**Problem:**

Job-shop scheduling problem in a setting where **m** machines are separated by **(m-1)** bounded buffers. The objective is to schedule the tasks of **n** available jobs such that the total makespan of the jobs is minimum.

Every job goes through every machine.   
The tasks in a job are constrained to be completed in a certain order.   
Every task of a job takes a fixed amount of duration.   
Every machine can carry out only one task at a time.   
The buffers do not maintain the FIFO order of the jobs.

The machines and jobs are as follows –

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **M1** | **M2** | **…** | **Mm** |
| **J1** | d11 | d12 | … | d1m |
| **J2** | d21 | d22 | … | d2m |
| **…** | … | … | … | … |
| **Jn** | dn1 | dn2 | … | dnm |

And the buffers are bounded by the maximum capacities b1, b2, b3, …, b(m-1).

**Solution:**

Variables –

1. t11, t12, …, tnm  
   Start times of the tasks   
   *O(nm)*
2. E11, E12, …, E1(m-1), …, En(m-1)  
   Enqueue operation times for the tasks  
   *O(nm)*
3. D12, D13, …, D1m, …, Dnm  
   De-queue operation times for the tasks  
   *O(nm)*
4. B11@E11, B12@E11, ..., B1(m-1)@E11, B21@E11, …, Bn(m-1)@E11,  
   B11@E12, …………….…, B1(m-1)@E12, B21@E12, …, Bn(m-1)@E12,  
   …,   
   B11@En(m-1), ……….…., B1(m-1)@En(m-1), B21@E11, …, Bn(m-1)@En(m-1)   
   *O(n­­2m2)*  
     
   and  
   B11@D12, B12@D12, ..., B1(m-1)@D12, B21@D12, …, Bn(m-1)@D12,  
   B11@D13, …………….…, B1(m-1)@D13, B21@D13, …, Bn(m-1)@D13,  
   …,   
   B11@Dnm, ……….….…., B1(m-1)@Dnm, B21@Dnm, …, Bn(m-1)@Dnm   
   *O(n­­2m2)*“Job-present-at-buffer” variables.   
   Bpq@Ers can be described as “whether job p is at buffer q, at the time of enqueue operation of task s of job r.   
   Bpq@Drs can be described as “whether job p is at buffer q, at the time of dequeue operation of task s of job r.

Constraints –

1. (t11 > 0) Λ (t12 >= t11 + d11) Λ (t13 >= t12 + d12) Λ … Λ (thorizon >= t1m + d1m)   
   Λ (t21 > 0) Λ (t22 >= t21 + d21) Λ (t23 >= t22 + d22) Λ … Λ (thorizon >= t2m + d2m)  
   Λ …  
   Λ (tn1 > 0) Λ (tn2 >= tn1 + dn1) Λ (tn3 >= tn2 + dn2) Λ … Λ (thorizon >= tnm + dnm)  
   Constraints to ensure the order of the tasks of job don’t overlap.  
   *O(nm)*
2. ((t11 >= t21 + d21) V (t21 >= t11 + d11)) Λ … Λ ((tn1 >= t(n-1)1 + d(n-1)1) V (t(n-1)1 >= tn1 + dn1))  
   Λ ((t12 >= t22 + d22) V (t22 >= t12 + d12)) Λ … Λ ((tn2 >= t(n-1)2 + d(n-1)2) V (t(n-1)2 >= tn2 + dn2))  
   Λ …  
   Λ ((t1m >= t2m + d2m) V (t2m >= t1m + d1m)) Λ … Λ ((tnm >= t(n-1)m + d(n-1)m) V (t(n-1)m >= tnm + dnm))  
   Constraints to ensure that a machine does not get shared between tasks.   
   *O(nm)*
3. (E11 == t11 + d11) Λ (E12 == t12 + d12) Λ … Λ (En(m-1) == tn(m-1) + dn(m-1))  
   Equality constraints on enqueue times.  
   *O(nm)*
4. (D12 == t12) Λ (D13 == t13) Λ … Λ (Dnm == tnm)  
   Equality constraints on dequeue times.  
   *O(nm)*
5. B11@E11 = 1 if E11 ≤ E11 < D12 (always true)  
    0 otherwise  
   B12@E11 = 1 if E12 ≤ E11 < D13 0 otherwise  
   …  
   ITE constraints on the “Job-present-at-buffer” variables at enqueue times.   
   *O(n2m2)*
6. B11@D12 = 1 if E11 ≤ D12 < D12 (always false)  
    0 otherwise  
   B12@D12 = 1 if E12 ≤ D12 < D13 0 otherwise  
   …  
   ITE constraints on the “Job-present-at-buffer” variables at dequeue times.  
   *O(n2m2)*
7. Σ (B11@E11, B21@E11, B31@E11, …, Bn1@E11) ≤ b1(i.e. Length of buffer 1 does not increase than its capacity at E11)  
   Σ (B11@E21, B21@E21, B31@E21, …, Bn1@E21) ≤ b1  
   (i.e. Length of buffer 1 does not increase than its capacity at E21)  
   …  
   Σ (B11@En1, B21@En1, B31@En1, …, Bn1@En1) ≤ b1  
   (i.e. Length of buffer 1 does not increase than its capacity at En1)  
   Σ (B12@E12, B22@E12, B32@E12, …, Bn2@E12) ≤ b2(i.e. Length of buffer 2 does not increase than its capacity at E12)  
   …  
   Σ (B12@En2, B22@En2, B32@En2, …, Bn2@En2) ≤ b2(i.e. Length of buffer 2 does not increase than its capacity at En2)  
   …  
   (i.e. Length of buffer m-1 does not increase than its capacity at En(m-1))  
   *O(mn)*
8. Σ (B11@D12, B21@D12, B31@D12, …, Bn1@D12) > 0  
   (i.e. Length of buffer 1 is not 0 at D12)  
   …  
   …  
   *O(mn)*