Probability and Statistics: MA6.101

Tutorial 2 Solutions

Topics Covered: Conditional Probability, Mutual Independence and Exclusion, Principles of Counting (Permutations and Combinations)

Q1: Suppose a student scores exactly 100 marks across three subjects, with the marks in each subject represented by s_1, s_2, and s_3. Find the number of distinct ways the student can achieve this, given that at least one subject has a passing mark, defined as more than 40 marks ($s_i > 40$ for at least one i). The marks in each subject are non-negative integers.

A: We need to find the number of solutions to the equation $s_1 + s_2 + s_3 = 59$ where at least one of the variables $s_i \geq 41$ (let's assume $s_1 \geq 41$).

This problem is equivalent to finding the number of solutions to $s_1' + s_2 + s_3 = 59$, where $s_1' = s_1 - 41$ and $s_1' \geq 0$. The number of solutions to this equation is given by ${}^{61}\mathrm{C}_2$ (using the "stars and bars" method, where $n+r-1$ is the total number of units plus dividers, and $r-1$ is the number of dividers).

Since $s_i \geq 41$ can apply to any one of the three variables s_1, s_2, or s_3, we need to multiply this result by 3.

However, this counts cases where two variables are greater than or equal to 41 more than once. So, we need to subtract double-counted solutions.

Let's assume $s_1 \geq 41$ and $s_2 \geq 41$. This is equivalent to finding the number of solutions to $s_1' + s_2' + s_3 = 18$.

Finally, the total number of solutions is:

$$3({}^{61}\mathrm{C}_2) - 3({}^{20}\mathrm{C}_2)$$

Q2: Consider a tetrahedral die with 3 of its sides painted red, blue and green and the fourth side has all the three colours. The colour of the side on which the die lands is to be considered. When the die lands on the fourth side, all 3 colours need to be considered.

(a) Find the probability of getting red $P(R)$, green $P(G)$ and blue $P(B)$.

(b) Find the probability of getting red and blue $P(R \cap B)$, red and green $P(R \cap G)$ and blue and green $P(B \cap G)$.

(c) Is the collection of these 3 events (getting red, blue and green) pairwise independent? Mutually independent?

A:

(a) There are 2 sides on which red appears hence $P(R) = 1/2$.
Similarly, $P(G) = 1/2$ and $P(B) = 1/2$.

(b) For getting red and blue, the die must land on the side containing all the 3 colours. Hence $P(R \cap B) = 1/4$.
Similarly, $P(R \cap G) = 1/4$ and $P(B \cap G) = 1/4$.

(c) The collection of events is pairwise independent but not mutually independent.

- A collection of events $\{A_i, i \in I\}$ are said to be mutually independent if the $P\left(\bigcap_{j\in J} A_j\right) = \prod_{j\in J} P(A_j)$ for any subset J of I.
- A collection of events $\{A_i, i \in I\}$ are said to be pairwise independent if any pair of events from the collection are independent.

From the events R, G and B, select any pair of events (P and Q) and check $P(P \cap Q) = P(P)P(Q)$ - it holds true and hence the events are pairwise independent.

But for the collection of events to be mutually independent this should hold for any subset. Consider the set itself:
$P(R \cap G \cap B) = \frac{1}{4} \neq P(R)P(G)P(B) = \frac{1}{8}$

Q3: A box contains 3 blue balls and some red balls. Your friend adds a blue ball or a red ball to the box with equal probability. After, your friend adds the ball, you randomly pick a ball from the box, and it is blue. What is the probability that the ball that your friend added was blue?

A:
Let initial set of balls be : 3 blue and r red.
Let B_n be the event that the new ball is a blue.
Let B be the event that a randomly selected ball is a blue.
Thus, total number of balls initially $= 3 + r$
Total balls after the new ball $= 4 + r$

Probability of selecting a blue ball before the new ball: $P(B) = \frac{3}{3+r}$

Probability of selecting a blue ball after the new ball:

- If the new ball is blue, total blue balls $= 4$ and total balls $= 4 + r$:

$$P(B|B_n) = \frac{4}{4+r}$$

- If the new ball is red, total blue balls $= 3$ and total balls $= 4 + r$:

$$P(B|B_n^c) = \frac{3}{4+r}$$

Using Bayes' theorem:

$$P(B_n|B) = \frac{P(B|B_n) \cdot P(B_n)}{P(B)}$$

- $P(B_n)$ (probability that the new ball is blue): $\frac{1}{2}$
- $P(B)$ (total probability of selecting a blue ball): [Law of Total Probability]

$$P(B) = P(B|B_n) \cdot P(B_n) + P(B|\neg B_n) \cdot P(\neg B_n)$$

$$P(B) = \left(\frac{4}{4+r}\right) \cdot \frac{1}{2} + \left(\frac{3}{4+r}\right) \cdot \frac{1}{2}$$
$$P(B) = \frac{4}{2(4+r)} + \frac{3}{2(4+r)}$$
$$P(B) = \frac{4+3}{2(4+r)}$$
$$P(B) = \frac{7}{2(4+r)}$$

Substituting the values back into Bayes' theorem:

$$P(B_n|B) = \frac{P(B|B_n) \cdot P(B_n)}{P(B)} = \frac{\left(\frac{4}{4+r}\right) \cdot \frac{1}{2}}{\frac{7}{2(4+r)}}$$

$$P(B_n|B) = \frac{\frac{4}{2(4+r)}}{\frac{7}{2(4+r)}}$$
$$P(B_n|B) = \frac{4}{7}$$

Q4: Percy Jackson and his two close friends, Annabeth Chase and Grover Underwood, are part of a group of ten demigods preparing for a quest. To decide who will go on the first mission, Chiron randomly splits the group into two teams of five. What is the probability that Percy, Annabeth, and Grover will end up on the same team for this mission? (Assume, teams are not labeled)

A: Suppose Percy's slot is fixed in one of the teams. Now, Annabeth has 4 remaining slots (in Percy's team) to pick from the leftover 9 slots. After Annabeth picks a slot, Grover has 3 remaining slots to pick from the leftover 8 slots. Note that, the order in which Percy, Annabeth, and Grover pick their slots doesn't matter, so we can assume any one order.

So, the probability that they end up together:

$$\text{Probability} = \frac{4}{9} \times \frac{3}{8} = \frac{1}{6}$$

Alternatively, this can also be expressed as:

$$\text{Probability} = \frac{\binom{7}{2}}{\frac{1}{2} \times \binom{10}{5}} = \frac{1}{6}$$

where $\binom{7}{2}$ represents choosing the 2 people (apart from Percy, Annabeth and Grover) from the remaining 7 for their team, and $\frac{1}{2} \times \binom{10}{5}$ represents the total number of ways to form two teams of 5.

Q5: Consider two fair six-sided dice. Let event A be the first die showing a 2 or 3. Let event B be the sum of the dice being less than equal to 3. Are these events independent? Now, you are given the event C that the first dice does not show a 4. Are the events A and B conditionally independent?

A:

$$\text{P(A)} = \frac{2}{6} = \frac{1}{3}$$

Now, we know that the number of possible cases for two dice is 36. Let's count the cases of sum being less than equal to 3 . The cases are

$$\{(1,1),(1,2),(2,1)\}$$

There are 5 such cases, so

$$\text{P(B)} = \frac{3}{36} = \frac{1}{12}$$

Now for event A $\cap$ B, we want the sum of dice less than equal to 3 and the first die showing 2 or 3, there is only one such case : {(2,1)}. Thus,

$$\text{P(A}\cap\text{B)} = \frac{1}{36}$$

While,

$$P(A) \times P(B) = \frac{1}{3} \times \frac{1}{12} = \frac{1}{36}$$

Clearly

$$P(A) \times P(B) = P(A \cap B)$$

Thus, events A and B are independent

Now, let's consider the event C, since the first dice cannot show a 4, the sample space for event A, reduces to {1,2,3,5,6}. So,

$$P(A|C) = \frac{2}{5}$$

For B the sample space remains unaffected So,

$$P(B|C) = \frac{3}{30} = \frac{1}{10}$$

(Since, sample space is now 30 elements and the 3 elements ({(1,1) , (2,1), (1,2)}) are still a part of the sample space)

Now,

$$P((A \cap B)|C) = \frac{1}{30}$$

(Since the sample is 30 elements and only one element {(2,1)} lies in the space)

While,

$$P(A|C) \times P(B|C) = \frac{2}{5} \times \frac{1}{10}$$

Thus,

$$P(A|C) \times P(B|C) = \frac{1}{25}$$

Clearly,

$$P((A \cap B)|C) \neq P(A|C) \times P(B|C)$$

Thus, the events A and B are independent.

But events A and B are not conditionally independent given C.

This, might seem counter intuitive since two independent events turn out not to be conditionally independent, but use this example to remember that we should use the formula to check conditional independance, we cannot assume that since the events are independent they will be conditionally independent.

Q6: Kushal and Medha play a game where they toss coins and compete to see who can get the most heads. Kushal gets to toss 100 coins and Medha gets to toss 101 coins, but if they get the same number of heads, Kushal wins the game. What is the probability that Kushal wins?

A: The answer is indeed 1/2.

Let's look at the probability that Medha wins the game.
So, let's assume both of them have tossed their 100 coins and Medha is about to toss the 101^{st} coin.
There are 3 possibilities here, either Kushal is ahead or Medha is ahead or they both have equal number of heads.
Let the probability that Kushal is ahead be p. Since there is no difference until the 100^{th} toss, probability that Medha is ahead is also p and they are tied is $1 - 2p$. Now, for Medha to win, either the total was tied and she has to get a heads in the 101^{st} or she has to be ahead of Kushal (last toss doesn't matter).
If Medha after 100 tosses has lesser number of heads than Kushal, she can't win as the best that can happen is a tie even then Kushal wins.
So, the probability that Medha wins is $(1 - 2p) \cdot 1/2 + p \cdot 1 = 1/2$
Hence, the probability that Kushal wins is also 1/2.

Q7: Consider a machine that produces a defective item with probability p and a non-defective item with probability $1 - p$. Suppose that items produced by the machine are selected at random and inspected one at a time until exactly five defective items have been obtained. Determine the probability p that exactly n items must be selected to obtain the five defectives.

A: To determine the probability that exactly n items must be selected to obtain exactly five defective items, we can use the following reasoning:

The fifth defective item will be the n-th item inspected if and only if there are exactly four defectives among the first $n - 1$ items and the n-th item is defective.

1. Probability of Exactly Four Defectives Among the First $n - 1$ Items:

The number of ways to choose 4 defective items out of the first $n - 1$ items is given by the binomial coefficient $\binom{n-1}{4}$. The probability of having exactly 4 defectives among these $n-1$ items, and the remaining $n-5$ items being non-defective, is:

$$\binom{n-1}{4} p^4 (1-p)^{n-5}$$

2. Probability That the n-th Item is Defective:

The probability that the n-th item is defective is p.

Since the event of having exactly four defectives among the first $n-1$ items and the event of the n-th item being defective are independent, the total probability that exactly n items are selected to obtain the five defectives is the product of these probabilities.

Therefore, the probability P is given by:

$$P = \binom{n-1}{4} p^5 (1-p)^{n-5}$$