Problem:

Job-shop scheduling problem in a setting where **m** machines are separated by **(m-1)** bounded buffers. The objective is to schedule the tasks of **n** available jobs such that the total makespan of the jobs is minimum.

Every job goes through every machine.
The tasks in a job are constrained to be completed in a certain order.
Every task of a job takes a fixed amount of duration.
Every machine can carry out only one task at a time.
The buffers do not maintain the FIFO order of the jobs.

The machines and jobs are as follows –

	M_1	**M_2**	**…**	**M_m**
J_1	d_{11}	d_{12}	…	d_{1m}
J_2	d_{21}	d_{22}	…	d_{2m}
…	…	…	…	…
J_n	d_{n1}	d_{n2}	…	d_{nm}

And the buffers are bounded by the maximum capacities $b_1, b_2, b_3, \ldots, b_{(m-1)}$.

Solution:

Variables –

1. $t_{11}, t_{12}, \ldots, t_{nm}$
   Start times of the tasks
   O(nm)
2. $E_{11}, E_{12}, \ldots, E_{1(m-1)}, \ldots, E_{n(m-1)}$
   Enqueue operation times for the tasks
   O(nm)
3. $D_{12}, D_{13}, \ldots, D_{1m}, \ldots, D_{nm}$
   De-queue operation times for the tasks
   O(nm)
4. $B_{11}@E_{11}, B_{12}@E_{11}, \ldots, B_{1(m-1)}@E_{11}, B_{21}@E_{11}, \ldots, B_{n(m-1)}@E_{11},$
   $B_{11}@E_{12}, \ldots\ldots\ldots\ldots\ldots\ldots, B_{1(m-1)}@E_{12}, B_{21}@E_{12}, \ldots, B_{n(m-1)}@E_{12},$
   …,
   $B_{11}@E_{n(m-1)}, \ldots\ldots\ldots\ldots, B_{1(m-1)}@E_{n(m-1)}, B_{21}@E_{11}, \ldots, B_{n(m-1)}@E_{n(m-1)}$
   $O(n^2m^2)$

and

$B_{11}@D_{12}$, $B_{12}@D_{12}$, ..., $B_{1(m-1)}@D_{12}$, $B_{21}@D_{12}$, ..., $B_{n(m-1)}@D_{12}$,

$B_{11}@D_{13}$, ………………., $B_{1(m-1)}@D_{13}$, $B_{21}@D_{13}$, ..., $B_{n(m-1)}@D_{13}$,

…,

$B_{11}@D_{nm}$, ………………., $B_{1(m-1)}@D_{nm}$, $B_{21}@D_{nm}$, ..., $B_{n(m-1)}@D_{nm}$

$O(n^2m^2)$

"Job-present-at-buffer" variables.

$B_{pq}@E_{rs}$ can be described as "whether job p is at buffer q, at the time of enqueue operation of task s of job r.

$B_{pq}@D_{rs}$ can be described as "whether job p is at buffer q, at the time of dequeue operation of task s of job r.

Constraints –

1. $(t_{11} > 0) \land (t_{12} >= t_{11} + d_{11}) \land (t_{13} >= t_{12} + d_{12}) \land \ldots \land (t_{horizon} >= t_{1m} + d_{1m})$
   $\land (t_{21} > 0) \land (t_{22} >= t_{21} + d_{21}) \land (t_{23} >= t_{22} + d_{22}) \land \ldots \land (t_{horizon} >= t_{2m} + d_{2m})$
   $\land \ldots$
   $\land (t_{n1} > 0) \land (t_{n2} >= t_{n1} + d_{n1}) \land (t_{n3} >= t_{n2} + d_{n2}) \land \ldots \land (t_{horizon} >= t_{nm} + d_{nm})$
   Constraints to ensure the order of the tasks of job don't overlap.
   $O(nm)$
2. $((t_{11} >= t_{21} + d_{21}) \lor (t_{21} >= t_{11} + d_{11})) \land \ldots \land ((t_{n1} >= t_{(n-1)1} + d_{(n-1)1}) \lor (t_{(n-1)1} >= t_{n1} + d_{n1}))$
   $\land ((t_{12} >= t_{22} + d_{22}) \lor (t_{22} >= t_{12} + d_{12})) \land \ldots \land ((t_{n2} >= t_{(n-1)2} + d_{(n-1)2}) \lor (t_{(n-1)2} >= t_{n2} + d_{n2}))$
   $\land \ldots$
   $\land ((t_{1m} >= t_{2m} + d_{2m}) \lor (t_{2m} >= t_{1m} + d_{1m})) \land \ldots \land ((t_{nm} >= t_{(n-1)m} + d_{(n-1)m}) \lor (t_{(n-1)m} >= t_{nm} + d_{nm}))$
   Constraints to ensure that a machine does not get shared between tasks.
   $O(nm)$
3. $(E_{11} == t_{11} + d_{11}) \land (E_{12} == t_{12} + d_{12}) \land \ldots \land (E_{n(m-1)} == t_{n(m-1)} + d_{n(m-1)})$
   Equality constraints on enqueue times.
   $O(nm)$
4. $(D_{12} == t_{12}) \land (E_{13} == t_{13}) \land \ldots \land (E_{nm} == t_{nm})$
   Equality constraints on dequeue times.
   $O(nm)$
5. $B_{11}@E_{11} = 1$ if $E_{11} \le E_{11} < D_{12}$ (always true)
   0 otherwise
   $B_{12}@E_{11} = 1$ if $E_{12} \le E_{11} < D_{13}$
   0 otherwise
   …

ITE constraints on the "Job-present-at-buffer" variables at enqueue times.
$O(n^2m^2)$

6. $B_{11}@D_{12} = 1$ if $E_{11} \leq D_{12} < D_{12}$ (always false)
   0 otherwise
   $B_{12}@D_{12} = 1$ if $E_{12} \leq D_{12} < D_{13}$
   0 otherwise
   …
   ITE constraints on the "Job-present-at-buffer" variables at dequeue times.
   $O(n^2m^2)$
7. Σ ($B_{11}@E_{11}$, $B_{21}@E_{11}$, $B_{31}@E_{11,}$ …, $B_{n1}@E_{11}$) $\leq b_1$
   (i.e. Length of buffer 1 does not increase than its capacity at E_{11})
   Σ ($B_{11}@E_{21}$, $B_{21}@E_{21}$, $B_{31}@E_{21,}$ …, $B_{n1}@E_{21}$) $\leq b_1$
   (i.e. Length of buffer 1 does not increase than its capacity at E_{11})
   …
   Σ ($B_{11}@E_{n1}$, $B_{21}@E_{n1}$, $B_{31}@E_{n1,}$ …, $B_{n1}@E_{n1}$) $\leq b_1$
   (i.e. Length of buffer 1 does not increase than its capacity at E_{n1})
   …
   …
   (i.e. Length of buffer m-1 does not increase than its capacity at $E_{1(m-1)}$ to $E_{n(m-1)}$)
   $O(mn)$
8. Σ ($B_{11}@D_{12}$, $B_{21}@D_{12}$, $B_{31}@D_{12,}$ …, $B_{n1}@D_{12}$) > 0
   (i.e. Length of buffer 1 is not 0 at D_{11})
   …
   …
   $O(mn)$