ASSIGNMENT-2

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(1) $\sum_{n=1}^{\infty} \frac{1}{e^n} - \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{e^n} - \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(i) $\sum_{n=1}^{\infty} \frac{1}{e^n}$, By ~~log~~ ratio test : $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{e^{n+1}}{e^n}$

$\Rightarrow \lim_{n\to\infty} e = e > 1$ $\therefore$ convergent.

(ii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, By p series test, it is divergent since $p \leq 1$.

since (i) is convergent and (ii) is divergent
$\therefore$ (i) − (ii) will be <u>divergent</u>.
OR Use Integral Test

(2) $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{2} \right)$

(i) $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \text{———}$

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = 1$ (Convergent)

(ii) $\sum_{n=1}^{\infty} \frac{1}{2} = \lim_{n\to\infty} \frac{n}{2} = \infty$ (Divergent)

since (i) is convergent and (ii) is divergent
$\therefore$ (i) − (ii) will be <u>divergent</u>.

(3) $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$, $x > 0$

(i) $\Rightarrow$ If $x = 1$: Series : $\frac{1}{2} + \frac{1}{2} + \text{———} \infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2} = \lim_{n\to\infty} \frac{n}{2} = \infty$ (Divergent)

(ii) $u_n = \frac{x^n}{x^{2n}+1}$, $\cancel{u_{n+1} = \frac{x^{n+1}}{x^{2(n+1)}+1}}$

$\cancel{\Rightarrow \text{By Ratio test, } \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{(x^{2(n+1)}+1)}{(x^{2n}+1)} \times \frac{1}{x}}$

$\Rightarrow$ Apply cauchy's root test,

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{x^n}{x^{2n}+1}\right)^{1/n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{x}{(x^{2n}+1)^{1/n}} = x \cdot \lim_{n\to\infty} \frac{1}{(x^{2n}+1)^{1/n}} = x$$

$\therefore$ Convergent if $x < 1$

Divergent if $x > 1$

(4) $\sum_{n=1}^{\infty} \frac{1}{n^p (n+1)^q}$: $u_n = \frac{1}{n^p (n+1)^q}$

Let $v_n = \frac{1}{n^{p+q}}$, Apply comparison test

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{\cancel{n^p}(n+1)^q} \times \cancel{n^p} \cdot n^q \Rightarrow \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^q} = 1$$

Finite non zero

But, $\sum v_n = \frac{1}{n^{p+q}}$ } p series

If $p+q > 1 \Rightarrow v_n$ and u_n both are convergent.

If $p+q \leq 1 \Rightarrow v_n$ and u_n both are divergent.

(5) Show that $\sum_{n=1}^{\infty} \frac{\log n}{n^q}$ converges if $q>1$ and diverges if $q \le 1$.

(i) When $q=1$, $u_n = \frac{\log n}{n}$, On applying integral test,

$$\int_1^{\infty} \frac{\log x}{x} \cdot dx = \left.\frac{(\log x)^2}{2}\right]_1^{\infty} = \infty.$$

$\therefore$ Divergent at $x=1$.

(ii) Applying integral test, $\log x = t \Rightarrow x = e^t$

$$\Rightarrow \int_1^{\infty} \frac{t \cdot e^t \cdot dt}{e^{qt}} \Rightarrow \int_1^{\infty} t \cdot e^{(1-q)t} \cdot dt$$

$$\Rightarrow \left[\frac{t \cdot e^{(1-q)t}}{1-q} - \frac{e^{(1-q)t}}{(1-q)^2} \cdot q \right]_1^{\infty}$$

Converges when $q>1$.

Diverges when $q \le 1$

(6) $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$

$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$, By cauchy's test

$$\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{(n+1)}{n}\right]^{-1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)\left[\left(\frac{n+1}{n}\right)^n - 1\right]} \Rightarrow \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)\left[\left(1+\frac{1}{n}\right)^n - 1\right]}$$

$$\Rightarrow \frac{1}{(e-1)} < 1$$

$\therefore$ convergent

(7) $\sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$; $u_n = \frac{n!\,(2^n)}{n^n}$

$$u_{n+1} = \frac{(n+1)!\,(2)^{n+1}}{(n+1)^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{\cancel{n!}\,(2^n)}{n^n} \times \frac{(n+1)^{n+1}}{\cancel{(n+1)!}\;2^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \times \frac{1}{2} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \times \frac{1}{2}$$

$$= \frac{e}{2} > 1$$

$\therefore$ Convergent

(8) $1 + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(1+2\alpha)}{(1+\beta)(1+2\beta)} + \ldots$

$$u_n = \frac{(1+\alpha)(1+2\alpha) \text{———} (1+(n-1)\alpha)}{(1+\beta)(1+2\beta) \text{———} (1+(n-1)\beta)}$$

$$u_{n+1} = \frac{(1+\alpha)(1+2\alpha) \text{———} (1+n\alpha)}{(1+\beta)(1+2\beta) \text{———} (1+n\beta)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1+n\beta}{1+n\alpha} = \frac{\beta}{\alpha} > 1$$

$\rightarrow$ $\beta > \alpha > 0$ For convergent

$0 < \beta < \alpha$ - For divergent

If $\alpha = \beta$ $\rightarrow$ series : $1 + 1 + \text{———} \infty$

$\therefore$ Divergent for $\alpha = \beta$.

(9) $\frac{a+x}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \text{———}$

$u_n = \frac{(a+nx)^n}{n!}$; $u_{n+1} = \frac{(a+(n+1)x)^{n+1}}{(n+1)!}$

By ratio test,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \left(\frac{a+(n+1)x}{a+nx}\right)^n \frac{(a+(n+1)x)}{(n+1)}$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{\frac{a}{n}+\left(1+\frac{1}{n}\right)x}{\frac{a}{n}+x}\right]^n \left(\frac{a}{n+1}+x\right) \Rightarrow e\cdot x < 1$$

$\therefore$ $x < 1/e \Rightarrow$ <u>Convergent</u>.

$x > 1/e \Rightarrow$ <u>Divergent</u>.

For $x = 1/e$, Applying log test,

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} \Rightarrow \frac{1}{2} < 1$$

$\therefore$ <u>Divergent</u>.

(10) $\sum_{n=1}^{\infty} \frac{(n!)^2 x^{2n}}{(2n)!}$: $u_n = \frac{(n!)^2 x^{2n}}{(2n)!}$

$$u_{n+1} = \frac{[(n+1)!]^2 x^{2(n+1)}}{[2(n+1)]!}$$

By ratio test,

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{(n!)^2 x^{2n}}{(2n)!} \times \frac{[2(n+1)]!}{[(n+1)!]^2 x^{2(n+1)}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{(n+1)\cancel{^2}} \times \frac{1}{x^2} \times (2n+1)\cancel{(n+1)} \times 2$$

$$\Rightarrow \lim_{n\to\infty} \frac{2}{x^2} \times \frac{(2+1/n)}{(1+1/n)} = \frac{4}{x^2} > 1$$

$$\Rightarrow x^2 - 4 < 0 \Rightarrow (x+2)(x-2) < 0$$

$\therefore$ <u>Convergent</u> for $-2 < x < 2$

<u>Divergent</u> for $x > 2$ and $x < 2$

For $x = \pm 2$, Applying Raabe's Test,

$$\lim_{n\to\infty} n\left(\frac{\cancel{2}(2n+1)}{\cancel{4}\,2(n+1)} - 1\right) = n\left(\frac{2n+1-2n-2}{2n+1}\right)$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{-1}{2n+1}\right) = -\frac{1}{2} < 1$$

$\therefore$ <u>Divergent</u> for $x = \pm 2$

(11) $\sin\pi + \frac{1}{4}\sin\frac{\pi}{2} + \frac{1}{9}\sin\frac{\pi}{3}$

$U_n = \frac{1}{n^2} \times \sin\frac{\pi}{n}$

$\int_1^\infty \frac{1}{x^2} \cdot \sin\frac{\pi}{x} \cdot dx \rightarrow$ let $\frac{1}{x} = t$

$\rightarrow -x^{-2} \cdot dx = dt$

$\rightarrow -\int_1^0 \sin\pi t \cdot dt \rightarrow \frac{1}{\pi} \cos\pi t \Big]_1^0$

$\rightarrow \frac{1}{\pi}\left[\cos 0 - \cos\pi\right]$

$\rightarrow \frac{1}{\pi}[2] = \frac{2}{\pi}$ (Finite)

$\therefore$ convergent.