^2), & (z\neq 0)\\ 0, & (z=0)\end{cases}$ is continuous function of x and y separately but is discontinuous at the origin.

Solution: It is obvious that the function $f(z)=\begin{cases}xy/(x^2+y^2), & (z\neq 0)\\ 0, & (z=0)\end{cases}$ is a continuous function of x

when y is constant and also a continuous function of y when x is constant. Thus $f(z)$ is continuous function of x and y separately.

To check the continuity of $f(x)$ at (0, 0), put $y = mx$, we have

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{xy}{x^2+y^2} = \lim_{z\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2} \text{ which depends on } m.$$

Thus the limit of $f(z)$ as z tends to zero does not exist and hence $f(z)$ is discontinuous function of z at $z = 0$.

Example 18.14: Show that the function $f(z) = z^2$ is differentiable everywhere in the finite complex plane.

Solution: For any point z in the complex plane,

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0} (2z + \Delta z) = 2z.$$

Hence $f(z)$ is differentiable everywhere in the finite complex plane.

In fact we can prove that z^n, where n is a positive integer, is differentiable everywhere in the finite complex plane and hence any polynomial function $f(z)$ of the form

$$f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n,$$

where a_i's are complex constants is continuous everywhere in the complex plane.

Example 18.15: Show that the function $f(z) = \bar{z}$ does not have a derivative at any point.

Solution: We observe that $\lim_{\Delta z\to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z} = \dfrac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \dfrac{\overline{\Delta z}}{\Delta z} = \dfrac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$.

Consider the path $y = mx$. We have $\Delta y = m\Delta x$. Here $\Delta x \to 0$ implies $\Delta y \to 0$ and thus $\Delta z \to 0$.

Consider $\lim_{z\to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta x\to 0} \dfrac{\Delta x - im\Delta x}{\Delta x + im\Delta x} = \dfrac{1-m}{1+im}$ which depends on m.

Thus the limit does not exist and, hence, the function $f(z) = \bar{z}$ is not differentiable anywhere.

Example 18.16: Show that the function $f(z) = |z|^2$ is differentiable only at $z = 0$.

Solution: We have $f(z) = |z|^2 = z\bar{z}$, therefore,

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{(z+\Delta z)\,\overline{(z+\Delta z)} - z\bar{z}}{\Delta z} = \lim_{\Delta z\to 0} \frac{z\overline{\Delta z} + \bar{z}\Delta z - \Delta z\Delta\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \left[z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \Delta\bar{z} \right] = 0 \text{ at } z = 0.$$

Hence the function $f(z) = |z|^2$ is differentiable at $z = 0$ and $f'(0) = 0$.

For $z \neq 0$, $\lim_{\Delta z\to 0} [\bar{z} + \overline{\Delta z}] = \bar{z}$ but, $\lim_{\Delta z\to 0} \dfrac{\overline{\Delta z}}{\Delta z}$ does not exist, refer to Example 18.7. Hence, the function $f(z) = |z|^2$ is not differentiable at $z \neq 0$.

Example 18.17: If $f(z)=\begin{cases} x^3y(y-ix)/(x^6+y^6) & z\neq 0 \\ 0 & z=0 \end{cases}$ prove that $[f(z)-f(0)]/z \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ along the curve $y = mx^3$.

Solution: Taking the radius vector $y = mx$, where m is arbitrary real constant. Then

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{z\to 0}\frac{x^3y(y-ix)/(x^6+y^2)}{(x+iy)}$$

$$=\lim_{x\to 0}\frac{mx^4(mx+ix)}{(x^6+m^2x^2)(x+imx)}=\lim_{x\to 0}\frac{mx^2(m+i)}{(x^4+m^2)(1+im)}=0$$

Next taking the curve $y = mx^3$, we have

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{x\to 0}\frac{mx^6(mx^2+ix)}{(x^6+m^2x^6)(x+imx^3)}=\lim_{x\to 0}\frac{m(mx^2+i)}{(1+m^2)(1+imx^2)}=\frac{-im}{(1+m^2)}$$

which depends on m. Hence the limit does not exist.

EXERCISE 18.2

1. Determine and sketch the following regions in the complex plane. Are these open/close, connected, bounded/unbounded? Determine the regions which are domain

   (a) $|z+2+5i|\le \frac{1}{2}$ (b) $|z-2i|\le|z+2i|$ (c) $(\pi/4)\le \arg(z)\le(\pi/3)$

   (d) $\text{Re}(z^2)\le 1$ (e) $-\pi < Im\, z < \pi$ (f) $|(1/z)|>2$

2. Find the domain of definition for the following functions:

   (a) $\frac{2z+3i}{z^2+(1-i)z-i}$ (b) $\frac{1}{z^4-1}$ (c) $\frac{1}{1-|z|^2}$

3. Find the range for the following functions

   (a) $z+2+i;\ 0<x<1, 0<y<1$ (b) $iz^2;\ 0<x<1, 0<y<1$

4. Find the value of real and imaginary parts of the following functions at the indicated points

   (a) $1/(1-z)$ at $z = 7+2i$ (b) $\frac{iz-2}{z+i}$ at $z = 1+i$ (c) $z\,(z-2i)$ at $z = 2+3i$

5. Using the definition of the limit, show that

   (a) $\lim_{z\to 1} 3iz = 3i$ (b) $\lim_{z\to z_0} z^2 = z_0^2$

   (c) $\lim_{z\to 2i}(3x+iy^2) = -4i$ (d) $\lim_{z\to\infty}\frac{1}{(1-z)^2}=0$

6. Find the following limits, if it exists

(a) $\lim_{z \to 1-i} [z^2 - \bar{z}^2]$ (b) $\lim_{z \to \infty} \sqrt{z}\left[\sqrt{z-2i} - \sqrt{z-i}\right]$

(c) $\lim_{z \to 0} \dfrac{z^2}{|z|^2}$ (d) $\lim_{z \to 0} \dfrac{\text{Re}(z)\text{Im}(y)}{|z|^2}$

7. If $f(x)$ is continuous for all real x, does it follow that $f(z)$ is continuous everywhere in the z-plane? Explain.

8. Check the continuity of the following functions at the origin

(a) $f(z) = \dfrac{xy^3}{x^2 + y^6}$ $(z \neq 0), f(0) = 0$ (b) $f(z) = \dfrac{xy}{x^2 + y^2}$ $(z \neq 0), f(0) = 0$

(c) $f(z) = \dfrac{x^2}{(x^2 + y^2)^{1/2}}$ $(z \neq 0), f(0) = 0$

9. Show that the function $f(z) = \bar{z}$ is continuous at the point $z = 0$ but not differentiable at $z = 0$.

10. Using the definition of derivative obtain $f'(z)$, if it exists, for the following functions.

(a) $f(z) = \dfrac{1}{z+1}$ $(z \neq -1)$ (b) $f(z) = \dfrac{1}{z^2}, z \neq 0$

(c) $f(z) = \bar{z}^2$ (d) $f(z) = \dfrac{1+z}{1-z}, z \neq 1$

11. Prove that if $f(z)$ is differentiable at z_0, then it must be continuous there.

18.4 BASIC ELEMENTARY COMPLEX FUNCTIONS

In this section we discuss a few basic elementary complex functions like the exponential functions, trigonometric functions, hyperbolic functions, logarithmic function, inverse trigonometric and inverse hyperbolic functions. They reduce to their counterparts in real calculus when $z = x$ is real and some of these have interesting properties and applications.

18.4.1 Exponential Function

We define the exponential function of the complex variable z as an extension of the corresponding function of the real variable x. We know that the Maclaurin series for the exponential function e^x for real x is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \text{...(18.13)}$$

Thus we define the exponential function e^z for complex z as

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \quad \text{...(18.14)}$$

Since (18.14) reduces to (18.13) when z is purely real, we formally write

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$= e^x\left[1 + (iy) + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \ldots\right] \quad \text{(using 18.14)}$$

$$= e^x\left\{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right)\right\}.$$

Using the Maclaurin series expansion of $\sin y$ and $\cos y$, this gives

$$e^z = e^x(\cos y + i \sin y) \qquad \ldots(18.15)$$

expression
$$e^{iy} = (\cos y + i \sin y) \qquad \ldots(18.16)$$

is known as *Euler's formula* and is a special case of (18.15) when z is purely imaginary and is very useful.

Since $|e^{iy}| = |\cos y + i \sin y| = 1$, thus $|e^z| = |e^{x+iy}| = e^x |e^{iy}| = e^x \neq 0$, for all finite x.

This implies that e^z is non-zero for all finite z. Also $\arg e^z = y \pm 2n\pi$, $\quad n = 0, 1, 2, \ldots$

Further as in real, we have $e^{z_1+z_2} = e^{z_1} e^{z_2}$, since for any $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have

$$e^{z_1}e^{z_2} = e^{x_1}[\cos y_1 + i \sin y_1]\, e^{x_2}[\cos y_2 + i \sin y_2]$$

$$= e^{x_1+x_2}[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\cos y_1 \sin y_2 + \cos y_2 \sin y_1)]$$

$$= e^{x_1+x_2}[\cos(y_1 + y_2) + i \sin(y_1 + y_2)] = e^{z_1+z_2}.$$

The domain of definition of e^z is the whole of the complex plane and the range is also the whole complex plane except the origin.

The function is periodic with period $2\pi i$, since $e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1$ and thus we can write $e^{z+2n\pi i} = e^z$. Also $e^z, z = x + iy$, is real for any x and $y = n\pi$ and e^z is pure imaginary for any x and $y = (2n + 1)\pi/2$, where n is any integer.

The polar form of a complex number $z = r[\cos\theta + i \sin\theta]$ in terms of exponential function is written as $z = re^{i\theta}$.

18.4.2 Trigonometric Functions

The Euler's formula (18.16) is $e^{iy} = \cos y + i \sin y$.

Changing y to $-y$, we obtain $e^{-iy} = \cos y - i \sin y$.

Solving for $\cos y$ and $\sin y$, we obtain

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \qquad \ldots(18.17)$$

$$\sin y = \frac{e^{iy} + e^{-iy}}{2i} \qquad ...(18.18)$$

Analogous to expressions (18.17) and (18.18), we define the cosine and sine functions of a complex variable z as

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad ...(18.19)$$

$$\sin z = \frac{e^{iz} + e^{-iz}}{2i} \qquad ...(18.20)$$

Besides $\sin z$ and $\cos z$, the other trigonometric functions are defined as

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \operatorname{cosec} z = \frac{1}{\sin z},$$

whenever the denominator is not zero.

Further (18.19) and (18.20) imply that Euler's formula (18.16) is valid in complex also, that is,

$$e^{iz} = \cos z + i \sin z \qquad ...(18.21)$$

Next, we investigate for the real and imaginary parts of $\cos z$ and $\sin z$. These are helpful in investigating the properties of these functions further. We have

$$\cos z = \frac{1}{2}[e^{iz} + e^{-iz}] = \frac{1}{2}[e^{i(x+iy)} + e^{-i(x+iy)}] = \frac{1}{2}[e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)]$$

$$= \left[\cos x \left(\frac{e^{y} + e^{-y}}{2}\right) - i \sin x \left(\frac{e^{y} - e^{-y}}{2}\right)\right]$$

Thus, $\cos z = \cos x \cosh y - i \sin x \sinh y$ ...(18.22)

Similarly, we have $\sin z = \sin x \cosh y + i \cos x \sinh y$. ...(18.23)

From (18.22)

$$|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$
$$= \cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y$$
$$= \cos^2 x + (\cos^2 x + \sin^2 x) \sinh^2 y.$$

Thus, $|\cos z|^2 = \cos^2 x + \sinh^2 y.$ ...(18.24)

Similarly, $|\sin z|^2 = \sin^2 x + \sinh^2 y.$ ...(18.25)

Formulae (18.24) and (18.25) point to an essential difference between the real and the complex cosine and sine. Since $\sinh y = \dfrac{e^{y} - e^{-y}}{2}$ tends to ∞ as y tends to ∞, thus $|\sin z|$, $|\cos z|$ and hence $\sin z$ and $\cos z$ are not bounded whereas their counterparts $\sin x$ and $\cos x$ are bounded.

Also we observe from (18.19) and (18.20) that $\cos z$ and $\sin z$ are periodic with period 2π just as in real and $\tan z$ is periodic with period π.

Further, $\sin z = 0$ implies that $|\sin z| = 0$, which gives

$$\sin^2 x + \sinh^2 y = 0, \text{ for all real } x, y.$$

or, $$\sin x = 0, \text{ and } \sinh^2 y = 0$$

or, $$x = n\pi, \text{ and } y = 0, \text{ that is, } z = n\pi.$$

Hence, $\sin z = 0$ only when z is purely real and $z = n\pi$, n any integer.

Similarly, $\cos z = 0$ only when z is real and $z = (2n+1)\pi/2$, n any integer.

The general formulae for the real trigonometric functions continue to hold for the corresponding complex valued functions also and can be easily verified using (18.19) and (18.20). We mention the following

(a) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

(b) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

(c) $\tan(z_1 \pm z_2) = (\tan z_1 + \tan z_2)/(1 \mp \tan z_1 \tan z_2)$

(d) $\sin 2z = 2 \sin z \cos z$

(e) $\cos 2z = \cos^2 z - \sin^2 z$

(f) $\sin^2 z + \cos^2 z = 1$

18.4.3 Hyperbolic Functions

The complex hyperbolic cosine and sine functions are defined as

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2} \qquad ...(18.26)$$

The other hyperbolic functions are defined by

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{cosech} z = \frac{1}{\sinh z}$$

whenever the denominator is not zero.

If we replace z by iz in (18.26), then we have

$$\cosh iz = \cos z \quad \text{and} \quad \sinh iz = i \sin z \qquad ...(18.27)$$

Again replacing z by iz in (18.27) and using the fact that $\cosh z$ is an even function of z and $\sinh z$ is an odd function of z, we obtain

$$\cos iz = \cosh z \quad \text{and} \quad \sin iz = i \sinh z \qquad ...(18.28)$$

Also, $$\sinh z = \frac{1}{i} \sin iz = -i \sin iz = -i \sin i(x+iy)$$

$$= i \sin(y - ix) = i[\sin y \cosh x - i \cos y \sinh x]$$

or, $$\sinh z = \sinh x \cos y + i \cosh x \sin y \qquad ...(18.29)$$

Similarly, $$\cosh z = \cosh x \cos y + i \sinh x \sin y \qquad ...(18.30)$$

Also from (18.29), we have

$$|\sinh z| = \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y = \sinh^2 x(1-\sin^2 y) + \cosh^2 x \sin^2 y$$
$$= \sinh^2 x + (\cosh^2 x - \sinh^2 x)\sin^2 y = \sinh^2 x + \sin^2 y, \text{ using } \cosh^2 x - \sinh^2 x = 1.$$

Similarly from (18.30), we have $|\cosh z| = \sinh^2 x + \cos^2 y$

Next, $\sinh z = 0$ gives $|\sinh z| = 0$, or $\sinh^2 x = 0$ and $\sin y = 0$, or $x = 0$ and $y = n\pi$.

Thus $\sinh z = 0$, only when $z = n\pi i$ is purely imaginary. Similarly, $\cosh z = 0$, only when $z = (2n+1)\frac{\pi}{2}i$, where n is any integer.

Further, as their counterparts $\sinh x$ and $\cosh x$, $\sinh z$ and $\cosh z$ are also not bounded. Also the general formulae for the real hyperbolic functions continue to hold for the corresponding complex valued functions also and can be easily verified. We mention the following

(a) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

(b) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$

(c) $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

(d) $\cosh^2 z - \sinh^2 z = 1$

(e) $\sinh 2z = 2\sinh z \cosh z$

(f) $\cosh 2z = \cosh^2 z + \sinh^2 z$

18.4.4 Logarithmic Function

The natural logarithm of $z = x + iy$ denoted by $\ln z$, or sometime by $\log_e z$, is defined as the inverse of the exponential function, that is, if $z = e^w$, then $w = \ln z$, $z \neq 0$, since $e^w \neq 0$ for all w.

Writing $z = re^{i\theta}$ and $w = u + iv$, we have $re^{i\theta} = e^{u+iv}$.

Since, $e^{i\theta} = e^{i(\theta + 2n\pi)}$ for any integer n, thus it can be written as

$$re^{i(\theta+2n\pi)} = e^{u+iv} \qquad ...(18.31)$$

Now e^{u+iv} has absolute value e^u and the argument v, then comparing these on both sides of (18.31) we have $e^u = r$, or $u = \ln r$ and $v = \theta + 2n\pi$, n any integer.

Hence for any complex $z \neq 0$, the solutions of equation $e^w = z$ are given by

$$w = \ln z = \ln r + i(\theta + 2n\pi), \quad n \text{ any integer}$$
$$= \ln r + i \arg(z), z \neq 0.$$

Thus the complex natural logarithm $\ln z$, $(z \neq 0)$ is infinitely multiple-valued function. For each n, we obtain a different branch of the multiple-valued function $\ln z$. If we restrict $\arg(z)$ to its principal value, $-\pi < \arg(z) \leq \pi$, denoted by $\mathrm{Arg}(z)$, then the corresponding branch $Ln\, z$ of $\ln z$ is

$$Ln\, z = \ln r + i\,\mathrm{Arg}(z)$$

or,
$$Ln\, z = \ln|z| + i\theta, -\pi < \theta \leq \pi$$

or,
$$Ln\, z = \ln\sqrt{x^2+y^2} + i\tan^{-1}(y/x).$$

The uniqueness of Arg (z) for given $z \neq 0$ implies that $Ln\, z$ is single-valued function and since different values of arg (z) differ by integer multiple of 2π, we have

$$\ln z = Ln\, z \pm 2n\pi i, \quad n = 1, 2, \ldots$$

They all have the same real part but their imaginary parts differ by an integral multiple of 2π.

If $z = x$ is positive real, then $|z| = x$ and Arg $(z) = 0$, and thus

$$Ln\, z = \ln|z| = \ln x, x > 0$$

which is the real natural logarithm.

If $z = x$ is negative real, then $|z| = |x|$ and Arg $(z) = \pi$ and thus

$$Ln(z) = \ln|z| + i\pi = \ln|x| + i\pi, x < 0.$$

Also the following properties can easily be verified for natural logarithm of complex values

(a) $e^{\ln z} = z$

(b) $\ln(z_1 z_2) = \ln z_1 + \ln z_2 \pm 2n\pi i$

(c) $\ln(z_1/z_2) = \ln z_1 - \ln z_2 \pm 2n\pi i$, where $n = 0, 1, 2, \ldots$.

18.4.5 General Powers

The general power of a complex number $z = x + iy$ is defined by the formula $z^c = e^{c \ln z}, z \neq 0$, where c is a complex constant.

Since $\ln z$ is infinitely many valued function, thus z^c will also be, in general, multiple valued function. The particular value $e^{c Ln z}$ is called the *principal value* of z^c.

If $c = n$ is a positive integer, then z^n is single valued and similarly for $z = -n$. But if $c = 1/n$, then z^c has exactly n values. If $c = p/q$, the quotient of two positive integers, then z^c has exactly q distinct values. If c is an irrational or a complex, then z^c is multivalued function.

18.4.6 Inverse Trigonometric Functions of a Complex Variable

If $z = \sin w$, then w is called the inverse sine function for a complex variable z and is written as $w = \sin^{-1} z$.

The function $\sin w = z$ implies $e^{iw} - e^{-iw} = 2iz$ or, $e^{2iw} - 2ize^{iw} - 1 = 0$. Solving for e^{iw} we get, $e^{iw} = iz \pm \sqrt{1-z^2}$. Since the double sign is covered by the double-valued function $\sqrt{1-z^2}$, thus

$$e^{iw} = iz + \sqrt{1-z^2} \text{ or, } w = -i \ln\left[iz + \sqrt{1-z^2}\right]. \text{ Hence, } \sin^{-1} z = -i \ln\left[iz + \sqrt{1-z^2}\right].$$

Now, $\sin^{-1} z$ is defined for all z except when $iz = -\sqrt{1-z^2}$ or, $-z^2 = 1 - z^2$ or, $1 = 0$, which is not possible. Thus $\sin^{-1} z$ is defined for all z.

The other complex inverse trigonometric functions are given by

$$\cos^{-1} z = -i \ln\left(z + \sqrt{z^2-1}\right), \tan^{-1} z = \frac{i}{2} \ln \frac{i+z}{i-z}, z \neq \pm i,$$

$$\operatorname{cosec}^{-1} z = \sin^{-1}\left(\frac{1}{z}\right) = -i \ln\left[\frac{1+\sqrt{z^2-1}}{z}\right], z \neq 0, \sec^{-1} z = \cos^{-1}\left(\frac{1}{z}\right) = -i \ln\left[\frac{1+\sqrt{1-z^2}}{z}\right], z \neq 0,$$

and, $$\cot^{-1} z = \tan^{-1}\left(\frac{1}{z}\right) = -\frac{i}{2}\ln\left[\frac{z+i}{z-i}\right], z \neq \pm i.$$

Clearly inverse trigonometric functions are multivalued functions but we shall consider only their principal values.

18.4.7 Inverse Hyperbolic Functions of a Complex Variable

If $z = \sinh w$, then w is called the inverse sine hyperbolic function of the complex variable z and is written as $w = \sinh^{-1} z$.

The function $\sinh w = z$ implies $e^w - e^{-w} = 2z$, or $e^{2w} - 2ze^w - 1 = 0$. Solving for e^w we get $e^w = z \pm \sqrt{z^2+1}$. Since the double sign is covered by the double valued function $\sqrt{z^2+1}$, thus

$$e^w = z + \sqrt{z^2+1} \text{ or, } w = \ln\left[z + \sqrt{z^2+1}\right]. \text{ Hence, } \sinh^{-1} z = \ln\left[z + \sqrt{z^2+1}\right].$$

The other inverse hyperbolic trigonometric functions of the complex variable are given by

$$\cosh^{-1} z = \ln\left(z + \sqrt{z^2-1}\right), \tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \quad z \neq \pm 1,$$

$$\operatorname{cosech}^{-1} z = \sinh^{-1}\left(\frac{1}{z}\right) = \ln\left\{\frac{1+\sqrt{1+z^2}}{z}\right\}, z \neq 0, \operatorname{sech}^{-1} z = \cosh^{-1}\left(\frac{1}{z}\right) = \ln\left\{\frac{1+\sqrt{1+z^2}}{z}\right\}, z \neq 0,$$

and, $$\coth^{-1} z = \tanh^{-1}\left(\frac{1}{z}\right) = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right), \quad z \neq \pm 1.$$

Clearly inverse hyperbolic trigonometric functions are also multivalued functions but we shall consider only their principal values.

Example 18.18: Solve $e^z = 4 + 3i$.

Solution: We have, $e^z = e^{x+iy} = e^x(\cos y + i \sin y) = 4 + 3i$. Therefore, $e^x \cos y = 4$ and $e^x \sin y = 3$. Squaring and adding, we have $e^{2x} = 25$ or, $e^x = 5$ or, $x = 1.609$.

Also, $$\tan y = \frac{3}{4}, \text{ or } y = \tan^{-1}\frac{3}{4}, \text{ or } y = 0.6435.$$

Hence, $$z = 1.609 + 0.6435\, i \pm 2n\pi i, \text{ where } n = 0, 1, 2, \ldots$$

Example 18.19: Solve $\cos z = 5$.

Solution: The equation $\cos z = 5$ gives $\dfrac{e^{iz} + e^{-iz}}{2} = 5$, or $e^{2iz} - 10e^{iz} + 1 = 0$.

Solving for e^{iz} we get $e^{iz} = 5 \pm \sqrt{25-1} = 9.899$, or 0.101

Now, $e^{iz} = 9.899$ gives $e^{-y+ix} = 9.899$, or $e^{-y}[\cos x + i\sin x] = 9.899$

This implies $e^{-y} = 9.899$, or $y = -\ln 9.899 = -2.292$. Also $\cos x = 1$ and $\sin x = 0$, this gives $x = \pm 2n\pi$. Similarly,

$e^{iz} = 0.101$ gives $e^{-y+ix} = 0.101$, which implies $y = -\ln(0.101) = 2.292$ and $x = \pm 2n\pi$, and hence $z = \pm 2n\pi \pm 2.292i$, where n is any integer.

Example 18.20: Solve $\cosh z = \dfrac{1}{2}$.

Solution: The equation $\cosh z = \dfrac{1}{2}$ gives $\dfrac{e^z + e^{-z}}{2} = \dfrac{1}{2}$, or $e^{2z} - e^z + 1 = 0$.

Solving for e^z, we get $e^z = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$, or, $e^x[\cos y + i\sin y] = (1 \pm i\sqrt{3})/2$.

Comparing we get $e^x \cos y = 1/2$, and $e^x \sin y = \pm\sqrt{3}/2$,

which gives $e^x = 1$, that is, $x = 0$ and, $\cos y = 1/2$, and $\sin y = \pm\sqrt{3}/2$, which give $y = \pm\pi/3$

Hence solutions are $z = i(\pm\pi/3 \pm 2n\pi)$, where n is any integer.

Example 18.21: Solve $\tan z = e^{i\alpha}$, where α is a real.

Solution: Let $z = x + iy$. The equation $\tan z = e^{i\alpha}$ gives

$$\tan(x+iy) = \cos\alpha + i\sin\alpha,\ \tan(x-iy) = \cos\alpha - i\sin\alpha.$$

Therefore, $\tan 2x = \tan[(x+iy)+(x-iy)] = \dfrac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$

$$= \frac{(\cos\alpha + i\sin\alpha) + (\cos\alpha - i\sin\alpha)}{1-(\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)} = \frac{2\cos\alpha}{1-(\cos^2\alpha + \sin^2\alpha)} = \frac{2\cos\alpha}{0} \to \infty$$

Thus, $2x = \dfrac{\pi}{2} + n\pi$ or $x = \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{2}$, n being any integer.

Also, $\tan 2iy = \tan[(x+iy)-(x-iy)] = \dfrac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$

$$= \frac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{1+(\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

or, $i\tanh 2y = \dfrac{2i\sin\alpha}{1+(\cos^2\alpha + \sin^2\alpha)} = i\sin\alpha$ or, $\dfrac{e^{2y}-e^{-2y}}{e^{2y}+e^{-2y}} = \dfrac{\sin\alpha}{1}$.

By componendo and dividendo, we get

$$\frac{e^{2y}}{e^{-2y}} = \frac{1+\sin\alpha}{1-\sin\alpha} = \frac{\cos^2\alpha/2+\sin^2\alpha/2+2\sin\alpha/2\cos\alpha/2}{\cos^2\alpha/2+\sin^2\alpha/2-2\sin\alpha/2\cos\alpha/2}$$

or, $$e^{4y} = \frac{(\cos\alpha/2+\sin\alpha/2)^2}{(\cos\alpha/2-\sin\alpha/2)^2}$$

or, $$e^{2y} = \frac{1+\tan\alpha/2}{1-\tan\alpha/2} = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \quad \text{or,} \quad y = \frac{1}{2}\ln\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

Hence solution is $z = \left(n+\frac{1}{2}\right)\frac{\pi}{2} + \frac{i}{2}\ln\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$.

Example 18.22: Separate $\cos^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts, where θ is a positive acute angle.

Solution: Let $\cos^{-1}(\cos\theta + i\sin\theta) = u + iv$, then

$$\cos\theta + i\sin\theta = \cos(u+iv) = \cos u\cosh v - i\sin u\sinh v$$

Therefore, $$\cos\theta = \cos u\cosh v \qquad ...(18.32)$$

and, $$\sin\theta = -\sin u\sinh v \qquad ...(18.33)$$

Squaring and adding, we have

$$1 = \cos^2 u\cosh^2 v + \sin^2 u\sinh^2 v = \cos^2 u\,(1+\sinh^2 v) + \sin^2 u\sinh^2 v$$
$$= \cos^2 u + (\cos^2 u + \sin^2 u)\sinh^2 v = \cos^2 u + \sinh^2 v$$

or, $$1-\cos^2 u = \sinh^2 v, \text{ or } \sin^2 u = \sinh^2 v \qquad ...(18.34)$$

Squaring (18.33) and using (18.34) we obtain $\sin\theta = \sin^2 u$, θ being a positive acute angle $\sin\theta$ is positive. Thus, the real part is $u = \sin^{-1}\sqrt{\sin\theta}$

Using $\sin u = \sqrt{\sin\theta}$ in (18.20), we have, $\sinh v = -\sqrt{\sin\theta}$; the imaginary part is

$$v = \sinh^{-1}\left(-\sqrt{\sin\theta}\right) = \ln\left[-\sqrt{\sin\theta}+\sqrt{\sin\theta+1}\right] = \ln\left[\sqrt{1+\sin\theta}-\sqrt{\sin\theta}\right]$$

Hence, $\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{\sin\theta} + i\ln\left[\sqrt{1+\sin\theta}-\sqrt{\sin\theta}\right]$.

Example 18.23: Compute i^i.

Solution: By definition $i^i = e^{i\ln i}$

But, $\ln i = Ln1 + i(\pi/2 + 2n\pi) = i(\pi/2 + 2n\pi)$, and hence

$i^i = e^{-(\pi/2+2n\pi)}$, where n is any integer.

Example, 18.24: Find all the roots of the equation

(a) $\sin z = \cosh 4$ (b) $\sinh z = i$.

Solution: (a) $\sin z = \cosh 4 = \cos 4i = \sin\left(\frac{\pi}{2} - 4i\right)$

increments in y and z are given in the second and third rows. When all the residuals have been almost reduced to zeros, the increments in x, y, z are added separately to obtain the requisite solution.

Example 21.21: Solve by relaxation method the system of equations

$$27x + 6y - z = 85,\ 6x + 15y + 2z = 72,\ x + y + 54z = 110$$

Solution: The residuals are given by

$$R_x = 85 - 27x - 6y + z$$
$$R_y = 72 - 6x - 15y - 2z$$
$$R_z = 110 - x - y - 54z$$

The operation table is

Increment	*Change in Residuals*		
	R_x	R_y	R_z
$\Delta x = 1$	−27	−6	−1
$\Delta y = 1$	−6	−15	−1
$\Delta z = 1$	1	−2	−54

Starting with $x = 0, y = 0, z = 0$, the relaxation table is given by

	R_x	R_y	R_z	Remark
$x = y = z = 0$	85	72	110	R_z is the largest
$\Delta z = 2$	87	68	2	R_x is the largest
$\Delta x = 3$	6	50	−1	R_y is the largest
$\Delta y = 3$	−12	5	−4	R_x is the largest
$\Delta x = -0.4$	−1.2	7.4	−3.6	R_y is the largest
$\Delta y = 0.5$	−4.2	0.1	−4.1	R_x is the largest
$\Delta x = -0.15$	0.05	1.0	−3.95	R_z is the largest
$\Delta z = -0.073$	−0.023	1.146	−.008	R_y is the largest
$\Delta y = .0764$	−0.4814	0	−.0844	R_x is the largest
$\Delta x = -0.0178$	−0.0008	0.1068	−0.0666	All are approaching zeros

Here $\Sigma\Delta x = 2.4322$, $\Sigma\Delta y = 3.5764$, and $\Sigma\Delta z = 1.927$.

Hence the approximate solution is $x = 2.43322$, $y = 3.5764$, and $z = 1.927$.

Remark: As a check the computed values are substituted in the equations, the residuals obtained should be negligible in case computations are correct.

EXERCISE 21.2

Solve the following system of equations by the Gauss-elimination method:

1. $x + 4y - z = -5, x + y - 6z = -12, 3x - y - z = 4$
2. $10x - 7y + 3z + 5u = 6, -6x + 8y - z - 4u = 5, 3x + y + 4z + 11u = 2, 5x - 9y - 2z + 4u = 7$

3. $x+\frac{1}{2}y+\frac{1}{3}z=1,\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=0,\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=0$

Solve the following system of equations by Gauss-Jorden method:

4. $x+2y+z=8, 2x+3y+4z=20, 4x+3y+2z=16$
5. $10x+y+z=12, x+10y+z=12, x+y+10z=12$
6. $2x+y+5z+w=5, x+y-3z+4w=-1, 3x+6y-2z+w=8, 2x+2y+2z-3w=2$

Solve the following system of equations by Crout's triangularization method:

7. $2x+3y+z=9, x+2y+3z=6, 3x+y+2z=8$
8. $10x+y+z=12, x+10y+z=12, x+y+10z=12$
9. $2x+y+5z+w=5, x+y-3z+4w=-1, 3x+6y-2z+w=8, 2x+2y+2z-3w=2$

Solve the following system of equations by Jacobi iterations correct up to two decimal points:

10. $5x+2y+z=12, x+4y+2z=15, x+2y+5z=20$
11. $4x+0.24y-0.08z=8, 0.09x+3y-0.15z=9, 0.04x-0.08y+4z=20$
12. $13x+5y-3z+w=18, 2x+12y+z-4w=13, 3x-4y+10z+w=29, 2x+y-3z+9w=31$

Solve the following system of equations by Gauss-Seidel iteration method:

13. $27x+6y-z=85, 6x+15y+2z=72, x+y+54z=110$
14. $5x+2y+z=9, x+4y+2z=15, x+2y+5z=20$
15. $x-0.25y-0.25z=50, -0.25x+y-0.25w=50, -0.25x+z-0.25w=25, -0.25y-0.25z+w=25$

Solve the following system of equations by relaxation method:

16. $9x-2y+2z=50, x+5y-3z=18, -2x+2y+7z=19$
17. $-9x+3y+4z+100=0, x-7y+3z+80=0, 2x+3y-5z+60=0$
18. $15x+4y-4z+2w=49.207, 3x+27y-5z-w=18.024,$
    $-4x-5y+16z+5w=-23.871, 2x-y+5z+19w=54.907$

21.4 EIGENVALUES BY ITERATION: THE POWER METHOD

As discussed in Section 2.11, an *eigenvalue* of a given $n \times n$ matrix $A = [a_{ij}]$ is a real or complex number λ such that the set of equations $Ax = \lambda x$ has a non-trivial solution $x \neq 0$, and the non-trivial solution x is then called an *eigenvector* of the matrix A corresponding to the eigenvalue λ. We have already studied a direct method for finding the eigenvalues and corresponding eigenvectors in Section 2.11. In this section, we shall study an iteration process for calculating the eigenvalues (in particular, numerically the largest and the smallest values) of a square matrix. *The process of determination of approximate values of eigenvalues and corresponding error bounds is called inclusion.* In this context, we state below without proof an *inclusion theorem* which gives a region consisting of closed circular disks in the complex plane and including all the eigenvalues of a given matrix.

Theorem 21.2 (Greschgorin's Theorem): *Let λ be an eigenvalue of an arbitrary $n \times n$ matrix $A = [a_{ij}]$. Then for some integer j, $1 \le j \le n$*

$$|a_{jj} - \lambda| \le |a_{j1}| + |a_{j2}| + \ldots |a_{jj-1}| + |a_{jj+1}| + \ldots + |a_{jn}| \quad \ldots(21.24)$$

Thus, all the eigenvalues lie within these n circular disks, some of these may be identicals, with centres at the diagonal elements a_{jj} and radii equal to the sum of the absolute values of the elements of jth row except that of a_{jj}. These n circular disks are called the *Gerschgorin disks*.

For example, for the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, the Greschgorin disks are:

D_1: centre 2, radius 2; D_2: centre 3, radius 4; D_3: centre 4, radius 6.

In fact, the exact eigenvalues of the matrix A are 1, 1 and 7 (refer, Example 7.4, Vol.1).

A standard method for finding the approximate values of the eigenvalues (particularly of the largest and of the smallest) of an $n \times n$ matrix is the *power method*. The main advantage of this method is that the corresponding eigenvector is also obtained simultaneously. The method applies to any $n \times n$ matrix A that has a *dominant eigenvalue*, that is, a λ is such that $|\lambda|$ is greater than the absolute values of the other eigenvalues.

Power method. We start from any non-zero vector x_0 with n components, normally $[1, 1, \ldots 1]^T$, or $[1, 0, \ldots 0]^T$, and compute successively

$$x_1 = Ax_0, x_2 = Ax_1, \ldots, x_k = Ax_{k-1}.$$

Denoting x_{k-1} by x and x_k by y, so that $y = Ax$.

Next, we state without proof the *Rayleigh's power method* to find the largest eigenvalue of a matrix A.

Theorem 21.3 (Rayleigh's power method): *If A is an $n \times n$ real matrix and x ($\neq 0$) is any real vector with n components and, further, let $y = Ax$, $m_0 = x^T x$, $m_1 = x^T y$, and $m_2 = y^T y$, then the quotient $q = \frac{m_1}{m_0}$ is an approximation of the largest eigenvalue λ of A and y is the corresponding eigenvector.*

Further, if A is symmetric and ϵ is the error in q, then

$$|\epsilon| \le \sqrt{\frac{m_2}{m_0} - q^2} \quad \ldots(21.25)$$

Remarks:

(1) To find the approximation for the smallest eigenvalue of A we take the reciprocal of the approximation of the largest eigenvalue of A^{-1}.

(2) To determine the intermediate eigenvalues, we proceed as follows.
Say λ_1 is the largest eigenvalue of A, then $(\lambda - \lambda_1)$ is an eigenvalue of the matrix $A - \lambda_1 I$ with some other eigenvalue λ_2 as the largest and this can be found by the power method and hence we can find the intermediate eigenvalues.

Example 21.22: Determine the absolutely largest eigenvalue and the corresponding eigenvector of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. Also find the error in the value of the eigenvalue computed.

Solution: Let $x_0 = [1, 0, 0]^T$ be the initial eigenvalue. Then

$$x_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ -14 \\ 6 \end{bmatrix}, \qquad x_4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 14 \\ -14 \\ 6 \end{bmatrix} = \begin{bmatrix} 42 \\ -48 \\ 26 \end{bmatrix}$$

Similarly,
$$x_5 = \begin{bmatrix} 131 \\ -164 \\ 100 \end{bmatrix}, \quad x_6 = \begin{bmatrix} 425 \\ -559 \\ 363 \end{bmatrix}, \quad x_7 = \begin{bmatrix} 1411 \\ -1906 \\ 1277 \end{bmatrix}$$

Take
$$x = x_6 = \begin{bmatrix} 425 \\ -559 \\ 363 \end{bmatrix}, \quad y = Ax_6 = x_7 = \begin{bmatrix} 1411 \\ -1906 \\ 1277 \end{bmatrix}$$

Thus,
$$m_0 = x^T x = [425 \;\; -559 \;\; 363]\begin{bmatrix} 425 \\ -559 \\ 363 \end{bmatrix} = 624875$$

$$m_1 = x^T y = [425 \;\; -559 \;\; 363]\begin{bmatrix} 1411 \\ -1906 \\ 1277 \end{bmatrix} = 2128680, \; m_2 = y^T y = [1411 \;\; -1906 \;\; 1277]\begin{bmatrix} 1411 \\ -1906 \\ 1277 \end{bmatrix} = 7254486.$$

Thus, $q = \dfrac{m_1}{m_0} = \dfrac{2128680}{624875} = 3.4066$ is the largest eigenvalue.

The corresponding eigenvector in the normalized form is, $y = \begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix}$.

If ϵ denotes the error in the value computed, then

$$|\epsilon| \le \sqrt{\frac{m_2}{m_0} - q^2} = \sqrt{\frac{7254486}{624875} - (3.41)^2} = 0.0676.$$

Example 21.23: Find the absolutely smallest eigenvalue of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ by power method.}$$

Solution: We find approximation to the largest eigenvalue of A^{-1}. We have,

$$B = A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Taking $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the initial approximation, we have

$$x_1 = Bx_0 = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

Similarly, $x_2 = \frac{1}{2}\begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}$, $x_3 = \frac{1}{4}\begin{bmatrix} 17 \\ 24 \\ 17 \end{bmatrix}$, $x_4 = \frac{1}{4}\begin{bmatrix} 29 \\ 41 \\ 29 \end{bmatrix}$, $x_5 = \frac{1}{8}\begin{bmatrix} 99 \\ 140 \\ 99 \end{bmatrix}$, and $x_6 = \frac{1}{8}\begin{bmatrix} 169 \\ 239 \\ 169 \end{bmatrix}$

Take $x = x_5$, $y = Ax = x_6$. Therefore,

$$m_0 = x^T x = \frac{1}{64}[99 \;\; 140 \;\; 99]\begin{bmatrix} 99 \\ 140 \\ 99 \end{bmatrix} = \frac{1}{64}(39202),\; m_1 = x^T y = \frac{1}{64}[99 \;\; 140 \;\; 99]\begin{bmatrix} 169 \\ 239 \\ 169 \end{bmatrix} = \frac{1}{64}(66922)$$

This gives, $q = \frac{m_1}{m_0} = 1.7071$. Thus, the smallest eigenvalue of the matrix A is $1/q = 0.5858$.

EXERCISE 21.3

1. Find by power method, the largest eigenvalue and the corresponding eigenvalue for the following matrices:

   (a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$

2. Find the largest eigenvalue and the corresponding eigenvector of the following matrices using power method.

(a) $\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

3. For the symmetric matrix given below, find the largest eigenvalue, corresponding eigenvector and the error in the value of the eigenvalue computed.

$$\begin{bmatrix} 0.49 & 0.02 & 0.22 \\ 0.02 & 0.28 & 0.20 \\ 0.22 & 0.20 & 0.40 \end{bmatrix}$$

4. Find the largest eigenvalue and the corresponding eigenvector of the matrix.

$$\begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}$$

21.5 FINITE DIFFERENCES

Suppose that we have a set of values (x_i, y_i), $i = 0, 1, 2, \ldots, n$ of any function $y = f(x)$, the values of x being equally spaced, that is, $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots n$. In case we need to obtain the values of $f(x)$ for some intermediates values of x, or we need to obtain the derivative of $f(x)$ for some x in the interval (x_0, x_n), then the methods of obtaining such values are based on the concept of *differences of a function*. In this section, we introduce a few basic *difference operators*.

21.5.1 Forward Differences

Let $y = f(x)$ be a function given by the values $y_0, y_1, \ldots, y_n$ at the equidistant values $x_0, x_1, \ldots, x_n$ of the independent variable x. The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ of the function y defined respectively by

$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \ldots, \Delta y_{n-1} = y_n - y_{n-1}$$

are called *first forward differences* and Δ is called the *forward difference operator*.

The differences of the first forward differences are called the *second forward differences* and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots$. Similarly, we can define the *third forward differences*, and so on. Thus,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

In general, we have

$$\Delta^r y_0 = C_0^r y_r - C_1^r y_{r-1} + C_2^r y_{r-2} - \ldots + (-1)^r C_r^r y_0 \quad \ldots(21.26)$$

and,

$$\Delta^r y_n = C_0^r y_{n+r} - C_1^r y_{n+r-1} + C_2^r y_{n+r-2} - \ldots + (-1)^r C_r^r y_n \quad \ldots(21.27)$$

A convenient method for displaying the successive forward differences of y is given below in the form of Table 21.4, called the *forward difference table.*

Table 21.4

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
$x_1 = x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2 = x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3 = x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4 = x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_5 = x_0 + 5h$	y_5					

The first term y_0 in the table is called the *leading term* and the differences $\Delta y_0, \Delta^2 y_0 \ldots$ are called the *leading differences*. The difference $h = x_i - x_{i-1}$ is called the *interval of differencing*.

The forward operator Δ satisfies the following simple properties:

1. If c is a constant, then $\Delta c = 0$.
2. For two functions f and g $\Delta[af(x) \pm bg(x)] = a\Delta f(x) \pm b\Delta g(x)$, where a and b are constants.
3. $\Delta[f(x)\, g(x)] = f(x + h)\, \Delta g(x) + g(x)\, \Delta f(x)$
4. $\Delta\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}$
5. For two positive integers m and n, $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$

21.5.2 Backward Differences

The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$, defined by

$$\nabla y_1 = y_1 - y_0,\ \nabla y_2 = y_2 - y_1, \ldots, \nabla y_n = y_n - y_{n-1}$$

are called the *first backward differences* and the operator ∇ is called the *backward difference operator*. The higher order backward differences can be defined in a similar way.

The *backward difference table* for the same values of x and y as in Table 21.4, is given below as Table 21.5.

21.5.3 Central Differences

The *first-order central differences* are defined as

$$\delta y_{1/2} = y_1 - y_0,\ \delta y_{3/2} = y_2 - y_1, \ldots, \delta y_{n-1/2} = y_n - y_{n-1}.$$

Table 21.5

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
$x_1 = x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_2 = x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_3 = x_0 + 3h$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_4 = x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_5 = x_0 + 5h$	y_5					

The operator δ is called the *central difference operator.*

The *second-order central differences* are defined as

$$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2},\ \delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}, \ldots$$

Similarly, we can define other higher order central differences.

The central differences table is given below as Table 21.6.

Table 21.6

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
x_0	y_0					
		$\delta y_{1/2}$				
$x_1 = x_0 + h$	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
$x_2 = x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
$x_3 = x_0 = 3h$	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
$x_4 = x_0 + 4h$	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
$x_5 = x_0 + 5h$	y_5					

From the Table 21.6, we observe that central differences on the same horizontal line have the same suffix. Also the differences of odd orders are only for half values of the suffix and of even orders are only for the integral values of the suffix.

Also we note that $y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}$.

21.5.4 Shift Operator *E* and Averaging Operator μ

The *shift operator E* is the operation of increasing the argument x by h, that is,

$$Ef(x) = f(x+h),\ E^2 f(x) = f(x+2h), \text{ etc.}$$

The inverse shift operator E^{-1} is defined by $E^{-1}f(x) = f(x-h)$.

The *averaging operator* μ is defined by $\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{1}{2}h\right) + f\left(x-\frac{1}{2}h\right)\right]$

Remark: In case we take $y_x = f(x)$, then $Ey_x = y_{x+h}$, $E^{-1}y_x = y_{x-h}$, and $\mu y_x = \frac{1}{2}[y_{x+h/2} + y_{x-h/2}]$.

21.5.5 Relations Between the Operators

We can easily establish the following identities between the different operators:

(i) $\Delta = E - 1$, or $E = 1 + \Delta$ (ii) $\nabla = 1 - E^{-1}$, or $\nabla = \frac{E-1}{E}$

(iii) $\delta = E^{1/2} - E^{-1/2}$ (iv) $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

(v) $\Delta = E\nabla = \nabla E = \delta E^{1/2}$ (vi) $E = e^{hD}$

where $D = \frac{d}{dx}$ is the differential operator and h is the interval of differencing.

For example, to prove $\Delta = E\nabla$ in (v), we have $E\nabla y_x = E(y_x - y_{x-h}) = Ey_x - Ey_{x-h} = y_{x+h} - y_x = \Delta y_x$.

Thus, $E\nabla = \Delta$ (or, $\nabla = E^{-1}\Delta$).

To prove (vi), consider

$$Ef(x) = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots$$

$$= f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \ldots = \left[1 + hD + \frac{h^2}{2!}D^2 + \ldots\right]f(x) = e^{hD}f(x).$$

Thus, $E = e^{hD}$.

Example 21.24: Prove the following identities:

(a) $(1+\Delta)(1-\nabla) = 1$ (b) $\Delta\nabla = \Delta - \nabla$

Solution:

(a) $(1+\Delta)(1-\nabla)f(x) = EE^{-1}f(x) = Ef(x-h) = f(x)$. Thus, $(1+\Delta)(1-\nabla) = 1$

(b)
$$\begin{aligned}\nabla\Delta f(x) &= (E-1)(1-E^{-1})f(x) = (E-1)[f(x) - f(x-h)] \\ &= Ef(x) - f(x) - Ef(x-h) + f(x-h) = [f(x+h) - f(x)] - [f(x) - f(x-h)] \\ &= \Delta f(x) - \nabla f(x) = (\Delta - \nabla)f(x)\end{aligned}$$

Thus, $\Delta\nabla = \Delta - \nabla$.

Example 21.25: Evaluate (i) $\Delta \tan^{-1} x$, (ii) $\Delta^2(e^{3x+5})$, (iii) $\dfrac{\Delta^2}{E}\sin(x+h) + \dfrac{\Delta^2 \sin(x+h)}{E \sin(x+h)}$, the interval of differencing being h.

Solution:

(i) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x = \tan^{-1}\left\{\dfrac{x+h-x}{1+(x+h)x}\right\} = \tan^{-1}\left\{\dfrac{h}{1+hx+x^2}\right\}$.

(ii) $\Delta^2(e^{3x+5}) = \Delta[\Delta(e^{3x+5})] = \Delta[e^{3x+3h+5} - e^{3x+5}] = (e^{3h}-1)\,\Delta e^{3x+5} = (e^{3h}-1)^2 e^{3x+5}$.

(iii) $$\frac{\Delta^2}{E}\sin(x+h) + \frac{\Delta^2 \sin(x+h)}{E\sin(x+h)} = \frac{(E-1)^2}{E}\sin(x+h) + \frac{(E-1)^2 \sin(x+h)}{E\sin(x+h)}$$

$$= (E-2+E^{-1})\sin(x+h) + \frac{(E^2-2E+1)\sin(x+h)}{\sin(x+2h)}$$

$$= \sin(x+2h) - 2\sin(x+h) + \sin x + \frac{\sin(x+3h) - 2\sin(x+2h) + \sin(x+h)}{\sin(x+2h)}$$

$$= \{\sin(x+2h) + \sin x\} - 2\sin(x+h) + \frac{\{\sin(x+3h) + \sin(x+h)\}}{\sin(x+2h)} - 2$$

$$= 2\sin(x+h)\cos h - 2\sin(x+h) + 2\cos h - 2$$

$$= 2\sin(x+h)(\cos h - 1) + 2(\cos h - 1) = 2[\sin(x+h) - 1](\cos h - 1).$$

Example 21.26: Evaluate

(a) (i) $\Delta^2\left[\dfrac{5x+12}{x^2+5x+6}\right]$ (ii) $\left(\dfrac{\Delta^2}{E}\right)x^3$, interval of differencing being unity.

(b) Show that $\displaystyle\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$.

Solution:

(a) (i) $$\Delta^2\left[\frac{5x+12}{x^2+5x+6}\right] = \Delta^2\left[\frac{2}{x+2} + \frac{3}{x+3}\right] = \Delta\left[\Delta\left(\frac{2}{x+2}\right) + \Delta\left(\frac{3}{x+3}\right)\right]$$

$$= \Delta\left[2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right)\right]$$

$$= -2\Delta\left(\frac{1}{(x+3)(x+2)}\right) - 3\Delta\left(\frac{1}{(x+4)(x+3)}\right)$$

$$= -2\left[\frac{1}{(x+4)(x+3)} - \frac{1}{(x+3)(x+2)}\right] - 3\left[\frac{1}{(x+5)(x+4)} - \frac{1}{(x+4)(x+3)}\right]$$

$$= \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}.$$

(ii) $\left[\frac{\Delta^2}{E}\right]x^3 = (\Delta^2 E^{-1})\, x^3 = (E-1)^2 E^{-1}x^3 = (E^2 - 2E + 1)\, E^{-1}x^3$

$$= (E - 2 + E^{-1})x^3 = (x+1)^3 - 2x^3 + (x-1)^3 = 6x.$$

(b) L.H.S. $= \sum_{k=0}^{n-1} \Delta(f_{k+1} - f_k) = \sum_{k=0}^{n-1} (\Delta f_{k+1} - \Delta f_k)$

$= \Delta f_1 - \Delta f_0 + \Delta f_2 - \Delta f_1 \;\ldots\; + \Delta f_n - \Delta f_{n-1} = \Delta f_n - \Delta f_0 =$ R.H.S.

Example 21.27: Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$ and $y_5 = 8$, find $\Delta^5 y_0$.

Solution: We have, $\Delta^5 y_0 = (E-1)^5 y_0 = (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_0$

$= y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 8 - 500 + 2000 - 810 + 60 - 3 = 755.$

Example 21.28: Evaluate $\Delta^{10}[(1-ax^4)(1-bx^3)(1-cx^2)(1-dx)]$.

Solution: $\Delta^{10}[(1-ax^4)(1-bx^3)(1-cx^2)(1-dx)] = \Delta^{10}[abcd\, x^{10}$ + terms of x^9 and lower degrees$]$

$$= abcd\, \Delta^{10}(x^{10}) = abcd\,(10!).$$

21.5.6 Differences of a Polynomial

We prove the following result.

Theorem 21.4: *The nth difference of a polynomial of the nth degree is constant and all higher order differences are zeros.*

Proof: Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$

be a polynomial of degree n with $a_0, a_1, a_2, \ldots a_n$ as constant coefficients.

$$\Delta f(x) = f(x+h) - f(x)$$
$$= a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + \ldots + a_{n-1}[(x+h) - x]$$
$$= a_0nhx^{n-1} + a_1'x^{n-2} + a_2'x^{n-3} + \ldots + a_{n-2}'x + a_{n-1}'$$

where $a_1', a_2', \ldots a_{n-1}'$ are new constant coefficients. Thus, the first-order differences of a polynomial of the nth degree is a polynomial of degree $(n-1)$.

Similarly, $\Delta^2 f(x) = a_0n(n-1)h^2x^{n-2} + a_1''x^{n-3} + \ldots + a_{n-3}''x + a_{n-2}''$

and continuing with this process, we obtain

$$\Delta^n f(x) = a_0 n!\, h^n, \text{ and, } \Delta^{n+1} f(x) = 0.$$

Remark: The converse of this result is also true, that is, *if the nth differences of a function, tabulated at equally spaced intervals are constant, then the function represents a polynomial of degree n*. This result is of vital importance since it enables us to approximate a function by a polynomial of degree k, if its kth differences are nearly constants.

Example 21.29: Find the first term of the series with second and subsequent terms as 8, 3, 0, –1, 0.

Solution: Given $f(2) = 8, f(3) = 3, f(4) = 0, f(5) = -1, f(6) = 0$, we are to find $f(1)$.

We construct the forward difference table with the value given as follows:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
2	8			
		–5		
3	3		2	0
		–3		
4	0		2	0
		–1		
5	–1		2	
		1		
6	0			

Since the second-order differences are constant, thus, $f(x)$ can be represented as a quadratic polynomial, say

$$f(x) = ax^2 + bx + c \qquad \text{...(21.28)}$$

Using $f(2) = 8, f(3) = 3$ and $f(4) = 0$ in (21.28), we obtain

$$4a + 2b + c = 8,\ 9a + 3b + c = 3, \text{ and } 16a + 4b + c = 0 \text{ respectively.}$$

Solving these for a, b, c, we obtain $a = 1, b = -10$ and $c = 24$ and substituting in (21.28), we have

$$f(x) = x^2 - 10x + 24. \text{ This gives } f(1) = 1 - 10 + 24 = 15.$$

Example 21.30: Following are the values of a polynomial of degree 5 in which $f(3)$ is in error. Correct the error.

x	0	1	2	3	4	5	6
y	1	2	33	254	1054	3126	7777

Solution: Let the correct entry be $f(3) = 254 + \epsilon$.

We construct the forward difference table with the values as follows:

Since the polynomial is of degree 5, hence the 5th differences must be constant. Thus,

$$-245 + 10\epsilon = 790 - 10\epsilon. \text{ This gives, } 20\epsilon = 1035, \text{ or } \epsilon = 51.75.$$

Hence, $\quad f(3) = 254 + \epsilon = 254 + 51.75 = 305.75.$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1					
		1				
1	2		30			
		31		$160+c$		
2	33		$190+c$		$309-4c$	
		$221+c$		$469-3c$		$-245+10c$
3	$254+c$		$659-2c$		$64+6c$	
		$880-c$		$533+3c$		$790-10c$
4	1054		$1192+c$		$854-4c$	
		2072		$1387-c$		
5	3126		2579			
		4651				
6	7777					

Example 21.31: Assuming that the following values of y belong to a polynomial of degree 4, compute the next three values

x :	0	1	2	3	4
y :	1	-1	1	-1	1

Solution: Let the next three values be y_5, y_6 and y_7. We construct the following difference table from the data given.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$y_0 = 1$				
		-2			
1	$y_1 = -1$		4	-8	
		2			
2	$y_2 = 1$		-4		16
		-2		8	
3	$y_3 = -1$		4		$\Delta^4 y_1$
		2		$\Delta^3 y_2$	
4	$y_4 = 1$		$\Delta^2 y_3$		$\Delta^4 y_2$
		Δy_4		$\Delta^3 y_3$	
5	y_5		$\Delta^2 y_4$		$\Delta^4 y_3$
		Δy_5		$\Delta^3 y_4$	
6	y_6		$\Delta^2 y_5$		
		Δy_6			
7	y_7				

Since the values of y belong to a polynomial of degree 4, hence, the 4th differences must be constant. Thus, $\Delta^4 y_1 = \Delta^4 y_2 = \Delta^4 y_3 = 16$ Now $\Delta^4 y_1 = 16$, gives

$$(E-1)^4 y_1 = 16$$

or,
$$(E^4 - 4E^3 + 6E^2 - 4E + 1)y_1 = 16$$

or, $$y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 = 16$$

or, $$y_5 = 16 + 4 + 6 + 4 + 1 = 31.$$

Similarly, $\Delta^4 y_2 = 16$ gives $y_6 = 129$, and $\Delta^4 y_3 = 16$ gives $y_7 = 351$.

Example 21.32: Show that $\Delta^n y_x = y_{x+n} - n\, y_{x+n-1} + \frac{n(n-1)}{2} y_{x+n-2} - \ldots + (-1)^n y_x$.

Solution: L.H.S. $= \Delta^n y_x = (E-1)^n y_x = [E^n - C_1^n E^{n-1} + C_2^n E^{n-2} - \ldots + (-1)^n] y_x$

$$= E^n y_x - n\, E^{n-1} y_x + \frac{n(n-1)}{2} E^{n-2} y_x - \ldots + (-1)^n y_x$$

$$= y_{x+n} - n y_{x+n-1} + \frac{n(n-1)}{2} y_{x+n-2} - \ldots + (-1)^n y_x = \text{R.H.S.}$$

Example 21.33: Show that $e^x \left[y_0 + x\Delta y_0 + \frac{x^2}{2!} \Delta^2 y_0 + \ldots \right] = y_0 + y_1 x + y_2 \frac{x^2}{2!} + \ldots$

Solution: L.H.S. $= e^x \left[y_0 + x\Delta y_0 + \frac{x^2}{2!} \Delta^2 y_0 + \ldots \right] = e^x \left[1 + x\Delta + \frac{x^2 \Delta^2}{2!} + \ldots \right] y_0$

$$= e^x e^{x\Delta} y_0 = e^{x(1+\Delta)} y_0 = e^{xE} y_0$$

$$= \left(1 + xE + \frac{x^2 E^2}{2!} + \ldots \right) y_0 = y_0 + x y_1 + \frac{x^2}{2!} y_2 + \ldots = \text{R.H.S.}$$

21.5.7 Detection of Errors by Use of Difference Tables

Difference tables can be used to check error in the tabular values. Suppose there is an error of ϵ units in a certain tabular value. As higher order differences are formed, the error spreads out uniformly and gets magnified also, as shown below in Table 21.7.

We make the following observations from the table:

1. The coefficients of the errors in any one column are the binomial coefficients with alternating sign and the algebraic sum of the errors is zero.
2. The maximum error occurs against the incorrect entry.
3. If an error is present in a given data, the differences of some order will become alternating in sign. Hence, in an actual problem higher order differences be formed till this pattern is revealed.

The above observations can be employed to detect the error in a specific entry of the table.

Table 21.7

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0				
	0			
0		0		
	0		0	
0		0		ϵ
	0		ϵ	
0		ϵ		-4ϵ
	ϵ		-3ϵ	
ϵ		-2ϵ		6ϵ
	$-\epsilon$		3ϵ	
0		ϵ		-4ϵ
	0		$-\epsilon$	
0		0		ϵ
	0		0	
0		0		
	0			
0				

Example 21.34: Locate and correct the error in the following table values:

x :	1	2	3	4	5	6	7	8
y :	3010	3424	3802	4105	4472	4771	5051	5315

Solution: Forming the difference table, we have

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	3010				
		414			
2	3424		−36		
		378		−39	
3	3802		−75		178
		303		139	
4	4105		64		−271
		367		−132	
5	4472		−68		181
		299		49	
6	4771		−19		−46
		280		3	
7	5051		−16		
		264			
8	5315				

The entries in the fourth order differences column are alternating positive and negative, and the term -271 in this column has fluctuations of 449 and 452 on either side of it. This suggests that there is error in the entry 4105 corresponding to $x = 4$.

Comparing the column 4 in this table with that of column 4 in Table 21.7. We can take $-10\epsilon = 450$ which gives $\epsilon = -45$. This gives the correct entry as $4105 - \epsilon = 4105 + 45 = 4150$.

EXERCISE 21.4

1. Show that

   (i) $\Delta^r y_x = \nabla^r y_{x+r} = \delta^r y_{x+r/2}$ (ii) $\Delta\nabla y_x = \nabla\Delta y_x = \delta^2 y_x$

   (iii) $\Delta(y_x^2) = (y_x + y_{x+1})\Delta y_x$ (iv) $\Delta\left(\dfrac{1}{y_x}\right) = -\Delta y_x/(y_x y_{x+1})$

   (vi) $\Delta \log y_x = \log\left[1 + \dfrac{\Delta y_x}{y_x}\right]$.

2. Show that

   (i) $\Delta = \nabla(1-\nabla)^{-1}$ (ii) $1 + \Delta = (E-1)\nabla^{-1}$.

3. Show that

   (i) $E^{1/2} = \mu + \dfrac{1}{2}\delta,\ \ E^{-1/2} = \mu - \dfrac{1}{2}\delta$ (ii) $\mu\delta = \dfrac{1}{2}\Delta E^{-1} + \dfrac{1}{2}\Delta$

4. Find (a) $\Delta^n(e^{ax+b})$, (b) $\Delta(x^2/\cos 2x)$, (c) $\Delta(e^x \log 2x)$, interval of differencing being h.

5. Taking unity as the interval of differencing show that

   (i) $\dfrac{\Delta^2 x^3}{Ex^3} = \dfrac{6}{(x+1)^3}$

   (ii) $\Delta^n \cos(a + bx) = \left(2\sin\dfrac{h}{2}\right)^n \cos\left[a + bx + \dfrac{n(b+\pi)}{2}\right]$

6. If $y_x = ax^2 + bx + c$, then show that $y_{2n} - C_1^n 2y_{2n-1} + C_2^n 2^2 y_{2n-2} - \ldots + (-2)^n y_n = (-1)^n(c - 2^n)$.

7. Evaluate the following when interval of differencing is 2

   (i) $\Delta^3[(1-x)(1-2x)(1-3x)]$

   (ii) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$.

8. Prove the following:

   (i) $y_1 x + y_2 x^2 + \ldots = \dfrac{x}{1-x}y_1 + \left(\dfrac{x}{1-x}\right)^2 \Delta y_1 + \left(\dfrac{x}{1-x}\right)^3 \Delta^2 y_1 + \ldots$

   (ii) $y_x = y_n - {}^{n-x}C_1\,\Delta y_{n-1} + {}^{n-x}C_2\,\Delta^2 y_{n-2} - \ldots (-1)^{n-x}\Delta^{n-x} y_x$.

(iii) $(y_1 - y_0) - x(y_2 - y_1) + x^3(y_3 - y_2) \ldots = \dfrac{\Delta y_0}{1+x} - x\dfrac{\Delta^2 y_0}{(1+x)^2} + x^2\dfrac{\Delta^3 y_0}{(1+x)^3} - \ldots$

9. Given that $y_0 + y_8 = 1.9243$, $y_1 + y_7 = 1.9540$, $y_2 + y_6 = 1.9823$ and $y_3 + y_5 = 1.9956$. Show that $y_4 = 0.9999557$.

10. Assuming that the following values of y belong to a polynomial of degree 3, compute the next two values using difference table:

x :	−1	0	1	2	3	4	5
y :	−13	−7	−1	11	35	77	143

11. Extend the following table to two more terms oneeither side by constructing the difference table:

x :	−0.2	0.0	0.2	0.4	0.6	0.8	1.0
y :	2.6	3.0	3.4	4.28	7.08	14.2	29.0

12. Find the missing values in the following table:

x :	45	50	55	60	65
y :	3.0	–	2.0	–	−2.4

13. Locate and correct the error in the following table of values:

x :	2.5	3.0	3.5	4.0	4.5	5.0	5.5
y :	4.32	4.83	5.27	5.47	6.26	6.79	7.23

14. From the following table, find the member of students who obtained less than 45 marks.

Marks:	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students:	31	42	51	35	31

15. Prove that $Dy = \dfrac{1}{h}\left[\Delta y - \dfrac{\Delta^2 y}{2} + \dfrac{\Delta^3 y}{3} - \dfrac{\Delta^4 y}{4} + \ldots\right]$

21.6 INTERPOLATION AND INTERPOLATION FORMULAE

Suppose we are given the following set of values of the function $y = f(x)$

x :	x_0	x_1	x_2	...	x_n
y :	y_0	y_1	y_2	...	y_n

Then the process of finding the value of y corresponding to any value of x between x_0 and x_n is called *interpolation*. Thus, *interpolation is a technique of estimating the value of a function for any intermediate value of the independent variable*. However, the process of computing the value of the function outside the given range is called *extrapolation*.

Generally, the tabulated values of the function $f(x)$ are given and the form of $f(x)$ is not known. It is very difficult to determine the exact form of $f(x)$ with the help of tabulated set of values (x_i, y_i). In such cases $f(x)$ is replaced by a simpler function $\phi(x)$ which assumes the same values as those of $f(x)$ at the tabulated set of points. Such a function $\phi(x)$ is known as the *interpolating function* or *smoothing function*. If $\phi(x)$ is a polynomial, then it is called the *interpolating polynomial* and the process is called the *polynomial interpolation*. Similarly, if $\phi(x)$ is a finite trigonometric series, we have *trigonometric interpolation*. Here we shall consider polynomial interpolation only.

21.6.1 Newton's Interpolation Formulae

The study of interpolation is based on the calculus of finite differences. We derive two important interpolation formulae based on forward and backward differences of a function.

1. Newton's forward interpolation formula. Let the function $y = f(x)$ take the values $y_0, y_1 ... y_n$ corresponding to the values $x_0, x_1, ..., x_n$ of x, such that $x_i = x_0 + ih$, $(i = 0, 1, 2, ..., n)$, that is, the values of x are equally spaced.

Assuming $y(x)$ to be a polynomial of the nth degree in x, satisfying

$$y(x_0) = y_0, \quad y(x_1) = y_1, ..., y(x_n) = y_n$$

we can write $y(x)$ as

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + ... + a_n(x - x_0) ... (x - x_{n-1}) \quad ...(21.29)$$

where a_is, $i = 0, 1, 2, ..., n$ are constants to be evaluated.

Substituting $x = x_0, x_1, ..., x_n$ successively in (21.29), we obtain

$$y_0 = a_0$$
$$y_1 = a_0 + a_1(x_1 - x_0),$$
$$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1), \text{ and so on.}$$

From these we find that, $a_0 = y_0$.

Next, $y_1 = y_0 + a_1(x_1 - x_0)$, or $a_1 h = y_1 - y_0$, or $a_1 = \frac{1}{h}\Delta y_0$.

Also, $\Delta y_1 = y_2 - y_1 = a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1) = a_1 h + a_2(2h)(h) = \Delta y_0 + 2h^2 a_2$

This gives, $\quad \Delta y_1 - \Delta y_0 = 2h^2 a_2$, or $a_2 = \frac{1}{2h^2}\Delta^2 y_0$

Similarly, $\quad a_3 = \frac{1}{3!h^3}\Delta^3 y_0$, and so on.

Substituting values of $a_0, a_1, a_2, a_3 ...$ in (21.29), we obtain

$$y(x) = y_0 + \frac{(x - x_0)}{h}\Delta y_0 + \frac{(x - x_0)(x - x_1)}{2!h^2}\Delta^2 y_0 + \frac{(x - x_0)(x - x_1)(x - x_2)}{3!h^3}\Delta^3 y_0 + ... \quad ...(21.30)$$

Let us suppose that we need to evaluate y for $x = x_0 + ph$. Then

$$x - x_0 = ph,\ x - x_1 = (x - x_0) - (x_1 - x_0) = ph - h = (p - 1)h$$
$$x - x_2 = (x - x_1) - (x_2 - x_1) = (p - 1)h - h = (p - 2)h, \text{ etc.}$$

Also writing, $y(x) = y(x_0 + ph) = y_p$, then (21.30) becomes

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ... + \frac{p(p-1)(p-2)...(p-\overline{n-1})}{n!}\Delta^n y_0$$

...(21.31)

This is *Newton's forward interpolation formula* as the expression consists of y_0 and the forward differences of y_0.

Alternatively, For any real number p, we have $E^p f(x) = f(x + ph)$. Thus,

$$y_p = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^p y_0$$

$$= \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right] y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \qquad \ldots(21.32)$$

If $y = f(x)$ is a polynomial of the nth degree, then Δy_0^{n+1} and higher order differences will be zero. Hence, the expression (21.32) becomes

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots + \frac{p(p-1)\ldots(p-\overline{n-1})}{n!}\Delta^n y_0, \text{ same as (21.31).}$$

2. *Newton's backward interpolation formula.* Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$, where p is any real number. Then we have

$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\right] y_n$$

$$= y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots \qquad \ldots(21.33)$$

It is called *Newton's backward interpolation formula* as it contains the backward differences of y_n.

Remarks:

(1) Newton forward interpolation formula (21.31) is applied for interpolating the values of y near the beginning of a set of tabulated values and also for extrapolating values of y a little before of y_0.

(2) Newton backward interpolation formula (21.33) is applied for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little after y_n.

Example 21.35: Following data gives the marks distribution of 190 students in a test. Find the number of students whose marks lie between 45 and 50.

Marks:	30–40	40–50	50–60	60–70	70–80
No. of Students:	31	42	51	35	31

Solution: Cumulative frequency table for the given data is

Upper limits of the class intervals (x):	40	50	60	70	80
Cumulative frequency (y):	31	73	124	159	190

The difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		–25	
60	124		–16		37
		35		12	
70	159		–4		
		31			
80	190				

We have, $x_0 = 40, x = 45, h = 10, y_0 = 73, \Delta y_0 = 42, \Delta^2 y_0 = 9, \Delta^3 y_0 = -25, \Delta^4 y_0 = 37$.

Take $$p = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

Newton's forward interpolation formula is

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^2 y_0 + \dots$$

Thus, $$f(45) = 31 + (0.5)(42) + \frac{(0.5)(-0.5)}{2}(9) + \frac{(0.5)(0.5-1)(0.5-2)}{6}(-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(37) + \dots$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4452 = 47.8673 = 48 \text{ (approximately)}$$

Thus, the number of students who obtained marks less than 45 is 48 and hence the number of students who scored marks between 45 and 50 is $73 - 48 = 25$.

Example 21.36: A second degree polynomial passes through the points (1, –1), (2, –1), (3, 1), (4, 5). Find the polynomial.

Solution: The difference table for the given values of x and y is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	–1			
		0		
2	–1		2	
		2		0
3	1		2	
		4		
4	5			

We have, $x_0 = 1, h = 1, y_0 = -1, \Delta y_0 = 0, \Delta^2 y_0 = 2, \Delta^3 y_0 = 0$. Take $p = \frac{x - x_0}{h} = (x - 1)$.

Newton's forward interpolation formula is

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots = -1 + (x-1)\,0 + \frac{(x-1)(x-1-1)}{2}\,2 = x^2 - 3x + 1.$$

Thus, $f(x) = x^2 - 3x + 1.$

Example 21.37: The following data gives the melting point of an alloy of lead and zinc, where T is the temperature and P is the percentage of lead in the alloy.

P:	40	50	60	70	80	90
T:	184	204	226	250	276	304

Find the melting point of the alloy containing 84 per cent lead.

Solution: The value 84 is near the end of the table, therefore, we use Newton's backward interpolation formula. The difference table is

P	T	∇T	$\nabla^2 T$	$\nabla^3 T$	$\nabla^4 T$	$\nabla^5 T$
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

We have, $x_n = 90$, $x = 84$, $h = 10$, $T_n = 304$, $\nabla T_n = 28$, $\nabla^2 y_n = 2$, and $\nabla^3 y_n = \nabla^4 y_n = \nabla^5 y_n = 0$,

$$p = \frac{x - x_n}{h} = \frac{84 - 90}{10} = -0.6$$

Newton's backward formula is

$$T(p) = T_n + p\nabla T_n + \frac{p(p+1)}{2}\nabla^2 T_n + \ldots$$

Thus, $$T(-0.6) = 304 - 0.6 \times 28 + \frac{(-0.6)(-0.6+1)}{2}2 = 304 - 16.8 - 0.24 = 286.96.$$

Example 21.38: The following are the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x (height):	100	150	200	250	300	350	400
y (distance):	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y, when x is (i) 218 ft, (ii) 410 ft.

Solution: The difference table for the given data is as under:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
100	10.63				
		2.40			
150	13.03		–0.39		
		2.01		0.15	
200	**15.04**		–0.24		–0.07
		1.77		0.08	
250	16.81		**–0.16**		– 0.05
		1.61		**0.03**	
300	18.42		–0.13		**–0.01**
		1.48		**0.02**	
350	19.90		**–0.11**		
		1.37			
400	**21.27**				

(i) If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 y_0 = 0.03$ and $\Delta^4 y_0 = -0.01$.

Also here $x = 218$ and $h = 50$, therefore, $p = \dfrac{x - x_0}{h} = \dfrac{18}{50} = 0.36$

Using Newton's forward interpolation formula,

$$y(p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots$$

We obtain, $$y(218) = 15.04 + (0.36)(1.77) + \frac{(.36)(-.64)(-0.16)}{2!} + \frac{(.36)(-.64)(-1.64)}{3!}(0.03) + \frac{(.36)(-.64)(-1.64)(-2.64)}{4!}(-0.01)$$

$$= 15.697 \simeq 15.7 \text{ nautical miles}$$

(ii) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

Taking $x_n = 400, p = \dfrac{x - x_n}{h} = \dfrac{410 - 400}{50} = \dfrac{10}{50} = 0.2.$

Using the line of backward differences, we have

$$y_n = 21.27,\ \nabla y_n = 1.37,\ \nabla^2 y_n = -0.11,\ \nabla^3 y_n = 0.02, \text{ and } \nabla^4 y_n = -0.01.$$

Using Newton's backward formula

$$y(p) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \ldots$$

We obtain, $$y(410) = 21.27 + (0.2)(1.37) + \frac{(0.2)(1.2)}{2}(-0.11) + \frac{(0.2)(1.2)(2.2)}{3!}(0.02) + \frac{(0.2)(1.2)(2.2)(3.2)}{4!}(-0.01) + \ldots$$

$$= 21.53 \text{ nautical miles.}$$

21.6.2 Central Difference Interpolation Formulae

Newton's forward and backward interpolation formulae are applicable for interpolation near the beginning and end of the tabulated values. In this section, we study *central difference formulae* which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$ and the corresponding values of $y = f(x)$ are y_{-2}, y_{-1}, y_0, y_1, y_2, then we can form the difference table as follows:

Table 21.8

x	y	1st diff	2nd diff	3rd diff	4th diff
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} (= \delta y_{-3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} (= \delta^2 y_{-1})$		
		$\Delta y_{-1} (= \delta y_{-1/2})$		$\Delta^3 y_{-2} (= \delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1} (= \delta^2 y_0)$		$\Delta^4 y_{-2} (= \delta^4 y_0)$
		$\Delta y_0 (= \delta y_{1/2})$		$\Delta^3 y_{-1} (= \delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0 (= \delta^2 y_1)$		
		$\Delta y_1 (= \delta y_{3/2})$			
$x_0 + 2h$	y_2				

1. Gauss forward interpolation formula. Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \quad \ldots(21.34)$$

We have, $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

This gives, $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$. Similarly, $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$, $\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$

Next, $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

This gives, $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$. Similarly, $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ ...(21.35)

Substituting for $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ in (21.34), we obtain

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

$$= y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \ldots$$

Using (21.35), it gives

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)(p)(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)(p)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(21.36)$$

This is *Gauss forward interpolation formula.*

In the central differences notation using Table 21.8, this formula becomes

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 y_0 + \ldots \quad \ldots(21.37)$$

2. *Gauss backward interpolation formula.* Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \quad \ldots(21.38)$$

We have, $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

This gives, $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$.

Similarly, $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$, $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$.

Next, $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

This gives, $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$ ...(21.39)

Similarly, $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ ...(21.40)

Substituting for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ in (21.38), we obtain

$$y_p = y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$
$$+ \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

$$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \ldots$$

$$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-2} + \Delta^4 y_{-2})$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \ldots$$

(using (21.39) & (21.40))

$$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(21.41)$$

This is *Gauss backward interpolation formula.*

In the central difference notations using Table 21.8, this formula is

$$y_p = y_0 + p\delta y_{-1/2} + \frac{(p+1)p}{2}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{-1/2} + \frac{(p+2)(p+1)p(p-1)}{4!}\delta^4 y_0 + \ldots \quad \ldots(21.42)$$

3. *Stirling's formula.* Taking the mean of the Gauss forward interpolation formula (21.36) and backward interpolation formula (21.41), we obtain

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(21.43)$$

the *Stirling formula*.

4. Bessel's formula. Gauss forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} \ldots \quad \ldots(21.44)$$

Also, $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

This gives, $\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1}$ ...(21.45)

Similarly, $\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$ ...(21.46)

Now (21.44) can be rewritten as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{p(p^2-1)(p-2)}{4!}\left(\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right) + \ldots$$

$$= y_0 + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}(\Delta^2 y_0 - \Delta^3 y_{-1}) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}(\Delta^4 y_{-1} - \Delta^5 y_{-2}) + \ldots$$

[using (21.45) & (21.46)]

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)}{2!}\left(\frac{p+1}{3} - \frac{1}{2}\right)\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots$$

Simplifying, we obtain

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-1/2)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots \quad \ldots(21.47)$$

the *Bessel's formula*.

5. Everett's formula. Gauss forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2}$$
$$+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \ldots \quad \ldots(21.48)$$

Substituting for odd differences in (21.48) using the relations

$$\Delta y_0 = y_1 - y_0,\ \Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1},\ \Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \text{ etc.},$$

we obtain
$$y_p = y_0 + p(y_1 - y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^2 y_0 - \Delta^2 y_{-1})$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}(\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \ldots$$
$$= (1-p)y_0 + py_1 - \frac{p(p-1)(p-2)}{3!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^2 y_0$$
$$- \frac{(p+1)p(p-1)(p-2)(p-3)}{5!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^4 y_{-1} - \ldots$$

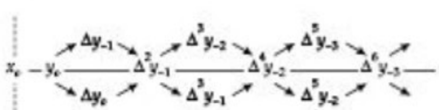
To change the terms with negative sign, substituting $p = 1 - q$, we obtain

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0$$
$$+ \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots, \quad \ldots(21.49)$$

the *Everett's formula.*

Remarks:

1. The Gauss forward interpolation formula (21.36) uses odd differences below the central line through y_0 and even differences on the central line whereas Gauss backward formula (21.41) uses odd differences above the central line through y_0 and even differences on the central line as indicated below:



2. The Gauss forward formula is used to interpolate the values of the function for the values of p, such that $0 < p < 1$ and the Gauss backward formula is used to interpolate the value of the function for the value of p, such that $-1 < p < 0$.

3. The Stirling formula (21.43) uses means of odd differences just above and below the central line and even differences on the central line as indicated below:

$$y_0 \to \begin{Bmatrix} \Delta y_{-1} \\ \Delta y_0 \end{Bmatrix} \to \Delta^2 y_{-1} \to \begin{Bmatrix} \Delta^3 y_{-2} \\ \Delta^3 y_{-1} \end{Bmatrix} \to \Delta^4 y_{-2} \to \begin{Bmatrix} \Delta^3 y_{-3} \\ \Delta^5 y_{-2} \end{Bmatrix} \to \Delta^6 y_{-3} \to$$

Here { } indicates the mean of the entries enclosed. This formula is suitable for $-0.25 \le p \le 0.25$.

4. The Bessel's formula (21.47) involves odd difference below the central line and means of the even difference on and below the central line as indicated below:

$$\begin{matrix} y_0 \\ y_0 \end{matrix} \to \Delta y_0 \to \begin{Bmatrix} \Delta^2 y_{-1} \\ \Delta^2 y_0 \end{Bmatrix} \to \Delta^3 y_{-1} \to \begin{Bmatrix} \Delta^4 y_{-2} \\ \Delta^4 y_{-1} \end{Bmatrix} \to \Delta^5 y_{-2} \to \begin{Bmatrix} \Delta^6 y_{-3} \\ \Delta^6 y_{-2} \end{Bmatrix} \to$$

This formula is suitable for $0.25 \le p \le 0.75$.

5. The Everett's formula (21.49) involves only even differences on and below the central line as indicated below:

$$\begin{matrix} y_0 & \Delta^2 y_{-1} & \Delta^4 y_{-2} & \Delta^6 y_{-3} \\ y_1 & \Delta^2 y_0 & \Delta^4 y_{-1} & \Delta^6 y_{-2} \end{matrix}$$

and converges rapidly.

6. Since the coefficients in the central difference formulas are smaller and converge faster than those in the Newton's formulae so, whenever possible, central differences formulae are used in preference to the Newton's formulae. However, the right choice of an interpolation formula depends on the position of the value to be interpolated in the table.

Example 21.39: Using Gauss forward formula, find $e^{1.17}$ from the following data:

x :	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x :	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Solution: Taking $x_0 = 1.15$, $h = 0.05$, we need to find the value of y for $x = 1.17$, that is, for

$$p = \frac{x - x_0}{h} = \frac{1.17 - 1.15}{0.05} = 0.25.$$

The difference table is given below see table on page 1228.

Gauss forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \ldots$$

$$= 3.1582 + 0.25(0.1619) + \frac{(0.25)(-0.75)}{2!}(0.0079) + \frac{(1.25)(0.25)(-0.75)}{3!}(0.004) + \ldots$$

$$= 3.1582 + 0.0405 - 0.0007 - 0.0002 + \ldots = 3.1978 \text{ (approx)}.$$

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$
1.00	–3	2.7183				
			0.1394			
1.05	–2	2.8577		0.0071		
			0.1465		0.004	
1.10	–1	3.0042		0.0075		0
			0.1540		0.004	
1.15	**0**	**3.1582**		**0.0079**		**0**
			0.1619		**0.004**	
1.20	1	3.3201		0.0083		0.001
			0.1702		0.005	
1.25	2	3.4903		0.0088		
			0.1790			
1.30	3	3.6693				

Example 21.40: Using Gauss backward formula, interpolate the population of a town for the year 1984, from the data given below:

Year :	1949	1959	1969	1979	1989	1999
Population (in lacs):	12	15	20	27	39	52

Solution: Taking $x_0 = 1979, h = 10$, the population of the town is to be found for 1984, which gives

$$p = \frac{1984 - 1979}{10} = 0.5.$$

The difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1949	–3	12					
			3				
1959	–2	15		2			
			5		0		
1969	–1	20		2		3	
			7		**3**		–10
1979	0	**27**		**–5**		**7**	
			12		**–4**		
1989	1	39		1			
			13				
1999	2	52					

Gauss's backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3} + \ldots$$

$$= 27 + (0.5)7 + \frac{(1.5)(0.5)}{2!}(5) + \frac{(1.5)(.5)(-.5)}{3!}(3) + \frac{(2.5)(1.5)(.5)(-.5)}{4!}(-7)$$
$$+ \frac{(2.5)(1.5)(.5)(-.5)(-1.5)}{5!}(-10) + \ldots$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2734 - 0.1172 + \ldots = 32.3437 \text{ lac (approx.)}$$

Example 21.41: Find the value of e^x when $x = 0.644$ using (a) Stirling's formula, (b) Bessel's formula, (c) Everett's formula, from the data given below:

x:	0.61	0.62	0.63	0.64	0.65	0.66	0.67
y:	1.840431	1.658928	1.877610	1.896481	1.915541	1.934792	1.954237

Solution: Taking $x_0 = 0.64, h = 0.01, x = 0.644$, we have $p = \dfrac{x - x_0}{h} = \dfrac{0.644 - 0.64}{0.01} = 0.4.$

The difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$
0.61	-0.3	1.840431				
			0.019497			
0.62	-0.2	1.858928		0.000185		
			0.018682		0.000004	
0.63	-0.1	1.877610		0.000189		-0.000004
			0.018871		0.000000	
0.64	0.0	1.896481		0.000189		0.000002
			0.019060		0.000002	
0.65	0.1	1.915541		0.000191		0.000001
			0.019251		0.000003	
0.66	0.2	1.934792		0.000194		
			0.019445			
0.67	0.3	1.954237				

The Stirling's formula is

$$y_p = y_0 + \frac{p}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots$$

$$= 1.896481 + \frac{0.4}{2!}(0.019060 + 0.018871) + \frac{(0.4)^2}{2!}(0.000189)$$
$$+ \frac{(0.4)\{(0.4)^2 - 1\}}{3!}(0.000002 + 0.000000) + \frac{(0.4)^2\{(0.4)^2 - 1\}}{4!}(0.0000002)$$
$$= 1.896481 + 0.0075862 + 0.00001512 - 0.000000112 - 0.0000000112 = 1.904082$$

The Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-1/2)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots$$

$$= 1.896481 + 0.4\,(0.019060) + \frac{(0.4)(0.4-1)}{2!}\frac{(0.000189 + 0.000191)}{2}$$
$$+ \frac{(0.4-0.5)(0.4)(0.4-1)}{3!}(0.000004)$$

$$= 1.896481 + 0.0076240 - 0.0000228 + 0.000000016 = 1.904082$$

The Everett's formula is

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0$$
$$+ \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots$$

where $q = 1 - p$.

Thus, $y_p = 0.6(1.896481) + (0.6)\dfrac{\{(0.6)^2 - 1^2\}}{3!}(0.000189) + \ldots + 0.4(1.915541)$

$$+ \frac{(0.4)\{(0.4)^2 - 1^2\}}{3!}(0.000191) + \ldots$$

$$= 1.1378886 - 0.000012096 + 0.7662164 - 0.000010696 = 1.904082.$$

Remark: Since the various interpolation formulae are the different forms of the same polynomial, thus the value interpolated should be the same irrespective of the formula used as observed in the above example.

EXERCISE 21.5

1. Compute $\cosh(0.56)$ from the following four values given below using Newton's forward difference formula:

x:	0.5	0.6	0.7	0.8
$\cosh x$:	1.127626	1.185465	1.255169	1.337435

2. Using Newton's interpolation formula, calculate the value of exp (1.85) from the values given below:

x:	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$\exp(x)$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

3. Find the cubic polynomial which takes the following values using Newton's forward difference formula:

$$y(0) = 1, \quad y(1) = 0, \quad y(2) = 1, \text{ and } y(3) = 10$$

Hence, or otherwise find $y(4)$.

4. The population of a town in the decennial census was as given below. Estimate the population for the years 1965 and 1995 using Newton's interpolation formulae:

Year (x):	1961	1971	1981	1991	2001
Population (in lacs) (y):	46	66	81	93	101

5. In the table below the values of y are consecutive terms of a series with interval of differencing being unity. Extrapolate the first and tenth terms of the series

x:	3	4	5	6	7	8	9
y:	2.7	6.4	12.5	21.6	34.3	51.2	72.9

6. Apply Newton's backward difference formula to the data given below to obtain a polynomial of degree 4 in x:

x:	1	2	3	4	5
y:	1	–1	1	–1	1

7. The probability integral $P(z) = \sqrt{\frac{2}{\pi}} \int_0^z e^{-1/2t^2} dt$ has the following values:

z:	1.00	1.05	1.10	1.15	1.20	1.25
$P(z)$:	0.682689	0.706282	0.728668	0.749856	0.769861	0.788700

Calculate P for $z = 1.235$, using Newton's backward interpolation formula.

8. Given the following score distribution of statistics:

Marks:	30-40	40-50	50-60	60-70
No. of students:	52	36	21	14

Find (i) number of students who secured below 35
(ii) number of students who secured above 65
(iii) number of students who secured between 35 – 45

9. Evaluate y at $x = 30$ from the data given below using Gauss forward formula:

x:	21	25	29	33	37
y:	18.4708	17.8144	17.1070	16.3432	15.5154

10. Using Gauss backward formula, find the value of $\sqrt{12516}$ given that

$\sqrt{12500} = 111.803399$, $\sqrt{12510} = 111.848111$ $\sqrt{12520} = 111.892806$, $\sqrt{12530} = 111.937483$.

11. Find the value of e^x for $x = 0.638$ from the following data using, (i) Stirling's, (ii) Bessel's, and (iii) Everett's formulae.

x:	0.61	0.62	0.63	0.64	0.65	0.66	0.67
y:	1.840431	1.658928	1.877610	1.896481	1.915541	1.934792	1.954237

12. Use Stirling formula to find $f(32)$ from the data:

$f(20) = 1435, f(25) = 13.674, f(30) = 13.257, f(35) = 12.734, f(40) = 12.089, f(45) = 11.309.$

13. Evaluate sin (0.197) from the following data:

x :	0.15	0.17	0.19	0.21	0.23
$\sin x$:	0.14944	0.16918	0.18886	0.20846	0.22798

14. Find the value of the elliptic integral $k(m) = \int_0^{\pi/2}(1 - m\sin^2\theta)^{-1/2}d\theta$ for $m = 0.25$ using Bessel's formula from the data given below:

m :	0.20	0.22	0.24	0.26	0.28	0.30
$k(m)$:	1.659624	1.669850	1.680373	1.691208	1.702374	1.713889

15. From the values given below, find log (337.5) using, (i) Gauss, (ii) Stirling's, (iii) Bessel's, and (iv) Everette's formulae.

x :	310	320	330	340	350	360
$\log x$:	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

21.6.3 Interpolation with Unequal Intervals

The various interpolation formulae discussed so far are applicable only to equally spaced values of the argument. In this section, we discuss two interpolation formulae applicable to unequally spaced values of x. These are:

1. *Lagrange's interpolation formula.*
2. *Newton's divided difference formula.*

1. Lagrange's interpolation formula. Let $y = f(x)$ be a function which take the values (x_0, y_0), (x_1, y_1) ..., (x_n, y_n). Since there are $n + 1$ pairs of values of x and y, we can represent the function $f(x)$ by a polynomial in x of degree n of the form

$$y = f(x) = a_0(x - x_1)(x - x_2) \ldots\ldots (x - x_n) + a_1(x - x_0)(x - x_2) \ldots\ldots (x - x_n)$$

$$+ a_2(x - x_0)(x - x_1)(x - x_3) \ldots\ldots (x - x_n) \quad \ldots \quad + a_n(x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1}) \qquad \ldots(21.50)$$

Here the coefficients a_i's are so chosen so as to satisfy the condition $y_i = f(x_i)$, $i = 0, 1, 2 \ldots n$. Substituting $x = x_0$ and $y = y_0$ in (21.50) yields

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

or,

$$a_0 = y_0/[(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)].$$

Similarly, by substituting $x = x_1$ and $y = y_1$ in (21.50) yields

$$a_1 = y_1/[(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)]$$

and, so on

$$a_n = y_n/[(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})]$$

Substituting the values of $a_0, a_1, \ldots, a_n$ in (21.50) we obtain

$$f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}y_1$$

$$+\ldots\ldots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\,y_n, \qquad \ldots(21.51)$$

the *Lagrange's interpolation formula*.

Remarks:

1. Lagrange's formula can be applied whether the values x_i are equally spaced or not.
2. The formula (21.51) can be rewritten as

$$\frac{f(x)}{(x-x_0)(x-x_1)\ldots(x-x_n)}=\frac{y_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\cdot\frac{1}{(x-x_0)}$$

$$+\frac{y_1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\frac{1}{(x-x_1)}+\ldots\ldots$$

$$+\frac{y_n}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\cdot\frac{1}{(x-x_n)}$$

Example 21.42: Find the Lagrange interpolation polynomial to fit the following data:

x_i:	0	1	2	3
$e^{x_i}-1$:	0	1.7183	6.3891	19.0855

Hence, find the value of $e^{1.5}$.

Solution: The Lagrange polynomial is given by

$$f(x)=\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}(0)+\frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}(1.7183)$$

$$+\frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)}(6.3891)+\frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}(19.0855)$$

$$=0+0.8591\,(x^3-5x^2+6x)-3.1945\,(x^3-4x^2+3x)+3.1809\,(x^3-3x^2+2x)$$

$$=0.8455\,x^3-1.0604\,x^2+1.9331\,x$$

For $x_i=1.5$, $f(1.5)=3.3677$. Thus, $e^{1.5}-1=3.3677$, which gives $e^{1.5}=4.3677$.

Example 21.43: Using Lagrange's interpolation formula, find the form of the function $y(x)$ from the following data:

x:	0	1	3	4
y:	–12	0	12	24

Solution: Since $y=0$ when $x=1$, thus $(x-1)$ is the factor of the desired polynomial. Let $y(x)=(x-1)\,r(x)$. Thus, $r(x)=y/(x-1)$, $x\neq 1$. We find $r(x)$ from the following tabulated values:

x:	0	3	4
$r(x)$:	12	6	8

Applying Lagrange's interpolation formula

$$r(x) = \frac{(x-3)(x-4)}{(0-3)(0-4)}(12) + \frac{x(x-4)}{3(3-4)}(6) + \frac{x(x-3)}{4(4-3)}(8)$$

$$= (x-3)(x-4) - 2x(x-4) + 2x(x-3) = x^2 - 5x + 12.$$

Thus, the desired polynomial is $y(x) = (x-1)(x^2 - 5x + 12)$.

2. *Newton's divided difference formula.* The Lagrange's formula has the shortcoming that if we want to insert an additional data point then the interpolation coefficients are required to be recalculated, and also, Lagrange's formula involves large number of arithmetic operations.

These shortcomings are overcomed in *Newton's general interpolation formula* which employs what are called *divided differences*. Before deriving this formula, we first define divided differences.

Divided differences. If $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ ... are the given points then the *first divided difference for the arguments* x_0, x_1 is defined by the relation

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}. \text{ Similarly, } [x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1} \text{ and } [x_2, x_3] = \frac{y_3 - y_2}{x_3 - x_2} \text{ etc.}$$

The *second divided difference for* x_0, x_1, x_2 is defined as $[x_0, x_1, x_2] = \dfrac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$

Similarly, the *third divided difference for* x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}, \text{ and so on.}$$

Next, we give two important results concerning the divided differences.

(i) *The divided differences are symmetrical in their arguments, that is, independent of the order of the arguments.* Since

$$[x_0, x_1] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0} = [x_1, x_0]$$

$$[x_0, x_1, x_2] = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$$= [x_1, x_2, x_0] \text{ or } [x_2, x_0, x_1],$$

and similarly for the higher order differences.

(ii) *The nth divided differences of a polynomial of the nth degree are constant.*

Let the arguments be equally spaced, so that, $x_1 - x_0 = x_2 - x_1 = ... = x_n - x_{n-1} = h$.

Then $[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{\Delta y_0}{h}$,

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\} = \frac{1}{2!h^2}\Delta^2 y_0$$

and, in general, $[x_0, x_1, x_2 ..., x_n] = \frac{1}{n!h^n}\Delta^n y_0$ ...(21.52)

If the tabulated function is an nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence, the nth divided differences are constants.

Now we are in a position to study the *Newton's divided differences formula.*

Let $y_0, y_1, ..., y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, ..., x_n$. Then from the definition of first divided difference,

$$[x, x_0] = \frac{y - y_0}{x - x_0}, \text{ we obtain}$$

$$y = y_0 + (x - x_0)[x, x_0] \quad ...(21.53)$$

Again from $[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$, we obtain $[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$

Substituting this value of $[x, x_0]$ in (21.53) yields

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x, - x_1)[x, x_0, x_1] \quad ...(21.54)$$

Next, $[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$, gives $[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$

Substituting this value of $[x, x_0, x_1]$ in (21.54) yields

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2].$$

Proceeding in this way, we obtain

$$\begin{aligned} y = f(x) = y_0 &+ (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + ... \\ &+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) ..... (x - x_n)[x, x_0, x_1, ..., x_n], \end{aligned} \quad ...(21.55)$$

the *Newton's divided differences formula.*

Obviously, when the values of x are equally spaced, this reduces to Newton's forward interpolation formula (21.31).

Example 21.44: Using Newton's divided difference formula, find $f(x)$ as a polynomial in x for the following data:

x:	–1	0	3	6	7
$f(x)$:	3	–6	39	822	1611

Hence, determine the value of $f(x)$ when $x = 1.5$.

Solution: The divided difference table is given on next page.

Thus, the desired polynomial is

$$\begin{aligned} f(x) &= 3 + (x + 1)(-9) + (x + 1)(x - 0)(6) + (x + 1)(x - 0)(x - 3)5 + (x + 1)(x - 0)(x - 3)(x - 6)(1) \\ &= 3 - (9x + 9) + 6(x^2 + x) + 5(x^3 - 2x^2 - 3x) + (x^4 - 8x^3 + 9x^2 + 18x) = x^4 - 3x^3 + 5x^2 - 6. \end{aligned}$$

When $x = 1.5$, $f(1.5) = 5.0625 - 10.125 + 11.25 - 6 = 0.1875$.

Table for Example 21.44.

x	$f(x)$	Ist divided differences	2nd divided differences	3rd divided differences	4th divided differences
–1	3				
		–9			
0	–6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

Example 21.45: The function $y = \sin x$ is tabulated below:

x:	0	$\pi/4$	$\pi/2$
$\sin x$:	0	0.70711	1.0

Find $\sin(\pi/6)$ using, (i) Lagrange's formula, (ii) Newton's divided difference formula.

Solution: (i) Value of $\sin(\pi/6)$ using Lagrange's interpolation is

$$\sin(\pi/6) = \frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}{\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}(0.70711) + \frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{4}\right)}{\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}(1) = \frac{8}{9}(0.70711) - \frac{1}{9} = 0.51743.$$

(ii) The divided differences table is

x	y	1st divided differences	2nd divided differences
0	0		
		0.90032	
$\pi/4$	0.7011		–0.33575
		0.37292	
$\pi/2$	1.0		

Using Newton's divided differences formula

$$\sin(\pi/6) = (\pi/6 - 0)(0.90032) + (\pi/6 - 0)(\pi/6 - \pi/4)(-0.33575) = 0.47141 + 0.04602 = 0.51743.$$

21.6.4 Inverse Interpolation

So far we have been interpolating the value of y corresponding to a specific value of x. The reverse process of estimating the value of x for a specific value of y is called the *inverse interpolation*. Lagrange's interpolation formula, merely by interchanging x and y, can be readily employed for inverse interpolation. Interchanging x and y in (21.51), we obtain the polynomial

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}x_1$$

$$+\ldots+\frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})}x_n \qquad \ldots(21.56)$$

to be used for estimating x for a specific y. However, it should be noted that inverse interpolation is in general meaningful only if the function is single-valued in the interval under consideration.

Example 21.46: Find the value of x when $y = 0.3$ applying Lagrange's inverse interpolation formula to the data:

x:	0.4	0.6	0.8
y:	0.3683	0.3332	0.2897

Solution: The Lagrange's inverse interpolation formula is

$$x=\frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}x_0+\frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}x_1+\frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}x_2$$

Substituting for the various values, we obtain

$$x=\frac{(0.3-0.3332)(0.3-0.2897)}{(0.3683-0.3332)(0.3683-0.2897)}(0.4)+\frac{(0.3-0.3683)(0.3-0.2897)}{(0.3332-0.3683)(0.3332-0.2897)}(0.6)$$

$$+\frac{(0.3-0.3683)(0.3-0.3332)}{(0.2897-0.3683)(0.2897-0.3332)}(0.8)=0.7574.$$

Example 21.47: Using Lagrange's polynomial, express $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ as sum of partial fractions.

Solution: Let $f(x) = 3x^2 + x + 1$. Tabulating the value of $f(x)$ for $x = 1, 2$ and 3, we obtain

x:	1	2	3
$f(x)$:	5	15	31.

Using Lagrange's formula, we obtain

$$f(x)=\frac{(x-2)(x-3)}{(1-2)(1-3)}(5)+\frac{(x-1)(x-3)}{(2-1)(2-3)}(15)+\frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$=\frac{5}{2}(x-2)(x-3)-15(x-1)(x-3)+\frac{31}{2}(x-1)(x-2)$$

Dividing both sides by $(x-1)(x-2)(x-3)$, we obtain

$$\frac{3x^2+x+1}{(x-1)(x-2)(x-2)}=\frac{5}{2(x-1)}-\frac{15}{x-2}+\frac{31}{2(x-3)}.$$

EXERCISE 21.6

1. From the following data giving the values of x and $\log_{10} x$, find $\log_{10} 301$ using Lagrange's polynomial:

x:	300	304	305	307
$\log_{10} x$:	2.4771	2.4829	2.4843	2.4871

2. Using Lagrange's interpolation formula, find a polynomial which passes through the points $(0, -12)$, $(1, 0)$ $(3, 6)$ and $(4, 12)$.

3. Using Lagrange's interpolation formula, find the value of y corresponding to $x = 10$ from the following data:

x:	5	6	9	11
$f(x)$:	12	13	14	16

4. Determine $f(x)$ as a polynomial in x for the following data using Newton's divided difference method:

x:	−4	−1	0	2	5
$f(x)$:	1245	33	5	9	1335

5. For the data

x:	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

evaluate $f(9)$ using, (i) Lagrange's method, (ii) Newton's divided difference method.

6. The following table gives the viscosity of an oil as a function of temperature. Using Newton's general formula find viscosity of oil at a temperature of 140°.

Temp. (in °C):	110	130	160	190
Viscosity (poise):	10.8	8.1	5.5	4.8

7. Using Newton's divided difference formula, evaluate $f(8)$ and $f(15)$ for the data:

x:	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

8. Using Lagrange's polynomial express $\dfrac{x^2+6x-1}{(x^2-1)(x-4)(x-6)}$ as a sum of partial fractions.

9. For the following data:

x:	1	2	7	8
$f(x)$:	4	5	5	4

find $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum.

10. The following table gives the values of the probability integral $P(x) = \dfrac{2}{\sqrt{2\pi}}\int_0^x e^{-t^2}\,dt$ for certain specific values of x. Find x for which $P(x) = 1/2$.

x:	0.46	0.47	0.48	0.49
$P(x)$:	0.484655	0.493745	0.502749	0.511668

21.7 NUMERICAL DIFFERENTIATION

Numerical differentiation is the process of finding the derivative of a function at some assigned values of the independent variable when we are given a set of values of that function. The problem of numerical differentiation consists of first approximating the function by an interpolating polynomial and then differentiating it. In case the values of the independent variable are equally spaced and we need to find the derivative of the function at a point near the start of the tabular values then we use Newton's forward formula. Similarly, Newton's backward formula is used if derivative is to be found near the end of the table. For finding the value of derivative near the middle of the table, we apply central differences formulae. When the values of the independent variable are not equally spaced then normally Newton's divided differences formula is applied.

21.7.1 Derivatives Using Newton's Forward Formula

Consider the function $y = f(x)$ which is tabulated for the values $x_i = x_0 + ih$, $i = 0, 1, 2, ..., n$. The Newton's forward difference formula is

$$y(p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ... \qquad ...(21.57)$$

where $p = (x - x_0)/h$.

Differentiating (21.57) w.r.t. x, we obtain

$$\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = \frac{1}{h}\frac{d}{dp}\left[y_0 + p\Delta y_0 + \frac{p^2 - p}{2!}\Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!}\Delta^3 y_0 + ...\right]$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!}\Delta^3 y_0 + ...\right] \qquad ...(21.58)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h}\frac{d}{dp}\left(\frac{dy}{dx}\right) = \frac{1}{h}\frac{d}{dp}\left[\frac{1}{h}\left(\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!}\Delta^3 y_0 + ...\right)\right]$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2 - 36p + 22}{4!}\Delta^4 y_0 + ...\right] \qquad ...(21.59)$$

The formulae (21.58) and (21.59) are used for computing the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for non-tabulated values of x. For tabular values of x, these formulae take simple forms, since by setting $x = x_0$, we obtain $p = 0$ and hence expressions (21.58) and (21.59) give respectively

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ...\right] \qquad ...(21.60)$$

and,

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + ...\right] \qquad ...(21.61)$$

Alternatively, we have $1 + \Delta = E = e^{hD}$, which implies that

$$D = \frac{1}{h}\log(1+\Delta) = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]$$

and,
$$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \ldots\right]$$

Applying these identities to y_0, we obtain (21.60) and (21.61).

21.7.2 Derivatives Using Newton's Backward Formula

Similarly, using Newton's backward difference formula, we obtain

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \ldots\right] \qquad \ldots(21.62)$$

and,
$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{12p^2+36p+22}{4!}\nabla^4 y_n + \ldots\right] \qquad \ldots(21.63)$$

for non-tabular values of x, where $x = x_n + ph$. For tabular values, we obtain

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \ldots\right] \qquad \ldots(21.64)$$

and,
$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \ldots\right] \qquad \ldots(21.65)$$

Alternatively, we have $1 - \nabla = E^{-1} = e^{-hD}$ which implies that

$$D = -\frac{1}{h}\log(1-\nabla) = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right]$$

and,
$$D^2 = \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right]^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots\right]$$

Applying these identities to y_n, we obtain (21.64) and (21.65).

21.7.3 Derivatives Using Stirling's Formula

The Stirling formula (21.43) is

$$y = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots, \qquad \ldots(21.66)$$

where $x = x_0 + ph$.

Differentiating (21.66) w.r.t. x, we obtain

$$\frac{dy}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4p^3-2p}{4!}\Delta^4 y_{-2} + \ldots\right] \qquad \ldots(21.67)$$

At $x = x_0$, $p = 0$, thus

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots\right] \quad \ldots(21.68)$$

Similarly, $$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \ldots\right] \quad \ldots(21.69)$$

Remark: Since the tabulated values define the function at the mentioned points only and does not completely define the function and hence the function may not be differentiable at all. Thus, numerical differentiation should be used only if the tabulated values are such that the differences of a specific order are constants, otherwise, errors are bound to creep in which may go on increasing as derivatives of higher orders are found. This is due to the fact that difference between the actual function $f(x)$ and the interpolating polynomial $\phi(x)$ may be small at the data points but $f'(x) - \phi'(x)$ may be large. Thus, numerical differentiation should be used with cautions.

Example 21.48: Given that

x:	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y:	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at, (i) $x = 1.2$, (ii) $x = 1.6$ and, (iii) $x = 2.2$.

Solution: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0194			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

(i) Here $x_0 = 1.2$, $y_0 = 3.3201$, $h = 0.2$. Using (21.60) and (21.61), we obtain respectively

$$\left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014)\right] = 3.3205$$

and, $\left[\dfrac{d^2y}{dx^2}\right]_{x=1.2} = \dfrac{1}{0.04}\left[0.1627 - 0.0361 + \dfrac{11}{12}(0.0080) - \dfrac{5}{6}(0.0014)\right] = 3.318$

(ii) Here $x_0 = 1.6$, $y_0 = 4.9530$ and $h = 0.2$. Using (21.68) and (21.69), we obtain respectively

$$\left[\frac{dy}{dx}\right]_{x=1.6} = \frac{1}{0.2}\left[\frac{0.8978 + 1.0966}{2} - \frac{1}{2}\left(\frac{0.0361 + 0.0441}{2}\right) + \frac{1}{30}\left(\frac{0.0013 + 0.0014}{2}\right)\right] = 4.9530$$

and, $\left[\dfrac{d^2y}{dx^2}\right]_{x=1.6} = \dfrac{1}{0.04}\left[0.1988 - \dfrac{1}{12}(0.0080) + \dfrac{1}{90}(0.0001)\right] = 4.9525$

(iii) Here $x_n = 2.2, y_n = 9.0250$ and $h = 0.2$. Using (21.64) and (21.65), we obtain respectively

$$\left[\frac{dy}{dx}\right]_{x=2.2} = \frac{1}{0.2}\left[1.6359 + \frac{1}{2}(0.2964) + \frac{1}{3}(0.0535) + \frac{1}{4}(0.0094) + \frac{1}{5}(0.0014) + \frac{1}{5}(0.0014)\right]$$

$$= 9.0228$$

and, $\left[\dfrac{d^2y}{dx^2}\right]_{x=2.2} = \dfrac{1}{0.04}\left[0.2964 + 0.0535 + \dfrac{11}{12}(0.0094) + \dfrac{5}{6}(0.0014)\right] = 8.992$

Example 21.49: A slider in a machine moves along a fixed straight rod. Its distance x cm along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when $t = 0.3$ second.

t:	0	0.1	0.2	0.3	0.4	0.5	0.6
x:	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Solution: The difference table is

t	x	Δx	$\Delta^2 x$	$\Delta^3 x$	$\Delta^4 x$	$\Delta^5 x$	$\Delta^6 x$
0	30.13						
		1.49					
0.1	31.62		–0.24				
		1.25		–0.24			
0.2	32.87		–0.48		0.26		
		0.77		0.02		–0.27	
0.3	33.64		–0.46		–0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		–0.45		0.01		
		–0.14		0.002			
0.5	33.81		–0.43				
		–0.57					
0.6	33.24						

Since the derivatives are required near the middle of the table, we use Stirling's formulae (21.68) and (21.69) with $h = 0.1$ and $t_0 = 0.3$, to obtain respectively

$$\left[\frac{dx}{dt}\right]_{t=0.3} = \frac{1}{0.1}\left[\frac{0.31 + 0.77}{2} - \frac{1}{6}\left(\frac{0.01 + 0.02}{2}\right) + \frac{1}{30}\left(\frac{0.02 - 0.27}{2}\right) - \ldots\right] = 5.33$$

and, $$\left[\frac{d^2x}{dt^2}\right]_{t=0.3} = \frac{1}{0.01}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29) - \ldots\right] = -45.6.$$

Hence, the required velocity is 5.33 cm/sec and acceleration is -45.6 cm/sec^2.

21.7.4 Maximum and Minimum Values of a Tabulated Function

Newton's forward difference formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \ldots \qquad \ldots(21.70)$$

Differentiating with respect to p, we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-3p+2}{6}\Delta^3 y_0 + \ldots$$

We know that for maxima or minima $\frac{dy}{dp} = 0$. Hence, for retaining the terms up to third order differences and equating right-hand side to zero, we obtain

$$a_0 + a_1 p + a_2 p^2 = 0, \qquad \ldots(21.71)$$

where $a_0 = \Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0$, $a_1 = \Delta^2 y_0 - \Delta^3 y_0$, and $a_2 = \frac{1}{2}\Delta^3 y_0$.

Evaluating the values of a_0, a_1, a_2 from the difference table, Eq. (21.71) being quadratic in p can be solved for p and the corresponding values of x are given by $x = x_0 + ph$, at which y is maximum or minimum.

Example 21.50: From the following data, find maximum value of y correct to two decimal places

x:	1.2	1.3	1.4	1.5	1.6
y:	0.9320	0.9636	0.9855	0.9975	0.9996

Solution: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.2	0.9320				
		0.0316			
1.3	0.9636		−0.0097		
		0.0219		−0.0002	
1.4	0.9855		−0.0099		0.0002
		0.0120		0.0000	
1.5	0.9975		−0.0099		
		0.0021			
1.6	0.9996				

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Ignoring $\Delta^3 y_0 = -0.002$ and substituting for y_0, Δy_0 and $\Delta^2 y_0$, we obtain

$$y = 0.9320 + 0.0316p - \frac{0.0097}{2}(p^2 - p)$$

$$\frac{dy}{dp} = 0, \text{ gives } 0.0316 - \frac{0.0097}{2}(2p - 1) = 0 \text{ or, } p = 3.80$$

Thus, $x = x_0 + ph = 1.2 + 3.80\,(.1) = 1.58.$

Also $\dfrac{d^2y}{dp^2} = -0.0097 < 1$, hence maxima exists at $x = 1.58$

For $x = 1.58$, using Newton's backward formula at $x_n = 1.6$, we obtain

$$y(1.58) = 0.9996 - 0.2(0.0021) + \frac{(-0.2)(-0.2+1)}{2}(-0.0099) - \ldots$$

$$= 1.0, \text{ as the maximum value.}$$

EXERCISE 21.7

1. Find $\dfrac{d}{dx}(J_0)$ at $x = 0.1$ from the following data:

x:	0.0	0.1	0.2	0.3	0.4
$J_0(x)$:	1.0000	0.9975	0.9900	0.9776	0.9604

2. The following table gives the angular displacement θ at different interval of time t.

θ (in radians):	0.052	0.105	0.168	0.242	0.327	0.408	0.489
t (in seconds):	0.00	0.02	0.04	0.06	0.08	0.10	0.12

3. Compute $f''(0)$ and $f'(0.2)$ from the following tabular data:

x:	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$:	1.00	1.16	3.56	13.96	41.96	101.00

4. Find $y'(2.2)$ and $y''(2.2)$ from the following data:

x:	1.4	1.6	1.8	2.0	2.2
$y(x)$:	4.0552	4.9530	6.0496	7.3891	9.0250

5. Find $y'(x)$ at $x = 0.6$ from the following data:

x:	0.4	0.5	0.6	0.7	0.8
$y(x)$:	1.5836494	1.7974426	2.0442376	2.3275054	2.6510818

6. The elevations above a datum line of seven points of a road 300 units apart are 135, 149, 157, 183, 201, 205 and 193 units. Find the gradient of the road at the middle point.

7. Find the force of mortality $u_x = -\frac{1}{l_x}\frac{dl_x}{dx}$ at $x = 50$ using the table below.

x:	50	51	52	53
l_x:	73499	72724	71753	70599

8. Using Bessel's formula find $f'(x)$ at $x = 0.04$ from the following data:

x:	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$:	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

9. From the table below find the value of x for which y is minimum, also find this value of y.

x:	3	4	5	6	7	8
y:	0.205	0.240	0.259	0.262	0.250	0.224

10. Find the maximum and minimum value of the function $y = f(x)$ from the following data:

x:	0	1	2	3	4	5
$f(x)$:	0	0.25	0	2.25	16.00	56.25

21.8 NUMERICAL INTEGRATION

Similar to numerical differentiation, the process of evaluating a definite integral from a set of tabulated values of the integrand $y = f(x)$ is called *numerical integration*. The process is applied when a definite integral cannot be solved analytically or when $f(x)$ is not known explicitly but only a set of observations is available. The process when applied to a function of a single variable is known as *quadrature*.

Let (x_i, y_i), $i = 0, 1, 2, \ldots, n$ be the set of data points of a function $y = f(x)$, (generally not known). It is required to compute the value of the definite integral

$$I = \int_a^b f(x)dx \qquad \ldots(21.72)$$

where $a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b$.

To compute (21.72), we replace $y = f(x)$ by a suitable interpolating polynomial. We derive a general quadrature formula by using Newton's forward difference formula.

21.8.1 Newton-Cote's Quadrature Formula

To compute the integral (21.72), the interval $[a, b]$ is divided into n subintervals of width h so that $a = x_0$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh = b$. Then

$$I = \int_{x_0}^{x_0+nh} f(x)dx = h\int_0^n f(x_0 + ph)dp \qquad [\text{using } x = x_0 + ph,\ dx = h\,dp]$$

$$= h\int_0^n\left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\right]dp$$

[using Newton's forward formula]

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \ldots\right] \qquad \ldots(21.73)$$

(integrating term by term and simplifying)

This is known as *Newton's-Cotes quadrature formula.*

For $n = 1, 2, 3, \ldots$ we obtain different quadrature formulae.

1. *Trapezoidal rule.* Substituting $n = 1$ in (21.73) and approximating the curve $y = f(x)$ through the points $A_0(x_0, y_0)$ and $A_1(x_1, y_1)$ by a straight line (that is a polynomial of degree one) so that second and higher order differences become zeros, we get

$$\int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_0+h} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = \frac{h}{2}[y_0 + y_1]$$

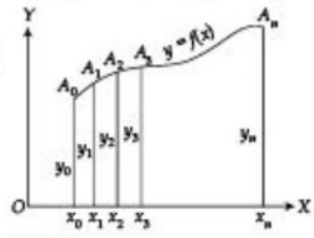


Fig. 21.5

Similarly, $\int_{x_1}^{x_2} f(x)dx = \frac{h}{2}[y_1 + y_2], \ldots, \int_{x_{n-1}}^{x_n} f(x)dx = \frac{h}{2}[y_{n-1} + y_n]$.

Adding these n integrals, we obtain

$$I = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n] \qquad \ldots(21.74)$$

This is known as *trapezoidal rule.*

Geometrically, the curve $y = f(x)$ is replaced by n straight lines joining the points (x_i, y_i), $i = 0, 1, 2, \ldots, n$, and the area of each trapezium $= \frac{h}{2}[y_{i-1} + y_i]$ is found separately and then the area under the curve and the ordinates at x_0 and $x_0 + nh$ is approximately equal to the sum of the areas of the n trapeziums as shown in Fig. 21.5.

2. *Simpson's 1/3 rule.* Substituting $n = 2$ in (21.73) and taking the curve through the points (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola (that is, a polynomial of degree two) so that third and higher order differences become zeros, we get

$$\int_{x_0}^{x_2} f(x)dx = \int_{x_0}^{x_0+2h} f(x)dx = 2h[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0] = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly, $\int_{x_2}^{x_4} f(x)\,dx = \int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$

$$\int_{x_{n-2}}^{x_n} f(x)dx = \int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n];\ \text{taking } n \text{ to be even.}$$

Adding all these integrals, for an even n, we obtain

$$I = \int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})] \qquad \ldots(21.75)$$

This is known as *Simpson's 1/3 rule*, or *Simpson's rule*. While applying this rule the given interval $[a, b]$ must be divided into even number of equal subintervals, since we find the area of two strips at a time. This rule is most commonly employed by civil engineers for estimating the amount of earth that must be moved to fill a depression or make a dam. The rule is also frequently used in some other scientific calculations as we shall see in the examples to follow.

3. *Simpson's 3/8 rule.* Substituting $n = 3$ in (21.73) and taking the curve through the points (x_i, y_i), $i = 0, 1, 2, 3$ as a polynomial of degree three so that the fourth and higher order differences become zeros, we get

$$\int_{x_0}^{x_3} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right] = \frac{3}{8}h(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly, $$\int_{x_3}^{x_6} f(x)\,dx = \int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3}{8}h(y_3 + 3y_4 + 3y_5 + y_6)$$

$\vdots$

$$\int_{x_{n-3}}^{x_n} f(x)\,dx = \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3}{8}h(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n);\ \text{taking } n \text{ as a multiple of 3.}$$

Adding all these integrals for an n multiple of 3, we have

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \ldots\ y_{n-3}\right] \quad \ldots(21.76)$$

This is known as *Simpson's 3/8 rule*. While applying this rule the given interval $[a, b]$ must be divided into number of subintervals equal to a multiple of 3, since we find the area of three strips at a time.

4. *Boole's rule.* Substituting $n = 4$ in (21.73) and proceeding on the same lines as above,

$$I = \int_{x_0}^{x_n} f(x)\,dx = \frac{2h}{45}\left[7(y_0 + y_n) + 32(y_1 + y_3 + y_5 + y_7 + \ldots) + 12(y_2 + y_6 + y_{10} + \ldots) + 14(y_4 + y_8 + y_{12} + \ldots)\right] \quad \ldots(21.77)$$

This is known as *Boole's rule*. Here, the number of subintervals should be multiple of 4.

5. *Weddle's rule.* Substituting $n = 6$ in (21.73) and proceeding as earlier, we obtain

$$I = \int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{10}\left[(y_0 + y_n) + 5(y_1 + y_5 + y_7 + y_{11} + \ldots) + (y_2 + y_4 + y_8 + y_{10} + \ldots)\right.$$
$$\left. + 6(y_3 + y_9 + y_{15} + \ldots) + 2(y_6 + y_{12} + y_{18} + \ldots)\right] \quad \ldots(21.78)$$

This is known as *Weddle's rule*. Here the number of subintervals should be multiple of 6.

21.8.2 Error Estimates in Integral Formulae

Regarding the error estimates in various integral formulae discussed above, we state without proof the following results:

1. *If y'' is continuous and M is any upper bound for the values of $|y''|$ over the interval $[a, b]$, then the error E_t in trapezoidal rule is such that*

$$|E_t| \le \frac{b-a}{180}h^2 M \quad \ldots(21.79)$$

2. *If $y^{(iv)}$ is continuous and M is an upper bound for the values of $|y^{(iv)}|$ over the interval [a, b], then the error E_s in Simpson's 1/3 rule is such that*

$$|E_s| \le \frac{b-a}{180}h^4 M \qquad ...(21.80)$$

3. *If $y^{(iv)}$ is continuous and M is any upper bound for the values of $|y^{(iv)}|$ over the interval [a, b], then the error E_s' in Simpson's 3/8 rule is such that*

$$|E_s'| \le \frac{(b-a)}{80}h^4 M \qquad ...(21.81)$$

Example 21.51: Find the approximate value of $I = \int_0^{\pi} \sin x dx$ using, (i) trapezoidal rule, (ii) Simpson's 1/3 rule by dividing the range of integration into six equal parts in both the cases.

Solution: Here $n = 6, a = 0, b = \pi$, thus, $h = \frac{b-a}{n} = \frac{\pi}{6}$. Hence, the function can be tabulated as

x:	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$y = \sin x$	0.0	0.5	0.8660	1.0	0.8660	0.5	0.0

Applying trapezoidal rule, we have

$$I = \int_0^{\pi} \sin x dx = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{12}[0 + 0 + 2(3.732)] = \frac{3.1415}{6}(3.732) = 1.9540$$

Applying Simpson's 1/3 rule, we have

$$\int_0^{\pi} \sin x\, dx = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{18}[0 + 0 + 4\,(2) + 2(1.732)] = \frac{3.1415}{18}(11.464) = 2.0008.$$

The exact value of the integral is $I = \int_0^{\pi} \sin x dx = -(\cos x)_0^{\pi} = -(\cos \pi - \cos 0) = 2.$

This example illustrates that, in general, Simpson's rule yields more accurate results than the trapezoidal rule.

Example 21.52: Evaluate the integral $I = \int_0^1 \frac{dx}{1+x^2}$ using, (i) trapezoidal rule, (ii) Simpson's 1/3 rule, taking $h = 1/4$. Further, compute the approximate value of π from Simpson's 1/3 rule.

Solution: Here $a = 0, b = 1, h = 1/4$. Thus, $n = \frac{b-a}{h} = 4$.

Hence, the function can be tabulated as

x:	0	1/4	1/2	3/4	1
$y = \frac{1}{1+x^2}$:	1	0.9412	0.8000	0.6400	0.5000

Using trapezoidal rule, $I = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)] = \frac{1}{8}[1.5 + 2(2.312)] = 0.7828$

Using Simpson's 1/3 rule, $I = \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2] = \frac{1}{12}[1.5 + 4(1.512) + 1.6] = 0.7854$

Also analytically $I = \int_0^1 \frac{dx}{1+x^2} + [\tan^{-1} x]_0^1 = \frac{\pi}{4}$

To compute π from Simpson's rule, we have, $\pi/4 = 0.7854$, which gives, $\pi = 0.31416$.

Example 21.53: Find the value of $\int_0^{0.6} e^x dx$ using Simpson's 1/3 and 3/8 rules by dividing the interval [0, 0.6] in suitable number of subintervals.

Solution: Since we are to evaluate the interval using both Simpson's 1/3 and 3/8 rules so the number of subintervals must be multiple of 2 as well as 3. Take $n = 6$, thus, $h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$, and hence the function can be tabulated as follows:

x:	0.0	0.1	0.2	0.3	0.4	0.5	0.6
$y = e^x$:	1.0000	1.10517	1.22140	1.34986	1.49182	1.64872	1.82212

Using Simpson's 1/3 rule, we have

$$I = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{0.1}{3}[(1.0000 + 1.82212) + 4(1.10517 + 1.34986 + 1.64872) + 2(1.22140 + 1.49182)] = 0.82212.$$

Using Simpson's 3/8 rule, we have

$$I = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{0.3}{8}[(1.0000 + 1.82212) + 3(1.10517 + 1.22140 + 1.49182 + 1.64872) + 2(1.34986)] = 0.82212$$

Example 21.54: A solid of revolution is formed by rotating about x-axis the area between the x-axis, the lines $x = 0$ and $x = 1$ and a curve through the points with the following coordinates:

x:	0.00	0.25	0.50	0.75	1.00
y:	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed.

Solution: If V is the volume of the solid formed, then $V = \pi \int_0^1 y^2 dx$.

The values of y^2 are tabulated as follows:

x:	0.00	0.25	0.50	0.75	1.00
y^2:	1.0000	0.9793	0.9195	0.8261	0.7081

Here $h = 0.25$, thus Simpson's 1/3 rule gives

$$V = \frac{\pi(0.25)}{3}[(1.0000 + 0.7081) + 4(0.9793 + 0.8261) + 2(0.9195)] = 2.8192.$$

Example 21.55: The velocity of a train which starts from rest is given by the following table:

t (minutes):	2	4	6	8	10	12	14	16	18	20
v (km/hr):	16.0	28.8	40.0	46.4	51.2	32.0	17.6	8.0	3.2	0.0

Estimate approximately the total distance travelled in twenty minutes using Simpson's 1/3 rule.

Solution: We have, $v = \frac{ds}{dt} \Rightarrow s = \int_0^{20} v dt$.

Since the train starts from rest, thus, $v = 0$ at $t = 0$, thus, the tabulated values are:

t:	0	2	4	6	8	10	12	14	16	18	20
v:	0	16.0	28.8	40.0	46.4	51.2	32.0	17.6	8.0	3.2	0.0

Here, $h = \frac{2}{60} = \frac{1}{30}$ hrs. Using Simpson's 1/3 rule, we have

$$s = \int_0^{20} v dt = \frac{h}{3}[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{1}{90}[(0 + 0) + 4(16 + 40 + 51.2 + 17.6 + 3.2) + 2(28.8 + 46.4 + 32.0 + 8.0)] = 8.25 \text{ km}$$

Example 21.56: Estimate the integral $\int_0^{\pi} x \sin x dx$ using (i) trapezoidal rule, (ii) Simpson's 1/3 rule, (iii) Simpson's 3/8 rule, by dividing the interval $[0, \pi]$ in six subintervals. Also find an upper bound for the error incurred in each case and confirm the results obtained by evaluating the integral analytically also.

Solution: We have, $h = \frac{\pi - 0}{6} = \pi/6$. The tabulated values of $y = x \sin x$ are

x:	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
y:	0	0.2618	0.9069	1.5708	1.8138	1.3090	0

(i) Using trapezoidal rule

$$I = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{12}[(0+0) + 2(0.2618 + 0.9069 + 1.5708 + 1.8138 + 1.3090)] = 3.0695$$

Next, if E_t is the error incurred in estimating the integral, then from (21.79)

$$|E_t| \le \frac{\pi}{12}\left(\frac{\pi}{6}\right)^2 M = \frac{\pi^3}{432} M, \text{ where } M \text{ is an upper bound to } y''(x) \text{ over the interval } [0, \pi].$$

Now, $\quad y''(x) = 2\cos x - x\sin x$. Thus, over $[0, \pi]$

$$|y''(x)| = |2\cos x - x\sin x| \le 2|\cos x| + |x|\,|\sin x| \le 2 + \pi, \text{ for } x \in [0, \pi]$$

Taking $M = 2 + \pi$, we have $|E_T| \le \dfrac{\pi^3}{432}(2 + \pi) = 0.3690$

(ii) Using Simpson's 1/3 rule

$$I = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{18}[(0+0) + 4(0.2618 + 1.5708 + 1.3090) + 2(0.9069 + 1.8138)] = 3.1429$$

Also if E_s is the error incurred in estimating the integral, then from (21.80)

$$|E_s| \le \left(\frac{\pi}{180}\right)\left(\frac{\pi}{6}\right)^4 M, \text{ where } M \text{ is an upper bound to } y^{(iv)}(x) \text{ over the interval } [0, \pi].$$

Now, $y^{(iv)}(x) = -4\cos x + x\sin x$. Thus, over $[0, \pi]$, $|y^{(iv)}(x)| \le 4|\cos x| + |x|\,|\sin x| \le 4 + \pi$.

Taking $M = 4 + \pi$, we have, $|E_s| \le \dfrac{\pi^5}{180 \times (6)^4}(4 + \pi) = 0.00937.$

(iii) Using Simpson's 3/8 rule

$$I = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\pi}{48}[(0+0) + 3(0.2618 + 0.9069 + 1.8138 + 1.3090) + 2(1.5708)] = 3.1447$$

Also if E_s' is the error incurred in estimating the integral, then from (21.81)

$$|E_s'| \le \frac{\pi}{80}\left(\frac{\pi}{6}\right)^4 M, \text{ where } M \text{ is same as in (ii).}$$

Thus, $$|E_s'| \le \frac{\pi^5}{80\times(6)^4}(4+\pi) = 0.02108.$$

Further evaluating $I = \int_0^{\pi} x \sin x \, dx$ analytically, we obtain $I = 3.1416$. This confirms the results obtained above in (i), (ii), and (iii).

Example 21.57: From the following data, estimate the value of $\int_1^5 \log x \, dx$ using Simpson's 1/3 rule. Also obtain the value of h, so that the value approximated will be accurate up to four decimal places.

x:	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$y = \log x$:	0.0000	0.4055	0.6931	0.9163	1.0986	1.2528	1.3863	1.5041	1.6094

Solution: Here $h = 0.5$ and $n = 8$. Using Simpson's 1/3 rule,

$$\int_1^5 \log x \, dx = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{0.5}{3}[(0 + 1.6094) + 4(4.0797) + 2(3.178)] = 4.0467.$$

The error E_s in Simpson's 1/3 rule is given by

$|E_s| \le \frac{b-a}{180} h^4 M$ where, M is an upper bound to $y^{(iv)}(x)$ over the interval [1, 5].

Here $y = \log x$, this gives $y^{(iv)}(x) = -\frac{6}{x^4}$. Thus, $M = \max y^{(iv)}(x), x \in [1, 5] = 6$

Hence, $|E_s| \le \frac{24h^4}{180}$.

Now for the result to be accurate up to four decimal places, $\frac{24h^4}{180} < 10^{-4}$, which gives, $h < .274$.

Example 21.58: Evaluate the integral, $I = \int_0^1 \int_0^1 e^{x+y} \, dx dy$ using trapezoidal rule.

Solution: We divide both the intervals for x and y in two equal parts thus if h and k denote the widths of their subintervals, then take $h = 0.5$ and $k = 0.5$. The tabulated values of $f(x, y) = e^{x+y}$ are given by

y \ x	0.0	0.5	1.0
0.0	1.0000	1.6487	2.7183
0.5	1.6487	2.7183	4.4817
1.0	2.7183	4.4817	7.3891

First we keep x fix and vary the variable y. The application of the trapezoidal rule to each column in the above table gives

$$\int_0^1 f(0, y)dy = \frac{0.5}{2}[(1 + 2.7183) + 2(1.6487)] = 1.7539$$

$$\int_0^1 f(0.5, y)dy = \frac{0.5}{2}[(1.6487 + 4.4817) + 2(2.7183)] = 2.8917$$

$$\int_0^1 f(1, y)dy = \frac{0.5}{2}[(2.7183 + 7.3891) + 2(4.4817)] = 4.7677$$

Therefore, $$\int_0^1\int_0^1 e^{x+y}dx\,dy = \frac{h}{2}\left[\left(\int_0^1 f(0, y)dy + \int_0^1 f(1, y)dy\right) + 2\int_0^1 f(0.5, y)dy\right]$$

$$= \frac{0.5}{2}[(1.7539 + 4.7677) + 2(2.8917)]. = 3.07625.$$

EXERCISE 21.8

1. Find from the following table the area bounded by the curve and the x-axis from $x = 7.47$ to $x = 7.52$.

x:	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$:	1.93	1.95	1.98	2.01	2.03	2.06

2. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using, (i) Trapezoidal rule, (ii) Simpson's 1/3 rule, and (iii) Simpson's 3/8 rule. Comment over the results obtained.

3. Evaluate $\int_0^1 \sqrt{1-x^2}\,dx$ using, (i) Trapezoidal rule, and (ii) Simpson's 1/3 rule, by dividing the interval in 10 subintervals.

4. Compute the integral $I = \sqrt{2/\pi}\int_0^1 e^{-x^2/2}dx$ using Simpson's 1/3 rule and taking $h = 0.125$.

5. Find the approximate value of $\int_0^{\pi/2} \sqrt{\cos\theta}\,d\theta$ by dividing the interval into six subintervals.

6. Determine maximum error in evaluating the integral $\int_0^{\pi/2} \cos x\,dx$ by both the trapezoidal and Simpson's rules using four subintervals.

7. Estimate the value of $\int_0^{\pi} \frac{\sin t}{t}dt$ using Simpson's 1/3 rule by taking $n = 6$.

8. Compute the integral $\int_1^2 \frac{dx}{x}$ using Simpson's 1/3 rule and also obtain an upper bound to the error by taking $h = 0.25$.

9. Evaluate the double integral $I = \int_1^2 \int_1^2 \frac{dx\,dy}{x+y}$ by using trapezoidal rule with $h = k = 0.25$.

10. Evaluate the integral $I = \int_0^1 \int_0^1 e^{x+y} dx\,dy$ by using Simpson's 1/3 rule with $h = k = 0.5$.

11. Estimate the length of the arc of the curve $3y = x^3$ from (0, 0) to (1, 1/3) using Simpson's 1/3 rule by taking $h = 0.125$.

12. A river is 80 ft. wide. The depth d, in feet, at a distance x ft. from one bank is given by the following table. Find approximately the area of cross-section

x:	0	10	20	30	40	50	60	70	80
d:	0	4	7	9	12	15	14	8	3

13. A curve is given by the table

x:	0	1	2	3	4	5	6
y:	0	2	2.5	2.3	2	1.7	1.5

The x-coordinate of C.G. of the area bounded by the curve, the end ordinates and the x-axis is given by $\bar{x} = \frac{1}{A}\int_0^6 xy\,dx$, where A is the area. Find $\bar{x}$, using Simpson's 1/3 rule.

14. A missle is launched from a ground station. The acceleration 'a' during its first 80 seconds of flight as recorded, is given in the following table:

t (in sec):	0	10	20	30	40	50	60	70	80
a (m/s^2):	30.0	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

Compute the velocity in km/sec when $t = 80$, using Simpson's 1/3 rule.

15. A tank is discharging water through an orifice at a depth of x metre below the surface of the water whose area is A metre sq. The following are the tabulated values of x and A

x	1.50	1.65	1.80	1.95	2.10	2.25	2.40	2.55	2.70	2.85	3.00
A	1.257	1.390	1.520	1.650	1.809	1.962	2.123	2.295	2.462	2.650	2.827

Using the formula $T = \frac{1}{0.018}\int_{1.5}^{3.0} \frac{A}{\sqrt{x}}dx$, calculate the time T in seconds for the water level to drop from 3.0 m to 1.5 m above the orifice.

ANSWERS

Exercise 21.1 (p. 1190)

1. 0.42949.
2. 1.30980.
3. 0.73909.
4. 0.3473.
5. 0.51520.
6. 0.5177.
7. 4.73004.
8. 3.0514.

9. 0.53918. 10. 4.493. 11. 2.74065 13. 1.03717.

14. 0.607. 15. 3.7893.

Exercise 21.2 (p. 1201)

1. $x = 117/71, y = -81/71, z = 148/71$
2. $x = 5, y = 4, z = -7, u = 1$
3. $x = 9, y = -36, z = 30$
4. $x = 1, y = 2, z = 3.$
5. $x = 1, y = 1, z = 1$
6. $x = 2, y = 1/5, z = 0, w = 4/5$
7. $x = 35/18, y = 29/18, z = 5/18$
8. $x = 1, y = 1, z = 1$
9. $x = 2, y = 1/5, z = 0, w = 4/5$
10. $x = 1.08, y = 1.95, z = 3.16$
11. $x = 1.90, y = 3.19, z = 5.04$
12. $x = 1, y = 2, z = 3, w = 4.$
13. $x = 2.42, y = 3.57, z = 1.92$
14. $x = 0.996, y = 2, z = 3$
15. $x = y = 87.5, z = w = 62.5$
16. $x = 6.13, y = 4.31, z = 3.23$
17. $x = 52.5, y = 44.5, z = 59.7$
18. $x = 2.031, y = 2.683, z = -1.118, w = 3.111$

Exercise 21.3 (p. 1205)

1. (a) $5.38, \begin{bmatrix} 0.46 \\ 1.0 \end{bmatrix}$ (b) $4.62, \begin{bmatrix} 1 \\ 0.62 \end{bmatrix}$
2. (a) $-2.0234, [1 \quad -1.0079, \quad 0.0015]^T$ (b) $5, [1 \quad 1 \quad 5]^T$
3. $0.72, [1 \quad 0.5 \quad 1]^T, 5.10^{-8}$
4. $30.29, [0.96 \quad 0.69 \quad 1 \quad 0.94]^T$

Exercise 21.4 (p. 1216)

1. (a) $(e^{ah} - 1)^n e^{ax+b}$ (b) $\dfrac{(2hx + h^2)\cos 2x + 2x^2 \sin h \sin(2x+h)}{\cos 2(x+h) \cos 2x}$

   (c) $e^x\left[e^h \log\left(1 + \frac{h}{x}\right) + (e^h - 1)\log 2x\right]$

7. (a) -36 (b) $24 \times 2^{10} \times 10!$

10. 239, 371.

11. 4.68, 2.68, 55.8, 99.88

12. 2.925, 0.225

13. Error in the entry corresponding to $x = 4.0$; correct value is 5.75.

14. 52

Exercise 21.5 (p. 1230)

1. 1.160944 2. 6.36 3. $y(x) = x^3 - 2x^2 + 1$, 33

4. 54.85, 96.84
5. 0.1, 100
6. $y = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31$
7. 0.783172
8. (i) 26, (ii) 7, (iii) 46
9. 16.9216
10. 111.874930
11. 1.892692
12. 13.026
13. 0.19573
14. 1.685750
15. 2.5283

Exercise 21.6 (p. 1238)

1. 2.4786
2. $x^3 - 7x^2 + 18x - 12$
3. 42/3
4. $3x^4 - 5x^3 + 6x^2 - 14x + 5$
5. 810
6. 7.03
7. 448; 3150
8. $\frac{1}{5(x-1)} + \frac{3}{35}\frac{1}{(x+1)} - \frac{13}{10}\frac{1}{x-4} + \frac{71}{10}\frac{1}{x-6}$
9. 5.66, maximum at $x = 4.5$
10. 0.476937

Exercise 21.7 (p. 1244)

1. −0.05
2. 4.054
3. 0.0, 32
4. 9.0215, 8.9629
5. 2.644225
6. 0.085222
7. 0.0099154
8. 0.2563
9. 5.6875, 0.2628
10. minima at $x = 0, 2$ and $f(0) = 0, f(2) = 0$
    maxima at $x = 1$ and $f(1) = 0.25$

Exercise 21.8 (p. 1253)

1. 0.0996
2. (i) 1.4108 (ii) 1.3662 (iii) 1.3571
3. (i) 0.77613 (ii) 0.78175
4. 0.6827
5. 1.1873
6. 0.02020, 0.000173
7. 0.7854
8. 0.6933, 0.00104
9. 0.3407
10. 2.9545
11. 1.0893
12. 710 sq. ft.
13. 3.032
14. 3.0861 km/sec
15. 110 sec

22 Numerical Methods for Differential Equations

CHAPTER

Numerical methods for differentials equations are of great practical importance, since the analytical methods are limited to certain special forms of equations, while numerical methods have no such limitations. The solution is obtained as tabulation of the values of the dependent variable at various values of the independent variable and not as a functional relationship. But if the initial conditions are changed, we need to recompute the entire table.

22.1 INTRODUCTION

Many of the practical problems governing some physical systems often lead to differential equations, ordinary or partial, which cannot be solved exactly through one of the methods as discussed in Chapters 10–12 and Chapters 16–17. In some other situations, there are differential equations in which exact closed form solutions exist but are too complicated to apply practically. In such cases, numerical approximation of the solution is considered as the possible approach.

In this chapter, we study various methods for numerical solution of ordinary differential equations and partial differential equations.

22.2 METHODS FOR FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

To describe the various numerical methods for the solution of first-order ordinary differential equations, we consider initial value problem (i.v.p) of the form

$$\frac{dy}{dx} = f(x, y), \ y(x_0) = y_0 \qquad ...(22.1)$$

where f is assumed to be such that the problem has a unique solution on some interval containing x_0. The solution of the i.v.p. (22.1) can be obtained in one of the following two forms:

1. The value of y as a power series in the independent variable x.
2. A set of tabulated values of x and y.

The method of Taylor's series and Picard's method of approximations belong to the form(1), whereas those of Euler, Runge-Kutta, Milne-Simpson and Adams-Bashforth belong to the form(2).

The methods included in the second form are called *step-by-step methods*. We start from the given initial condition $y(x_0) = y_0$ and proceed ahead stepwise computing approximate values of the solution $y(x)$ at the equally spaced *mesh points* $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ..., $x_n = x_0 + nh$ of width h. Next, we discuss the various methods.

22.2.1 Taylor's Series Method

Let $y = y(x)$ be the exact solution of the initial value problem

$$y' = f(x, y),\ y(x_0) = y_0 \qquad ...(22.2)$$

Expanding $y(x)$ by Taylor's series about the point $x = x_0$, we obtain

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2!}\, y''(x_0) + ... \qquad ...(22.3)$$

In case the values of y_0', y_0'' ... are known, then the series (22.3) becomes a power series in x for y. From (22.2), we have

$$\begin{aligned} y'' &= f_x + y'f_y = f_x + ff_y \\ y''' &= f_{xx} + f_{xy}\, y' + f\,[f_{yx} + f_{yy}\, y'] + f_y[f_x + f_y\, y'] \\ &= f_{xx} + f_{xy} f + ff_{yx} + f_{yy} f^2 + f_y f_x + (f_y)^2 f \\ &= f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + ff_y^2 \end{aligned}$$

Similarly, we go for higher order derivatives and evaluate these at $x = x_0$. Substituting in (22.3), we find $y(x)$ as series in x. We illustrate the method through the examples considered next.

Example 22.1: Solve $y' = x^2 + y^2$, $y(0) = 1$ using Taylor's series. Also find $y(0.1)$ correct to four decimal places.

Solution: The Taylor's series for $y(x)$ about $x = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{iv}(0) + ... \qquad ...(22.4)$$

We have, $y' = x^2 + y^2$, $\quad y'' = 2x + 2yy'$, $\quad y''' = 2 + 2(y')^2 + 2yy''$,

$y^{iv} = 4y'y'' + 2y'y'' + 2yy''' = 6y'y'' + 2yy'''$ etc.

At $x = 0, y(0) = 1$ $\quad y'(0) = (y(0))^2 = 1$, $\quad y''(0) = 2y(0)\, y'(0) = 2$,

$y'''(0) = 2 + 2(y'(0))^2 + 2y(0)\, y''(0) = 8$ $\quad y^{iv}(0) = 6y'(0)\, y''(0) + 2y(0)\, y'''(0) = 28$.

Substituting these values in (22.4), the series becomes

$$\begin{aligned} y(x) &= 1 + x + x^2 + \frac{8}{3!}x^2 + \frac{28}{4!}x^4 + ... \\ &= 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + ... \end{aligned}$$

Thus,

$$\begin{aligned} y(0.1) &= 1 + 0.1 + 0.01 + \frac{4}{3}(0.001) + \frac{7}{6}(0.0001) + ... \\ &= 1 + 0.1 + 0.01 + 0.0013 + 0.0001 + ... = 1.1114 \end{aligned}$$

Remark: If in the Taylor's series method, we neglect the terms of the powers $(n+1)$ and highers, then the truncation error is of the order $(x-x_0)^{n+1}$. So in case $|x-x_0|$ is large, then the error can also become large. Hence, to improve accuracy, we can divide the entire interval in two (or, more) subintervals say (x_0, x_1) and (x_1, x) and complete the desired value in two iterations using Taylor series. Here $y(x_0)$ is used as an initial condition in the first iteration and $y(x_1)$ in the next iteration.

Example 22.2: Using Taylor's series method, find the solution of the initial value problem

$$\frac{dy}{dx} = x + y,\ y(1) = 0$$

at $x = 1.2$, taking $h = 0.1$ and compare the result obtained with the closed form solution.

Solution: From the given differential equation, we have

$$y' = x + y, \quad y'' = 1 + y', \quad y''' = y'', \quad y^{iv} = y''', \ldots$$

Using the initial condition, $x_0 = 1$, $y_0 = y(x_0) = 0$, we have $y_0' = 1$, $y_0'' = 2$, $y_0''' = y_0^{iv} = 2$.

The Taylor's series expansion of $y(x)$ about $x = x_0$ is

$$y(x) = y(x_0) + (x - x_0)\, y_0' + \frac{(x-x_0)^2}{2} y_0'' + \frac{(x-x_0)^3}{6} y_0''' + \frac{(x-x_0)^4}{24} y^{iv} + \ldots$$

Substituting the values of the derivatives and the initial condition, we obtain

$$y(1.1) = 0 + (0.1)(1) + \frac{0.01}{2}(2) + \frac{0.001}{6}(2) + \frac{0.0001}{24}(2) + \ldots$$

$$= 0.1 + 0.01 + \frac{0.001}{3} + \frac{0.0001}{12} + \ldots$$

$$= 0.1 + 0.01 + 0.000333 + 0.000008 = 0.110341 \simeq 0.1103.$$

Taking $y_1 = 0.1103$ and $x_1 = 1.1$, the values of the derivatives as computed at $x_1 = 1.1$ are

$$y_1' = 1.1 + 0.1103 = 1.2103,\ y_1'' = 1 + 1.2103 = 2.2103,\ y_1''' = y_1^{iv} = 2.2103.$$

Next, the Taylor's series expansion of $y(x)$ about $x = x_1$ is

$$y(x) = y_1 + (x - x_1) y_1' + \frac{(x-x_1)^2}{2} y_1'' + \frac{(x-x_1)^3}{6} y_1''' + \frac{(x-x_1)^4}{24} y_1^{iv} + \ldots$$

Substituting the value of y_1 and its derivatives in the Taylor's series expansion, we obtain

$$y(1.2) = 0.1103 + 0.12103 + 0.0110515 + 0.0003683 + 0.000184 = 0.242934 \simeq 0.2429.$$

To obtain the closed form solution, we rewrite the given equation as

$$\frac{dy}{dx} - y = x, \text{ or } d(ye^{-x}) = xe^{-x}.$$

On integration, we get

$$y = -e^x(xe^{-x} + e^{-x}) + ce^x = ce^x - x - 1;\ c \text{ being constant.}$$

Using the initial condition $y(1) = 0$, we get

$$0 = ce - 2, \text{ or, } c = 2/e.$$

Therefore, the closed form solution is

$$y = -x - 1 + 2e^{x-1}.$$

When $x = 1.2$, this gives

$$y(1.2) = -1.2 - 1 + 2(1.2214028) = -2.2 + 2.4428056 = 0.2428$$

Thus, the results obtained numerically and in closed form agree upto three decimal places.

22.2.2 Picard's Method of Successive Approximations

Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \qquad ...(22.5)$$

Integrating it in the interval $[x_0, x]$, we obtain

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x, y)\,dx$$

or,

$$y(x) = y(x_0) + \int_{x_0}^{x} f(x, y)\,dx. \qquad ...(22.6)$$

Equation (22.6) in which the unknown function y appears under the integral sign is called an *integral equation*. To solve this, we apply the method of successive approximations. To obtain the first approximation, we put y_0 for y on the right side of Eq. (22.6), and write

$$y^{(1)} = y(x_0) + \int_{x_0}^{x} f(x, y_0)\,dx \qquad ...(22.7)$$

Integral on the right side of Eq. (22.7) is solved and the resulting $y^{(1)}$ is substituted for y on the right side of the Eq. (22.6) to obtain the second approximation, and write

$$y^{(2)} = y(x_0) + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

and so on. In general, the iterative equation is

$$y^{(i)} = y(x_0) + \int_{x_0}^{x} f(x, y^{(i-1)})\,dx, \; i = 1, 2, \ldots \qquad ...(22.8)$$

with $y^{(0)} = y_0$.

Hence, the Picard's method yields a sequence of approximations $y^{(1)}, y^{(2)}, \ldots$ and in case the given *i.v.p.* has a unique solution, this sequence converges to the same.

Remark: Since Picard's method involves actual integration, sometimes it may not be possible to solve the integration. In fact, both Taylor's series method and Picard method are *semi-numeric methods*.

Example 22.3: Using Picard's method, obtain a solution upto the fifth approximation of the equation $dy/dx = y + x$, such that $y = 1$ when $x = 0$. Check your answer by finding the analytical solution.

Solution: Intergating the given differential equation in the interval [0, x], we have

$$y = 1 + \int_0^x (y + x)\,dx.$$

First approximation: Put $y = 1$ in $y + x$, we obtain

$$y^{(1)} = 1 + \int_0^x (1 + x)\,dx = 1 + x + x^2/2.$$

Second approximation: Put $y = 1 + x + x^2/2$ in $y + x$, we obtain

$$y^{(2)} = 1 + \int_0^x (1 + 2x + x^2/2)\,dx = 1 + x + x^2 + x^3/6.$$

Third approximation: Put $y = 1 + x + x^2 + x^3/6$ in $y + x$, we obtain

$$y^{(3)} = 1 + \int_0^x (1 + 2x + x^2 + x^3/6)\,dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Fourth approximation: Put $y = y^{(3)}$ in $y + x$, we obtain

$$y^{(4)} = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right)dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

Fifth approximation: Put $y = y^{(4)}$ in $y + x$, we obtain

$$y^{(5)} = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right)dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}.$$

The integrating function for $\frac{dy}{dx} - y = x$ is e^{-x}. Hence, the analytical solution is

$$ye^{-x} = \int xe^{-x}\,dx + c = -xe^{-x} - e^{-x} + c$$

or, $y = ce^x - x - 1$. Using $y(1) = 0$, we obtain $c = 2$.

Thus, the desired analytical solution is $y = 2e^x - x - 1$.

Using $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \infty$, we get

$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \ldots \infty$$

We observe that the solution obtained by the Picard's method approximates with the exact solution upto terms in x^5.

Example 22.4: For the initial value problem $\frac{dy}{dx} = \frac{x^2}{y^2 + 1}$, $y(0) = 0$ obtain y for $x = 0.25$, 0.5 and 1.0 correct to three decimal places using Picard's successive approximations.

Solution: Integrating the given differential equation in the interval $[0, x]$, we obtain

$$y = y(0) + \int_0^x \frac{x^2}{y^2+1}\,dx.$$

First approximation: Put $y = 0$ in the integrand, giving

$$y^{(1)} = \int_0^x x^2 dx = \frac{1}{3}x^3.$$

Second approximation: Put $y = \frac{1}{3}x^3$ in the integrand, giving

$$y^{(2)} = \int_0^x \frac{x^2}{\frac{x^6}{9}+1}\,dx = \int_0^x \frac{x^2}{(x^3/3)^2+1}\,dx = \tan^{-1}\left(\frac{1}{3}x^3\right) = \frac{1}{3}x^3 - \frac{1}{81}x^9 + \ldots$$

Now the range of values of x, so that the series with the term $\frac{1}{3}x^3$ alone gives the result correct to three decimal places, is given by $(x^9/81) \le 0.0005$ which implies that $x \le 0.7$.

Hence, $y(0.25) = \frac{1}{3}(0.25)^3 = 0.005$ $y(0.5) = \frac{1}{3}(0.5)^3 = 0.042$ and, $y(1.0) = \frac{1}{3} - \frac{1}{81} = 0.321$.

EXERCISE 22.1

1. Given $y' = 1 + xy$ with $y(0) = 0$, obtain the Taylor series for $y(x)$.
2. From the Taylor series for $y(x)$, find $y(0.1)$ correct to four decimal places if $y(x)$ satisfies $y' = x - y^2$ with $y(0) = 1$.
3. Find by Taylor's series method the value of y at $x = 0.1$ and $x = 0.2$ to five places of decimals from $y' = x^2y - 1$, $y(0) = 1$.
4. Find by Taylor's series method the value of y at $x = 0.2$ for the differential equation $y' = 2y + 3e^x$, $y(0) = 0$, taking $h = 0.1$ and compare the result obtained with the exact solution.
5. Using Picard's method obtain $y(0.25)$, $y(0.5)$ and $y(1.0)$ correct to three decimal places when $y' = x^2/(y^2+1)$, $y(0) = 0$.
6. Solve $y' = xe^y$, $y(0) = 0$ by Picard's method and estimate $y(0.1)$, $y(0.2)$ and $y(1)$.
7. Find $y(0.1)$, if $y' = x - y^2$, $y(0) = 1$, using (a) Picard's method, and (b) Taylor's series.

22.2.3 Euler's Method

Consider the differential equation $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ and suppose we want to solve the given equation for the value of y at $x = x_n$. We divide the interval $[x_0, x_n]$ in n subintervals of each of equal length, say h, with the division points $x_r = x_0 + rh$, $r = 0, 1, 2, \ldots n$.

Consider the Taylor's series expansion

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \ldots$$

Neglecting terms containing h^2, h^3, etc., we obtain

$$y(x+h) \approx y(x) + hy'(x) = y(x) + hf(x, y), \text{ using } \frac{dy}{dx} = f(x, y)$$

Thus, in first step, we compute

$$y_1 = y(x_0 + h) = y_0 + hf(x_0, y_0). \quad \ldots(22.9)$$

Similarly, in the second step $y_2 = y_1 + hf(x_1, y_1)$ and, in general, we obtain

$$y_{n+1} = y_n + hf(x_n, y_n), \; n = 0, 1, 2, \ldots \quad \ldots(22.10)$$

the Euler's formula

Geometrically, in Euler's method the solution curve $y = y(x)$ is approximated by a polygon whose first side is tangent to the curve $y(x)$ at $x = x_0$, as shown in Fig. (22.1).

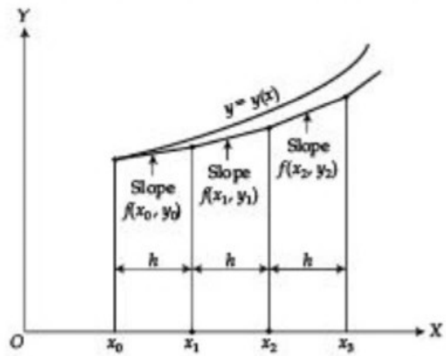


Fig. 22.1

Example 22.5: Using Euler's method solve the initial value problem $\frac{dy}{dx} = x + y$, $y(0) = 1$ in the interval [0, 1], taking $h = 0.2$. Compare the values obtained with the exact solution at each step.

Solution: Here $f(x, y) = x + y$, and $n = (b - a)/h = 1/0.2 = 5$. The Euler's method gives

$$y_{n+1} = y_n + h(x_n + y_n), \; n = 0, 1, 2, \ldots, 5$$

Thus, the successive applications of the Euler's method, with $h = 0.2$, gives

$$y(0.2) = 1 + 0.2\,(0 + 1) = 1 + 0.2 = 1.2$$
$$y(0.4) = 10.2 + 0.2\,(0.2 + 1.2) = 1.48$$
$$y(0.6) = 1.48 + 0.2\,(0.4 + 1.48) = 1.856$$
$$y(0.8) = 1.856 + 0.2(0.6 + 1.856) = 2.3472$$
$$y(1.0) = 2.3472 + 0.2(0.8 + 2.3472) = 2.97664$$

The analytical solution of the given i.v.p., (refer to Example 22.3), is $y = 2e^x - x - 1$.

We formulate the following table:

x	Numerical solution $y(x)$	Exact solution $y(x)$	Error
0.0	0.0	0.00	0.00
0.2	1.2	1.2428055	0.0428055
0.4	1.48	1.5836494	0.1036494
0.6	1.856	2.0442376	0.1882376
0.8	2.3472	2.6510819	0.3038819
1.0	2.97664	3.4365637	0.4599237

Example 22.6: Given $\dfrac{dy}{dx} = 3x^2 + 1, y(1) = 2$, estimate $y(2)$ by Euler's method using, (i) $h = 0.50$, and (ii) $h = 0.25$. Also comment over the results obtained.

Solution: Here $f(x, y) = 3x^2 + 1$. Thus, Euler's method gives

$$y_{n+1} = y_n + h(3x_n^2 + 1), n = 0, 1, 2, ...$$

(i) $h = 0.5$

$$y(1) = 2, y(1.5) = 2 + 0.5[3(1.0)^2 + 1] = 4.0, y(2.0) = 4 + 0.5[3(1.5)^2 + 1] = 7.875.$$

(ii) $h = 0.25$

$$y(1) = 2, y(1.25) = 2 + 0.25[3(1.0)^2 + 1] = 3.0, y(1.50) = 3 + 0.25[3(1.25)^2 + 1] = 5.42188$$

$$y(1.75) = 5.42188 + 0.25[3(1.5)^2 + 1] = 7.35938, y(2.0) = 7.35938 + 0.25[3(1.75)^2 + 1] = 9.90626$$

Also the analytical solution of the given initial value problem is $y = x^3 + x$. Thus, $y(2) = 10.0$, and we observe from the results obtained that accuracy is improved considerably when h is reduced from 0.5 to 0.25.

22.2.4 Modified Euler's Method

In general, Euler's method is very slow and to obtain reasonable accuracy, we need to take a smaller value for h and thus the method as such is not appropriate for practical purpose. Here we describe a modification of the Euler's method, known as *modified Euler's method* which gives more accurate result.

In Euler's method, in first step, refer to Eq. (22.9), we compute

$$y_1 = y_0 + hf(x_0, y_0) \quad ...(22.11)$$

As a modification to (22.11), we consider

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})] \quad ...(22.12)$$

where $y_1^{(0)} = y_0 + hf(x_0, y_0)$ is computed from the Euler's formula.

Similarly,

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

and, in general,

$$y_1^{(k+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(k)})], k = 0, 1, 2, \ldots \quad \ldots(22.13)$$

where $y_1^{(k)}$ is the kth approximation of y_1 and $y(x_0) = y_0$.

Remark: The modified Euler's method is a *one-step predictor-corrector method*, because in each step, first we predict the value by (22.11) and then correct it by (22.13).

Example 22.7: Given $\frac{dy}{dx} = x + y, y(0) = 1$. Use modified Euler's method to find an approximate value of y, when $x = 0.2$ by taking $h = 0.1$, correct to four decimal places.

Solution: Here $f(x, y) = x + y, x_0 = 0, y_0 = 1, h = 0.1$. Thus,

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.1)(0 + 1) = 1.10$$

Euler's modified formula is $y_1^{(k+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(k)})]$ where $x_1 = x_0 + h = 1.1$.

We formulate the following table:

x	y	$f(x, y) = x + y$	$y^{(k+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x, y^{(k)})]$
0	1	1	$1 + (0.1)(1) = 1.10$
0.1	1.10	1.2	$1 + \frac{0.1}{2}(1 + 1.2) = 1.11$
0.1	1.11	1.21	$1 + \frac{0.1}{2}(1 + 1.21) = 1.1105$
0.1	1.1105	1.2105	$1 + \frac{0.1}{2}(1 + 1.2105) = 1.1105$

Hence, $y(0.1) = 1.1105$

Next we take $x_0 = 0.1, y_0 = 1.1105, x_1 = x_0 + h = 0.2$ and formulate the following table:

x	y	$f(x, y) = x + y$	$y^{(k+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x, y_1^{(k)})]$
0.1	1.1105	1.2105	$1.1105 + 0.1(1.2105) = 1.2316$
0.2	1.2316	1.4316	$1.1105 + \frac{0.1}{2}[1.2105 + 1.4316] = 1.2426$
0.2	1.2426	1.4426	$1.1105 + \frac{0.1}{2}[1.2105 + 1.4426] = 1.2432$
0.2	1.2432	1.4432	$1.1105 + \frac{0.1}{2}[1.2105 + 1.4432] = 1.2432$

Hence, $y(0.2) = 1.2432.$

Example 22.8: Using modified Euler's method obtain the solution of the differential equation

$$\frac{dy}{dx} = x + \sqrt{y},\ y(0) = 1$$

over the interval [0, 0.4] by taking $h = 0.2$.

Solution: Here $f(x, y) = x + \sqrt{y}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$ and $x_1 = x_0 + h = 0.2$.

We formulate the following table:

x	y	$f(x, y) = x + \sqrt{y}$	$y^{(k+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(k)})]$
0	1	1	$1 + 0.2(1) = 1.2$
0.2	1.2	1.2954	$1 + \frac{0.2}{2}[1 + 1.2954] = 1.2295$
0.2	1.2295	1.3088	$1 + \frac{0.2}{2}[1 + 1.3088] = 1.2309$
0.2	1.2309	1.3095	$1 + \frac{0.2}{2}[1 + 1.3095] = 1.2309$
0.2	1.2309	1.3095	$1.2309 + 0.2(1.3095) = 1.4928$
0.4	1.4928	1.6218	$1.2309 + \frac{0.2}{2}(1.3095 + 1.6218) = 1.5240$
0.4	1.5240	1.6345	$1.2309 + \frac{0.2}{2}(1.3095 + 1.6345) = 1.5253$
0.4	1.5253	1.6350	$1.2309 + \frac{0.2}{2}(1.3095 + 1.6350) = 1.5254$
0.4	1.5254	1.6351	$1.2309 + \frac{0.2}{2}(1.3095 + 1.6351) = 1.5254$

Hence, $y(0.2) = 1.2309$ and $y(0.4) = 1.5254$.

Example 22.9: Using modified Euler's method, find the value of y when $x = 0.15$, given that

$$\frac{dy}{dx} = x^2 + y,\ y(0) = 1$$

in three steps.

Solution: Here $f(x, y) = x^2 + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.05$ and $x_1 = 0.05$.

We formulate the following table:

x	y	$f(x, y) = x^2 + y$	$y^{(k+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(k)})]$
0	1	1	$1 + 0.5(1) = 1.05$
0.05	1.05	1.0525	$1 + \frac{0.05}{2}[1.05 + 1.0525] = 1.0526$
0.05	1.0526	1.0551	$1 + \frac{0.05}{2}[1.05 + 1.0551] = 1.0526$
0.05	1.0526	1.0551	$1.0526 + 0.05(1.0551) = 1.1054$
0.10	1.1054	1.1154	$1.0526 + \frac{0.05}{2}[1.0526 + 1.1154] = 1.1068$
0.10	1.1068	1.1168	$1.0526 + \frac{0.05}{2}[1.0526 + 1.1168] = 1.1068$
0.10	1.1068	1.1168	$1.1068 + 0.05(1.1168) = 1.1626$
0.15	1.1626	1.1851	$1.1626 + \frac{0.05}{2}(1.1626 + 1.1851) = 1.2213$
0.15	1.2213	1.2438	$1.1626 + \frac{0.05}{2}(1.1626 + 1.2438) = 1.2228$
0.15	1.2228	1.2453	$1.1626 + \frac{0.05}{2}(1.1626 + 1.2453) = 1.2228$

Hence, $y(0.15) = 1.2228$.

22.2.5 Error Analysis of Euler's Methods

Normally, there are two sources of error, namely *roundoff error* and *truncation error*. Roundoff error is always present and can be minimized by increasing the precision of calculations. The major source is the truncation error which arises because of the use of a truncated Taylor's series in the Euler's method. Since in the Euler's method, we use only

$$y(x + h) = y(x) + hy'(x)$$

and neglect terms containing h^2 and higher powers in h, *the truncation error in each step* (or, *local truncation error*) is proportional to h^2. Further over a fixed x-interval over which we solve the differential equation, the number of steps is proportional to $1/h$. Hence, the *total error* (or, *global error*) is proportional to $h^2(1/h) = h$. Hence, *Euler's method is called a first-order method.*

It can be shown that *modified Euler's method is a second-order method*, that is, the total error here is proportional to h^2. In fact, *a method which agrees with Taylor's series solution up to the terms in h^r, where r differs from method to method is called the rth-order method.*

EXERCISE 22.2

1. Using Euler's method, find $y(0.04)$, when $y' = -y$, $y(0) = 1$ taking $h = 0.01$.
2. Using Euler's method, obtain $y(0.1)$, $y(0.2)$ and $y(0.3)$, when $y' = 1 + y^2$, $y(0) = 0$

3. Using Euler's modified method, solve $y' = \ln(x + y)$, $y(0) = 2$ at $x = 1.2$ and 1.4, with $h = 0.2$ correct to four decimal places.
4. Using Euler's modified method, find $y(2)$ in steps of $h = 0.2$ when $y' = 2 + \sqrt{xy}$, $y(1) = 1$ correct up to three decimal places.
5. Solve $y' = 2x^{-1}\sqrt{y - \ln x} + x^{-1}$, $y(1) = 0$, for $1 \le x \le 1.3$, by Euler's method with $h = 0.1$. Verify that the exact solution is $y = (\ln x)^2 + \ln x$ and compute the error.

22.2.6 Runge-Kutta Methods

Runge-Kutta (R-K) methods, designed by two German mathematicians Runge and Kutta, are the most efficient methods in terms of accuracy. In addition to give the greater accuracy they possess the advantage of requiring only the function values at some selected points on the subinterval. The different R-K methods are distinguished by their orders in the sense that they agree with Taylor's series solution up to terms of h^r, where r is the order of the method. Mostly the *fourth-order Runge-Kutta method* is widely used for finding the numerical solution of the differential equations but the development of this method is quite complicated algebraically. However, to understand the basic idea of these methods, we develop the *second-order Runge-Kutta method.*

From the modified Euler's method, refer to (22.12)

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \qquad ...(22.14)$$

Substituting $y_1 = y_0 + hf(x_0, y_0)$ for $y_1(0)$ on the right side and writing $f_0 = f(x_0, y_0)$, (22.14) gives

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + f(x_0 + h, y_0 + hf_0)\right] \qquad ...(22.15)$$

If we set $\quad k_1 = hf_0$ and $k_2 = hf(x_0 + h, y_0 + k_1)$, then (22.15) becomes

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2] \qquad ...(22.16)$$

which is the *second-order Runge-Kutta* method. The error in (22.16) can be shown to be of order h^3 by expanding both sides by Taylor's series.

In fact, there are several second-order Runge-Kutta formulae and the formula (22.16) is the one which is commonly used.

Other higher order Runge-Kutta formulae and their different forms also exist, out of these, we describe the most commonly used form of the *fourth-order Runge-Kutta formula*, as given below.

If $\quad k_1 = hf(x_0, y_0)$,

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right),$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right),$$

and, $\quad k_4 = hf(x_0 + h, y_0 + k_3)$, then $$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \qquad ...(22.17)$$

The error in (22.17) can be shown to be of order h^5 by expanding both sides by Taylor's series.

Example 22.10: Find the numerical solution of the initial value problem $\frac{dy}{dx} = \frac{y+x}{y-x}$, $y(0) = 1$ at $x = 0.2$ and 0.4 using the second-order Runge-Kutta method.

Solution: Here $f(x, y) = \frac{y+x}{y-x}$.

To find $y(0.2)$:

We have, $x_0 = 0$, $y_0 = 1$, $h = 0.2$, thus

$$k_1 = hf(x_0, y_0) = 0.2\left(\frac{1+0}{1-0}\right) = 0.2,\ k_2 = hf(x_0 + h, y_0 + k_1) = 0.2\left(\frac{1.2+0.2}{1.2-0.2}\right) = 0.28.$$

Hence, $$y(0.2) = y_0 + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2}(0.48) = 1.24.$$

To find $y(0.4)$:

We have, $x_1 = 0.2$, $y_1 = 1.24$, $h = 0.2$, thus

$$k_1 = hf(x_1, y_1) = 0.2\left(\frac{1.24+0.2}{1.24-0.2}\right) = 0.2769,\ k_2 = hf(x_1 + h, y_1 + k_1) = 0.2\left(\frac{1.5169+0.4}{1.5169-0.4}\right) = 0.3433$$

Hence, $$y(0.4) = 1 + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2}(0.6202) = 1.3101.$$

Example 22.11: Using Runge-Kutta method of fourth-order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, $y(0) = 1$, at $x = 0.2, 0.4$.

Solution: Here $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

To find $y(0.2)$: we have $x_0 = 0$, $y_0 = 1$, $h = 0.2$, thus,

$$k_1 = hf(x_0, y_0) = 0.2\, f(0, 1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 1.1967) = 0.1891$$

Hence, $$y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891] = 1.19599 \approx 1.196$$

To find $y(0.4)$: we have $x_1 = 0.2$, $y_1 = 1.196$, $h = 0.2$, thus,

$$k_1 = hf(x_1, y_1) = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2f(0.3, 1.2906) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2f(0.3, 1.2858) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.3753) = 0.1688$$

Hence, $$y(0.4) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.196 + \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] = 1.3752.$$

Example 22.12: Solve $\frac{dy}{dx} = x + y$, $y(0) = 1$ using fourth-order Runge-Kutta method over the interval [0, 0.4], taking $h = 0.1$.

Solution: Here $f(x, y) = x + y$

To find $y(0.1)$: $x_0 = 0$, $y_0 = 1$, $h = 0.1$, thus,

$$k_1 = hf(x_0, y_0) = 0.1(1) = 0.1$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = hf(0.05, 1.05) = 0.1[0.05 + 1.05] = 0.11$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1(0.05, 1.055) = 0.1105$$

$$k_4 = hf(x_0, h, y_0 + k_3) = 0.1(0.1, 1.1105) = 0.12105$$

Hence, $$y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105) = 1.11034$$

To find $y(0.2)$: $x_1 = 0.1$, $y_1 = 1.11034$, $h = 0.1$, thus,

$$k_1 = hf(x_1, y_1) = 0.1\ (0.1 + 1.11034) = 0.121034$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1[0.15 + (1.11034 + 0.060517)] = 0.13208$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1[0.15 + (1.11034 + 0.06604)] = 0.132638$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1[0.2 + (1.11034 + 0.132638)] = 0.1442978$$

Hence, $$y(0.2) = 1.11034 + \frac{1}{6}[0.121034 + 2(0.13208) + 2(0.132638) + 0.1442978] = 1.2428$$

To find $y(0.3)$: $x_2 = 0.2$, $y_2 = 1.2428$, $h = 0.1$, thus,

$$k_1 = hf(x_2, y_2) = 0.1\,[0.2 + 1.2428] = 0.14428$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.1[0.25 + (1.2428 + 0.07214)] = 0.156494$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.1[0.25 + (1.2428 + 0.078247)] = 0.1571047$$

$$k_4 = hf\,(x_2 + h, y_2 + k_3) = 0.1[0.3 + (1.2428 + 0.1571047)] = 0.16999047$$

Hence, $$y(0.3) = 1.2428 + \frac{1}{6}[0.14428 + 2(0.156494) + 2(0.1571047) + 0.16999047] = 1.399711$$

Finally, to find $y(0.4)$: $x_3 = 0.3$, $y_3 = 1.3997$, $h = 0.1$, thus,

$$k_1 = hf(x_3, y_3) = 0.1\,[0.3 + 1.3997] = 0.16997$$

$$k_2 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = 0.1[0.35 + (1.3997 + 0.084985)] = 0.1834685$$

$$k_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right) = 0.1[0.35 + (1.3997 + 0.091734)] = 0.1841434$$

$$k_4 = hf(x_3 + h, y_3 + k_3) = 0.1[0.4 + (1.3997 + 0.1841434)] = 0.19838434$$

and, hence $$y(0.4) = 1.3997 + \frac{1}{6}\,[0.16997 + 2(0.1834685) + 2(.1841434) + 0.19838434] = 1.58363$$

EXERCISE 22.3

1. Given $y' = y - x$, $y(0) = 2$, apply Runge-Kutta second-order formula to find $y(0.1)$, $y(0.2)$ and $y(0.3)$ correct to four decimal places.
2. Find $y(0.1)$ and $y(0.2)$ in the above problem using Runge-Kutta fourth-order method.
3. Consider the i.v.p. $y' = 3x + y/2$, $y(0) = 1$. Find $y(0.2)$ in two steps using (i) Euler's method, (ii) modified Euler's method (iii) Runge-Kutta fourth-order method correct up to five places of decimal. Find $y(0.2)$ from the exact solution, give your comments.
4. Apply Runge-Kutta method to find $y(0.2)$ in steps of 0.1, if $y' = x + y^2$, $y(0) = 1$.
5. Apply Runge-Kutta method to find $y(0.8)$ give that $y' = \sqrt{x + y}$, $y(0.4) = 0.41$ in two steps, correct up to four decimal places.

22.3 MULTISTEP METHODS (PREDICTOR-CORRECTOR METHODS)

So far, for finding the numerical solution of an ordinary differential equation, we have discussed the methods which are all *one step methods* in the sense that they use information only from the preceding computed point x_i to compute the value at the next point x_{i+1}. However, in *multistep methods* at each step, the values from more than one of the preceding steps is used to compute the information at a specific point. Normally, a pair of formulae is used in conjunction with each other, one for predicting the value of y at x_{i+1} and the other for correcting the predicted value y. Such methods are also termed as *predictor-corrector* methods. The main disadvantage of multistep method is that they are not self-starting, they need additional information more than merely the initial value condition $y(x_0) = y_0$. In this section, we explain two such methods the *Milne's method,* and *Adams-Bashforth method*. The methods are of order h^4 and their error terms are of order h^5.

22.3.1 Milne's Method

Given $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ and say, we are to find an approximate value of y for $x = x_0 + kh$. By Milne's method, we proceed as follows.

Using $y(x_0) = y_0$, we compute $y_1 = y(x_0 + h)$, $y_2 = y(x_0 + 2h)$ and $y_3 = y(x_0 + 3h)$ by applying some single-step method, say Picard's, Taylor's series or Runge-Kutta method. Then we calculate $f_0 = f(x_0, y_0)$, $f_1 = f(x_0 + h, y_1)$, $f_2 = f(x_0 + 2h, y_2)$, and $f_3 = f(x_0 + 3h, y_3)$.

To find the next value $y_4 = (x_0 + 4h)$, we consider the relation

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x, y)\,dx \qquad ...(22.18)$$

and substitute for $f(x, y)$ in (22.18) using Newton's forward difference formula in the form

$$f(x, y) = f_0 + k\Delta f_0 + \frac{k(k-1)}{2!}\Delta^2 f_0 + \frac{k(k-1)(k-2)}{3!}\Delta^3 f_0 + ...$$

Thus, (22.18) becomes

$$y_4 = y_0 + \int_{x_0}^{x_0+4h}\left[f_0 + k\Delta f_0 + \frac{k(k-1)}{2!}\Delta^2 f_0 + ...\right]dx$$

$$= y_0 + h\int_0^4\left[f_0 + k\Delta f_0 + \frac{k(k-1)}{2!}\Delta^2 f_0 + ...\right]dk; \text{ using } x = x_0 + kh$$

$$= y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + ...\right]$$

$$= y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3] \qquad ...(22.19)$$

neglecting the fourth and higher order differences and simplifying.

The formula (22.19) is called *predictor* and is applied to predict the value of y_4 when those of y_0, y_1, y_2 and y_3 are known. Using this value of y_4, we obtain $f_4 = f(x_0 + 4h, y_4)$.

Next to obtain corrector, consider the relation

$$y_4 = y_2 + \int_{x_2}^{x_2+2h} f(x, y)\, dx$$

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \qquad ...(22.20)$$

using Simpson's 1/3rd rule.

The formula (22.20) is called *corrector* and is applied to obtain a better approximation of y_4. Then an improved value of f_4 is computed using this better approximation and again the corrector (22.20) is applied to further improve the value of y_4, and so on. The steps are repeated till y_4 remains unchanged upto the desired number of decimal places. Once this is achieved, f_4 is obtained and the value of y_5 is obtained from the *predictor* as

$$y_5 = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4). \qquad ...(22.21)$$

and, then $f_5 = f(x_0 + 4h, y_5)$ is calculated. Using this, a better approximation to the value of y_5 is obtained from the corrector

$$y_5 = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5) \qquad ...(22.22)$$

and the process is repeated as before till y_5 is obtained upto the desired degree of accuracy and then we proceed to calculate y_6, etc.

We illustrate all this in the examples to follow.

Example 22.13: Given $y' = \frac{1}{2}(1 + x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$. Evaluate $y(0.4)$ by Milne's predictor-corrector method.

Solution: We have, $f(x, y) = \frac{1}{2}(1 + x^2)y^2$ and $h = 0.1$. Thus,

$$f_0 = f(0,1) = \frac{1}{2}, f_1 = f(0.1, 1.06) = 0.5674, f_2 = f(0.2, 1.12) = 0.6552, \text{ and } f_3 = f(0.3, 1.21) = 0.7980$$

The Milne's predictor formula is

$$y_4 = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3] \qquad ...(22.23)$$

and corrector formula is

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \qquad ...(22.24)$$

Substituting for y_0, h, f_1, f_2 and f_3 in (22.23), we get

$$y_4 = 1 + \frac{4(.1)}{3}[\,2(0.5674) - (0.6552) + 2(0.7980)\,] = 1.2772.$$

Thus, $$f_4 = f(0.4, 1.2772) = 0.9458.$$

Substituting for y_2, h, f_2, f_3 and f_4 in (22.24), we get the corrected value for y_4 as

$$y_4 = 1.12 + \frac{0.1}{3}[0.6552 + 4(0.7980) + 0.9458] = 1.2797$$

The next improved value of f_4 is $f_4 = f(0.4, 1.2797) = 0.9498$

and, thus, the corrected value of y_4 from (22.24) is

$$y_4 = 1.12 + \frac{0.1}{3}[0.6552 + 4(0.7980) + 0.9498] = 1.2799$$

The further improved f_4 is $f_4 = f(0.04, 1.2799) = 0.9501$

and still corrected value of y_4 from (22.24) is

$$y_4 = 1.12 + \frac{0.1}{3}[0.6552 + 4(0.7980) + 0.9501] = 1.2799$$

which is the same as obtained in the preceding step. Hence, $y(0.4) = 1.2799$, correct to four places of decimals.

Example 22.14: Apply Milne's method to solve $y' = x - y^2$ in the interval [0,1], when $y = 0$ at $x = 0$, taking $h = 0.2$.

Solution: To determine the starting values, we use Picard's method. We have

$$y = y(0) + \int_0^x f(x, y)dx, \text{ where } f(x, y) = x - y^2.$$

First approximation: Put $y = 0$ in $f(x, y)$, this gives $y_1 = 0 + \int_0^x x\,dx = x^2/2.$

Second approximation: Put $y = x^2/2$ in $f(x, y)$, this gives $y_2 = \int_0^x \left(x - \frac{x^4}{4}\right)dx = \frac{x^2}{2} - \frac{x^5}{20}.$

Third approximation:
$$y_3 = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right]dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400}. \quad ...(22.25)$$

Thus, the starting values for the Milne's method obtained from (22.25), for $h = 0.2$, are given as follows:

$$\begin{aligned}
x_0 &= 0.0, & y_0 &= 0.0000, & f_0 &= 0.0000\\
x_1 &= 0.2, & y_1 &= 0.0200, & f_1 &= 0.1996\\
x_2 &= 0.4, & y_2 &= 0.0795, & f_2 &= 0.3937\\
x_3 &= 0.6, & y_3 &= 0.1762 & f_3 &= 0.5689
\end{aligned}$$

Using Milne's predictor formula

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) = 0.0000 + \frac{4(0.2)}{3}[2(0.1996) - 0.3937 + 2(0.5689)] = 0.3049.$$

This gives, $\quad f_4 = f(0.8, 0.3049) = 0.7070$

Using the corrector formula

$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7070] = 0.3046$, and this gives $f_4 = 0.7072$.

Again using the *corrector*, we obtain $y_4 = 0.3046$, which is same as the preceding one.

Next, using the *predictor*, $y_5 = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$, we obtain, $y_5 = 0.4554$ and this further gives, $f_5 = 0.7926$.

Next, the *corrector* $y_5 = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5)$, gives $y_5 = 0.4555$, and this further gives, $f_5 = 0.7925$.

Again using the *corrector*, we obtain $y_5 = 0.4555$, a value which is the same as the preceding one.

Hence, $y(1) = 0.4555$.

Example 22.15: Apply Milne's method to find a solution of the differential equation $y' = x + y$, $y(0) = 1$, in the interval $[0, 0.4]$, with $h = 0.1$. Use Runge-Kutta method of fourth order to find the starting values.

Solution: Applying Runge–Kutta method of fourth order, refer to Example 22.12, the starting values are: $y(0) = 1.0000$, $y(0.1) = 1.1103$, $y(0.2) = 1.2428$, and $y(0.3) = 1.3997$.

Thus, for $f(x, y) = x + y$, we have

$$\begin{aligned} x_0 &= 0.0, & y_0 &= 1.0000, & f_0 &= 1.0000 \\ x_1 &= 0.1, & y_1 &= 1.1103, & f_1 &= 1.2103 \\ x_2 &= 0.2, & y_2 &= 1.2428, & f_2 &= 1.4428 \\ x_3 &= 0.3, & y_3 &= 1.3997, & f_3 &= 1.6997 \end{aligned}$$

Using Milne's predictor formula

$$y_4 = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3] = 1 + \frac{4(0.1)}{3}[2(1.2103) - 1.4428 + 2(1.6997)] = 1.5836.$$

This gives, $f_4 = f(0.4, 1.5836) = 1.9836$.

Next, using corrector formula

$$y_4 = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4] = 1.2428 + \frac{0.1}{3}[1.4428 + 4(1.6997) + 1.9836] = 1.5836,$$

which is the same as the preceding one and hence $y(0.4) = 1.5836$.

22.3.2 Adams-Bashforth Method

In this method, we use the fact that the solution to the given initial value problem at the four equispaced past points, say $x_0 - 3h$, $x_0 - 2h$, $x_0 - h$ and x_0 is available and utilize this information to compute y at $x = x_1 = x_0 + h$. We calculate

$$f_{-1} = f(x_0 - h, y_{-1}), \quad f_{-2} = f(x_0 - 2h, y_{-2}), \quad f_{-3} = f(x_0 - 3h, y_{-3}).$$

To find y_1, we consider

$$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y)\,dx \qquad ...(22.26)$$

and substitute for $f(x, y)$ in (22.26) using the Newton's backward formula in the form

$$f(x, y) = f_0 + k\nabla f_0 + \frac{k(k+1)}{2!}\nabla^2 f_0 + \frac{k(k+1)(k+2)}{3!}\nabla^3 f_0 + ...$$

Thus, (22.26) becomes

$$y_1 = y_0 + \int_{x_0}^{x_0+h}\left[f_0 + k\nabla f_0 + \frac{k(k+1)}{2!}\nabla^2 f_0 + \frac{k(k+1)(k+2)}{3!}\nabla^3 f_0 + ...\right]dx$$

$$y_1 = y_0 + h\int_0^1\left[f_0 + k\nabla f_0 + \frac{k(k+1)}{2}\nabla^2 f_0 + \frac{k(k+1)(k+2)}{3!}\nabla^3 f_0 + ...\right]dk;\ x_0 = x + kh$$

$$= y_0 + h\left[f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^3 f_0 + ...\right]$$

$$= y_0 + \frac{h}{24}[55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}]; \qquad ...(22.27)$$

neglecting the fourth and higher order differences and simplifying.

The formula (22.27) is called *Adams–Bashforth predictor formula.*

To improve y further, we consider Newton's backward formula at f_1, that is,

$$f(x, y) = f_1 + k\nabla f_1 + \frac{k(k+1)}{2!}\nabla^2 f_1 + \frac{k(k+1)(k+2)}{3!}\nabla^3 f_1 + ...$$

and substitute in (22.26), and then using $x = x_1 + kh$, we obtain

$$y_1 = y_0 + h\int_{-1}^{0}\left(f_1 + k\nabla f_1 + \frac{k(k+1)}{2!}\nabla^2 f_1 + \frac{k(k+1)(k+2)}{3!}\nabla^3 f_0 + ...\right)dk$$

$$= y_0 + h\left[f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_1 - \frac{1}{24}\nabla^3 f_1 + ...\right]$$

$$= y_0 + \frac{h}{24}[9f_1 + 19f_0 - 5f_{-1} + f_{-2}]; \qquad ...(22.28)$$

neglecting the fourth and higher order differences and simplifying.

The formula (22.28) is called *Adams–Bashforth corrector formula*.

From the value of y_1 obtained from (22.28) an improved value of f_1 is calculated and again applied to corrector to find a still improved value of y_1 and the process is repeated till y_1 remains unchanged upto the desired number of decimal places and then we proceed to calculate y_2, etc. on the same lines.

In general, Adams-Bashforth Method, when starting values are computed using Runge-Kutta method of fourth order, has been found to be the most efficient one.

Example 22.16: Solve using Adams-Bashforth method $\frac{dy}{dx} = (x + y)$ on the interval $0 \le x \le 1$, choosing $h = 0.2$ given that $y(0) = 0.0000$, $y(0.2) = 0.0214$, $y(0.4) = 0.0918$ and $y(0.6) = 0.2221$.

Solution: Here $f(x, y) = x + y$, thus the starting values of the Adams-Bashforth method with $h = 0.2$ are given by

$$\begin{array}{lll} x = 0.0 & y_{-3} = 0.0000 & f_{-3} = 0.0000 \\ x = 0.2 & y_{-2} = 0.0214 & f_{-2} = 0.2214 \\ x = 0.4 & y_{-1} = 0.0918 & f_{-1} = 0.4918 \\ x = 0.6 & y_0 = 0.2221 & f_0 = 0.8221 \end{array}$$

Using the *predictor*,

$$y_1 = y_0 + \frac{h}{24}[55f_0 - 59f_{-1} + 37f_{-2} + 9f_{-3}]$$

$$= 0.2221 + \frac{0.2}{24}[55(0.8221) - 59(0.4918) + 37(0.2214) + 9(0.0000)]$$

$$= 0.4253$$

Thus, $x = 0.8$, $y_1 = 0.4253$, $f_1 = 1.2253$

Using the *corrector*,

$$y_1 = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$= 0.2221 + \frac{0.2}{24}[9(1.2253) + 19(0.8221) - 5(0.4918) + (0.2214)] = 0.4255$$

This gives improved $f_1 = 0.8 + 0.4255 = 1.2255$.

Using the corrector again, we obtain

$$y_1 = 0.2221 + \frac{0.2}{24}[9(1.2255) + 19(0.8221) - 5(0.4918) + (0.2214)] = 0.4255,$$

which is the same as the preceding one. Hence, $y(0.8) = 0.4255$. Proceding on the same lines and using $y(0.2)$, $y(0.4)$, $y(0.6)$, $y(0.8)$, we find that $y(1.0) = 0.7182$.

Example 22.17: Given $f(x, y) = x - y^2$, $y(0) = 1$ find $y(0.4)$ by Adams-Bashforth method, starting solutions are required to be obtained using Runge-Kutta method of order 4 using $h = 0.1$.

Solution: We have, $f(x, y) = x - y^2$.

To find $y(0.1)$: $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$$k_1 = hf(x_0, y_0) = (0.1)\,f(0, 1) = -0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)\,f(0.05, 0.95) = -0.08525$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)\,f(0.05, 0.9574) = -0.0867$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\,f(0.1, 0.9137) = -0.07341$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0883$$

Thus, $y(0.1) = y_1 = y_0 + k = 1 - 0.0883 = 0.9117$

To find $y(0.2)$: $x_1 = 0.1, y_1 = 0.9117, h = 0.1$

$$k_1 = hf(x_1, y_1) = (0.1)\,f(0.1, 0.9117) = -0.0731$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1)\,f(0.15, 0.8751) = -0.0616$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1)\,f(0.15, 0.8809) = -0.0626$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)\,f(0.2, 0.8491) = -0.0521$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0623$$

Thus, $y(0.2) = y_2 = y_1 + k = 0.9117 - 0.0623 = 0.8494.$

To find $y(0.3)$: $x_2 = 0.2, y_2 = 0.8449, y = 0.1$

$$k_1 = hf(x_2, y_2) = (0.1)\,f(0.2, 0.8494) = -0.0521$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1)\,f(0.25, 0.08233) = -0.0428$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)\,f(0.25, 0.828) = -0.0436$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)\,f(0.3, 0.88058) = -0.0349$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0438$$

Thus, $y(0.3) = y_3 = y_2 + k = 0.8449 - 0.04338 = 0.8061$

Thus, the starting values for Adams Bashforth methods with $h = 0.1$ and $f(x, y) = x - y^2$ are:

$$\begin{array}{lll} x = 0.0, & y_{-3} = 1.0000, & f_{-3} = -1.0000, \\ x = 0.1, & y_{-2} = 0.9117, & f_{-2} = -0.7312, \\ x = 0.2, & y_{-1} = 0.8494, & f_{-1} = -0.5215, \\ x = 0.3, & y_0 = 0.8061, & f_0 = -0.3498, \end{array}$$

Using the *predictor*,

$$y_1 = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$= 0.8061 + \frac{0.1}{24}[55(-0.3498) - 59(-0.5215) + 37(-0.7312) - 9(-1) = 0.7789$$

Thus, $x = 0.4,\ \ y_1 = 0.7789,\ \ f_1 = -0.2067$

Using the *corrector,*

$$y_1 = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$= 0.8061 + \frac{0.1}{24}[9(-0.2067) + 19(0.3498) - 5(-0.5215) - 0.73121] = 0.7785$$

This gives improved $f_1 = 0.4 - (0.7785)^2 = -0.2061$

Using the corrector again

$$y_1 = 0.8061 + \frac{0.1}{24}[9(-0.2061) + 19(-0.3498) - 5(-0.5215) - 0.7312] = 0.7785,$$

the same as the preceding one, hence $y(0.4) = 0.7785$.

EXERCISE 22.4

1. Given $y' = (x + y)y$, $y(0) = 1$, $y(0.1) = 1.3591$, $y(0.2) = 1.8869$ and $y(0.3) = 2.7132$, find the solution at $x = 0.4$ using Milne's method.
2. Solve the initial value problem $y' = \frac{1}{2}(x + y)$, $y(0) = 2$ over the interval [0, 2] by taking $h = 0.5$, using Milne's method.
3. Use Milne's method to find $y(0.3)$ from $y' = x^2 + y^2$, $y(0) = 1$, find $y(-0.1)$, $y(0.1)$ and $y(0.2)$ using Taylor's series method.
4. Solve $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$ over the interval [0, 0.5] using Milne's method by obtaining the starting values using Runge-Kutta method.
5. Using Adams-Bashforth method, evaluate $y(1.4)$, when $y' = (1 - xy)/x^2$ and $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, and $y(1.3) = 0.972$.
6. Given $y' = 2y/x$, $y(1) = 2$. Estimate $y(2)$ using Adams-Bashforth method by taking $h = 0.25$. Derive the starting values using R-K fourth-order method.

22.4 METHODS FOR SIMULTANEOUS AND HIGHER ORDER DIFFERENTIAL EQUATIONS

22.4.1 Methods For Simultaneous Differential Equations

The simultaneous differential equation of the form

$$\frac{dx}{dt} = f(t, x, y) \text{ and, } \frac{dy}{dt} = \phi(t, x, y)$$

with initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$ can be solved numerically by any of the methods discussed earlier, specially by Picard's, Taylor's series or Runge-Kutta method, as explained below.

1. Picard's method. This gives

$$x_1 = x_0 + \int f(t, x_0, y_0)dt, \quad y_1 = y_0 + \int \phi(t, x_0, y_0)dt$$

$$x_2 = x_0 + \int f(t, x_1, y_1)dt, \quad y_2 = y_0 + \int \phi(t, x_1, y_1)dt$$

and so on for x_3, y_3, etc.

2. Taylor's series method. If h is the step-size and $x_1 = x(t_0 + h)$, $y_1 = y(t_0 + h)$, then using Taylor's series algorithm, we obtain

$$x_1 = x_0 + hx_0' + \frac{h^2}{2!}x_0'' + \frac{h^3}{3!}x_0''' + \ldots \qquad \ldots(22.29)$$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots \qquad \ldots(22.30)$$

From the given differential equation, we get x'', y'', x''', y''', etc. and so the values x_0', x_0'', x_0''', etc. and y_0', y_0'', y_0''', etc. can be calculated. Substituting these in (22.29) and (22.30), we obtain x_1 and y_1 and then we proceed to evaluate x_2 and y_2, etc. step by step.

3. Runge-Kutta fourth-order method. If h, k, l, are the step-size for t, x and y respectively, then this method gives

$$k_1 = hf(t_0, x_0, y_0) \qquad l_1 = h\phi(t_0, x_0, y_0)$$

$$k_2 = hf\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_1, y_0 + \frac{1}{2}l_1\right) \qquad l_2 = h\phi\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_1, y_0 + \frac{1}{2}l_1\right)$$

$$k_3 = hf\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_2, y_0 + \frac{1}{2}l_2\right) \qquad l_3 = h\phi\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_2, y_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf(t_0 + h, x_0 + k_3, y_0 + l_3) \qquad l_4 = h\phi(t_0 + h, x_0 + k_3, y_0 + l_3)$$

Thus, $x_1 = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, and $y_1 = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

The next approximations x_2, y_2 can be computed on the similar lines from the above formulae simply replacing t_0, x_0, y_0 by t_1, x_1, y_1 respectively, and so on.

22.4.2 Method For Higher Order Differential Equations

Consider the second-order differential equation of the form

$$y'' = f(x, y, y') \qquad \ldots(22.31)$$

with initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$.

Set $z = y'$, the above problem can be reduced to the problem of solving the system of first-order differential equations

$$y' = z,\ z' = f(x, y, z) \qquad \text{...(22.32)}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = y'_0$

This can be solved by the methods described in case of simultaneous differential equations. Similarly, any higher order differential equation can be reduced to a system of first-order differential equations and can be solved accordingly.

Example 22.18: Using Picard's method find approximate values of x and y corresponding to $t = 0.1$, given that $x' = ty + 1$, $y' = -tx$ and $x(0) = 0$, $y(0) = 1$.

Solution: Here $t_0 = 0$, $x_0 = 0$, $y_0 = 1$, $x' = ty + 1 = f(t, x, y)$ and $y' = -tx = \phi(t, x, y)$.

Thus, $\quad x = x_0 + \int_{t_0}^{t} f(t, x, y)dt$ and $y = y_0 + \int_{t_0}^{t} \phi(t, x, y)dt$

This gives,

$$x_1 = x_0 + \int_{t_0}^{t} f\phi(t, x_0, y_0)dt = 0 + \int_0^t (t+1)dt = \frac{t^2}{2} + t$$

$$y_1 = y_0 + \int_{t_0}^{t} f\,\phi(t, x_0, y_0)dt = 1 + \int_0^t 0.dt = 1$$

$$x_2 = x_0 + \int_{t_0}^{t} f(t, x_1, y_1)dt = 0 + \int_0^t (t+1)dt = \frac{t^2}{2} + t$$

$$y_2 = y_0 + \int_{t_0}^{t} \phi(t, x_1, y_1)dt = 1 + \int_0^t -t\left(\frac{t^2}{2} + t\right)dt = 1 - \frac{t^2}{2} - \frac{t^3}{6}$$

$$x_3 = x_0 + \int_{t_0}^{t} f(t, x_2, y_2)dt = 0 + \int_0^t (t+1)dt = \frac{t^2}{2} + t$$

$$y_3 = y_0 + \int_{t_0}^{t} \phi(t, x_2, y_2)dt = 1 + \int_0^t -t\left(1 - \frac{t^2}{2} - \frac{t^3}{6}\right)dt = 1 - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} \text{ and so on.}$$

Thus, when $t = 0.1$, $x(0.1) = 0.105$ and $y(0.1) = 0.99517$.

Example 22.19: Solve the differential equations $\dfrac{dy}{dx} = 1 + xz$, $\dfrac{dz}{dx} = -xy$, for $x = 0.3$, using fourth-order Runge-Kutta method in one step when the initial values are $x = 0$, $y = 0$, $z = 1$.

Solution: Here $f(x, y, z) = 1 + xz$, $\phi(x, y, z) = -xy$, $x_0 = 0$, $y_0 = 0$, $z_0 = 1$ and $h = 0.3$.

Therefore, R-K formulae become

$$\therefore\ k_1 = hf(x_0, y_0, z_0) = 0.3f(0, 0, 1) = 0.3(1 + 0) = 0.3 \qquad l_1 = h\phi(x_0, y_0, z_0) = 0.3[-(0)(0)] = 0$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \qquad l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= (0.3)\,f(0.15, 0.15, 1) = 0.3(1 + 0.15) = 0.345 \qquad = 0.3\,[-(0.15)\,(0.15)] = -0.00675.$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= (0.3)\,f(0.15, 0.1725, 0.996625)$$
$$= 0.3[1 + (0.996625)(0.15)] = 0.34485$$

$$l_3 = h\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.3[-(0.15)(0.1725)] = -0.007762$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= (0.3)\,f(0.3, 0.34485, 0.99224) = 0.3893$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.3[-(0.3)(0.34485)] = -0.03104$$

Hence, $y(0.3) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0 + \frac{1}{6}[0.3 + 2(0.345) + 2(0.34485) + 0.3893)] = 0.34483$

$$z(0.3) = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4] = 1 + \frac{1}{6}[0 + 2(-0.00675) + 2(-0.0077625) + (-0.03104)] = 0.98999$$

Example 22.20: Use Runge-Kutta method to solve $y'' = xy'^2 - y^2$, correct to four decimal places, subject to the initial conditions $x = 0, y = 1, y' = 0$.

Solution: The given equation is $y'' = xy'^2 - y^2$.

Let $\frac{dy}{dx} = z$, so that $\frac{dz}{dx} = xz^2 - y^2$.

We have, $f(x, y, z) = z,\ \phi(x, y, z) = xz^2 - y^2,\ x_0 = 0,\ y_0 = 1,\ z_0 = 0$ and $h = 0.2$

Therefore, Runge-Kutta formulae become

$$k_1 = hf(x_0, y_0, z_0) = 0.2(0) = 0, \qquad l_1 = h\phi(x_0, y_0, z_0) = 0.2(-1) = -0.2$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) = -0.02, \qquad l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) = -0.1998$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) = -0.02, \qquad l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) = -0.1958$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = -0.0392, \qquad l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3) = -0.1905$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0199, \qquad l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = -0.1970$$

Hence, at $x = 0.2$, we obtain

$$y = y_0 + k = 1 - 0.0199 = 0.9801,\ y' = z = z_0 + l = 0 - 0.1970 = -0.1970$$

as the required solution.

Example 22.21: Use Picard's method to obtain the second approximation to the solution of the initial value problem $\frac{d^2y}{dx^2} = x^3\frac{dy}{dx} + x^3y,\quad y(0) = 1.0,\ y'(0) = \frac{1}{2}$.

Solution: The given equation is $\dfrac{d^2y}{dx^2} = x^3\dfrac{dy}{dx} + x^3y$

Let $\dfrac{dy}{dx} = z = f(x, y, z)$ so that $\dfrac{dz}{dx} = x^3z + x^3y = x^3(y + z) = \phi(x, y, z)$, say.

We have, $x_0 = 0$, $y_0 = 1$, and $z_0 = 1/2$, thus

$$y = y_0 + \int_{x_0}^{x} f(x, y, z)dx \quad \text{and} \quad z = z_0 + \int_{x_0}^{x} \phi(x, y, z)dx$$

First approximations:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)dx = 1 + \int_0^1 (1/2)dx = 1 + \frac{x}{2}$$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)dx = \frac{1}{2} + \int_0^1 x^3\left(1 + \frac{1}{2}\right)dx = \frac{1}{2} + \frac{3}{2}\frac{x^4}{4} = \frac{1}{2} + \frac{3}{8}x^4$$

Second approximations:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3}{8}x^4\right)dx = 1 + \left|\frac{x}{2} + \frac{3}{40}x^5\right|_0^x = 1 + \frac{x}{2} + \frac{3}{40}x^5$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)dx = \frac{1}{2} + \int_0^x x^3\left(1 + \frac{x}{2} + \frac{1}{2} + \frac{3}{8}x^4\right)dx = \frac{1}{2} + \int_0^x \left(\frac{3}{2}x^3 + \frac{x^4}{2} + \frac{3}{8}x^7\right)dx$$

$$= \frac{1}{2} + \frac{3}{2}\frac{x^4}{4} + \frac{x^5}{10} + \frac{3}{8}\cdot\frac{x^8}{8} = \frac{1}{2} + \frac{3}{8}x^4 + \frac{x^5}{10} + \frac{3}{64}x^8$$

Hence, the required second approximation to the solution of given equation is

$$y_2 = 1 + \frac{x}{2} + \frac{3}{40}x^5.$$

EXERCISE 22.5

1. Use Runge-Kutta method of second-order to compute values of $x(0.1)$ and $y(0.1)$ when $x' = x + y$, $y' = -xy$, given that, $x(0) = 0$ and $y(0) = 1$.
2. Use Picard method to find approximate values of x and y corresponding to $t = 0.1$, given that $x(0) = 2$, $y(0) = 1$ and $x' = t + y$, $y' = t - x^2$.
3. Apply Runge-Kutta method of order 4 to find approximate values of $x(0.1)$, $x(0.2)$ and $y(0.1)$, $y(0.2)$ as solution of pair of equations, $x' = t + y$, $y' = x - t$ with $x(0) = 1$, $y(0) = -1$, taking $h = 0.1$.

4. Use Runge-Kutta method of second-order to approximate $y(1.1)$ and $y'(1.1)$ for the equation

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 3y = x^2 + 2 \text{ with } y(1) = 1, y'(1) = 2, \text{ taking } h = 0.1.$$

5. The angular displacement θ of a simple pendulum is given by the equation $\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$, where $l = 98$ cm and $g = 980$ cm/sec^2. If $\theta = 0$ and $\frac{d\theta}{dt} = 4.472$ at $t = 0$, use Runge-Kutta method of order 4 to find θ and $\frac{d\theta}{dt}$ when $t = 0.2$ sec.

6. Compute $y(1.4)$ and $y'(1.4)$ for the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = -2x + 3$, using Adams-Bashforth method with $h = 0.1$, given that

x:	1	1.1	1.2	1.3
y:	0.95	2.0	3.0	4.8
y':	6.5	11.0	14.2	18.3

22.5 METHODS FOR BOUNDARY VALUE PROBLEMS

We shall consider simple two-point linear boundary value problems of the following two forms

1. $y''(x) + p(x)\,y'(x) + q(x)\,y(x) = r(x),\ x_0 \le x \le x_n$ ...(22.33)

   subject to $y(x_0) = a,\ y(x_n) = b.$

2. $y^{(iv)}(x) + p(x)\,y(x) = r(x),\ x_0 \le x \le x_n$ ...(22.34)

   subject to $y(x_0) = y'(x_0) = a,\ y(x_n) = y'(x_n) = b.$

There exist many numerical methods like *shooting method*, *finite-difference method* and *finite element method* for solving the boundary value problems. However, we shall discuss here only the finite difference method.

22.5.1 Finite-Difference Method

This method consists of replacing the derivatives occurring in the boundary value problem by means of their finite-difference approximations and solving the resulting linear system of equations by a standard procedure. To obtain the finite-difference approximations to the derivatives, we proceed as follows:

If $y(x)$ has single-valued continuous derivatives then by Taylor's series

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \ldots, \qquad \ldots(22.35)$$

which gives $\frac{y(x+h)-y(x)}{h} = y'(x) + \frac{h}{2}y''(x) + \frac{h^2}{6}y'''(x) + \dots$ Thus, we have

$$y'(x) = \frac{1}{h}[y(x+h) - y(x)] + o(h) \qquad ...(22.36)$$

as the *forward difference approximation to* $y'(x)$ with an error of order h.

Similarly, the Taylor's series expansion of $y(x-h)$,

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \dots \qquad ...(22.37)$$

gives

$$y'(x) = \frac{1}{h}[y(x) - y(x-h)] + o(h) \qquad ...(22.38)$$

as the *backward difference approximation to* $y'(x)$ with an error of the order h.

Subtracting (22.37) from (22.35), we obtain

$$y'(x) = \frac{1}{2h}[y(x+h) - y(x-h)] + o(h^2) \qquad ...(22.39)$$

the *central difference approximation to* $y'(x)$ with an error of the order h^2. This approximation is preferred over the forward or backward difference approximations.

Next, adding (22.35) and (22.37), we obtain

$$y''(x) = \frac{1}{h^2}[y(x+h) - 2y(x) + y(x-h)] + 0(h^2) \qquad ...(22.40)$$

the *central difference approximation to* $y''(x)$, and similarly we can derive finite-difference approximations to higher order derivatives.

To solve the boundary value problem over the interval $R = [x_0, x_n]$, we divide R into n equal subintervals of width h by the nodal points $x_i = x_0 + ih$, $i = 0, 1, 2, \dots, n$, and denote $y_i = y(x_i) = y(x_0 + ih)$, $i = 0, 1, 2, \dots, n$. Hence, we obtain

$$y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1}) \qquad ...(22.41)$$

$$y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \qquad ...(22.42)$$

$$y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} - y_{i-2}) \qquad ...(22.43)$$

$$y_i^{(iv)} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) \qquad ...(22.44)$$

and similar results in case of higher order derivatives. The accuracy is improved as the width h decreases.

Example 22.22: Solve the boundary value problem $y'' + y + 1 = 0$, $y(0) = 0$ and $y(1) = 0$, using the method of finite differences by taking the width $h = 1/4$.

Solution: Since $h = 1/4$, thus $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$, $x_4 = 1$, and $n = 4$. The given differential equation is approximated as $\frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + y_i + 1 = 0$. This gives,

$$y_{i+1} - \frac{31}{16}y_i + y_{i-1} = -\frac{1}{16}, \quad i = 1, 2, 3$$

which together with the boundary conditions $y_0 = 0$ and $y_4 = 0$ comprise a system of five linear equations in y_i's, $i = 0, 1, 2, 3, 4$ given by

$$y_2 - \frac{31}{16}y_1 + y_0 = -\frac{1}{16}, y_3 - \frac{31}{16}y_2 + y_1 = -\frac{1}{16}, y_4 - \frac{31}{16}y_3 + y_2 = -\frac{1}{16} \text{ and, } y_0 = y_4 = 0.$$

Solving for y_1, y_2, y_3 it gives the solution at the nodal points $x_1 = 1/4$, $x_2 = 1/2$ and $x_3 = 3/4$, as $y_1 = 0.1047$, $y_2 = 0.1403$, and $y_3 = 0.1047$ respectively.

Example 22.23: Solve the boundary value problem $y'' = y + x$, $y(0) = 1$ and $y(1) = 0$ using the method of finite differences with $h = 1/4$.

Solution: Since $h = 1/4$, thus, $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$, $x_4 = 1$ and $n = 4$.

The given differential equation is approximated as

$$\frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) = y_i + x_i, i = 1, 2, 3.$$

This gives for $i = 1, 2, 3$ respectively

$$-16y_0 + 33y_1 - 16y_2 = -0.25$$
$$-16y_1 + 33y_2 - 16y_3 = -0.50$$
$$-16y_2 + 33y_3 - 16y_4 = -0.75$$

Using the boundary conditions $y_0 = 0$ and $y_4 = 0$, we get the system of equations

$$33y_1 - 16y_2 = -0.25$$
$$-16y_1 + 33y_2 - 16y_3 = -0.50$$
$$-16y_2 + 33y_3 = -0.75.$$

Solving for y_1, y_2, y_3, it gives the solution at the nodal points $x_1 = 1/4$, $x_2 = 1/4$ and $x_3 = 3/4$ as

$$y_1 = -0.03488, y_2 = -0.05633, \text{ and } y_3 = -0.05004.$$

Example 22.24: The deflection of a beam is governed by the equation

$$\frac{d^4y}{dx^4} + 81y = 243x, \quad y(0) = y'(0) = y''(1) = y'''(1) = 0.$$

Evaluate the deflection at the pivot points of the beam using $h = 1/3$.

Solution: Since $h = 1/3$, thus the pivot points are $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$, and let $y_0 = 0$, y_1, y_2 and y_3 be the corresponding values.

The given differential equation can be approximated to

$$\frac{1}{h^4}[y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}] + 81y_i = 243x_i,\ i = 1, 2, 3.$$

This gives,

$$\left.\begin{aligned} y_3 - 4y_2 + 7y_1 - y_0 + y_{-1} &= 1 \\ y_4 - 7y_3 + 7y_2 - 4y_1 + y_0 &= 2 \\ y_5 - 4y_4 + 7y_3 - 4y_2 + y_1 &= 3 \end{aligned}\right\} \qquad \text{...(22.45)}$$

where $y_0 = 0$.

Also, $y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1})$. For $i = 0$, it gives

$$0 = y_0' = \frac{1}{2h}(y_1 - y_{-1}), \text{ or } y_{-1} = y_1 \qquad \text{...(22.46)}$$

Further, $y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$. For $i = 3$, it gives

$$0 = y_3'' = \frac{1}{h^2}(y_4 - 2y_3 + y_2), \text{ or } y_4 = 2y_3 - y_2 \qquad \text{...(22.47)}$$

Again $y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2})$. For $i = 3$, it gives

$$0 = y_3''' = \frac{1}{2h^3}(y_5 - 2y_4 + 2y_2 - y_1), \text{ or } y_5 = 2y_4 - 2y_2 + y_1 \qquad \text{...(22.48)}$$

Using (22.46), (22.47) and (22.48) in (22.45), we obtain

$$\begin{aligned} y_3 - 4y_2 + 8y_1 &= 1 \\ -y_3 + 3y_2 - 2y_1 &= 1 \\ 3y_3 - 4y_2 + 2y_1 &= 3 \end{aligned}$$

Solving these for y_1, y_2, y_3, we get the solution at the nodal points $x_1 = 1/3$, $x_2 = 2/3$, and $x_3 = 1$ respectively as $y_1 = 0.6154$, $y_2 = 1.6923$, and $y_3 = 2.8462$.

EXERCISE 22.6

1. Solve the boundary value problem, $y'' = 64y + 10 = 0$ with $y(0) = y(1) = 0$ by taking $h = 1/4$. Compute $y(0.5)$ and compare it with the true value.
2. Solve the boundary value problem, $y'' = xy$, $y(0) + y'(0) = 1$, $y(1) = 1$, with $h = 1/3$.
3. Solve the boundary value problem, $y'' + 8(\sin^2 \pi x)y = 0$, $y(0) = y(1) = 1$, with $h = 1/4$.

4. Solve the boundary value problem, $y'' + xy' + y = 3x^2 + 2$, $y(0) = y(1) = 0$, with $h = 1/4$.
5. Solve the boundary value problem $y^{iv} + 81y = 729x^2$, $y(0) = y'(0) = y''(1) = y'''(1) = 0$, with $h = 1/3$.

22.6 METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

Many physical phenomena in engineering and science when formulated into mathematical models results in partial differential equations (PDE). We have already studied PDE and various analytical treatments for solving these equations in Chapters 16 & 17. The most of the problems of practical interest can be formulated as second-order partial differential equations which can be expressed in the form

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = F(x, y), \qquad ...(22.49)$$

where u represents some function of the independent variables x and y, and a, b, c, d, e, f may be constants or functions of x and y.

The Eq.(22.49) is classified as *elliptic, parabolic* or *hyperbolic* respectively, when $b^2 - 4ac < 0$, $b^2 - 4ac = 0$, or $b^2 - 4ac > 0$.

The analytical methods for the solution of PDEs sometimes become too complex, while it is generally easier to obtain suitable approximate solution by numerical methods. There are basically two numerical techniques, namely *finite-difference methods* and *finite-element methods* that can be used to solve PDEs, however we shall consider here only the finite-difference methods. Further, we shall consider, in particular, the numerical methods for the solution of the *Laplace, Poisson, heat* and *wave equations* which are of vital importance from applications point of view and are also modal examples of elliptic, parabolic and hyperbolic type of partial differential equations. Before proceeding further for the actual methods, first we find the finite-difference approximations to various derivatives appearing in these equations.

22.6.1 Finite-difference Approximations to Derivatives

Let the domain R in the x–y plane be divided into rectangular grids of width $\Delta x = h$ and height $\Delta y = k$ as shown in Fig. 22.2. The points of intersection of grid lines are called *mesh points,* or *nodal points,* or *grid points.*

The finite-difference approximations for the partial derivatives w.r.t. x, refer Section 22.5.1, are:

$$\frac{\partial u}{\partial x} = \frac{u(x+h,y) - u(x,y)}{h} + o(h)$$

$$= \frac{u(x,y) - u(x-h,y)}{h} + o(h)$$

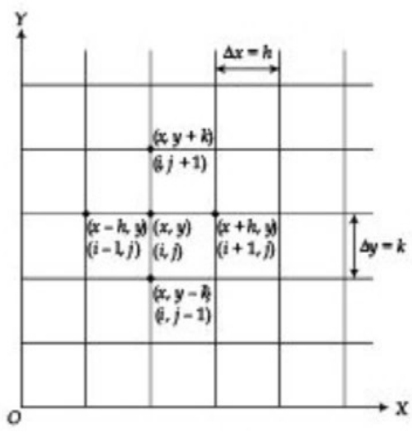


Fig. 22.2

$$= \frac{u(x+h,y) - u(x-h,y)}{2h} + o(h^2)$$

and,
$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x-h,y) - 2u(x,y) + u(x+h,y)}{h^2} + o(h^2)$$

Writing $u(x, y) = u(ih, jk) = u_{i,j}$, the above approximations can be expressed as

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{h} + o(h) \quad ...(22.50)$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} + o(h) \quad ...(22.51)$$

$$= \frac{u_{i+1,j} - u_{i-1,j}}{2h} + o(h^2) \quad ...(22.52)$$

and,
$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + o(h^2) \quad ...(22.53)$$

Similarly, the approximations for the partial derivatives w.r.t. y are:

$$\frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j}}{k} + o(k) \quad ...(22.54)$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} + o(k) \quad ...(22.55)$$

$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k} + o(k^2) \quad ...(22.56)$$

and,
$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + o(k^2) \quad ...(22.57)$$

In the finite-difference method, we replace the partial derivatives that occur in the PDE by their finite-difference approximations and solve the system of equations to obtain the solution at the grid points.

22.7 SOLUTION OF LAPLACE EQUATION

The Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(22.58)$$

is one of the most important example of elliptic equations. We consider it in a bounded region R with boundary C on which the value of u is specified. We consider R to be a square region and divide this into a network of square mesh of width h. Replacing the partial derivatives in (22.58) by their respective difference approximations given by (22.53) and (22.57) and using $h = k$, we obtain

$$\frac{1}{h^2}[u_{i-1,j} - 2u_{ij} + u_{i+1,j}] + \frac{1}{h^2}[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = 0$$

or,
$$u_{ij} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}] \quad ...(22.59)$$

Thus, the value of $u_{i,j}$ at any interior grid point is the average of its value at the four neighbouring points as illustrated in Fig. 22.3.

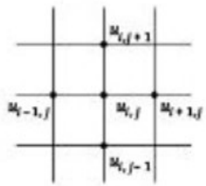


Fig. 22.3

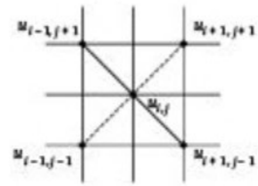


Fig. 22.4

The formula (22.59) is called the *standard five point formula.*

An alternative formula obtained from the right side of (22.59) using Taylor series is

$$u_{i,j} = \frac{1}{4}[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}] \quad ...(22.60)$$

which uses the grid values at the diagonal points as shown in Fig. 25.3. The formula (22.60) is called the *diagonal five-point formula.*

Although formula (22.60) is less accurate than the formula (22.59), yet it serves as a good approximation whenever found suitable. However, we always prefer to use standard five-point formula whenever feasible.

Next, let the values of $u(x, y)$ on the boundary C be given by b_i and let u_j denote the values at the interior grid points as shown in Fig. 22.5.

The approximate values of u at the interior grid points can now be computed as follows:

First using diagonal five-point formula (22.60), we compute u_5, u_7, u_9, u_1 and u_3, and then, using standard five-point formula (22.59), we compute u_8, u_4, u_6 and u_2. Thus, we have

$$u_5 = \frac{1}{4}[b_1 + b_9 + b_5 + b_{13}]$$

$$u_7 = \frac{1}{4}[b_{15} + b_{11} + u_5 + b_{13}]$$

$$u_9 = \frac{1}{4}[u_5 + b_9 + b_7 + b_{11}]$$

$$u_1 = \frac{1}{4}[b_1 + u_5 + b_3 + b_{15}]$$

$$u_3 = \frac{1}{4}[b_3 + b_7 + b_5 + u_5]$$

$$u_8 = \frac{1}{4}[u_5 + b_{11} + u_7 + u_9]$$

$$u_4 = \frac{1}{4}[u_1 + u_7 + b_{15} + u_5]$$

$$u_6 = \frac{1}{4}[u_3 + u_9 + u_5 + b_7]$$

$$u_2 = \frac{1}{4}[u_1 + u_3 + b_3 + u_5]$$

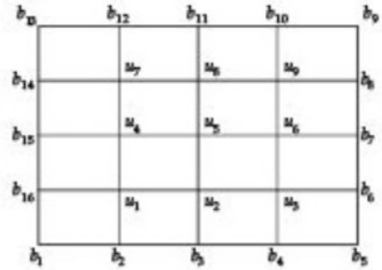


Fig. 22.5

When all the u_i ($i = 1, 2, ..., 9$) are computed, their accuracy can be improved either by applying the Jacobi's method using the iterative formula

$$u_{ij}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}] \quad ...(22.61)$$

or, by Gauss-Seidel method, using the iterative formula

$$u_{ij}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)}, u_{i,j+1}^{(n)}] \quad ...(22.62)$$

Here $u_{ij}^{(n)}$ denotes the nth iterative value of $u_{i,j}$.

It can be shown that Gauss-Seidel scheme converges twice as fast as the Jacobi scheme since it utilises the latest iterative values available.

This process is known as *Leibmann's iteration process.*

Example 22.25: Given the value of $u(x, y)$ on the boundary of the square region as shown in Fig. 22.6. Find the steady state temperature at the interior grid points using, (i) direct method, (ii) iterative process upto five iterations.

Solution:

(i) The steady state temperature distribution $u(x, y)$ is given by the Laplace's equation $\nabla^2 u = 0$. Using the standard five-point formula, we have

$$u_1 = \frac{u_2 + u_3 + 200}{4}, \quad \text{or} \quad -4u_1 + u_2 + u_3 + 0 = -200$$

$$u_2 = \frac{u_1 + u_4 + 100}{4}, \quad \text{or} \quad u_1 - 4u_2 + 0 + u_4 = -100$$

$$u_3 = \frac{u_1 + u_4 + 100}{4}, \quad \text{or} \quad u_1 + 0 - 4u_3 + u_4 = -100$$

$$u_4 = \frac{u_2 + u_3}{4}, \quad \text{or} \quad 0 + u_2 + u_3 - 4u_4 = 0.$$

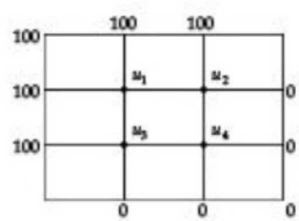


Fig. 22.6

Because of the symmetry in the temperature distribution, we have $u_2 = u_3$, and thus the system of equations become $-4u_1 + 2u_2 = -200$, $u_1 - 4u_2 + u_4 = -100$, $u_2 - 2u_4 = 0$.

Solving for u_1, u_2 and u_4, we obtain $u_1 = 75$, $u_2 = u_3 = 50$, $u_4 = 25$.

(ii) To get the appropriate initial solution, assuming $u_4 = 0$, we have

$$u_1 = \frac{1}{4}(100 + 100 + 100 + 0) = 75 \qquad \text{(diagonal formula)}$$

$$u_2 = \frac{1}{4}(75 + 100 + 0 + 0) = 43.75 \qquad \text{(standard formula)}$$

$$u_3 = \frac{1}{4}(100 + 75 + 0 + 0) = 43.75 \qquad \text{(standard formula)}$$

$$u_4 = \frac{1}{4}(43.75 + 43.75 + 0 + 0) = 21.88 \qquad \text{(standard formula)}$$

We note that u_2, u_3 and u_4 are being computed using the latest values on the right-hand side. Next, we carry out the successive iterations using the formulae obtain from the equations derived in the direct method.

$$u_1^{(n+1)} = \frac{1}{4}[u_2^{(n)} + u_3^{(n)} + 200]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + u_4^{(n)} + 100]$$

$$u_3^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + u_4^{(n)} + 100]$$

$$u_4^{(n+1)} = \frac{1}{4}[u_2^{(n+1)} + u_3^{(n+1)}]$$

Performing successive iterations, taking initial solution corresponding to $n = 0$, we obtain the values as shown in the table given below.

u_i	*Initial solution*	*Iterations*				
		1	2	3	4	5
u_1	75.00	71.88	74.22	74.81	74.95	74.99
u_2	43.75	48.44	49.61	49.90	49.98	49.99
u_3	43.75	48.44	49.61	49.90	49.98	49.99
u_4	21.88	24.22	24.81	24.95	24.99	24.99

Example 22.26: Solve the Laplace equation $\nabla^2 u = 0$ in two-dimensional square region for the boundary value problem as described in Fig. 22.7, using Gauss-Seidel iterations upto four times.

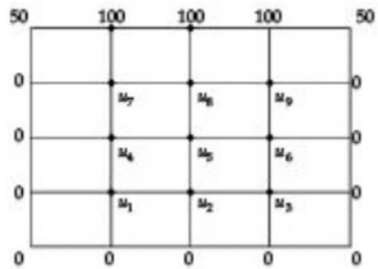


Fig. 22.7

Solution: From the symmetry in Fig. 22.7, we have $u_1 = u_3$, $u_4 = u_6$, and $u_7 = u_9$, so we will evaluate only u_1, u_2, u_4, u_5, u_7 and u_8, which we compute as follows:

$$u_5 = \frac{1}{4}(0 + 50 + 50 + 0) = 25.00 \qquad \text{(diagonal formula)}$$

$$u_9 = u_7 = \frac{1}{4}(0 + 100 + 50 + 25) = 43.75 \qquad \text{(diagonal formula)}$$

$$u_3 = u_1 = \frac{1}{4}(0 + 25 + 0 + 0) = 6.25 \qquad \text{(diagonal formula)}$$

$$u_8 = \frac{1}{4}(43.75 + 43.75 + 25.00 + 100) = 53.125 \qquad \text{(standard formula)}$$

$$u_6 = u_4 = \frac{1}{4}(0 + 25.00 + 6.25 + 43.75) = 18.75 \qquad \text{(standard formula)}$$

$$u_2 = \frac{1}{4}(6.25 + 6.25 + 0 + 25.00) = 9.375 \qquad \text{(standard formula)}$$

This gives the initial solution. Next we improve the solution obtained by performing successive iterations using the Gauss-Seidel iteration formula

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n+1)}]$$

Taking the initial solution corresponding to the iteration zero, we carry out the iterations using the formulae

$$u_1^{(n+1)} = \frac{1}{4}[0 + u_2^{(n)} + u_4^{(n)} + 0]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + u_1^{(n+1)} + u_5^{(n)} + 0]$$

$$u_4^{(n+1)} = \frac{1}{4}[0 + u_5^{(n)} + u_1^{(n+1)} + u_7^{(n)}]$$

$$u_5^{(n+1)} = \frac{1}{4}[u_4^{(n+1)} + u_4^{(n+1)} + u_2^{(n+1)} + u_8^{(n)}]$$

$$u_7^{(n+1)} = \frac{1}{4}[0 + u_8^{(n)} + u_4^{(n+1)} + 100]$$

$$u_8^{(n+1)} = \frac{1}{4}[u_7^{(n+1)} + u_7^{(n+1)} + u_5^{(n+1)} + 100]$$

We obtain the following table:

u_i	*Initial solution*	*Iterations*			
		1	2	3	4
$u_3 = u_1$	6.25	7.031	7.178	7.184	7.166
u_2	9.375	9.765	9.887	9.866	9.845
$u_6 = u_4$	18.75	18.945	18.848	18.798	18.774
u_5	25.00	25.195	25.098	25.048	25.024
$u_9 = u_7$	43.75	43.018	42.914	42.883	42.869
u_8	53.125	52.808	52.732	52.703	52.691

22.8 SOLUTION OF POISSON EQUATION

The Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \qquad \text{...(22.63)}$$

like Laplace equation, is an elliptic equation. We solve it in a bounded square region R with boundary C on which the values of u are specified. The standard five-point formula (22.59) takes the form

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = h^2 f(ih, jh) \qquad \text{...(22.64)}$$

and the diagonal five-point formula (22.60) takes the form

$$u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1} - 4u_{ij} = h^2 f(ih, jh) \qquad \text{...(22.65)}$$

By applying these formulae, as per the suitability at each mesh point, we obtain the linear equation in terms of the values at the nodal points, which can be solved by Gauss- Seidel iteration method.

Example 22.27: Solve the Poisson equation $\nabla^2 u = 2x^2y^2$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$, with $u = 0$ on the boundary, by dividing the domain into squares of one unit size.

Solution: The domain is divided into squares of one unit size as shown in Fig. 22.8.

Here $h = 1$ and $f(x, y) = 2x^2y^2$, the standard five-point formula (22.64) gives

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = 2i^2j^2$$

For $u_1 (i = 1, j = 1)$ gives

$$0 + u_2 + u_3 + 0 - 4u_1 = 2$$

or, $$u_1 = \frac{1}{4}(u_2 + u_3 - 2) \quad \text{...(i)}$$

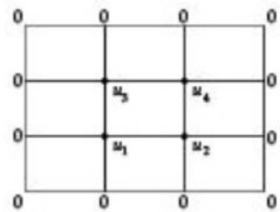


Fig. 22.8

Similarly, for u_2, we obtain

$$u_1 + 0 + u_4 + 0 - 4u_2 = 8$$

or, $$u_2 = \frac{1}{4}(u_1 + u_4 - 8) \quad \text{...(ii)}$$

For u_3, we obtain $0 + u_4 + 0 + u_1 - 4u_3 = 8$ or, $u_3 = \frac{1}{4}(u_1 + u_4 - 8)$ ...(iii)

For u_4, we obtain $u_3 + 0 + 0 + u_2 - 4u_4 = 32$ or, $u_4 = \frac{1}{4}(u_2 + u_3 - 32)$ ...(iv)

Form (ii) and (iii), we observe that $u_2 = u_3$. Hence, the equations to be solved become

$$u_1 = \frac{1}{2}(u_2 - 1) \quad \text{...(v)}$$

$$u_2 = \frac{1}{4}(u_1 + u_4 - 8) \quad \text{...(vi)}$$

$$u_4 = \frac{1}{2}(u_2 - 16) \quad \text{...(vii)}$$

Let the initial solution be $u_1 = u_2 = u_4 = 0$, then applying the successive iterations using Gauss-Seidel method to Eqs. (v), (vi) and (vii), we obtain the following table.

u_i	*Initial values*	*Iterations*						
		1	2	3	4	5	6	7
u_1	0	–0.5	–1.563	–2.828	–3.145	–3.224	–3.245	–3.248
$u_2 (=u_3)$	0	–2.125	–4.656	–5.289	–5.447	–5.489	–5.497	–5.499
u_4	0	–9.0630	–10.328	–10.645	–10.724	–10.745	–10.749	–10.749

Since the difference between the values obtained at 6th and 7th iterations are very small, hence,

$$u_1 = -3.248,\ u_2 = -5.499,\ u_3 = -5.499 \text{ and } u_4 = -10.749.$$

EXERCISE 22.7

1. Given the values of $u(x, y)$ on the boundary of the square as shown in Fig. 22.9. Evaluate the function $u(x, y)$ satisfying the Laplace equation $\nabla^2 u = 0$ at the mesh points when meshes are square of one unit size.

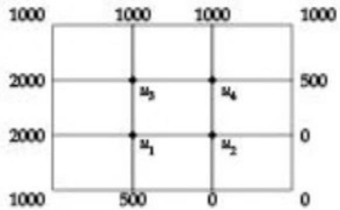


Fig. 22.9

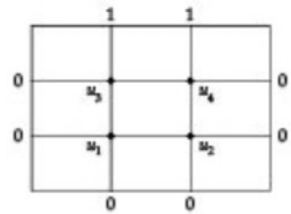


Fig. 22.10

2. Solve the Laplace equation $\nabla^2 u = 0$ in the domain, as shown in Fig. 22.10, at the mesh points, where meshes are square of one unit size using (up to four iterations),
   (i) Jacobi method
   (ii) Gauss-Seidel method.
3. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mesh of Fig. 22.11 with boundary values as shown using Gauss-Seidel iteration method up to four iterations.
4. Solve the Laplace equation $\nabla^2 u = 0$ for the region given by $0 \le x \le 4$ and $0 \le y \le 4$ with boundary conditions

$$u = 0 \text{ at } x = 0 \text{ and } u = 8 + 2y \text{ at } x = 4$$

$$u = \frac{1}{2}x^2 \text{ at } y = 0 \text{ and } u = x^2 \text{ at } y = 4$$

   using Gauss-Seidel method with $h = k = 1.0$.
5. Solve the Poisson equation $\nabla^2 u = 8x^2y^2$ for the square mesh given below in Fig. 22.12 with, $u(x, y) = 0$ on the boundary, and mesh length one unit.

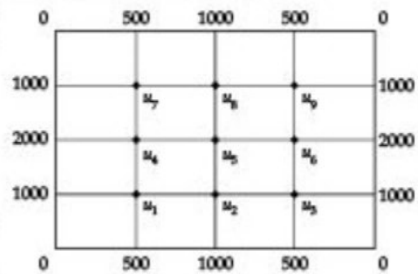


Fig. 22.11

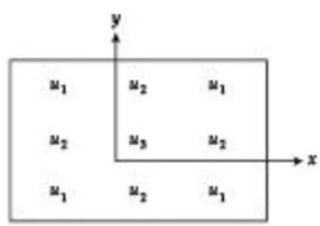


Fig. 22.12

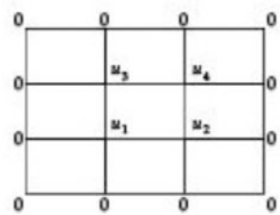


Fig. 22.13

6. Solve the Poisson equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with sides $x = 0 = y$, $x = 3 = y$ and $u = 0$ on the boundary as showing in Fig. 22.13; use Gauss-Seidel method upto five iterations and take $h = k = 1$.

22.9 SOLUTION OF ONE-DIMENSIONAL HEAT FLOW EQUATION

The one-dimensional heat flow equation given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \text{...(22.66)}$$

is an example for the parabolic type of partial differential equation. It describes a problem that depends on both space and time variables. Equation (22.66) arises in the study of heat flow in one-dimensional direction in an insulated rod and is governed by both boundary and initial conditions. Here c^2 is the diffusivity of the bar depending on the material, refer to Section 17.5.

There are various numerical methods for the solution of heat flow Eq. (22.66); we shall describe here only *Schmidt method*, and *Crank-Nicholson method*.

22.9.1 Schmidt Method

Consider a rectangular region in the x-t plane with spacing h along x-axis and k along t-axis. Denoting a grid point $(x, t) = (ih, jk)$ as (i, j), we have

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2},$$

refer to Section 22.5.1. Substituting these in Eq. (22.66), we obtain

$$u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}].$$

Solving for $u_{i,j+1}$, it gives

$$u_{i,j+1} = (1 - 2r)u_{i,j} + r(u_{i-1,j} + u_{i+1,j}), \qquad \text{...(22.67)}$$

where $r = \dfrac{kc^2}{h^2}$ is called the *grid-ratio parameter*.

The relation (22.67) is a relation between the function values at the two time levels $j + 1$ and j and hence is called *two-level formula*. It enables us to determine the value of u at the $(i, j + 1)$th grid point in terms of the known function values at the points x_{i-1}, x_i and x_{i+1} at the instant t_j.

The formula (22.67) is called the *Schmidt explicit formula* and converges only for $0 < r \le ½$. When $r = ½$, it simplifies to

$$u_{i,j+1} = \frac{1}{2}(u_{i+1,j} + u_{i-1,j}), \qquad \text{...(22.68)}$$

known as the *Bender-Schmidt recurrence relation*. This gives the value of u at the $(i, j + 1)$th grid point as the average of values right and left of x_i at the jth time step.

Example 22.28: Solve $\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}$, $0 < t < 1.5$ and $0 < x < 4.0$, given the initial condition $u(x, 0) = 50(4 - x), 0 \le x \le 4$, and the boundary conditions $u(0, t) = 0$, and $u(4, t) = 0, 0 \le t \le 1.5$, taking $\Delta x = 1$ and $\Delta t = 0.25$.

Solution: Here $c^2 = 2$, $h = 1$ and $k = 0.25$. Thus, $r = \frac{0.25 \times 2}{1} = 0.5$. Hence, we use the Bender-Schmidt recurrence relation (22.68) $u_{i,j+1} = \frac{1}{2}(u_{i+1,j} + u_{i-1,j})$,

which gives that the temperature at each interior point is the average of the values at the adjacent point of the previous time value. The temperature values at the boundary and interior grid points are tabulated in the table given below.

t \ x	$x = 0.0$	$x = 1.0$	$x = 2.0$	$x = 3.0$	$x = 4.0$
0.00	0.0	150.0	100.0	50.0	0.0
0.25	0.0	50.0	100.0	50.0	0.0
0.50	0.0	50.0	50.0	50.0	0.0
0.75	0.0	25.0	50.0	25.0	0.0
1.00	0.0	25.0	25.0	25.0	0.0
1.25	0.0	12.5	25.0	12.5	0.0
1.50	0.0	12.5	12.5	12.5	0.0

The first and the last columns in the body of the table correspond respectively to the boundary conditions $u(0, t) = 0$ and $u(4, t) = 0$ and the first row corresponds to the initial condition $u(x, 0) = 50(4 - x)$. Remaining function values have been obtained using the recurrence relation (22.68).

Esample 22.29: Solve the p.d.e. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(x, 0) = \sin \pi x$, $0 \le x \le 1$; $u(0, t) = u(1, t) = 0$ using Schmidt method, taking $\Delta x = 1/3$ and $\Delta t = 1/36$.

Solution: Here $c^2 = 1$, $h = 1/3$, $k = 1/36$. Thus, $r = \frac{kc^2}{h^2} = \frac{1}{4}$.

Thus, the Schmidt's formula (22.67) becomes

$$u_{i,j+1} = \frac{1}{2}u_{i,j} + \frac{1}{4}(u_{i-1,j} + u_{i+1,j}) = \frac{1}{4}(u_{i-1,j} + 2u_{i,j} + u_{i+1,j})$$

The various values for u, including the initial and the boundary, are tabulated as follows:

t \ x	$x = 0$	$x = 1/3$	$x = 2/3$	$x = 1$
0	0	0.866	0.866	0
1/36	0	0.649	0.649	0
2/36	0	0.487	0.487	0
3/36	0	0.365	0.365	0

The computations may continue similarly for higher time steps 4/36 onward.

22.9.2 Crank-Nicholson Method

In the Crank-Nicholson method $\frac{\partial^2 u}{\partial x^2}$ in Eq. (22.66) is replaced by the average of the finite-difference approximations on the jth and $(j+1)$th level. Thus,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right] \quad ...(22.69)$$

Hence, Eq. (22.66) becomes

$$\frac{1}{k}[u_{i,j+1} - u_{i,j}] = \frac{c^2}{2h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}]$$

Substituting $r = \frac{kc^2}{h^2}$ and rearranging the terms, we obtain

$$-ru_{i-1,j+1} + 2(1+r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + 2(1-r)u_{i,j} + ru_{i+1,j} \quad ...(22.70)$$

This equation is called the *Crank-Nicholson formula* and converges if $0 < r \le 1$. When $r = 1$, this reduces to

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j} \quad ...(22.71)$$

The terms on the right-hand side are all known. The grid points used are shown in Fig. 22.14.

In Crank-Nicholson formula (22.70) values to be computed are not just a function of the values at the preceding time step but also involve the values at the same time step, as shown in Fig. (22.14) which are not readily available and this requires us to solve a system of simultaneous equations at each time step, thus, in this sense, it is an *implicit scheme*.

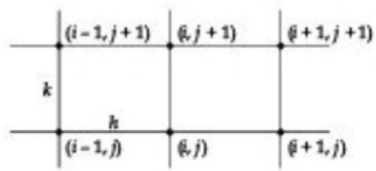


Fig. 22.14

Example 22.30: Solve Example 22.29 by Crank-Nicholson method.

Solution: For $r = 1/4$, Eq. (22.70) becomes

$$-\frac{1}{4}u_{i-1,j+1} + \frac{5}{2}u_{i,j+1} - \frac{1}{4}u_{i+1,j+1} = \frac{1}{4}u_{i-1,j} + \frac{3}{2}u_{i,j} + \frac{1}{4}u_{i+1,j} \quad ...(22.72)$$

where $u_{i,j}$ is the function value at the (i, j)th grid point.

From the initial condition $u(x, 0) = \sin \pi x$, we have

$$u_{00} = 0,\ u_{10} = 0.866,\ u_{20} = 0.866, \text{ and } u_{30} = 0$$

Also the boundary conditions $u(0, t) = u(1, t) = 0$ give

$$u_{00} = u_{01} = u_{02} = ... = 0 \quad \text{and} \quad u_{30} = u_{31} = u_{32} = ... = 0.$$

For $i = 1, 2; j = 0$ and using the initial and boundary values Eq. (22.72) gives

$$10u_{11} - u_{21} = 6.062 \quad \text{and} \quad -u_{11} + 10u_{21} = 6.062$$

Solving for u_{11}, u_{21}, we obtain $u_{11} = u_{21} = 0.674$

Next, for $i = 1, 2; j = 1$, we obtain $10u_{12} - u_{22} = 4.718$ and $-u_{12} + 10u_{22} = 4.718$

Solving for u_{12}, u_{22}, we obtain $u_{12} = u_{22} = 0.524$.

Similarly, we can find the values for $u_{i,j}$ corresponding to $j = 2, 3$, etc.

22.10 SOLUTION OF ONE-DIMENSIONAL WAVE EQUATION

The wave equation given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad ...(22.73)$$

is an example for the hyperbolic type of partial differential equation. The equation may be considered to describe the motion of a vibrating string that is fixed at both the ends, and is governed by both boundary and initial conditions, refer to Section 17.2.

We shall consider the problem under the *boundary conditions*

$$u(0, t) = u(a, t) = 0, \qquad 0 \le t \le b$$

and *initial conditions* $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $\quad 0 \le x \le a$.

Using the difference approximations for the derivatives

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \text{ and, } \frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

refer to Section 22.5.1, in Eq. (22.73) we obtain

$$\frac{1}{k^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) = \frac{c^2}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

which gives
$$u_{i,j+1} = -u_{i,j-1} + \alpha^2(u_{i-1,j} + u_{i+1,j}) + 2(1 - \alpha^2)\, u_{i,j} \quad ...(22.74)$$

where, $\alpha^2 = k^2c^2/h^2$.

The formula (22.74) is *three time step formula* and converges only if $\alpha^2 \le 1$.

When $\alpha^2 = 1$, (22.74) simplifies to

$$u_{i,j+1} = -u_{i,j-1} + u_{i-1,j} + u_{i+1,j}. \quad ...(22.75)$$

This gives that the function value at $(i, j+1)$th grid point is the sum of the values at $(i-1, j)$th and $(i+1, j)$th grid points minus the value at $(i, j-1)$th grid point, as indicated in grid diagram of Fig. 22.15.

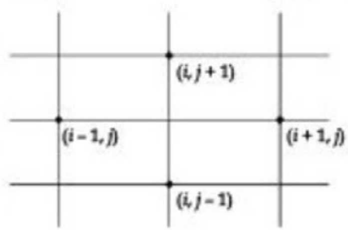


Fig. 22.15

Example 22.31: Solve the equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions $u(0, t) = u(1, t)$

$= 0$, $t \geq 0$ and initial condition $u_t(x, 0) = 0$ and $u(x, 0) = \sin^3 \pi x$, for $0 \leq x \leq 1$, taking $\Delta x = 0.25$ and $\Delta t = 0.25$.

Solution: We have, $h = k = 0.25$, $c^2 = 1$. Thus, $\alpha^2 = c^2k^2/h^2 = 1$, and we have the recurrence relation

$$u_{i,j+1} = -u_{i,j-1} + u_{i+1,j} + u_{i-1,j} \quad ...(22.76)$$

From the boundary conditions, we have $u_{00} = u_{01} = u_{02} = ... = 0$, $u_{40} = u_{41} = u_{42} = ... = 0$

The initial condition $u(x, 0) = \sin^3 \pi x$ gives $u_{00} = 0$, $u_{10} = 0.353$, $u_{20} = 1$, $u_{30} = 0.353$, $u_{40} = 0$

Further, the initial condition $u_t(x, 0) = 0$ gives $\dfrac{u_{i,0+1} - u_{i,0-1}}{2(0.25)} = 0 \Rightarrow u_{i,1} = u_{i,-1}$.

Using this in Eq. (22.76) for $j = 0$, we obtain $u_{i,1} = \dfrac{1}{2}(u_{i+1,0} + u_{i-1,0})$. This gives,

$$u_{11} = \frac{1}{2}(u_{20} + u_{00}) = 0.5,\ u_{21} = \frac{1}{2}(u_{30} + u_{10}) = 0.353,\ u_{31} = \frac{1}{2}(u_{40} + u_{20}) = 0.5$$

We formulate the following table for the function values

t \ x	$x = 0$	$x = 0.25$	$x = 0.50$	$x = 0.75$	$x = 1.0$
0	0	0.353	1.0	0.353	0
0.25	0	0.5	0.353	0.5	0
0.50	0	0	0	0	0
0.75	0	–0.5	–0.353	–0.5	0
1.0	0	–0.853	–0.5	–0.353	0

The first two rows in the body of the table correspond to initial values. The first and the last columns correspond to boundary values. Remaining function values are obtained using the formula (22.76).

Similarly, we can compute the function values for the higher time-steps.

EXERCISE 22.8

1. Find the solution of the parabolic equation $u_{xx} = 2u_t$, $0 \leq x \leq 4$, $t \geq 0$ by Schmidt method given that $u(0, t) = u(4, t) = 0$ and $u(x, 0) = x(4 - x)$, taking $\Delta x = \Delta t = 1$ and the value up to $t = 5$.
2. In a laterally insulated bar of length 1, let the initial temperature be $f(x) = x$, if $0 \leq x \leq 0.2$, and $f(x) = 0.25(1 - x)$, if $0.2 \leq x \leq 1$. Let $u(0, t) = u(1, t) = 0$ for all t. Apply Schmidt method to find $u(x, t)$ with $h = 0.2$, $k = 0.01$ for $t = 0.05$.
3. Consider a laterally insulated metal bar of length 1 and such that $c^2 = 1$ in the heat equation. Suppose that the ends of the bar are kept at temperature $u = 0$°C and the temperature in the bar at instant $t = 0$ is $f(x) = \sin \pi x$. Apply Crank-Nicholson method with $h = 0.2$ and $r = 1$, find the temperature $u(x, t)$ in the bar for $0 \leq t \leq 0.2$.
4. Solve $u_t = u_{xx}$ for the initial condition $f(x) = x$, if $0 \leq x \leq 0.5$, $f(x) = 1 - x$, if $0.5 \leq x \leq 1.0$ and boundary conditions $u(0, t) = u(1, t) = 0$ by Crank-Nicholson method with $h = 0.2$ and $r = 1$ for $0 \leq t \leq 0.20$.

5. Solve $u_{tt} = u_{xx}$ given that $u(0, t) = u(1, t) = 0$, $u(x, 0) = 0.5x(1-x)$ and $u(x, 0) = 0$ taking $h = k = 0.1$ for $0 \le t \le 0.4$. Compare your solution with the exact solution at $x = 0.5$ and $t = 0.3$.
6. Solve $u_{tt} = u_{xx}$ with boundary conditions $u(0, t) = u(1, t) = 0$, initial conditions $u_t(x, 0) = 0$ and $u(x, 0) = \sin \pi x$, take $h = 0.2$ and $k = 0.2$ up to $t = 1$.
7. Illustrate the starting procedure for solving the wave equation $u_{tt} = u_{xx}$ with boundary conditions $u(0, t) = u(1, t) = 0$, initial conditions $u_t(x, 0) = x - x^2$ and $u(x, 0) = 1 - \cos 2\pi x$. Choose $h = k = 0.1$ and compute 2 time steps.

ANSWERS

Exercise 22.1 (p. 1262)

1. $y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{15}x^5 + \ldots$
2. 0.9138
3. 0.90033, 0.80227
4. 0.8110
5. 0.005, 0.042, 0.321
6. 0.0050125, 0.0202013, 0.6487213.
7. (a) 0.9138, 0.9138

Exercise 22.2 (p. 1267)

1. 0.9606
3. 2.5351, 2.6531
4. 5.051
5. 0, 0.1000, 0.2034, 0.3109; errors: 0, 0.0044, 0.0122, 0.0203

Exercise 22.3 (p. 1271)

1. 2.2050, 2.4210, 2.6492
2. 2.2052, 2.4214
3. 1.13250, 1.15000, 1.16722; Exact 1.6722
4. 1.2736. 5. 0.8489

Exercise 22.4 (p. 1279)

1. 1.8392
2. 6.8734
3. 1.4392
4. 1.0101, 1.0207, 1.0318, 1.0438, 1.0569, 1.0713
6. 8.0079

Exercise 22.5 (p. 1283)

1. 0.105, 0.995
2. 2.0845, 0.5867
4. 1.21, 2.1985
5. 0.8367, 3.6545
6. 6.8716, 23.4332

Exercise 22.6 (p. 1287)

1. Computed $y(0.5) = 0.1470$, True $y(0.5) = 0.1505$
2. $y_0 = -0.98795$, $y_1 = -0.32530$, $y_2 = 0.32530$
3. $y_1 = 2.4$, $y_2 = 3.2$, $y_3 = 2.4$
4. $y_1 = 0.062$, $y_2 = 0.25$, $y_3 = 0.568$
5. $y_1 = 1.1539$, $y_2 = 3.9231$, $y_3 = 7.4615$

Exercise 22.7 (p. 1295)

1. $u_1 = 1042, u_2 = 458, u_3 = 1208, u_4 = 792$
2. (i) $u_1 = u_2 = 0.13281, u_3 = u_4 = 0.38281$
   (ii) $u_1 = u_2 = 0.13086, u_3 = u_4 = 0.37793$
3. $u_1 = u_3 = u_7 = u_9 = 956, u_2 = u_8 = 1019, u_4 = u_6 = 1269, u_5 = 1144$
5. $u_1 = -3, u_2 = -2, u_3 = -2$
6. $u_1 = 67.5, u_2 = 75, u_3 = 75, u_4 = 82.5$

Exercise 22.8 (p. 1301)

1.

t \ x	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

2. 0, 0.063, 0.093, 0.047, 0

3.

t \ x	0	0.2	0.4	0.6	0.8	1.0
0.00	0	0.588	0.951	0.951	0.988	0
0.04	0	0.399	0.646	0.646	0.399	0
0.08	0	0.271	0.439	0.439	0.271	0
0.12	0	0.184	0.298	0.298	0.184	0
0.16	0	0.125	0.202	0.202	0.125	0
0.20	0	0.085	0.138	0.138	0.085	0

5. At $t = 0.3$;

	$x =$	0.1	0.2	0.3	0.4	0.5
Calculated	$u =$	0.02	0.04	0.06	0.075	0.08
Exact	$u =$	0.02	0.04	0.06	0.075	0.08

6.

t	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
0.0	0	0.588	0.951	0.951	0.588	0
0.2	0	0.476	0.769	0.769	0.476	0
0.4	0	0.182	0.294	0.294	0.182	0
0.6	0	−0.182	−0.294	−0.294	−0.182	0
0.8	0	−0.476	−0.769	−0.769	−0.476	0
1.0	0	−0.588	−0.951	−0.951	−0.588	0

7. For $t = 0.1$; 0, 0.354, 0.766, 1.271, 1.679, 1.834 ...
   For $t = 0.2$; 0, 0.575, 0.935, 1.135, 1.296, 1.357 ...

23 CHAPTER Linear Difference Equations and Z-Transforms

The difference equations appear in discrete time dynamical systems equivalent to differential equations in continuous time and find applications in discretization of differential equations, the computation of special functions, the discrete modeling of economic or biological phenomena. An important tool to solve the difference equations is Z-transforms which have properties similar to Laplace transforms, a tool for solving differential equations.

23.1 INTRODUCTION

Difference equations arise out of the sampling process. If an analog signal is sampled, then the differential equation describing the analog signal becomes a difference equation. In general, mathematerial computations are based on equations that allow us to compute the value of a function recursively from a given set of values. Such an equation is called a 'difference equation' or 'recurrence equation'. These equations occur in numerous contexts both in mathematics itself and its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology and other fields. Analogous to the Laplace transform used to solve differential equations, the difference equations are solved using the Z-transform. As an example, consider the sequence of the numbers

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

called Fibonacci sequence. It is generated by the rule (difference equation)

$$y_{n+2} = y_{n+1} + y_n, \quad n = 0, 1, 2, \ldots \qquad \ldots(23.1)$$

with $y_o = y_1 = 1$.

Specifically we can define *a difference equation as an equation involving an independent variable n and the successive differences of the dependent variable* y_n (or $y(n)$).

For example, the equations

$$\Delta^2 y_n - 3\Delta y_n + 5y_n = 2n + 3 \qquad \ldots(23.2)$$

$$y_{n+1} + y_{n-1} = 1 \qquad \ldots(23.3)$$

are difference equations.

Since $\Delta = E - 1$, assuming the unit of increment to be 1, Eq. (23.2) can be written as

$$(E-1)^2 y_n - 3(E-1)y_n + 5y_n = 2n+3$$

or,
$$(E^2 - 5E + 9)\, y_n = 2n + 3 \quad \ldots(23.4)$$

or,
$$y_{n+2} - 5y_{n+1} + 9y_n = 2n + 3. \quad \ldots(23.5)$$

Thus a particular difference equation can be expressed in the form (23.2), (23.4) or (23.5).

The *order* of a difference equation is the difference between the greatest and the smallest subsuffixes of y appearing in it divided by unit of increment of the independent variable n.

For example, the difference Eq. (23.5) is of order $[(n+2) - n]/1 = 2$.

The *degree* of a difference equation expressed in a form free from Δ's is the highest power of y. For example, the difference equation $y_{n+1}^4 + 2y_n y_{n-2} + 3y_{n-1}^2 = n + 3$ is of degree 4 and order 3.

A *solution* of a difference equation is an expression that satisfies the difference equation for all permissible values of the integral variable n.

For example, a solution of the difference equation $y_{n+1} = a\, y_n + b$ is

$$y_n = \begin{cases} y_o + nb, a = 1 \\ a^b y_o + \left(\dfrac{1-a^n}{1-a}\right), a \neq 1 \end{cases}$$

The *general solution* of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation, and a solution obtained from the general solution by giving particular values to the constants is called a *particular solution*.

23.2 FORMATION OF DIFFERENCE EQUATIONS

We consider the following examples to illustrate the formation of a difference equation.

Example 23.1: Form the difference equation generated by the family of curves $y_x = ax + b2^x$.

Solution: We obtain the difference equation by eliminating the arbitrary constants a and b from the given equation $y_x = ax + b2^x$. We have

$\Delta y_x = a(x+1-x) + b(2^{x+1} - 2^x) = a + b2^x$, and $\Delta^2 y_x = b(2^{x+1} - 2^x) = b2^x$.

Thus,
$$b = \frac{\Delta^2 y_x}{2^x}, \text{ and } a = \Delta y_x - \Delta^2 y_x.$$

By substituting these values in the given equation, the desired difference equation is obtained as

$y_x = (\Delta y_x - \Delta^2 y_x)x + \Delta^2 y_x$ or, $(1-x)\,\Delta^2 y_x + x\,\Delta y_x - y_x = 0$

or, $(1-x)\,y_{x+2} + (3x-2)\,y_{x+1} - 2xy_x = 0$.

Example 23.2: Derive a difference equation by eliminating the constants a and b from the equation $y_n = a2^n + b(-2)^n$.

Solution: We have, $y_n = a2^n + b(-2)^n$, $y_{n+1} = 2a2^n - 2b(-2)^n$ and, $y_{n+2} = 4a\,2^n + 4b(-2)^n$.

Eliminating a and b, we obtain

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -2 \\ y_{n+2} & 4 & 4 \end{vmatrix} = 0, \text{ or } y_{n+2} - 4y_n = 0, \text{ as the desired difference equation.}$$

EXERCISE 23.1

1. Find the sequence represented by the recurrence formula $y_1 = 2, y_n = 3\,y_{n-1}, n \geq 2$.
2. Find the recurrence formula for the sequence 87, 82, 77, 72, 67, ...
3. Write the following difference equations in subscript notation.
   (a) $\Delta^3 y_n + \Delta^2 y_n + \Delta y_n + y_n = 0$ (b) $\delta^2 y_n = \alpha y_n$
   where δ is the central operator and α is a constant.
4. Form the difference equation by eliminating the arbitrary constants from the family of curves following.
   (a) $y_n = a/n + b$ (b) $y_n = (a + bn)3^n$
   (c) $y_n = a3^n + b\,5^n$ (d) $y_n = a2^n + b\,(-2)^n$.

23.3 LINEAR DIFFERENCE EQUATIONS

A *general linear difference equation* with constant coefficients is of the form

$$a_0\, y_{n+r} + a_1\, y_{n+r-1} + a_2\, y_{n+r-2} + \ldots + a_r y_n = f(n); \quad \ldots(23.6)$$

where $a_0, a_1, a_2, \ldots$ are constants and f is a function of the integral variable n. We must note that in a linear difference equation $y_n, y_{n+1}, \ldots$ occur only in first degree and are not multiplied together.

The general linear difference equation (23.6) can be written as

$$(a_0 E^r + a_1 E^{r-1} + \ldots + a_{r-1} E + a_r) y_n = f(n). \quad \ldots(23.7)$$

Here E is the shift operator similar to the D operator in an ordinary differential equation and r is positive integer that denotes the order of the difference equation.

Moreover, if $f(n) = 0$, then the Eq. (23.7) given by

$$(a_0 E^r + a_1 E^{r-1} + \ldots + a_{r-1} E + a_r) y_n = 0 \quad \ldots(23.8)$$

is referred to as a *homogeneous difference equation*, and if $f(n) \neq 0$, it is referred to as a *non-homogeneous difference equation*.

If $u_1(n), u_2(n), \ldots, u_r(n)$ are r independent solutions of the homogenous linear difference equation (23.8), then the linear combination

$$y_c(n) = k_1 u_1(n) + k_2 u_2(n) + \ldots + k_r u_r(n) \quad \ldots(23.9)$$

where k_i's are arbitrary constants, is also its solution.

For the non-homogeneous difference Eq. (23.7), where $f(n) \neq 0$, if $y_p(n)$ is a particular solution of it, then

$$y_n = y_c(n) + y_p(n), \quad \ldots(23.10)$$

is the complete solution of the Eq. (23.7). The part $y_c(n)$ is called the *complimentary function* (C.F) and the part $y_p(n)$ is called the *particular solution* of Eq. (23.7).

23.3.1 Rules to Find Complimentary Function

The Eq. (23.7) can be symbolically written as

$$F(E)\, y_n = f(n), \quad \ldots(23.11)$$

where $F(E)$ is a polynomial in E of degree r, the order of the difference equation. The corresponding homogeneous equation is

$$F(E)\, y_n = 0. \quad \ldots(23.12)$$

The equation $F(E) = 0$ is called the *auxiliary equation* (A.E.) of (23.11). If the roots of the *auxiliary equation* are $m_1, m_2, \ldots, m_r$ then $F(E) = (E - m_1)(E - m_2) \ldots (E - m_r)$.

Suppose y_n satisfies the subsidiary equation $(E - m_1)y_n = 0$, then it will also satisfy $(E - m_1)(E - m_2) \ldots (E - m_r)y_n = 0$. Hence, the complete solution of (23.12) is composed of the solutions of the r component equations $(E - m_1)\, y_n = 0$, $(E - m_2)\, y_n = 0$, ..., $(E - m_r)\, y_n = 0$.

If $r = 1$, then (23.12) becomes

$$(E - m_1)\, y_n = 0 \quad \ldots(23.13)$$

or,

$$y_{n+1} - m_1 y_n = 0.$$

On dividing throughout by m_1^{n+1}, we have

$$\frac{y_{n+1}}{m_1^{n+1}} - \frac{y_n}{m_1^n} = 0 \quad \text{or,} \quad \Delta\left(\frac{y_n}{m_1^n}\right) = 0 \quad \text{or,} \quad \frac{y_n}{m_1^n} = k_1, \text{ where } k_1 \text{ is a constant.}$$

Thus, the solution of $(E - m_i)y_n = 0$ is $y_n = k_i m_i^n$, $i = 1, 2, \ldots r$.

Case(i): If $m_1, m_2, \ldots, m_r$ are all real and distinct, then the complete solution of Eq. (23.12) is

$$y_c(n) = k_1 m_1^n + k_2 m_2^n + \ldots + k_r m_r^n$$

If $r = 2$ and $m_1 = m_2$, then (23.12) becomes

$$(E - m_1)^2 y_n = 0 \quad \ldots(23.14)$$

or,

$$y_{n+2} - 2m_1 y_{n+1} + m_1^2 y_n = 0.$$

Let $y_n = z_n m_1^n$ be a solution of (23.14). On substituting the value, we have

$$(z_{n+2} - 2z_{n+1} + z_n)m_1^{n+2} = 0$$

Since $m_1^{n+2} \neq 0$, thus

$$(z_{n+2} - 2z_{n+1} + z_n) = 0 \quad \text{or,} \quad (E - 1)^2 z_n = 0 \quad \text{or,} \quad \Delta^2 z_n = 0 \quad \text{or,} \quad z_n = k_1 + k_2 n.$$

Thus, the solution of Eq. (23.14) is $y_n = (k_1 + k_2 n)\, m_1^n$.

Similarly, the solution of equation $(E - m_1)^\lambda\, y_n = 0$ is

$$y_n = (k_1 + k_2 n + \ldots + k_\lambda n^{\lambda-1})\, m_1^n.$$

Case(ii): If all the roots $m_1, \ldots, m_r$ are real, but some are equal, say $m_1 = m_2 = \ldots = m_\lambda$, and the rest $m_{\lambda+1}, \ldots, m_r$ are distinct, then the complete solution of (23.12) is

$$y_c(n) = (k_1 + k_2 n + \ldots + k_\lambda n^{\lambda-1})m_1^n + k_{\lambda+1}\, m_{\lambda+1}^n + \ldots + k_r m_r^n$$

Case (iii): If $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the complete solution of $(E - m_1)(E - m_2)\, y_n = 0$ is

$$y_c(n) = A(\alpha + i\beta)^n + B(\alpha - i\beta)^n$$
$$= Am^n (\cos n\theta + i \sin n\theta) + Bm^n (\cos n\theta - i \sin n\theta);\ \alpha = m\cos\theta,\ \beta = m \sin\theta$$

Thus $$y_c(n) = m^n (k_1 \cos n\theta + k_2 \sin n\theta), \text{ where } m = \sqrt{\alpha^2 + \beta^2}, \text{ and } \theta = \tan^{-1}\frac{\beta}{\alpha}.$$

Thus, if m_1 and m_2 are complex conjugates given by $\alpha \pm i\,\beta$, and rest of the roots are real and distinct, then the complete solution of Eq. (23.11) is

$$y_c(n) = m^n (k_1 \cos n\,\theta + k_2 \sin n\theta) + k_3 m_3^n + \ldots + k_r m_r^n.$$

In case the complex conjugate roots are repeated, then

$$y_c(n) = m^n[(k_1 + nk_2)\cos n\theta + (k_3 + nk_4)\sin n\theta] + k_5 m_5^n + \ldots$$

The results obtained can be summarized in the form of the following table.

Roots of A.E. $(m_1, m_2, \ldots m_r)$	*Complimentary Function*
Real and distinct	$y_c(n) = k_1 m_1^n + k_2 m_2^n + \ldots + k_r m_r^n$
Real and equal $m_1 = m_2$	$y_c(n) = (k_1 + k_2 n) m_1^n + k_3 m_3^n + \ldots + k_r m_r^n$
Complex conjugates $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$	$y_c(n) = (k_1 \cos n\theta + k_2 \sin n\theta) m^n + k_3 m_3^n + \ldots + k_r m_r^n$ where $m = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\frac{\beta}{\alpha}$

Example 23.3: Solve the difference equation $y_{n+2} - 8y_{n+1} + 12y_n = 0$.

Solution: The given equation can be expressed as $(E^2 - 8E + 12)y_n = 0$

Its auxiliary equation is $E^2 - 8E + 12 = 0$, or, $(E - 2)(E - 6) = 0$.

Thus $E = 2, 6$, hence the complete solution of the given equation is $y_n = k_1 (2)^n + k_2 (6)^n$, where k_1 and k_2 are two arbitrary constants.

Example 23.4: Solve the difference equation $y_{n+3} - 4y_{n+2} + 5y_{n+1} - 2y_n = 0$.

Solution: The given equation can be expressed as $(E^3 - 4E^2 + 5E - 2)y_n = 0$.

Its auxiliary equation is $E^3 - 4E^2 + 5E - 2 = 0$ or, $(E - 1)^2 (E - 2) = 0$.

Thus, $E = 1, 1, 2$, hence complete solution of given equation is $y_n = (k_1 + k_2 n) + k_3(2)^n$, where k_1, k_2 and k_3 are arbitrary constants.

Example 23.5: Solve the difference equation $y_{n+1} - y_n + y_{n-1} = 0$

Solution: The given equation is $(E^2 - E + 1)\, y_{n-1} = 0$.

Its auxiliary equation is $E^2 - E + 1 = 0$ which gives $E = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

Hence, the complete solution is $y_{n-1} = [k_1 \cos(n-1)\theta + k_2 \sin(n-1)\theta]\, m^{n-1}$

or, $y_n = [k_1 \cos n\theta + k_2 \sin n\theta]\, m^n$, where

$$m=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1,$$

and,
$$\theta=\tan^{-1}\left(\frac{\sqrt{3}}{2}\Big/\frac{1}{\sqrt{2}}\right)=\tan^{-1}\sqrt{3}=\pi/3.$$

Thus $y_n=\left(k_1\cos\frac{n\pi}{4}+k_2\sin\frac{n\pi}{3}\right)$, where k_1 and k_2 are two arbitrary constants.

Example 23.6: Form difference equation for Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ... and solve it.

Solution: Each number beyond the second number in the Fibonacci sequence is the sum of its two preceding numbers. Hence, if y_n is the nth number, then

$$y_n=y_{n-1}+y_{n-2}, \qquad n\geq 3, \text{ with } y_1=y_2=1$$

or,
$$y_{n+2}-y_{n+1}-y_n=0, \qquad n\geq 1,$$

or,
$$(E^2-E-1)\,y_n=0$$

The auxiliary equation is $E^2-E-1=0$. This gives $E=\frac{1}{2}\left(1\pm\sqrt{5}\right)$, hence the solution is

$$y_n=k_1\left(\frac{1+\sqrt{5}}{2}\right)^n+k_2\left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n\geq 1.$$

When $n=1$, $y_1=1$; thus, $k_1\left(\frac{1+\sqrt{5}}{2}\right)+k_2\left(\frac{1-\sqrt{5}}{2}\right)=1.$

When $n=2$, $y_2=1$; thus, $k_1\left(\frac{1+\sqrt{5}}{2}\right)^2+k_2\left(\frac{1-\sqrt{5}}{2}\right)^2=1.$

Solving these two equations, we obtain $k_1=\frac{1}{\sqrt{5}}$ and $k_2=-\frac{1}{\sqrt{5}}$,

and hence the solution is $y_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n\geq 1.$

23.3.2 Rules to Find Particular Solution

Consider the linear difference equation

$$y_{n+r}+a_1\,y_{n+r-1}+\ldots+a_n\,y_n=f(n) \qquad \ldots(23.15)$$

which symbolicallly can be represented as $F(E)y_n=f(n)$, where $F(E)=E^r+a_1\,E^{r-1}+\ldots+a_{n+1}\,E+a_n$, and a_i's are constants.

Analogous to the case of linear differential equation, refer to Section 11.5.2, the particular solution (P.S.) of the Eq. (23.15) is given by $\frac{1}{F(E)} f(n)$, where $\frac{1}{F(E)}$, that is, $[F(E)]^{-1}$ represents the inverse operator of $F(E)$.

Next we give procedure for finding the particular solution (P.S.) for some specific forms of $f(n)$.

Case I: $f(n) = a^n$, where a is a constant. We have P.S. $= \frac{1}{F(E)} a^n = \frac{1}{F(a)} a^n$, provided $F(a) \neq 0$.

If $F(a) = 0$, then for the equation

(i) $(E - a)\, y_n = a^n$, P.S. $= \frac{1}{E-a} a^n = na^{n-1}$

(ii) $(E - a)^2 y_n = a^n$, P.S. $= \frac{1}{(E-a)^2} a^n = \frac{n(n-1)}{2!} a^{n-2}$, and so on.

Case II: $f(n) = \sin kn$. We have P.S. $= \frac{1}{F(E)} \sin kn = \frac{1}{F(E)} \frac{e^{ikn} - e^{-ikn}}{2i} = \frac{1}{2i}\left[\frac{1}{F(E)} a^n - \frac{1}{F(E)} b^n\right]$, where $a = e^{ik}$ and $b = e^{-ik}$, and now this can be treated as in Case I above.

Similarly, in case of $f(n) = \cos kn$, we have P.S. $= \frac{1}{2}\left[\frac{1}{F(E)} a^n + \frac{1}{F(E)} b^n\right]$.

Case III: $f(n) = n^p$; p being positive integer. We have P.S. $= \frac{1}{F(E)} n^p = \frac{1}{F(1+\Delta)} n^p$.

Expand $[F(1+\Delta)]^{-1}$ in ascending powers of Δ by using Binomial theorem as far as the terms up to Δ^p, and then proceed as illustrated in Example 23.8.

Case IV: $f(n) = a^n \phi(n)$, $\phi(n)$ being a polynomial of finite degree in n. We have

$$\text{P.S.} = \frac{1}{F(E)} a^n \phi(n) = a^n \frac{1}{F(aE)} \phi(n).$$

Since $\phi(n)$ is a polynomial of finite degree in n, now proceed as in Case III above.

Example 23.7: Solve the difference equation $y_{n+2} - 4y_{n+1} + 3y_n = 2^n$, $y_0 = y_1 = 1$.

Solution: The given equation can be written as $(E^2 - 4E + 3)y_n = 2^n$.

The auxiliary equation is $E^2 - 4E + 3 = 0$. This gives $E = 1, 3$.

Hence, the complementary function is C.F. $= k_1 + k_2(3)^n$.

Next, the particular solution is P.S. $= \frac{1}{(E-1)(E-3)} 2^n = \frac{1}{(2-1)(2-3)} 2^n = -2^n$.

Thus the general solution is $y_n = k_1 + k_2(3)^n - 2^n$.

Using $y_0 = 1$ and $y_1 = 1$, we obtain $k_1 = 3/2$ and $k_2 = 1/2$ and hence the solution is $y_n = \frac{3}{2} + \frac{3^n}{2} - 2^n$.

Example 23.8: Solve $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$.

Solution: The given equation can be expressed as $(E^2 - 6E + 9)y_n = 3^n$.

The auxiliary equation is $(E^2 - 6E + 9) = 0$, or $(E - 3)^2 = 0$.

This gives $E = 3, 3$. Hence, the complementary function is C.F. $= (k_1 + k_2 n)\, 3^n$

The particular solution is P.S. $= \dfrac{1}{(E-3)^2} 3^n = \dfrac{n(n-1)}{2!} 3^{n-2} = \dfrac{n(n-1)}{2} 3^{n-2}$.

Hence, the general solution is $y_n = (k_1 + k_2 n)\, 3^n + \dfrac{n(n-1)}{2} 3^{n-2}$.

Example 23.9: Solve the difference equation $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = \cos \alpha n$.

Solution: The given equation can be expressed as $(E^2 - 2\cos\alpha\, E + 1)y_n = \cos\alpha n$.

The auxiliary equation is $E^2 - 2\cos\alpha\, E + 1 = 0$. This gives

$$E = \frac{2\cos\alpha \pm (4\cos^2\alpha - 4)}{2} = \cos\alpha \pm i\sin\alpha.$$

Hence, the complementary function is

$$\text{C.F.} = (1)^n [k_1 \cos\alpha n + k_2 \sin\alpha n] = k_1 \cos\alpha n + k_2 \sin\alpha n.$$

Next, the particular solution is

$$\text{P.S.} = \frac{1}{E^2 - 2E\cos\alpha + 1}\cos\alpha n = \frac{1}{E^2 - E(e^{i\alpha} + e^{-i\alpha}) + 1}\left(\frac{e^{i\alpha n} + e^{-i\alpha n}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})} e^{i\alpha n} + \frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})} e^{-i\alpha n}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(E - e^{i\alpha})}\cdot\frac{1}{e^{i\alpha} - e^{i\alpha}} e^{i\alpha n} + \frac{1}{E - e^{-i\alpha}}\cdot\frac{1}{e^{-i\alpha} - e^{i\alpha}} e^{-i\alpha n}\right];\ E = e^{i\alpha}, e^{-i\alpha}$$

$$= \frac{1}{4i\sin\alpha}\left[\frac{1}{E - e^{i\alpha}} e^{i\alpha n} - \frac{1}{E - e^{-i\alpha}} e^{-i\alpha n}\right] = \frac{1}{4i\sin\alpha}\left[n e^{i\alpha(n-1)} - n e^{-i\alpha(n-1)}\right]$$

$$= \frac{n}{2\sin\alpha}\left[\frac{e^{i\alpha(n-1)} - e^{-i\alpha(n-1)}}{2i}\right] = \frac{n\sin(n-1)\alpha}{2\sin\alpha}.$$

Hence the complete solution is $y_n = k_1 \cos\alpha n + k_2 \sin\alpha n + \dfrac{n\sin(n-1)\alpha}{2\sin\alpha}$.

Example 23.10: Solve the difference equation $y_{n+2} - 4y_n = n^2 + n$.

Solution: The given equation is $(E^2 - 4)\, y_n = n^2 + n$.

The auxiliary equation is $E^2 - 4 = 0$, which gives $E = \pm 2$. Hence the complementery function is C.F. $= k_1 (2)^n + k_2 (-2)^n$, where k_1 and k_2 are two arbitrary constants.

The particular solution is

$$\text{P.S.} = \frac{1}{E^2 - 4}(n^2 + n) = \frac{1}{(1+\Delta)^2 - 4} + ([n]^2 + 2[n]), \quad \text{where } [n]^r = n(n-1)\ \dots\ (n-r+1)$$

$$= -\frac{1}{3}\left[1 - \left(\frac{2\Delta}{3} + \frac{\Delta^2}{3}\right)\right]^{-1} ([n]^2 + 2[n])$$

$$= -\frac{1}{3}\left[1 + \left(\frac{2\Delta}{3} + \frac{\Delta^2}{3}\right) + \left(\frac{2\Delta}{3} + \frac{\Delta^2}{3}\right)^2 + \dots\right] ([n]^2 + 2[n])$$

$$= -\frac{1}{3}\left[1 + \frac{2\Delta}{3} + \frac{7}{9}\Delta^2 + \dots\right] ([n]^2 + 2[n]) = -\frac{1}{3}\left[[n]^2 + 2[n] + \frac{2}{3}(2[n] + 2) + \frac{7}{9}(2)\right]$$

$$= -\frac{1}{3}\left[n(n-1) + 2n + \frac{2}{3}(2n+2) + \frac{14}{9}\right] = -\frac{1}{3}\left[n^2 + \frac{7n}{3} + \frac{26}{9}\right]$$

Thus the complete solution is $y_n = k_1(2)^n + k_2(-2)^n - \frac{1}{3}\left[n^2 + \frac{7n}{3} + \frac{26}{3}\right]$.

Example 23.11: Solve the difference equation $y_{n+2} - 3y_{n+1} + 2y_n = 2^n$.

Solution: The given equation is $(E^2 - 3E + 2)\, y_n = 2^n$. The auxiliary equation is $E^2 - 3E + 2 = 0$. This gives $E = 1, 2$.

Hence the complimentary function is

$$\text{C.F.} = k_1 + k_2 2^n.$$

The particular solution is

$$\text{P.S.} = \frac{1}{(E^2 - 3E + 2)} 2^n = 2^n \frac{1}{4E^2 - 6E + 2} 1$$

$$= 2^{n-1} \frac{1}{\Delta}\left[\frac{1}{1 + 2\Delta}\right] 1 = 2^{n-1} \frac{1}{\Delta}[1 - 2\Delta + 4\Delta^2 \dots] 1 = n\, 2^{n-1}.$$

Thus the complete solution is $y_n = k_1 + k_2 2^n + n2^{n-1}$.

Example 23.12: Solve the simultaneous difference equations

$$x_{n+1} - 7x_n - 10y_n = 0, \text{ and } y_{n+1} - x_n - 4y_n = 0, \text{ with } x_0 = 3 \text{ and } y_0 = 2.$$

Solution: Rewriting the equations as

$$(E - 7)x_n - 10y_n = 0 \qquad \dots(23.16)$$

$$-x_n + (E-4)y_n = 0 \qquad \text{...(23.17)}$$

Operating (23.16) by $(E-4)$ and (23.17) by 10, adding and simplifying, we obtain

$$(E^2 - 11E + 18)\, x_n = 0.$$

Its solution is

$$x_n = c_1\, 2^n + c_2\, 9^n. \qquad \text{...(23.18)}$$

Substituting (23.18) in (23.16), we obtain

$$(c_1\, 2^{n+1} + c_2\, 9^{n+1}) - 7(c_1\, 2^n + c_2\, 9^n) - 10\, y_n = 0$$

or,

$$y_n = \frac{1}{10}[-5c_1\, 2^n + 2c_2\, 9^n]. \qquad \text{...(23.19)}$$

Using the initial condition $x_0 = 3$ in (23.18) and $y_0 = 2$ in (23.19) and from the equations obtained solving for c_1 and c_2, we get $c_1 = -2$, $c_2 = 5$. Thus the required solution is

$$x_n = (-2)2^n + (5)\, 9^n, \text{ and } y_n = 2^n + 9^n.$$

EXERCISE 23.2

1. Solve the following difference equations
   (a) $16\, y_{n+2} - 8y_{n+1} + y_n = 0$ (b) $y_{n+3} - 2y_{n+2} - y_{n+1} + 2y_n = 0$
   (c) $y_{n+1} - 2y_n \cos\alpha + y_{n-1} = 0$
2. Solve the following difference equations
   (a) $y_n = 2y_{n-1} - y_{n-2}$; $y_1 = 1.5, y_2 = 3$
   (b) $y_{n+2} - y_{n+1} - 6y_n = 0$; $y_0 = 4, y_1 = -13$
   (c) $y_{n+3} - 5y_{n+2} + 8y_{n+1} - y_n = 0$; $y_0 = 3, y_1 = 2, y_2 = 4.$
3. Solve the difference equation $\delta^2 y_n = \alpha y_n$, where δ is the central operator.
4. Solve the difference equations
   (a) $y_n - 4y_{n-1} + 5y_{n-2} = 2$ (b) $y_n - y_{n-1} - 2y_{n-2} = 2n^2$
   (c) $y_n + 5y_{n-1} + 4y_{n-2} = (56)\, 3^n$ (d) $y_n - y_{n-1} - 2y_{n-2} = (3n)\, 4^n$
   (e) $y_{n+2} - 5y_{n+1} + 6y_n = 2^n + n$ (f) $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = \cos\alpha n$
   (g) $y_{n+2} - 2y_{n+1} + y_n = (n^2)2^n$ (h) $y_{n+2} - y_n = e^n.$
5. Solve the simultaneous difference equations
   (a) $x_{n+1} - y_n - 1 = 0$, $y_{n+1} + x_n = 0$; $x_0 = 0, y_0 = -1$
   (b) $x_{n+1} + y_n - 3x_n = n$, $3x_n + y_{n+1} - 5y_n = 4^n$; $x_1 = 2, y_1 = 0$
   (c) $x_{n+1} + y_n + z_n = 1$, $x_n + y_{n+1} + z_n = n$, $x_n + y_n + z_{n+1} = 2n$

23.4 THE Z-TRANSFORM

In communication engineering we come across sequences of discrete signals which can be represented as $\{f_n\}_{n=0}^{\infty}$ where the elements $f_n = f(n)$ are functions of discrete non-negative integer valued arguments. The operation of such discrete system is governed by difference equations and Z-transform is one of the important tools for solving difference equations, as Laplace transform is for differential equations. It has many properties similar to Laplace transform.

We define the Z-*transform* of the sequence $\{f_n\}$, $n \geq 0$, where f_n is a real or complex number as

$$Z\{f_n\} = \sum_{n=0}^{\infty} f_n z^{-n} = F(z) \qquad \text{...(23.20)}$$

whenever the series is convergent. Here the permeter z may be real or complex parameter.

We may view (23.20) as a power series in $\left(\frac{1}{z}\right)$ and its radius of convergence R can be determined using the test of convergence as discussed in Section 20.2.

We must note that it is possible to consider Z-transform of the sequence $\{f_n\}$ defined on integer valued argements, $n = 0, \pm 1, \pm 2, \ldots$ as

$$Z\{f_n\} = \sum_{n=-\infty}^{\infty} f_n z^{-n}. \qquad \text{...(23.21)}$$

The sequence $\{f_n\}_{n=-\infty}^{n=\infty}$ is called a *two-sided sequence* and the transform (23.21) as *two-sided Z-transform* or *bilateral transform*. However, here we shall normally concentrate on Z-transform of the sequence $\{f_n\}$ defined for non-negative integral values only.

23.4.1 Z-Transforms of Some Standard Sequences

1. **Unit impulse sequence:** It is defined by $f_n = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n > 0 \end{cases}$

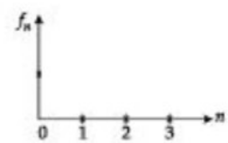


Fig. 23.1

Its graphical representation is shown in Fig. 23.1. A sequence of this type plays a role similar to Dirac-delta function $\delta(t)$ of continuous argument t. The Z-transform is $Z\{f_n\} = 1$.

Normally the unit impulse function is denoted by $\delta(n)$.

2. **Unit-step sequence:** It is defined by $f_n = 1$, for $n = 0, 1, 2, \ldots$

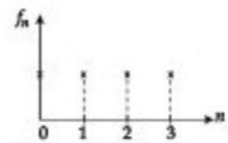


Fig. 23.2

Its graphical representation is shown in Fig. 23.2.

The Z-transform is

$$Z\{f_n\} = \sum_{n=0}^{\infty} z^{-n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots = \frac{z}{z-1}.$$

The region of convergence is $\left|\frac{1}{z}\right| < 1$, or $|z| > 1$.

Normally unit step function is denoted by $u(n)$.

3. **Unit-ramp sequence:** It is defined by $f_n = n$, $n \geq 0$.

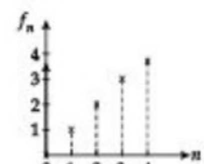


Fig. 23.3

Its graphical representation is shown in Fig. 23.3.

The Z-transform is

$$Z\{n\} = \sum_{n=0}^{\infty} nz^{-n} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^2} + \ldots = \frac{z}{(z-1)^2}.$$

The region of convergence is $\left|\frac{1}{z}\right| < 1$, or $|z| > 1$.

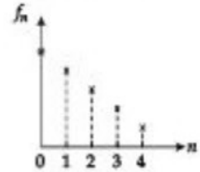


Fig. 23.4

4. Exponential sequence: It is defined by $f_n = a^n$, $n \geq 0$
where a is any real or complex number.

Its graphical representation for $a < 1$ is shown in Fig. 23.4.

The Z-transform is

$$Z\{a^n\} = \sum_{n=0}^{\infty} nz^{-n} = \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \frac{z}{z-a}.$$

The region of convergence is $\left|\frac{a}{z}\right| < 1$, or $|z| > a$.

In case a is negative real number $a = -b$, $b > 0$, the Z-transform is $Z\{(-b)^n\} = \frac{z}{z+b}$.

For $f_n = a^{-n}$, the Z-transform is obtained from that of a^n by replacing a with $\frac{1}{a}$. Thus,

$$Z\{a^{-n}\} = \frac{az}{az-1}, \text{ with region of convergence } |z| > 1/a.$$

For $f_n = e^{an}$, the Z-transform is obtained from that of a^n by replacing a by e^a. Thus,

$$Z\{e^{an}\} = \frac{z}{z-e^a}, \text{ with region of convergence } |z| > e^a.$$

5. The sine sequence: It is defined by $f_n = \sin n\theta$, $n \geq 0$.

Its graphical representation is shown in Fig. 23.5. To find the Z-transform, we write $\sin n\theta = \operatorname{Im} e^{in\theta}$, where Im denotes the imaginary part. We have

$$Z\{e^{in\theta}\} = \sum_{n=0}^{\infty} e^{in\theta} z^{-n} = \sum_{r=0}^{\infty} (e^{i\theta}/z)^n = \frac{z}{z-e^{i\theta}}$$

$$= \frac{z}{(z-\cos\theta) - i\sin\theta} = \frac{z[(z-\cos\theta)+i\sin\theta]}{(z-\cos)^2 + \sin^2\theta}$$

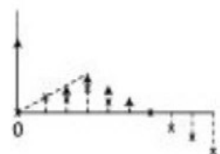


Fig. 23.5

Thus, $Z[\cos n\theta + i\sin n\theta] = \frac{z[(z-\cos\theta)+i\sin\theta]}{z^2 - 2\cos\theta\, z + 1}$.

Comparing the real and imaginary parts, using the linearity property to be stated in Section 23.5, we obtain

$$Z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2 - 2\cos\theta\, z + 1}.$$

and,
$$Z\{\sin n\theta\} = \frac{(\sin\theta)z}{z^2 - 2(\cos\theta)z + 1}.$$

Obviously the region of convergence is $|z| > |e^{i\theta}| = 1$.

Example 23.13: Obtain the Z-transform of $\{f_n\}$, where f_n is defined as

(i) $\frac{a^n}{n!}$ (ii) e^{-n} (iii) $\frac{1}{\Gamma(n+1)}$

Solution: (i) By definition $Z\left\{\frac{a^n}{n!}\right\} = \sum_{n=0}^{\infty} \frac{a^n z^{-n}}{n!} = 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \ldots = e^{a/z}.$

The region of convergence is $|z| > a$.

(ii) By definition $Z\{e^{-n}\} = \sum_{n=0}^{\infty} e^{-n} z^{-n} = 1 + \frac{1}{ez} + \left(\frac{1}{ez}\right)^2 + \ldots = \frac{ez}{ez-1}.$

The region of convergence is

$$\left|\frac{1}{ez}\right| < 1, \text{ or } |z| > \frac{1}{e}.$$

(iii) Here $f_n = \frac{1}{\Gamma(n+1)} = \frac{1}{n!}$. The Z-transform for this may be obtained from (i) by replacing a with 1. Thus, $Z\left(\frac{1}{n!}\right) = e^{1/z}$.

The region of convergence is $|z| > 1$.

Remark: A necessary and sufficient condition for the *existence of Z-transform* of a sequence $\{f_n\}$ is that it is of exponential type, that is, there exist numbers $N > 0$, $t_0 \geq 0$ and $n_0 \geq 0$ such that $|f_n| < Ne^{nt_0}$ for all $n \geq n_0$.

23.4.2 Properties of The Z-Transforms

The Z-transforms have properties similar to Laplace transforms. Some of the important properties are given as follows.

1. **Linearity property:** Let $\{f_n\}$ and $\{g_n\}$ be two sequences such that $Z\{f_n\} = F(z)$ and $Z\{g_n\} = G(z)$ with region of convergence $|z| > \frac{1}{R_1}$, and $|z| > \frac{1}{R_2}$ respectively. Then, for arbitrary constants a and b

$$Z\{af_n + bg_n\} = aZ\{f_n\} + b\,Z\{g_n\} = a\,F(z) + bG(z) \qquad \ldots(23.22)$$

for all $|z| > 1/R$, where $1/R = \max\{1/R_1, 1/R_2\}$.

The proof for this follows immediately from the definition of the Z-transform.

Example 23.14: Obtain the Z-transform of $\{f_n\}$, where f_n is defined as

(i) $\cosh(n\theta)$ (ii) $\sinh(n\theta)$ $\theta > 0$.

Solution: (i) We have $\cosh(n\theta) = (e^{n\theta} + e^{-n\theta})/2$. Using linearity property, this gives

$$Z\{\cosh(n\theta)\} = \frac{1}{2}\left[Z\{e^{n\theta}\} + Z\{e^{-n\theta}\}\right] = \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(\frac{e^{\theta}}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{ze^{\theta}}\right)^n\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} + \frac{z}{z-e^{-\theta}}\right] = \frac{1}{2}\left[\frac{2z^2-(e^{\theta}+e^{-\theta})z}{z^2-(e^{\theta}+e^{-\theta})z+1}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2-2(\cosh\theta)z}{z^2-2(\cosh\theta)z+1}\right] = \frac{z(z-\cosh\theta)}{z^2-2(\cosh\theta)z+1}.$$

Its region of convergence is $|z| > \max.\left\{e^{\theta}, \frac{1}{e^{\theta}}\right\} = e^{\theta}$.

(ii) On the same lines as above in (i), we can show that

$$Z\{\sinh(n\theta)\} = \frac{(\sinh\theta)z}{z^2-2(\cosh\theta)z+1}$$

with region of convergence $|z| > e^{\theta}$.

2. **Shifting property:** Let $F(z)$ be the Z-transform of the sequence $\{f_n\}$ convergent in the region $|z| > (1/R)$ and m be a positive integer. Then

(i) $Z\{f_{n-m}\} = z^{-m}F(z), \quad n \ge m$ (shift to the right) ...(23.23)

(ii) $Z\{f_{n+m}\} = z^{m}\left[F(z) - \sum_{n=0}^{m-1} f_n z^{-n}\right]$ (shift to the left) ...(23.24)

convergent in the region $|z| > (1/R)$.

Proof: (i) By definition of the Z-transform

$$Z\{f_{n-m}\} = \sum_{n=0}^{\infty} f_{n-m}z^{-n} = \sum_{n=m}^{\infty} f_{n-m}z^{-n} \qquad (\text{since } f_k < 0, \quad \text{for } k<0)$$

$$= z^{-m}\sum_{n=m}^{\infty} f_{n-m}z^{-(n-m)} = z^{-m}\sum_{k=0}^{\infty} f_k z^{-k} \qquad (\text{writing } n-m=k)$$

$$= z^{-m}F(z).$$

(ii)

$$Z\{f_{n+m}\} = \sum_{n=0}^{\infty} f_{n+m}z^{-n} = z^{m}\sum_{n=0}^{\infty} f_{n+m}z^{-(n+m)}$$

$$= z^{m}\left[\sum_{n=0}^{m-1} f_n z^{-n} + \sum_{n=0}^{\infty} f_{n+m}z^{-(n+m)} - \sum_{n=0}^{m-1} f_n z^{-n}\right]$$

$$= z^{m}\left[\sum_{n=0}^{\infty} f_n z^{-n} - \sum_{n=0}^{m-1} f_n z^{-n}\right] = z^{m}\left[F(z) - \sum_{n=0}^{m-1} f_n z^{-n}\right].$$

Example 23.15: Determine the Z-transform of $\{f_n\}$, when f_n is

(i) $\frac{1}{(n+p)!}$, $p = 1, 2, 3 \ldots$ (ii) $\cosh(n-p)\theta$, $p = 1, 2, \ldots$

Solution: (i) We know that $Z\left\{\frac{1}{n!}\right\} = e^{1/z}$. Using the shifting property

$$Z\left\{\frac{1}{(n+p)!}\right\} = z^p\left[Z\left\{\frac{1}{n!}\right\} - \sum_{n=0}^{p-1}\frac{1}{n!}z^{-n}\right] = z^p\left[e^{1/z} - \sum_{n=0}^{p-1}\frac{1}{n!}z^{-n}\right]$$

For $p = 1, 2, 3$, we obtain

$$Z\left\{\frac{1}{(n+1)!}\right\} = z[e^{1/z} - 1], \; Z\left\{\frac{1}{(n+2)!}\right\} = z^2\left[e^{1/z} - 1 - \frac{1}{z}\right]$$

and,
$$Z\left\{\frac{1}{(n+3)!}\right\} = z^3\left[e^{1/z} - 1 - \frac{1}{z} - \frac{1}{z^2}\right] \text{ respectively.}$$

(ii) We know that $Z[\cosh n\theta] = \dfrac{z(z - \cosh\theta)}{z^2 - 2(\cosh\theta)z + 1}$

Using the shifting property

$$Z\{\cosh (n-p)\theta\} = z^{-p}\, Z[\cosh n\theta] = z^{-p}\frac{z(z - \cosh\theta)}{z^2 - 2(\cosh\theta)z + 1}, \; n \geq p.$$

For $p = 1, 2$, we obtain

$$Z\{\cosh (n-1)\theta\} = \frac{z - \cosh\theta}{z^2 - 2(\cosh\theta)z + 1}, \; n \geq 1, \; Z\{\cosh (n-2)\theta\} = \frac{z - \cosh\theta}{z[z^2 - 2(\cosh\theta)z + 1]}, \; n \geq 2.$$

3. **Scaling property (or Damping property):** If $Z\{f_n\} = F(z)$ exists and is convergent in the region $|z| > 1/R$, and a is any real or complex constant, then

$$Z\{a^{-n} f_n\} = F(az) \quad \text{...(23.25)}$$

$$Z\{a^n f_n\} = F(z/a) \quad \text{...(23.26)}$$

Proof: By definition of Z-transform

(i) $Z\{a^{-n} f_n\} = \sum_{n=0}^{\infty} a^{-n} f_n z^{-n} = \sum_{n=0}^{\infty} f_n (az)^{-n} = F(az)$

(ii) $Z\{a^n f_n\} = \sum_{n=0}^{\infty} a^n f_n z^{-n} = \sum_{n=0}^{\infty} f_n \left(\frac{z}{a}\right)^{-n} = F\left(\frac{z}{a}\right).$

Example 23.16: Determine the Z-transforms of $\{f_n\}$, when f_n is

(i) $a^n \sinh n\theta$ (ii) $a^{-n} \cosh n\theta$.

Solution: We have

(i) $$Z\{\sinh n\theta\} = \frac{(\sinh\theta)z}{z^2-(2\cosh\theta)z+1} = F(z), \text{ say}$$

By scaling property $Z\{a^n \sinh n\theta\} = F(az) = \dfrac{a(\sinh\theta)z}{a^2z^2-(2\,a\cosh\theta)z+1}$.

(ii) $$Z\{\cosh n\theta\} = \frac{z(z-\cosh\theta)}{z^2-(2\cosh\theta)z+1} = F(z), \text{ say}.$$

By scaling property, $Z\{a^{-n}\cosh n\theta\} = F(z/a) = \dfrac{(\sinh\theta)z/a}{(z/a)^2-(2\cosh\theta)z/a+1} = \dfrac{(a\sinh\theta)z}{z^2-(2a\cosh\theta)z+a^2}$.

4. ***Multiplication by n:*** If $Z\{f_n\} = F(z)$ exists in the region $|z| > 1/R$, then

$$Z\{nf_n\} = -z\frac{d}{dz}[F(z)] \qquad \dots(23.27)$$

which is also convergent in the region $|z| > 1/R$.

Proof: By definition

$$Z\{nf_n\} = \sum_{n=0}^{\infty} nf_n z^{-n} = (-z)\sum_{n=0}^{\infty} f_n(-n)z^{-n-1} = -Z\sum_{n=0}^{\infty} f_n\frac{d}{dz}(z^{-n}) = -z\frac{d}{dz}\left(\sum_{n=0}^{\infty} f_n z^{-n}\right) = -z\frac{d}{dz}[F(z)].$$

In general, for $p = 1, 2, 3, \dots$, we have $Z\{n^p f_n\} = (-z)^p \dfrac{d}{dz}[F(z)]$ $\qquad \dots(23.28)$

Example 23.17: Find the Z-transforms of the following:

(i) 1 (ii) n (iii) n^2 (iv) n^3

Solution: (i) By definition of Z-transform

$$Z\{1\} = \sum_{n=0}^{\infty} 1\,z^{-n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \dots = z/(z-1),\ |z| > 1.$$

(ii) Using the multiplication by n property

$$Z\{n\} = -z\frac{d}{dz}[Z\{1\}] = -z\frac{d}{dz}\left[\frac{z}{z-1}\right] = \frac{z}{(z-1)^2}.$$

(iii) $$Z\{n^2\} = -z\frac{d}{dz}[Z\{n\}] = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = \frac{z^2+z}{(z-1)^3}$$

(iv) $$Z\{n^3\} = -z\frac{d}{dz}[Z\{n^2\}] = -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right] = \frac{z^3+4z^2+z}{(z-1)^4}.$$

23.5 TWO BASIC THEOREMS ON Z-TRANSFORM

In applications sometimes we need to find the value of f_n for some small values of n, say 0, 1, 2, etc. or as $n \to \infty$ from $F(z)$, the Z-transform of $\{f_n\}$. In this context, we have the following two theorems:

Theorem 23.1 (Initial Value Theorem): If $Z\{f_n\} = F(z)$ exists in the region $|z| > 1/R$, then

$$f_o = \lim_{z \to \infty} F(z). \quad \text{...(23.29)}$$

Proof: By definition of Z-transform $F(z) = \sum_{n=0}^{\infty} f_n z^{-n} = f_o + \frac{f_1}{z} + \frac{f_2}{z^2} + \ldots$

Taking limit as z tends to ∞, we obtain (23.29). Similarly f_1, f_2 etc. can be obtained as

$$f_1 = \lim_{z \to \infty} [z(F(z) - f_0)] \quad \text{...(23.30)}$$

$$f_2 = \lim_{z \to \infty} [z^2(F(z) - f_o - f_1)] \quad \text{...(23.31)}$$

and so on.

Theorem 23.2: (Final Value Theorem) If $Z\{f_n\} = F(z)$ exists in the region $|z| > 1/R$, then

$$\lim_{n \to \infty} f_n = \lim_{z \to 1} (z - 1) F(z). \quad \text{...(23.32)}$$

Proof: By the linearity property of Z-transform

$$\begin{aligned} Z\{f_{n+1} - f_n\} &= Z\{f_{n+1}\} - Z\{f_n\} \\ &= z[Z\{f_n\} - f_0] - Z\{f_n\}, \quad \text{using the shifting property (23.24)} \\ &= (z-1) F(z) - zf_o \end{aligned}$$

or
$$\begin{aligned} (z-1) F(z) - zf_o &= \sum_{n=0}^{\infty} (f_{n+1} - f_n) z^{-n} \\ &= (f_1 - f_0) + (f_2 - f_1) z^{-1} + (f_3 - f_2) z^{-2} + \ldots + (f_{n+1} - f_n) z^{-n} + \ldots \end{aligned}$$

Taking limits on both sides as $z \to 1$, we obtain

$\lim_{z \to 1} [(z-1) F(z)] - f_o = \lim_{n \to \infty} f_n - f_o$, which gives $\lim_{n \to \infty} f_n = \lim_{z \to 1} [(z-1) F(z)]$.

Example 23.18: If $F(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$, find f_o, f_1, f_2 and $\lim_{n \to \infty} f_n$.

Solution: Rewriting $F(z)$ as $F(z) = \dfrac{1}{z^2} \dfrac{2 + 5/z + 14/z^2}{(1 - 1/z)^4}$. By initial value theorem

$$f_o = \lim_{z \to \infty} F(z) = 0$$

$$f_1 = \lim_{z \to \infty} [z(F(z) - f_0)] = \lim_{z \to \infty} [zF(z)] = 0$$

$$f_2 = \lim_{z\to\infty}\left[z^2\left(F(z) - f_0 - \frac{f_1}{z}\right)\right] = \lim_{z\to\infty}\,[z^2F(z)] = 2$$

$$f_3 = \lim_{z\to\infty}\left[z^3\left(F(z) - f_0 - \frac{f_1}{z} - \frac{f_2}{z^2}\right)\right] = \lim_{z\to\infty} z^3\left[\frac{2z^2+5z+14}{(z-1)^4} - \frac{2}{z^2}\right]$$

$$= \lim_{z\to\infty}\left[\frac{13+2/z+8/z^2-2/z^3}{(1-1/z)^4}\right] = 13.$$

Next, $\lim_{n\to\infty} f_n = \lim_{z\to 1}\,[(z-1)\,F(z)] = \lim_{z\to 1}\left[\frac{2z^2+5z+14}{(z-1)^3}\right] \to \infty.$

23.6 INVERSE Z-TRANSFORM

If $F(z) = Z\{f_n\}$, then *the inverse Z-transform* of $F(z)$, denoted by $Z^{-1}[F(z)]$ is defined as

$$f_n = Z^{-1}\,[F(z)]. \qquad \text{...(23.33)}$$

We can obtain the inverse Z-transform using any of the following methods:

I. Power series method: If $F(z)$ can be expressed as $\sum_{n=0}^{\infty} f_n z^{-n}$, then $Z^{-1}\,[F(z)] = f_n$.

Example 23.19: Find inverse Z-transform of $\log(z/z+1)$ by power series method.

Solution: We have $F(z) = \log\left(\frac{z}{z+1}\right) = -\log\left(1+\frac{1}{z}\right) = -\frac{1}{z} + \frac{1}{2}\frac{1}{z^2} - \frac{1}{3}\frac{1}{z^3} + \ldots$

Thus,
$$f_n = \begin{cases} 0, & \text{for } n = 0 \\ (-1)^n/n, & n \geq 1 \end{cases}$$

Example 23.20: Find inverse Z-transform of $F(z) = \dfrac{z^2-1}{z^3+2z+4}$

Solution: We have $F(z) = \dfrac{z^2-1}{z^3+2z+4}$. Dividing the numerator by denominator, we obtain

$$F(z) = \frac{1}{z} - \frac{3}{z^3} - \frac{4}{z^4} + \ldots$$

Comparing the R.H.S. with $\sum_{n=0}^{\infty} f_n z^{-n}$, we obtain $f_0 = 0, f_1 = 1, f_2 = 0, f_3 = -3, f_4 = -4, \ldots$

II. Partial fraction method: This method of finding the inverse Z-transform is similar to that of finding the inverse Laplace transform using partial fraction.

In case $F(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials having no factor in common and degree of $P(z)$ is less than or equal to that of $Q(z)$, we can express $F(z) \left(\text{or } \frac{F(z)}{z}\right)$ into partial fractions and then can find the inverse Z-transform of each term. Method is illustrated in the example given next.

Example 23.21: Find the inverse Z-transforms of

(i) $\frac{2z^2+3z}{(z+2)(z-4)}$ (ii) $\frac{1}{z(z-2)^2}$.

Solution: (i) We express $F(z)/z$ in partial fractions as follows

$$\frac{F(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}.$$

It is easy to see that $A = 1/6$ and $B = 11/6$. Thus,

$$\frac{F(z)}{z} = \frac{1}{6(z+2)} + \frac{11}{6(z-4)} \quad \text{or,} \quad F(z) = \frac{z}{6(z+2)} + \frac{11z}{6(z-4)}.$$

Therefore, $$Z^{-1}[F(z)] = \frac{1}{6} Z^{-1}\left[\frac{z}{z+2}\right] + \frac{11}{6} Z^{-1}\left[\frac{z}{z-4}\right]$$

$$= \frac{(-2)^n}{6} + \frac{11(4)^n}{6}, \text{ using } Z^{-1}\left[\frac{z}{z-a}\right] = a^n \text{ and assuming that } |z| > 4.$$

(ii) We express $F(z)/z$ in partial fractions as follows

$$\frac{F(z)}{z} = \frac{1}{z^2(z-2)^2} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-2} + \frac{D}{(z-2)^2}.$$

It is easy to see that $A = B = -C = D = 1/4$. Thus,

$$\frac{F(z)}{z} = \frac{1}{4}\left[\frac{1}{z} + \frac{1}{z^2} - \frac{1}{z-2} + \frac{1}{(z-2)^2}\right] \quad \text{or,} \quad F(z) = \frac{1}{4}\left[1 + \frac{1}{z} - \frac{z}{z-2} + \frac{z}{(z-2)^2}\right].$$

This gives $$Z^{-1}[F(z)] = \frac{1}{4} Z^{-1}[1] + \frac{1}{4} Z^{-1}\left[\frac{1}{z}\right] - \frac{1}{4} Z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{4} Z^{-1}\left[\frac{z}{(z-2)^2}\right],$$

where $$Z^{-1}[1] = f_n, \text{ such that } f_0 = 1, f_n = 0, n = 1, 2, \ldots$$

$$Z^{-1}\left[\frac{1}{z}\right] = f_n, \text{ such that } f_0 = 0, f_1 = 1, f_n = 0, n = 2, 3, \ldots$$

$$Z^{-1}\left[\frac{z}{z-2}\right]=2^n,\ n=0,1,\ldots$$

and $$Z^{-1}\left[\frac{z}{(z-2)^2}\right]=\frac{1}{2}\,Z^{-1}\left[-z\frac{d}{dz}\left[\frac{z}{z-2}\right]\right]=n\,2^{n-1},$$

using (23.27), and $\frac{d}{dz}\left[\frac{z}{z-2}\right]=\frac{-2}{(z-2)^2}$, and $Z^{-1}\left[\frac{z}{z-2}\right]=2^n$.

III. Contour integral method: If $F(z)=\sum_{n=0}^{\infty} f_n z^{-n}$, then $f_n=\frac{1}{2\pi i}\int_C F(z)z^{-1}dz$ = sum of the residues at the poles of $F(z)\,z^{n-1}$ inside the contour C drawn in anticlockwise direction within the given region of convergence and enclosing the origin. (For details of contour integration refer to Section 20.7)

In case there is no pole of $F(z)\,z^{n-1}$ inside C for one or more values of n, then $f_n=0$ for those values.

Example 23.22: Find the inverse Z-transform of $\frac{2z}{(z-1)(z^2-1)}$, $|z|>1$ using contour integration method.

Solution: Using the contour integration, we have

$$f_n=\frac{1}{2\pi i}\int_C F(z)\,z^{n-1}dz,\quad n\geq 0=\frac{1}{2\pi i}\int_C\frac{2z^n}{(z-1)(z^2+1)}\,dz$$

where C is a closed contour inside the region of convergence $|z|>1$, enclosing the origin and the poles of the function $\frac{2z^n}{(z-1)(z^2+1)}$.

Now for $n\geq 0$, poles are $z=1,\pm i$ and all are simple poles, thus

$$\operatorname{Res}\left[\frac{2z^n}{(z-1)(z^2+1)}\right]_{z=1}=\lim_{z\to 1}\left[(z-1)\frac{2z^n}{(z-1)(z^2+1)}\right]=1;$$

$$\operatorname{Res}\left[\frac{2z^n}{(z-1)(z^2+1)}\right]_{z=i}=\lim_{z\to i}\left[(z-i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right]=\frac{-i^n}{i+1};$$

$$\operatorname{Res}\left[\frac{2z^n}{(z-1)(z^2+1)}\right]_{z=-i}=\lim_{z\to i}\left[(z+i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right]=\frac{-(-i)^n}{1-i}.$$

The f_n given by the sum of residues of $F(z)\,z^{n-1}$ is $f_n = 1 - \frac{i^n}{1+i} - \frac{(-i)^n}{1-i}$.

Example 23.23: Using contour integration method, find the inverse Z-transform of

$$\frac{1}{z(z-1)(z+0.5)},\ |z|>1.$$

Solution: Using contour integration, we have

$$f_n = \frac{1}{2\pi i}\int_C F(z) z^{n-1}\,dz,\quad n \ge 0 = \frac{1}{2\pi i}\int_C \frac{z^{n-2}}{(z-1)(z+0.5)}\,dz,$$

where C is a closed contour inside the region of convergence $|z|>1$, enclosing the origin and poles of $\frac{z^{n-2}}{(z-1)(z+0.5)}$.

For $n = 0$, the poles of $F(z)\,z^{n-2}$ are at $z = 0, 1, -0.5$.

Here $z = 0$ is pole of order 2, and 1 and -0.5 are simple poles. Thus

$$\text{Res } f(z)\,|_{z=0} = \frac{d}{dz}\left[(z-0)^2\cdot\frac{1}{z^2(z-1)(z+0.5)}\right]_{z=0} = \frac{1}{1.5}\frac{d}{dz}\left[\frac{1}{z-1}-\frac{1}{z+0.5}\right]_{z=0}$$

$$= \frac{2}{3}\left[\frac{-1}{(z-1)^2}+\frac{1}{(z+0.5)^2}\right]_{z=0} = \frac{2}{3}\left[-1+\frac{4}{1}\right] = 2.$$

Next, $$\text{Res } f(z)\,|_{z=1} = \lim_{z\to 1}\left[(z-1).\frac{1}{z^2(z-1)(z+0.5)}\right]_{z=0} = \frac{1}{1.5} = \frac{2}{3}, \text{ and}$$

$$\text{Res } f(z)\,|_{z=0.5} = \lim_{z\to 0.5}\left[(z+0.5)\frac{1}{z^2(z-1)(z+0.5)}\right]_{z=-0.5} = \frac{1}{(-0.5)^2(-1.5)} = -\frac{8}{3}.$$

This gives $$f_0 = 2 + \frac{2}{3} - \frac{8}{3} = 0;$$

For $n = 1$, the poles of $F(z)\,z^{n-2}$ are at $z = 0, 1, -0.5$ and all are simple poles, thus

$$\text{Res } f(z)\,|_{z=0} = \lim_{z\to 0}\left[z\frac{1}{z(z-1)(z+0.5)}\right] = -\frac{1}{0.5} = -2,$$

$$\text{Res } f(z)\,|_{z=1} = \lim_{z\to 1}\left[(z-1)\frac{1}{z(z-1)(z+0.5)}\right] = \frac{1}{1.5} = \frac{2}{3},$$

$$\text{Res } f(z)\,|_{z=-0.5} = \lim_{z\to 0.5}\left[(z+0.5)\frac{1}{z(z-1)(z+0.5)}\right] = \frac{1}{-(0.5)(-1.5)} = \frac{4}{3}.$$

This gives $$f_1 = -2 + \frac{2}{3} + \frac{4}{3} = 0.$$

For $n \geq 2$, the poles of $f(z)\, z^{n-2}$ are at $z = 1, -0.5$, thus

$$\text{Res} f(z)\,|_{z=0} = \lim_{z\to 1}\left[(z-1)\frac{z^{n-2}}{(z-11)(z+0.5)}\right] = \frac{1}{1.5} = \frac{2}{3};$$

$$\text{Res } f(z)\,|_{z=-0.5} = \lim_{z\to -0.5}\left[(z+0.5)\frac{z^{n-2}}{(z-1)(z+0.5)}\right] = \frac{(-0.5)^{n-2}}{-1.5} = -\frac{2}{3}\left(-\frac{1}{2}\right)^{n-2}.$$

This gives $$f_n = \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^{n-2}, \quad n \geq 2.$$

Thus, $Z^{-1}\left[\frac{1}{z(z-1)(z+0.5)}\right]$ is $f_0 = 0, f_1 = 0$, and $f_n = \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^{n-2}, \quad n \geq 2.$

23.7 CONVOLUTION THEOREM

The convolution theorem for Z-transform is a result concerning the inverse Z-transform of the product of two functions which plays an important role in the solution of difference equations.

Theorem 23.3: (Convolution Theorem): If $Z\{f_n\} = F(z)$ with region of convergence $|z| > (1/R_1)$, and $Z\{g_n\} = G(z)$ with region of convergence $|z| > (1/R_2)$, then

$$Z\{f_n \bullet g_n\} = F(z)\, G(z), \qquad \text{...(23.34)}$$

or equivalently

$$Z^{-1}\,[F(z)\, G(z)] = \{f_n \bullet g_n\} \qquad \text{...(23.35)}$$

valid in the region $1/R = \max\,\{1/R_1, 1/R_2\}$.

Here $\{f_n \bullet g_n\}$ is the convolution of the two sequences $\{f_n\}$ and $\{g_n\}$ defined by

$$\{f_n\} \bullet \{g_n\} = \{f_n \bullet g_n\} = \sum_{k=0}^{n} f_k g_{n-k} = f_0 g_n + f_1 g_{n-1} + \ldots + f_n g_0.$$

Proof: We have $F(z)\, G(z) = Z\{f_n\} Z\{g_n\} = \left[\sum_{n=0}^{\infty} f_n z^{-n}\right]\left[\sum_{n=0}^{\infty} g_n z^{-n}\right]$

$$= [f_0 + f_1 z^{-1} + \ldots + f_n z^{-n} + \ldots]\,[g_0 + g_1 z^{-1} + \ldots + g_n z^{-n} + \ldots]$$

$$= f_0g_0 + (f_0g_1 + f_1g_0)z^{-1} + (f_0g_2 + f_1g_1 + f_2g_0)z^{-2} + \dots + (f_0g_n + f_1g_{n-1} + \dots f_ng_0)z^{-n} + \dots$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} f_k g_{n-k}\right]z^{-n} = \sum_{n=0}^{\infty}\{f_n \bullet g_n\}z^{-n} = Z\{f_n \bullet g_n\}.$$

Example 23.24: Using convolution theorem find the inverse Z-transform of $\frac{z^2}{(z-2)(z-3)}$, $|z| > 3$.

Solution: We have $Z^{-1}\left\{\frac{z}{z-2}\right\} = 2^n$, and $Z^{-1}\left\{\frac{z}{z-3}\right\} = 3^n$. Thus by convolution theorem

$$Z^{-1}\left[\frac{z^2}{(z-2)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-2}\cdot\frac{z}{z-3}\right] = 2^n \bullet 3^n$$

$$= \sum_{k=0}^{n} 2^k 3^{n-k} = 3^n \sum_{k=0}^{n}\left(\frac{2}{3}\right)^k = 3^n\,\frac{(1-(2/3)^{n+1})}{1-2/3} = 3^{n+1} - 2^{n+1}.$$

Example 23.25: Verify convolution theorem for Z-transforms when $\{f_n\} = n^2$ and $\{g_n\} = n$.

Solution: We have $f_n = n^2$, and $g_n = n$. Therefore,

$$f_n \bullet g_n = \sum_{k=0}^{n} f_k g_{n-k} = \sum_{k=0}^{n} k^2(n-k) = n\sum_{k=0}^{n} k^2 - \sum_{k=0}^{n} k^3$$

$$= n\left[\frac{n(n+1)(2n+1)}{6}\right] - \frac{n^2(n+1)^2}{4} = \frac{1}{12}(n^4 - n^2), \text{ (after simplifying)}.$$

Next, we have $\quad Z\{1\} = \sum_{n=0}^{\infty} 1\cdot z^{-n} = \frac{z}{z-1},\ |z| > 1.$

Using the 'multiplication by n' property

$$Z\{n\} = -z\frac{d}{dz}[Z\{1\}] = -z\frac{d}{dz}\left[\frac{z}{z-1}\right] = \frac{z}{(z-1)^2},$$

$$Z\{n^2\} = -z\frac{d}{dz}[Z\{n\}] = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = \frac{z^2+z}{(z-1)^3},$$

$$Z\{n^3\} = -z\frac{d}{dz}[Z\{n^2\}] = -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right] = \frac{z^3+4z^2+z}{(z-1)^4},$$

$$Z\{n^4\} = -z\frac{d}{dz}[Z\{n^3\}] = -z\frac{d}{dz}\left[\frac{z^3+4z^2+z}{(z-1)^4}\right] = \frac{z^4+11z^3+11z^2+z}{(z-1)^5}.$$

Thus,
$$F(z) = Z\{f_n\} = \frac{z^2+z}{(z-1)^3},\ G(z) = Z\{g_n\} = \frac{z}{(z-1)^2},\text{ and}$$

$$Z\{f_n \bullet g_n\} = \frac{1}{12}Z[\{n^4-n^2\}] = \frac{1}{12}[Z\{n^4\} - Z\{n^2\}]$$

$$= \frac{1}{12}\left[\frac{z^4+11z^3+11z^2+z}{(z-1)^5} - \frac{z^2+z}{(z-1)^3}\right]$$

$$= \frac{z^3+z^2}{(z-1)^5} = \frac{z^2+z}{(z-1)^3}\,\frac{z}{(z-1)^2} = F(z)\,G(z).$$

EXERCISE 23.3

1. Obtain the Z-transform of the sequence $\{f_n\}$, where f_n is given as follows:

(a) -1 (b) $\delta(n) = \begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}$ (c) $u(n) = \begin{cases}0, & n<0\\ 1, & n\geq 0\end{cases}$

(d) na^n (e) a^{2n} (f) $e^{ni\theta}$

(g) $\cos(n\pi/3)$ (h) $a2^n + b3^n$ (i) $(n+1)^2$

(j) $\alpha^n \cos(\beta n)$ (k) $n\sin(\beta n)$ (l) $\dfrac{\cos(\beta n)}{n!}$

(m) $\dfrac{2^n e^{-n}}{n!}$ (n) $\sin(n+1)\theta$ (o) $\cos(n+1)\theta$

2. Using $Z\{n^2\} = \dfrac{z^2+z}{(z-1)^3}$, show that $Z\{n+1\}^2 = \dfrac{z^3+z^2}{(z-1)^3}$.

3. Show that $Z\{\delta(n+1)\} = 1/z$.

4. Show that $Z\{{}^{n+p}C_p\} = (1-1/z)^{-(p+1)}$.

5. If $Z\{f_n\} = \dfrac{z}{(z-1)} + \dfrac{z}{z^2+1}$, find $Z\{f_{n+2}\}$.

6. Find f_0, f_1, f_2 when $Z\{f_n\} = F(z)$ is given by

(a) $\dfrac{1}{(z-1)(z-3)}$ (b) $\dfrac{z^2}{z^2+1}$

(c) $\dfrac{(z-1)^2(z+2)}{(z+3)(z+5)^2}$ (d) $\dfrac{2z^2+5z+14}{(z-1)^4}$.

7. Find $\lim_{n\to\infty} f_n$ when $Z\{f_n\} = F(z) = \dfrac{z^2 - 3z + 5}{(z-1)(z+2)}$.

8. Find the inverse Z-transform of the following:

   (a) $\dfrac{4z}{z-a}$, $|z| > |a|$ (b) $\dfrac{z}{(z-1)^2}$.

9. Find the inverse Z-transform of $\dfrac{1}{(z-2)(z-3)}$ for

   (a) $|z| < 2$ (b) $2 < |z| < 3$ (c) $|z| > 3$.

10. Find the inverse Z-transform of

    (a) $\dfrac{z^3 + 2z^2 + 29z}{(z-1)(z+3)^2}$ (b) $\dfrac{z^3 + 5z^2 + 6z}{(z-2)(z-3)^3}$

11. Using convolution theorem find the inverse Z-transform of

    (a) $\dfrac{z^2}{(z-a)(z-b)}$ (b) $\left(\dfrac{z}{z-a}\right)^2$ (c) $\dfrac{z^2}{(z+2)(z-1)^2}$

12. Verify convolution theorem on Z-transform when

    (a) $f_n = a^n,\ g_n = b^n$ (b) $f_n = e^n,\ g_n = e^n$

    (c) $f_n = 1,\ g_n = n(2^n)$ (d) $f_n = \dfrac{1}{n!}, g_n = \dfrac{1}{n!}$

13. Using contour integration find the inverse Z-transform of

    (a) $\dfrac{10z}{(z-1)(z-2)}$ (b) $\dfrac{z^2 + z}{(z-1)(z^2+1)}$

23.8 APPLICATIONS TO DIFFERENCE EQUATIONS

Analogous to the Laplace transform in case of differential equation, the Z-transforms are useful for solving difference equations arising out of the performance of discrete systems. Consider a general linear difference equation with constant coefficients of order k given by

$$y_{n+k} + a_1 y_{n+k-1} + \dots + a_k y_n = f(n). \qquad \dots(23.36)$$

To solve Eq. (23.36) using Z-transforms, we take the following steps:

1. Let $F(z) = Z\{y_n\}$.
2. Apply Z-transform to both sides of Eq. (23.36).
3. Using linearity property and shifting theorem, simplify and obtain $F(z)$.
4. Take inverse Z-transform to obtain y_n, the required solution.

Example 23.26: Solve the following difference equations using Z-transforms

(i) $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$, $y_0 = 0$ and $y_1 = 1$.

(ii) $y_{n+2} - 2y_{n+1} + y_n = n$, $y_0 = 1$, $y_1 = 1$

Solution: (i) Let $F(z) = Z\{y_n\}$. Applying the Z-transform and using the linearity property, we obtain $Z\{y_{n+2}\} + 4Z\{y_{n+1}\} + 3Z\{y_n\} = Z\{3^n\}$.

Using the shifting property, we have $z^2[F(z) - y_0 - y_1 z^{-1}] + 4z\,[F(z) - z_0)] + 3F(z) = z(z-3)$.

Using $y_0 = 0$, $y_1 = 1$ and simplifying, we obtain

$$\frac{F(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)}$$

$$= \frac{3}{8}\,\frac{1}{z+1} + \frac{1}{24}\,\frac{1}{z-3} - \frac{5}{12}\,\frac{1}{z+3}$$

or,

$$F(z) = \frac{3}{8}\,\frac{z}{z+1} + \frac{1}{24}\,\frac{z}{z-3} - \frac{5}{12}\,\frac{z}{z+3}.$$

Applying inverse Z-transform,

$$y_n = \frac{3}{8}\,Z^{-1}\left[\frac{z}{z+1}\right] + \frac{1}{24}\,Z^{-1}\left[\frac{z}{z-3}\right] - \frac{5}{12}\,Z^{-1}\left[\frac{z}{z+3}\right] = \frac{3}{8}(-1)^n + \frac{1}{24}\,3^n - \frac{5}{12}(-3)^n$$

is the desired solution.

(ii) Applying the Z-transform and using the linearity property, we obtain

$$Z\{y_{n+2}\} - 2Z\{y_{n+1}\} + Z\{y_n\} = Z\{n\}.$$

Using the shifting property and let $F(z) = Z\{y_n\}$, we obtain

$$z^2[F(z) - y_0 - y_1 z^{-1}] - 2z\,[F(z) - y_0] + F(z) = \frac{z}{(z-1)^2}.$$

Using $y_0 = 1$, $y_1 = 1$ and simplifying, we obtain $F(z) = \dfrac{z^4 - 3z^3 + 3z^2}{(z-1)^4}$.

Since $Z\{1\} = \dfrac{z}{z-1}$, $Z\{n\} = \dfrac{z}{(z-1)^2}$, $Z\{n^2\} = \dfrac{z^2+z}{(z-1)^3}$, $Z\{n^3\} = \dfrac{z^3+4z^2+z}{(z-1)^4}$, writing $F(z)$ as a linear combination of $Z\{1\}$, $Z\{n\}$, $Z\{n^2\}$ and $Z\{n^3\}$, that is,

$$F(z) = \frac{z^4 - 3z^3 + 3z^2}{(z-1)^4} = \frac{A(z^3+4z^2+z)}{(z-1)^4} + \frac{B(z^2+z)}{(z-1)^3} + \frac{Cz}{(z-1)^2} + \frac{Dz}{(z-1)}.$$

This gives $(z^4 - 3z^3 + 3z^2) = A(z^3 + 4z^2 + z) + B(z^2 + z)(z-1) + Cz(z-1)^2 + Dz(z-1)^3$

$$= Dz^4 + (A + B + C - 3D)\,z^3 + (4A - 2C + 3D)\,z^2 + (A - B + C - D)z$$

Comparing the coefficients of z^4, z^3, z^2, z on both sides, we obtain

$$D = 1,\quad A + B + C - 3D = -3,\ 4A - 2C + 3D = 3, \text{ and } A - B + C - D = 0.$$

Solving these for A, B, C and D, we obtain $A = 1/6$, $B = -1/2$, $C = 1/3$, and $D = 1$.

Thus, $$\frac{z^4-3z^2+3z^2}{(z-1)^4}=\frac{1}{6}\left[\frac{z^3+4z^2+z}{(z-4)^4}\right]-\frac{1}{2}\left[\frac{z^2+z}{(z-1)^3}\right]+\frac{1}{3}\left[\frac{z}{(z-1)^4}\right]+\frac{z}{(z-1)}.$$

Taking the inverse Z-transform, $y_n=\frac{1}{6}n^3-\frac{1}{2}n^2+\frac{1}{3}n+1$ is the desired solution.

Example 23.27: Solve the difference equation $y_n+\frac{1}{4}y_{n-1}=f_n+\frac{1}{3}f_{n-1}$, where f_n is the unit impulse function defined by $f_n=\begin{cases}1, n=0\\0, n>0\end{cases}$

Solution: Let $F(z) = Z\{y_n\}$. Applying the Z-transform and using the linearity and shifting properties, we obtain $F(z)+\frac{1}{4}z^{-1}F(z)=1+\frac{1}{3}z^{-1}$

Solving for $F(z)$, we have $F(z)=\left(z+\frac{1}{3}\right)\Big/\left(z+\frac{1}{4}\right)=\frac{z}{z+\frac{1}{4}}+\frac{1}{3}\frac{1}{z+\frac{1}{4}}.$

Taking the inverse Z-transform, we obtain

$$y_n=\left(-\frac{1}{4}\right)^n+\frac{1}{3}\left(-\frac{1}{4}\right)^{n-1}u(n-1), \text{where } u(n-1)=\begin{cases}0, n=0\\1, n\geq 0\end{cases}$$

Example 23.28: Solve the Fibonacci sequence $y_{n+1}=y_n+y_{n-1}$, $n\geq 1$ with $y_o=y_1=1$ using the Z-transform.

Solution: Let $F(z) = Z\{y_n\}$. Applying the Z-transform on both sides and using linearity and shifting properties, we obtain $Z[F(z)-y_o]=F(z)+z^{-1}F(z)$.

Using $y_o=1$ and simplifying,

$$\frac{F(z)}{z}=\frac{z}{z^2-z-1}=\frac{z}{(z-a)(z-b)}=\frac{a}{a-b}\frac{1}{z-a}+\frac{b}{b-a}\frac{1}{z-b},$$

where $a=\frac{1}{2}(1+\sqrt{5}), b=\frac{1}{2}(1-\sqrt{5})$. Hence,

$$F(z)=\frac{a}{a-b}\left(\frac{z}{z-a}\right)+\frac{b}{b-a}\left(\frac{z}{z-b}\right).$$

Taking inverse Z-transform,

$$y_n=\frac{a}{a-b}a^n+\frac{b}{b-a}b^n=\frac{a^{n+1}-b^{n+1}}{a-b}, n=0,1,2, \ldots \text{ is the desired solution.}$$

EXERCISE 23.4

Solve the following difference equations using Z-transforms:

1. $y_{n+1} - 5y_n = 0$
2. $y_{n+2} - 3y_{n+1} + 2y_n = 0;$ $y_0 = -1, y_1 = 2$
3. $y_{n+2} - 2y_{n+1} + y_n = 3n + 2;$ $y_0 = y_1 = 0$
4. $y_{n+2} + 5y_{n+1} + 4y_n = 2^n;$ $y_0 = 1, y_1 = -4$
5. $y_{n+3} - 3y_{n+1} + 2y_n = 0;$ $y_0 = 4, y_1 = 0, y_2 = 8$
6. $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$
7. Using residue method, solve $y_n + \frac{1}{9} y_{n-2} = \frac{1}{3^n} \cos \frac{n\pi}{2}, n \geq 0.$
8. Find the impulse response of the system given by $y_{n+1} + 2y_n = \delta(n);$ $y_0 = 0.$
9. Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u(n); y_0 = y_1 - 1 = 0$ and $u(n) = 1$ for $n \geq 0.$

ANSWERS

Exercise 23.1 (p. 1306)

1. 2, 6, 18, 54, ...
2. $y_1 = 87, y_n = y_{n-1} - 5, n \geq 2$
3. (a) $y_{n+3} - 2y_{n+2} + 2y_{n+1} = 0$ (b) $y_{n+2} - (2+\alpha)\, y_{n+1} + y_n = 0$
4. (a) $(n+2)\, y_{n+2} - 2(n+1)\, y_{n+1} + ny_n = 0.$ (b) $y_{n+2} - 6y_{n+1} + 9y_n = 0$
   (c) $y_{n+2} - 8y_{n+1} + 15y_n = 0$ (d) $y_{n+2} - 4y_n = 0$

Exercise 23.2 (p. 1313)

1. (a) $y_n = \frac{1}{4^n}(c_1 + c_2 n)$ (b) $y_n = c_1 + c_2(-1)n + c_3 2^n$
   (c) $y_n = c_1 \cos n\alpha + c_2 \sin n\alpha$
2. (a) $y_n = 1.5\, n$
3. (a) $y_n = c_1 \left(\alpha + 2 + \sqrt{\alpha^2 + 4\alpha}\right)^n + c_2 \left(\alpha + 2 - \sqrt{\alpha^2 + 4\alpha}\right)^n$
4. (a) $y_n = c_1 (2+i)^n + c_2 (2-i)^n$ (b) $y_n = c_1 2^n + c_2 (-1)^n - n^2 - 5n - 8$
   (c) $y_n = c_1 (-4)^n + c_2 (-1)^n + (18)3^n$ (d) $y_n = c_1 2^n + c_2 (-1)^n + \left(\frac{24}{5} n - \frac{96}{25}\right) 4^n$

(e) $y_n = c_1 2^n + c_2 3^n + \frac{1}{2}(n+1) - n\,2^{n-1}$ (f) $y_n = c_1 \cos \alpha n + c_2 \sin \alpha n + \dfrac{n \sin (n-1)\alpha}{2 \sin \alpha}$

(g) $y_n = c_1 + c_2 n + 2^n (n^2 - 8n + 20)$

5. (b) $x_n = (1.33)\, 2^n - (0.0167)\, 6^n - 0.8n - 0.76 + 4^{n-1}$

$y_n = (1.33)\, 2^n - (0.05)\, 6^n - 0.6n - 1.36 - 4^{n-1}$

(c) $x_n = -2c_1 + c_2 (-2)^n - c_3 + \frac{1}{2} n (3 - n)$

$y_n = c_1 + c_2 (-2)^n + c_3$

$z_n = c_1 + c_2 (-2)^n + \frac{1}{2} n (n-1)$

Exercise 23.3 (p. 1327)

1. (a) $z/z+1$ (b) 1 (c) $\dfrac{z}{z-1}$

(d) $\dfrac{az}{(z-a)^2}$ (e) $z/(z-a^2)$ (f) $z/(z-e^{i\theta})$

(g) $\dfrac{z[z-(1/2)]}{z^2-z+1}$ (h) $\dfrac{az}{z-2} + \dfrac{bz}{z-3}$ (i) $z^2(z+1)/(z-1)^3$

(j) $\dfrac{z(z-\alpha\cos\beta)}{z^2 - 2\alpha(\cos\beta)z + \alpha^2}$ (k) $\dfrac{z(z^2-1)\sin\beta}{[z^2-2(\cos\beta)z+1]^2}$ (l) $e^{(\cos\beta)/z} \cos\left(\dfrac{\sin\beta}{z}\right)$

(m) $e^{2/(az)}$ (n) $\dfrac{z^2 \sin\theta}{z^2 - 2z\cos\theta + 1}$ (o) $\dfrac{z^2\cos\theta - z}{z^2 - 2z\cos\theta + 1}$

5. $\dfrac{z^2(1+3z^2)}{(1-z)(1+z^2)}$

6. (a) 0, 1, 4 (b) 1, 0, −1 (c) 1, −13, 111 (d) 0, 0, 2

7. 1

8. (a) $4a^n$ (b) n

9. (a) $-\frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \dots\right) + \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \dots\right)$

(b) $(-2^{n-1})2^{-n}, \quad n > 0$ (c) $\begin{matrix} (3^{n-1} - 2^{n-1})z^{-n}, & n \geq 1, \\ 0 \quad , & n = 0 \end{matrix}$

10. (a) $2 + \frac{1}{3}(8n-3)(-3)^n$ (b) $-20(2^n) + \frac{1}{3}(60 - 24n + 5n^2)3^n$

11. (a) $\frac{a^{n+1} - b^{n+1}}{a-b}$ (b) $(n+1)a^n$ (c) $\frac{1}{9}[2 + 3n - 2(-2)^n]$

13. (a) $10(2^n - 1)$ (b) $1 + \frac{1}{2}[(i)^{n-2} + (-i)^{n-2}]$

Exercise 23.4 (p. 1331)

1. $y_n = (5^n)\, y_0$
2. $y_n = 3(2^n) - 4$
3. $y_n = \frac{1}{2}n(n-1)(n+3)$
4. $y_n = \frac{19}{18}(-4)^n - \frac{1}{9}(-1)^n + \frac{1}{18}(2^n)$
5. $y_n = \frac{4}{3}[2(-1)^n + 2^n]$
6. $y_n = c_1 y^n + \left(c_2 - \frac{n}{4}\right)2^n + 2^n - \frac{8}{3}$.
7. $y_n = \frac{1}{2}(n+2)\left(\frac{1}{3}\right)^n \cos\frac{k\pi}{2}$
8. $y_n = (-2)^{n-1}, n \geq 1$.
9. $y_n = \frac{1}{2} - 2(2^n) + \frac{3}{2}(3n)$

PART H

Statistical Methods and Linear Programming

24 CHAPTER Descriptive Statistics, Probability and Distributions

The aspects of collection, processing, analysis and interpretation of data are covered under the domain of statistics. Processing of data is important to study their properties and extract information from them. If data are random in nature they are governed by probability laws and also may follow some specific pattern which makes their study quite systematic and informative.

24.1 BASIC CONCEPTS

Statistics is the art of learning from data. *It is a science which deals with the collection, processing, analysis and interpretation of the numerical data.* Normally, we are interested in obtaining information about the total collection of elements, which we refer to as '*population*' e.g., all the cars produced by a particular company during the last year, or all the students enrolled in an institution this year. We try to learn about the population by selecting and then examining a subgroup of its elements (or members), called a '*sample*'. Population may, or may not be finite in size but sample is always finite.

Variables and Data: *A variable is a characteristic that changes or varies over time, or varies for different individuals or objects under consideration.* For example, body temperature is a variable which changes with time for an individual and also it varies from individual to individual at a particular time.

A variable is measured on an individual or object under consideration called the *experimental unit*. The collection of such measurements form the *data*, and the set of all measurements of interest for every experimental unit in the entire collection form the *population*. Any smaller subset of the measurements forms the *sample*.

For example, say we are interested in the body weights of the trainees registered in a sports academy, then the set of measurements of the body weights of all the trainees, say 500 form the population. If we select say 50 trainees out of the 500 registered by adopting certain methodology, then the set of measurements of the body weights of these selected 50 trainees forms the sample.

If a single variable is measured on a single experimental unit, then data obtained is *univariate data*, e.g., blood pressures of employees working in an organization. When two variables are measured on a single experimental unit then data obtained is *bivariate data*, e.g., blood pressure and weight of employees. Similarly, if our interest lies in more than two characteristics then data obtained is *multivariate data*.

Categories of variables: Variables can be classified in two categories: *qualitative* and *quantitative*.

A *qualitative variable* measures a quality or characteristic on each experimental unit. For example, colour of the skin: fair, wheatish, black; performance of an individual: excellent, good, fair, average, poor.

A *quantitative variable* measures a quantity or amount on each experimental unit. For example, weight, height, or marks obtained by the experimental unit under observation. Further quantitative variables are of two types: *discrete* and *continuous*. *If a variable can assume only finite or at the most countabley infinite number of values then it is said to be discrete. For example, number of students in a class. However, if a variable can assume infinite number of values between two specific limits, then it is said to be continuous*, e.g., time, temperature, weight all are continuous variables. In case of continuous variable, for any two values selected a third value can always be found between the two.

24.2 DATA REPRESENTATION

The numerical findings of a study should be presented in such a systematical manner so that an observer is in a position to grasp the essential characteristics of the data. The data can be represented numerically or graphically in various ways. In this section we present some common graphical and tabular ways for data presentation.

Frequency Tables and Graphs

A data set having a relatively small number of distinct values can be presented in the form of a frequency table. For example, the following frequency table gives the starting salary per month for graduate students of a class of 50 students.

Starting salary (in 000' Rs.)	*Frequency*
40	2
42	3
44	4
45	5
47	5
48	7
49	8
51	7
52	4
54	3
55	2
Total	50

Data can be graphically represented by a line graph by plotting data values along x-axis and corresponding frequencies along y-axis as shown in Fig. 24.1.

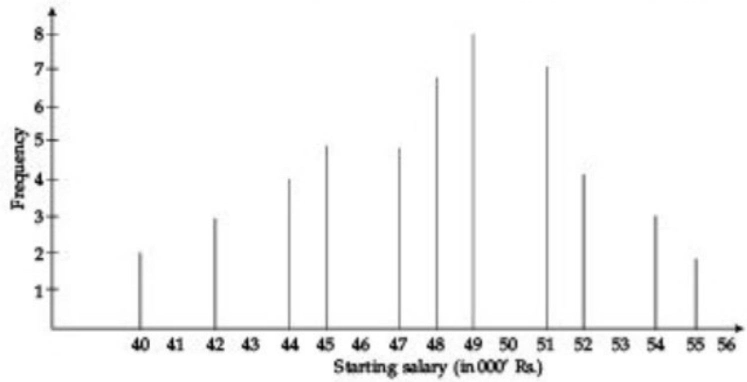


Fig. 24.1

The above frequency distribution is called the *ungrouped frequency distribution*. This frequency distribution, particularly when the data is large can be condensed in the form of the following *grouped distribution*.

Salary-group (class)	*No. of candidates (frequency)*
40 – 44, (40 and under 44)	05
44 – 48	14
48 – 52	22
52 – 56	09
Total	50

This grouped frequency distribution can be represented graphically by *histogram*, *frequency polygon*, *frequency curve* and *cumulative frequency curve (s)*.

Histogram

A histogram is composed of a set of rectangles, one over each class interval on the horizontal scale. The area of the rectangles are taken proportional to the frequencies of the classes. Thus, in case of equal class-interval, the heights of the rectangles are proportional to the frequencies while for classes of unequal width, the heights are taken proportional to the ratio of the frequencies to the width of the corresponding classes. Figure 24.2 is the histogram for the grouped frequency distribution given above.

The basis of the rectangles in Fig. 24.2 are the class-intervals 40 – 44, 44 – 48, 48 – 52, 52 – 56 whose mid-points (class marks) are $x = 42, 46, 50, 54$, respectively. Since the class widths are equal, heights of the rectangles are proportional to respective class frequencies 5, 14, 22 and 9. Thus, areas of these rectangles are proportional to these class frequencies.

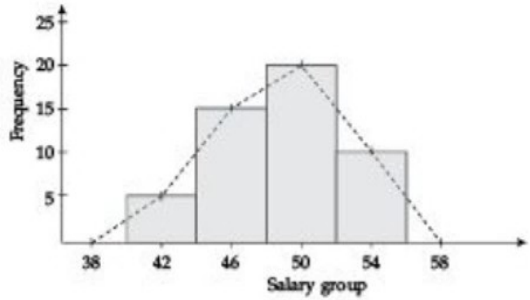


Fig. 24.2

Frequency Polygon and Frequency Curve

The *frequency polygon* of a grouped frequency distribution is obtained by joining the points whose abscissae are the mid-points, that is, *class marks* of the classes and the ordinate are the corresponding frequencies by means of straight lines. This can be obtained from a histogram by joining the mid-points of the upper sides of the adjacent rectangles by means of straight lines. The points on the horizontal axis at the mid points of the intervals, immidiately preceeding and immidiatey proceding the intervals that contain observations are also joined, as shown in Fig. 24.2.

If the class-intervals are of small width the frequency polygon can be approximated by a smooth *frequency curve* obtained by drawing a smooth free-hand curve through the vertices of the frequency polygon.

Cumulative Frequency Curve(s), or Ogive

Sometimes we are interested in plotting a cumulative frequency curve, or an ogive. There are two types of cumulative frequencies *'less than'* and *'greater than'*. For example, the grouped frequency distribution for the salary of group of 50 students can be expressed as

Salary (less than)	*Cumulative frequency*
44	05
48	19
52	41
56	50

or, it can be expressed as

Salary (greater than)	*Cumulative frequency*
40	50
44	45
48	31
52	09

To plot cumulative frequency curve the limits of the classes are taken along the horizontal line and corresponding cumulative frequencies along the vertical line. The two cumulative frequency curves for the data given above are shown in Fig. 24.3.

The various frequency curves are useful in getting the idea about the nature of the distribution.

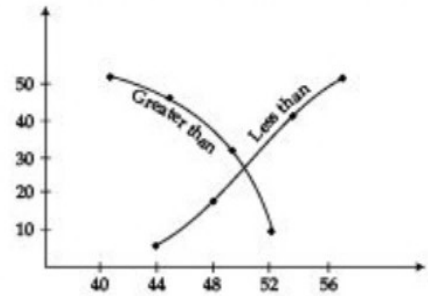


Fig. 24.3

24.3 DESCRIPTIVE MEASURES

In this section, we develop numerical measures to describe the data set. These measures can provide a great deal of information about the set of observation under consideration.

24.3.1 Measures of Central Tendency, or Averages

These are the measures that are used for describing the central location of a set of data values and are representatives of the entire distribution. The commonly used measures are:

I. Arithmetic mean, or simple mean,

II. Median,

III. Mode,

IV. Geometric mean,

V. Harmonic mean.

An *ideal measure of central tendency* should be rigidly defined, easy to calculate, based on all observations, least affected by fluctuation of sampling and by extreme values. In addition to these, it should be amenable for further mathematical treatment.

I. Arithmetic mean: *Given a set of n observations $x_1, x_2, \ldots, x_n$, the arithmetic mean, or simply mean, is defined as the sum of the observations divided by the sample size.* It is given by

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

In case of ungrouped frequency distribution where the value x_i occurs with frequency f_i, (normally represented as $x_i | f_i$, $i = 1, 2, \ldots n$), it is given by

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i, \text{ where } N = \sum_{i=1}^{n} f_i. \qquad \ldots(24.1)$$

In case of grouped frequency distribution, x_i is taken as the mid-point of the ith class-interval.

II. Median: *It is that value of the variable which divides the data set into two equal parts when arranged in increasing (or, decreasing) order.* The median is thus a positional average. If n is odd the median is the value in position $(n+1)/2$; if n is even it is the average of the values in position $n/2$ and $(n/2)+1$.

In case of ungrouped frequency distribution median is obtained by considering cumulative frequencies ($c.f.$).

In the case of grouped frequency distribution the median is obtained by the formula

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - C\right), \qquad ...(24.2)$$

where, l is the lower limit of the *median class*, that is, the class corresponding to which the *c.f.* is just greater than $N/2$, f is the frequency of the median class, h is the width of the median class, and C is the *c.f.* of the pre-median class.

III. Mode: *It is that value of the variable which occurs with the greatest frequency.* If no single value occurs most frequently, then all the values that occur with the highest frequency are considered as the modal values.

In case of grouped frequency distribution the mode is given by the formula

$$\text{Mode} = l + \frac{h(f_m - f_1)}{2f_m - (f_1 + f_2)}, \qquad ...(24.3)$$

where, l is the lower limit of the *modal class*, that is, the class corresponding to which the frequency is maximum, f_m is the frequency of the modal class, f_1 and f_2 respectively are the frequencies of the *pre-modal class* and *post-modal class*, and h is the width of the modal class.

Example 24.1: The following frequency distribution gives the values obtained in 60 rolls of a dice. Find (a) the mean, (b) the median, and (c) the mode.

Value (x)	1	2	3	4	5	6
Frequency (f)	9	8	12	11	13	7

Solution: (a) The mean is $\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i = \frac{9 + 16 + 36 + 44 + 65 + 42}{60} = \frac{212}{60} \simeq 3.5$

(b) The median is the average of 30th and 31st values (data is arranged in the increasing order). It is thus 4.

(c) The mode is 5 corresponding to which frequency 13 is the maximum.

Before introducing an example to find arithmetic mean for a grouped frequency distribution, we would introduce shifting of origin and scale which helps to reduce a lot of arithmetic while calculating mean, particularly when the class frequencies and class marks are large enough.

Shifting of origin and scale: Let us suppose that origin in x is shifted to a point a called the *assumed mean* and scale be made h times the original scale, then if u is the new variable we have

$$u = \frac{x - a}{h}, \text{ or } x = a + hu, \text{ and thus, } \bar{x} = \frac{1}{N}\Sigma fx = \frac{1}{N}\Sigma f(a + hu) = a + h\bar{u}.$$

We can choose a and h as any suitable values depending upon the data given. Normally, a is the mid-point of the class-interval corresponding to which frequency is maximum and h is taken as the *H.C.F.* of the widths of the various class-intervals.

Example 24.2: The following data gives the frequencies of serum cholesterol level of 1000 males aged between 25 to 35 years arrived at a particular city hospital during the last one year.

Chostestrol level (mg/100 ml.)	*Number of males*
80 – 120	12
120 – 160	145
160 – 200	380
200 – 240	292
240 – 280	118
280 – 320	35
320 – 360	11
360 – 400	07

Calculate (a) the mean, (b) the median, and (c) the mode for the data.

Solution: (a) To calculate mean take $a = 180$ and $h = 40$ and formulate the following table

Class-interval	*Mid-value (x)*	*Frequency (f)*	$u = \frac{x-180}{40}$	*fu*
80 – 120	100	12	–2	–24
120 – 160	140	145	–1	–145
160 – 200	180	380	0	0
200 – 240	220	292	1	292
240 – 280	260	118	2	236
280 – 320	300	35	3	105
320 – 360	340	11	4	44
360 – 400	380	07	5	35
Total		1000		591

The mean, $\bar{x} = a + h\,\bar{u} = 180 + 40\left(\frac{591}{1000}\right) = 180 + 23.64 = 203.64$ mg/100 ml.

(b) To calculate median formulate the cumulative frequencies table:

Class-interval	*Frequency (f)*	*Cumulative frequency (C)*
80 – 120	12	12
120 – 160	145	157
160 – 200	380	537
200 – 240	292	829
240 – 280	118	947
280 – 320	35	982
320 – 360	11	993
360 – 400	07	1000

We have, $\frac{N}{2} = 500$. Thus 160 – 200 is the median class. Therefore,

$$l = 160, \qquad f = 380, \qquad h = 40, \qquad C = 157$$

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - C\right) = 160 + \frac{40}{380}(500 - 157) = 160 + 36.11 = 196.11 \text{ mg/100 ml.}$$

(c) Since, the maximum frequency is 380, thus 160 – 200 is the modal class. Therefore,

$$l = 160, \quad f_m = 380, \quad f_1 = 145, \quad f_2 = 292, \quad h = 40.$$

$$\text{Mode} = l + \frac{(f_m - f_1)h}{2f_m - (f_1 + f_2)} = 160 + \frac{(380 - 145)40}{2(380) - (145 + 292)} = 160 + 29.10 = 189.10 \text{ mg/100 ml.}$$

Properties of arithmetic mean

We state below a few properties satisfied by the arithmetic mean which can be proved very easily.

1. *Algebraic sum of the deviations of a set of observations from their arithmetic mean is zero, that is, for the frequency distribution* $x_i | f_i$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} f_i(x_i - \bar{x}) = 0$, *where* $\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i$.

2. *The sum of the squares of the deviations of a set of observations is minimum about its mean, that is,* $\sum_{i=1}^{n} f_i(x_i - a)^2$ *is minimum at* $a = \bar{x}$.

3. *The mean of the combined sample is the weighted mean of the individual sample means, that is, if* $\bar{x}_i$ *is the mean of the ith sample of size* n_i, $i = 1, 2, \ldots, k$, *then the mean* $\bar{x}$ *of the combined sample of size* $\sum_{i=1}^{k} n_i$ *is,*

$$\bar{x} = \sum_{i=1}^{k} n_i \bar{x}_i \Big/ \sum_{i=1}^{k} n_i.$$

IV. Geometric mean: *The geometric mean G of a set of N observations is the Nth root of their product, that is, if* $x_i | f_i$, $i = 1, 2, \ldots, n$ *is the frequency distribution, then*

$$G = \left(x_1^{f_1} x_2^{f_2} \ldots x_n^{f_n}\right)^{\frac{1}{N}}, \text{ where } \quad N = \Sigma f_i. \qquad \ldots(24.4)$$

This can be written as

$$\ln G = \frac{1}{N}\sum_{i=1}^{n} f_i \ln x_i, \quad \text{or} \quad G = \text{antilog}\left(\frac{1}{N}\sum_{i=1}^{n} f_i \ln x_i\right),$$

an expression normally used for computational purpose.

Obviously it is not practicable to calculate G when one of the values is zero or non-negative.

V. Harmonic mean: *The harmonic mean H of a set of N non-zero observations is the reciprocal of the arithmetic mean of the reciprocals of the data values, that is, if $x_i | f_i$, $i = 1, 2, \ldots, n$ is the frequency distribution, then*

$$H = \frac{1}{\frac{1}{N}\sum_{i=1}^{n}(f_i/x_i)}; \quad N = \sum_{i=1}^{n} f_i. \qquad \ldots(24.5)$$

There are specific situations when a particular measure of central tendency is appropriate to use. For example, both mean and median each provide a single number to represent an entire set of data, the mean is usually preferred in problems of estimation and statistical inference since it depends upon all the information contained in the data set. The main advantage of median over mean is that it is least affected by the extreme values, for example, median salary is the more representative one than the mean salary in a small company where the top management is highly paid.

Mode is normally most suitable to use to find the ideal size, for example, in manufacturing the readymade garments, shoes, etc.

Geometric mean is used to find the average rate of population growth and construction of index numbers. Harmonic mean attaches greater importance to numerically small observation and hence is useful when situation demands so. In addition to this, in certain situations true average of a data set is given by the harmonic mean of observation values and not by the arithmetic mean. For example, if a motorist travels first 50 km with a speed of 60 km/hr, next 50 km with a speed of 75 km/hr and another 50 km with a speed of 80 km/hr, then his true average speed will be H.M. of 60, 75 and 80 and not the A.M. of 60, 75 and 80. However, barring any particular situation the arithmetic mean is the most suitable measure.

Partition Values: The median value divides a frequency distribution in two equal parts. Partition values divide the distribution into a number of equal parts.

The three values Q_i, $i = 1, 2, 3$ which divide the distribution in four equal parts are called ***quartiles****. Obviously Q_2 is the median.*

The nine values D_i, $i = 1, 2, \ldots, 9$ which divide the distribution in ten equal parts are called ***deciles*** *and the ninety nine values P_i, $i = 1, 2, \ldots, 99$ which divide it in hundred equal parts are called* ***percentiles****.*

For a grouped data quartiles can be calculated by using the formula

$$Q_i = l + \frac{h}{f}\left(\frac{iN}{4} - C\right), i = 1, 2, 3. \qquad \ldots(24.6)$$

where, l, h, f and C have their usual meanings in respect to a particular quartile class. Similar formulae hold for deciles and percentiles.

24.3.2 Measures of Dispersion or Variability

The measures of central tendency do not provide a proper summary of the nature of a set of observations. We are always interested in the scatteredness (or, dispersion) of the data values also. So in order to get greater insight, the measures of averages must be supported and supplemented by the measures of dispersion, which give the degree to which numerical data tend to spread about an average value.

The commonly used measures of dispersion are:

I. Range,

II. Inter-quartile range,

III. Variance and standard deviation,

IV. Coefficient of variation.

I. Range: *The range of a set of observations is defined as the difference between the largest and the smallest observations.* Although range is easy to compute but its usefulness is limited since it depends only on two extreme values. Normally, it is employed in statistical quality control.

II. Inter-quartile range: It is given by $(Q_3 - Q_1)$, where Q_1 and Q_3 are the first and third quartiles of the set of observations. It is definitely better than the range since it makes use of the 50% of the data set.

III. Variance and standard deviation: Variance is the most commonly used measure of dispersion for the given set of observations. *It quantifies the amount of variability, or scatteredness around the mean of the measurements.*

If $x_i \mid f_i$, $i = 1, 2, ..., n$ is the frequency distribution, then its variance denoted by σ^2 is defined as

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2. \qquad ...(24.7)$$

It is also called the *mean square deviation of x_i's from the mean $\bar{x}$.*

The standard deviation (S.D.) *of a set of observations is the positive square root of the variance and is denoted by* σ. Thus,

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}. \qquad ...(24.8)$$

It is also called the root mean square deviation of x_i's from the mean $\bar{x}$.

We note that the unit for the variance is the square of the unit in which x_i's are measured and for S.D. it is the same as that of x_i's.

Computation of the variance. The computation of the variance can be eased by shifting of origin and scale, that is, by considering $u = (x - a)/h$, or $x = a + hu$; and so

$$\sigma_x^2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 = \frac{1}{N}\sum_{i=1}^{n} f_i\,[(a + hu_i) - (a + h\bar{u})]^2 = \frac{h^2}{N}\sum_{i=1}^{n} f_i(u_i - \bar{u})^2,$$

or, $$\sigma_x^2 = h^2\sigma_u^2 \qquad ...(24.9)$$

which gives $$\sigma_x = |h|\,\sigma_u. \qquad ...(24.10)$$

Thus, there is no effect of change of origin on the variance but there is effect of change of scale on the variance as given by (24.9).

Also consider

$$\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 = \sum_{i=1}^{n} f_i(x_i^2 + \bar{x}^2 - 2x_i\bar{x}) = \sum_{i=1}^{n} f_i x_i^2 + N\bar{x}^2 - 2N\bar{x}^2 = \sum_{i=1}^{n} f_i x_i^2 - N\bar{x}^2.$$

Thus, the variance given by (24.7) can be written as

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \left(\frac{1}{N}\Sigma f_i x_i\right)^2. \quad ...(24.11)$$

We use the expression (24.11) for computation of the variance, since it reduces the calculations to a great extent.

Example 24.3: Following data give the daily emission (in tonnes) of sulphur oxides from an industrial plant observed for 80 days. Calculate the mean, variance and S.D. of the daily emission.

Emission (in tonnes)	*No. of days*
5 – 9	3
9 – 13	10
13 – 17	14
17 – 21	25
21 – 25	17
25 – 29	9
29 – 33	2
Total	80

Solution: We take $a = 19$ and $h = 4$ and formulate the following table:

Class interval	*Mid-point (x)*	*Frequency (f)*	$u = \frac{x-19}{4}$	f_iu_i	$f_iu_i^2$
5 – 9	7	3	–3	– 9	27
9 – 13	11	10	–2	– 20	40
13 – 17	15	14	–1	– 14	14
17 – 21	19	25	0	0	0
21 – 25	23	17	1	17	17
25 – 29	27	9	2	18	36
29 – 33	31	2	3	6	18
Total		80		– 2	152

Thus, the mean $\quad \bar{x} = a + h\bar{u} = 19 + 4\left(\frac{-2}{80}\right) = 18.90$ tonnes

The variance $\quad \sigma_u^2 = \frac{1}{N}\Sigma f_i u_i^2 - \left(\frac{1}{N}\Sigma f_i u_i\right)^2 = \frac{152}{80} - \left(\frac{-2}{80}\right)^2 = 1.9 - .000625 = 1.899375$

Hence $\quad \sigma_x^2 = 16\ \sigma_u^2 = 30.39$ sq. tonnes.

Thus, S.D. $\quad \sigma_x = 5.51$ tonnes.

IV. Coefficient of variation: It does not make any sense to compare the variability of the two frequency distributions which differ widely in their averages or measured in different units. In such a situation we use *coefficient of variation (C.V.), which is the ratio of the standard deviation σ to the mean $\bar{x}$ of a data set multiplied by 100. It is a measure of relative variability and is a dimensionless number.* The C.V. for a data set is given by

$$\text{C.V.} = \frac{\sigma}{\bar{x}} \times 100. \qquad \text{...(24.12)}$$

The ratio $\sigma/\bar{x}$ is called the *coefficient of dispersion based upon S.D.*

Example 24.4: The following data gives the weight and chest size of 10 infants at birth in a city hospital. Compare the variability of the two characteristics in the infants

Weight (in kg)	*Chest size (in cm.)*
2.75	29.1
3.12	30.1
4.15	32.1
5.50	36.1
3.20	30.2
4.32	33.1
2.31	28.2
5.12	35.1
4.12	31.9
3.72	31.1

Solution: We formulate the following table.

	Weight (x)	*Chest size (y)*	x^2	y^2
	2.75	29.1	7.56	846.81
	3.12	30.1	9.73	906.01
	4.15	32.1	16.93	1030.41
	5.50	36.1	30.25	1303.21
	3.20	30.2	10.24	912.04
	4.32	33.1	18.66	1095.61
	2.31	28.2	5.34	795.24
	5.12	35.1	26.21	1232.01
	4.12	31.9	16.97	1017.61
	3.72	31.1	13.83	967.21
Total	38.31	317.0	155.72	10106.16

We have, $\quad n = 10, \quad \bar{x} = \frac{1}{n}\Sigma x_i = 3.83, \quad \bar{y} = \frac{1}{n}\Sigma y_i = 31.7$

$$\sigma_x^2 = \frac{1}{n}\Sigma x_i^2 - (\bar{x})^2 = 15.57 - 14.67 = 0.9$$

$$\sigma_y^2 = \frac{1}{n}\Sigma y_i^2 - (\bar{y})^2 = 1010.61 - 1004.89 = 5.72$$

Thus, C.V. $(x) = \frac{\sigma_x}{\bar{x}} \times 100 = 24.77$, and C.V. $(y) = \frac{\sigma_y}{\bar{y}} \times 100 = 7.54$.

Since C.V. for infant weight is greater than the C.V. for infant chest size, thus infant weight is more variable than infant chest size.

24.3.3 Moments

The rth moment of a variable x about any point $x = a$, also called the ordinary moment, denoted by μ_r', is defined by

$$\mu_r' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - a)^r; \quad N = \sum_{i=1}^{n} f_i. \qquad ...(24.13)$$

In case $a = \bar{x}$, then the rth moment about mean, also called the central moment denoted by μ_r, is defined by

$$\mu_r = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^r \qquad ...(24.14)$$

In particular, we have $\mu_0 = 1$, $\mu_1 = 0$, $\mu_2 = \sigma^2$.

Also $\mu_0' = 1$, $\mu_1' = \bar{x} - a$. Thus if $a = 0$, then $\mu_1' = \bar{x}$, hence *first moment about the origin is the mean.*

Relation between μ_r and μ_r'. We have

$$\mu_r = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^r = \frac{1}{N}\sum_{i} f_i[(x_i - a) - (\bar{x} - a)]^r$$

$$= \frac{1}{N}\sum_{i} f_i\left[C_0^r(x_i - a)^r - C_1^r(x_i - a)^{r-1}(\bar{x} - a) + C_2^r(x_i - a)^{r-2}(\bar{x} - a)^2 + \ldots + (-1)^r C_r^r(\bar{x} - a)^r\right]$$

$$= \mu_r' - C_1^r\mu_{r-1}'\,\mu_1' + C_2^r\,\mu_{r-2}'(\mu_1')^2 + \ldots + (-1)^r(\mu_1')^r. \qquad ...(24.15)$$

In particular, for $r = 2, 3, 4$ we have

$$\left.\begin{aligned} \mu_2 &= \mu_2' - (\mu_1')^2 \\ \mu_3 &= \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 \\ \mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 \end{aligned}\right\} \qquad ...(24.16)$$

These formulae give us moments about mean once the moments about any arbitrary point a are known. Since moments about mean are particularly employed to know the nature of the frequency distribution, hence the above noted formulae are quite useful.

We can check very easily that *there is no effect of change of origin on the central moments but there is effect of change of scale. The rth moment of the variable x about mean is h^r times the rth moment of the variable u about mean when $u = (x - a)/h$.*

Pearson's β and γ coefficients. The four important coefficients are defined as follows:

$$\left.\begin{aligned} \beta_1 &= \frac{\mu_3^2}{\mu_2^3}, \quad \gamma_1 = \sqrt{\beta_1} \\ \beta_2 &= \frac{\mu_4}{\mu_2^2}, \quad \gamma_2 = \beta_2 - 3 \end{aligned}\right\} \qquad \text{...(24.17)}$$

These four coefficients are pure numbers, that is, they are independent of origin and scale of the measurement of the variate and are widely used to know the nature of the frequency distribution.

24.3.4 Skewness and Kurtosis

Skewness means *lack of symmetry. A distribution is 'symmetric', or 'normal' when the frequencies are symmetrically distributed about mean,* refer to Fig. 24.4a. Any data set which is not approximately symmetric about its mean is said to be *skewed.* It is skewed to the right, called the *positively skewed,* if it has longer tail to the right, refer to Fig. 24.4b, and skewed to the left, called the *negatively skewed,* if it has longer tail to the left, refer to Fig. 24.4c.

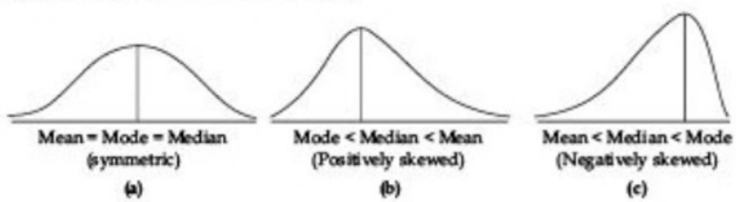


Fig. 24.4

Next we discuss the various *measures of skewness.*

For a symmetrical distribution mean = median = mode.

Skewness is positive, if mean > mode, and skewness is negative if mean < mode. The expression, $\dfrac{\text{mean} - \text{mode}}{\text{S.D.}}$ called the *Karl Pearson's coefficient of skewness,* gives a measure of skewness. It is a dimensionless number and is zero for a symmetric distribution.

In the case of symmetric distribution, all the odd order moments about the mean are equal to zero, hence $\beta_1 = \mu_3^2/\mu_2^3$ is also equal to zero. Thus, β_1 gives a measure of skewness.

***Kurtosis** is a measure which gives us an idea about the flatness or peakedness of the frequency curve.*

For example, as shown in Fig. 24.5, all the three curves A, B and C are symmetric about mean and are spread over the same range but are different in respect of their flatness.

Curve B which is neither flat nor peaked is called the *mesokurtic or normal curve.* Curve A which is peaked than the normal curve is called the *leptokurtic curve* and the curve C which is flat than the normal curve is called the *platykurtic curve.*

The value of β_2 gives a *measure of kurtosis of a curve*. We have

$\beta_2 = 3$ or $\gamma_2 = 0$, for a normal curve,

$\beta_2 > 3$ or $\gamma_2 > 0$, for a leptokurtic curve,

$\beta_2 < 3$ or $\gamma_2 < 0$, for a platykurtic curve.

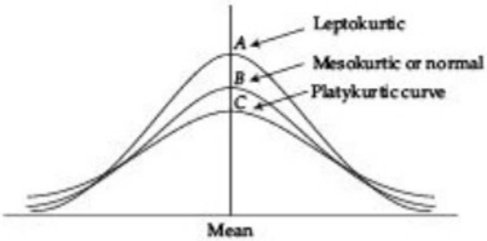


Fig. 24.5

Example 24.5: The first four moments of a distribution about the value 4 of the variable are -1.5, 17, -30 and 108. Find the moments about the mean. Calculate β_1 and β_2 and comment upon the nature of the frequency distribution.

Solution: We have, $a = 4$, $\mu_1' = -1.5$, $\mu_2' = 17$, $\mu_3' = -30$ and $\mu_4' = 108$.

The moments about mean are

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 17 - (-1.5)^2 = 17 - 2.25 = 14.75$$

$$\mu_3 = \mu_3' - \mu_2'\mu_1' + 2(\mu_1')^3 = -30 - (17)(-1.5) + 2(-1.5)^3 = 39.75$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = 108 - 4(-30)(-1.5) + 6(17)(-1.5)^2 - 3(-1.5)^4 = 142.31$$

Thus, $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{(39.75)^2}{(14.75)^3} = 0.49$, and $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{142.31}{(14.75)^2} = 0.65$

Since $\beta_1 > 0$, the distribution is positively skewed. Also $\beta_2 < 3$, so the distribution is platykurtic, that is, flat.

Example 24.6: Calculate the first four moments for the following frequency distribution about the mean and comment upon the nature of the frequency distribution

x:	-4	-3	-2	-1	0	1	2	3	4
f:	3	4	5	7	12	7	5	4	3

Solution: We formulate the following table:

x	f	fx	fx^2	fx^3	fx^4
-4	3	-12	48	-192	768
-3	4	-12	36	-108	324
-2	5	-10	20	-40	80
-1	7	-7	7	-7	7
0	12	0	0	0	0
1	7	7	7	7	7
2	5	10	20	40	80
3	4	12	36	108	324
4	3	12	48	192	768
Total	50	0	222	0	2358

The moments about $x = 0$ are

$$\mu_1' = 0, \qquad \mu_2' = \frac{222}{50} = 4.44, \qquad \mu_3' = 0, \qquad \mu_4' = \frac{2358}{50} = 47.16$$

Since $\mu_1' = 0$, therefore, $\bar{x} = 0$. Hence these are the moments about mean also. Thus,

$$\mu_1 = 0, \qquad \mu_2 = 4.44, \qquad \mu_3 = 0, \qquad \mu_4 = 47.16.$$

Also, $$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0, \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{47.16}{(4.44)^2} = 2.39$$

Since, $\beta_1 = 0$ and $\beta_2 < 3$ thus, the frequency curve is symmetric but is platykurtic.

EXERCISE 24.1

1. The following numbers give the weight of 55 students of a class

42	74	40	60	82	115	41	61	75	83	63
53	110	76	84	50	67	65	78	77	56	95
68	69	104	80	79	79	54	73	59	81	100
66	49	77	90	84	76	42	64	69	70	80
72	50	79	52	103	96	51	86	78	94	71

(a) Form a grouped frequency distribution having classes 40-50, 50-60, ... 100-110.

(b) Construct a histogram.

(c) Draw cumulative frequency curves 'less than' and 'greater than' both.

2. The following data gives the ages of the 40 coins in circulation, where

Age = Current year – Year on coin

5	1	9	1	2	25	0	25
1	4	4	3	0	25	3	3
5	21	19	9	0	5	0	2
0	1	19	0	2	0	20	16
19	36	23	0	1	17	6	0

Draw a histogram to describe the distribution of coin ages. How do you describe the shape of the distribution?

3. Determine the mean, median and mode for the following data values:

(a) 3, 10, 8, 7, 5, 14, 2, 9, 8

(b) 73.8, 126.4, 40.7, 141.7, 28.5, 237.4, 157.9

4. The gain of 90 similar transistors is measured and the results are recorded as given below:

Gain:	83.5-85.5	86.5-88.5	89.5-91.5	92.5-94.5	95.5-97.5
No. of transistors:	6	39	27	15	3

Determine the mean, median and the modal values of the distribution.

5. An incomplete frequency distribution is given as follows.

Class-interval	Frequency	Class-interval	Frequency
10 – 20	12	50 – 60	?
20 – 30	30	60 – 70	25
30 – 40	?	70 – 80	18
40 – 50	65		
		Total	229

If the median value is 46, determine the missing frequencies.

6. The nicotine content in milligrams for 40 cigarettes of a certain brand are given as follows:

1.09	1.92	2.31	1.79	2.28
1.74	1.47	1.97	0.85	1.24
1.58	2.03	1.70	2.17	2.55
2.11	1.86	1.90	1.68	1.51
1.64	0.72	1.69	1.85	1.82
1.79	2.46	1.88	2.08	1.67
1.37	1.93	1.40	1.64	2.09
1.75	1.63	2.37	1.75	1.69

Find the mean, median and S.D.

7. Find the mean and variance of the first n natural numbers.

8. Find the median, lower and upper quartiles, 4th decile and 60th percentile for the following data.

Marks	*No. of Students*	*Marks*	*No. of Students*
0-4	10	14-18	5
4-8	12	18-20	8
8-12	18	20-25	4
12-14	7	25 and above	6

9. Following data gives the daily emission (in tonnes) of SO_2 observed for 80 days from an industrial unit. Find Q_1, Q_3 and the inter-quartile range.

Class-interval :	5-9	9-13	13-17	17-21	21-25	25-29	29-33
Frequency :	3	10	14	25	17	9	2

10. In an electrical engineering class, there are 20 juniors, 15 seniors, and 5 graduate students. If the juniors averaged 71 in the mid-term examination, the seniors averaged 80 and the mean for the entire class is 76.625, then find the average for the graduates.

11. Calculate Karl Pearson's coefficient of skewness for the following frequency distribution

Class-interval :	383-387	388-392	393-397	398-402	403-407
Frequency :	8	10	15	17	8

12. Calculate the coefficients of skewness and kurtosis for the following data and comment upon the nature of the distribution.

Class-interval :	0-4	4-8	8-12	12-16	16-20
Frequency :	4	10	6	12	8

13. The first four moments of a distribution about the value 5 are –4, 22, –117 and 560. Obtain the different characteristics of the distribution on the basis of the information provided.

24.4 PROBABILITY: CLASSICAL, STATISTICAL AND AXIOMATIC CONCEPTS

So far we have studied to organize and summarize a data set. However, we are always interested in utilizing the information contained in the sample to infer the characteristics of the population from which it was drawn. The theory of probability forms the basis for statistical inference. In this section, we develop the concept of probability. Before introducing the definition(*s*) of probability, we explain the various terms to be employed.

Random experiment: An experiment is the process by which an observation is obtained. *A random experiment is an experiment if in each trial of which, conducted under identical conditions, the outcome is not unique, but may be one of the possible outcomes*. For example, tossing a coin, throwing a dice or a set of dice, etc.

When a random experiment is performed, the outcome observed is called a *simple event*. An *event* is a collection of one or more simple events. For example, when a dice is tossed once there are six possible simple events, observing face 1, 2, 3, 4, 5, or 6. However, we can define the event A as 'observing face less than 4' which is a collection of three simple events, observing face 1, 2 or 3. The total number of possible outcomes of a random experiment is known as *exhaustive events*. For example, in tossing a coin the set $\{H, T\}$ forms the set of exhaustive cases while, in case of throw of a dice the exhaustive set is $\{1, 2, 3, 4, 5, 6\}$.

Mutually exclusive events: *Events are called mutually exclusive, if the occurrence of one of them precludes the occurrence of all the others*. For example, in tossing a coin the two outcomes head and tail are mutually exclusive.

Equally likely events: *Events are called equally likely when we have no reason to expect one in preference to the others*. For example, as the result of drawing a card from a well-shuffled pack of 52 cards may appear as the result of the draw.

Favourable cases: *Cases which entail or favour the happening of an event A are said to be favourable cases to the event A*. For example, if A is an event that in a throw of a dice face obtained is less than 4, then 1, 2, 3 forms a set of three favourable cases to the event A.

Now we are in a position to discuss the concept of probability.

24.4.1 Classical Probability Concept

If a trial may result in one of the 'n' exhaustive, mutually exclusive and equally likely cases out of which m are favourable to the happening of an event A, then the probability 'p' that the event A will happen as the result of the trial, is given by

$$p = P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{m}{n}$$

Clearly it follows from this definition that $0 \le P(A) \le 1$ and also, if $\overline{A}$, or A^c is the event '*not* *happening'* of A, then $q = P(\overline{A}) = \dfrac{n-m}{n} = 1 - \dfrac{m}{n} = 1 - P(A)$, so that $P(A) + P(\overline{A}) = 1$.

An event A with $P(A) = 0$ is called an *impossible event* and an event A with $P(A) = 1$ is called a *sure event*.

Example 24.7: From a pack of 52 cards two are drawn at random. Find the chance that one is a king and the other a queen.

Solution: Let A be the event that the two cards drawn are a king and a queen.

$$\text{No. of exhaustive cases} = {}^{52}C_2 = \frac{52!}{50!2!} = 26 \times 51 = 1326$$

No. of cases favourable to $A = 4 \times 4$.

Therefore, $$P(A) = \frac{16}{1326} = .01207.$$

Example 24.8: A five-figure number is fomed by the numbers 0, 1, 2, 3, 4, without repetition. Find the probability that number formed is divisible by 4.

Solution: The five digits can be arranged in 5! ways and out of these 4! will begin with zero. Thus, number of five digits formed, that is, exhaustive cases are $= 5! - 4! = 96$.

A number will be divisible by 4 which will have two extreme right digits divisible by 4, that is, the number ending with 04, 12, 20, 24, 32 and 40.

Numbers ending with $04 = 3! = 6$
Numbers ending with $12 = 3! - 2! = 4$
Numbers ending with $20 = 3! = 6$
Numbers ending with $24 = 3! - 2! = 4$
Numbers ending with $32 = 3! - 2! = 4$
Numbers ending with $40 = 3! = 6$.

Thus total numbers divisible by 4, that is, favourable cases are $= 6 + 4 + 6 + 4 + 4 + 6 = 30$.

Hence the required probability $= \dfrac{30}{96} = 0.3125$.

Example 24.9: If 5 of 20 fuses in a box are defective and 5 of them are randomly chosen for inspection, what is the probability that two of the defective fuses will be included?

Solution: Let A be event that out of 5 selected, two will be defective and three will be non-defective.

$$\text{Exhaustive cases} = {}^{20}C_5 = \frac{20!}{15!5!} = 15504.$$

$$\text{Cases favourable to } A = C_2^5 \times C_3^{15} = \frac{5!}{3!2!} \times \frac{15!}{12!3!} = 4550.$$

Therefore, $$P(A) = \frac{4550}{15504} = 0.293.$$

Example 24.10: A committee of 4 members is to be appointed from 3 officers of the production department, 4 officers of the purchase department, 2 officers of the sales department and 1 Chartered Accountant. Find the probability of forming the committee in the following manners

(a) There must be one from each category,

(b) It should have at least one from the purchase department,

(c) The Chartered Accountant must be in the committee.

Solution: Total number of persons out of which four members are to be selected

$$= 3 + 4 + 2 + 1 = 10.$$

Exhaustive number of cases $= {}^{10}C_4 = \dfrac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.$

(a) Favourable number of cases for committee to consist of one member from each category

$$= C_1^3 \times C_1^4 \times C_1^2 \times C_1^1 = 3 \times 4 \times 2 \times 1 = 24.$$

Therefore desired probability $= \dfrac{24}{210} = 0.114$ (approx.).

(b) Let A be the event that committee has at least one purchase officer, then A^c is event that committee has no purchase officer.

Cases favourable to $A^c = C_4^6 = \dfrac{6 \times 5}{2 \times 1} = 15$

Therefore, $P(A^c) = \dfrac{15}{210} = 0.071$ (approx).

Hence, $P(A) = 1 - P(A^c) = 0.929$ (approx).

(c) Favourable number of cases to include one Chartered Accountant out of 4 are

$$1 \times C_3^9 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84.$$

Hence, the desired probability $= \dfrac{84}{210} = 0.40.$

Example 24.11: A class in probability theory consists of 6 boys and 4 girls. An examination is conducted and the students are ranked according to their performance. Assume that no two students obtain the same score, what is the probability that girls receive the top 4 scores?

Solution: Since each ranking corresponds to a particular ordered arrangement of the 10 students, thus total number of different rankings = 10!.

Since there are 4! possible rankings among the girl students and 6! possible rankings among the boy students, so the total number of ways in which the four girls can receive the top rankings

$$= 4! \times 6!$$

Hence the desired probability $= \dfrac{4! \times 6!}{10!} = \dfrac{4 \times 3 \times 2 \times 1}{10 \times 9 \times 8 \times 7} = \dfrac{1}{210}.$

Example 24.12: From a set of n items a random sample of size k is to be selected. What is the probability a given item will be among the k selected?

Solution: Number of exhaustive cases $= C_k^n$.

Number of different selections that contain the given item $= C_1^1 \times C_{k-1}^{n-1} = C_{k-1}^{n-1}$

Hence, the desired probability $= C_{k-1}^{n-1}/C_k^n = \dfrac{(n-1)!}{(k-1)!(n-k)!} \times \dfrac{(n-k)!}{n!}k! = \dfrac{k}{n}$.

Major shortcomings of the classical concept is that it is applicable only to equally likely possibilities. In addition to this, it is applicable only when the number of exhaustive cases are finite; and also it gives probability of an event only as a rational number in [0, 1]. There are numerous situations in which various possibilities cannot be regarded as equally likely, e.g., if we are concerned with the question whether it will rain tomorrow, whether the flight will have the safe landing under a particular weather condition, etc., then the different possibilities are not equally likely and so the classical concept fails in such situations.

Another concept of probability, which is most widely used is the frequency interpretation concept, called the *statistical, or empirical probability concept*, as discussed next.

24.4.2 Statistical (or Empirical) Probability

The probability of an event is the proportion of times the event occurs in a long run of repeated experiments performed under essentially homogeneous and identical conditions.

Symbolically if in n trials an event A happens m times, then $P(A)$ the probability of happening of an event A is given by $P(A) = \lim\limits_{n\to\infty} \dfrac{m}{n}$.

In accordance with this concept, we estimate the probability of an event by observing what fraction of the time similar events have happened in past. For example, if our record shows that in past out of 1250 flights 1200 had the safe landings under the similar weather conditions then the probability of the safe landing will be $1200/1250 = 24/25 = 0.96$.

The shortcomings of the statistical concept of the probability is that if an experiment is repeated a large number of times, the experimental conditions may not remain identical. Also the limit m/n, $n \to \infty$ may not be unique.

24.4.3 Axiomatic Approach to Probability

In axiomatic approach probability is defined as a *set function*. Consider a random experiment E. The set of all possible outcomes of the experiment E is called the *sample space* of the experiment and is denoted by S. Any subset A of the sample space is known as an *event*. That is, an event is a set consisting of possible outcomes of an experiment. If the outcome of the experiment is contained in A we say that A has happened.

For example, let the experiment be throw of a dice and the event A be face is even.

Then, $S = \{1, 2, 3, 4, 5, 6\}$ and $A = \{2, 4, 6\}$.

If face as a result of the throw is 2 or 4 or 6 we say A has happened.

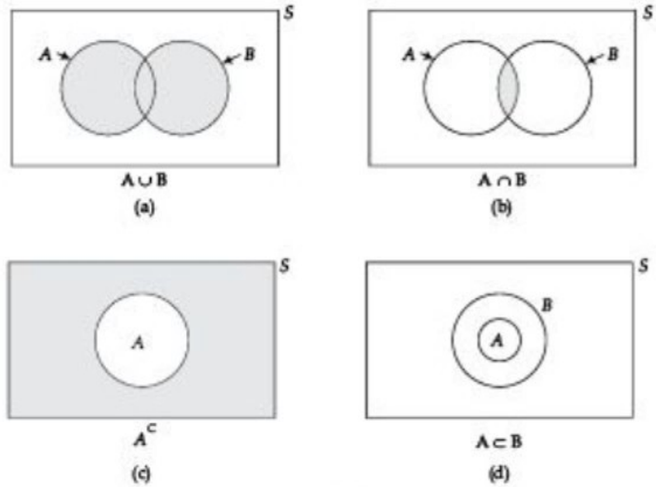


Fig. 24.6

For any two events A and B of a sample space S the event $A \cup B$, called the *union of the events A and B*, consists of all outcomes that are either in A or in B or in both A and B, refer Fig. 24.6a. Thus, $A \cup B$ would occur if either A or B occurs. Similarly the event $A \cap B$ or AB, called the *intersection of the events A and B*, occurs if both A and B occur, refer to Fig. 24.6b.

If $A \cap B = \phi$, then the events A and B are said to *mutually exclusive*, or *mutually disjoint*. The empty set ϕ is said to be the *null event* or *impossible event*. The sample space S defines the *sure event*.

The event A^c, referred as *complement* of A, consists of all the outcomes in the sample space S that are not in A, refer to Fig. 24.6c. Thus A^c occurs if, and only if A does not occur. We note that $S^c = \phi$ and $\phi^c = S$.

For any two events A and B, if all the outcomes of A are also in B, then we say that A is contained in B and we write $A \subset B$. In this case occurrence of A implies the occurrence of B, refer to Fig. 24.6d.

If $A \subset B$ and $B \subset A$, we define the two events A and B to be *equal* and write $A = B$.

Now we give the ***axiomatic probability concept***.

Given a finite sample space S and an event $A \in S$ we define P(A), the probability of an event A, as a set function that satisfies the following three axioms:

Axiom 1: $0 \le P(A) \le 1$, *for each* $A \in S$

Axiom 2: $P(S) = 1$

Axiom 3: *If A and B are mutually exclusive events in S, then* $P(A \cup B) = P(A) + P(B)$.

We can show that the axiomatic probability concept is in consistent with the classical and empirical concepts of the probability. On the basis of the axiomatic approach the theory of probability is developed which forms the basis of statistical inference.

Remarks:

1. Using mathematical induction, Axiom 3 can be extended to any number of mutually exclusive events $A_1, A_2, \ldots, A_n$ in S, that is,

$$P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n). \quad \ldots(24.18)$$

2. *The probability of the impossible event* ϕ *is zero.* Since, $S \cup \phi = S$ and $S \cap \phi = \phi$, thus from Axiom 3, $P(S) + P(\phi) = P(S)$ which implies $P(\phi) = 0$.

3. *If A is in S, then* $P(A^c) = 1 - P(A)$. Since, $A \cup A^c = S$ and $A \cap A^c = \phi$, thus from Axiom 3, $P(A) + P(A^c) = P(S)$, which implies $P(A^c) = 1 - P(A)$.

The results 2 and 3 are in conformity with the results already studied.

4. The *odds of an event A* is defined by $P(A)/P(A^c) = P(A)/(1 - P(A))$. It tells how much more likely it is that A occurs than that it does not occur. If $P(A) = 2/5$, then $P(A)/(1 - P(A)) = \frac{2}{5}\Big/\frac{3}{5} = 2/3$. So the odds is $2/3$, hence it is $2/3$ times as likely that A occurs as it is that it does not, that is $P(A) : P(A^c) :: 2/3 : 1$, or $2 : 3$.

Example 24.13: If an experiment has four possible and mutually exclusive outcomes A, B, C and D specify in the following cases whether the assignment of probability is permissible.

(a) $P(A) = 1/3, \quad P(B) = 1/6, \quad P(C) = 1/4, \quad P(D) = 1/4.$

(b) $P(A) = 1/4, \quad P(B) = 1/6, \quad P(C) = 1/3, \quad P(D) = 1/3.$

Solution: (a) All the assignments of probabilities are in the interval $[0, 1]$ and $P(A) + P(B) + P(C) + P(D) = 1$. Hence the assignments are permissible.

(b) Assignments lie in the interval $[0, 1]$, but $P(A) + P(B) + P(C) + P(D) = 13/12 > 1$. Hence the assignments are not permissible.

Example 24.14: A, B, and C are three mutually exclusive and exhaustive events associated with a random experiment. Find $P(A)$, if $P(B) = \frac{3}{2}P(A)$ and $P(C) = \frac{1}{2}P(B)$.

Solution: Since, A, B and C are mutually exclusive and exhaustive events, thus

$$P(A) + P(B) + P(C) = 1.$$

or, $$P(A) + \frac{3}{2}P(A) + \frac{1}{2}\left(\frac{3}{2}P(A)\right) = 1 \quad \text{or,} \quad \frac{13}{4}P(A) = 1 \quad \text{or,} \quad P(A) = 4/13.$$

24.5 ADDITION AND MULTIPLICATION LAWS OF PROBABILITY

In this section, we consider two basic laws of probability, addition law and multiplication law.

24.5.1 Addition Law of Probability, or Theorem of Total Probability

Theorem 24.1 **(Addition law):** *If A and B are any two events in the sample space S, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \quad \ldots(24.19)$$

Proof. From Fig. (24.7), we have

$$A \cup B = A \cup (A^c \cap B),$$

where A and $A^c \cap B$ are mutually exclusive.

Therefore, by Axiom 3

$$P(A \cup B) = P(A) + P(A^c \cap B) \qquad ...(24.20)$$

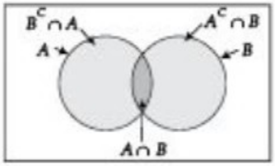


Fig. 24.7

Also from Fig. 24.7

$$B = (A \cap B) \cup (A^c \cap B),$$

and again $A \cap B$ and $A^c \cap B$ are mutually exclusive. Therefore,

$$P(B) = P(A \cap B) + P(A^c \cap B).$$

or,
$$P(A^c \cap B) = P(B) - P(A \cap B). \qquad ...(24.21)$$

Substituting for $P(A^c \cap B)$ in (24.20), we obtain (24.19).

In case the events A and B are mutually exclusive, then $A \cap B = \phi$ and hence (24.19) gives

$$P(A \cup B) = P(A) + P(B),$$

which is Axiom 3.

The result (24.19) can be extended to more than two events, say for three events A, B and C in S, we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C), \qquad ...(24.22)$$

and so on.

Example 24.15: A total of 21 per cent male employees of a company smoke cigarettes, 5 per cent smoke cigar and 3 percent smoke both cigar and cigarette. What percentage of males smokes neither cigar nor cigarette?

Solution: Let a male employee of the company is selected at random and A be the event that selected individual smokes cigarette and B be the events that he smokes cigar. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.21 + 0.05 - 0.03 = 0.23.$$

Thus 0.23 is the probability that employee is a smoker. Hence $1 - 0.23 = 0.77$ is the probability of employee being non-smoker. Thus 77% employees are non-smokers.

Example 24.16: In tossing an unbiased dice, what is the probability of getting an odd number or a number less than 4?

Solution: Let A be the event 'number is odd' and B the event 'number is less than 4'. Then,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}.$$

Example 24.17: What is the probability of getting a total of 7 or 11, when a pair of unbiased dice are tossed?

Solution: Let A be the event that 'total is 7' and B be the event that 'total is 11.'

No. of exhaustive cases $= 36$.

Number of cases favourable to $A = 6$

Number of cases favourable to $B = 2$

Thus, $P(A) = 6/36 = 1/6$ and $P(B) = 2/36 = 1/18$.

Further, A and B are mutually exclusive, therefore,

$$P(A\cup B) = P(A) + P(B) = 1/6 + 1/18 = 2/9.$$

Example 24.18: A graduate student applies for a job in two companies X and Y. The probability of being selected in X is 0.6 and being rejected in Y is 0.4. The probability of at least one of his applications being rejected is 0.5. What is the probability of getting job?

Solution: Let A be the event getting job in X and B be the event getting job in Y, then $A\cup B$ is the event getting job.

Here $\qquad P(A) = 0.6,\quad P(\bar{B}) = 0.4,\quad P(\bar{A}\cup\bar{B}) = 0.5,$

We have, $\qquad P(\bar{A}\cup\bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A}\cap\bar{B}).$

It gives $\qquad P(\bar{A}\cap\bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A}\cup\bar{B}) = 0.4 + 0.4 - 0.5 = 0.3$

Thus, $\qquad P(A\cup B) = 1 - P(\bar{A}\cap\bar{B}) = 1 - 0.3 = 0.7.$

Example 24.19: A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

Solution: Let A be the event 'getting a king', B be the event 'getting a heart' and, C be the event 'getting a red card'.

Obviously A, B, C are not mutually disjoint events. We are interested in the event $A\cup B\cup C$. We have,

$$P(A) = 4/52 = 1/13 \quad P(B) = 13/52 = 1/4 \quad P(C) = 26/52 = 1/2$$

$$P(A\cap B) = 1/52 \qquad P(B\cap C) = P(B) = 1/4$$

$$P(C\cap A) = 2/52 \quad P(A\cap B\cap C) = P(A\cap B) = 1/52.$$

Thus, $\qquad P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

$$= \frac{1}{13} + \frac{1}{4} + \frac{1}{2} - \frac{1}{52} - \frac{1}{4} - \frac{2}{52} + \frac{1}{52} = \frac{1}{13} + \frac{1}{2} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}.$$

Compound events. Independent events. Conditional probability

When two or more events occur in connection with each other, their simultaneous occurrence is called a compound event.

Events are said to be independent, if the probability of the occurrence of one does not depend on the occurrence or non-occurrence of the others, otherwise, the events are said to be dependent.

For example, let A be the event that first draw from a pack of 52 cards is queen and B be the event that second draw is a king. Then $P(A) = 4/52 = 1/13$ and $P(B) = 3/51$ or $4/51$ depending upon whether the first draw was 'a king' or 'not a king'. Hence, A and B are dependent events.

In case the second card has been drawn after replacing the first, then $P(B) = 1/13$. It does not depend upon whether the first draw was a king or not. Hence A and B are independent events in this case.

The probability for the event A to occur, when it is known that the event B has already occurred is called the conditional probability of A given B and is denoted by $P(A \mid B)$.

As an example, consider that a pair of unbiased dice is tossed. Then there are 36 possible outcomes given by $S = \{(i, j): i = 1, 2, ..., 6, j = 1, 2, ..., 6\}$.

Let A be the event that 'sum of the two dice equals 8', then the cases favourable to A are: (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) and hence $P(A) = 5/36$.

Let B be the event that 'face of the first dice is 3'. Then $A \mid B$ is the event that 'sum of the two dice equals 8 when the face of the first dice is 3'.

Exhaustive cases are (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6) and favourable case is (3, 5). Hence, the conditional probability of A given B denoted by $P(A \mid B) = 1/6$.

We observe that $P(A \mid B) \neq P(A)$ hence the event A is dependent on B.

If $P(A \mid B) = P(A)$, then event A is called independent of the event B.

24.5.2 Multiplication Law of Probability

Theorem 24.2 **(Multiplication law):** For two events A and B in S

$$\left.\begin{aligned} P(A \cap B) &= P(A)\, P(B \mid A), \quad P(A) > 0 \\ &= P(B)\, P(A \mid B), \quad P(B) > 0 \end{aligned}\right\} \qquad ...(24.23)$$

Proof. Let $n(A)$ and $n(B)$ be respectively the number of favourable cases to the events A and B and let $n(S)$ be the total number of possible outcomes in S. Then

$$P(A) = \frac{n(A)}{n(S)}, \qquad P(B) = \frac{n(B)}{n(S)} \quad \text{and} \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)}.$$

For the conditional event $A \mid B$, the sample space is B and out of $n(B)$ cases in B, $n(A \cap B)$ cases are favourable to A, and hence

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)} = \frac{n(A \cap B)}{n(S)} \frac{n(S)}{n(B)} = \frac{P(AB)}{P(B)}$$

or, $$P(AB) = P(B)\, P(A \mid B).$$

Since, $AB = BA$, replacing A with B and vice versa, we obtain

$$P(AB) = P(A)\, P(B \mid A).$$

Since, $P(A \mid B) = \dfrac{P(AB)}{P(B)}$, thus $P(A \mid B)$ is defined only when $P(B) > 0$.

Similarly, $P(B \mid A)$ is defined only when $P(A) > 0$.

In case of two independent events A and B with $P(A) \neq 0$, $P(B) \neq 0$, the multiplication law (24.23) becomes

$$P(AB) = P(A)\, P(B), \qquad ...(24.24)$$

and the condition (24.24) is taken as *the necessary and sufficient condition for the two events A and B to be independent.*

The result (24.23) can be extended to more than two events, for instance, in case of three events A, B, and C in S. The multiplication law is

$$P(A \cap B \cap C) = P(A)P(B \mid A)P(C \mid AB), \qquad ...(24.25)$$

where $P(C \mid AB)$ represents the conditional probability of the event C given that the events A and B have already occurred.

When the events A, B and C are independent, then (24.25) becomes

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C) \qquad ...(24.26)$$

and so on.

Remark: We must note that two mutually exclusive events with non-zero probabilities are always dependent events; and two independent events with non-zero probabilities cannot be mutually exclusive.

Example 24.20: A bag contains 4 red and 3 blue balls. Two draws of 2 balls each are made. Find the chance that the first draw gives 2 red balls and the second draw 2 blue balls, if

(a) the balls are returned to the bag after the first draw.

(b) the balls are not returned.

Solution: (a) The number of ways in which two balls out of 7 may be drawn = $\overset{7}{C}_2$.

The number of ways in which 2 red balls out of 4 may be drawn = $\overset{4}{C}_2$.

Thus, the probability of drawing two red balls $= \overset{4}{C}_2 / \overset{7}{C}_2 = \frac{4!}{2!2!} \times \frac{5!2!}{7!} = \frac{2}{7}$.

Similarly, the probability of drawing two blue balls at the second draw = 2/7.

Therefore the desired probability $= \frac{2}{7} \times \frac{2}{7} = \frac{4}{49}$.

(b) As in (a) the probability of drawing two red balls at the first draw = 2/7.

Since the balls are not returned, the probability of drawing two blue balls at the second draw

$$= \overset{3}{C}_2 / \overset{5}{C}_2 = 3/10.$$

Therefore, the desired probability in this case $= \frac{2}{7} \times \frac{3}{10} = \frac{3}{35}$.

Example 24.21: Mr. X works out that there is 50 per cent chance that his company will set up a branch office in New Delhi. If it is so, he is 80 per cent certain that he will be assigned the responsibilities of manager of the new set up. What is the probability that Mr. X will be a New Delhi branch office manager?

Solution: Let D be the event that branch office will be set up at New Delhi and M the event that Mr. X will be made manager there. Then

$$P(DM) = P(D)\,P(M \mid D) = \frac{1}{2} \times \frac{4}{5} = \frac{2}{5}.$$

Hence there are 40% chance that Mr. X will be a New Delhi branch officer manager.

Example 24.22: Two cards are drawn from a pack of 52 cards. Find the probability that draw includes an ace and a ten.

Solution: Let the event A: Draw an ace and a ten. Then $A = B \cup C$, where

B : First draw an ace and second draw a ten

C : First draw a ten and second draw an ace

Now, $\quad P(B) = \frac{4}{52} \times \frac{4}{51}$ and $P(C) = \frac{4}{52} \times \frac{4}{51}$.

Also we observe that B and C are mutually exclusive events, thus applying addition rule

$$P(A) = P(B) + P(C) = \frac{32}{52 \times 51} = \frac{8}{663}.$$

Example 24.23: A system composed of k separate components is said to be a parallel system if it functions when at least one of the k component functions. For such a system, if p_i, independent of others, is the probability that ith component will function $i = 1, 2, \ldots, k$, then what is the probability that system will function?

Solution: Let A_i be the event that the ith component functions. Then,

$$\begin{aligned} P[\text{System functions}] &= 1 - P[\text{System does not function}] \\ &= 1 - P[\text{No component functions}] \\ &= 1 - P[A_1^c A_2^c \ldots A_k^c] = 1 - \prod_{i=1}^{k} (1 - p_i). \end{aligned}$$

Example 24.24: A system consists of four components as shown in Fig. 24.8. System functions if components A and B both function and at least one of the components C or D functions. If the probabilities of functioning components A, B, C and D, respectively are 0.8, 0.8, 0.6 and 0.6, find the probability that, (a) entire system functions and, (b) the component C does not function given that the system functions. Assume that the components function independently.

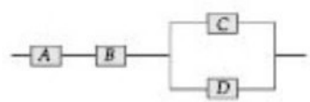


Fig. 24.8

Solution: (a) Let A be the event that component A functions, and so on. Then the event that entire system functions is $A \cap B \cap (C \cup D)$. Therefore,

$$\begin{aligned} P(A \cap B \cap (C \cup D)) &= P(A)P(B)P(C \cup D) = P(A)P(B)\,[1 - P(\overline{C} \cap \overline{D})] \\ &= P(A)\,P(B)\,[1 - P(\overline{C})\,P(\overline{D})] = (0.8)\,(0.8)\,[1 - (1 - 0.6)\,(1 - 0.6)] \\ &= (0.64)\,(.84) = 0.5376. \end{aligned}$$

(b) $P[C \text{ does not function} \mid \text{System functions}]$

$$= \frac{P[\text{System functions and C does not function}]}{P[\text{System functions}]}$$

$$= \frac{P[A \cap B \cap \bar{C} \cap D]}{P[\text{System functions}]} = \frac{P(A)P(B)P(\bar{C})P(D)}{P[\text{System functions}]} = \frac{(0.8)(0.8)(0.4)(0.6)}{0.5376} = 0.2857.$$

Example 24.25: The odds that a research monograph will be accepted by 3 independent referees are 3 to 2, 4 to 3 and 2 to 3, respectively. Find the probability that of the three reports,

(a) all will be favourable,

(b) majority of the reports will be favourable,

(c) at least one of the reports will be favourable.

Solution: Let A, B and C be the events that monograph is accepted favourably by the referee I, II and III, respectively. Then

$$P(A) = 3/5, \qquad P(B) = 4/7, \qquad P(C) = 2/5,$$
$$P(\bar{A}) = 2/5, \qquad P(\bar{B}) = 3/7, \qquad P(\bar{C}) = 3/5.$$

(a) $A \cap B \cap C$ is the event all will be favourable.

$$P(A \cap B \cap C) = P(A)P(B)P(C) = 3/5 \times 4/7 \times 2/5 = \frac{24}{175}.$$

(b) The event that majority, that is, at least two will be favourable happens when (i) $A \cap B \cap \bar{C}$, or (ii) $A \cap \bar{B} \cap C$, or (iii) $\bar{A} \cap B \cap C$, or (iv) $A \cap B \cap C$ happens; and all these are mutually exclusive.

Hence the desired probability $= P(AB\bar{C}) + P(A\bar{B}C) + P(\bar{A}BC) + P(ABC)$

$$= \frac{3}{5} \times \frac{4}{7} \times \frac{3}{5} + \frac{3}{5} \times \frac{3}{7} \times \frac{2}{5} + \frac{2}{5} \times \frac{4}{7} \times \frac{2}{5} + \frac{3}{5} \times \frac{4}{7} \times \frac{2}{5} = \frac{36}{175} + \frac{18}{175} + \frac{16}{175} + \frac{24}{175} = \frac{94}{175}.$$

(c) $$P(A \cup B \cup C) = 1 - P(\bar{A} \cap \bar{B} \cap \bar{C}) = 1 - \frac{2}{5} \times \frac{3}{7} \times \frac{3}{5} = 1 - \frac{18}{175} = \frac{157}{175}.$$

Example 24.26: Suppose an assembly plant receives its voltage regulators from three different sources, 60% from B_1, 30% from B_2 and 10% from B_3. Let 95%, 80% and 65% of the supply received respectively from the sources B_1, B_2 and B_3 perform as per specifications laid. If A is the event that a voltage regulator received at the plant performs as per specification, then find $P(A)$.

Solution: We can express $A = A \cap (B_1 \cup B_2 \cup B_3) = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$.

Since B_1, B_2, B_3 are mutually exclusive, therefore, $(A \cap B_1)$, $(A \cap B_2)$ and $(A \cap B_3)$ are also so, and hence

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) = P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)$$
$$= (0.60)(0.95) + (0.30)(0.80) + (0.10)(0.65) = 0.57 + 0.24 + 0.065 = 0.875,$$

is the probability that any one voltage regulator received at the company will perform as per specifications laid.

24.6 BAYES' RULE

Let us suppose we want to extend the forgoing problem as discussed in Example 24.26. We want to know the probability that a particular voltage regulator which is performing as per specifications came from some specific source, say B_1. Thus, we want to know $P(B_1|A)$. We have

$$P(B_1|A) = \frac{P(A \cap B_1)}{P(A)}, \quad P(A) > 0.$$

$$= \frac{P(B_1)P(A|B_1)}{\sum_{i=1}^{3} P(B_i)P(A|B_i)} = \frac{(0.60)(0.95)}{0.875} = 0.651,$$

using result from Example 24.26.

We observe that the probability that a voltage regulator is supplied by B_1 increases from 0.60 to 0.651 once it is known that it is performing as per specifications.

This method can be extended to yield the result called Bayes' rule as stated below.

Theorem 24.3 (Bayes' rule): *If $B_1, B_2, \ldots, B_k$ are mutually exclusive events in the sample space S of which one must occur and $P(B_i) \neq 0$, for $i = 1, 2, \ldots k$, then for any event A in S such that $P(A) \neq 0$,*

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}, \quad for = 1, 2, \ldots, k. \qquad \ldots(24.27)$$

Proof. We have, $A = \bigcup_{i=1}^{k} (A \cap B_i)$, where the events $A \cap B_i$, $i = 1, 2, \ldots, k$, are mutually exclusive.

Hence by addition and multiplication laws of probability

$$P(A) = \sum_{i=1}^{k} P(A \cap B_i) = \sum_{i=1}^{k} P(B_i)P(A|B_i),$$

Also, $$P(A \cap B_i) = P(A)P(B_i|A)$$

Hence, $$P(B_i|A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}, \; i = 1, 2, \ldots k.$$

Remark: The probabilities $P(B_i)$ are known as the *a priori probabilities* and the probabilities $P(B_i|A)$ are known as the *posterior probabilities*.

Example 24.27: The probabilities of X, Y and Z becoming managers of a company are 4/9, 2/9 and 1/3, respectively. The probabilities that the Bonus Scheme will be introduced if X, Y and Z becomes managers are 3/10, 1/2 and 4/5, respectively.

(a) Find the probability that Bonus Scheme will be introduced.

(b) If the Bonus Scheme has been introduced, find the probability that manager appointed was X or Y.

Solution: Let B_1, B_2, B_3 be the events that respectively X, Y and Z become manager, and A the event that Bonus Scheme is introduced. We have,

$$P(B_1) = 4/9, \quad P(B_2) = 2/9, \quad P(B_3) = 1/3,$$
$$P(A \mid B_1) = 3/10, \quad P(A \mid B_2) = 1/2, \quad P(A \mid B_3) = 4/5.$$

(a) We have, $A = \bigcup_{i=1}^{3} (A \cap B_i)$, where $A \cap B_i$, $i = 1, 2, 3$ are mutually exclusive, thus

$$P(A) = \sum_{i=1}^{3} P(B_i)\, P(A \mid B_i) = \frac{4}{9} \times \frac{3}{10} + \frac{2}{9} \times \frac{1}{2} + \frac{1}{3} \times \frac{4}{5} = \frac{2}{15} + \frac{1}{9} + \frac{4}{15} = \frac{23}{45}.$$

(b) $B_1 \cup B_2$ is the event that manager appointed was X_1 or Y, also $B_1 \cap B_2 = \phi$. Thus,

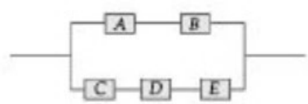
$$P(B_1 \cup B_2 \mid A) = P(B_1 \mid A) + P(B_2 \mid A)$$

$$= \frac{P(B_1)P(A|B_1) + P(B_2)P(A|B_2)}{P(A)} = \frac{\frac{2}{15} + \frac{1}{9}}{\frac{23}{45}} = \frac{22}{90} \times \frac{45}{23} = \frac{11}{23}.$$

EXERCISE 24.2

1. Two balls are randomly drawn from a bowl containing 6 white and 5 black balls. What is the probability that one of the drawn balls is white and the other black?
2. If n people are present in a room what is the probability that no two of them celebrate their birthday on the same day of the year. (Ignore the possibility of someone having been born on 29th of Feb.) How large need n be, so that, this probability is less than $1/2$?
3. A dice is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.
4. In a class of 100 students, 54 studied mathematics, 69 studied physics, and 35 studied both mathematics and physics. If one of these students is selected at random, find the probability that
   (a) the student took mathematics or physics,
   (b) the student did not take either of these subjects,
   (c) the student took physics but not mathematics.
5. One shot is fired from each of the three guns. G_1, G_2 and G_3 denote the events that target is hit by the first, second and third gun, respectively. If $P(G_1) = 0.5$, $P(G_2) = 0.6$ and $P(G_3) = 0.8$ and G_1, G_2, G_3 are independent events, find the probability that (a) exactly one hit is registered (b) at least two hits are registered.
6. The odds that X speaks the truth are 3:2 and the odds that Y speaks the truth are 5:3. What is the probability that they are likely to contradict each other on an identical point?

7. A and B cast each with a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. Find their respective chances of winning, if A begins.
8. A smoke-detector system uses two devices A and B. If smoke is present the probability that it will be detected by device A is 0.95; by device B, 0.98 and by both devices, 0.94. If smoke is present find the probability that it will be get detected.
9. Six cards are drawn with replacement from an ordinary pack. What is the probability that each of the four suits will be represented at least once among the six cards?
10. The probability of n independent events are $p_1, p_2, \ldots, p_n$. Find an expression for the probability that at least one of the events will happen. Use this result to find the probability of obtaining at least one 6 in a throw of four dice.
11. A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, what is the probability of getting 2 tails and 1 head?
12. A town has 2 fire engines operating independently. The probability that a specific engine is available when needed is 0.96.
    (a) What is the probability that neither engine is available when needed?
    (b) What is the probability that a fire engine is available when needed?
13. A circuit system with five components A, B, C, D and E is shown in the figure given If the probabilities that components A, B, C, D, E will function are 0.7, 0.7, 0.8, 0.8, 0.8 respectively, find the probability that system will function.



14. A consulting firm rents cars from three agencies, 30% from agency A, 20% from B and 50% from C. If 10% of the cars from A, 15% of the cars from B, and 5% of cars from C have bad tires, what is the probability that firm will get a car with bad tires.
15. At a nuclear plant tests are performed to check corrosion inside the cooling pipes. The test has probability 0.7 of detecting corrosion when it is present but it has probability 0.2 of falsely indicating internal corrosion. Suppose the probability that any section of pipe has internal corrosion is 0.1.
    (a) Find the probability that a section of pipe has internal corrosion, given that the test indicates its presence.
    (b) Find the probability that a section of pipe has internal corrosion, given that the test is negative.
16. While answering a question on a multiple-choice test, a student either knows the answer or he guesses. Let 1/2 be the probability that he knows the answer. Assuming that a student who guesses the answer will be correct with probability 1/5, when 5 is the number of multiple-choice alternative. What is the conditional probability that a student knew the answer correct to a question given that he answers it correctly?
17. A laboratory blood test is 99 per cent effective in detecting a certain disease when it is, in fact, present. However, test also yields a "false positive" result for 1 per cent of the healthy persons tested. If 0.5 percent of the population actually has the disease, what is the probability a person has the disease given that his test result is positive.

18. A plane is missing and it is presumed that it was equally likely to have gone down in any of the three possible regions. Let $1-\alpha_i$, $i=1,2,3$ denote the probability the plane will be found upon a search of the ith region when the plane is, in fact, in that region. What is the conditional probability that plane is in, (a) region 1, (b) region 2, (c) region 3, given that a search of region 1 is unsuccessful.
19. A worker operated machine produces a defective item with probability 0.01, if he follows the machine's operating instructions exactly, and with probability .03, if he does not. If the worker follows the instructions 90% of the time, what proportion of all items produced by the machine will be defective?
20. Ashok speaks truth 4 out of 5 times. A dice is tossed. He reports that there is 6. What is the chance that actually there was 6?

24.7 RANDOM VARIABLE. DISTRIBUTION FUNCTION

In this section we introduce random variable, its probability density and distribution function.

24.7.1 Random Variable

Consider an experiment having sample space S. *A random variable X is a function that assigns a real value to each outcome in the sample space S.*

Let the experiment be the toss of a pair of similar coins and let the random variable X be the number of heads obtained.

Outcome :	*TT*	*HT*	*TH*	*HH*
Value of X :	0	1	1	2

Thus to each outcome w in S there corresponds a real number $X(w)$.

Random variables are generally denoted by capital letters like X, Y, etc. and their possible values are denoted respectively by small letters x, y, etc.

A random variable is said to be discrete if its set of possible values is either finite or countably infinite.

For a discrete random variable X, we define its *probability mass function* $p(x)$ by

$$p(x) = P\{X = x\}.$$

If x_i, $i \geq 1$, represent, the possible values of X, then

$$\sum_{i=1}^{\infty} p(x_i) = 1. \qquad ...(24.28)$$

A random variable is said to be continuous if it can take each and every value between two specified limits.

In case of continuous random variable probability is not concentrated on specific points, rather we define a function $f(x)$, called the *'probability density function'* such that

$$P\left(x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}\right) = f(x)dx,$$

and, $$\int_{-\infty}^{\infty} f(x)dx = 1. \qquad ...(24.29)$$

24.7.2 Distribution Function

The *distribution function* F of the random variable X is defined for all real numbers by

$$F(x) = P\{X \le x\} = P\{X \in (-\infty, x]\}. \qquad ...(24.30)$$

Sometimes $\bar{F}(x)$ is used to represent $1 - F(x)$, that is $\bar{F}(x) = 1 - F(x) = P\{X > x\}$.

In case of discrete random variable, the relation between probability mass function and distribution function is

$$F(x) = \sum_{i: x_i \le x} p(x_i) \qquad ...(24.31)$$

and, in case of continuous random variable, the relation is

$$F(x) = P\{X \in (-\infty, x]\} = \int_{-\infty}^{x} f(x)dx. \qquad ...(24.32)$$

Differentiating (24.32) w.r.t. x, we have

$$f(x) = \frac{dF(x)}{dx}. \qquad ...(24.33)$$

Also we observe that

$$P(a \le X \le b) = F(b) - F(a) = \int_{a}^{b} f(x)dx; \qquad ...(24.34)$$

and, $$F(-\infty) = 0 \text{ and } F(\infty) = 1.$$

In many situations we are interested in relationship between two random variables X and Y. If X and Y are both discrete random variables, then their *joint probability mass function* is defined by

$$p(x, y) = P\{X = x, Y = y\};\ \sum_{x}\sum_{y} p(x, y) = 1, \qquad ...(24.35)$$

and *joint distribution function* by

$$F(x, y) = P\{X \le x, Y \le y\}. \qquad ...(24.36)$$

The individual distribution functions are obtained by using

$$F_X(x) = \lim_{y \to \infty} F(x, y) \text{ and } F_Y(y) = \lim_{x \to \infty} F(x, y). \qquad ...(24.37)$$

The two discrete random variables X and Y are said to be independent, provided

$$P\{X = x, Y = y\} = P\{X = x\}P\{Y = y\}$$

for all x and y. Also we have in this case

$$F(x, y) = F_X(x)F_Y(y) \qquad ...(24.38)$$

for all x and y.

Similarly, we can define joint distribution in case of continuous random variables X and Y.

Example 24.28: A consignment of 10 similar PCs contains 4 defective PCs. If an institution makes a random purchase of 3 PCs from this consignment, find the probability distribution for the number of defective PCs purchased and the distribution function.

Solution: Let X be the number of defective PCs purchased, then X can take the values $x = 0, 1, 2, 3$. We have

$$p(0) = P(X = 0) = \frac{C_0^4 \times C_3^6}{C_3^{10}} = \frac{20}{120} = 1/6, \qquad p(1) = P(X = 1) = \frac{C_1^4 \times C_2^6}{C_3^{10}} = \frac{60}{120} = 1/2$$

$$p(2) = P(X = 2) = \frac{C_2^4 \times C_1^6}{C_3^{10}} = \frac{36}{120} = 3/10, \qquad p(3) = P(X = 3) = \frac{C_3^4 \times C_0^6}{C_3^{10}} = \frac{4}{120} = 1/30.$$

Thus, the probability distribution $p(x)$ of X and distribution function $F(x)$ are given by

x :	0	1	2	3
$p(x)$:	1/6	1/2	3/10	1/30
$F(x)$:	1/6	2/3	29/30	1.

Example 24.29: If the life of a component X has the probability density function

$$f(x) = \begin{cases} 2e^{-2x}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases}$$

find the probabilities that it will take on a value, (a) between 1 and 3, (b) greater than 0.5, (c) find the distribution function.

Solution: (a) $P\{1 < X < 3\} = \int_1^3 2e^{-2x}dx = e^{-2} - e^{-6} = 0.133.$

(b) $P\{X > 0.5\} = \int_{0.5}^{\infty} 2e^{-2x}dx = e^{-1} = 0.368.$

(c) Performing the necessary integration, we obtain the distribution function as

$$F(x) = \begin{cases} 0, & x \le 0 \\ \int_0^x 2e^{-2x}\,dx = 1 - e^{-2x}, & x > 0 \end{cases}$$

Example 24.30: Suppose that 3 batteries are randomly chosen from a group of 3 new and 4 used but still working, and 5 defective batteries. If X and Y denote respectively the number of new and used but still working batteries chosen, then find the joint probability mass function of X and Y.

Solution: Let $p_{ij} = P\{X = i, Y = j\}$, $i, j = 0, 1, 2, 3$ be the probability that out of 3 selected i are new and j are used but still working. Then

$$p_{ij} = \frac{c_i^3 \times c_j^4 \times c_{3-(i+j)}^5}{c_3^{12}}, \quad i, j = 0, 1, 2, 3, \ i + j \leq 3.$$

We find that

$p_{00} = 10/220$ $\quad p_{01} = 40/220$ $\quad p_{02} = 30/220$ $\quad p_{03} = 4/220$

$p_{10} = 30/220$ $\quad p_{11} = 60/220$ $\quad p_{12} = 18/220$ $\quad p_{13} = 0$

$p_{20} = 15/220$ $\quad p_{21} = 12/220$ $\quad p_{22} = p_{23} = 0$

$p_{30} = 1/220$ $\quad p_{31} = p_{32} = p_{33} = 0.$

The distribution can be conveniently put in the form of the following two-way table given as below

$$[p_{ij} = P_r\{X = i, Y = j\}]$$

i \ j	0	1	2	3	$P\{x = i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
$P\{y = j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	1

Remarks:

1: The distribution

X :	0	1	2	3
$p(x)$:	$\frac{84}{220}$	$\frac{108}{220}$	$\frac{27}{220}$	$\frac{1}{220}$

is called the *marginal distribution* of X; and the distribution

$Y =$	0	1	2	3
$p(y) =$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$

is called the *marginal distribution* of Y.

2: We observe in this case that $P_r\{X = i, Y = j\} \neq P_r\{X = i\}\, P_r\{Y = j\}$.

Hence the variables X and Y are not independent.

Example 24.31: The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y}, & 0 < x < \infty,\ 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

Find, (a) $P\{X > 1, Y < 1\}$, (b) $P\{X < Y\}$ and (c) $P\{x < a\}$.

Solution: (a) $P\{x > 1, Y < 1\} = \int_0^1 \int_1^\infty 2e^{-x}e^{-2y}dx\,dy = \left(\int_0^1 2e^{-2y}dy\right)\left(\int_1^\infty e^{-x}dx\right)$

$$= \left(-e^{-2y}\right)_0^1 \left(-e^{-x}\right)_1^\infty = (1 - e^{-2})e^{-1} = e^{-1} - e^{-3}.$$

(b) $P\{X < Y\} = \iint_{(x,y):x<y} 2e^{-x}e^{-2y}dx\,dy = \int_0^\infty \int_0^y 2e^{-x}e^{-2y}dx\,dy$

$$= \int_0^\infty 2e^{-2y}(1 - e^{-y})dy = 2\int_0^\infty e^{-2y}dy - 2\int_0^\infty e^{-3y}dy = 1 - \frac{2}{3} = \frac{1}{3}.$$

(c) $P\{X < a\} = \int_0^a \int_0^\infty 2e^{-2y}e^{-x}dy\,dx = \int_0^a e^{-x}dx = 1 - e^{-a}.$

24.8 EXPECTATION. MOMENT GENERATING FUNCTIONS

The concept of expectation and of moment generating functions are extensively used in studying the probability distribution of a random variable X. We introduce these two in this section.

24.8.1 Expected Value

If X is a discrete random variable that takes value x_i, $i \geq 1$ with $P[X = x_i] = p_i$, then *expected value*, or *expectation of* X, denoted by $E(X)$, is defined by

$$E(X) = \sum_i x_i p_i. \quad ...(24.39)$$

If X is a continuous random variable with probability density function $f(x)$, then expected value of X is defined by

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx. \qquad ...(24.40)$$

In general, expectation of any function $g(X)$ of a r.v. X is defined by

$$E(g(X)) = \sum_i g(x_i) p_i \quad \text{or} \quad E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx, \text{ as the case may be.}$$

We note that $E(X)$ is the mean of the r.v. X and is generally denoted by μ.

The *variance* of X in term of expectation is given by

$$\sigma^2 = E(X-\mu)^2 = \sum_i (x_i - \mu)^2 p_i, \quad \text{or} \quad = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx.$$

In general, the *rth moment about the mean* μ, denoted by μ_r, is given by

$$\mu_r = E(X-\mu)^r = \sum_i (x_i - \mu)^r p_i \quad \text{or} \quad = \int_{-\infty}^{\infty} (x-\mu)^r f(x) dx.$$

Example 24.32: A lot containing 4 good components and 3 defective components is sampled by an inspector by taking a sample of 3 components. Find the expected value of the number of good components in this sample.

Solution: Let X be the number of good components in the sample of 3. The probability mass function $p(x)$ is $p(x) = \dfrac{C_x^4 \times C_{3-x}^3}{C_3^7}$, $x = 0, 1, 2, 3.$

We can check that, $p(0) = 1/35$, $p(1) = 12/35$, $p(2) = 18/35$, $p(3) = 4/35$

Thus, $\mu = E(X) = 0.\dfrac{1}{35} + 1.\dfrac{12}{35} + 2.\dfrac{18}{35} + 3.\dfrac{4}{35} = \dfrac{12}{35} + \dfrac{36}{35} + \dfrac{12}{35} = \dfrac{12}{7}.$

Example 24.33: Suppose that a message is expected at sometime past 5 P.M. From past experience it is known that X, the waiting period in hrs. after 5 P.M. is a r.v. with p.d.f.

$$f(x) = \begin{cases} \dfrac{1}{1\cdot 5}, & 0 < x < 1\cdot 5 \\ 0, & \text{otherwise} \end{cases}$$

Find the expected waiting time.

Solution: The expected waiting hrs. after 5 P.M. until message is received, is given by

$$E(X) = \int_0^{1.5} x f(x)\,dx = \int_0^{1.5} \frac{x}{1.5}\,dx = 0.75 \text{ hrs.}$$

Hence, on the average message would be received by 5.45 p.m.

Some results on expectation

We state the following intuitively reasonable results on expectation.

1. If a and b are constants, then
$$E(aX + b) = aE(X) + b. \qquad ...(24.41)$$
2. If X and Y are two random variables, then
$$E(X \pm Y) = E(X) \pm E(Y), \text{ (addition law)} \qquad ...(24.42)$$
3. If X and Y are two independent random variables, then
$$E(XY) = E(X)\,E(Y), \text{ (multiplication law)} \qquad ...(24.43)$$

These results can be proved simply applying the definition of expectation.

Covariance

The covariance of two random variables X and Y, denoted by Cov(X, Y), is defined by

$$\text{Cov}(X, Y) = E(X - \bar{X})(Y - \bar{Y}) = E(XY - X\bar{Y} - \bar{X}Y + \bar{X}\,\bar{Y})$$
$$= E(XY) - \bar{Y}E(X) - \bar{X}E(Y) + \bar{X}\,\bar{Y} = E(XY) - \bar{X}\,\bar{Y} = E(XY) - E(X)\,E(Y) \qquad ...(24.44)$$

When X and Y are independent, then $E(XY) = E(X)\,E(Y)$ and hence (24.44) gives Cov $(X, Y) = 0$.

The covariance of two random variables is an indicator of the relationship between them. A positive value of the Cov (X, Y) is an indication that Y tends to increase as X increases while a negative value indicates that Y tends to decrease as X increases. We will return to this aspect again in Section 24.13 while discussing correlation and regression.

Example 24.34: The weekly demand for a certain drink in thousand of litres at chain of retail stores is continuous r.v. $g(X) = X^2 + X - 2$, where X has the density function

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the expected weekly demand.

Solution: We have, $E[g(X)] = E[X^2 + X - 2] = E(X^2) + E(X) - 2$

$$= \int_1^2 2x^2(x-1)\,dx + \int_1^2 2x(x-1)\,dx - 2 = \frac{17}{6} + \frac{5}{3} - 2 = \frac{5}{2}.$$

Thus on the average demand is 2500 litres of drink.

Example 24.35: A secretary has typed n letters along with their respective envelopes. The envelopes get mixed up when they fall on the floor. If the letters are placed in the mixed-up envelopes in a completely random manner, what is the expected number of letters that are placed in the correct envelopes?

Solution: Let X denote the number of letters that are placed in the correct envelope, we compute X by considering

$$X=\sum_{i=1}^{n} X_i\text{, where } X_i=\begin{cases}1, & \text{if the } i\text{th letter is in its proper envelope.}\\ 0, & \text{otherwise}\end{cases}$$

We note that $P[X_i=1]=1/n$.

Hence, $E(X_i)=1.P[X_i=1]+0.P[X_i=0]=\frac{1}{n}$. Therefore, $E(X)=\sum_{i=1}^{n} E(X_i)=n.\frac{1}{n}=1$.

Hence irrespective of the number of letters, on the average exactly one letter will be in its proper envelope.

24.8.2 Moment Generating Function

The moment generating function (m. g. f.) of a r.v. X about origin, denoted by $M_0(t)$, is defined by

$$M_0(t)=E[e^{tX}]=\begin{cases}\Sigma\, e^{tx}p(x), & \text{if } X \text{ is discrete,}\\ \int_{-\infty}^{\infty} e^{tx} f(x)dx, & \text{if } X \text{ is continuous.}\end{cases} \qquad ...(24.45)$$

$M_0(t)$ is called the m.g.f. because all the moments of X can be obtained from $M_0(t)$, for

$$M_0(t)=E\left[1+tX+\frac{t^2X^2}{2!}+...+\frac{t^rX^r}{r!}+...\right]=1+tE(X)+\frac{t^2}{2!}E(X^2)+...+\frac{t^r}{r!}E(X^r)+...$$

$$=1+t\mu_1'+\frac{t^2}{2!}\mu_2'+...+\frac{t^r}{r!}\mu_r'+...$$

and we observe that

$$\mu_r'=\frac{d^r}{dt^r}M_0(t)\Big|_{t=0} \qquad ...(24.46)$$

The moment generating function of the r.v. X about an arbitrary point a, (a may be $\overline{X}$ also), is defined by

$$M_a(t)=E[e^{t(X-a)}]=e^{-at}E(e^{tX}).$$

An important property of moment generating functions is that the moment generating function of the sum of two independent random variables is the product of their moment generating functions. Also the moment generating function determines a distribution uniquely.

If X is a r.v which takes only non-negative integral values 0, 1, 2, ..., then the expression

$$P_X(t) = E(t^X) = \sum_{x=0}^{\infty} p_i t^x = p_0 + p_1 t + p_2 t^2 + \dots$$

is called the *probability generating function* (p.g.f.) of X.

Example 24.36: Find the m.g.f. of the exponential distribution $f(x) = ae^{-ax}, 0 \le x < \infty, a > 0$. Hence, find its mean and S.D.

Solution: By definition $M_0(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} a e^{-ax} dx = a \int_0^{\infty} e^{(t-a)x} dx, \ t < a$

$$= \left(1 - \frac{t}{a}\right)^{-1} = 1 + \frac{t}{a} + \frac{t^2}{a^2} + \frac{t^3}{a^3} + \dots$$

Therefore, $\mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \frac{1}{a}$, and $\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = \frac{2}{a^2}$.

Hence, mean $= \frac{1}{a}$ and S.D. $= \sqrt{\mu_2' - (\mu_1')^2} = \sqrt{\frac{2}{a^2} - \frac{1}{a^2}} = \frac{1}{a}$.

Example 24.37: Find the m.g.f. of a Bernoulli variate X with probability mass function $P\{x = 1\} = p, P\{x = 0\} = q, p + q = 1$. Hence find its mean and variance.

Solution: By definition $M_0(t) = E(e^{tX}) = \Sigma e^{tx_i} p(x_i) = pe^t + q$.

Therefore, $\mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = p$, and $\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = p$.

Hence, mean $= p$, and variance $= p - p^2 = pq$.

24.9 CHEBYSHEV'S INEQUALITY

The variance and hence standard deviation of a random variable gives us idea about the variability of the observations about mean. If σ is large there is correspondingly higher probability of getting values farther away from the mean. The Chebyshev's inequality gives us bound on probability that how far a random variable is deviated when both mean and variance of the distribution are known. The result is helpful when the actual distribution of X is not known. It is stated as follows.

Theorem 24.4 (Chebyshev's inequality): If X is a random variable with mean μ and variance σ^2, then for any value $k > 0$

$$P\{|X - \mu| \ge k\} \le \frac{\sigma^2}{k^2}. \qquad \dots(24.47)$$

Proof. We prove it for the case when X is continuous with density function $f(x)$.

For a non-negative r.v.X,

$$E(X) = \int_0^{\infty} xf(x)dx = \int_0^{a} xf(x)dx + \int_a^{\infty} xf(x)dx, \text{ for any } a > 0$$

$$\geq \int_a^{\infty} xf(x)dx \geq a\int_a^{\infty} f(x)dx = aP\{X \geq a\}.$$

Thus, $$P\{X \geq a\} \leq \frac{E(X)}{a}. \quad ...(24.48)$$

Replacing X by $(X-\mu)^2$ and a by k^2, (24.48) gives

$$P\{(X-\mu)^2 \geq k^2\} \leq \frac{E(X-\mu)^2}{k^2} \text{ or, } P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}, \text{ the Chebyshev's inequality.}$$

Remarks:

1. By replacing k by $k\sigma$ in (24.47), Chebyshev's inequality can be expressed as

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}. \quad ...(24.49)$$

2. Sometimes result is applicable in the form

$$P\{|X-\mu| < k\sigma\} \geq 1 - \frac{1}{k^2}. \quad ...(24.50)$$

3. The inequality (24.48) is known as *Markov inequality* and is of practical importance.

Example 24.38: The number of items cleared by an assembly line during a week is a random variable with mean 50 and variance 25. (a) What is the probability that this week items cleared will exceed 75? (b) What can be said about the probability that this week's clearance will be between 40 to 60?

Solution: Let X be the r.v. denoting the number of items cleared in a week. We have

$$E(X) = 50 \text{ and } \operatorname{Var}(X) = 25.$$

(a) By (24.48), $P\{X > a\} \leq \frac{E(X)}{a}$, $a > 0$; thus, $P\{x > 75\} \leq \frac{50}{75} = \frac{2}{3}$.

(b) Chebyshev's inequality (24.47) is $P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$.

Take $k = 10$, it gives, $P\{|X-50| \geq 10\} \leq \frac{25}{100} = \frac{1}{4}$

or, $$P\{|X-50|<10\} \geq \frac{3}{4}, \text{ which gives } P\{40<X<60\} \geq \frac{3}{4} = 0.75.$$

Example 24.39: Number of customers who visit a car dealer's showroom on weekend is a random variable with mean 18 and S.D. 2.5. What can be said about the probability that on a weekend the customers will be between 8 to 28?

Solution: Let X be the number of customers visiting on weekend, then

$$E(X) = 18 \text{ and } \operatorname{var}(X) = (2.5)^2 = 6.25.$$

Chebyshev's inequality is $P\{|X-\mu| \geq k\} \leq \dfrac{\sigma^2}{k^2}$.

Take $k = 10$, the inequality gives, $P\{|X-18| \geq 10\} \leq \dfrac{6.25}{100} = \dfrac{1}{16}$

or, $$P\{|X-18|<10\} \geq \frac{15}{16}, \text{ or } P\{8<X<28\} \geq \frac{15}{16} = 0.9375.$$

EXERCISE 24.3

1. A fair dice is tossed once, find the probability function and distribution function for X, the number obtained.
2. In tossing a fair coin, obtain the probability function for the number of trials until the first head appears.
3. In a lottery 8000 tickets are to be sold at Rs. 5 each. The prize is a Rs. 12,000 T.V. If two tickets are purchased what is the expected gain?
4. A Rs. 5,000 item can be insured for its total value by paying an yearly premium of Rs. N. If the probability of theft in a given year is estimated to be .01, what premium should the insurance company charge if it wants the expected gain to equal Rs. 1000?
5. In a lottery m tickets are drawn at a time out of n tickets numbered from 1 to n. Find the expected value of the sum of the number on the tickets drawn.
6. Let X be the life in hours of an electronic device with p.d.f.

$$f(x) = \begin{cases} \dfrac{20,000}{x^3}, & x > 100 \\ 0, & \text{elsewhere} \end{cases}$$

   Find the expected life of this type of devices.
7. Let X be the number of cars washed between 10 A.M. to 11 A.M. on any working day with probability distribution

x	4	5	6	7	8	9
$p(x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

If the attendant is paid an amount $g(X) = 2X - 1$ Rs, then what is his expected earning during this time period.

8. Suppose that X is a continuous r.v. with p.d.f. given by

$$f(x) = \begin{cases} k(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

   (a) What is the value of k?
   (b) Find $P\{x > 1\}$.
   (c) Find the distribution function for X.

9. Suppose that the shelf-life, in years, of a certain perishable food product packaged in cardboard containers is a r.v whose p.d.f. is given by

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

   Let X_1, X_2 and X_3 represent the shelf-lives for three of these containers selected independently, find $P(X_1 < 2, 1 < X_2 < 3, X_3 > 2)$.

10. From a box containing 3 defective, 2 partially defective and 3 good fuses, a random sample of 4 fuses is selected. If X is the number of defective and Y is the number of partially defective fuses in the sample selected, find
   (a) the joint probability distribution of X and Y
   (b) $P\{(X, Y) \in A\}$, when $A = \{(x, y) \mid x + y \le 2\}$
   (c) whether the variables X and Y independent?

11. Consider a sequence of independent trials, each of which is a success with probability $p, 0 < p < 1$, or a failure with probability $1 - p$. If X represents the number of trials preceding the first success, compute its expected value.

12. Find the m.g.f. for the 'geometric distribution' with parameter p, given by $P\{x = n\} = pq^{n-1}$, $n = 1, 2, \ldots$, where $q = 1 - p$. Hence, find its mean and variance.

13. Over the range of cylindrical parts manufactured on a computer-controlled lathe, the S.D. of the diameters is 0.002 millimetres. What about the probability that a new part will be within 0.006 units of the mean μ for that run? If 400 parts are made during the run, about what portion do you expect will lie in the interval found?

14. A r.v. X with unknown probability distribution has a mean $\mu = 8$ and S.D. $\sigma = 3$. Find, (a) $P\{-4 < X < 20\}$, (b) $P\{|X - 8| \ge 6\}$.

15. Show that for 40,000 tosses of a balanced coin, the probability is at least 0.99 that proportion of heads will fall between 0.475 and 0.525.

24.10 SPECIAL DISCRETE PROBABILITY DISTRIBUTIONS

In this section, we consider some special discrete probability distributions which occur frequently in applications. We shall study uniform, binomial, hypergeometric, Poisson, geometric and multinomial distributions.

24.10.1 Discrete Uniform Distribution

If a r.v. X assumes the values $x_1, x_2, \ldots, x_k$ with equal probability, then the random variable is said to follow 'uniform distribution' with probability distribution

$$P\{X = x_i\} = \frac{1}{k}, \quad i = 1, 2, \ldots, k. \qquad \ldots(24.51)$$

Thus for a throw of an unbiased die or random draw of a card from a pack, this distribution is the suitable one. We can easily calculate that in this case

$$E(X) = \frac{k+1}{2}, \quad E(X^2) = \frac{(k+1)(2k+1)}{6}, \quad \text{and} \quad V(X) = \frac{(k+1)(k-1)}{12}.$$

24.10.2 Binomial Distribution

First we define *Bernoulli trials.*

Repeated independent trials in which there are only two possible outcomes say 'success' or 'failure' and the probability of success remains constant throughout the trials, are called 'Bernoulli trials'.

For example, repeated tosses of a coin, and say, falling head is classified as 'success' and falling tail as 'failure'. Repeated draws of a card from a pack with replacement and classifying 'success' as the event getting a card of heart on a draw, otherwise 'failure'.

Next, we derive binomial distribution.

Consider a set of n independent Bernoulli trials in which the probability of success is 'p' and of failure is $q = 1 - p$. We wish to find the probability of x successes in n such trials.

First consider the probability of x successes and $(n - x)$ failures in a specified order. It is obviously $p^x q^{n-x}$. Next, these x successes in n trials can occur in C^n_x ways and all these are mutually exclusive. Hence, the requisit probability is $C^n_x p^x q^{n-x}$.

The probability distribution of the number of successes so obtained is called the *Binomial probability distribution*, and the r.v. X giving the number of successes is called the *binomial variate*. Thus,

A r.v. X taking non-negative integral values 0, 1, 2, ... with probability function

$$p(x) = P\{X = x\} = C^n_x p^x q^{n-x}, \; x = 0, 1, 2, \ldots n; \; q = 1 - p \qquad \ldots(24.52)$$

is called Binomial variate and the distribution is called the *Binomial distribution.*

The expression (24.52) defines a probability distribution, since

$$\sum_{x=0}^{n} P\{X = x\} = \sum_{x=0}^{n} c^n_x p^x q^{n-x} = (p+q)^n = 1.$$

The two independent constants n, p are called the parameters of the distribution. The probability function (24.52) is sometimes denoted by $b(x; n, p)$.

Frequency function of the binomial distribution

Let us suppose that n trials constitute an experiment and let this experiment is repeated N times, then

$$f(x) = Np(x) = N.c^n_x p^x q^{n-x}, \; x = 0, 1, 2, \ldots, n \qquad \ldots(24.53)$$

are the expected frequencies of 0, 1, 2, ... n successes out of N. These are the successive terms of the expression $N(p+q)^n$.

Binomial distribution is important not only because of its practical applications and also because, it leads to many other special distributions. In industrial applications p is often taken as the probability that a component is defective and q the probability that component is appropriate one.

Example 24.40: A machine is producing a large number of bolts. In a box of these bolts, 95% are within the permissible limits with respect to diameter. Seven bolts are drawn at random from the box. Determine the probability that, (a) two, and (b) more than or equal to two, of the seven bolts are not within the permissible limits with respect to diameter.

Solution: Let p be the probability of a bolt not being within permissible limits, then $p = 0.05$, and $q = 0.95$.

If X denotes the number of bolts with non-permissible limits out of seven selected, then

(a) $$P\{x=2\} = C_2^7(0.05)^2(0.95)^5 = 21(0.0025)(0.95)^5 = 0.0406$$

(b) $$P\{x \geq 2\} = 1 - P\{x=0\} - P\{x=1\}$$
$$= 1 - C_0^7(0.05)^0(0.95)^7 - C_1^7(0.05)(0.95)^6 = 1 - 0.630 - 0.232 = 0.138.$$

Example 24.41: A multiple choice test consists of 8 questions with 3 choices to each question of which only one is correct. A student answers each question by tossing a fair dice and marking the first choice if he get 1 or 2, the second choice if he gets 3 or 4 and the third choice if he gets 5 or 6. To get admission the students must mark at least 75% answers correct. What is the probability that the students gets admission if there are no negative marking?

Solution: Since there are three choices to a question and probability of getting 1, 2 or 3, 4 or 5, 6 are equally likely, and hence probability of marking correct answer $p = 1/3$ and thus $q = 2/3$.

So the probability of getting x correct answers out of 8 is given by

$$P\{X=x\} = p(x) = C_x^8\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{8-x}, x = 0, 1, 2, ..., 8.$$

Since to get admission student must mark 75% of 8, that is 6 or more correct choices. Thus its probability is $P\{x \geq 6\} = p(6) + p(7) + p(8) = C_6^8\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^2 + C_7^8\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right) + C_8^8\left(\frac{1}{8}\right)^8 = 0.0197.$

Constants of a binomial variate.

The mean, $$E(x) = \sum_{x=0}^{n} x.c_x^n p^x q^{n-x} = \sum_{x=0}^{n} x.\frac{n!}{x!(n-x)!}p^x q^{n-x}$$

$$= np\sum_{x=1}^{n}\frac{(n-1)!}{(x-1)!(n-x)!}p^{x-1}q^{n-x} = np(p+q)^{n-1} = np. \quad ...(24.54)$$

Similarly, $$E(x^2) = E(x(x-1)+x)$$

$$= \sum_{x=0}^{n} x(x-1)c_x^n p^x q^{n-x} + \sum_{x=0}^{n} x c_x^n p^x q^{n-x}$$

$$= n(n-1)p^2(p+q)^{n-2} + np = n(n-1)p^2 + np.$$

Hence, the variance, Var $(x) = E(x^2) - [E(x)]^2$

$$= n(n-1)p^2 + np - n^2p^2 = np - np^2 = npq. \quad ...(24.55)$$

The constants of a binomial variate X can be found from its *m. g. f.* also given by

$$M_0(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} c_x^n p^x q^{n-x} = \sum_{x=0}^{n} c_x^n (pe^t)^x q^{n-x} = (pe^t + q)^n. \quad ...(24.56)$$

Therefore, $\mu_1' = [n(pe^t + q)^{n-1} pe^t]_{t=0} = np$

$$\mu_2' = [n(n-1)(pe^t + q)^{n-2} p^2 e^{2t} + n(pe^t + q)^{n-1} pe^t]_{t=0} = n(n-1)p^2 + np.$$

Hence, $\mu_2 = \mu_2' - (\mu_1')^2 = npq$, as obtained earlier.

We can obtain moments about the mean from the m.g.f. about the mean given by

$$M_{\bar{x}}(t) = e^{-npt}(pe^t + q)^n. \quad ...(24.57)$$

Differentiating it successively w.r.t. t and substituting $t = 0$, we find that

$$\mu_1 = 0, \quad \mu_2 = npq, \quad \mu_3 = npq(q-p), \quad \mu_4 = npq[1 + 3(n-2)pq].$$

The coefficients of skewness and kurtosis are

and,

$$\left.\begin{aligned} \beta_1 &= \frac{\mu_3^2}{\mu_2^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq} \\ \beta_2 &= \frac{\mu_4}{\mu_2^2} = 3 + \frac{1-6pq}{npq} \end{aligned}\right\} \quad ...(24.58)$$

Hence, the binomial curve is symmetrical when $p = 1/2$. It is positively skewed for $p < 1/2$ and negatively skewed for $p > 1/2$. Also as $n \to \infty$, β_1 tends to zero and β_2 tends to 3, hence the distribution tends to be symmetric.

The mode is the most probable number of successes in a series of n independent trials of constant probability.

The chance for x successes will be greater than that of $x - 1$ and also that of $(x + 1)$, if

$$c_{x-1}^n p^{x-1} q^{n-x+1} < c_x^n p^x q^{n-x} > c_{x+1}^n p^{x+1} q^{n-x-1}$$

that is, if

$$\frac{x}{n-x+1}\frac{q}{p} < 1 > \frac{n-x}{x+1}\frac{p}{q}.$$

It can be simplified to

$$np - q < x < np + p \text{ or, } (n+1)p - 1 < x < (n+1)p.$$

Hence, the most probable number of successes is the integral part of $(n + 1)p$. *If $(n + 1)p$ is an integer, then there are two modes $(n + 1)p$ and $(n + 1)p - 1$ and the distribution is said to be bimodal.*

Example 24.42: In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least three defective parts.

Solution: If p is the probability of defective part, then we have $np = 2$, thus $p = 2/20 = 0.1$. Therefore, probability of a non-defective part $= 0.9$.

Let X denote the number of defective parts in a sample of size 20, then

$$\begin{aligned} P\{X \geq 3\} &= 1 - [P\{X=0\} + P\{X=1\} + P\{X=2\}] \\ &= 1 - [c_0^{20}(0.9)^{20} + c_1^{20}(0.1)(0.9)^{19} + c_2^{20}(0.1)^2(0.9)^{18}] \\ &= 1 - (0.9)^{18}[0.81 + 0.09 \times 20 + 190 \times .01] = 1 - 0.15\,[4.51] = 0.324. \end{aligned}$$

Thus, the expected number of samples having at least three defective parts out of 1,000 samples

$$= 1{,}000 \times 0.324 = 324.$$

Example 24.43: In a precision bombing attack there is a 50% chance that any bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better of completely destroying the target?

Solution: If p is the probability that the bomb hits the target, then $p = 1/2$.

Let n be the number of bombs needed to be dropped to ensure 99% or better chance to completely destroy the target. Thus the probability that out of n, at least two strike the target is greater than 0.99.

Let X be the *r.v.* representing the number of bombs striking the target, then X is a binomial variate with parameters n and $1/2$.

Now, $$p(x \geq 2) \geq 0.99$$

if $$[1 - p(x \leq 1)] \geq 0.99$$

or, if $$[1 - p(0) - p(1)] \geq 0.99$$

or, if $$1 - (1+n)\left(\frac{1}{2}\right)^n \geq 0.99$$

or, if $$2^n \geq 100 + 100n.$$

This inequality is satisfied for $n \geq 11$, hence minimum of 11 bombs are needed to achieve the goal.

Example 24.44: Seven coins are tossed and number of heads are noted. The experiment is repeated 128 times and the following distribution is obtained

No. of heads	0	1	2	3	4	5	6	7	Total
Frequency	7	6	19	35	30	23	7	1	128

Fit a binomial distribution assuming the coin to be unbiased.

Solution: We have, $p = q = \frac{1}{2}$ and $N = 128$. Let X be a r.v. denoting the number of heads out of the seven, then X is a binomial variate with parameter 7 and $1/2$, hence

$$p(x) = C_x^7 p^x q^{7-x} = C_x^7 \left(\frac{1}{2}\right)^7 ; \quad x = 0, 1, ..., 7.$$

Thus, expected frequencies are given by $f(x) = Np(x) = 128 C_x^7 \left(\frac{1}{2}\right)^7 = C_x^7; \quad x = 0, 1, ..., 7.$

Hence, $f(0) = 1, \quad f(1) = 7, \quad f(2) = 21, \quad f(3) = 35,$
$f(4) = 35, \quad f(5) = 21, \quad f(6) = 7, \quad f(7) = 1.$

24.10.3 Multinomial Distribution

In case each trial have more than 2 possible outcomes the binomial distribution becomes multinomial. For example, the drawing of a card from a pack with replacement is a multinomial distribution when we are interested in the 4 suits. Similarly selecting items with replacement from a manufactured product classified as being good, average, not acceptable is a multinomial distribution.

In general, if a given trial can result in one of the k outcomes $E_1, E_2, ..., E_k$ with probabilities $p_1, p_2, ..., p_k$; $\sum_{i=1}^{k} p_i = 1$ and, if $X_1, X_2, ..., X_k$ are the r.v representing the number of occurrences of $E_1, E_2, ..., E_k$, then the probability that in n trials $X_1, X_2, ..., X_k$ take respectively the values $x_1, x_2, ..., x_k$, is

$$p(x_1, x_2, ..., x_k; p_1, p_2, ..., p_k, n) = \frac{n!}{x_1! x_2! ... x_k!} p_1^{x_1} p_2^{x_2} ... p_k^{x_k}, \qquad ...(24.59)$$

where $\sum_{i=1}^{k} x_i = n, \; 0 \le x_i \le n.$

The distribution defined by (24.59) is called *multinomial distribution*. It defines a probability distribution, since

$$\Sigma\, p(x_1, x_2, ..., x_k; p_1, p_2, ..., p_k, n) = \sum \frac{n!}{x_1! x_2! ... x_k!} p_1^{x_1} p_2^{x_2} ... p_k^{x_k}$$
$$= (p_1 + p_2 + ... + p_k)^n = 1.$$

Example 24.45: Out of a lot containing 5 good, 4 faulty and 3 partially faulty but working batteries, three have been selected at random with replacement. Find the probability that selection consists of exactly one of each type.

Solution: Let p, q and r be the probabilities of selecting good, faulty and partially faulty batteries at a single draw, then $p = 5/12, \; q = 4/12, \; r = 3/12$. If X_1, X_2 and X_3 be the r.v. giving the number of good, faulty and partially faulty batteries out of 3, then

$$P_r\{X_1 = 1, X_2 = 1, X_3 = 1\} = \frac{3!}{1!\,1!\,1!}\left(\frac{5}{12}\right)\left(\frac{4}{12}\right)\left(\frac{3}{12}\right) = \frac{5}{24}.$$

24.10.4 Hypergeometric Distribution

The difference between binomial distribution and hypergeometric distribution lies in the procedure sampling is made. When the population is finite and sampling is done without replacement so the events although random, become stochastically dependent, then the resulting distribution is no more binomial. It leads to hypergeometric distribution. Let us suppose that a random sample of n items is selected without replacement from N items, k of which are classified as successes and $N - k$ as failures. If X is the r. v giving the number of successes out of the n selected, then

$$P_r\{X = x\} = \frac{C_x^k \times C_{n-x}^{N-k}}{C_n^N}, x = 1, 2, \ldots, \min(n; k). \qquad \ldots(24.60)$$

The r.v. X defined so is called hypergeometric variable and the distribution defined by (24.60) is called hypergeometric probability distribution. We can see easily that

$$\sum_x P_r\{X = x\} = \frac{\sum_x C_x^k \times C_{n-x}^{N-k}}{N_{C_n}} = N_{C_n} / N_{C_n} = 1.$$

Also, we can show that $E(X) = \frac{nk}{N}$, and $\mathrm{Var}(X) = \frac{nk(N-k)(N-n)}{N^2(N-1)}$ ...(24.61)

Hypergeometric distribution tends to binomial distribution when $N \to \infty$ and $\frac{k}{N} = p$, in fact, in this case the sampling becomes equivalent to sampling with replacement.

Applications of the hypergeometric distribution are found in electronic testing and quality assurance when the item tested is destroyed and can't be replaced in the sample.

Example 24.46: A lot consisting of 100 fuses is inspected by the following procedure. Five of these fuses are chosen at random and tested; if 4 or more work at the correct amperage, the lot is accepted. If there are 20 defective fuses in the lot, find the probability of acceptance.

Solution: Let X be the number of fuses working out of the five selected, then

$$P\{X \geq 4\} = P\{X = 4\} + P\{X = 5\} = \frac{C_4^{80} \times C_1^{20}}{C_5^{100}} + \frac{C_5^{80} \times C_0^{20}}{C_5^{100}} = 0.42 + 0.32 = 0.74.$$

24.10.5 Geometric Distribution

In a Bernoulli sequence of trials with probability of success p, let the r.v. X denote the number of failures preceding the first success, then

$$P_r\{X = x\} = q^x p, \quad x = 0, 1, 2, \ldots; \quad q = 1 - p. \qquad \ldots(24.62)$$

The r.v. X is called *geometric variable* and the distribution defined by (24.62) is called *geometric distribution*. It defines a probability distribution, since

$$\sum_{x} P_r(X = x) = \sum_{x=0}^{\infty} q^x p = p(1 + q + q^2 + \ldots) = \frac{p}{1-q} = \frac{p}{p} = 1.$$

We can calculate very easily that

$$E(X) = \frac{q}{p}, \quad \text{Var }(X) = \frac{q}{p^2}, \text{ and } \quad M_0(t) = \frac{p}{1-qe^t}. \qquad \ldots(24.63)$$

Example 24.47: If the probability is 0.10 that a certain kind of measuring device will show excessive drift. What is the probability that the fifth measuring device tested will be the first to show excessive drift?

Solution: We have, $p = 0.10$ and $q = 0.90$, thus $Pr\{X = 4\} = (0.9)^4(0.10) = 0.066$

24.10.6 The Poisson Distribution

Consider the situation when in binomial distribution n is large and p is small such that average number of successes np is a finite constant, say equal to λ.

The probability of x successes is given by $p(x) = C_x^n p^x q^{n-x}$. Rewriting it as

$$p(x) = \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}\cdot\frac{n!}{n^x(n-x)!}$$

$$= \frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right).$$

Taking limit as $n \to \infty$, it gives $p(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots \infty$. ...(24.64)

A r.v. X, taking non-negative integral values, with probability distribution (24.64) is called Poisson variate and the distribution is called Poisson distribution. It defines a probability distribution, since

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} e^{-\lambda}.\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}.e^{\lambda} = 1.$$

Besides approximating binomial distribution for large n and small p (normally we apply Poisson distribution when $n \geq 20$ and $p \leq 0.05$), the Poisson distribution has numerous applications. A few situations where Poisson variate is applied are:

1. The number of printing errors per page in a printed book.
2. The number of defective fuses in a pack of 100
3. The number of accidents per year at a busy crossing.
4. The number of wrong telephone numbers dialed in a day.

5. The number of α particles discharged in a fixed period of time from some radioactive material.
6. The number of customers arriving at a service counter on a given day.

Constants of a Poisson variate

The *mean*, $E(x) = \sum_{x=0}^{\infty} x.e^{-\lambda}\frac{\lambda^x}{x!} = \lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} = \lambda e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = \lambda e^{-\lambda}e^{\lambda} = \lambda.$ ...(24.65)

Similarly, $E(x^2) = E[x(x-1)+x] = E[x(x-1)] + E(x) = \sum_{x=0}^{\infty} x(x-1)e^{-\lambda}\frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} xe^{-\lambda}\frac{\lambda^x}{x!}$

$$= \lambda^2\sum_{x=2}^{\infty}e^{-\lambda}\frac{\lambda^{x-2}}{(x-2)!} + \lambda\sum_{x=1}^{\infty}e^{-\lambda}\frac{\lambda^{x-1}}{(x-1)!} = \lambda^2 + \lambda.$$

Hence the *variance*, $\mathrm{Var}(x) = E(x^2) - (E(x))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$ ...(24.66)

Thus in case of Poisson variate mean is equal to variance.

The *m.g.f about origin* is given by

$$M_0(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx}.e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = \exp[\lambda(e^t-1)]. \quad ...(24.67)$$

The *m.g.f. about mean* is given by

$$M_{\bar{x}}(t) = e^{-\lambda t}M_0(t) = e^{-\lambda t}e^{-\lambda}e^{\lambda e^t} = \exp[\lambda(e^t - (1+t))]. \quad ...(24.68)$$

From this we can calculate the moments about the mean. We can very easily check that

$$\mu_2 = \frac{d^2}{dt^2}[M_{\bar{x}}(t)]_{t=0} = \lambda,\ \mu_3 = \frac{d^3}{dt^3}[M_{\bar{x}}(t)]_{t=0} = \lambda, \text{ and } \mu_4 = \frac{d^4}{dt^4}[M_{\bar{x}}(t)]_{t=0} = 3\lambda^2 + \lambda \quad ...(24.69)$$

Thus the *coefficients of skewness and kurtosis* are given by

$$\left.\begin{aligned}\beta_1 &= \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}\\ \beta_2 &= \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2+\lambda}{\lambda^2} = 3 + \frac{1}{\lambda}\end{aligned}\right\} \quad ...(24.70)$$

and,

Since $\beta_1 > 0$, so the Poisson distribution is a positively skewed distribution, and also since $\beta_2 > 3$, the distribution is leptokurtic. As $\lambda \to \infty$, then $\beta_1 = 0$ and $\beta_2 = 3$, the distribution tends to be symmetric.

It is easy to verify that the *mode* of Poisson variate is integral part of λ, when λ is not an integer. In case λ is an integer, then λ and $\lambda - 1$ both are the mode, thus distribution is *bimodal*.

Example 24.48: Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 2% of such fuses are defective.

Solution: Here $\lambda = np = 200(.02) = 4$.

Hence, the requisite probability is

$$P(x \le 5) = \sum_{x=0}^{5} e^{-4} \frac{4^x}{x!} = e^{-4}\left[1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right]$$

$$= (0.0183)\,[1 + 4 + 8 + 10.6667 + 10.6667 + 8.5333] = (0.0183)(42.8667) = 0.7845.$$

Example 24.49: Consider an experiment that consists of counting the number of α particles given off in a one-second interval by one gram of radioactive material. If past experience shows that on the average 3.2 such α-particles are given off, find the probability that no more than 2 α-particles will appear?

Solution: If r.v X denotes the number of α-particles given off in a second interval, then X will be a Poisson variate with mean $\lambda = 3.2$. Hence, the requisite probability is

$$P[X \le 2] = \sum_{x=0}^{2} e^{-3.2} \frac{(3.2)^x}{x!} = e^{-3.2} + 3.2e^{-3.2} + \frac{(3.2)^2}{2!} e^{-3.2} = 0.041 + 0.130 + 0.209 = 0.38.$$

Example 24.50: If the average number of road accidents reported daily in a township is 5, what proportion of days have less than 3 accidents reported? What is the probability that there will be 4 accidents reported per day in exactly three days out of five, assuming that the number accidents on different days is independent?

Solution: Let X be the number of accidents reported daily, then X is Poisson variate with mean 5. Hence the probability that there will be less than three accidents reported is

$$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) = e^{-5} + e^{-5}\frac{5^1}{1!} + e^{-5}\frac{5}{2!} \approx 0.125.$$

Thus over the long run on about 12.5% days the number of accidents reported will be less than or equal to 3.

Since, it has been assumed that number of accidents on different days is independent, thus number of days in a five-day duration that has exactly 4 accidents reported is a binomial distribution with parameter $n = 5$ and probability of success p, given by

$$p = P[X = 4] = e^{-5}\frac{5^4}{4!} \approx 0.175.$$

Thus, the probability that exact 3 out of the next five days will report 4 accidents daily

$$= C_3^5 (0.175)^3 (0.825)^2 = 0.0365.$$

Example 24.51: A manufacturer who produces medicine bottles finds that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes. Using Poisson distribution find how many boxes will contain (a) no defective, (b) at least two defectives.

Solution: We have,

$N = 100$, $n = 500$, p = Probability of a defective bottle = 0.001, $\lambda = np = 500 \times 0.001 = 0.5$.

If the r.v. X denotes the number of defective bottles in a pack of 500, then by Poisson distribution $P(X = x) = e^{-0.5}\dfrac{(0.5)^x}{x!} = \dfrac{0.6065(0.5)^x}{x!}$; $x = 0, 1, 2, \ldots$

Hence in a lot of 100 boxes the frequency of boxes with x defective bottles is given by

$$f(x) = NP(X = x) = \frac{100 \times 0.6065 \times (0.5)^x}{x!}.$$

(i) Number of boxes with no defective $= 100P(X = 0) = 100 \times 0.6065 \approx 61$.

(ii) Number of boxes with at least two defectives

$$= 100\,[P(X \geq 2)] = 100\,[1 - P(X = 0) - P(X = 1)]$$
$$= 100\,[1 - 0.6065 - 0.6065 \times 0.5] = 100 \times 0.0903 \approx 9.$$

Example 24.52: Fit a Poisson distribution to the following data which gives the number of yeast cells per square for 400 squares

No. of cells per square (x) :	0	1	2	3	4	5	6	7	8	9	10
No. of squares (f) :	103	143	98	42	8	4	2	0	0	0	0

Solution: The parameter λ of the Poisson distribution is given by

$$\lambda = \frac{1}{N}\Sigma f_i x_i = \frac{529}{400} = 1.32.$$

If the r.v. X denotes the number of yeast cells per square then expected frequencies on the basis of Poisson distribution are given by

$$f(x) = NP(X = x) = 400\,e^{-1.32}\frac{(1.32)^x}{x!},\ x = 0, 1, 2, \ldots, 10.$$

It gives the following frequencies:

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	107	141	93	41	14	4	1	0	0	0	0

EXERCISE 24.4

1. It is known that disks produced by a certain company will be defective with probability .01 independently of each other. The company sells the disk in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective. What proportion of packages is returned. If someone buys three packages, what is the probability that exactly one of them will be returned?
2. Over a long period of time it has been observed that a given shooter can hit a target on a single trial with probability equal to 0.8. Suppose he fires four shots at the target. (a) What

is the probability that he will hit the target exactly two times? (b) What is the probability that he will hit the target at least once?

3. The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) at least 10 survive, (b) from 3 to 8 survive, and (c) exactly 5 survive?
4. If the probability that a fluorescent light has a useful life of at least 800 hours is 0.9, find the probabilities that among 20 such lights
   (a) exactly 18 will have a useful life of at least 800 hours;
   (b) at least 15 will have a useful life of at least 800 hours;
   (c) at least 2 will not have a useful life of at least 800 hours.
5. Among the 15 cities that a professional society is considering for next 3 annual conventions, 5 are in the northern part of India. To avoid arguments, the selection is left to chance. If none of the cities can be chosen more than once, what are the probabilities that
   (a) none of the conventions will be held in the northern part,
   (b) all of the conventions will be held in the northern part?
6. A sortie of 20 aeroplanes is sent on operational flight. The chances that an aeroplane fails to return is 5%. Find the probability that (a) one plane does not return (b) at the most five planes do not return, (c) what is the most probable number of returns?
7. The following data shows the results of throwing 12 fair dice 4096 times; throw of 4, 5 or 6 being called success:

Success (x)	0	1	2	3	4	5	6	7	8	9	10	11	12
Frequency (f)	0	7	60	198	430	731	948	847	536	257	71	11	0

   Fit a binomial distribution and find the expected frequencies.
8. If a fair coin is tossed an even number of $2n$ times, show that the probability of obtaining more heads than tails is $\frac{1}{2}\left\{1 - C_n^{2n}\left(\frac{1}{2}\right)^{2n}\right\}$.
9. According to a genetic theory, a certain cross of guinea pigs will result in red, black, white and gray offspring in the ratio 4: 2: 3: 1. Find the probability that among 9 offspring, 3 will be red, 2 black, 3 white and 1 gray.
10. As a student goes to school, he encounters a traffic signal which stays green for 35 seconds, yellow for 5 sec and red for 60 seconds. Assume that he goes to school each week day between 8:00 and 8:30 A.M. and X_1, X_2, X_3 be the number of times he encounters green, yellow and red signal, respectively. Find the joint distribution for (X_1, X_2, X_3).
11. In a test a light switch is turned on and off until it fails. If the probability that switch will fail any time it is turned on or off is 0.001, what is the probability that the switch will fail after it has been turned on or off 1,200 times? Assuming that the conditions for the geometric distribution are met.
12. The average number of accidents on a certain section of highway is two per week. Assuming it to follow Poisson distribution find the probability of (a) no accident on this section during a week period, (b) at most three accidents on this section during a two week period.

13. In a book of 520 pages, 390 typographical errors occur. Assuming Poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.
14. Suppose that in the production of radio resistors the probability of a resistor being defective is 0.1%. The resistors are sold in lots of 200, with the guarantee that all resistors are non-defective. What is probability that a given lot will violate this guarantee?
15. The probability that a person dies when he contracts a respiratory infection is 0.002. Of the next 2000 so infected, what is the mean number that will die? What is the S.D.?
16. The probability that a student pilot passes the written test for a pilot's licence is 0.7. Find the probability that the student will pass the test, (a) on the third try, (b) before the fourth try.
17. Twenty firms are under suspicion for violation of pollution norms but all cannot be inspected. Suppose that 3 of the firms are in violation. What is the probability that, (a) inspection of 5 firms find no violation. (b) will find two violations?
18. After correcting 50 pages of the proof of a book, the proof-reader finds that there are on an average 2 errors per 5 pages. How many pages would one expect to find with 0, 1, 2, 3 and 4 errors in 1000 pages of the first print of the book?

24.11 SPECIAL CONTINUOUS PROBABILITY DISTRIBUTIONS

In this section we consider some special continuous probability distributions like uniform, normal, exponential, gamma and beta, and study their various characteristics and applications.

24.11.1 Continuous Uniform Distribution

It is one of the simplest continuous distributions with constant (uniform) probability in a closed interval, say $[a, b]$.

If X is a continuous uniform r.v. defined on the interval $[a, b]$, then its p.d.f. is defined as

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise.} \end{cases} \qquad ...(24.71)$$

It is clear that $\int_a^b f(x)dx = 1$. We can easily see that

$$E(X) = \frac{a+b}{2}, \quad \text{Var }(X) = \frac{(b-a)^2}{12}, \quad M_0(t) = \frac{e^{bt}-e^{at}}{t(b-a)}, t \neq 0. \qquad ...(24.72)$$

Example 24.53: If X is uniformly distributed with mean 1 and variance 4/3, find $P(X < 0)$.

Solution: Let X is defined over $[a, b]$, then p.d.f is $f(x) = \dfrac{1}{b-a}, \quad a \le x \le b$

Also, $E(X) = \frac{a+b}{2}$ and $\text{Var}(X) = \frac{(b-a)^2}{12}$. Thus, $\frac{a+b}{2} = 1$ and $\frac{(b-a)^2}{12} = \frac{4}{3}$.

Solving for a and b and using the fact that $a < b$, we get $a = -1$ and $b = 3$. Therefore,

$$f(x) = \frac{1}{4};\quad -1 \le x \le 3. \text{ Hence, } P(X<0) = \int_{-1}^{0} f(x)dx = \frac{1}{4}[x]_{-1}^{0} = 1/4.$$

Example 24.54: The metro trains on a certain section run every 10 minutes between 5 A.M. to 10 P.M. What is the probability that a commuter entering the station at a random time during this period will have to wait at least five minutes?

Solution: Let X be the waiting time in minutes, then X is distributed uniformly over [0, 10] with

$$\text{p.d.f. } f(x) = \begin{cases} \frac{1}{10}, & 0 \le x \le 10 \\ 0, & \text{otherwise.} \end{cases}$$

The probability that the waiting time will be at least five minutes is

$$P(X \ge 5) = \int_{5}^{10} \frac{1}{10} dx = \frac{1}{2}.$$

24.11.2 Normal Distribution

It is the most important continuous probability distribution in the field of statistics since in applications many random variables are normal random variables or they are approximately normal, particularly when the population size is large.

A continuous r.v. X with two parameters μ and σ having the p.d.f.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty \qquad ...(24.73)$$

is called the normal variate, and the distribution defined by (24.73) is called the normal distribution. It defines a p.d.f., since

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz, \quad \left(z = \frac{x-\mu}{\sigma}\right)$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz, \text{ integrand being an even function in } z$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t} t^{-\frac{1}{2}} dt, \quad \left(t = \frac{1}{2}z^2\right)$$

$$= \frac{1}{\sqrt{\pi}}\Gamma(1/2) = 1, \text{ since } \Gamma(1/2) = \sqrt{\pi}.$$

The normal probability curve, $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is a bell-shaped curve symmetrical about the line $x = \mu$ and attains its maximum value of $1/\sigma\sqrt{2\pi} \approx 0.399/\sigma$ at $x = \mu$, refer to Fig. 24.9.

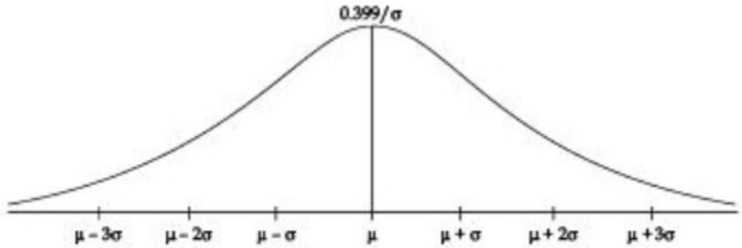


Fig. 24.9

The normal probability curve describes approximately many phenomena that occur in nature, industry and research. Physical measurements such as meteorological experiments, rainfall studies, error made in measuring a physical quantity are approximately normal in their behaviour. This distribution is often referred to as the *Gaussian distribution* also.

Constants of a normal variate

The mean, $$E(x) = \int_{-\infty}^{\infty} x f(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \mu\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}dz + \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ze^{-\frac{1}{2}z^2}dz = \mu, \qquad ...(24.74)$$

since $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = 1$ and $\int_{-\infty}^{\infty} ze^{-\frac{1}{2}z^2} dz = 0; \left(z = \frac{x-\mu}{\sigma}\right)$.

The variance, $$E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} \, z dz, \left(z = \frac{x-\mu}{\sigma} \right)$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \left[\left[-z e^{-\frac{1}{2}z^2} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz \right] = \sigma^2. \qquad ...(24.75)$$

Hence the *standard deviation* (S.D.) is σ.

The mode is that value of x for which $f(x)$ is maximum, that is, mode is the solution of $f'(x) = 0$ and $f''(x) < 0$.

For a normal distribution the p.d.f. is, $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$. It gives,

$$\ln f(x) = k - \frac{1}{2\sigma^2}(x-\mu)^2,$$

where $k = \ln\left(\frac{1}{\sqrt{2\pi}\sigma}\right)$ is a constant. Differentiating w.r.t. x, we obtain

$$f'(x) = -\frac{1}{\sigma^2}(x-\mu)f(x), \text{ and } f''(x) = \frac{-f(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right].$$

Now $\quad f'(x) = 0$ gives $x = \mu$ and at $x = \mu$, we have

$$f''(x) = -\frac{1}{\sigma^2}[f(x)]_{x=\mu} = -\frac{1}{\sigma^2}\frac{1}{\sqrt{2\pi}\sigma} < 0.$$

Hence, $x = \mu$ is the mode of the distribution.

The median is that value of x which divides the distribution in two equal parts.

Thus for x to be median $\int_{-\infty}^{x} f(x)dx = \frac{1}{2}$. This gives $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$

or, $$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \frac{1}{\sqrt{2\pi}\sigma}\int_{\mu}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}. \qquad ...(24.76)$$

But, $$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz = \frac{1}{2}.$$

Hence, from (24.76), we have $\frac{1}{\sqrt{2\pi}\sigma}\int_{\mu}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$, which gives $x = \mu$.

Hence for normal distribution: *mean = mode = median.*

Thus, the normal distribution is symmetrical.

Moments about the mean: Odd order moments are given by

$$\mu_{2n+1} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)^{2n+1} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{\sigma^{2n+1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^{2n+1} e^{-\frac{1}{2}z^2} dz, \quad z = \frac{x-\mu}{\sigma}$$

$$= 0, \text{ integrand being an odd function of } z.$$

Hence, $\mu_{2n+1} = 0.$

Thus in case of normal variate all odd order moments about the mean are zeros.

Even order moments are given by

$$\mu_{2n} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)^{2n} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{\sigma^{2n}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^{2n-1} e^{-\frac{1}{2}z^2}\, z dz$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}}\left[\left[-z^{2n-1} e^{-\frac{1}{2}z^2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (2n-1) z^{2n-2} e^{-\frac{1}{2}z^2} dz\right]$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}}(0-0) + (2n-1)\sigma^2 \frac{\sigma^{2n-2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^{2n-2} e^{-\frac{1}{2}z^2} dz$$

$$= (2n-1)\sigma^2 \mu_{2n-2}.$$

Hence, $\mu_{2n} = (2n-1)\sigma^2 \mu_{2n-2}.$ ...(24.77)

It gives, $\mu_{2n} = (2n-1)(2n-3) \ldots 5.\,3.\,1.\,\sigma^{2n}.$

In particular, $\mu_2 = \sigma^2$, $\mu_4 = 3\sigma^4$ etc.

Hence, $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$ and $\beta_2 = \frac{\mu_4}{\mu_2^2} = 3.$ ...(24.78)

Thus the normal probability curve is symmetric and mesokurtic.

Standard normal variate

If X is normally distributed with mean μ and variance σ^2, generally written as $X \sim N(\mu, \sigma^2)$ and if we define $Z = \frac{X-\mu}{\sigma}$, then

$$E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}E(X-\mu) = 0 \text{ and } \operatorname{Var}(Z) = E(Z-\overline{Z})^2 = \frac{1}{\sigma^2}E(X-\overline{X})^2 = 1.$$

The variable Z defined so is called *standard normal variate* and its p.d.f. is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}, \ -\infty < z < \infty. \qquad \text{...(24.79)}$$

Obviously the mean and variance of the standard variable are respectively zero and one, and it is denoted by $Z - N(0, 1)$. The distribution function of a standard normal variate is given by

$$\Phi(z) = P\{Z < z\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}z^2}dz. \qquad \text{...(24.80)}$$

The area property of normal probability integral

The probability of a normal variate lying between two values x_1 and x_2 is given by the area under the normal curve from x_1 to x_2, that is

$$P(x_1 \le X \le x_2) = \frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma}}dx = \frac{1}{\sqrt{2\pi}}\int_{z_1}^{z_2} e^{-\frac{1}{2}z^2}dz, \left(z = \frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{z_2} e^{-\frac{1}{2}z^2}dz - \int_0^{z_1} e^{-\frac{1}{2}z^2}dz\right] = P(z_2) - P(z_1),$$

where the definite integral $P(z) = \frac{1}{\sqrt{2\pi}}\int_0^{z} e^{-\frac{1}{2}z^2}dz$ is known as *normal probability integral* and gives the *area under standard normal curve between the ordinates at* $Z = 0$ *and* $Z = z$. These areas have been tabulated for different values of z at intervals of 0.01 and are given at Table I.

In particular, $P(\mu - \sigma < X < \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma} f(x)dx$ can be evaluated as

$$P(-1 < Z < 1) = \int_{-1}^{1}\phi(z)dz = \frac{2}{\sqrt{2\pi}}\int_0^1 e^{-\frac{1}{2}z^2}dz = 2\times 0.3413 = 0.6826, \text{ from Table } I.$$

Similarly, $P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-2 < Z < 2) = \frac{2}{\sqrt{2\pi}}\int_0^2 e^{-\frac{1}{2}z^2}dz = 2(0.4772) = 0.9544$

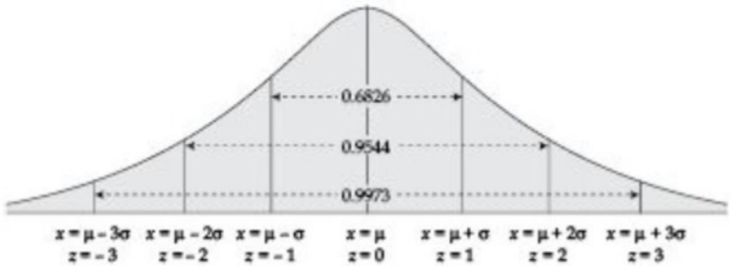


Fig. 24.10

and, $P(\mu - 3\sigma < X < \mu + 3\sigma) = P(-3 < Z < 3) = \dfrac{2}{\sqrt{2\pi}}\int_0^3 e^{-\frac{1}{2}z^2}\,dz = 2(0.49865) = 0.9973.$

Hence, the probability that a normal variate X lies outside the region $\mu \pm 3\sigma$ is given by

$$P(|x - \mu| > 3\sigma) = P(|z| > 3) = 1 - P(-3 \le z \le 3) = 1 - 0.9973 = 0.0027.$$

Thus though theoretically normal variate ranges from $-\infty$ to ∞, yet in all probability we should expect it to lie within the range $\mu \pm 3\sigma$, as shown in Fig. 24.10.

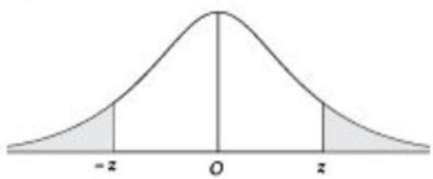


Fig. 24.11

Remarks:

1. Since in table we are given the areas under standard normal curve thus in numerical problems we convert the variable in its standard form.
2. From the symmetry of the normal probability curve we have $P(Z > z) = P(Z < -z)$, as shown in Fig. 24.11.
3. From Table *I* we observe that $P(-1.96 < Z < 1.96) = 0.95$ and $P(-2.58 < Z < 2.58) = 0.99$, and these are two important values to remember.

Example 24.55: If the amount of cosmic radiations to which a person exposed while flying across a specific continent is a normal random variable with mean 4.35 units and S.D. 0.59 units, then find the probabilities that the amount of exposure during such a flight is

(a) between 4.00 and 5.00 units, (b) at least 5.50 units.

Solution: Let X be the amount of cosmic radiations exposed, we define

$$Z = \frac{X - 4.35}{0.59} \sim N(0, 1).$$

We have, $P(4 < X < 5) = P(-0.59 < Z < 1.10)$

$= P(1.10) + P(0.59) = 0.3643 + 0.2224 = 0.5867$, from Table *I*.

The area is as shown in Fig. 24.12a.

(b) $P(X \geq 5.50) = P(Z \geq 1.95)$

$= 0.5 - P(1.95) = 0.5 - 0.4744 = .0256$, from Table *I*.

The area is as shown in Fig. 24.12b.

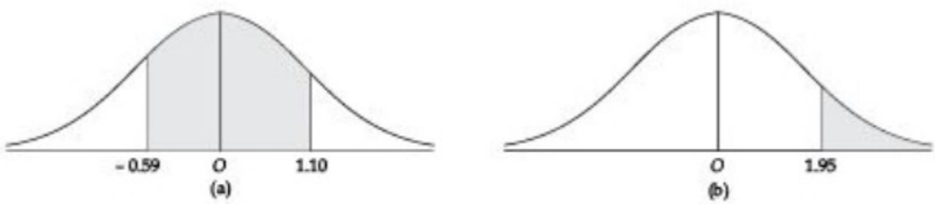


Fig. 24.12

Example 24.56: In a production of iron rods the diameter X can be approximated to be normally distributed with mean 2 inches and S.D. 0.008 inches.

(a) What percentage of defectives can we expect if we set the acceptance limits at 2 ± 0.02 inches?

(b) How should we set the acceptance limits to allow for 4% defectives?

Solution: (a) Define $Z = \dfrac{X - 2}{0.008} \sim N(0, 1)$.

Then $P(1.98 \leq X \leq 2.02) = P(-2.5 \leq Z \leq 2.5) = 2P(0 \leq Z \leq 2.5) = 2P(2.5)$.

From Table *I*, $P(2.5) = 0.4938$, hence $P(1.98 \leq x \leq 2.02) = 0.9876$.

This gives $P(|X - 2| > 0.02) = 1 - 0.9876 = .0124$.

Hence the percentage of defectives expected is 1.24%.

(b) Let the acceptance limits be fixed at $2 \pm k$, then $P(2 - k \leq X \leq 2 + k) = 0.96$

or, $$P\left(\frac{-k}{0.008} < Z < \frac{k}{0.008}\right) = 0.96, \quad \text{or} \quad P\left(\frac{k}{0.008}\right) = 0.48,$$

which gives $\dfrac{k}{0.008} = 2.054$, or $k = .016432$.

The acceptance limits should be set at 2 ± 0.0164, that is, the interval [1.9836, 2.0164].

Example 24.57: A company has installed 10,000 electric lamps in a metro. If these lamps have an average life of 1,000 burning hours with a S.D. of 200 hours. Assuming normality, what number of lamps might be expected to fail:

(a) in the first 800 burning hours.
(b) between 800 and 1200 burning hours.
After what period of burning hours would you expect that
(c) 10% of the lamps would fail?
(d) 10% of the lamps would survive?

Solution: Let X be the life of a bulb in burning hours. Define $Z = \dfrac{X - 1000}{200}$, then $Z \sim N(0, 1)$.

(a) $P(X < 800) = P(Z < -1) = P(Z > 1) = 0.5 - P(0 < Z < 1) = 0.5 - 0.3413 = 0.1587$, form Table *I*.

Therefore, out of 10,000 bulbs it is expected that 1587 will fail in the first 800 hours.

(b) $P(800 < X < 1200) = P(-1 < Z < 1) = 2P(0 < Z < 1) = 0.6826.$

Therefore, out of 10,000 bulbs it is expected that 6826 will burn between 800 and 1200 hours.
(c) If 10% of the bulbs fail after x_1 hours of burning, then x_1 be such that $P(X < x_1) = 0.10$. When $x = x_1$, then $z = \dfrac{x_1 - 1000}{200} = -z_1$, say. We find z_1 such that

$$P(Z < -z_1) = 0.10, \text{ or } P(Z > z_1) = 0.10, \text{ or } P(0 < Z < z_1) = 0.40.$$

From Table *I*, $z_1 = 1.28$. Thus, $\dfrac{x_1 - 1000}{200} = -1.28.$

It gives $x_1 = 1000 - 256 = 744.$

Therefore after 744 hours of burning life, 10% of the bulbs are likely to fail.
(d) If 10% of the bulbs are still burning after x_2 hours of burning, then x_2 be such that $P(X > x_2) = 0.10$. When $x = x_2$, then $z = (x_2 - 100)/200 = z_2$, say.

It gives $P(Z > z_2) = 0.10$ or, $P(0 < Z < z_2) = 0.40.$

From Table *I* $z_2 = 1.28$. Hence, $\dfrac{x_2 - 100}{200} = 1.28$, which gives, $x_2 = 1256.$

Thus, 10% of the bulbs are likely to burn after 1256 hours of the burning life.

Fitting of normal distribution.

To fit normal distribution to the given data we first calculate the mean μ and S.D. σ from the given data. Then the normal probability curve to be fitted to the given data is given by

$$y = f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty.$$

Example 24.58: Obtain the equation of the normal curve that may be fitted to the following data. Also obtain the expected normal frequencies.

Class	: 60–65	65–70	70–75	75–80	80–85	85–90	90–95	95–100
Frequency :	3	21	150	335	326	135	26	4

Solution: First we calculate the mean and S.D. for the given data.

Class	Frequency	Mid pt. (x)	$u = \frac{x-77.5}{5}$	fu	fu^2
60 – 65	3	62.5	–3	–9	27
65 – 70	21	67.5	–2	–42	84
70 – 75	150	72.5	–1	–150	150
75 – 80	335	77.5	0	0	0
80 – 85	326	82.5	1	326	326
85 – 90	135	87.5	2	270	540
90 – 95	26	92.5	3	78	234
95 – 100	4	97.5	4	16	64
	1000			489	1425

$$\bar{u} = \frac{489}{1000} = 0.489, \qquad \sigma_u^2 = \frac{1425}{1000} - (0.489)^2 = 1.425 - 0.239 = 1.186.$$

Thus, $\bar{x} = a + h\bar{u} = 77.5 + 5(0.489) = 77.5 + 2.445 = 79.945.$

and, $\sigma_x = h\sigma_u = 5(1.089) = 5.445.$

Hence, the equation of the normal curve to be fitted to the given data is given by

$$f(x) = \frac{1}{5.445\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-79.945}{5.445}\right)^2} \qquad ...(24.81)$$

To calculate the theoretical frequencies we calculate the area under the probability curve (24.81) in the interval (x_1, x_2), or (z_1, z_2) as given by

$$\Delta z = \frac{1}{\sqrt{2\pi}} \int_0^{z_2} e^{-z^2/2} dz - \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2} dz, \text{ where } z = \frac{x - 79.945}{5.445}.$$

By using the values from Table *I*, we form the following table:

(x_1, x_2)	(z_1, z_2)	Area $\Delta z = P(z_2) - P(z_1)$	Expected Frequency = $N\Delta z$
(–∞, 60)	(–∞, –3.663)	0.00011	0.11 ≈ 0
(60, 65)	(–3.663, –2.745)	0.00291	2.91 ≈ 3
(65, 70)	(–2.745, –1.826)	0.03104	31.04 ≈ 31
(70, 75)	(–1.826, –0.908)	0.14787	147.87 ≈ 142
(75, 80)	(–0.908, –0.010)	0.32205	322.05 ≈ 322
(80, 85)	(–0.010, 0.928)	0.31930	319.30 ≈ 319
(85, 90)	(0.928, 1.487)	0.14407	144.07 ≈ 144
(90, 95)	(1.487, 2.675)	0.02979	29.79 ≈ 30
(95, 100)	(2.675, 3.683)	0.00273	2.73 ≈ 3
(100, ∞)	(3.683, ∞)	0.00011	0.11 ≈ 0
Total			1000

The normal approximation as a limiting case of binomial

If in case of binomial distribution n is large and neither p nor q is small enough to use the Poisson approximation, then distribution can be approximated to normal. In fact, we have the following result which we state without proof.

If x is a binomial variate with parameters n and p, then the limiting form of the p.d.f. of the standardized r.v. $z = \dfrac{x - np}{\sqrt{np(1-p)}}$ *as* $n \to \infty$ *is the standard normal distribution given by* $\phi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, $-\infty < z < \infty$.

We should note that although x takes on only the values 0, 1, 2, ..., n, but in the limiting case as $n \to \infty$, the standardized variable z is continuous and takes value from $-\infty$ to ∞, with probability density as standard normal density.

Example 24.59: A 20% of the memory chips made in a certain plant are defective. What are the probabilities that in a lot of 100 randomly chosen for inspection

(a) at most 15 will be defective? (b) exactly 15 will be defective?

Solution: Here mean $\mu = 100(0.20) = 20$ and S.D. $\sigma = \sqrt{100(0.20)(0.80)} = 4$, thus the binomial variable may be approximated by $X - N(20, 16)$, a normal variate with mean 20 and variance 16.

Since the variable under consideration is discrete to spread its values over a continuous scale we represent each value k by the interval $(k - ½, k + ½)$. Thus 15 is represented as 14.5 – 15.5.

(a) $$P(X < 15.5) = P(Z < -1.13) = P(Z > 1.13)$$
$$= 0.5 - P(0 < Z < 1.13) = 0.5 - 0.3708 = 0.1292, \text{using Table } I$$

(b) $$P(14.5 < X < 15.5) = P(-1.38 < Z < -1.13) = P(1.13 < Z < 1.38)$$
$$= 0.4162 - 0.3708 = .0454, \text{ using Table } I.$$

Remark: In case calculations are worked out using binomial distribution, for part (a) we arrive at 0.1285 and for part (b) 0.0481, both in close with the respective approximations obtained. In general, the normal approximation to the binomial distribution is advisable when both np and $n(1 - p)$ are greater than 15.

Example 24.60: A multiple-choice quiz has 200 questions each with 4 possible answers of which only 1 is the correct answer. What is the probability that sheer guess-work yields from 25 to 30 correct answer for 80 of the 200 problems about which the student has no knowledge.

Solution: We have, $n = 80$, $p = \dfrac{1}{4}$, thus $\mu = np = 20$ and $\sigma = \sqrt{npq} = \sqrt{80 \times \dfrac{1}{4} \times \dfrac{3}{4}} = 3.873$.

Let X be the normal approximation to the underlying binomial variate, then $X - N(20, 15)$, a normal variate with mean 20 and variance 15.

To spread the variable over the continuous scale we find the probability that X lies in the interval 24.5 – 30.5. We define $z = \dfrac{x - 20}{3.873}$. Thus,

$$P(24.5 < x < 30.5) = P(1.16 < z < 2.71) = 0.4966 - 0.3770 = 0.1196.$$

EXERCISE 24.5

1. A conference room can be reserved for no more than five hours. Assuming that duration X of a conference has a uniform distribution over the interval [0, 5]. What is the p.d.f.? What is the probability that any given conference lasts at most 4 hours?
2. The daily amount of coffee in litres dispensed by a machine at a plaza is uniformly distributed between 7 litres to 10 litres. Find the probability that on a given day the amount of coffee dispensed by this machine will be
   (a) at most 8.8 litres,
   (b) more than 7.4 litres but less than 9.5 litres,
   (c) at least 8.5 litres.
3. If X is normally distribution with mean 18 and S.D. 2.5, find
   (a) $P(X < 15)$ (b) $P(17 < X < 21)$
   (c) the value of k such that $P(X < k) = 0.2236$,
   (d) the value of k such that $P(X > k) = 0.1814$.
4. The actual amount of instant coffee that a filling machine put into '4-ounce' jar may be approximated as a normal random variable with S.D. 0.04 ounces. If only 2% of the jars are to contain less than 4 ounces what should be the mean fill of these jars?
5. If in a normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and S.D. of the distribution.
6. In a test on 200 electric bulbs, it was found that the life of a particular make was normally distributed with mean 2040 hours and S.D. 60 hours. Estimate the number of bulbs likely to burn for
   (a) more than 2150 hours,
   (b) less than 1950 hours, and
   (c) more than 1920 hours but less than 2160 hours.
7. The average life of an inverter is 10 year with a S.D. of 2 year. The manufacturer replaces free all inverters that fail while under guarantee. If he is willing to replace only 3% of the inverters that fail, how long a guarantee should he offer, assuming that the lifetime follows a normal distribution.
8. If the lifetime of a certain kind of automobile battery is normally distributed with a mean of 5 years and a S.D. of 1 year, and the manufacturer wishes to guarantee the battery for 4 years, what percentage of the batteries will he have to replace under the guarantee?
9. If the mathematics score of an entrance exam are normally distributed with mean 480 and S.D. 100 and if an institution sets 500 as the minimum score for new students, what per cent of students would not reach that score?
10. Cerebral blood flow (CBF) in the brains of healthy people is normally distributed with a mean of 74 and a S.D. of 16.
    (a) What percentage of healthy people will have CBF readings between 60 and 80?
    (b) If a person has a CBF reading below 40 he is classified at risk for a stroke. What proportion of healthy people will mistakenly be diagnosed as "at risk"?

11. Suppose that the amount of money spent by shoppers at a mall between 4 P.M. to 6 P.M. on Sunday is normally distributed with mean of Rs. 4250 and a S.D. of Rs. 500. A shopper is randomly selected on a Sunday between 4-6 P.M. and asked about his spending pattern
    (a) What is the probability that he has spent more than Rs. 4500 at the mall?
    (b) What is the probability that he has spent between Rs. 4500 and Rs. 5000 at the mall?
    (c) If two shoppers are randomly selected, what is the probability that both have spent more than Rs. 5000 at the mall?
12. Fit a normal distribution to the following data and calculate the expected frequencies

Class	1-3	3-5	5-7	7-9	9-11
Frequency	1	4	6	4	1.

13. The following table gives baseball throw for a distance by 303 first year students of a college
    (a) Fit a normal distribution and find the theoretical frequencies.
    (b) Find the expected number of students throwing baseballs at a distance exceeding 105 feet on the basis that the data fits a normal distribution.

Distance in feet	*Number of students*
15 – 25	1
25 – 35	2
35 – 45	7
45 – 55	25
55 – 65	33
65 – 75	53
75 – 85	64
85 – 95	44
95 – 105	31
105 – 115	27
115 – 125	11
125 – 135	4
135 – 145	1

14. A sample of 100 items is taken from a batch known to contain 40% defectives. Using normal approximations find the probability that sample contains:
    (a) at least 44 defectives, (b) exactly 44 defectives.
15. A certain drug is effective in 72% of cases. Given 2,000 patients are treated with drug, what is the probability that it will be effective for, (a) at least 1,400 patients, (b) less than 1,300 patients, (c) exactly 1,420 patients.
16. A process for manufacturing an electronic component is 1% defective. A quality control plan is to select 100 items from the process, and if none is defective the process continues. Using normal approximation, find the probability that the process continues
    (a) for the sample plan described
    (b) even if the process has gone bad to produce 5% defective.

24.11.3 Exponential Distribution

A continuous random variable X with probability density function $f(x)$ defined by

$$f(x) = \begin{cases} ae^{-ax}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0, \end{cases} \quad ...(24.82)$$

for some $a > 0$, is called an *exponential variate with parameter a* and the distribution is said to be *exponential distribution*. The function (24.82) defines a probability density function, since

$$\int_{-\infty}^{\infty} f(x)dx = a\int_{0}^{\infty} e^{-ax}dx = [-e^{-ax}]_0^{\infty} = 1.$$

The distribution function $F(x)$ of an exponential variate is given by

$$F(x) = P(X \leq x) = \int_0^x ae^{-ax}dx = (1 - e^{-ax}), \quad x \geq 0. \quad ...(24.83)$$

Also we can calculate that

$$E(X) = 1/a, \quad \text{Var}(X) = 1/a^2 \text{ and } \quad M_0(t) = \frac{a}{a-t}, \; t < a \quad ...(24.84)$$

Thus, the mean is reciprocal of the parameter a and the variance is equal to the square of the mean in case of an exponential variate.

The exponential distribution arises as the distribution of amount of time until some specific event occurs. For example, the amount of time starting from now a car comes for service at a service station, a new war breaks out, a patient comes at an emergency reception, etc. are all random variables that behave exponentially.

An important property of the exponential distribution is that it *lacks memory*, that is, if X has an exponential distribution, then

$$P(X > s + t \mid X > t) = P(X > s) \quad ...(24.85)$$

for all $s, t > 0$. Since (24.85) can be written as

$$\frac{P(X > s + t \text{ and } X > t)}{P(X > t)} = P(x > s)$$

or,
$$P(X > s + t) = P(X > s)P(X > t),$$

which is satisfied when X has exponential distribution (24.82).

In case we interpret x as the lifetime of some equipment in hours, then (24.85) simply means that the probability that the equipment survives for at least $(s + t)$ hours given that it has survived t hours is the same as the initial probability that it survives for at least s hours.

Example 24.61: A system contains a certain type of component whose life-time X is exponentially distributed with mean of 5 years. If 8 such components are installed in different systems, then what is the probability that at least 3 are still working at the end of 7 years?

Solution: The p.d.f for r.v. X is given by

$$f(x) = \frac{1}{5}e^{-x/5}, \quad x \geq 0. \text{ Thus, } P(X>7) = \frac{1}{5}\int_7^{\infty} e^{-x/5}dx = e^{-7/5} = 0.1827.$$

If n represents the number of components out of 8 working after 7 years of instalment, then

$$P(n \geq 3) = \sum_{n=3}^{8} C_n^8 (0.1827)^n (0.8173)^{8-n}$$
$$= 1 - [C_0^8(0.8173)^8 + C_1^8(0.1827)(0.8173)^7 + C_2^8(0.1827)^2(0.8173)^6]$$
$$= 1 - [0.1991 + 0.3560 + 0.2786] = 0.1663.$$

Example 24.62: If on the average three trucks arrive per hour to be unloaded at a warehouse, using exponential distribution find the probabilities that the time between the arrival of successive trucks will be, (a) less than 5 minutes, (b) at least 45 minutes.

Solution: Let the r.v. t denote the time in hrs between arrival of successive trucks then its p.d.f is

$$f(t) = 3e^{-3t}, \quad 0 \leq t < \infty$$

(a) $$P(0 < t < 1/12) = \int_0^{1/12} 3e^{-3t}dt = 1 - e^{-1/4} = 0.221.$$

(b) $$P\left(\frac{3}{4} < t < \infty\right) = \int_{3/4}^{\infty} 3e^{-3t}dt = e^{-9/4} = 0.105.$$

24.11.4 The Gamma Distribution

A continuous random variable X with probability density function, for some $\alpha > 0, \beta > 0$, defined by

$$f(x) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad ...(24.86)$$

is called *gamma variate with parameters* (α, β) and the distribution is called *gamma distribution*. Here $\Gamma(\alpha)$ is the value of the gamma function with parameter $\alpha > 0$, given by

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx.$$

We have $\Gamma(\alpha) = (\alpha - 1)\,\Gamma(\alpha - 1)$ and when α is positive integer, then $\Gamma(\alpha) = (\alpha - 1)!$

The function (24.86) defines a p.d.f., since

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty} x^{\alpha-1}e^{-\beta x}dx = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}dt = 1, \quad t = \beta x.$$

We can show that $E(X) = \alpha/\beta$, $\text{Var}(X) = \alpha/\beta^2$ and $M_0(t) = \left(\dfrac{\beta}{\beta - t}\right)^{\alpha}$. ...(24.87)

We note that exponential p.d.f defined by (24.82) is a special case of gamma p.d.f (24.86) for $\alpha = 1$.

The relationship between the gamma and the exponential distribution allows the gamma function to find applications similar to that of exponential in particular in the field of queuing theory and reliability problems. In addition to this, gamma distribution is frequently used in life-testing, the waiting time until death probability models.

Remark: Sometimes $f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} x^{\alpha-1}e^{-x}, & x \geq 0,\ \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$ ...(24.88)

is also defined as the p.d.f of the gamma variate x with parameter α.

Example 24.63: The daily consumption of milk in a city, in excess of 20,000 litres, is approximately distributed as a gamma variate with parameters $\alpha = 2$ and $\beta = 1/10{,}000$. The city has a daily stock of 30,000 litres. What is the probability that the stock is insufficient on a particular day?

Solution: If X denotes the daily consumption in excess of 20,000 litres, then p.d.f of X is

$$f(x) = \frac{1}{(10,000)^2\,\Gamma(2)} x^{2-1}e^{-x/10,000}, \quad x > 0.$$

The stock of 30,000 litres will be insufficient on a particular day, if the excess consumption is more than 10,000 litres.

Thus, $$P(X > 10,000) = \int_{10,000}^{\infty} \frac{xe^{-x/10,000}}{(10,000)^2}dx = \int_1^{\infty} te^{-t}dt, \quad t = x/10,000$$

$$= 2/e = 0.736$$

24.11.5 The Beta Distribution

A continuous random variable X with probability density function for some $\alpha > 0$, $\beta > 0$, defined by

$$f(x) = \begin{cases} \dfrac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \qquad ...(24.89)$$

is called *beta variable with parameters α and β* and the distribution is called *beta distribution*, sometimes *beta distribution of the first kind*. Here $B(\alpha, \beta)$ is the value of the beta function given by

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx.$$

The relation between beta and gamma function is $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$. Thus, for positive integral values of α and β, we have $B(\alpha, \beta) = (\alpha-1)!(\beta-1)!/(\alpha+\beta-1)!$.

Obviously the function defined by (24.89) is p.d.f., since

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{B(\alpha,\beta)}{B(\alpha,\beta)} = 1.$$

Using the properties of beta and gamma functions we can show that

$$E(X) = \frac{\alpha}{\alpha+\beta} \text{ and } \operatorname{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}. \quad ...(24.90)$$

Example 24.64: If the proportion of a brand of television set requiring service during the first year of operation is a random variable having a beta distribution with $\alpha = 3$ and $\beta = 2$, what is the probability that at least 80% of the new models sold this year of this brand will require service during the first year of operation?

Solution: If the r.v. X denotes the proportion of T.V. set requiring service during the first year of operation, then its p.d.f. is $f(x) = \frac{1}{B(3,2)}x^2(1-x),\ 0 < x < 1.$

Thus, $$P(x > 0.8) = \frac{1}{B(3,2)}\int_{0.8}^1 x^2(1-x)dx = 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{0.8}^1 = 0.1808.$$

EXERCISE 24.6

1. The amount of time that a surveillance camera will run without having to be reset is random variable having exponential distribution with an average of 60 days. Find the probability that such a camera will have to be reset, (a) in less than 60 days, (b) at least 50 days.
2. The length of time for one individual to be served at a canteen is a random variable having an exponential distribution with mean of 4 minutes. What is the probability that a person is served in less than 3 minutes on at least 4 of the next 6 visits?
3. A continuous r.v X has the p.d.f. $f(x) = \begin{cases} Ae^{-x/5}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$

   Find the value of A and show that for any two positive numbers s and t,
   $$P[X > s + t \mid X > s] = P[X > t].$$
4. Following data gives the burning hours of 200 bulbs. Calculate the theoretical frequencies on the basis that burning hours of a bulb is exponentially distributed random variable.

Burning hrs	0-20	20-40	40-60	60-80	80-100
Number of bulbs	104	56	24	12	4

5. For a certain dose of the toxicant, a study on mice determines that the survival time, in weeks, has a gamma distribution with parameter $\alpha = 5$. What is the probability that a mouse survives no longer than 60 weeks?
6. The survival time in weeks of an animal when subjected to certain exposure of gamma radiation has a gamma distribution with $\alpha = 5$ and $\beta = 1/10$.
   (a) What is the mean survival time of a randomly selected animal of the type used in the experiment?
   (b) What is the probability that an animal survives more than 30 weeks?
7. Suppose that proportion of defectives, supplied by a vendor from lot to lot may be looked upon as a random variable having the beta distribution with $\alpha = 2$ and $\beta = 3$.
   (a) Find the average proportion of defectives in a lot from this vendor.
   (c) Find the probability that a lot from this vendor will contain 30% or more defectives.
8. The response time of a certain computer system in seconds has an exponential distribution with a mean of 3 seconds.
   (a) What is the probability that response time exceeds 5 seconds?
   (b) What is the probability that response time is between 5-10 seconds?

24.12 METHOD OF LEAST SQUARES AND CURVE FITTING

Fitting of curve to a given bivariate data is important both from the point of view of theoretical and practical statistics. Theoretically, concept is useful in the study of correlation and regression and practically, functional relationship between x and y enables us to predict the response y for a specific input x. The appropriate relationship to be fitted may be polynomial, algebraic, exponential or logarithmic depending upon the nature of the data. Method of least square is an excellent technique for fitting an appropriate relationship to the given data.

24.12.1 Method of Least Squares

Suppose we have n paired observations (x_i, y_i), $i = 1, 2, ..., n$, say, marks in mathematics and physics for a group of n individuals in the end-semester examination. Let the nature of the problem under consideration suggests a linear relation between x and y. We want to determine the line which, in some sense, provides the *best-fit*. Such a line may be useful for estimating values for y for some specific values of x. This line of *best-fit* is obtained by applying the *method of least squares* which in this case may be stated as follows.

The straight line

$$y = a + bx \qquad ...(24.91)$$

should be fitted through the given points (x_i, y_i), $i = 1, 2, \ldots, n$ so that sum of the squares of deviations (errors) of these points from the straight line is minimum, where the deviations are measured in the vertical direction.

Referring to Fig. 24.13, the point M_i on the straight line with abscissa x_i has the ordinate $a + bx_i$. Hence, the deviation from P_i is $M_iP_i = (y_i - a - bx_i)$ and the sum of the squares of these deviations is

$$S = \sum_{i=1}^{n} (y_i - a - bx_i)^2. \quad ...(24.92)$$

We are to determine a and b such that S is minimum.

Differentiating (24.92) w.r.t a and b, respectively we obtain

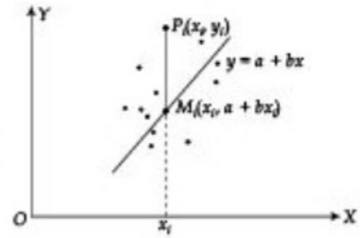


Fig. 24.13

$$\frac{\partial S}{\partial a} = -2\sum_{i=1}^{n} (y_i - a - bx_i)$$

$$\frac{\partial S}{\partial b} = -2\sum_{i=1}^{n} x_i(y_i - a - bx_i).$$

Equating these separately to zero and rearranging the terms we obtain

$$\left.\begin{aligned} \sum_{i=1}^{n} y_i &= na + b\sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i y_i &= a\sum x_i + b\sum_{i=1}^{n} x_i^2 \end{aligned}\right\} \quad ...(24.93)$$

These equations are called the *normal equations* for fitting the straight line (24.91). Solving (24.93) for a and b and substituting the values obtained, the line (24.91) is the line of *best-fit* in the sense of least square.

24.12.2 Fitting of Polynomial of Degree k

The method discussed above can be generalized to fit a polynomial of degree k, $(k \le n-1)$, given by

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k,\ a_k \ne 0 \quad ...(24.94)$$

We need to determine a_i's such that sum of the squares of the error is minimum, that is

$$S = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2 - \dots - a_k x_i^k)^2$$

is minimum.

Proceeding on the same lines as in case of fitting of a straight line, the $(k+1)$ normal equations obtained are:

$$\Sigma\, y_i = na_0 + a_1 \Sigma\, x_i + a_2 \Sigma\, x_i^2 + \dots + a_k \Sigma\, x_i^k$$
$$\Sigma\, x_i y_i = a_0 \Sigma\, x_i + a_1 \Sigma\, x_i^2 + a_2 \Sigma\, x_i^3 + \dots + a_k \Sigma\, x_i^{k+1}$$
$$\dots\dots$$
$$\dots\dots$$
$$\Sigma\, x_i^k y_i = a_0 \Sigma\, x_i^{k+1} + a_1 \Sigma\, x_i^{k+2} + a_2 \Sigma\, x_i^{k+3} + \dots + a_k \Sigma\, x_i^{2k}.$$

The normal equations form a linear system of $(k+1)$ equations in $(k+1)$ unknown and can possibly be solved for $a_0, a_1, \ldots a_k$. Substituting these $(k+1)$ values in (24.94), we get the desired polynomial.

Example 24.65: Find the line of best fit relating y to x for the following data, plot the line and the data points.

x	2	3	4	5	6	7
y	3.0	5.0	5.5	6.0	8.0	9.5

Solution: Let the line be $y = a + bx$. The normal equations are

$$\Sigma y = 6a + b\Sigma x, \quad \text{and} \quad \Sigma xy = a\Sigma x + b\Sigma x^2.$$

We formulate the following table:

	x	y	x^2	xy
	2	3.0	4	6.0
	3	5.0	9	15.0
	4	5.5	16	22.0
	5	6.0	25	30.0
	6	8.0	36	48.0
	7	9.5	49	66.5
Total	27	37.0	139	187.5

Substituting for $n, \Sigma x, \Sigma y, \Sigma x^2, \Sigma xy$ in the normal equations, we have

$$6a + 27b = 37.0$$
$$27a + 139b = 187.5$$

Solving these for a and b, we obtain $a = 0.767, \quad b = 1.2$

Thus, the line of best fit is $y = 0.767 + 1.2x$

Fitted line and the points are shown in Fig. 24.14

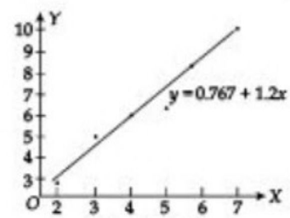


Fig. 24.14

Example 24.66: Fit a parabola of second degree to the following data:

x :	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y :	1.1	1.3	1.6	2.0	2.7	3.4	4.1

Solution: Let the parabola be $y = a + bu + cu^2$, where $u = \dfrac{x - 2.5}{0.5}$.

The normal equations are

$$\Sigma y = 7a + b\Sigma u + c\Sigma u^2$$
$$\Sigma uy = a\Sigma u + b\Sigma u^2 + c\Sigma u^3$$
$$\Sigma u^2 y = a\Sigma u^2 + b\Sigma u^3 + c\Sigma u^4$$

We formulate the following table:

x	$u=\dfrac{x-2.5}{0.5}$	y	u^2	u^3	u^4	uy	u^2y
1.0	−3	1.1	9	−27	81	−3.3	9.9
1.5	−2	1.3	4	−8	16	−2.6	5.2
2.0	−1	1.6	1	−1	1	−1.6	1.6
2.5	0	2.0	0	0	0	0	0
3.0	1	2.7	1	1	1	2.7	2.7
3.5	2	3.4	4	8	16	6.8	13.6
4.0	3	4.1	9	27	81	12.3	36.9
Total	0	16.2	28	0	196	14.3	69.9

Substituting for $\Sigma u, \Sigma u^2, \Sigma u^3, \Sigma u^4, \Sigma y, \Sigma uy$ and Σu^2y in the normal equations, we obtain

$$7a + 28c = 16.2, \quad 28b = 14.3, \quad \text{and} \quad 28a + 196c = 69.9.$$

Solving for a, b and c, we obtain $a = 2.07$, $b = 0.511$ and $c = 0.061$.

Hence the parabolic curve of 'best-fit' to the given data is

$$y = 2.07 + 0.511\left(\frac{x-2.5}{0.5}\right) + 0.061\left(\frac{x-2.5}{0.5}\right)^2 \quad y = 1.04 - 0.2x + 0.24x^2.$$

24.12.3 Fitting of Non-polynomial Curves

Using the concept of fitting of straight line, we can fit the curves of the form

$$y = ax^b, \quad y = ae^{bx}, \quad \text{or} \quad xy^a = b.$$

For example, to fit the curve $y = ax^b$ take logarithm both sides, we obtain

$$\ln y = \ln a + b \ln x.$$

or, $$Y = A + bX, \text{ where } Y = \ln y, \quad A = \ln a, \quad \text{and} \quad X = \ln x.$$

We obtain the normal equations for this straight line and solve those equations for A and b. From A we calculate for a. Substituting the values for a and b we obtain the desired curve. Similarly, we can proceed for other curves.

Example 24.67: Determine the constants a and b such that $y = ae^{bx}$ is the 'best-fit' curve for the following data.

x	2	4	6	8	10
y	4.077	11.084	30.128	81.897	222.62

Solution: The curve is $y = ae^{bx}$. Taking logarithm both sides, we obtain

$$\ln y = \ln a + bx,$$

or, $$Y = A + bx,$$

where $$Y = \ln y \text{ and } A = \ln a$$

We formulate the following table.

	x	$Y = \ln y$	x^2	xY
	2	1.405	4	2.810
	4	2.405	16	9.620
	6	3.405	36	20.430
	8	4.405	64	35.240
	10	5.405	100	54.050
Total	30	17.025	220	122.150

Substituting for $\Sigma x, \Sigma Y, \Sigma x^2$ and ΣxY in the normal equations for the straight line $Y = A + bx$, we obtain

$$5A + 30b = 17.025$$

$$17.025A + 220b = 122.50$$

Solving for A and b, we obtain $A = 0.405$ and $b = 0.5$. Hence, $a = e^{0.405} = 1.499$.

Thus the curve is $y = 1.499x^{0.5}$.

EXERCISE 24.7

1. Fit a straight line of the form $y = a + bx$ to the data

x :	1	2	3	4	6	8
y :	2.4	3.1	3.5	4.2	5.0	6.0

2. If the straight line $y = a + bx$ is the best-fit line to the set of points (x_i, y_i), $i = 1, 2, ..., n$, then

show that $\begin{vmatrix} x & y & 1 \\ \Sigma x_i & \Sigma y_i & n \\ \Sigma x_i^2 & \Sigma x_i y_i & \Sigma x_i \end{vmatrix} = 0.$

3. The result of measurement of electric resistance R of a copper bar at various temperature $t°c$ are given below. Fit a relation of the form $R = a + bt$.

t :	19	25	30	36	40	45	50
R :	76	77	79	80	82	83	85

4. The weight of a calf taken at weekly intervals is given below. Fit a straight line using the method of least squares:

Age(x) :	1	2	3	4	5	6	7	8	9	10
Weight(y) :	52.5	58.7	65.0	70.2	75.4	81.1	87.2	95.5	102.2	108.4

5. Fit a second degree parabola to the following data:

x :	1	2	3	4	5	6	7	8	9
y :	2	6	7	8	10	11	11	10	9

6. Following data is for the measurement of train resistance R(lbs./ton) with the velocity V(mph). If $R = a + bV + cV^2$, find a, b, c.

V:	20	40	60	80	100	120
R:	5.5	9.1	14.9	22.8	33.3	46.0.

7. Fit the curve $y = ka^{bx}$ to the following giving the number of children (x) in a family and number of families (y) in a survey conducted over 390 families.

x:	1	2	3	4	5	6
y:	151	100	61	50	20	8

8. Fit a curve of the form $PV^a = b$, where a and b are constants in the following data:

P(kg/cm^2) :	0.5	1.0	1.5	2.0	2.5	3.0
V(litres) :	1.62	1.00	0.75	0.62	0.52	0.46.

9. The voltage V across a capacitor at time t seconds is given by the following data. Fit a curve of the form $V = ae^{bt}$ to this data

t:	0	2	4	6	8
V:	150	63	28	12	5.6.

10. Fit a curve of the form $y = ax^b$ to the following data:

x:	2	4	7	10	20	40	60	80
y:	43	25	18	13	8	5	3	2.

24.13 CORRELATION AND REGRESSION

We are often concerned with data set that consist of pairs of values that may have some relationship to each other. If each element in the data has an x value and a y value, then the data for the set of n individual is represented as (x_i, y_i), $i = 1, 2, ..., n$. For example, heights and weights of a group of men, demand and price of a commodity during the different months, the force on a wire and resulting extension, etc. In such a bivariate date we may be interested to find out if there is any relationship between the two variables under study. The concept of correlation is related to this aspect of the bivariate data.

24.13.1 Correlation

Correlation is a measure of the degree of association existing between two variables in a bivariat data. The simplest way to get an idea whether the variables are correlated is to obtain the diagram of the points (x_i, y_i), $i = 1, 2, ..., n$ called the *scatter diagram*, refer to Fig. 24.15.

If all the points on the scatter diagram lie on a straight line, then *perfect linear correlation* is said to exist. When the points tend to concentrate about a straight line with positive gradient, then *positive linear correlation* is said to exist, refer to Fig. 24.15a and when the points tend to concentrate about a straight line with negative gradient, then *negative linear correlation* is said to exist, refer to Fig. 24.15b. In case the points are widely scattered, as shown in Fig. 24.15c, then a *poor correlation* or *no correlation* is expected. In positive linear correlation, increase (decrease) in x is accompanied with the increase (decrease) in y and vice versa; while in negative linear correlation increase (decrease) in x is accompanied with decrease (increase) in y and vice versa.

Karl Pearson's coefficient of linear correlation

The Karl Pearson's correlation coefficient between two random variables X and Y, denoted by r_{xy}, is a numerical measure of *linear relationship* between them and is defined as

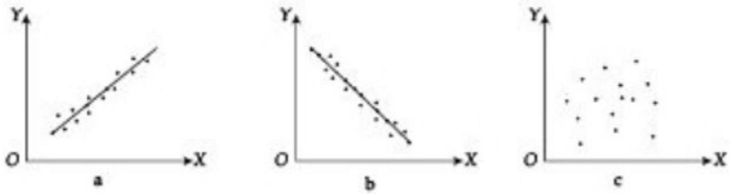


Fig. 24.15

$$r_{xy} = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y} = \frac{E(x - \bar{x})(y - \bar{y})}{\sigma_x \sigma_y} \quad ...(24.95)$$

If (x_i, y_i), $i = 1, 2, ..., n$ is the bivariate distribution, then

$$\text{Cov}(x, y) = \frac{1}{n}\Sigma (x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n}\Sigma x_i y_i - \left(\frac{1}{n}\Sigma x_i\right)\left(\frac{1}{n}\Sigma y_i\right),$$

$$\sigma_x^2 = \frac{1}{n}\Sigma (x_i - \bar{x})^2 = \frac{1}{n}\Sigma x_i^2 - \left(\frac{1}{n}\Sigma x_i\right)^2,$$

$$\sigma_y^2 = \frac{1}{n}\Sigma (y_i - \bar{y})^2 = \frac{1}{n}\Sigma y_i^2 - \left(\frac{1}{n}\Sigma y_i\right)^2.$$

Thus a convenient expression for the Karl-Pearson's coefficient of correlation for calculation purpose is given by

$$r_{xy} = \frac{\Sigma (x - \bar{x})(y - \bar{y})}{\sqrt{\Sigma (x - \bar{x})^2}\sqrt{\Sigma (y - \bar{y})^2}} = \frac{\Sigma xy - \dfrac{(\Sigma x)(\Sigma y)}{n}}{\sqrt{\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}}\sqrt{\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}}}. \quad ...(24.96)$$

Remark. The coefficient r_{xy} provides a measure of only the linear relationship between X and Y. It is not suitable for non-linear relationship.

Limits for correlation coefficient

We have,
$$E\left[\left(\frac{x - \bar{x}}{\sigma_x}\right) \pm \left(\frac{y - \bar{y}}{\sigma_y}\right)\right]^2 \geq 0.$$

This gives
$$E\left(\frac{x - \bar{x}}{\sigma_x}\right)^2 + E\left(\frac{y - \bar{y}}{\sigma_y}\right)^2 \pm 2\frac{E(x - \bar{x})(y - \bar{y})}{\sigma_x \sigma_y} \geq 0$$

or,
$$1 + 1 \pm 2r_{xy} \geq 0, \text{ or } -1 \leq r_{xy} \leq 1.$$

Thus r_{xy} lies between −1 and + 1. If $r = +1$, the correlation is perfect and positive, and if $r = -1$, the correlation is perfect and negative. If $r = 0$, then there is no linear correlation between x and y and variables are said to be uncorrelated.

Effect of change of origin and scale

Let $u = \dfrac{x-a}{h}$ and $v = \dfrac{y-b}{k}$, where a, b, h, k are constants with $h, k > 0$.

We have $x = a + hu, y = b + kv$. We can very easily show that

$$\text{Cov}(x, y) = hk\,\text{Cov}(u, v).$$

Also, $\sigma_x = h\sigma_u$ and $\sigma_y = k\sigma_v$. Hence,

$$r_{xy} = \frac{\text{Cov}(x, y)}{\sigma_x\,\sigma_y} = \frac{\text{Cov}(u, v)}{\sigma_u\,\sigma_v} = r_{uv} \qquad ...(24.97)$$

Thus correlation coefficient r is independent of the change of origin and scale.

Also we note that correlation coefficient is a dimensionless number independent of the units in which variables X and Y are measured.

Example 24.68: In an experiment to determine the relationship between force on a wire and the resulting extension, the following data is obtained:

Force (N) :	10	20	30	40	50	60	70
Extension (mm.) :	0.22	0.40	0.61	0.85	1.20	1.45	1.70

Find the coefficient of linear correlation for this data.

Solution: Let x be the variable force applied and y be the extension, then

$$r_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2}\,.\sqrt{\Sigma(y-\bar{y})^2}}$$

We have, $n = 7$, $\Sigma x = 280$, $\Sigma y = 6.43$, $\bar{x} = 40$, $\bar{y} = 0.919$.

We formulate the following table:

x	y	$(x-\bar{x})$	$(y-\bar{y})$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$	$(y-\bar{y})^2$
10	0.22	−30	−0.699	20.97	900	0.489
20	0.40	−20	−0.519	10.38	400	0.269
30	0.61	−10	−0.309	3.09	100	0.095
40	0.85	0	−0.069	0	0	0.005
50	1.20	10	0.281	2.81	100	0.079
60	1.45	20	0.531	10.62	400	0.282
70	1.70	30	0.781	23.43	900	0.610
280	6.43			71.30	2800	1.829

Hence, $$r_{xy} = \frac{71.30}{\sqrt{2800 \times 1.829}} = 0.996.$$

It shows almost perfect positive correlation between the force applied and the resultant extension.

Example 24.69: Calculate the coefficient of correlation between the age of husband(x) and of wife(y) for the data given below:

x (yrs)	:	23	27	28	28	29	30	31	33	35	36
y (yrs)	:	18	20	22	27	21	29	27	29	28	29.

Solution: We take $u = x - 29$, $v = y - 25$ and form the following table.

x	$u = x - 29$	u^2	y	$v = y - 25$	v^2	uv
23	−6	36	18	−7	49	42
27	−2	4	20	−5	25	10
28	−1	1	22	−3	9	3
28	−1	1	27	2	4	2
29	0	0	21	−4	16	0
30	1	1	29	4	16	4
31	2	4	27	2	4	4
33	4	16	29	4	16	16
35	6	36	28	3	9	18
36	7	49	29	4	16	28
Total	10	148		0	164	123

Since there is no effect of change of origin on r, thus

$$r_{xy} = r_{uv} = \frac{\Sigma uv - \frac{(\Sigma u)(\Sigma v)}{n}}{\sqrt{\Sigma u^2 - \frac{(\Sigma u)^2}{n}}\sqrt{\Sigma v^2 - \frac{(\Sigma v)^2}{n}}}$$

$$= \frac{123 - 0}{\sqrt{148 - \frac{100}{10}}\sqrt{164 - 0}} = \frac{123}{\sqrt{138 \times 164}} = 0.818.$$

This shows a good positive correlation between ages of husbands and wives.

Example 24.70: Prove that two independent variables are uncorrelated, that is, $r = 0$ but the converse is not true.

Solution: Let X and Y are two independent random variables, then

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0,$$

since in case of two independent variables $E(XY) = E(X)\,E(Y)$.

Thus, $r = 0$ when X and Y are independent.

But converse is not true. Consider the following example:

X:	−3	−2	−1	0	1	2	3
Y:	9	4	1	0	1	4	9

Here $\Sigma X = 0$, $\Sigma Y = 28$, $\Sigma XY = 0$.

Thus, $$\text{Cov}(X, Y) = \frac{1}{n}\Sigma XY - \left(\frac{1}{n}\Sigma X\right)\left(\frac{1}{n}\Sigma Y\right) = 0.$$

In this case variables X and Y are dependent and are related by an explicit relationship $Y = X^2$ but $r = 0$. Hence, two uncorrelated variables may not necessary be independent. The reason for this is that r_{xy} *is the numerical measure of only linear relationship between the two r.v. X and Y, r = 0 means absence of the linear relationship between X and Y. There may be non-linear relationship between the variables X and Y.*

Example 24.71: Two variables X and Y are related by the linear equation $aX + bY + c = 0$. Find r_{XY}.

Solution: The relation is $aX + bY + c = 0$. It gives $aE(X) + bE(Y) + c = 0$.

Hence, $$a[X - E(X)] + b[Y - E(Y)] = 0 \text{ or, } [Y - E(Y)] = -\frac{a}{b}[X - E(X)].$$

Thus, $\text{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))] = -\frac{a}{b}E[X - E(X)]^2 = -\frac{a}{b}\sigma_x^2.$

Also, $\sigma_y^2 = E[Y - E(Y)]^2 = \frac{a^2}{b^2}E[X - E(X)]^2 = \frac{a^2}{b^2}\sigma_x^2$, which gives, $\sigma_y = \left|\frac{a}{b}\right|\sigma_x$. Hence,

$$r_{xy} = \frac{\text{Cov}(X,Y)}{\sigma_x\,\sigma_y} = -\frac{a}{b}\Big/\left|\frac{a}{b}\right| = \begin{cases} +1, & \text{if } a \text{ and } b \text{ are of the opposite signs.} \\ -1, & \text{if } a \text{ and } b \text{ are of the same signs.} \end{cases}$$

Example 24.72: The joint probability distribution of (X, Y) is given below. Find the correlation coefficient between X and Y.

Y \ X	−1	+1
0	1/8	3/8
1	1/4	1/4

Solution: Let $p(x)$ and $q(y)$ be the marginal probabilities for X and Y. We form the following table:

Y \ X	−1	+1	$q(y)$
0	1/8	3/8	1/2
1	1/4	1/4	1/2
$p(x)$	3/8	5/8	1

Thus $E(X) = \Sigma\, xp(x) = -3/8 + 5/8 = 1/4,\ E(X^2) = \Sigma\, x^2 p(x) = 3/8 + 5/8 = 1,$
$E(Y) = \Sigma\, yq(y) = 0 + 1/2 = 1/2,\ E(Y^2) = \Sigma\, y^2 q(y) = 0 + 1/2 = 1/2,$

and,
$$E(XY) = \Sigma\, xyp(x, y) = 0 + 0 + \left(-\frac{1}{4}\right) + \left(\frac{1}{4}\right) = 0.$$

Hence, $\text{Cov}(X, Y) = E(XY) - E(X)\, E(Y) = -1/8,$
$\sigma_x^2 = E(X^2) - [E(X)]^2 = 1 - 1/16 = 15/16$, and, $\sigma_y^2 = E(Y^2) - [E(Y)]^2 = 1/2 - 1/4 = 1/4.$

Therefore,
$$r_{xy} = \frac{-1/8}{\sqrt{15/16}\,\sqrt{1/4}} = -0.2582.$$

24.13.2 Spearman's Rank Correlation Coefficient

Let us suppose that a group of n individuals is arranged in order of merit (rank) regarding two characteristics A and B. These ranks in respect of the two characteristics, in general, will be different.

Let $x_i, y_i; i = 1, 2, \ldots, n$ be the ranks of the ith individual in respect to characteristics A and B respectively. The Spearman's rank correlation coefficient, denoted by ρ, is simply Pearson's coefficient of correlation calculated for the ranks x_i's and y_i's.

Assuming that no two individuals have the same rank, each of the two variables x and y takes the values $1, 2, \ldots, n$.

Hence, $\bar{x} = \bar{y} = \dfrac{n+1}{2}$, and,

$$\Sigma (x_i - \bar{x})^2 = \Sigma (y_i - \bar{y})^2 = \Sigma y_i^2 - n\bar{y}^2 = \frac{n(n+1)(2n+1)}{6} - n\left(\frac{n+1}{2}\right)^2 = \frac{1}{12}(n^3 - n).$$

Let
$$d_i = x_i - y_i = (x_i - \bar{x}) - (y_i - \bar{y}), \text{ therefore}$$

$$\frac{1}{n}\Sigma d_i^2 = \frac{1}{n}\Sigma (x_i - \bar{x})^2 + \frac{1}{n}\Sigma (y_i - \bar{y})^2 - \frac{2}{n}\Sigma (x_i - \bar{x})(y_i - \bar{y}).$$

It gives $2\,\text{Cov}(X, Y) = \dfrac{2}{n}\Sigma (x_i - \bar{x})(y_i - \bar{y}) = \dfrac{1}{n}\Sigma (x_i - x)^2 + \dfrac{1}{n}\Sigma (y_i - \bar{y})^2 - \dfrac{1}{n}\Sigma d_i^2$

$$= \frac{2}{n}\,\frac{1}{12}(n^3 - n) - \frac{1}{n}\Sigma d_i^2$$

or, $\text{Cov}(X, Y) = \dfrac{1}{12}(n^2 - 1) - \dfrac{1}{2n}\Sigma d_i^2.$

Also, $\sigma_x \sigma_y = \sqrt{\dfrac{1}{n}\Sigma (x_i - \bar{x})^2}\,\sqrt{\dfrac{1}{n}\Sigma (y_i - \bar{y})^2} = \dfrac{1}{n}\,\dfrac{1}{12}(n^3 - n) = \dfrac{1}{12}(n^2 - 1).$

Hence the correlation coefficient between the two characteristics A and B is

$$\rho = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y} = 1 - \frac{6\Sigma d_i^2}{n(n^2-1)}, \quad ...(24.98)$$

called the *Spearman's rank correlation coefficient*.

Remarks

1. If any two or more individuals are bracketed, then these individuals are given the common rank equal to the average of the ranks they would have assumed in case of marginal differences. The formula (24.98) is corrected by adding the factor $\frac{m(m^2-1)}{12}$ to Σd^2, where m is the number of items tied up. This factor is to be compensated for each tied group in both the x's and y's. The procedure is illustrated in Example (24.74).

2. Spearman's rank correlation coefficient ρ also lies between -1 to 1. ρ is maximum when Σd_i^2 is minimum, that is, zero. In this case, $x_i = y_i$; $i = 1, 2, ..., n$. Hence, *maximum value of* ρ *is* 1. Further ρ is minimum when x_i and y_i have opposite ranks, that is, $x_i + y_i = n + 1$; $i = 1, 2, ..., n$. In this case,

$$\Sigma d_i^2 = \Sigma (x_i - y_i)^2 = \Sigma [2x_i - (n+1)]^2 = 4\Sigma x_i^2 + n(n+1)^2 - 4(n+1)\Sigma x_i$$

$$= 4\frac{n(n+1)(2n+1)}{6} + n(n+1)^2 - 2n(n+1)^2 = n(n+1)\left[\frac{2(2n+1)}{3} - (n+1)\right] = \frac{n(n^2-1)}{3}$$

Hence, min. $\rho = 1 - \dfrac{6\frac{n(n^2-1)}{3}}{n(n^2-1)} = 1 - 2 = -1$. Thus ρ lies between -1 and 1.

Example 24.73: Ten students got the following marks in mathematics and economics in end-sem examination. Calculate the rank correlation coefficient.

Students	1	2	3	4	5	6	7	8	9	10
Marks in Economics	78	36	98	25	75	82	90	62	65	39
Marks in Mathematics	81	51	91	60	68	62	86	58	53	47

Solution: Alloting the ranks to the students, we have

Students	1	2	3	4	5	6	7	8	9	10
Rank in Eco.	4	9	1	10	5	3	2	7	6	8
Rank in Maths.	3	9	1	6	4	5	2	7	8	10
Difference (d_i)	1	0	0	4	1	−2	0	0	−2	−2
d_i^2	1	0	0	16	1	4	0	0	4	4

We have, $\Sigma d_i^2 = 30$

Therefore, $$\rho = 1 - \frac{6\Sigma d_i^2}{n(n^2-1)} = 1 - \frac{180}{10(100-1)} = 0.8182,$$

which shows a good positive correlation between the two characteristics.

Example 24.74: Obtain the rank correlation coefficient for the following data:

$$\begin{array}{lllllllllll} x: & 68 & 64 & 75 & 50 & 64 & 80 & 75 & 40 & 55 & 64 \\ y: & 62 & 58 & 68 & 45 & 81 & 60 & 68 & 48 & 50 & 70 \end{array}$$

Solution: We form the following table:

x	y	Rank x	Rank y	$d = x - y$	d^2
68	62	4	5	–1	1
64	58	6	7	–1	1
75	68	2.5	3.5	–1	1
50	45	9	10	–1	1
64	81	6	1	5	25
80	60	1	6	–5	25
75	68	2.5	3.5	–1	1
40	48	10	9	1	1
55	50	8	8	0	0
64	70	6	2	4	16
				Total: 0	72

Due to common ranks in the x-series, the average rank 2.5 has been assigned twice and the average rank 6 has been assigned thrice. Similarly in y-series, the average rank 3.5 has been assigned twice. Hence, the correction in Σd^2 due to common ranks corresponds to $m = 2$, 3 and 2 and the correction factors is $\frac{m(m^2-1)}{12}$.

Thus the total correction $= \frac{2(4-1)}{12} + \frac{3(9-1)}{12} + \frac{2(4-1)}{12} = \frac{1}{2} + 2 + \frac{1}{2} = 3.$

Hence, $$\rho = 1 - \frac{6(\Sigma d^2 + 3)}{n(n^2-1)} = 1 - \frac{6(72+3)}{10 \times 99} = 0.545.$$

24.13.3 Regression

In regression analysis one of the two variables, say x is regarded as an independent variable, the other variable y as the dependent variable which may be random in nature. We are interested in the dependence of y on x. For example, dependence of blood pressure (y) on the age (x) of a person, or the dependence of height of son (y) on the height of father (x), etc. In general, we specify $x_1, x_2, ..., x_n$, and then observe the corresponding values $y_1, y_2, ..., y_n$ of the r.v. Y, so that we get a bivariate sample (x_i, y_i); $i = 1, 2, ..., n$. We are interested in finding a mathematical measure of the average relationship between the two (or, more in case of multivariate data) variables in terms of the original units of the data.

Linear Regression: If the variables in the bivariate distribution (x_i, y_i); $i = 1, 2, ..., n$, are related, then the points in the scatter diagram will converge around some curve, called the *curve of* *regression*. In case the curve happens to be a straight line, the regression is said to be *linear* and the line is called the *line of regression. It is the line which gives the best estimate of the dependent variable for any specific value of the independent variable.* The best-fit line is obtained by the principle of least squares, which consists in minimising the sum of the squares of the deviations of the actual values of y from their estimated values as given by the line of best-fit.

Let the line of regression of Y on x be $Y = a + bx$.

Then the error of estimate for $Y = y_i$, refer to Fig. 24.13, is $P_iM_i = y_i - (a + bx_i)$.

According to the principle of least squares, we need to find a and b so that

$$S = \sum_{i=1}^{n} (y_i - a - bx_i)^2 \text{ is minimum.}$$

Thus the normal equations for estimating a and b are

$$\Sigma\, y_i = na + b\Sigma\, x_i \qquad ...(24.99)$$

and,
$$\Sigma\, x_iy_i = a\Sigma\, x_i + b\Sigma\, x_i^2 \qquad ...(24.100)$$

Dividing (24.99) by n, we obtain

$$\bar{y} = a + b\bar{x}. \qquad ...(24.101)$$

Thus the line of regression passes through $(\bar{x}, \bar{y})$.

Next, dividing (24.100) by n, we obtain

$$\frac{1}{n}\Sigma\, x_iy_i = \frac{a}{n}\Sigma\, x_i + \frac{b}{n}\Sigma\, x_i^2$$

or,
$$[\text{Cov}(x, y) + \bar{x}\,\bar{y}] = a\bar{x} + b[\sigma_x^2 + (\bar{x})^2], \qquad ...(24.102)$$

$$\text{since Cov}(x, y) = \frac{1}{n}\Sigma\, x_iy_i - \bar{x}\,\bar{y}, \text{ and } \sigma_x^2 = \frac{1}{n}\Sigma\, x_i^2 - (\bar{x})^2.$$

To obtain b, multiply (24.101) by $\bar{x}$ and subtract from (24.102), we obtain b, *the slope of the line of regression of y on x*, as

$$b = \frac{\text{Cov}(x, y)}{\sigma_x^2}. \qquad ...(24.103)$$

Hence the *equation of the line of regression of y on x* is

$$y - \bar{y} = b(x - \bar{x}) = \frac{\text{Cov}(x, y)}{\sigma_x^2}(x - \bar{x})$$

or,
$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}), \qquad ...(24.104)$$

where r is the coefficient of correlation between x and y.

On interchanging the variables, we arrive at the *equation of the line of regression of x on y* given by

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}). \quad ...(24.105)$$

The two coefficients b_{yx} and b_{xy}, given respectively by

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} \text{ and } b_{xy} = r\frac{\sigma_x}{\sigma_y}, \quad ...(24.106)$$

are called the *regression coefficients* of y on x and of x on y respectively.

We observe that $\quad r = \pm\sqrt{b_{yx}b_{xy}}$.

Thus the coefficient of correlation r between the two variables X and Y is the geometric mean of the two coefficients of regression and the sign of r is taken as the common sign of b_{yx} and b_{xy}.

Also we should note that the regression coefficients are independent of the change of origin but depends upon the change of scale.

Angle between two lines of regression

If θ is the acute angle between the two regression lines (24.104) and (24.105), then

$$\tan\theta = \left|\frac{r\frac{\sigma_y}{\sigma_x} - \frac{\sigma_y}{r\sigma_x}}{1 + r\frac{\sigma_y}{\sigma_x}\cdot\frac{\sigma_y}{r\sigma_x}}\right| = \frac{1-r^2}{|r|}\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}. \quad ...(24.107)$$

If $r = 0$, then $\tan\theta = \infty$, that is, $\theta = \pi/2$. *Thus, if two variables are uncorrelated, then the regression lines are perpendicular to each other.* If $r = \pm 1$, then $\tan\theta = 0$, that is, $\theta = 0$. *Thus, in case of perfect correlation, positive or negative, the two regression lines coincide.*

Remark: There are always two regression lines, one of y on x and other of x on y. The line of regression of y on x gives the best estimate of y for given specific value of x and is used when x is an independent variable and y is a dependent variable. If the situation is otherwise, y being independent and x dependent, then the line of regression of x on y is used.

Example 24.75: The following are measurements of the air velocity(x) and evaporation coefficient (y) of burning fuel droplets in an impulse engine. Find the coefficient of correlation r and the line of regression of y on x. Also estimate the evaporation coefficient when the air velocity is 210 cm/sec.

Air velocity (cm/sec) (x)	: 20	60	100	140	180	220	260	300	340	380
Evaporation coefficient (mm²/sec) (y)	: 0.18	0.37	0.35	0.78	0.56	0.75	1.18	1.36	1.17	1.65

On the basis of the data available what should be the expected air velocity in case evaporation of 1.50 mm^2/sec. is needed?

Solution: For this set of observations

$$n = 10, \quad \Sigma x_i = 2000, \quad \Sigma x_i^2 = 532{,}000$$
$$\Sigma y_i = 8.35, \quad \Sigma x_i y_i = 2175.40, \quad \Sigma y_i^2 = 9.11.$$

Thus, $\bar{x} = 200$, $\bar{y} = 0.835$, and, $\sigma_x^2 = \frac{1}{n}\Sigma x_i^2 - (\bar{x})^2 = 53200 - 40000 = 13200$

Therefore, $\sigma_x = 114.89$.

Also $$\sigma_y^2 = \frac{1}{n}\Sigma y_i^2 - (\bar{y})^2 = 0.911 - 0.6997 = 0.214, \text{ Therefore, } \sigma_y = 0.463.$$

Further $$\text{Cov}(x, y) = \frac{1}{n}\Sigma x_i y_i - (\bar{x})(\bar{y}) = 217.54 - (200)(0.835) = 50.54.$$

Thus, $$r = \frac{\text{Cov}(x,y)}{\sigma_x \sigma_y} = \frac{50.54}{(114.89)(0.463)} = 0.95.$$

Also, $$b_{yx} = r\frac{\sigma_y}{\sigma_x} = (0.95)\frac{0.463}{114.89} = 0.00383.$$

Thus the equation of the line of regression of y on x is

$$y - 0.835 = 0.00383\,(x - 200) \text{ or, } y = 0.00383\,x + 0.069.$$

When $x = 210$ cm/sec, then estimated y is, $y = 0.00383\,(210) + 0.0069 \simeq 0.8733$ mm/sec^2

To find the expected air-velocity (x) corresponding to the evaporation coefficient (y) of 1.50 mm^2/sec, we need to find the regression equation of x on y.

We have, $$b_{xy} = r\frac{\sigma_x}{\sigma_y} = 0.95.\frac{114.89}{0.463} = 235.735$$

Hence, the regression line of x on y is

$$x - 200 = (235.735)\,(y - 0.835) \text{ or, } x = 235.735\,y + 3.161.$$

Thus, the expected air-velocity corresponding to evaporation coefficient of 1.50 mm^2/sec is $x = (235.735)\,(1.50) + 3.61 \approx 357.21$ cm/sec.

Example 24.76: Obtain the equation of the line of regression of y on x for the following data. Also estimate y when $x = 32$.

x :	23	27	28	28	29	30	31	33	35	36
y :	18	20	22	27	21	29	27	29	28	29

Solution: Let $u = x - 29$, $v = y - 25$. Refer to Example 24.69, we have

$n = 10$, $\Sigma u = 10$, $\Sigma u^2 = 148$, $\Sigma v = 0$

$\Sigma v^2 = 164$, $\Sigma uv = 123$, and $r_{xy} = r_{uv} = 0.86$

Also, $$\bar{x} = a + \bar{u} = 29 + \frac{10}{10} = 30, \ \bar{y} = b + \bar{v} = 25 + 0 = 25$$

and $$\sigma_x = \sigma_u = \sqrt{\frac{148}{10} - (1)^2} = 3.71, \ \sigma_y = \sigma_v = \sqrt{\frac{164}{10} - 0} = 4.05.$$

Hence, the line of regression of y on x is

$$y-25=(0.86)\frac{4.05}{3.71}(x-30) \text{ or, } y=0.94x-3.2.$$

Thus, the estimated value of y for $x=32$ is $y=(0.94)(32)-3.2=26.88$.

Example 24.77: Can $y=5+2.8\,x$ and $x=3-0.5y$ be the estimated regression equations of y on x and x on y respectively? Explain.

Solution: In case the given equations happen to be the equations of the regression lines, then the coefficient of regression of y on x is $b_{yx}=2.8$, and the coefficient of regression of x on is $b_{xy}=-0.5$, which is not possible since the two coefficients of regression are always of the same sign that of the sign of Cov (x, y). Hence, the given equations cannot be the regression equations.

EXERCISE 24.8

1. In an experiment to determine the relationship between the current flowing in an electrical circuit and the voltage applied, the results obtained are

Current (mA) :	5	11	15	19	24	28	33
Applied voltage (V) :	2	4	6	8	10	12	14

Find the Karl-Pearson's coefficient of correlation.

2. A gas is being compressed in a closed cylinder and the values of pressures and corresponding volumes at constant temperature are as shown:

Pressure (kP$_a$) :	160	180	200	220	240	260	280	300
Volume (m^3) :	0.034	0.036	0.030	0.027	0.024	0.025	0.020	0.019

Find the coefficient of correlation for these values.

3. The following marks have been obtained by a class of students in statistics

Paper I :	45	55	56	58	60	65	68	70	75	80	85
Paper II :	56	50	48	60	62	64	65	70	74	82	90

Compute the coefficient of correlation for the above data.

Find also the equations of the lines of regression.

4. For the following data, compute the coefficient of correlation between X and Y.

	X-series	Y-series
No. of items :	15	15
Arithmetic mean :	25	18
Sum of square of deviations from mean :	136	138

Also the sum of products of deviations of X and Y from their respective arithmetic means is 122.

5. For the bivariate probability distribution given below, find the coefficient of correlation between X and Y.

Y \ X	−1	0	1
0	1/15	2/15	1/15
1	3/15	2/15	1/15
2	2/15	1/15	2/15

6. Calculate rank correlation coefficient for the following measurements of the air velocity (x) and evaporation coefficient (y) of burning fuel droplets in an impulse engine. Compare it with Karl-Pearson's coefficient of correlation.

x(cm/sec) :	20	60	100	140	180	220	260	300	340	280
y(mm^2/sec) :	0.18	0.37	0.35	0.78	0.56	0.75	1.18	1.36	1.17	1.65

7. Ten competitors in a musical contest were ranked by the three judges A, B and C in the following order:

Ranks by A:	1	6	5	10	3	2	4	9	7	8
Ranks by B:	3	5	8	4	7	10	2	1	6	9
Ranks by C:	6	4	9	8	1	2	3	10	5	7

Discuss which pair of judges has the nearest approach to common likings in music.

8. Calculate the rank correlation coefficient for the following data:

x :	65	63	67	64	68	62	70	66	68	67	69	71
y :	68	66	68	65	69	66	68	65	71	67	68	70

9. If x_1, x_2, x_3 are uncorrelated variables each having the same S.D.'s, obtain the coefficient of correlation between $x_1 + x_2$ and $x_2 + x_3$.

10. Find the coefficient of correlation between the number of heads and number of tails obtained in n throws of a coin.

12. Find the coefficient of correlation from the following table giving the ages of husbands(x) and wives(y) in case of 100 couples surveyed,

x \ y	20-30	30-40	40-50	50-60	60-70	Total
15-25	5	9	3	0	0	17
25-35	0	10	25	2	0	37
35-45	0	1	12	2	0	15
45-55	0	0	4	16	5	25
55-65	0	0	0	4	2	6
Total	5	20	44	24	7	100

13. For the following data giving the effect of extraction time on the efficiency of an extraction process obtained from a chemical industry, find the extraction efficiency expected when the extraction time is 40 minutes.

Extraction time (minutes) x	:	27	45	41	19	35	39	19	49	15	31
Extraction efficiency (%) y	:	57	64	80	46	62	72	52	77	57	68

14. The pressure P of a gas corresponding to various volumes V is recorded as follows:

$V(cm^3)$	:	50	60	70	90	100
$P(kg/cm^3)$	:	64.7	51.3	40.5	25.9	7.8

The ideal gas law is given by the function form $PV^{\gamma} = C$, where γ and C are constants. Apply regression analysis to estimate γ and C.

ANSWERS

Exercise 24.1 (p. 1352)

1. (a) The class frequencies are 5, 8, 10, 15, 8, 4, 3, 1 and 1.
2. Skewed right
3. (a) mean 7.33, median 8, mode 8
   (b) mean 115.2, median 126.4, no mode
4. mean 89.5, median 89, mode 88.2
5. 34, 45   6. 1.774, 1.77, 0.385   7. $\frac{n+1}{2}$, $\frac{1}{12}(n^2-1)$
8. Median = 10.9, $Q_1 = 6.5$, $Q_2 = 18.25$, $D_4 = 10$ and $P_{60} = 12.57$.
9. $Q_1 = 14.95$, $Q_3 = 22.83$, $Q_3 - Q_1 = 7.88$
10. 89.   11. −0.421
12. $\beta_1 = 0.038$, $\beta_2 = 1.806$, slightly positively skewed and platykurtic.
13. mean = 1, $\mu_2 = 6$, $\beta_1 = 1.67$, $\beta_2 = 0.89$, positively skewed and platykurtic curve.

Exercise 24.2 (p. 1367)

1. 6/11   2. $(365)(364)(363)\ldots(365-n+1)/(365)^n$, 23
3. 4/9,   4. (a) 22/25   (b) 3/25   (c) 17/50
5. (a) 0.26   (b) 0.70
6. 0.475   7. 30/61   8. 0.99
9. 0.381   10. $1 - \prod_{i=1}^{n}(1-p_i)$, $\frac{671}{1296}$
11. 2/9   12. (a) 0.0016   (b) 0.9984
13. 0.75112   14. 0.0850   15. 0.28
16. 5/6   17. 0.3322

18. (a) $\alpha_1/(\alpha_1+2)$ (b) $1/(\alpha_1+2)$ (c) $1/(\alpha_1+2)$
19. .012
20. 4/9.

Exercise 24.3 (p. 1379)

1.

x :	1	2	3	4	5	6
$p(x)$:	1/6	1/6	1/6	1/6	1/6	1/6
$F(x)$:	1/6	2/6	3/6	4/6	5/6	6/6

2. $P_r\{X=k\}=\dfrac{1}{2^k}$,
3. Rs. –7.0
4. Rs. 1050
5. $\dfrac{1}{2}m(n+1)$
6. 200 hrs.
7. Rs. 12.67
8. $k=3/8,\ P\{x>1\}=1/2,\ F(x)=\dfrac{3x^2}{4}\left[1-\dfrac{x}{3}\right]$
9. 0.0372
10. (a)

y \ x	0	1	2	3
0	0	3/70	9/70	3/70
1	2/70	18/70	18/70	2/70
2	3/70	9/70	3/70	0

(b) 1/2 (c) No

11. $\dfrac{q}{p}$
12. $pe^t/(1-qe^t)$, mean $(x)=1/p$, var $(x)=\dfrac{1-p}{p^2}$
13. 8/9
14. (a) $\geq 15/16$ (b) $\leq 1/4$.

Exercise 24.4 (p. 1390)

1. 0.5%, .015
2. (a) 0.1536 (b) 0.9984
3. (a) 0.0338 (b) 0.8779 (c) 0.1859
4. (a) 0.2852 (b) 0.9887 (c) 0.6083
5. (a) 0.2637 (b) 0.0220
6. (a) 0.3774 (b) 0.9997 (c) 19
7. $f(0)=1$, $f(1)=12$, $f(2)=66$, $f(3)=220$, $f(4)=495$, $f(5)=792$ $f(6)=924$, $f(7)=792$, $f(8)=495$,

$f(9) = 220$ $f(10) = 66,$ $f(11) = 12,$ $f(12) = 1.$

10. $\frac{n!}{x_1!x_2!x_3!}(0.35)^{x_1}(0.05)^{x_2}(0.60)^{x_3}$ 11. 0.3010

12. (a) 0.1353 (b) 0.4335 13. 0.0235

14. 0.1813 15. 4, 2 16. (a) 0.0630 (b) 0.9730

17. (a) 0.3991 (b) 0.1315 18. 670, 268, 54, 7, 1.

Exercise 24.5 (p. 1403)

1. 1/5, 4/5 2. (a) 0.6 (b) 0.7 (c) 0.5

3. (a) 0.1151 (b) 0.5403 (c) 16.1 (d) 20.275

4. 4.082 ounces 5. mean 50, S.D. 10, 6. (a) 67 (b) 184 (c) 1909

7. 6.24 yrs 8. 15.9% 9. 58%

10. (a) 0.4586 (b) 0.0170

11. (a) 0.3085 (b) 0.2417 (c) 0.0045

12. 1, 4, 6, 4, 1, $y = \frac{\sqrt{2}}{4\sqrt{\pi}}e^{-\frac{1}{2}\left(\frac{x-6}{2}\right)^2}$ 14. (a) 0.2376 (b) 0.0576

15. (a) 0.9782 (b) 0.0059 (c) 0.0121

16. (a) 0.3085 (b) 0.0197.

Exercise 24.6 (p. 1408)

1. (a) 0.3935 (b) 0.4346 2. 0.3968 5. 0.715

6. (a) mean $\frac{\alpha}{\beta} = 50$ (b) S.D. $= \sqrt{\frac{\alpha}{\beta^2}} = 22.36$ (c) 0.8155

7. (a) 0.4 (b) 0.4537 8. (a) 0.1889 (b) 0.1532.

Exercise 24.7 (p. 1413)

1. $y = 2.016 + 0.503y$ 3. $R = 70.052 + 0.292t$

4. $y = 45.74 + 6.16x$ 5. $y = -0.98 + 3.55x - 0.27x^3$

6. $a = 3.48,\ b = -0.002,\ c = 0.0029.$

7. $y = (309.0)(0.5754)^x$ 8. $PV^{1.276} = 1.039.$

9. $v = 146.13e^{-0.412t}$ 10. $y = 78x^{-0.8}.$

Exercise 24.8 (p. 1425)

1. 0.999
2. −0.962
3. $r = 0.912,\ y = 0.99x + 1,\ x = 0.85y + 9.48$
4. $r = 0.891$
6. $\rho = 0.915,\ r = 0.95$
7. $\rho_{AB} = -0.212,\ \rho_{BC} = 0.636,\ \rho_{AC} = -0.297$, A and C
9. 1/2
10. −1
12. 0.796
13. 69.61%
14. $\gamma = 2.6535,\ C = 2568862.88$

25 Sampling Distributions and Hypothesis Testing

CHAPTER

The sample statistics describe the sample and are used to make inference about the sampled population parameters. In hypothesis testing our primary concern is to develop a procedure for determining whether or not the values of the random sample from the population are consistent with the hypothesis. Distribution of the sample statistic facilitates to formulate that procedure.

25.1 BASIC CONCEPTS

In any particular study the number of observations recorded may be finite or infinite. For example, number of defective screws in a box of 1000 results in a finite number of observations, while if we could toss a pair of dice indefinitely and record the total obtained, then we obtain an infinite set of observations. A *population* consists of the totality of the observations under study. In case the number of observations are finite, population is called *finite population* otherwise *infinite population*. When it is not desirable to take into account all the observations, then we take a finite subset of the population called *sample*. In this chapter, we focus on sampling from population and study constants of the sample drawn called *statistics* like, sample mean ($\bar{x}$), sample variance (s^2), etc. and see how the information drawn from the sample is utilized to draw some conclusion about the population constants like, mean (μ), variance (σ^2), etc. called *parameters*.

The main *advantages of sampling* over complete enumeration consist of reduced cost, greater speed and scope and sometimes even better precision. When testing is destructive in nature, then sampling becomes necessary.

Methods of sampling: Some of the commonly employed methods of sampling are:

Random sampling: If each unit of the population has the same chance of being selected in the sample then sampling is said to be random sampling. Suppose we take a sample of size n from a population of finite size N, then in case of random sampling each of the ${}^{N}C_{n}$ samples has the same probability, that is, $1/{}^{N}C_{n}$ of being selected.

The simplest method of drawing a random sample is the lottery method, assigning numbers 1 to N to each unit of the population; writing these numbers on N identical slips; putting these slips in a box and then drawing n slips one by one from this well-shuffled lot. Then n units, corresponding to the numbers on the slips drawn constitute the random sample. Since this method is always not very practical, a simpler and more reliable method is the use of random numbers. *Random numbers* are the digits generated so that values 0 to 9 occur randomly with equal frequency. These numbers can be generated by computer, or alternatively, available from *random numbers tables* due to L.H.C. Tippett. These tables consist of 10400 four-digit numbers giving in all 41600 digits taken from British Census Reports. Random numbers help in obtaining samples that are in fact random samples.

Simple random sampling: Simple random sampling is random sampling in which each unit of the population has an equal probability 'p' of being included in the sample and this probability is independent of the previous drawings. Thus random sampling becomes simple, if either the units are drawn with replacement, or when the population is infinite. A simple sample of size n from a population may be identified with a series of n Bernoulli trials with constant probability 'p' of success for each trial.

Stratified sampling: In case the population is heterogeneous, then it is divided into homogeneous stratas (groups) of various sizes. Then units are sampled at random from each of these strata according to the relative importance of the stratas in the population. The sample drawn thus will be more representative than the simple random sample, since each strata will be represented in the sample drawn.

Systematic sampling: Let there be population with $N = nk$ ordered units from 1 to N. In systematic sampling if we are to draw a sample of size n, then we draw a unit at random from the first k ordered units and, then every kth unit is drawn to form the sample.

Purposive sampling: Here the units are selected with definite purpose in view. Usually, the selected units do not form a representative sample of the population and yield results which are generally biased.

In general, not all sampling plans involve random selection but any sampling plan used for drawing inferences must involve randomization.

25.2 STATISTICS AND SAMPLING DISTRIBUTIONS

The numerical descriptive measures calculated from the sample are called *statistics* and the numerical descriptive measures of the population, (generally unknown), are called *parameters*. These statistics vary for each different random sample selected and hence they are random variables. The probability distributions for statistics e.g., for sample mean, sample variance, etc. are called *sampling distributions*.

For example, consider a population consisting of 5 numbers 3, 5, 7, 9, 11. If a random sample of size $n = 2$ is selected without replacement, then we can find the sampling distribution of the sample mean $\bar{x}$ as follow.

There are 10 possible equally likely random samples of size $n = 2$. The values of $\bar{x}$ for random sampling when $n = 2$ and $N = 5$ are tabulated below.

Sample	*Sample units*	*Sample mean* $\bar{x}$
1	3, 5	4
2	3, 7	5
3	3, 9	6
4	3, 11	7
5	5, 7	6
6	5, 9	7
7	5, 11	8
8	7, 9	8
9	7, 11	9
10	9, 11	10

Hence, sampling distribution of the sample mean $\bar{x}$ is

$$\begin{array}{llllllll} \bar{x} = & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ f(\bar{x}) = & 1 & 1 & 2 & 2 & 2 & 1 & 1 \\ p(\bar{x}) = & 0.1 & 0.1 & 0.2 & 0.2 & 0.2 & 0.1 & 0.1 \end{array}$$

$$\text{Population mean, } \mu = \frac{3+5+7+9+11}{5} = 7.$$

$$\text{Population variance, } \sigma^2 = \frac{(-4)^2+(-2)^2+0+(2)^2+(4)^2}{5} = 8.$$

$$\text{Mean of sample means} = 0.4 + 0.5 + 1.2 + 1.4 + 1.6 + 0.9 + 1.0 = 7.0$$

$$\begin{aligned}\text{Variance of sample means} &= (-3)^2(0.1) + (-2)^2(0.1) + (-1)^2(0.2) + 0 + (1)^2(0.2) + (2)^2(0.1) + (3)^2(0.1) \\ &= 0.9 + 0.4 + 0.2 + 0.2 + 0.4 + 0.9 = 3.0\end{aligned}$$

We observe that mean of the sample means is the same as the population mean but variance of the sample means is not the same as the population variance.

The standard deviation of sampling distribution of a statistic is called its *standard error* (S.E.). In this case S.E. is $\sqrt{3}$. Normally, we use statistical theory to derive sampling distribution of a statistic or use simulation to derive the sampling distribution empirically.

An important result which describes the sampling distribution of a statistic which are sum or averages of the sample observations is the *Central limit theorem* stated as below:

Theorem 25.1 **(Central limit theorem):** *If random samples of n observations are drawn from a non-normal population with finite mean μ and standard deviation σ, then when n is large, the sampling distribution of the sample mean $\bar{x}$ is approximately normally distributed with mean μ and S.E. $\sigma/\sqrt{n}$.*

This approximation becomes more accurate as n becomes large.

The central limit theorem has important contribution in statistical inference since many estimators that are used to make inference about population parameters are sum or averages of the sample observations, and when the sample size n is large then these estimators can be approximated as normal variates.

In case the population itself is normal, then sampling distribution of $\bar{x}$ is always normal, irrespective of the size n of the sample selected. But when the population is skewed then the sample size n must be large, say $n > 30$ to approximate the distribution of $\bar{x}$ as normal.

25.2.1 The Sampling Distribution of the Sample Mean

If the population mean μ is unknown, then the statistic sample mean $\bar{x}$, in general, is chosen as the natural estimate of the population mean. The following theorem gives sampling distribution of the sample mean $\bar{x}$.

Theorem 25.2 (Sampling distribution of the sample mean): *If a random sample of size n is selected from a population with mean μ and S.D. σ, then the sampling distribution of the sample mean $\bar{x}$ will have mean μ and standard error (S.E.) $\sigma/\sqrt{n}$.*

Proof: Let $x_1, x_2, ..., x_n$ be a random sample of size n drawn from a population of size N with mean μ and variance σ^2. Then the sample mean, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and the sample variance, $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

Consider $$E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu \qquad ...(25.1)$$

$$\text{var}(\bar{x}) = \text{var}\left(\frac{x_1 + x_2 + ... + x_n}{n}\right) = \frac{1}{n^2}\,[\text{var}(x_1) + \text{var}(x_2) + ... + \text{var}(x_n)],$$

since x_i's are independent, thus the co-variances terms are absent. Hence

$$\text{var}(\bar{x}) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n} \qquad ...(25.2)$$

and, $$\text{S.E.}(\bar{x}) = \sigma/\sqrt{n}. \qquad ...(25.3)$$

Hence $\bar{x}$ is distributed with mean μ and S.E. $\sigma/\sqrt{n}$.

In case the sampled population is normal, the distribution of $\bar{x}$ will be exactly normal irrespective of the size n, however, if the population is non-normal, then the distribution of $\bar{x}$ will be approximately normal for large n by central limit theorem. Thus the statistic z is

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1). \qquad ...(25.4)$$

Remark: A statistic θ is called an *unbiased estimator* of a polulation parameter γ, if $E(\theta) = \gamma$. Since, $E(\bar{x}) = \mu$, thus *sample mean $\bar{x}$ is an unbiased estimate of the population mean* μ. But sample variance, $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is not an unbiased estimate of the population variance as shown next.

We have, $$E(s^2) = E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right) = E\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-(\bar{x})^2\right] = \frac{1}{n}\sum_{i=1}^{n}E(x_i^2)-E(\bar{x}^2). \qquad ...(25.5)$$

Now $E(x_i^2) = \text{var}(x_i)+[E(x_i)]^2 = \sigma^2+\mu^2,$

and, $E(\bar{x}^2) = \text{var}(\bar{x})+[E(\bar{x})]^2 = \frac{\sigma^2}{n}+\mu^2,$ using (25.1) and (25.2).

Using these in (25.5), we obtain

$$E(s^2) = \frac{1}{n}\sum_{i=1}^{n}(\sigma^2+\mu^2)-\left(\frac{\sigma^2}{n}+\mu^2\right)=\left(1-\frac{1}{n}\right)\sigma^2 \neq \sigma^2.$$

However, if we define $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$, then

$$E(S^2) = E\left(\frac{n}{n-1}s^2\right) = \frac{n}{n-1}E(s^2) = \frac{n}{n-1}\cdot\frac{n-1}{n}\sigma^2 = \sigma^2. \qquad ...(25.6)$$

Hence, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$ *is an unbiased estimate of the population variance* σ^2.

25.2.2 The Sampling Distribution of the Sample Proportion

In some practical situations we need to estimate the proportion p of people in the population who have a specified characteristic say smoking, computer literacy, etc. If x out of the n sampled people have this characteristic, then the sample proportion $\hat{p} = x/n$ can be taken as an estimate of the population proportion p. We observe that the distribution of the random variable x is binomial with mean np and S.D. $\sqrt{npq}$, and thus $\hat{p} = \frac{x}{n}$ will also be distributed like a binomial variate with mean

$$E(\hat{p}) = E\left(\frac{x}{n}\right) = \frac{1}{n}E(x) = p, \qquad ...(25.7)$$

variance, $$\text{var}(\hat{p}) = \text{var}\left(\frac{x}{n}\right) = \frac{1}{n^2}\text{var}(x) = \frac{pq}{n} \qquad ...(25.8)$$

and, standard error as $\text{S.E.}(\hat{p}) = \sqrt{\frac{pq}{n}}$, ...(25.9)

where $q = 1-p$.

Further, since the binomial distribution can be approximated to normal for large n, thus the statistic z given by

$$z = \frac{\hat{p} - p}{\sqrt{pq/n}} \qquad ...(25.10)$$

will be a standard normal variate for large n, that is, $z \sim N(0, 1)$.

Example 25.1: An electrical firm manufactures light bulbs that have burning life normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a random sample of 16 bulbs will have an average burning life of less than 775 hours.

Solution: Let $\bar{x}$ denote the average burning life of 16 bulbs, then $\bar{x}$ is a normal variate with mean 800 hrs and S.E. $= 40/\sqrt{16} = 10$. Thus $z = \dfrac{\bar{x} - 800}{10} \sim N(0, 1)$.

Hence, $P(\bar{x} < 775) = P(z < -2.5) = P(z > 2.5) = 0.5 - P(0 < z < 2.5) = 0.5 - 0.4938 = 0.0062$.

Example 25.2: The duration of Alzheimer's disease from the appearance of symptoms until death, is distributed with an average 9 years and a S.D. of 4 years. The medical records of 36 randomly selected deceased patients from a large medical database has been taken. Find the approximate probability that average duration lies within 7 and 11 years.

Solution: Let $\bar{x}$ be the average duration of survival after the appearance of symptoms. Since the sample size n is 36, $\bar{x}$ can be approximated as a normal variate with mean $\mu = 9$ and standard error $= \sigma/\sqrt{n} = 4/\sqrt{36} = 2/3$. Thus $z = \dfrac{\bar{x} - 9}{2/3} \sim N(0, 1)$.

Hence, $\quad P(7 < \bar{x} < 11) = P(-3 < z < 3) = 2P(0 < z < 3) = 2(0.4987) = 0.9974$

Example 25.3: A random sample of 100 students was taken from a campus and 12 were found to be smokers. Estimate the proportion of smokers in the campus as well as the S.E. of the estimate. Find the almost certain limits to the percentage of smokers in the campus.

Solution: The proportion of smokers in the sample is $\hat{p} = \dfrac{12}{100} = 0.12, \quad \hat{q} = 0.88$.

Hence the S.E. $(\hat{p}) = \sqrt{\dfrac{(0.12)\,(0.88)}{100}} = 0.0325$, using (25.9).

Hence, the proportion of smokers lies certainly between

$$\hat{p} \pm 3(\text{S. E.}) = 0.12 \pm 3(0.0325) = 0.12 \pm 0.0975,$$

that is between 0.0225 and 0.2175. Therefore the percentage of smokers almost certainly lies between 2.25 and 21.75.

EXERCISE 25.1

1. A soft drink machine is being regulated so that the amount of drink dispensed averages 240 ml. with a S.D. 15 ml. The machine is checked periodically by taking a sample of 40

drinks and if the mean amount $\bar{x}$ for the sample taken lies within $E(\bar{x}) \pm 2$ S.E. $(\bar{x})$, the machine is certified O.K., otherwise is rectified. An apprentice from the company found the mean of 40 drinks to be 236 ml and certified O.K. Was that a reasonable decision? Justify your answer.

2. A random sample of 500 fuses was taken from a large consignment and 65 were found to be defective. Show that the percentage of defectives in the consignment almost certainly lies between 8.5 and 17.5.
3. A coin is tossed 1000 times and the head comes out 550 times. Can the deviation from expected value be due to fluctuations of sampling?
4. A large batch of electric bulbs have a mean time to failure of 800 hours and the S.D. of 60 hours. For a random sample of 64 electric bulbs determine the probability that mean time to failure will be
   (a) less than 785 hours, (b) more than 820 hours.
5. The contents of a consignment of 1200 tins of a product have a mean mass of 5040 gm with a S.D. of 2.3 gm. Find the probability that a random sample of 40 tins drawn from the consignment will have a combined mass of
   (a) less than 20.13 kg, (b) between 20.13 kg and 20.17 kg, and (c) more than 20.17 kg.
6. It has been observed that almost 75% of customers visiting a fabric mall prefer natural fabrics in comparison to man-made fabrics. A random sample of 200 customers has been selected and the number who like natural fabrics is recorded.
   (a) What is the approximate sampling distribution for the sample proportion $\hat{p}$?
   (b) What is the probability that the sample proportion is greater than 80%?
   (c) Within what limits the sample proportion are expected to lie about 95% of the time?

25.3 NULL AND ALTERNATIVE HYPOTHESIS. TYPES OF ERRORS AND LEVEL OF SIGNIFICANCE

An important aspect of sampling theory is to make decision about the parameter value. The tests of hypothesis enable us to decide on the basis of the statistic obtained, that whether the deviation between the observed and the theoretical value is significant or might be attributed to fluctuations of sampling. Since for large n the sampling distribution of the statistic under study can be approximated to normal, so for large sample testing normal distribution is applied. However, in case of small sample testing we employ specific variates like t, χ^2, F, etc.

25.3.1 Null and Alternative Hypotheses

A statistical hypothesis is an assertion concerning one or more populations. A hypothesis we wish to test, is called the 'null hypothesis' and is denoted by H_0; and any hypothesis, complementary to the null hypothesis, is called an 'alternative hypothesis' and is usually denoted by H_1. The null hypothesis H_0 is usually a hypothesis of 'no-difference' and is tested for possible rejection under the assumption that it is true.

For example, if we want to test that average daily wages of workers in a construction company is different from Rs. 170, the national average, then we can set up the null hypothesis as

$$H_0 : \mu = 170$$

and, the alternative hypothesis could be any of

(a) $H_1: \mu \neq 170$ (b) $H_1: \mu < 170$ (c) $H_1: \mu > 170$.

The alternative hypothesis, (a) is known as *two-tailed alternative*; (b) is known as *left-tailed alternative*; and (c) as *right-tailed alternative*.

The hypothesis (a) is *composite alternative*, while hypotheses (b) and (c) are *simple alternatives*.

25.3.2 Acceptance and Rejection Regions

The decision to reject or accept the null hypothesis is based on the information contained in a sample drawn from the population under study. On the basis of the data in the sample a *test statistic* is formulated and using this test statistic a probability value is calculated (generally, obtained from the tables available). On the basis of these measures obtained the hypothesis H_0 is rejected or accepted. Now the important question arises: *How to decide whether to reject or accept H_0?* This is answered as follows.

Since, our decision is based on the value of the test statistic obtained, thus entire set of values that the test statistic may attain is divided into two regions, acceptance region and rejection region.

The region consisting of the values which support the null hypothesis, leading to acceptance of H_0, is called the *acceptance region*, and the region consisting of values which support the alternative hypothesis, leading to rejection of H_0, is called the *rejection region* or the *critical region*. The value (*s*) that separate the acceptance and rejection region is (are) called the *critical value* (*s*).

For example, in case we wish to test the null hypothesis $H_0: \mu = 170$ against the alternative $H_1: \mu \neq 170$, then the acceptance and rejection regions are as shown in Fig. 25.1a.

In case the alternative is $H_1: \mu < 170$, or $H_1: \mu > 170$, then acceptance and rejection regions are as shown in Figs. 25.1b and 25.1c, respectively.

The type of test in case of (a) is called *two-tailed test* and, in case of (b) and (c) is called *left-tailed test* and *right-tailed test*, (jointly, as *single-tailed tests*), respectively.

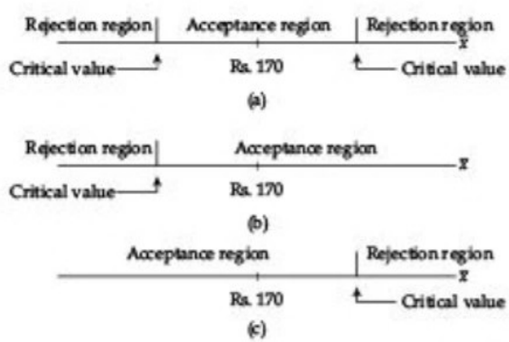


Fig. 25.1

25.3.3 Types of Errors and Level of Significance

The decision procedure described above can lead to either of the following two types of errors:

Type I error : Rejection of the null hypothesis H_0 when it is true (*Rejection error*).

Type II error : Acceptance of the null hypothesis H_0 when it is false (*Acceptance error*).

The probability of Type I error, that is,

$P\{\text{Rejection of } H_0 \text{ when it is true}\} = P\{\text{Reject } H_0 \mid H_0\}$,

is called the *level of significance*, or *size of the test* and is denoted by α.

The probability of Type II error, that is,

$P\{\text{Acceptance of } H_0 \text{ when it is false}\} = P\{\text{Accept } H_0 \mid H_1\}$

is denoted by β.

Remarks:

1. In statistical quality control Type I error amounts to rejecting a lot when it is good and Type II error may be regarded as accepting the lot when it is bad, thus α and β are often referred to as *producer's risk* and *consumer's risk* respectively.
2. For a fixed sample size, a decrease in the probability of one type error will usually result in an increase in the probability of the other type of error. Both types of errors can be reduced only by increasing the sample size n.
3. For applying the test of significance, the level of significance, that is, the size of Type I error is kept fixed normally at 5% or 1% and the sampling is so designed that for given α, the size of the Type II error β is minimum.
4. The factor $1 - \beta$, the probability of rejecting H_0 when a specific alternative H_1 is true is called the *power of a test*.

25.4 LARGE SAMPLES TESTING

In case of large sampling the test statistic z, say $z = \dfrac{\bar{x} - E(\bar{x})}{S.E.(\bar{x})}$, is approximated to $N(0, 1)$. If z_α is the critical value of the test statistic at level of significance α, then for a *two-tailed test* it is given by $P(|z| > z_\alpha) = \alpha$, that is, z_α is the value so that the total area of the critical region on both tails is α and since the standard normal probability curve is symmetrical about its mean $z = 0$, thus

$$P(z > z_\alpha) = P(z < -z_\alpha) = \alpha/2, \text{ as shown in Fig. 25.2a.}$$

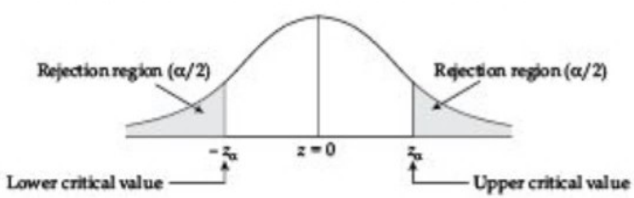


Fig. 25.2a

In case of left-tailed test or right-tailed test the total area to the left of $-z_\alpha$, or to the right of z_α, as the case is, is α. That is, for the left-tailed test $P(z < -z_\alpha) = \alpha$ and for the right-tailed test $P(z > z_\alpha) = \alpha$, as shown in Figs. 25.2b and 25.2c, respectively.

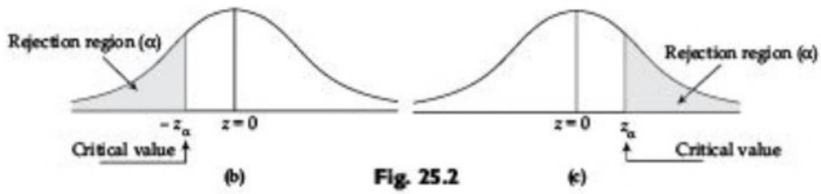


Fig. 25.2

Remarks:

1. In case of large sample testing, the critical value z_α of z for two-tailed test at level of significance α is numerically same as the critical value for a single-tailed test at level of significance $\alpha/2$, but this does not hold good when the sample size is small.
2. When test statistic is approximated to standard normal distribution, for two-tailed test critical values at 1% and 5% level of significance are 2.58 and 1.96 respectively. For left-tailed tests, these values are -2.33 and -1.645 respectively and, for right-tailed tests, the values are 2.33 and 1.645 respectively, see Table I.
3. In case the sample size n is small, say <30, and sampled population is not normal then distribution of the test-statistic cannot be approximated to normal and thus these critical values don't hold good. In such cases the values based on the exact sampling distribution of the test statistic are used e.g., t, χ^2, F, etc.

Procedure for testing: We set up the null hypothesis H_0 and compute the test statistic z under the assumption that H_0 is true. If $|z| > 3$, H_0 is rejected outright. In case $|z| \le 3$, we test its significance at a specified level usually at 5% or 1% level of significance.

For a two-tailed test, if $|z| > 1.96$, H_0 is rejected at 5% level of significance and if $|z| > 2.58$ also, it is rejected even at 1% level of significance also. In case $|z| < 2.58$, H_0 may be accepted at 1% level of significance, and if $|z| < 1.96$, then H_0 may be accepted at 5% level of significance also.

For a single-tailed test, the test statistic $|z|$ is compared with 1.645 at 5% level and with 2.33 at 1% level and H_0 is accepted and rejected accordingly.

25.5 SIMPLE SAMPLING OF ATTRIBUTES. TESTS FOR SINGLE PROPORTION AND DIFFERENCE BETWEEN TWO PROPORTIONS

The theory of sampling can be studied under two heads: *sampling of attributes* and *sampling of variables*. In this section we discuss the sampling of attributes.

In the sampling of attributes we are concerned only with the possession or non-possession of some specified attribute or, characteristic say, smoking, swimming, inoculated against a disease, etc. by the individuals selected in the sample. The possession of the specified attribute by the individual selected in the sample is termed as '*success*' while the non-possession as *failure*.

In this case simple sampling of n observations may be identified with that of a series of n independent Bernoulli trials with constant probability p of success for each trial and so the mean number of successes is np and the S.E. of the number of successes is $\sqrt{npq}$, where $q = (1 - p)$.

In case we consider the proportion of successes $\hat{p} = \frac{x}{n}$, then mean and S. E. of proportion of successes are given by respectively

$$\text{Mean } (\hat{p}) = E\left(\frac{x}{n}\right) = \frac{1}{n}E(x) = \frac{np}{n} = p$$

and,
$$\text{S. E. } (\hat{p}) = \sqrt{\text{var}\left(\frac{x}{n}\right)} = \frac{1}{n}\text{S. E. }(x) = \frac{\sqrt{npq}}{n} = \sqrt{\frac{pq}{n}}$$

The factor $\sqrt{n}/\sqrt{pq}$ is called the *precision of the proportion of successes* and varies as $\sqrt{n}$, since $\sqrt{pq}$ is constant for the specific population under study.

25.5.1 Test for Single Proportion

If x is the number of successes in n independent trials and suppose we wish to test the hypothesis H_0 that proportion of success in each trial is p, then under H_0 $E(x) = np$ and S.E. $(x) = \sqrt{npq}$, and for large n, the test statistic z given by

$$|z| = \frac{|x - np|}{\sqrt{npq}} \qquad \text{...(25.11)}$$

is a standard normal variate, that is, $z - N(0, 1)$ and so we can apply the normal test.

Similarly, for large n the test statistic z for proportion of successes $\hat{p}$ given by

$$z = \frac{\hat{p} - p}{\sqrt{pq/n}} \qquad \text{...(25.12)}$$

is a standard normal variate, that is, $z \sim N(0, 1)$.

Further, the limits

$$x = np \pm 1.96\sqrt{npq} \qquad \text{...(25.13)}$$

are called *the 95% confidence limits*, and the limits

$$x = np \pm 2.58\sqrt{npq} \qquad \text{...(25.14)}$$

are called *the 99% confidence limits* for the number of successes x.

Similarly we can write the 95% and 99% confidence limits for the proportion of successes.

Example 25.4: A dice is thrown 9000 times and a throw of 3 or 4 is observed 3240 times. Can the dice be regarded as unbiased? Also find the limits between which the probability of a throw of 3 or 4 is most likely to lie.

Solution: Let the null hypothesis H_0 be that dice is unbiased. Under H_0, if p is the probability of getting a throw of 3 or 4, then we test H_0: $p = 1/3$, against the alternative $H_1: p \neq 1/3$.

Here $n = 9000$, $x = 3240$, $np = 3000$, $q = 2/3$. Thus $\sqrt{npq} = \sqrt{9000 \times 1/3 \times 2/3} = 44.72$.

Under H_0 the test statistic z, given by $|z| = \dfrac{|x - np|}{\sqrt{npq}} = \dfrac{240}{44.72} = 5.37 > 3$ is highly significant and hence the hypothesis H_0 is rejected and we regard that dice is almost certainly biased.

Since, the dice is not unbiased the most likely limits in which the probability of a throw of 3 or 4 lie are given by $\hat{p} \pm 3\sqrt{\dfrac{\hat{p}\hat{q}}{n}}$, where $\hat{p} = \dfrac{x}{n} = \dfrac{3240}{9000} = 0.36$, and $\hat{q} = 0.64$.

Hence the limits are, $0.36 \pm 3\sqrt{\dfrac{(0.36)(0.64)}{9000}} = 0.345$ and 0.375.

Example 25.5: During testing in a sample of 300 chips, 10 have been found to be defective. Can the manufacturer's claim that 2% of the chips are defective may be accepted?

Solution: Let the null hypothesis H_0 be that 2% of the chips are defective, thus we test

$$H_0: p = .02 \text{ against } H_1: p \neq .02.$$

We have, $n = 300$, $np = 6$, $\sqrt{npq} = \sqrt{300(.02)(.98)} = 2.42$

Under H_0, the test statistic z, given by $|z| = \dfrac{|x - np|}{\sqrt{npq}} = \dfrac{|10 - 6|}{2.42} = 1.65 < 1.96$ is not significant and hence H_0 is accepted at 5% level of significance; manufacturer's claim may be accepted.

Example 25.6: Long-term database indicates that 5% of the components produced at a certain manufacturing facility are defective. A training program for the workforce employed has been conducted with the aim to reduce the percentage of defective produced. After this if a random sample of 500 items consists of 16 defectives, can we conclude that training was effective?

Solution: Let the null hypothesis H_0 be that the training was not effective in reducing the proportion p of defectives. Thus we test $H_0: p = 0.05$ against the left-tailed alternative $H_1: p < 0.05$.

We have, $n = 500$, $np = 25$, $\sqrt{npq} = \sqrt{500(0.5)(0.95)} = 4.87$.

Under H_0, the test statistic z, given by $|z| = \dfrac{|x - np|}{\sqrt{npq}} = \dfrac{|16 - 25|}{4.87} = 1.848 > 1.645$ is significant and hence, using the left-tailed test, hypothesis is rejected at 5% level of significance. However, since $|z| = 1.848 < 2.33$, by left-tailed test the hypothesis may be accepted at 1% level of significance.

Example 25.7: Out of the twenty persons who were reported to be attacked by brain fever only eighteen survived. Using the large sample test, test the hypothesis at 5% level that if, attacked by brain fever survival rate is 85% against the alternative that it is more.

Solution: Let the null hypothesis be that survival rate is 85%, that is, p = 0.85. Thus we test $H_0: p = 0.85$, against right-tail alternative $H_1: p > 0.85$.

We have, $n = 20$, $x = 18$, $\hat{p} = x/n = 0.9$ and $\sqrt{pq/n} = \sqrt{(0.85)(0.15)/20} = 0.0798$

Under H_0, the test statistic is

$$|z| = \frac{|\hat{p} - p|}{\sqrt{pq/n}} = \frac{|0.90 - .85|}{.0798} = 0.627,$$

which is less than 1.645. Using the right-tailed test it is not significant at 5% level and hence the hypothesis may be accepted at 5% level of significance.

Example 25.8: In a random sample of 525 families owning television set in the region of New Delhi it is found that 370 subscribe to Star Plus. Find a 95% confidence interval for the actual proportion of such families in New Delhi which subscribe to Star Plus.

Solution: We have, $\hat{p} = x/n = 370/525 = 0.705$.

Therefore, the 95% confidence limits for the actual proportion p are

$$\hat{p} \pm 1.96\sqrt{\frac{\hat{p}\hat{q}}{n}} = 0.705 \pm 1.96\sqrt{\frac{(0.705)(0.295)}{525}}, \text{ that is, between 0.666 to 0.744.}$$

Hence, 95% confidence interval for the actual proportion p is $0.666 < p < 0.744$.

25.5.2 Test for Difference Between Two Proportions

Suppose we want to compare two distinct populations with respect to the prevalence of a specific attribute. For example, we may be interested in comparing the prevalence of lung cancer among smokers (population I) and non-smokers (population II). Let x_1, x_2 be the number of persons with this attribute in random samples of size n_1 and n_2 selected from population I and population II, respectively. Then sample proportions are $\hat{p}_1 = x_1/n_1$ and $\hat{p}_2 = x_2/n_2$.

If p_1 and p_2 are proportion for the two populations, then

$$E(\hat{p}_1) = p_1, \quad E(\hat{p}_2) = p_2, \quad \text{and} \quad \text{var}(\hat{p}_1) = \frac{p_1q_1}{n_1}, \quad \text{var}(\hat{p}_2) = \frac{p_2q_2}{n_2}.$$

Since for large samples, $\hat{p}_1$ and $\hat{p}_2$ are each approximately normally distributed with means p_1 and p_2 and variances p_1q_1/n_1 and p_2q_2/n_2 respectively also the samples being independent, thus $\hat{p}_1 - \hat{p}_2$ is also normally distributed with mean

$$E(\hat{p}_1 - \hat{p}_2) = p_1 - p_2, \text{ and, } \text{var}(\hat{p}_1 - \hat{p}_2) = \text{var}(\hat{p}_1) + \text{var}(\hat{p}_2) = \frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2}.$$

Thus,

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2}}} \sim N(0, 1) \quad \text{...(25.15)}$$

Let H_0: There is no difference between the population proportions, that is, $p_1 = p_2 = p$, say.

Under H_0, $E(\hat{p}_1 - \hat{p}_2) = p_1 - p_2 = 0$, and, var $(\hat{p}_1 - \hat{p}_2) = \frac{pq}{n_1} + \frac{pq}{n_1} = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$.

Hence under H_0, the test statistic (25.15) becomes

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}. \quad ...(25.16)$$

Normally p is unknown so we use $\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2}$, an unbiased estimate of p, in place of p. Thus, in this case the required test statistic z under H_0 is

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1). \quad ...(25.17)$$

Example 25.9: Suppose that a method A results in 20 unacceptable transistors out of 100 produced, whereas another method B results in 12 unacceptable transistors out of 100 produced. Can we conclude at 5% level that the two methods are equivalents.?

Solution: Let p_1 and p_2 be the true proportions of unacceptable and let the null hypothesis be that method I and method II are equivalent, that is, we test H_0: $p_1 = p_2$ against the alternative H_1: $p_1 \neq p_2$

We have, $\hat{p}_1 = \frac{20}{100} = 0.20$, $\hat{p}_2 = \frac{12}{100} = 0.12$, $\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2} = \frac{20 + 12}{100 + 100} = 0.16$.

Under H_0 the test statistic z given by

$$|z| = \frac{|\hat{p}_1 - \hat{p}_2|}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{|0.20 - 0.12|}{\sqrt{(0.16)(0.84)\left(\frac{1}{100} + \frac{1}{100}\right)}} = \frac{0.08}{0.052} = 1.538 < 1.96.$$

Thus it is not significant at 5% level and hence the null hypothesis may be accepted.

Example 25.10: An alternate manufacturing mechanism is being tested. Samples are taken using both the existing and the alternate mechanism so as to determine if the alternate mechanism results in an improvement. If 50 of 1000 items from the existing mechanism and 60 of 1500 items from the alternate mechanism were found to be defective, find a 90% confidence interval for the true difference of defectives between the two mechanisms. Can you conclude that alternate mechanism decreases the proportion of defectives significantly?

Solution: Let p_1 and p_2 be the true proportions of defectives in the existing and alternate mechanism respectively.

We have, $\hat{p}_1 = 50/1000 = 0.05$ and $\hat{p}_2 = 60/1500 = 0.04$ and thus

$$\hat{p}_1 - \hat{p}_2 = 0.05 - 0.04 = 0.01.$$

Also, $$|z| = \frac{|(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)|}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} \sim N(0, 1), \text{ refer to (25.15).}$$

From Table *I*, $z_{0.05} = 1.645$, therefore 90% confidence limits for $p_1 - p_2$ are

$$(\hat{p}_1 - \hat{p}_2) \pm 1.645 \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}.$$

Since p_1, p_2 are unknown, using $\hat{p}_1$, and $\hat{p}_2$ unbiased estimators for p_1 and p_2, the limits obtained are

$$0.01 \pm 1.645 \sqrt{\frac{(0.05)(.95)}{1000} + \frac{(0.04)(.96)}{1500}}, \text{ or, } 0.01 \pm .0141, \text{ or } \; -0.041 \text{ to } .0241.$$

Hence, 90% confidence interval for true difference in the fraction of defectives between the two mechanisms is, $-0.041 < p_1 - p_2 < .0241$. Since the interval contains the value zero, so we cannot conclude that alternate mechanism decreases the proportion of defective being produced by the existing mechanism significantly.

Example 25.11: A tea company claims that its premium tea brand outsells its normal brand by 10%. If it is found that 46 out of a sample of 200 tea-users prefer premium brand and 19 out of another independent sample of 100 tea-users prefer normal brand, test the validity of the claim made by the company.

Solution: Let p_1 and p_2 be the true proportions of the premium and normal brands, and let we set up the null hypothesis such that company's claim is valid one, that is,

$H_0 : p_1 - p_2 = 0.1$ against the alternative $H_1 : p_1 - p_2 \neq 0.1$.

We have, $n_1 = 200, \quad x_1 = 46, \quad \hat{p}_1 = x_1/n_1 = 46/200 = 0.23$

$n_2 = 100, \quad x_2 = 19, \quad \hat{p}_2 = x_2/n_2 = 19/100 = 0.19.$

Under H_0, the test statistic is $z = \dfrac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1)$,

where $$\hat{p} = \frac{n_1 \hat{p}_1 + n_2 \hat{p}_2}{n_1 + n_2} = \frac{46 + 19}{200 + 100} = \frac{65}{300} = 0.217, \text{ and thus } \hat{q} = 0.783.$$

Thus, $$|z| = \frac{|(0.04) - (0.1)|}{\sqrt{(0.217)(0.783)\left(\frac{1}{200} + \frac{1}{100}\right)}} = \frac{0.06}{0.0505} = 1.18.$$

Since $|z| = 1.18 < 1.96$, it is not significant at 5% level of significance and hence null hypothesis may be accepted and thus company's claim may be considered to be valid one.

Example 25.12: The percentage of officials in two big PSU's with computer knowledge is 30 and 25, respectively. Is this difference likely to be hidden in samples of 1000 and 800 officials, respectively from the two PSU's?

Solution: Let p_1 and p_2 be the true proportions of the officials in the two PSU's and let the null hypothesis be that difference is likely to be hidden, that is,

$H_0 : \hat{p}_1 = \hat{p}_2$ against the alternative $H_1 : \hat{p}_1 \neq \hat{p}_2$

We have, $n_1 = 1000,\ n_2 = 800,\ p_1 = 0.30,\ p_2 = 0.25.$

Under H_0, the test statistic z given by

$$z = \frac{|p_1 - p_2|}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} = \frac{0.30 - 0.25}{\sqrt{\frac{(0.3)(0.7)}{1000} + \frac{(0.25)(0.75)}{800}}} = \frac{0.05}{0.021} = 2.37 > 1.96,$$

is significant at 5% level and hence the hypothesis is rejected and thus the difference is likely to be revealed in the samples drawn at 5% level of significance.

We observe that, since $|z| = 2.37 < 2.58$ is not significant at 1% level and so that hypothesis may be accepted at 1% and hence the samples are unlikely to reveal the difference at 1% level of significance.

25.6 SAMPLING OF VARIABLES. TESTS FOR SINGLE MEAN AND DIFFERENCE BETWEEN TWO MEANS

In this section, we discuss the sampling of the values of a variable such as height, weight, marks obtained in a test, weekly wages, etc. Each unit of the population provides a specific value (measurement) of the variable under study and the aggregate of these values forms the frequency distribution of the population. From this population, (that is, the aggregate of the values), a random sample of size n is selected to estimate and draw the conclusions about the population parameters, generally unknown.

25.6.1 Test for Single Mean

For large sample size n, sample mean is distributed normally with its mean as sampled population mean μ and S.E. as $\sigma/\sqrt{n}$, where σ is the S.D. of the sampled population. Thus under the null hypothesis H_0, that the sample has been drawn from a population with mean μ and S.D. σ, the test statistic z given by

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1), \qquad \text{...(25.18)}$$

is a standard normal variate with mean zero and S.E. one, when n is large.

If $|z| < 1.96$ the deviation is not significant at 5% and the hypothesis is accepted, otherwise, it is rejected.

The limits $\bar{x} \pm 1.96\left(\sigma/\sqrt{n}\right)$ are the *95% confidence limits for the population mean* μ; and

$$\bar{x} - 1.96\left(\sigma/\sqrt{n}\right) < \mu < \bar{x} + 1.96\left(\sigma/\sqrt{n}\right) \quad \text{...(25.19)}$$

is the *95% confidence interval.*

Similarly, $\bar{x} \pm 2.58\left(\sigma/\sqrt{n}\right)$ are the *99% confidence limits for the population mean* μ.

Remark: In case population S.D. σ is unknown we use s, the sample S.E. as its estimate, for $s^2 = \frac{n-1}{n}S^2 = \left(1 - \frac{1}{n}\right)S^2$ and thus for large n, $s^2 \to S^2$ and further $E(S^2) = \sigma^2$ justifies s^2 as an estimate of σ^2 in case latter is unknown.

Example 25.13: Sugar is packed in bags by an automatic machine with mean contents of bags as 1.000 kg. A random sample of 36 bags is selected and mean mass has been found to be 1.003 kg. If a S.D. of 0.01 kg, is acceptable on all the bags being packed, determine on the basis of sample test whether the machine requires adjustment.

Solution: Let the null hypothesis H_0 be that the machine does not require any adjustment, that is,

$$H_0: \mu = 1.000 \text{ kg. against } H_1: \mu \neq 1.000 \text{ kg.}$$

Under H_0, the statistic z given by $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{1.003 - 1.000}{.01/\sqrt{36}} = 1.8 < 1.96.$

Thus, it is not significant at 5% level and hence H_0 may be accepted, that is, machine does not require any adjustment.

Example 25.14: The daily collection of milk at a plant has averaged 850 kilolitres for the last several years. An observer wants to know whether the average has changed in recent months. He randomly selects 40 days from the database and finds the average collection as $\bar{x} = 840$ kilolitres with a S.D. $s = 18$ kilolitres. Test the appropriate hypothesis at $\alpha = 0.05$.

Solution: We test the null hypothesis $H_0: \mu = 850$, against $H_1: \mu \neq 850$.

Under H_0 the test statistic z, given by $|z| = \frac{|\bar{x} - \mu|}{s/\sqrt{n}} = \frac{|840 - 850|}{18/\sqrt{40}} = 3.51 > 1.96.$

Thus it is significant at 5% (even it is significant at 1% level) and hence hypothesis is rejected, that is, daily average collection of milk has changed.

Example 25.15: If e is the permissible error for estimating the population parameter μ, then prove that the minimum sample size n required for estimating μ with 95% confidence is given by $n = (1.96\sigma/e)^2$, where σ^2 is the population variance.

Solution: For large n, the test statistic z is given by $z = \dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}} - N(0, 1)$.

Thus, $P\left\{|x-\mu| \le 1.96\dfrac{\sigma}{\sqrt{n}}\right\} = 0.95$. We need n such that $P\{|x-\mu| < e\} > 0.95$.

Comparing these two, we obtain

min. $e = \dfrac{1.96\sigma}{\sqrt{n}}$, thus $\dfrac{1.96\sigma}{\sqrt{n}} \le e$, which gives, $n \ge \left(\dfrac{1.96\sigma}{e}\right)^2$. Hence, min. $n = \left(\dfrac{1.96\sigma}{e}\right)^2$.

Remark. For 99% confidence, min. $n = \left(\dfrac{2.58\sigma}{e}\right)^2$.

Example 25.16: The average zinc concentration recovered from a sample of zinc measurements in 40 different locations is found to be 2.54 gm per millitre. Find the 95% confidence intervals for the mean zinc concentration in the river assuming the population S.D. to be 0.32 gm. Find the minimum sample size required at 95% confidence if the permissible error is 0.05 gm.

Solution: We have, $\bar{x} = 2.54$, $n = 40$, $\sigma = 0.32$.

For large n the statistic z is given by $z = \dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}} - N(0, 1)$, where μ is the population mean.

Thus, 95% confidence interval for μ is

$$\bar{x} - (1.96)\sigma/\sqrt{n} < \mu < \bar{x} + 1.96\,\sigma/\sqrt{n}$$

or, $$2.54 - (1.96)\frac{0.32}{\sqrt{40}} < \mu < 2.54 + 1.96\frac{0.32}{\sqrt{40}},$$ which simplifies to $2.44 < \mu < 2.63$.

The minimum sample size n required at 95% confidence is

$$n = \left(\frac{1.96\sigma}{e}\right)^2 = \left(\frac{(1.96)(0.32)}{0.05}\right)^2 = 157.35 \simeq 158.$$

Example 25.17: The average monthly earnings for women in executive positions is Rs. 33,500. A random sample of $n = 40$ men in the executive positions showed average monthly earning $\bar{x} =$ Rs. 36,250, with S.E. $s =$ Rs. 5100. Do men in the same position have average monthly earnings higher than those for women?

Solution: We test H_0: $\mu = 33{,}500$ against the right-tailed alternate $H_1 : \mu > 33{,}500$.

Under H_0, the test statistic z, is given by $z = \dfrac{\bar{x}-\mu}{s/\sqrt{n}} - N(0, 1)$. We have, $z = \dfrac{36250 - 33500}{5100/\sqrt{40}} = 3.41$ > 1.645, the value of z from Table I at $\alpha = .05$, for right-tailed test.

Hence calculated z is significant and thus hypothesis is rejected. Thus men in the same positions have higher salary then their females counterparts.

25.6.2 Test for Difference Between Two Means

In many situations we are concerned with the comparison of two population means. For example, our problem of concern may be the comparison of the lead levels in drinking water in two different sections of a city.

Let $\bar{x}_1$ be the mean of a sample of size n_1 from a population with mean μ_1 and variance σ_1^2, and let $\bar{x}_2$ be the mean of an independent sample of size n_2 from a popution with mean μ_2 and variance σ_2^2. Since the two samples are independent for large values of n_1 and n_2, $\bar{x}_1 - \bar{x}_2$ can be approximated to a normal variate with mean and variance respectively as

$$E(\bar{x}_1 - \bar{x}_2) = E(\bar{x}_1) - E(\bar{x}_2) = \mu_1 - \mu_2; \text{ and, } \operatorname{var}(\bar{x}_1 - \bar{x}_2) = \operatorname{var}(\bar{x}_1) + \operatorname{var}(\bar{x}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}.$$

Thus the statistic z given by

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \qquad \text{...(25.20)}$$

is a standard normal variate with mean zero and S.E. one, that is, $z \sim N(0, 1)$.

Under the null hypothesis H_0: $\mu_1 = \mu_2$, that is, there is no significant difference between the two population means, the statistic (25.20) becomes

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1). \qquad \text{...(25.21)}$$

and hence can be tested accordingly.

Remarks.

1. In case σ_1^2 and σ_2^2 are not known, then $\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$ is used as an estimate of the S.E. of $(\bar{x}_1 - \bar{x}_2)$

for calculating the test statistic z; and so z becomes

$$z = \frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}. \qquad \text{...(25.22)}$$

2. If the samples have been drawn from the two populations with common variance σ^2, then under H_0 the test statistic (25.21) becomes

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}. \qquad \text{...(25.23)}$$

When σ is unknown, then $\sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}}$ is taken as an estimate of σ.

Example 25.18: Two random samples of 100 male students each of car-owners and non-owners of cars were drawn from a college. The grade point average for the non-owners of cars had an average equal to 2.82 with S.E. 0.63, while for the car-owners average equals to 2.43 with S.E. 0.65. Do the data present sufficient evidence to indicate a difference in the average achievements between car-owners and non-owners?

Solution: Let μ_1 and μ_2 be the true averages for non-owners of cars and car-owners respectively.

We set up the null hypothesis of no-difference, that is, H_0: $\mu_1 - \mu_2 = 0$ against H_1: $\mu_1 - \mu_2 \neq 0$.

Under H_0 the test statistic z given by

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{2.82 - 2.43}{\sqrt{\frac{(0.63)^2}{100} + \frac{(0.65)^2}{100}}} = \frac{0.39}{0.0905} = 4.30 > 3.$$

Thus z is highly significant and hence the hypothesis is rejected. Hence, there is difference in the average achievements between car-owners and non-owners students.

Examples 25.19: Two types of engines A and B were compared in respect of mileage in miles per litre of the petrol under identical conditions. The average of 50 trials in case of engine A was 34 miles and the average of 60 trials in case of engine B was 42 miles per litre. If μ_A and μ_B are population mean mileages for engines A and B respectively, find the 95% confidence interval for $\mu_B - \mu_A$ assuming that population S.D. for A and B engines are respectively 6 and 8 miles. What do you conclude about the difference in the population mean mileages from the confidence interval?

Solution: The statistic z given by, $z = \frac{(\bar{x}_B - \bar{x}_A) - (\mu_B - \mu_A)}{\sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}}} \sim N(0, 1)$ for large n_A and n_B.

Hence the 95% confidence interval for $\mu_B - \mu_A$ is

$$(\bar{x}_B - \bar{x}_A) - 1.96\sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}} < \mu_B - \mu_A < (\bar{x}_B - \bar{x}_A) + 1.96\sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}}.$$

We have, $n_A = 50$, $\bar{x}_A = 34$, $n_B = 60$, $\bar{x}_B = 42$, $\sigma_A = 6$ and $\sigma_B = 8$.

Using these values, 95% confidence interval becomes

$$8 - 1.96\sqrt{\frac{36}{50} + \frac{64}{60}} < \mu_B - \mu_A < 8 + 1.96\sqrt{\frac{36}{50} + \frac{64}{60}} \text{ or, } 5.38 < \mu_B - \mu_A < 10.62.$$

Since, the interval does not include zero, $\mu_B - \mu_A > 0$ throughout the interval so we can conclude that, at $\alpha = .05$ there is difference in the population mean mileage of the engines *A* and *B*.

Example 25.20: The mean heights in two large samples of 1000 and 2000 men are 67.5 inches and 68.0 inches respectively. Can the two samples be regarded as drawn from the same population of S.D. 2.5 inches?

Solution: We have, $n_1 = 1000$, $\bar{x}_1 = 67.5$, $n_2 = 2000$, $\bar{x}_2 = 68.0$. Let the null hypothesis be that the samples have been drawn from the same population with S.D. 2.5 inches, that is, we test $H_0 : \mu_1 = \mu_2$ and $\sigma = 2.5$ inches against $H_1 : \mu_1 \neq \mu_2$.

Under H_0, the test statistic z is given by $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim N(0, 1)$.

We have
$$z = \frac{67.5 - 68.0}{2.5\sqrt{\dfrac{1}{2000} + \dfrac{1}{1000}}} = \frac{-0.5}{.0968} = -5.165.$$

Since, $|z| = 5.165 > 3$, the value is highly significant and so the null hypothesis is rejected. The two samples cannot be regarded as drawn from the same population with S.D. 2.5 inches.

Example 25.21: A random sample of 500 coins has the mean weight 28.57 gm with S.D. 1.25 gm. Another random sample of 400 coins has the mean weight 29.62 gm with S.D. of 1.42 gm. Can the samples be considered to be drawn from the same population?

Solution: We have, $n_1 = 500$, $\bar{x}_1 = 28.57$, $s_1 = 1.25$, $n_2 = 400$, $\bar{x}_2 = 29.62$, $s_2 = 1.42$.

Let the null hypothesis be that the samples have been drawn from the same population with S.D. σ, that is, we test H_0: $\mu_1 = \mu_2$ with S.D. σ against the alternative $H_1 : \mu_1 \neq \mu_2$.

Under H_0, the test statistic z is given by, $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim N(0, 1)$.

Since σ^2 is not given we use $\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2} = \dfrac{500(1.25)^2 + 400(1.42)^2}{500 + 400} = 1.764$, as an estimate to σ^2.

Thus, we have
$$z = \frac{28.57 - 29.62}{1.328\sqrt{\dfrac{1}{500} + \dfrac{1}{400}}} = \frac{-1.05}{0.0891} = -11.78.$$

Since, $|z| = 11.78 > 3$, the value is highly significant, and hence the null hypothesis is rejected. Samples cannot be assumed to be drawn from the population.

EXERCISE 25.2

1. In a survey of 100 adults over 40 years old, a total of 15 people were found to be participating in a fitness activity at least twice a week. Test the hypothesis at $\alpha = .05$ that the participation rate for adult over 40 years of age is not less than the 20% figure.
2. A sleep inducing tablet when administered to 50 insomniacs was found to be effective on 37 patients. Test the hypothesis at $\alpha = 0.05$ that tablet was effective in at least 80% cases.
3. A random sample of 500 fuses was taken from a large consignment and out of these 60 were found to be defective. Obtain the 98% confidence limits for the percentage of defective fuses in the consignment.
4. In a locality containing 18000 families, a sample of 840 families was selected at random. Of these 840 families, 206 families were found to have a daily earning of Rs. 250 or less. Estimate the almost certain limits within which such families are likely to lie.
5. In a city A, 20% of a random sample of 900 Sr. Sec. school boys had computer knowledge, while in another city B, 18.5% of a random sample of 1600 Sr. Sec. school boys had computer knowledge. Test the hypothesis that there is no difference in proportions of boys with computer knowledge among Sr. Sec. students in the two cities.
6. In two large cities out of the houses with cable connections 30% and 25% respectively have Discovery channel. Is this difference likely to be hidden in samples of 1200 and 900 such houses from the two cities?
7. A study shows that 16 of 200 tractors produced on one assembly line required extensive adjustment before they could be shipped, while the same was true for 14 of 400 tractors produced on another assembly line. At the 0.01 level of significance does this support the claim that the second production line does superior work?
8. An airline claims that only 6% of all lost luggage is never found. If in a random sample, 17 of 200 pieces of lost luggage are not found, test the null hypothesis $p = 0.06$ against the alternative hypothesis $p > 0.06$ at the 0.05 level of significance.
9. To compare two different types of paints, eighteen specimens are painted using type A and the drying time in hours is recorded on each. Then the same is done with type B. If $\bar{x}_A$ and $\bar{x}_B$ are the mean drying times in hours for types A and B respectively, find $p[\bar{x}_A - \bar{x}_B > 1.0]$ under the assumption that population mean drying times for two types A and B are same with S.D. of 1.0.
10. An insurance agent claims that the average age of policy-holders who insure through him is less than the average for all agents which is 30.5 years. A random sample of 100 policy-holders insured through him gave the following age distribution.

Age as on last birthday :	16-20	21-25	26-30	31-35	36-40
No. of persons :	12	22	20	30	16

Test his claim at the 5% level on the basis of the data obtained.

11. A survey is proposed to be conducted to estimate the monthly income of the alumni of a technical institution. How large should the sample be taken in order to estimate the annual earning within plus and minus Rs. 10,000 at 95% confidence level, assuming the S.D. of the annual earnings of the entire alumni is known to be Rs. 30,000?

12. A taxi company is to decide whether to purchase brand *A* or brand *B* tires for its fleet of taxis. To estimate the difference in the two brands, an experiment is conducted using 30 tires of each brand. The tires are run until they wear out. The results are

$$\bar{x}_A = 36{,}300 \text{ kilometre} \qquad \bar{x}_B = 38{,}100 \text{ kilometre}$$
$$s_A = 5{,}000 \text{ kilometre} \qquad s_B = 6{,}100 \text{ kilometre}$$

Compute a 95% confidence interval for $\mu_B - \mu_A$ assuming the populations to be normal. What do you conclude from confidence interval obtained?

13. A random sample of 100 pieces was immersed in a bath for 24 hrs yielding an average of 12.2 millimetres of metal removed and a sample S.D. of 1.1 millimetre. A second sample of 200 pieces was exposed to some treatment, followed by the 24 hours immersion in the bath, resulting in an average removal of 9.1 millimetre with a sample S.D. of 0.9 millimetre. Compute a 98% confidence interval for the difference between the population means. Does the treatment appear to reduce the mean amount of metal removed?
14. The mean breaking strength of cables supplied by a manufactures is 1800 with a S.D. 100. By a new technique in the manufacturing process it is claimed that the breaking strength has increased. In order to test this claim a sample of 50 cables is tested. It is found that the mean breaking strength is 1850. Can we support the claim at $\alpha = 0.01$?
15. The electric light tubes of type *A* have a lifetime of 1400 hrs with a S.D. of 200 hrs, while of type *B* have mean lifetime of 1200 hrs with a S.D. of 100 hrs. If random samples of 125 tubes of each batch are tested, what is the probability that the type *A* tubes will have a mean time which is at least, (a) 160 hrs. more than the type *B* tubes, and (b) 250 hrs. more than the type *B* tubes?

25.7 SMALL SAMPLES TESTING. STUDENT'S t-VARIATE AND ITS APPLICATIONS

So far we have discussed the large samples testing. All these tests were based on central limit theorem to justify the normality of the test statistic derived. In case we are unable to collect a large sample, the test procedures described so far are of no use. Here we study equivalent procedures which can be employed when the sample size is small. However, we shall assume that the population (*s*) from which the samples are drawn is normal.

25.7.1 Student's t-Variate

When the sampled population is normal, the statistic $z = (\bar{x} - \mu)/(\sigma/\sqrt{n})$ has normal distribution for any sample size *n*, small or large. In case the population S.D. σ is unknown and the sample size n is small, the statistic $(\bar{x} - \mu)/(S/\sqrt{n})$, where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is no longer distributed as a normal variate and we define this statistic as t, that is,

$$t = \frac{(\bar{x} - \mu)}{S/\sqrt{n}}. \qquad \text{...(25.24)}$$

The statistic t defined as in (25.24) and its distribution, was mathematically derived by W.S. Gosset in 1908. He published his work under the pen name 'Student' and hence the distribution is known as Student's t, distribution. The probability density function of t-variate (25.24) is given by

$$f(t) = \frac{1}{\sqrt{\nu}\, B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \frac{1}{\left(1+\frac{t^2}{\nu}\right)^{(\nu+1)/2}}, \quad -\infty < t < \infty, \qquad ...(25.25)$$

where $\nu = n-1$.

The probability curves for some specific values of $\nu = n-1 = 2, 5$ are shown in Fig. 25.3

The chief characteristics of the t-statistic are:

1. It is mound shape and symmetric about $t = 0$, but is more flat on the top than the normal curve.

2. From Fig. 25.3 we observe that the t curve does not approach the horizontal axis as fast as standard normal curve approaches, in fact for small n, $p(|t| \geq t_0) \geq p(|z| \geq t_0)$; $z - N(0, 1)$.

3. The shape of the t curve depends on the sample size n and when $n \to \infty$ the distribution of t tends to standard normal.

4. Since, $f(t)$ is symmetrical about the line $t = 0$, all the odd order moments of about $t = 0$ are zeros, that is, $\mu'_{2r+1} = 0$. In particular μ, the mean is zero. Hence the central moments, that is, moments about mean coincide with moment about zero and so $\mu_{2r+1} = 0$.

Also we can show that the even order moments are given by

$$\mu_{2r} = \nu^r \frac{(2r-1)(2r-3)\ldots 3.1}{(\nu-2)(\nu-4)\ldots(\nu-2r)}, \quad \nu > 2r. \qquad ...(25.26)$$

In particular

$$\mu_2 = \frac{\nu}{\nu-2}, \quad \nu > 2 \text{ and } \mu_4 = \frac{3\nu^2}{(\nu-2)(\nu-4)}, \quad \nu > 4.$$

Hence,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^2} = 0, \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2} = 3\left(\frac{\nu-2}{\nu-4}\right), \quad \nu > 4. \qquad ...(25.27)$$

As $\nu \to \infty$, $\beta_2 \to 3$, that is, the curve tends to normal.

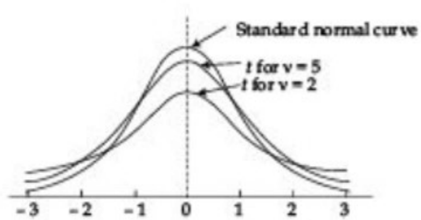


Fig. 25.3

5. The distribution of the random variable t as defined in (25.24) is called the t distribution with $(n-1)$ *degrees of freedom* (df) and the variate is represented as t_{n-1} or t_ν, where $\nu = n-1$.

Degrees of freedom $\nu = n-1$ is the quantity by which $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is divided in order to obtain an unbiased estimate of σ^2 and refers to the amount of information available in the data used for estimating σ^2. For each possible value of df $\nu = n-1$, there is a different t distribution and as ν approaches large, the t-variate tends to z-variate, the standard normal variate.

In particular, for $\nu = 1$, that is, for $n = 2$, (25.25) gives

$$f(t) = \frac{1}{\beta\left(\frac{1}{2}, \frac{1}{2}\right)} \frac{1}{(1+t^2)}$$

$$= \frac{1}{\pi(1+t^2)}; \quad -\infty < t < \infty, \qquad \text{...(25.28)}$$

the p.d.f. of a standard *Cauchy variate*.

The t-variate has vital applications in statistical inference. Although distribution of t is asymptotically normal for large n, but for small n it considerably differs from normal. The discovery of t variate have led to many important contributions in the development of small sample theory.

Significant values of t: The significant values of t at level of significance α and degrees of freedom ν for two-tailed test, denoted by $t_{\nu(\alpha)}$, are given by

$$P[|t| > t_{\nu(\alpha)}] = \alpha \qquad \text{...(25.29)}$$

or,
$$P[|t| < t_{\nu(\alpha)}] = 1 - \alpha. \qquad \text{...(25.30)}$$

The Table II gives the values $t_{\nu(\alpha)}$ of t-distribution (two-tail areas) for different values of α and ν, as shown in Fig. 25.4.

Since the distribution is symmetrical about $t = 0$, (25.29) gives

$$P[t > t_{\nu(\alpha)}] + P[t < -t_{\nu(\alpha)}] = \alpha$$

which implies,
$$2P[t > t_{\nu(\alpha)}] = \alpha, \text{ or } P[t > t_{\nu(\alpha)}] = \frac{\alpha}{2}, \text{ or } P[t > t_{\nu(2\alpha)}] = \alpha.$$

Thus, the significant values of t at level of significance α for single-tailed tests (left or right) are those of two-tailed test at level of significance 2α.

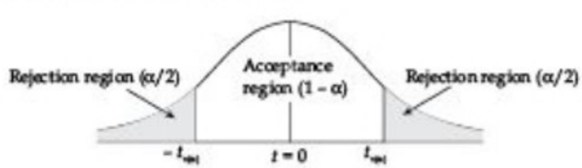


Fig. 25.4

For example, from Table II

$$t_{10(0.05)} \text{ for single-tail test} = t_{10(1.10)} \text{ for two-tailed test} = 1.81$$

We should note that the significant values of t lead us to reliable inferences about the sampled population if the sample drawn meets the following requirements.

1. The sample must be randomly selected.
2. The sampled population must be normally distributed.

25.7.2 Test for Single Mean

Given a random sample $x_1, x_2, \dots x_n$ from a normal population. We need to test the hypothesis that the mean of the sampled population is μ. Under the null hypothesis H_0, the statistic

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}}, \qquad \text{...(25.31)}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ follows t-distribution with $\nu = n - 1$ df.

We compare the calculated value of t with the tabulated value at the desired level of significance α. If calculated $|t|$ is greater than the tabulated t, null hypothesis is rejected and if calculated $|t|$ is less than the tabulated t, the null hypothesis is accepted at the level of significance α.

If $t_{\nu(0.05)}$ is the tabulated value of t for $\nu = n - 1$ df at 5% level of significance, then 95% *confidence limits for the population mean* μ are given by $\bar{x} \pm t_{\nu(0.05)}\left(S/\sqrt{n}\right)$. Similarly, 99% *confidence limits for the population mean* μ are given by $\bar{x} \pm t_{\nu(0.01)}\left(S/\sqrt{n}\right)$.

Example 25.22: Ten individuals were chosen at random from a normal population and their heights were found to be in inches as 63, 63, 66, 67, 68, 69, 70, 70, 71 and 71. Test the hypothesis that the mean height of the population is 66 inches. Also find the 95% confidence limits for the true population mean μ.

Solution: We have $\bar{x} = 67.8$ inches, and

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{(\Sigma x_i)^2}{n}\right] = \frac{1}{n-1}\left[\sum_{i=1}^{n} d_i^2 - \frac{(\Sigma d_i)^2}{n}\right], \quad d_i = x_i - 68$$

$$= \frac{1}{9}\left[82 - \frac{4}{10}\right] = 9.067 \text{ or, } S = 3.011 \text{ inches.}$$

We test the null hypotheses H_0: $\mu = 66$ against two-tailed alternative H_1: $\mu \neq 66$.

Under H_0, $$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} \sim t_\nu; \text{ the } t\text{-variate with } \nu = n - 1 = 9 \text{ df}.$$

We have, $$t = \frac{67.8 - 66}{3.011/\sqrt{10}} = 1.89.$$

From Table II, $t_{9[.05]} = 2.26$. Since t calculated is less than t tabulated hence the null hypotheses H_0 may be accepted at 5% level of significance.

Also the 95% confidence limits for the population mean μ are:

$$\mu = \bar{x} \pm t_{v[0.05]}\left(S/\sqrt{n}\right) = 67.8 \pm (2.26)\left(3.011/\sqrt{10}\right) = 67.8 \pm 2.53.$$

Example 25.23: The mean weekly sales of TVs of a particular brand in company's showrooms was 14.6 TV per showroom. After announcing a few incentives the mean weekly sales in 22 stores for a typical week increased to 15.4 with S.D. of 1.7. Were the incentives announced effective in boosting the sale?

Solution: We have, $n = 22$, $\bar{x} = 15.4$, $s = 1.7$. Let the null hypothesis H_0 be that incentive announced were not effective. Thus we test

$H_0 : \mu = 14.6$ against the right tailed alternative $H_1 : \mu > 14.6$.

Under H_0, $$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} - t_v$$

We have, $$t = \frac{15.4 - 14.6}{1.7/\sqrt{21}} = \frac{\sqrt{21}(0.8)}{1.7} = 2.16.$$

From Table II, $t_{21[0.05]}$ for single-tail test = $t_{21[0.10]}$ for two-tailed test = 1.72.

Since calculated value of t is greater than the tabulated value so hypothesis H_0 is rejected at 5% level of significance, that is, incentives announced were effective.

Example 25.24: The specifications for a certain kind of ribbon call for a mean breaking strength of 180 pounds. If five randomly selected pieces of the ribbon have a mean breaking strength of 169.5 pounds with a S.D. of 5.7 pound, test the null hypothesis $\mu = 180$ against the alternative hypothesis $\mu < 180$ at 5% level of significance.

Solution: We have, $n = 5$, $\bar{x} = 169.5$, $s = 5.7$. We test the null hypothesis $H_0 : \mu = 180$ against the left-tailed alternative $H_1 : \mu < 180$.

Under H_0, $$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} \sim t_v.$$

We have $$|t| = \frac{|169.4 - 180|}{5.7/\sqrt{n-1}} = |-3.72| = 3.72.$$

From Table II, $t_{4[0.05]}$ for single-tailed test = $t_{4[0.1]}$ for double-tailed test = 2.13.

Since t calculated is greater than the t tabulated, the null hypothesis H_0 is rejected at 5% level of significance.

25.7.3 Test for Difference Between Two Means

We shall consider tests of significance for following different cases:

Case I: Given two independent random samples $x_1, x_2, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ with means $\bar{x}$ and $\bar{y}$ and standard deviations s_x and s_y from normal populations with means μ_x and μ_y and with the same variances. We need to test the hypothesis that the population means are the same, that is, samples have been drawn from the same normal population.

Under the null hypothesis $H_0: \mu_x = \mu_y$, the test statistic t given by

$$t = \frac{\bar{x} - \bar{y}}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}, \qquad \text{...(25.32)}$$

where $\bar{x} = \dfrac{1}{n_1}\sum_{i=1}^{n_1} x_i, \quad \bar{y} = \dfrac{1}{n_2}\sum_{i=1}^{n_2} y_i,$

and, $S^2 = \dfrac{1}{n_1 + n_2 - 2}[(n_1 - 1)]S_1^2 + (n_2 - 1)S_2^2] = \dfrac{1}{n_1 + n_2 - 2}\left[\sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y})^2\right]$

follows t-distribution with $\nu = (n_1 + n_2 - 2)$ degrees of freedom and thus the hypothesis H_0 can be tested accordingly.

Remark: We use this test under the assumption that the sampled populations have the same variance and use S^2 as an unbiased estimate of the common variance σ^2. In case this assumption is not justified the test becomes invalid.

Case II: When the observations are paired like (x_i, y_i), $i = 1, 2, \ldots, n$, refer to Example 25.27, then sample sizes n_1 and n_2 become same and two samples are not independent, then in this case under the null hypothesis H_0 the test statistic t given by

$$t = \frac{\bar{d}}{S/\sqrt{n}}, \qquad \text{...(25.33)}$$

where $\bar{d} = \dfrac{1}{n}\sum_{i=1}^{n} d_i, \quad S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n} (d_i - \bar{d})^2; \quad d_i = (x_i - y_i)$

follows t-distribution with $(n - 1)$ degrees of freedom.

Example 25.25: The following random samples are measurements of the heat producing capacity in millions of calories per ton of specimens of coal from two mines:

Mine I:	8,260	8,130	8,350	8,070	8,340	
Mine II:	7,950	7,890	7,900	8,140	7,920	7,840

Test at 5% level of significance whether the difference between the means of these two samples is significant.

Solution: Let μ_1, μ_2 be the two population means for Mine I and Mine II, respectively. We test the null hypothesis, $H_0: \mu_1 = \mu_2$ against the alternative $H_1: \mu_1 \neq \mu_2$.

$$\text{We have, } \bar{x} = \frac{1}{5}\sum_{i=1}^{5} x_i = \frac{41150}{5} = 8{,}230, \quad \bar{y} = \frac{1}{6}\sum_{i=1}^{6} y_i = \frac{47640}{6} = 7{,}940$$

$$S_1^2 = \frac{1}{4}\sum_{i=1}^{5} (x_i - \bar{x})^2 = \frac{63{,}000}{4} = 15{,}750, \quad S_2^2 = \frac{1}{5}\sum_{i=1}^{6} (y_i - \bar{y})^2 = \frac{54{,}600}{5} = 10{,}920$$

Under H_0,

$$t = \frac{\bar{x} - \bar{y}}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t_\nu; \; \nu = n_1 + n_2 - 2$$

Also,

$$S^2 = \frac{\sum_{i=1}^{5}(x_i - \bar{x})^2 + \sum_{i=1}^{6}(y_i - \bar{y})^2}{n_1 + n_2 - 2} = \frac{63000}{9} + \frac{54600}{9} = 13066.67$$

Thus

$$t = \frac{8230 - 7940}{114.31\sqrt{\dfrac{1}{5} + \dfrac{1}{6}}} = 4.19.$$

From Table II, $t_{9(0.05)} = 2.26$. Since t calculated is greater than t tabulated, hence hypothesis is rejected at 5% level of significance.

Example 25.26: Samples of two types of electric light bulbs were tested for length of life and following data were obtained

	Type I	Type II
Sample sizes	$n_1 = 8$	$n_2 = 7$
Sample means	$\bar{x}_1 = 1234$ hrs	$\bar{x}_2 = 1036$ hrs
Sample S.D.'s	$s_1 = 36$ hrs	$s_2 = 40$ hrs.

Does the data support the hypothesis that Type I is superior to Type II regarding length of life?

Solution: Let μ_1, μ_2 be the population means, we test the null hypothesis,

$$H_0: \mu_1 = \mu_2 \text{ against right-tailed alternative } H_1: \mu_1 > \mu_2$$

Under H_0,

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t_\nu; \; \nu = n_1 + n_2 - 2$$

Here

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{1}{13}[8(36)^2 + 7(40)^2] = \frac{1}{13}[10368 + 11200] = 1659.08.$$

Thus
$$t = \frac{1234 - 1036}{40.73\left(\sqrt{\frac{1}{8} + \frac{1}{7}}\right)} = \frac{198}{21.08} = 9.39.$$

From Table II, $t_{13(0.05)}$ for single-tailed test = $t_{13(0.10)}$ for two-tailed test = 1.77.

Since t calculated is greater than t tabulated, the hypothesis H_0 is rejected at 5% level of significance.

Example 25.27: The yields of two types Type I and Type II of grains in pounds per acre in 6 replications are given below. Give your comments on the difference in the mean yields.

Replication	*Type I*	*Type II*
1	205	248
2	246	263
3	230	282
4	300	308
5	304	300
6	238	220

Solution: Let the null hypothesis H_0 be that there is no difference in the mean yields of Type I and Type II. We test the null hypothesis $H_0: \mu_1 = \mu_2$ against the two-tailed alternative $H_1: \mu_1 \neq \mu_2$. Under H_0, the test statistic t in case of paired observation is, $t = \dfrac{\bar{d}}{S/\sqrt{n}} - t_v;\ v = n - 1$.

We have,
$$\bar{d} = \frac{1}{6}\sum_{i=1}^{6} (x_i - y_i) = \frac{-43 - 17 - 52 - 8 + 4 + 18}{6} = -16.3 \text{ and,}$$

$$S^2 = \frac{1}{5}\sum_{i=1}^{6} (d_i - \bar{d})^2 = \frac{(-26.7)^2 + (0.49)^2 + (-35.7)^2 + (8.3)^2 + (20.3)^2 + (34.3)^2}{5} = 729.07.$$

Thus,
$$|t| = \frac{|\bar{d}|}{S/\sqrt{n}} = \frac{|(-16.3)\sqrt{6}|}{27} = 1.48.$$

From Table II, $t_{5(0.05)} = 2.57$. Since t calculated is less than t tabulated hypothesis may be accepted at 5% level of significance.

Case III: Given two random samples $x_1, x_2, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ from normal populations with the same variance we need to test the hypothesis that the population means are μ_x and μ_y, respectively. In this case under H_0 the test statistic t given by

$$t = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad ...(25.34)$$

follows t distribution with $\nu = (n_1 + n_2 - 2)$ degrees of freedom and thus we test the hypothesis accordingly.

Example 25.28: To test the claim that the resistance of electric wire can be reduced by at least 0.05 ohm by alloying, 25 measurements obtained for each alloyed wire and standard wire produced the following results:

	Mean	S.D.
Alloyed wire (x) :	0.083 ohm	0.003 ohm
Standard wire (y) :	0.136 ohm	0.002 ohm

Test the claim at 5% level of significance.

Solution: Let the claim be valid, so we test the null hypothesis

$H_0 : \mu_x - \mu_y \geq 0.05$ against the left-tailed alternative $H_1 : \mu_x - \mu_y < 0.05$.

Under H_0,
$$t = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_\nu \quad \nu = n_1 + n_2 - 2$$

Here
$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{(25)(.003)^2 + 25(.002)^2}{48} = .0000067.$$

Therefore,
$$|t| = \frac{|(0.083 - 0.136) - 0.05|}{(0.00260)\sqrt{\left(\frac{1}{25} + \frac{1}{25}\right)}} = \frac{|-.103|}{.00073} = |-144.09| = 144.09.$$

From Table II, $t_{48(.05)}$ for single-tailed test $= t_{48(.1)}$ for double-tailed test $= 1.65$.

Since $|t|$ calculated is greater than t tabulated, hypothesis is rejected at 5% level of significance.

25.7.4 Testing the Significance of an Observed Correlation Coefficient

Let r be the observed correlation coefficient in random sample of n observations (x_i, y_i) from a bivariate normal population; we need to test the hypothesis H_0 that the sampled population correlation coefficient ρ is zero.

We can show that under H_0, the test statistic t given by

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \qquad ...(25.35)$$

is a t variate with $\nu = (n - 2)$ degrees of freedom and thus we test the hypothesis accordingly.

Example 25.29: A random sample of fifteen paired observations from a bivariate normal population gives a correlation coefficient of –0.5. Does this signify the existence of correlation in the sampled population?

Solution: Let the null hypothesis H_0 be that sampled population is uncorrelated. Thus, we test

$$H_0 : \rho = 0 \text{ against the alternate } H_1 : \rho \neq 0.$$

Under H_0, $$t = r\sqrt{n-2}\Big/\sqrt{1-r^2} \sim t_v;\ v = n-2.$$

We have, $n = 15$, $r = -0.5$, thus $|t| = \dfrac{|r|\sqrt{n-2}}{\sqrt{1-r^2}} = \dfrac{0.5\sqrt{13}}{\sqrt{0.75}} = 2.08.$

From Talbe II, $t_{13(.05)} = 2.16$. Since t calculated is less than the t tabulated, hence null hypothesis may be accepted at 5% level of significance.

EXERCISE 25.3

1. A new process for producing synthetic diamonds is viable only if the average weight of the diamond is greater than 0.5 karat. The weights of the six diamonds generated are 0.46, 0.61, 0.52, 0.48, 0.57 and 0.54 karat. Test the viability of the process.
2. For a random sample of 16 observations with mean 41.5 inches and the sum of the squares of the deviation from the mean 135 $(\text{inches})^2$, drawn from a normal population, find the 95% confidence limits for the population mean μ.
3. The following ten observations are from a normal population:

   7.4, 7.1, 6.5, 7.5, 7.6, 6.3, 6.9, 7.7, 6.5, 7.0

   (a) Find 99% one-sided confidence bound for the population mean μ.

   (b) Test $H_0 : \mu = 7.5$ against $H_1 : \mu < 7.5$ at $\alpha = .01$.

   (c) Do the results of part (a) support your conclusion in part (b)?
4. A certain tablet administered to each of the 12 patients resulted in the following increase in blood pressure 5, 2, 8, −1, 3, 0, −2, 1, 5, 0, 4 and 6. Can it be concluded that tablet will, in general, be accompanied by an increase in blood pressure ?
5. Below are given the gain in weight (in lbs.) of pigs fed on two diets A and B

   Diet A: 25, 32, 30, 34, 24, 14, 32, 24, 30, 31, 35, 25

   Diet B: 44, 34, 22, 10, 47, 31, 40, 36, 32, 35, 18, 21, 35, 29, 22.

   Test if the two diets differ significantly as regards their effect on increase in weight.
6. The following are the average weekly losses of worker-hours due to accidents in 10 industrial plants before and after a certain safety programme was put into operation

Before:	45	73	46	124	33	57	83	34	26	17
After :	36	60	44	119	35	51	77	29	24	11

   At 5% level of significance test whether the safety programme was effective. Also find the 90% confidence interval for the mean improvement in lost worker-hours.
7. Measuring specimens of nylon yarn taken from two spinning machines, it was found that 7 specimens from the first machine had a mean denier of 8.62 with a S.D. of 2.8, while 9 specimens from the second machine had a mean denier of 6.38 with a S.D. of 2.4. Assuming that population sampled are normal with the same variance, test the null hypothesis $\mu_1 - \mu_2 = 1.0$ against the alternative $\mu_1 - \mu_2 > 1.0$ at $\alpha = 0.5$.

8. In a certain experiment to compare two types of animal feed *A* and *B*, the following results of increase in weights were observed in two independent samples of animals each of size 8. Test the hypothesis that food *B* is better than food *A*.

	Increase in weight in lbs							
Food A	49	53	51	52	47	50	52	53
Food B	52	55	52	53	50	54	54	53

9. A coefficient of correlation of 0.2 is obtained from a random sample of 625 pairs of observations.
   (i) Is this value of r significant?
   (ii) Obtain the 95% confidence limits to the correlation coefficient in population; use that when n is large t-variate is distributed like a standard normal variate.
10. Find the least value of r in a sample of 18 pairs of observations from a bivariate population significant at 5% level.

25.8 CHI-SQUARE VARIATE AND TEST FOR POPULATION VARIANCE

While studying the applications of t-variates in tests of significance we have seen that an estimate of the population variance σ^2 is usually required to make inferences about the population mean. Otherwise also in some practical situations the knowledge of the variance of the sampled population may be more important than the population mean. For example, our concern may be to know the precision of a measuring instrument being used, or we may be much concerned about the variation of the water level at different points during flood.

Suppose we want to test if a random sample $x_1, x_2, ..., x_n$ has been drawn from a normal population with a specified variance σ^2. Then under the null hypothesis that the population variance is σ^2, the statistic χ^2 called chi-square variable, defined by

$$\chi^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2} = \frac{1}{\sigma^2}\left[\sum_{i=1}^{n} (x_i - \bar{x})^2\right] = \frac{(n-1)S^2}{\sigma^2}, \quad ...(25.36)$$

where $S^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is an unbiased estimate of σ^2, follows sampling distribution with probability distribution given by

$$\frac{1}{2^{\nu/2}\Gamma(\nu/2)}\cdot\left[\exp\left(-\frac{1}{2}\chi^2\right)\right](\chi^2)^{\frac{\nu}{2}-1} \quad 0 < \chi^2 < \infty. \quad ...(25.37)$$

The distribution defined by (25.37) is called χ^2 probability distribution with $\nu = n - 1$ degrees of freedom (*df*). The probability curve for a chi-square distribution is shown in Fig. 25.5. The curve is skewed towards right and its shape varies with the degrees of freedom $\nu = n - 1$. The variate tends to standard normal variate as $n \to \infty$.

Critical values and test of significance. Let $\chi^2_{\nu[\alpha]}$ denote the value of chi-square variate for ν *df* such that the area to the right of this point is α, that is, $P[\chi^2 > \chi^2_{\nu[\alpha]}] = \alpha$, as shown in Fig. 25.5. The Table III gives the critical values or significant values of $\chi^2_{v(\alpha)}$ for the right-tailed test for different degrees of freedom v and significant level α. We observe that value of $\chi^2_{v(\alpha)}$ increases with increase in v and decrease in α. At a specific level α and *df* v, the null hypothesis $H_0: \sigma^2 = \sigma_0^2$ is rejected against the alternate hypothesis

(i) $H_1: \sigma^2 > \sigma_0^2$ if calculated $\chi^2 > \chi^2_{v(\alpha)}$, refer to Fig. 25.5

(ii) $H_1: \sigma^2 < \sigma_0^2$; if calculated $\chi^2 < \chi^2_{v(1-\alpha)}$, refer to Fig. 25.6

(iii) $H_1: \sigma^2 \neq \sigma_0^2$; if calculated $\chi^2 > \chi^2_{v(\alpha/2)}$ or $\chi^2 < \chi^2_{v(1-\alpha/2)}$, refer to Fig. 25.7

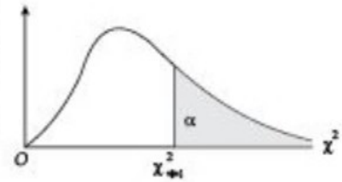


Fig. 25.5

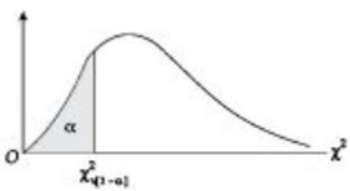


Fig. 25.6

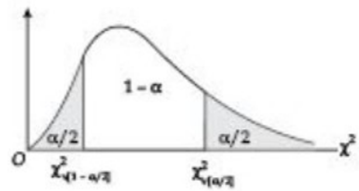


Fig. 25.7

Equal tails are used for the two-tailed χ^2 test as a matter of mathematical convenience only otherwise, chi-square distribution is not symmetric. However, normally in practice right-tailed test is applicable.

Example 25.30: A manufacturer of car batteries claims that the life of his batteries is approximately normally distributed with a S.D. of 0.9 years. If a random sample of 10 of these batteries has a S.D. of 1.1 years, do you think $\sigma > 0.9$ years at $\alpha = 0.05$?

Solution: We test $H_0: \sigma^2 = 0.81$ against the right-tailed alternative $H_1: \sigma^2 > 0.81$.

We have $n = 10, \quad \sigma^2 = 0.81, \quad s^2 = (1.1)^2 = 1.21$

Under H_0, the test statistic

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \frac{ns^2}{\sigma^2} = \frac{10(1.21)}{0.81} = 14.94$$

follows χ^2 distribution with df $v = n - 1 = 10 - 1 = 9$.

From Table III, $\chi^2_{9(0.5)} = 16.92$. Since χ^2 calculated is less than χ^2 tabulated so value is not significant and hence hypothesis H_0 may be accepted at 5% level of significance.

Example 25.31: Following data give the 11 measurements of the same object on the same instrument:

2.5, 2.3, 2.4, 2.3, 2.5, 2.7, 2.5, 2.6, 2.6, 2.7, 2.5.

At 1% level, test the hypothesis that the variance of the instrument is no more than 0.16.

Solution: We test the null hypothesis $H_0 : \sigma^2 = 0.16$ against alternative $H_1 : \sigma^2 > 0.16$.

Here $\bar{x} = \dfrac{27.6}{11} = 2.51$, $\Sigma (x - \bar{x})^2 = 0.1891$. Under H_0 the test statistic χ^2 given by

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \frac{\Sigma (x - \bar{x})^2}{\sigma^2} = \frac{0.1891}{0.16} = 1.182$$

follows χ^2 distribution with degrees of freedom $\nu = n - 1 = 11 - 1 = 10$.

From Table III, $\chi^2_{10[.01]} = 23.2$, and since the χ^2 calculated is less than the χ^2 tabulated, so hypothesis may be accepted at 1% level of significance.

25.9 F-VARIATE AND TEST FOR THE EQUALITY OF TWO POPULATION VARIANCES

Sometimes we need to compare two population variances. For example, we may be interested to compare the precisions of the two measuring instruments, or we may by interested in the stability of measurement on a manufactured product from two assembly lines. Suppose we want to test whether the two independent samples $x_1, x_2, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ have been drawn from the normal populations with the same variance σ^2. Under the null hypothesis that the population variances σ_x^2 and σ_y^2 are the same, that is, $\sigma_x^2 = \sigma_y^2 = \sigma^2$, the *variance ratio statistic* F, defined by

$$F = S_x^2 / S_y^2, \qquad \text{...(25.38)}$$

where $S_x^2 = \dfrac{1}{n_1 - 1}\displaystyle\sum_{i=1}^{n_1} (x_i - \bar{x})^2$ and $S_y^2 = \dfrac{1}{n_2 - 1}\displaystyle\sum_{i=1}^{n_2} (y_i - \bar{y})^2$

follows sampling distribution with probability density function given by

$$\frac{(\nu_1/\nu_2)^{\nu_1/2}}{B\left(\dfrac{\nu_1}{2}, \dfrac{\nu_2}{2}\right)} \cdot \frac{F^{(\nu_1/2)-1}}{\left(1 + \dfrac{\nu_1}{\nu_2} F\right)^{(\nu_1+\nu_2)/2}}, \quad 0 < F < \infty, \qquad \text{...(25.39)}$$

where $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$.

The distribution defined by (25.39) is called *Snedecor's F-distribution with* (ν_1, ν_2) *degrees of freedom* and the variate F is denoted by $F_{(\nu_1, \nu_2)}$. Generally, the greater of the two variances S_x^2 and S_y^2 is taken as numerator and ν_1 corresponds to the variance in the numerator. The probability curve for the F-distribution is shown in Fig. 25.8.

The curve is not symmetric and the shape depends on the degrees of freedom ν_1 and ν_2 and their order. The curve is completely described by ν_1 and ν_2.

Critical values and test of significance. Let $F_{(\nu_1, \nu_2)[\alpha]}$ denote the value of F for (ν_1, ν_2) degrees of freedom such that the area to the right of this point is α, that is, $P[F > F_{(\nu_1, \nu_2)[\alpha]}] = \alpha$, as shown in Fig. 25.9. The Tables IV A & B, give critical values or significant values of $F_{(\nu_1, \nu_2)[\alpha]}$ for the right-tailed test for different *d.f.* (ν_1, ν_2) and significant level $\alpha = 0.05$ and .01, respectively.

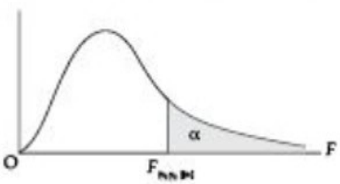


Fig. 25.8

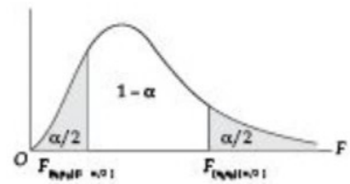


Fig. 25.9

For an F-variate the reciprocal relation $F_{(\nu_1, \nu_2)(\alpha)} = \dfrac{1}{F_{(\nu_2, \nu_1)(1-\alpha)}}$ also holds.

At a specific level α and degrees of freedom (ν_1, ν_2) the null hypothesis H_0: $\sigma_x^2 = \sigma_y^2$ is rejected against the alternate hypothesis,

(i) $H_1 : \sigma_x^2 > \sigma_y^2$ and $F = \dfrac{S_x^2}{S_y^2}$, if $F > F_{(\nu_1, \nu_2)(\alpha)}$

(ii) $H_1 : \sigma_x^2 < \sigma_y^2$ and $F = \dfrac{S_y^2}{S_x^2}$, if $F > F_{(\nu_2, \nu_1)(\alpha)}$

(iii) $H_1 : \sigma_x^2 \neq \sigma_y^2$ and $F = \dfrac{S_x^2}{S_y^2}$, if $F > F_{(\nu_1, \nu_2)(\alpha/2)}$

We must note that F-distribution is not symmetric but as in chi-square test, here also equal tails are used in two-tailed test as a matter of mathematical convenience only.

Example 25.32: There are two different choices to stimulate a certain chemical process. To test whether the variance of the yield is the same no matter which catalyst is used, a sample of 10 batches is produced using the first catalyst, and of 12 using the second. If the resulting data is $S_1^2 =$ 0.14 and $S_2^2 = 0.28$, test the hypothesis of equal variance at 2% level.

Solution: We have, $n_1 = 10$, $n_2 = 12$, $S_1^2 = 0.14$, $S_2^2 = 0.28$. We test H_0: $\sigma_1^2 = \sigma_2^2$ against two-tail alternative H_1: $\sigma_1^2 \neq \sigma_2^2$.

Under H_0, the test statistic F is given by $F = \dfrac{S_2^2}{S_1^2} = \dfrac{0.28}{0.14} = 2.$

The statistic F follows F-distribution with (11, 9) degress of freedom.

From Table IVB, $F_{(11, 9)(.02)}$ two-tail alternative $= F_{(11, 9)(.01)}$ at right-tail alternative = 5.20.

Since, F calculated is less than F-tabulated, it is not significant and hence hypothesis may be accepted at 2% level of significance.

Example 25.33: With reference to the data given in Example 25.25, test at 10% level of significance whether it is reasonable to assume that the two population variances are the same against the alternative that they are different.

Solution: We have, $n_1 = 5,\ n_2 = 6,\ S_1^2 = 15{,}750,\ S_2^2 = 10{,}920.$

We test the null hypothesis $H_0: \sigma_1^2 = \sigma_2^2$ against two-tailed alternative $H_1: \sigma_1^2 \neq \sigma_2^2$.

Under H_0, the variance ratio statistic F, is

$$F = \frac{S_1^2}{S_2^2} = \frac{15750}{10920} = 1.44 - F_{(4,5)}$$

From Table IVA, $F_{(4,5)\,[.10]}$ for two-tailed test $= F_{(4,5)\,[.05]}$ for single tailed test $= 5.19$.

Since, the calculated value is less than the tabulated so hypothesis may be accepted at 10% level of significance.

Example 25.34: Two random samples gave the following results:

Sample	*Size*	*Sample mean*	*Sum of squares of deviations from the mean*
1	10	15	90
2	12	14	108

Test whether the sample come from the same normal population at 5% level of significance.

Solution: Since, a normal population is specified by two parameters : mean μ and variance σ^2, thus to test that two independent samples have been drawn from the same population, we need to test (i) the equality of population means using t-test (ii) the equality of population variances, using F-test.

Since t-test is applied under the assumption that population variances are the same, so first we shall test for the equality of population variances.

Here, we have

$n_1 = 10,\ n_2 = 12,\ \bar{x} = 15,\ \bar{y} = 14,\ \Sigma(x_i - \bar{x})^2 = 90,\ \Sigma(y_i - \bar{y})^2 = 108,$

$$S_1^2 = \frac{90}{9} = 10,\quad S_2^2 = \frac{108}{11} = 9.82, \text{ and } S^2 = \frac{1}{n_1+n_2-2}[\Sigma(x_i - \bar{x})^2 + \Sigma(y_i - \bar{y})^2] = \frac{90+108}{20} = 9.9.$$

We test $H_0: \sigma_1^2 = \sigma_2^2$ against the right-tailed alternative $H_1: \sigma_1^2 > \sigma_2^2$. Under H_0, statistic F given by

$$F = \frac{S_1^2}{S_2^2} = \frac{10}{9.82} = 1.018 - F_{(9,11)}$$

From Table IV A, $F_{(9,11)\,[.05]} = 2.90$. Since F calculated is less than the F tabulated, hence H_0 is accepted.

Since $H_0: \sigma_1^2 = \sigma_2^2$ is established, we can now apply t test for testing $H_0: \mu_1 = \mu_2$ against the alternative $H_1: \mu_1 \neq \mu_2$.

Under H_0, the statistic t given by

$$t = \frac{\bar{x} - \bar{y}}{S\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{15 - 14}{3.15\sqrt{\frac{1}{10} + \frac{1}{12}}} = \frac{1}{1.349} = 0.74 \sim t_{20}$$

From Table II, $t_{20(.05)} = 2.086$. Since t calculated is less than the t-tabulated hence the hypothesis $H_0: \mu_1 = \mu_2$ may be accepted. Since both the null hypotheses $H_0: \sigma_1^2 = \sigma_2^2$ and $H_0: \mu_1 = \mu_2$ are accepted so samples may be considered to come from the same normal population.

EXERCISE 25.4

1. A manufacturer claims that his measuring instrument has a variability measured by S.D. $\sigma = 2$. During a test the measurements recorded are 4.1, 5.2 and 10.2. Do these data confirm or disprove his claim? Construct a 90% confidence interval to estimate the true population variance.
2. A precision instrument is guaranteed to read accurately to within 2 units. A sample of four instrument readings on the same object yield the measurements 353, 351, 351 and 355. Test the null hypothesis that $\sigma = 0.7$ against the alternative $\sigma > 0.7$ at $\alpha = .05$.
3. Playing 10 rounds of golf on his home course, a golf professional averaged 71.3 with a S.D. 1.32. Test the null hypothesis at $\alpha = .05$ that consistency of his game on his home course is actually measured by $\sigma = 1.20$ against the alternative that he is less consistent.
4. Following data gives the amounts of sulphur monoxide recorded by two instruments A and B in the atmosphere. Assuming the populations of measurements to be normal, test the hypothesis $H_0: \sigma_A = \sigma_B$ against $H_1: \sigma_A \neq \sigma_B$ at $\alpha = .02$,

Instruments	*Amounts of sulphur monoxide*								
A	0.86	0.82	0.75	0.61	0.89	0.64	0.81	0.68	0.65
B	0.87	0.74	0.63	0.55	0.76	0.70	0.69	0.57	0.53.

5. The following are the values in thousands of an inch obtained by two technicians in 10 successive measurements with the same micrometer. Is one technician significantly more consistent than the other at $\alpha = 0.05$?

Technician A :	503	505	497	505	495	502	499	493	510	501
Technician B :	502	497	492	498	499	495	497	496	498.	

6. The nicotine contents in milligrams of two samples of tobacco were found to be as follows:

Sample A :	24	27	26	21	25	
Sample B :	27	30	28	31	22	36

Can it be claimed that two samples come from the same normal population?

7. For the two samples

105 108 86 103 103 107 124 105 and 89 92 84 97 103 107 111 97

giving the relative output of tin plate workers under two different working conditions, test the hypothesis at $\alpha = .05$, $H_0: \sigma_1^2 = \sigma_2^2$ against the alternative $H_1: \sigma_1^2 > \sigma_2^2$ assuming the two populations to be normal.

25.10 CHI-SQUARE TEST OF GOODNESS-OF-FIT, CONTINGENCY TABLES AND YATE'S CORRECTION FOR CONTINUITY

In many random experiments we are interested to know whether a particular probabilistic model is appropriate or not. For example, we may hypothesize that number of industrial accidents occurring monthly at a particular industrial plant follow Poission distribution. This hypothesis can be tested by observing the number of accidents over a sequence of months; finding the theoretical or expected number of accidents on the basis of the hypothesis made and then testing whether the deviations between the observed and the expected number of accidents in each category can be attributed to fluctuations of sampling or not.

The statistical tests that determine whether a given theoretical probability distribution is appropriate in case of the random phenomena under study are called 'goodness of fit' tests.

Suppose that $O_1, O_2 \ldots, O_k$ are the observed frequencies and $E_1, E_2, \ldots, E_k$ are the corresponding expected frequencies in the k categories on the basis of the hypothesis made. If hypothesis is correct, the observed cell frequency O_i should not be much different from the expected frequency E_i. The larger the difference, the more likely it is that the hypothesis is incorrect. The chi-square statistic to test the *goodness-of-fit* between the observed and the expected frequencies is defined as

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}, \qquad \ldots(25.40)$$

where $\sum_{i=1}^{k} O_i = \sum_{i=1}^{k} E_i = n$, the total cell frequency.

When n is large the statistic χ^2 defined by (25.40) follows chi-square probability distribution with $\nu = k - m$ degrees of freedom, where k is the number of categories and m is the number of constraints applied to the observed data to calculate the expected frequencies.

If the theoretical cell frequencies are correct then χ^2 is close to zero but if theoretical cell frequencies are incorrect then χ^2 is large and thus we use right-tailed statistical test to find the significant value of χ^2 for the specified degrees of freedom ν and level of significance α.

Also to apply χ^2-test of goodness-of-fit, we pool some of the data so that no expected frequency is less than 5 and we change the degrees of freedom accordingly. This is done to avoid irregularities due to discontinuity since distribution of χ^2 is continuous but distribution of frequencies by nature is discontinuous.

Example 25.35: Suppose that a dice is tossed 120 times and the recorded data is as follows:

Face :	1	2	3	4	5	6
Observed frequency :	20	22	17	18	19	24

Test the hypothesis that the dice is unbiased at $\alpha = 0.05$.

Solution: On the basis of the null hypothesis that dice is unbiased the probability p_i for the face i is 1/6. So we test the hypothesis

$$H_0 : p_1 = p_2 = \ldots = p_6 = 1/6.$$

Thus, expected frequencies E_i for the face i is $np_i = 120 \times 1/6 = 20$, $i = 1, 2, \ldots, 6$

Under the hypothesis H_0, the statistic $\chi^2 = \sum_{i=1}^{6} \frac{(O_i - E_i)^2}{E_i}$ follows χ^2 distribution with degrees of freedom $\nu = k - m = 6 - 1 = 5$.

We have, $$\chi^2 = \frac{0+4+9+4+1+16}{20} = \frac{34}{20} = 1.7.$$

From Table III, $\chi^2_{\nu(.05)} = 11.07$. Since χ^2 calculated is less than the χ^2 tabulated the hypothesis may be accepted , that is, dice may be considered to be unbiased.

Example 25.36: The proportion of blood phenotypes A, B, AB and O in a population are expected to be 0.41, 0.10, 0.04 and 0.45, respectively. To determine whether or not the actual proportions fit this set of probabilities, a random sample of size 200 is selected from this population and blood phenotypes of the units selected are recorded. The observed data is given as follows:

Phenotypes :	A	B	AB	O
No. of units :	89	18	12	81

Test the goodness-of-fit of these blood phenotype proportions at $\alpha = .05$.

Solution: The hypothesis to be tested is

$H_0 : p_A = 0.41, \quad p_B = 0.10, \quad p_{AB} = 0.04, \quad p_O = 0.45.$

Under H_0, the expected cell frequencies are

$$E(A) = 200(0.41) = 82, \quad E(B) = 200\,(0.10) = 20$$
$$E(AB) = 200\,(0.04) = 8, \quad E(O) = 200\,(0.45) = 90.$$

Thus $$\chi^2 = \sum_{i=1}^{4} \frac{(O_i - E_i)^2}{E_i} = \frac{49}{82} + \frac{4}{20} + \frac{16}{8} + \frac{81}{90} = 3.70$$

Number of degrees of freedom $= 4 - 1 = 3$.

From Table III, $\chi^2_{3(.05)} = 7.82$. Since χ^2 calculated is less than χ^2 tabulated, thus null hypothesis H_0 may be accepted at 5% level of significance.

Example 25.37: During 400 five-minute interval the air traffic control of an airport received 0, 1, 2,..., or 13 radio messages with respective frequencies of 3, 15, 47, 76, 68, 74, 46, 39, 15, 9, 5, 2, 0 and 1. Test at $\alpha = 0.05$, the hypothesis that the number of radio messages received during a 5 minute interval follows Poisson distribution with $\lambda = 4.6$.

Solution: Let the r.v. X be the number of radio messages received during a 5-minute interval and p_x be the probability of receiving x messages. Set the null hypothesis that X follows a Poisson distribution with parameter 4.6. Thus, we test the hypothesis.

$$H_0 : p_x = e^{-4.6} \frac{(4.6)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Form the following table:

No. of radio messages (x)	Observed frequencies (O)	Poisson probabilities (p_x)	Expected frequencies ($E = 400\, p_x$)
0	3	0.010	4.0
1	15	0.046	18.4
2	47	0.107	42.8
3	76	0.163	65.2
4	68	0.187	74.8
5	74	0.173	69.2
6	46	0.132	52.8
7	39	0.087	34.8
8	15	0.050	20.0
9	9	0.025	10.0
10	5	0.012	4.8
11	2	0.005	2.0
12	0	0.002	0.8
13	1	0.001	0.4

To apply χ^2-test of goodness-of-fit, since no expected frequency should be less than 5, so we pool the first two expected frequencies and the last four expected frequencies. The modified frequencies are:

Observed (O) :	18	47	76	68	74	46	39	15	9	8
Expected (E) :	22.4	42.8	65.2	74.8	69.2	52.8	34.8	20.0	10.0	8.0.

Thus $\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$

$$= \frac{(4.4)^2}{22.4} + \frac{(4.2)^2}{42.8} + \frac{(10.8)^2}{65.2} + \frac{(6.8)^2}{74.8} + \frac{(4.8)^2}{69.2} + \frac{(5.8)^2}{52.8} + \frac{(4.2)^2}{34.8} \frac{(5)^2}{20} + \frac{(1)^2}{10} + \frac{(0)^2}{8}$$

$= 6.749$. Number of degrees of freedom $= 10 - 1 = 9$.

From Table III, $\chi^2_{9[.05]} = 16.919$. Since χ^2 calculated is less than the χ^2 tabulated so null hypothesis may be accepted at 5% level of significance.

25.10.1 Chi-Square Test of Independence in Contingency Tables

Sometimes experimental units are classified according to two characteristics generating a bivariate data. The resulting observations are displayed in the form of a two-way table, called a *contingency table*, consisting of finite numbers of rows and columns. One characteristic varies along the rows and the second characteristic varies along the columns.

For example, a random sample of 500 employees of a PSU are classified whether they are in a low, medium, or high income bracket and whether or not they favour the new salary structure announced. The data can be presented in the form of the following 2×3 contingency table.

Salary structure	Income level			Total
	Low	Medium	High	
For	91	106	98	295
Against	80	72	53	205
Total	171	178	151	500

A contingency table with r rows and c columns is referred to as an $r \times c$ table. The row and column totals are called the *marginal frequencies*.

In two categorical variable data, our interest may be to know whether or not the two characteristics are independent. The chi-square test procedure can be used to test the hypothesis for independence of two characteristics of classification. We test the null hypothesis

H_0: The two characteristics of classification are independent, against the alternative

H_1: The two characteristics of classification are dependent.

Let O_{ij} be the observed cell frequency in row i and column j of the contingency table and if we know E_{ij} the expected cell frequency under H_0, then we can use χ^2 to compare the observed and expected frequencies.

Let p_{ij} be the probability of falling observation in the ith row and jth column and if n is the total number of observations, then $E_{ij} = np_{ij} = n\, p_i\, q_j$ since under hypothesis of independence, $p_{ij} = p_i q_j$, where p_i and q_j are the marginal probabilities of falling observations in the ith row and jth column respectively.

We approximate p_i with $\hat{p}_i = \frac{n_i}{n}$ and q_j with $\hat{q}_j = \frac{m_j}{n}$, where n_i, m_j respectively are the ith row and jth column totals. Thus estimated expected cell frequencies under H_0 become

$$E_{ij} = n\frac{n_i}{n}\,\frac{m_j}{n} = \frac{n_i m_j}{n},$$

and the statistic χ^2 is given by

$$\chi^2 = \sum_i \sum_j \frac{(O_{ij} - E_{ij})^2}{E_{ij}}. \qquad \text{...(25.41)}$$

This test statistic χ^2 can be shown to have an approximate χ^2 probability distribution with $(r-1)(c-1)$ degrees of freedom and the hypothesis H_0 is tested accordingly using the right-tailed test.

Example 25.38: A company operates four machines on three separate shifts daily. The following table presents the data for machine breakdowns resulted during a 6-month time period.

Shift	Machine				Total
	A	B	C	D	
1	10	12	6	7	35
2	10	24	9	10	53
3	13	20	7	10	50
Total	33	56	22	27	138

Test the hypothesis that for an arbitrary breakdown the machine causing the breakdown and the shift on which the breakdown occurred are independent.

Solution: Let H_0: For an arbitrary breakdown the machine and the shift are independent.

Under H_0, the twelve expected frequencies are given by

$$E_{11} = \frac{35 \times 33}{138} = 8.37, \quad E_{12} = \frac{35 \times 56}{138} = 14.20, \quad E_{13} = \frac{35 \times 22}{138} = 5.58,$$

$$E_{14} = \frac{35 \times 27}{138} = 6.85, \quad E_{21} = \frac{53 \times 33}{138} = 12.67, \quad E_{22} = \frac{53 \times 56}{138} = 21.50,$$

$$E_{23} = \frac{53 \times 22}{138} = 8.45, \quad E_{24} = \frac{53 \times 27}{138} = 10.37, \quad E_{31} = \frac{50 \times 33}{138} = 11.96,$$

$$E_{32} = \frac{50 \times 56}{138} = 20.29, \quad E_{33} = \frac{50 \times 22}{138} = 7.79, \quad E_{34} = \frac{50 \times 27}{138} = 9.78.$$

Under H_0 the statistic χ^2 is given by $\chi^2 = \sum_i \sum_j \frac{(O_{ij} - E_{ij})^2}{E_{ij}} - \chi^2_{(r-1)(c-1)}$

Thus,
$$\chi^2 = \frac{(10-8.37)^2}{8.37} + \frac{(12-14.20)^2}{14.20} + \frac{(6-5.58)^2}{5.58} + \frac{(7-6.85)^2}{6.85}$$
$$+ \frac{(10-12.67)^2}{12.67} + \frac{(24-21.50)^2}{21.50} + \frac{(9-8.45)^2}{8.45} + \frac{(10-10.37)^2}{10.37}$$
$$+ \frac{(13-11.96)^2}{11.96} + \frac{(20-20.29)^2}{20.29} + \frac{(7-7.97)^2}{7.97} + \frac{(10-9.78)^2}{9.78} = 1.78$$

The *d.f.* are $(2-1)(4-1) = 3$.

From Table III, $\chi^2_{3[.05]} = 7.8$. Since χ^2 calculated is less than χ^2 tabulated for 3 *df* at $\alpha = .05$, hence null hypothesis may be accepted.

Example 25.39: To know about the student's response over the proposed evaluation system it was decided to select 200 1st year, 150 2nd year and 150 3rd year students from a college and record whether they are for, against or undecided for the proposed evaluation system. The observed responses were:

Evaluation System	*Students*			*Total*
	1st year	*2nd year*	*3rd year*	
For	82	70	62	214
Against	93	62	67	222
Undecided	25	18	21	64
Total	200	150	150	500

Test the hypothesis that the three categories of the students are homogeneous with respect to their opinions on the proposed evaluation system.

Solution: Let H_0: Students are homogeneous with respect to their opinions on the proposed evaluation system.

Assuming homogeneity, the expected cell frequencies are:

$$E_{11} = \frac{200\times 214}{500} = 85.6, \quad E_{12} = \frac{150\times 214}{500} = 64.2, \quad E_{13} = \frac{150\times 214}{500} = 64.2,$$

$$E_{21} = \frac{200\times 222}{500} = 88.8, \quad E_{22} = \frac{150\times 222}{500} = 66.6, \quad E_{23} = \frac{150\times 222}{500} = 66.6,$$

$$E_{31} = \frac{200\times 64}{500} = 25.6, \quad E_{32} = \frac{150\times 64}{500} = 19.2, \quad E_{33} = \frac{150\times 64}{500} = 19.2.$$

Thus, $$\chi^2 = \sum_i \sum_j \frac{(O_{ij}-E_{ij})^2}{E_{ij}}$$

$$= \frac{(3.6)^2}{85.6} + \frac{(5.8)^2}{64.2} + \frac{(2.2)^2}{64.2} + \frac{(4.2)^2}{88.8} + \frac{(4.6)^2}{66.6} + \frac{(0.4)^2}{66.6} + \frac{(0.6)^2}{25.6} + \frac{(1.2)^2}{19.2} + \frac{(1.8)^2}{19.2}$$

$$= 1.53.$$

Degrees of freedom $v = (r-1)(c-1) = (3-1)(3-1) = 4$.

From Table III $\chi^2_{4(0.05)} = 9.488$. Since χ^2 calculated is less than the χ^2 tabulated, hypothesis may be accepted at 5% level of significance.

Example 25.40: Two different sampling techniques were adopted while investigating the same group of students to find the number of students falling in different intelligence level. The results are tabulated as follows:

Techniques	*No. of Students*				*Total*
	Below average	*Average*	*Above average*	*Genius*	
X	86	60	44	10	200
Y	40	33	25	2	100
Total	126	93	69	12	300

Are the sampling techniques adopted significantly different?

Solution: Let us assume that sampling techniques are not significantly different thus we test H_0: Data obtained is independent of the sampling techniques adopted.

Under H_0, the expected frequencies are

$$E_{11} = \frac{126 \times 200}{300} = 84, \quad E_{12} = \frac{93 \times 200}{300} = 62, \quad E_{13} = \frac{69 \times 200}{300} = 46, \quad E_{14} = \frac{12 \times 200}{300} = 8,$$

$$E_{21} = \frac{126 \times 100}{300} = 42, \quad E_{22} = \frac{93 \times 100}{300} = 31, \quad E_{23} = \frac{69 \times 100}{300} = 23, \quad E_{24} = \frac{12 \times 100}{300} = 4.$$

Since to apply χ^2 test no expected cell frequency should be less than 5 but here $E_{24} = 4$, so we pool E_{24} with E_{23} (or, with E_{14}) and accordingly O_{24} with O_{23} (or with O_{14}). We have

$$\chi^2 = \sum_i \sum_j \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

$$= \frac{(86-84)^2}{84} + \frac{(60-62)^2}{62} + \frac{(44-46)^2}{46} + \frac{(10-8)^2}{8} + \frac{(40-42)^2}{42} + \frac{(33-31)^2}{31} + \frac{[(25+2)-(23+4)]^2}{23+4}$$

$= 0.92.$

Degrees of freedom $\nu = (2-1)(4-1) - 1 = 2.$

From Table III, $\chi^2_{2(.05)} = 5.991$. Since χ^2 calculated is less than χ^2 tabulated, thus the null hypothesis H_0 is accepted at 5% level. Hence there is no significant difference in the sampling techniques adopted.

25.10.2 Yate's Correction for Continuity

The 2×2 contingency tables are of great practical importance. However, in a 2×2 table there is only one degrees of freedom and the frequency of only one cell can be assigned arbitrarily, but in case any of the expected frequency is less than 5, then pooling that cell frequency results in χ^2 with zero degree of freedom which is meaningless. In this case we apply a correction due to F. Yates known as *Yate's correction for continuity*. It consists in adding 0.5 to the cell frequency which is less than 5 and then adjusting for the remaining cell frequencies accordingly. The χ^2 test is applied in the resultant table without making any further correction.

Example 25.41: Two batches each of 12 experimental animals 'inoculated' and the other, 'not inoculated', were exposed to the infection of a disease. The following frequencies of dead and surviving animals were noted in the two cases, can the inoculation be regarded as effective against the disease?

Animals	*Dead*	*Survived*	*Total*
Inoculated	2	10	12
Not inoculated	8	4	12
Total	10	14	24

Solution: Let H_0: Inoculation is not effective against the disease.

Since, the cell frequencies are less than 5, applying Yate's correction for continuity the corrected table is

Animals	*Dead*	*Survived*	*Total*
Inoculated	2.5	9.5	12
Not inoculated	7.5	4.5	12
Total	10	14	24

Under H_0 the expected frequencies are

$$E_{11} = \frac{10 \times 12}{24} = 5, \quad E_{12} = \frac{14 \times 12}{24} = 7, \quad E_{21} = \frac{10 \times 12}{24} = 5, \quad E_{22} = \frac{14 \times 12}{24} = 7.$$

$$\text{Hence, } \chi^2 = \frac{(2.5)^2}{5} + \frac{(2.5)^2}{7} + \frac{(2.5)^2}{5} + \frac{(2.5)^2}{7} = 4.286.$$

Degrees of freedom, $\nu = (2-1)(2-1) = 1$.

From Table III, $\chi^2_{1[.05]} = 3.841$. Since χ^2 calculated is greater than χ^2 tabulated, thus null hypothesis is rejected at 5% level of significance, that is, inoculation may be considered to be effective against the disease.

EXERCISE 25.5

1. The following figures show the distribution of digits in numbers chosen at random from a telephone directory:

Digits	0	1	2	3	4	5	6	7	8	9	Total
Frequency	1026	1107	997	966	1075	933	1107	972	964	853	10,000

Test the hypothesis that digits occur with equal frequency in the directory.

2. A survey of 800 families with four children each recorded the following distribution:

No. of boys	0	1	2	3	4
No. of girls	4	3	2	1	0
No. of families	32	178	290	236	64

Test the hypothesis that male and female births are equally likely.

3. The following data give the life of 40 similar car batteries recorded to the nearest length of years

Class	1.5-1.9	2.0-2.4	2.5-2.9	3.0-3.4	3.5-3.9	4.0-4.4	4.5-4.9
Frequency	2	1	4	15	10	5	3

Test the hypothesis that the frequency distribution of battery lives may be approximated by a normal distribution with mean $\mu = 3.5$ and S.D. $\sigma = 0.7$.

4. The distribution of printing mistakes in the proof of first 392 pages of a book under publishing were found to be as follows:

No. of mistakes in a page (x)	0	1	2	3	4	5	6
No. of pages (f)	275	72	30	7	5	2	1

Fit a Poisson distribution to this data and test the goodness of fit.

5. To determine whether there really is a relationship between an employee's performance in the company's training programme and his or her ultimate success in the job, the company takes a sample of 400 cases and obtains the results shown in the following table:

Success in job	*Performance during training*			*Total*
	Below average	*Average*	*Above average*	
Poor	23	60	29	112
Average	28	79	60	167
Very good	9	49	63	121
Total	60	188	152	400

At $\alpha = 0.01$, test the null hypothesis that performace in the training program and success in the job are independent.

6. To determine the response about student's uniform in the professional colleges, a survey was conducted in four colleges in a metro. The following table gives the response of 200 students from college *A*, 150 students from college *B* and 100 students each from colleges *C* and *D*:

Response	*College*				*Total*
	A	*B*	*C*	*D*	
For	65	66	40	34	205
Against	42	30	33	42	147
Undecided	93	54	27	24	198
Total	200	150	100	100	550

Test for homogeneity of responses among the four colleges concerning student's uniform in the professional colleges.

7. A random chosen group of 20,000 non-smokers and one of 10,000 smokers were observed over a 10-year period. The following data relate the numbers of them that developed lung cancer during that period.

	Smokers	*Non-smokers*	*Total*
Lung cancer	62	14	76
No lung cancer	9,938	19,986	29,924
Total	10,000	20,000	30,000

At $\alpha = 0.01$, test the hypothesis that smoking and lung cancer are independent.

8. To study whether or not the level of earning is affected by educational attainment, a social scientist randomly selected 100 people from each of three income categories 'low', 'middle', 'high' and then recorded their educational attainment as in the following table:

	Income categories			
Educational attainment	*Low*	*Middle*	*High*	*Total*
No college	32	20	23	75
UG	13	16	1	30
PG	43	51	60	154
Doctoral	12	13	16	41
Total	100	100	100	300

Do these data indicate that the level of earning is affected by educational attainment? Test at $\alpha = 0.01$.

9. In an experiment on immunization of cattle from tuberculosis the following results were obtained:

	Affected	*Unaffected*
Inoculated	12	26
Not inoculated	16	06

Examine the effect of vaccine in controlling susceptibility to tuberculosis.

10. In an experiment on the immunization of goats from anthrax the following results were obtained:

	Died	*Survived*
Inoculated	2	10
Not inoculated	6	6

Give your conclusion on the efficiency of the vaccine.

25.11 TESTING HOMOGENEITY OF MEANS: ANALYSIS OF VARIANCE

In Section 25.7, we have studied how t-test enables us to test the significance of the difference between two sample means of two independent populations. Sometimes we encounter situations in which we wish to test the differences among three or more independent sample means rather than just two. In such a situation, the technique known as *Analysis of Variance* (ANOVA) developed by Prof. R.A. Fisher is applied. It consists of splitting the total variation into component variations attributed to independent factors of interest to the experimenter. If the experiment has been properly designed, then these component variations can be used to study the effects of the various factors on the variable of interset.

The assumptions required for an analysis of variance are similar to those required for the Student's t and Snadecor's F-tests. We assume that underlying populations are normally distributed with common variance σ^2.

25.11.1 One-Way Analysis of Variance

In this experimental design, independent random samples of sizes $n_1, n_2, \ldots, n_k$ are selected from k normal populations with means $\mu_1, \mu_2, \ldots, \mu_k$ and common variance σ^2. The k different populations are classified on the basis of a *single criterion* only such as different treatments or groups, e.g. different regions, different seeds or different machines, etc. The null hypothesis to be tested is

$$H_0: \mu_1 = \mu_2 = \ldots = \mu_k$$

against the alternative

H_1: At least two of the means are not equal.

Let x_{ij} be the jth observation ($j = 1, 2, \ldots, n_i$) of the ith sample ($i = 1, 2, \ldots, k$). The total variation in the experiment, measured by the quantity *total sum of squares* (TSS), is given by

$$TSS = \sum_i \sum_j (x_{ij} - \bar{x})^2, \qquad \ldots(25.42)$$

where $N = n_1 + n_2 + \ldots + n_k$ is the total number of observations and $\bar{x} = \frac{1}{N}\sum_i \sum_j x_{ij}$.

This total variation among x_{ij}'s may be attributed to the following two components.

The first component of variation is the variation *between k sample means* (SSB) and is given by

$$SSB = \sum_i n_i(\bar{x}_i - \bar{x})^2 \qquad \ldots(25.43)$$

where $\bar{x}_i = \frac{1}{n_i}\sum_j x_{ij}$ is the mean of the sample from the ith population.

The second component of variation is the variation *within the k samples* (SSW) and is given by

$$SSW = \sum_i \sum_j (x_{ij} - \bar{x}_i)^2. \qquad \ldots(25.44)$$

We can show very easily that the three sums given by (25.42), (25.43) and (25.44) are not independent and

$$TSS = SSB + SSW;$$

hence we need to calculate only two of these three variations.

The degrees of freedom (df) for TSS are $(N - 1)$, since it involves N squared observations. Similarly the degrees of freedom for SSB are $k - 1$ and the degrees of freedom for SSW are

$$\sum_{i=1}^{k}(n_i - 1) = N - k, \text{ and, we observe that}$$

$$df(TSS) = df(SSB) + df(SSW).$$

Thus the degrees of freedom are additive. Further each mean square (MS), obtained by dividing each sum of square by its respective df, that is,

$$MSS = \frac{TSS}{N-1}, \quad MSB = \frac{SSB}{K-1} \text{ and } \quad MSW = \frac{SSW}{N-K}$$

provides an unbiased estimate of the *total variation in the experiment*.

The ratio of the mean square between the class means (MSB) to the mean square within the classes (MSW) can be tested using the F-test, where the statistic F given by

$$F = \frac{\sum_i n_i(\bar{x}_i - \bar{x})^2}{\sum_i \sum_j (x_{ij} - \bar{x}_i)^2} \frac{N-K}{K-1} \qquad ...(25.45)$$

is distributed like a F-variate with $\upsilon_1 = k-1$ and $\upsilon_2 = N-k$ degrees of freedom.

The ANOVA table for the one-way classification is given below.

ANOVA table for one-way classification

Source of variation	df	SS	MS	F
Between classes	$k-1$	SSB	$MSB = SSB/(k-1)$	$\frac{MSB}{MSW} \sim F_{(k-1, N-k)}$
Within classes	$N-k$	SSW	$MSW = SSW/(N-k)$	
Total	$N-1$	TSS		

In case the tabulated value of F for $(k-1)$, $(N-k)$ degrees of freedom at the specified level α is less than the calculated value of F, then the hypothesis is accpeted at level α, otherwise it is rejected.

Remarks:

1. Since all the three sum of squares are independent of change of origin and hence origin may be shifted to any convenient point.
2. The TSS in (25.42) can be expressed as

$$TSS = \sum_i \sum_j x_{ij}^2 - \frac{G^2}{N} \qquad ...(25.46)$$

where $G = \sum_i \sum_j x_{ij}$ is the grand total of all the N observations.

3. The SSB in (25.43) may be expressed as

$$SSB = \sum_i \frac{T_i^2}{n_i} - \frac{G^2}{N},\ T_i = \sum_j x_{ij} \qquad ...(25.47)$$

4. The underlying experimental design in case of one-way classification is known as *completely randomized design* (CRD).

Example 25.42: The three drying techniques for curing a glue were studied and the following times were observed:

Formula *A*:	13	10	8	11	8	
Formula *B*:	13	11	14	14		
Formula *C*:	4	1	3	4	2	4

At $\alpha = 0.01$, test the hypothesis that the average times for the three formulae are same.

Solution: We form the following table:

Techniques	Drying Times						$T_i = \sum_j x_{ij}$	$\sum_j x_{ij}^2$
I	13	10	8	11	8		50	518
II	13	11	14	14			52	682
III	4	1	3	4	2	4	18	62
Total							120	1262

Here, $k = 3, n_1 = 5, n_2 = 4, n_3 = 6$, and thus $N = 15$. From the table,

$$G = \sum_i \sum_j x_{ij} = 120 \text{ and } \sum_i \sum_j x_{ij}^2 = 1262. \text{ Thus}$$

$$TSS = \sum_i \sum_j x_{ij}^2 - \frac{G^2}{N} = 1262 - \frac{(120)^2}{15} = 1262 - 960 = 302.$$

$$SSB = \sum_i \frac{T_i^2}{n_i} - \frac{G^2}{N} = \frac{(50)^2}{5} + \frac{(52)^2}{4} + \frac{(18)^2}{6} - \frac{(120)^2}{15} = 500 + 676 + 54 - 960 = 270$$

and,
$$SSW = TSS - SSB = 302 - 270 = 32.$$

The ANOVA table is

Source of variation	df	SS	MS	F
Between classes	2	270	135	50.56
Within classes	12	32	2.67	
Total	14	302		

From Table IVB, $F_{(2,12)(.01)} = 6.93$. Since the value of the test statistic calculated is greater than the value tabulated, the hypothesis that average drying timings are equal in case of the three techniques is rejected.

Example 25.43: Following data represents the total mileages obtained by the vehicles of the same type run on three different gas fuels

Gas 1:	220	251	226	246	260
Gas 2:	244	235	232	242	225
Gas 3:	252	272	250	238	256

At $\alpha = 0.05$, test the hypothesis that the average mileage obtained is not affected by the type of the gas used.

Solution: Shifting the origin to the point 220, we form the following table:

Gas	Mileage					$T_i = \sum_j x_{ij}$	$\sum_j x_{ij}^2$
1	0	31	6	26	40	103	3273
2	24	15	12	22	05	78	1454
3	32	52	30	18	36	168	6248
Total						349	10975

Here, $k = 3$, $n_1 = n_2 = n_3 = 5$, and thus $N = 15$

From the table, $G = \sum_i \sum_j x_{ij} = 349$, and $\sum_i \sum_j x_{ij}^2 = 10975$. Thus,

$$TSS = \sum_i \sum_j x_{ij}^2 - \frac{G^2}{N} = 10975 - \frac{(349)^2}{15} = 10975 - 8120.07 = 2854.93.$$

$$SSB = \sum_i \frac{T_i^2}{n_i} - \frac{G^2}{N} = \frac{(103)^2 + (78)^2 + (168)^2}{5} - \frac{(349)^2}{15} = 8983.40 - 8120.07 = 863.33$$

and,
$$SSW = TSS - SSB = 2854.93 - 863.33 = 1991.60$$

The ANOVA table is

Source of variation	df	SS	MS	F
Between classes	2	863.33	431.67	2.74
Within classes	12	1991.60	157.72	
Total	14	2854.93		

From Table IVB, $F_{(2,\ 12)(.05)} = 3.88$. Since the value of the test statistic calculated does not exceed the tabulated value, thus the hypothesis that the average mileage is not affected by the type of gas used may be accepted.

25.11.2 Two-Way Analysis of Variance

One-way analysis of variance is employed when the experimental units are homogeneous in respect to their configuration and there is only one factor which might influence the response and any other variation in the response is due to random chances or experimental errors. But sometimes the units under study are not homogeneous and are likely to add their own variability to the response. To isolate this source of variation units are divided into relatively homogeneous *classes* (*blocks*) and one unit within each class is randomly subjected to one specific *factor* (*treatment*). Thus the response of each experimental unit is affected by two considerations, one, because of block and, second, because of treatment, and the analysis is called *two-way analysis of variance*.

The complete data can be arranged in m rows and n columns and ANOVA is employed to test the independence of row or/and column factor levels as explained next.

Consider a set of $N = mn$ experimental units arranged in m rows and n columns. Let x_{ij} $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ be the response of the (i, j)th experimental unit. The *total sum of square* (*TSS*) is given by

$$TSS = \sum_i \sum_j (x_{ij} - \bar{x})^2, \qquad \ldots(25.48)$$

where $\bar{x} = \frac{1}{N} \sum_i \sum_j x_{ij}$ is the mean response.

The total variation among x_{ij}'s in this case is attributed to the following three components:

1. The variation *between the rows (SSR)*, given by

$$SSR = n \sum_{i=1}^{m} (\bar{x}_{i.} - \bar{x})^2; \quad \bar{x}_{i.} = \frac{1}{n} \sum_{j=1}^{n} x_{ij}. \qquad \ldots(25.49)$$

2. The variation *between the columns (SSC)*, given by

$$SSC = m \sum_{j=1}^{n} (\bar{x}_{.j} - \bar{x})^2; \quad \bar{x}_{.j} = \frac{1}{m} \sum_{i=1}^{m} x_{ij}. \qquad \ldots(25.50)$$

3. The variation due to *random chances* or *experimental errors (SSE)*, given by

$$SSE = TSS - SSR - SSC. \qquad \ldots(25.51)$$

The degrees of freedom for *TSS* are $mn - 1$, for *SSR* are $m - 1$, for *SSC* are $n - 1$, and finally since the degrees of freedom are additive, therefore, the degrees of freedom for *SSE* are

$$(mn - 1) - (m - 1) - (n - 1) = (m - 1)(n - 1).$$

Further, each mean square (*MS*) obtained by dividing a sum of square with its *df* provides an unbiased estimate of the variation in the experiment. Thus the ANOVA table for the two-way classification is given by

Source of variation	df	SS	MS	F
Row	$m-1$	SSR	$MSR = SSR/(m-1)$	$F_R = MSR/MSE \sim F_{(m-1),(m-1)(n+1)}$
Column	$n-1$	SSC	$MSC = SSC/(n-1)$	$F_C = MSC/MSE \sim F_{(n-1),(m-1)(n-1)}$
Errors	$(m-1)(n-1)$	SSE	$MSE = SSE/(m-1)(n-1)$	
Total	$mn-1$	TSS		

In case the tabulated value of F for $(m-1), (m-1)(n-1)$ degrees of freedom at level α is greater than the calculated $F_R = MSR/MSF$, then there is no significant difference between the row factor levels. Similarly, by comparing $F_C = MSC/MSE$ with the corresponding tabulated value, we can draw conclusion about column's factor levels.

Remarks:

1. To simplify the calculation various sum of squares can be expressed as follows.

(a) $TSS = \sum_i \sum_j x_{ij}^2 - \frac{G^2}{N}$, ...(25.52)

(b) $SSR = \sum_{i=1}^{m} \frac{R_i^2}{n} - \frac{G^2}{N}$, ...(25.53)

(c) $SSC = \sum_{j=1}^{n} \frac{C_j^2}{m} - \frac{G^2}{N}$, ...(25.54)

where $G = \sum_i \sum_j x_{ij}$ is the grand total of $N = mn$ observations, $R_i = \sum_{j=1}^{n} x_{ij}$, $i = 1, 2, \ldots, m$ are the row totals, and $C_j = \sum_{i=1}^{m} x_{ij}$, $j = 1, 2, \ldots n$ are the column totals.

2. The underlining experimental design in case of two-way classification is known as *randomized block design (RBD)*.

Example 25.44: In an experiment to study the performance of 4 different detergents for cleaning fuel injectors of 3 different models of engines, following data was obtained:

Detergent	*Engine*		
	1	*2*	*3*
A	45	43	51
B	47	46	52
C	48	50	55
D	42	37	49

Obtain the ANOVA table and test at $\alpha = .05$ whether there are difference in the detergent or in the engines.

Solution: Shifting the origin at the point 45, we obtain the table as

Detergent	*Engine*			*Row total* (R_i)
	1	*2*	*3*	
A	0	−2	6	4
B	2	1	7	10
C	3	5	10	18
D	−3	−8	4	−7
Col. total (C_j)	2	−4	27	25