Forced Oscillation - Resonance.

If we apply external force to damped oscillator, then the applied force tends to keep the oscillator oscillating. The force needs to be periodic and not necessarily equal to the natural freq. ω_0.

If the freq. of applied force is same as the natural freq. of the oscillator, the phenomenon of resonance or resonante absorption occurs. In such cases, the amp. of oscillation increases enormously.

Differential eqn. of forced oscillations.

Consider a body of mass 'm' under-going damped harmonic motion. Suppose ext. force $F_e = F_0 \sin pt$ is applied on it.

Freq. of applied force $= p/2\pi$

Amp " " " $= F_0$.

Forces acting on the oscillator :-

1) Restoring force $= -kx$

2) Damping force $= -b\, dx/dx$

3) Applied force $= F_0 \sin pt$

$\therefore$ Net force $F = F_0 \sin pt - b\frac{dx}{dt} - kx$

or $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \sin pt$

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt$$

where $\frac{b}{m} = 2r$, $\frac{k}{m} = \omega_0^2$, $\frac{F_0}{m} = f_0$

(1)

If RHS $= 0$, then eqn. becomes same as damped oscillator.

(2) :-

Steady state function -

Let $x = A\sin(pt - \phi)$

be a particular soln. where ϕ is the phase difference b/w the applied force and displacement of the oscillator.

$$\frac{dx}{dt} = pA\cos(pt-\phi)$$

$$\frac{d^2x}{dt^2} = -p^2 A\sin(pt-\phi)$$

Substituting in diff. eqn :-

$$-p^2 A\sin(pt-\phi) + 2\gamma pA\cos(pt-\phi) + \omega_0^2 A\sin(pt-\phi) = f_0\sin(pt-\phi+\phi)$$

$$A(\omega_0^2-p^2)\sin(pt-\phi) + 2\gamma pA\cos(pt-\phi) = f_0\sin(pt-\phi)\cos\phi + f_0\cos(pt-\phi)\sin\phi$$

If such an eqn. has to be satified, then each term has to be equal on both sides as :-

$$A(\omega_0^2-p^2) = f_0\cos\phi \quad \text{—} \; (1)$$

$$2\gamma pA = f_0\sin\phi \quad \text{—} \; (2)$$

Squaring and adding the above two :-

$$A^2\{(\omega_0^2-p^2)^2 + 4\gamma^2p^2\} = f_0^2$$

or

$$A = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2 + 4\gamma^2p^2}} \quad \text{—} \; (3)$$

This is the amplitude of the forced oscillator.

Again dividing (1) by (2)

$$\tan\phi = \frac{2rp}{\omega_0^2 - p^2}$$

∴ phase difference b/w forced oscillator and the applied force

$$\phi = \tan^{-1}\left(\frac{2rp}{\omega_0^2 - p^2}\right)$$

And finally.

$$x = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4r^2p^2}} \sin(pt - \phi)$$

which is the soln. of forced osc. having same freq $= p/2\pi$ but lagging behind in phase by ϕ.

Amplitude Resonance :

Acc. to eqn (3), the amplitude depends on $\omega_0 - p$, the difference in freq. b/w oscillator and force applied. Smaller the differences larger the amplitude.

CASE-I : At very low driving freq $p << \omega_0$, then

$$A \simeq \frac{f_0}{\omega_0} = \frac{F_0/m}{k/m} = \frac{F_0}{k}$$

which means amp depends only on force const.

CASE-II : At very high driving freq $p >> \omega_0$, then

$$A \simeq \frac{f_0}{p^2} = \frac{F_0/m}{p^2} = \frac{F_0}{mp^2}$$

amp decreases ~~with~~ ~~as~~ p inc. and depends on mass.

CASE-III :-

Resonant driving frequency

The freq. at which the amp. of the oscillator is maximum is k/a resonant frequency. For this denominator of Amp. should be minimum :-

$$\frac{d}{dp}\left[(\omega_0^2 - p^2)^2 + 4r^2p^2\right] = 0$$

$$2(\omega_0^2 - p^2)(-2p) + 4r^2(2p) = 0$$

or $$\omega_0^2 - p^2 = 2r^2$$

$$\boxed{p = p_R = \sqrt{\omega_0^2 - 2r^2}}$$

$\therefore$ freq $= \dfrac{p_R}{2\pi} = \dfrac{\sqrt{\omega_0^2 - 2r^2}}{2\pi}$

which is slightly less than natural frequency $\omega_0 / 2\pi$ and than damped freq $\dfrac{\sqrt{\omega_0^2 - r^2}}{2\pi}$.

$$A_{max} = \frac{f_0}{2r(\omega_0^2 - r^2)^{1/2}}$$

Also $\omega_0^2 - 2r^2 = p^2$ ~~or~~ so,

$$A_{max} = \frac{f_0}{2r(p^2 + r^2)^{1/2}}$$

showing that smaller the value of r, greater the value of A_{max}.

When damping is small; $r \to 0$, the resonant freq. $\simeq$ natural freq.. In ideal case, when there is no damping, amp will become infinite but since damping is never zero, so amp finite and controls the amp.

Phase difference :-

$$\tan\phi = \frac{2rp}{\omega_0^2 - p^2}$$

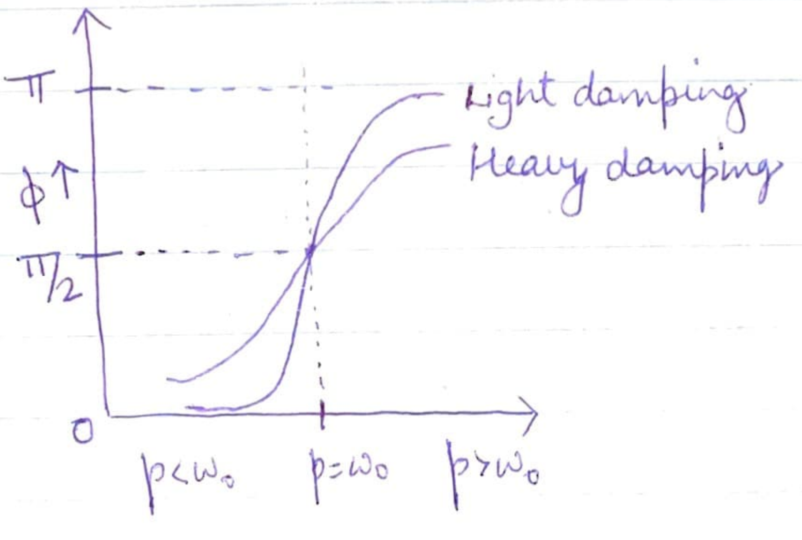
1) At low driving freq, $p << \omega_0$, $\tan\phi$ is small and positive, which means osc. is nearly in phase with driving force.

2) At high freq $p >> \omega_0$, $\tan\phi$ = ~~∞~~ ~~or $\phi = \pi/2$, lag~~. small and negative. ∴ out of phase $\simeq \pi$

3. At resonance $p = \omega_0$ $\tan\phi = \infty$ or $\phi = \pi/2$. Displacement lags behind the driving force by $\pi/2$.

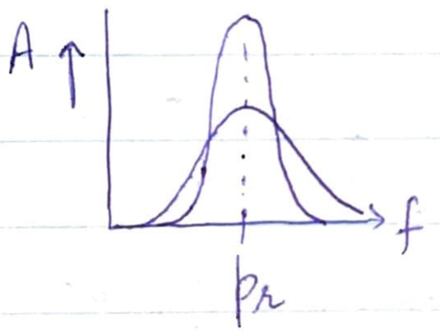
So phase angle changes from 0 to π.

Variation of phase lag with increase of driving frequency :-



Sharpness of resonanace -

Amp. of forced oscillations attains peak value when freq. of applied force = resonance freq. But below and above it, the amp falls. If fall in amp. for a small change in freq. from resonant value is high, then the resonance is said to be sharp, while if the fall in amp is small, then the resonance is said to be flat.

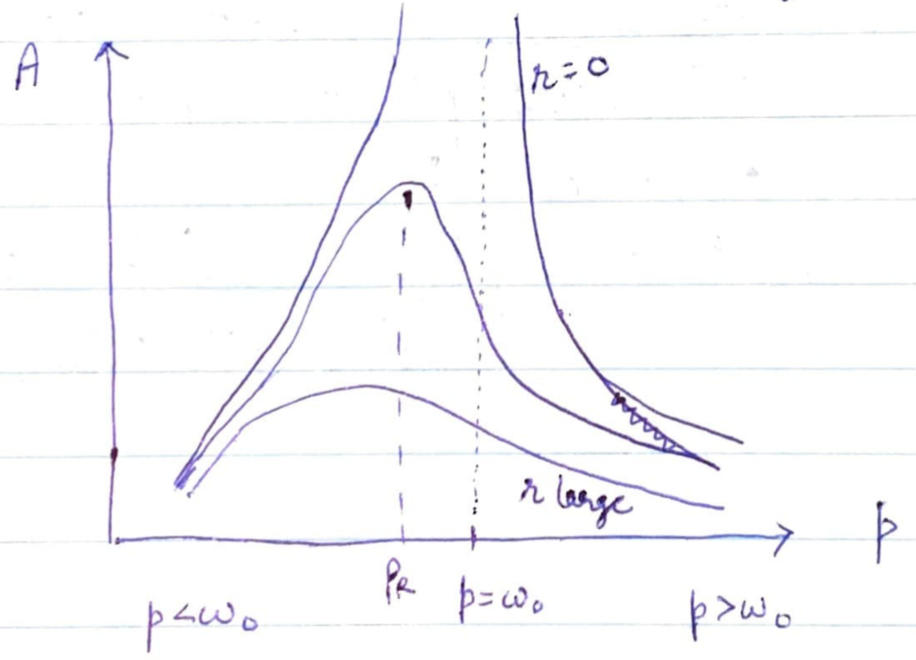


Effect of damping on the sharpness of resonance -

$$A = \frac{f_0^2}{\sqrt{(\omega_0^2 - p^2)^2 + 4r^2p^2}}$$

1. Smaller the damping, sharper the resonance and vice-versa

2. At low driving freq, amp is nearly the same for all values of damping.
3. As p increases, A increases. A increases more for low 'r'. At $r=0$, $A=\infty$ at resonance.
4. Further increase in p beyond p_r reduces the amp.
5. With increase in damping, peak moves to the left.
6. With inc in 'r', peak flattens.



Power absorption by forced oscillator

Avg. power absorbed per cycle is equal to the avg. power dissipated after steady state is reached. To find the expression:-

$$x = \frac{f_0}{[(\omega_0^2 - p^2)^2 + 4r^2p^2]^{\frac{1}{2}}} \sin(pt - \phi)$$

where $f_0 = \frac{F_0}{m}$ and so on ---

Instantaneous velocity :-

$$v = \frac{dx}{dt} = \frac{f_0 p}{[(\omega_0^2 - p^2)^2 + 4r^2p^2]^{1/2}} \cos(pt - \phi)$$

Power absorbed :

$$P = Fv = (F_0 \sin pt) \cdot \frac{f_0 p}{[(\omega_0^2 - p^2)^2 + 4r^2p^2]^{1/2}} \cos(pt - \phi)$$

$$= \frac{m f_0^2 p}{[(\omega_0^2 - p^2)^2 + 4r^2p^2]^{1/2}} \left(\sin pt \cos pt \cos\phi + \sin^2 pt \sin\phi \right)$$

Taking integral from 0 to T and find avg.

$$P_{av} = \frac{m f_0^2 p}{[(\omega_0^2 - p^2)^2 + 4r^2p^2]^{1/2}} \left(\frac{1}{2} \sin\phi \right)$$

Since $\tan\phi = \frac{2rp}{\omega_0^2 - p^2} \Rightarrow \sin\phi = \frac{2rp}{\sqrt{(\omega_0^2 - p^2)^2 + 4r^2p^2}}$

from eqn 1

$\therefore P_{av} = \dfrac{m f_0^2 r p^2}{(\omega_0^2 - p^2)^2 + 4r^2 p^2}$

which is the exp. for avg. power absorbed by the oscillator.

Power dissipation by Driven Oscillator:

The power absorbed from the driving force is dissipated in doing work against the damping force $-b\dfrac{dx}{dt}$ or $-2mr\dfrac{dx}{dt}$.

The rate of doing work or instantaneous power P' against the damping force is given by:

$$P' = \left(2mr\frac{dx}{dt}\right)\cdot\frac{dx}{dt} = 2mr\left(\frac{dx}{dt}\right)^2$$

$$= 2mr \frac{f_0^2 p^2}{(\omega_0^2 - p^2)^2 + 4r^2p^2} \cos^2(pt-\phi)$$

and

$$P'_{av} = \frac{m f_0^2 r p^2}{(\omega_0^2 - p^2)^2 + 4r^2p^2}$$

$$\therefore \boxed{P_{av} = P'_{av}}$$

Maximum power absorption

The denominator should be min and for that

$$(\omega_0^2 - p^2)^2 = 0$$

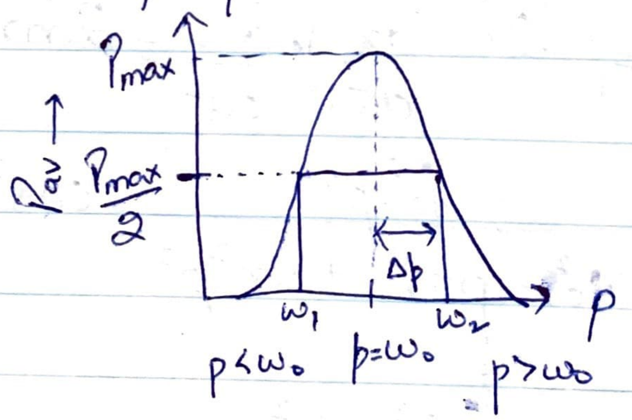
$$p = \omega_0 .$$

And $P_{av}(max) = \dfrac{m f_0^2 r p^2}{4 r^2 p^2} = \dfrac{m f_0^2}{4r}$

$$= \frac{m f_0^2}{2b/m} = \frac{m^2 f_0^2}{2b}.$$

Bandwidth of resonance curve :

As seen above max power is absorbed at ~~the~~ natural freq. ω_0. So, Power vs. freq. curve is as :



ω_1 and ω_2 correspond to the value $= \dfrac{P_{max}}{2}$

$$P_{av}(max) = \frac{mf_0^2}{4r}$$

At ω_1 and ω_2 $P_{av} = \frac{1}{8} P_{ar}(max)$

or

$$\frac{mf_0^2 r p^2}{(\omega_0^2 - p^2)^2 + 4r^2p^2} = \frac{1}{2} \cdot \frac{mf_0^2}{4r}.$$

$$(\omega_0^2 - p^2)^2 + 4r^2p^2 = 8r^2p^2$$

$$(\omega_0^2 - p^2)^2 = 4r^2p^2$$

$$\omega_0^2 - p^2 = \pm 2rp$$

$$\omega_0 - p = \pm \frac{2rp}{\omega_0 + p}.$$

If $\omega_2 - \omega_1$ is the bandwidth of oscillator then

$$\omega_2 - p = \frac{2rp}{\omega_0 + p}$$

$$\omega_1 - p = -\frac{2rp}{\omega_0 + p}$$

$$\omega_2 - \omega_1 = \frac{4rp}{\omega_0 + p} = \frac{4r}{\frac{\omega_0}{p} + 1} = 2r$$

or $\omega_2 - \omega_1 = \frac{1}{\tau} =$ Relaxation time

Smaller the bandwidth, sharper is the resonance & vice-versa.

Since $\omega_2 - \omega_1 = 2\gamma$.

$$|\omega_0 - \omega_1| = |\omega_2 - \omega_0| = \Delta p = \gamma$$

This change Δp in the value of driving freq. p for which the avg. power absorbed by the driven oscillator falls from its max value P_{max} to half this value is the half width of the avg. power absorbed.

$$\therefore \Delta p = \frac{1}{2\tau}$$

<u>Quality factor in Terms of half-bandwith.</u>

$$Q = \frac{2\pi E}{PT}$$

$$\text{Energy} = K.E + P.E = \frac{1}{2} m p^2 A^2 \cos^2(pt - \phi) + \frac{1}{2} m \omega_0^2 A^2 \sin^2(pt - \phi)$$

$$= \frac{1}{4} m p^2 A^2 + \frac{1}{4} m \omega_0^2 A^2$$

$$\therefore Q = \frac{2\pi \left[\frac{1}{4} m A^2 (p^2 + \omega_0^2)\right]}{\left(\frac{m A^2 p^2}{2\tau}\right) \times \underset{2\pi/p}{T}}$$

$$Q = \frac{1}{2}\frac{(p^2+\omega_0^2)}{p}\cdot\tau = \frac{1}{2}\left(1+\frac{\omega_0^2}{p^2}\right)p\tau$$

At resonance $p = p_R = \omega_0$

$$Q = \frac{1}{2}\times 2p\tau = p\tau = \omega_0\tau = \frac{\omega_0}{2\Delta p}.$$

At low damping, τ is large so Q-factor is large, making resonance sharp.

Velocity Resonance

$$x = A\sin(pt-\phi) = \frac{f_0\sin(pt-\phi)}{\sqrt{(\omega_0^2-p^2)^2+4r^2p^2}}$$

$$v = \frac{dx}{dt} = \frac{f_0 p}{\sqrt{(\omega_0^2-p^2)^2+4r^2p^2}}\cos(pt-\phi)$$

$$v = pA\cos(pt-\phi) = v_0\cos(pt-\phi)$$

where $v_0 = pA = \dfrac{f_0 p}{\sqrt{(\omega_0^2-p^2)^2+4r^2p^2}}$

This is the velocity amp. of a driven harmonic oscillator.

Using v_0, the avg. Power

$$P_{av} = mrv_0^2 = \frac{mv_0^2}{2\tau}$$

$$P_{av} = \frac{mp^2A^2}{2\tau}$$