Probability and Statistics: MA6.101

Homework 3

Topics Covered: Random Variables, Expectation, Functions on Random Variables, Discrete Random Variables.

Q1: St. Petersburg Paradox: "A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The pot starts at 1 dollar and is doubled every time a head appears. The first time a tail appears, the game ends and the player wins whatever is in the pot. Thus the player wins 1 dollar if a tail appears on the first toss, 2 dollars if a head appears on the first toss and a tail on the second, 4 dollars if a head appears on the first two tosses and a tail on the third, 8 dollars if a head appears on the first three tosses and a tail on the fourth, and so on. In short, the player wins 2^{k-1} dollars if the coin is tossed k times until the first tail appears. What would be a fair price to pay the casino for entering the game?"

a. Let X be the amount of money (in dollars) that the player wins. Find $E[X]$.

b. Now suppose that the casino only has a finite amount of money. Specifically, suppose that the maximum amount of the money that the casino will pay you is 2^{30} dollars (around 1.07 billion dollars). That is, if you win more than 2^{30} dollars, the casino is going to pay you only 2^{30} dollars. Let Y be the money that the player wins in this case. Find $E[Y]$.

A:

(a)

$$P(X = 2^{k-1}) = P(\text{coin is tossed } k \text{ times until the first tail appears})$$

$$\implies P(X = 2^{k-1}) = \left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2} \text{ (Since this is a Geometric distribution with } p = \frac{1}{2})$$

$$\implies P(X = 2^{k-1}) = \left(\frac{1}{2}\right)^{k}, \quad k = 1, 2, \ldots$$

$$\implies E[X] = \sum_{k=1}^{\infty} 2^{k-1} \cdot \left(\frac{1}{2}\right)^{k} = \sum_{k=1}^{\infty} \frac{1}{2} = \infty$$

(b) Define Y as the random variable denoting the money that the player wins in this case. Then

$$Y = \begin{cases} 2^{k-1} & \text{for } k = 1, 2, \ldots, 30 \\ 2^{30} & \text{for } k = 31, 32, \ldots \end{cases}$$

$$p_Y(Y = 2^{k-1}) = \frac{1}{2^k}$$

$$p_Y(Y = 2^{30}) = \sum_{k=31}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2^{30}}}{1 - \frac{1}{2}} = \frac{1}{2^{29}}$$

$$\implies E[Y] = \sum_{k=1}^{30} 2^{k-1} \cdot \frac{1}{2^k} + 2^{30} \cdot \frac{1}{2^{29}}$$
$$\implies E[Y] = 15 + 2 = 17$$

Q2: Let X be a random variable with mean $\mathrm{E}[X] = \mu$. Define the function $f(\alpha)$ as

$$f(\alpha) = \mathrm{E}[(X - \alpha)^2].$$

Find the value of α that minimizes f.

A:

$$f(\alpha) = \mathrm{E}[(X - \alpha)^2] = \mathrm{E}[X^2 + \alpha^2 - 2X\alpha]$$
$$\implies f(\alpha) = \mathrm{E}[X^2] + \mathrm{E}[\alpha^2] + \mathrm{E}[-2X\alpha] \quad \text{(Linearity of Expectation)}$$
$$\implies f(\alpha) = \mathrm{E}[X^2] + \alpha^2 - 2\alpha\mathrm{E}[X]$$
$$\text{To minimize } f(\alpha), \quad f'(\alpha) = 0 \implies 2\alpha - 2\mathrm{E}[X] = 0$$
$$\implies \alpha = \mathrm{E}[X] = \mu$$

Q3: : Let X be a binomial random variable with parameters (n, p). What value of p maximizes $P\{X = k\}, k = 0, 1, 2, ..., n$?

A: The pmf of a binomial random variable X is given by

$$P_X(k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

To find the value of p maximizing this, we differentiate w.r.t p and set the derivative to 0

$$\frac{\mathrm{d}P_X(k)}{\mathrm{d}p} = \binom{n}{k} \left(kp^{k-1}(1 - p)^{n-k} - (n - k)p^k(1 - p)^{n-k-1}\right) = 0$$
$$\implies k(1 - p) - (n - k)(1 - p) = 0$$

Solving, we get $p = \frac{k}{n}$

Q4: For each of the following random variables, find $P(X > 7)$ and $P(3 < X \leq 8)$:

(a) $X \sim \text{Geometric}(0.25)$

(b) $X \sim \text{Binomial}(12, 0.4)$

A:

PART (a)

Given $X \sim \text{Geometric}(0.25)$, we want to find:

$$P(X > 7) \quad \text{and} \quad P(3 < X \leq 8)$$

The probability mass function for a Geometric distribution is:

$$P(X = k) = (1 - p)^{k-1}p$$

where $p = 0.25$.

Solution for $P(X > 7)$

$$P(X > 7) = 1 - P(X \leq 7) = 1 - \sum_{k=1}^{7} P(X = k)$$

Since $P(X \leq 7) = 1 - (1-p)^7$, we calculate:

$$P(X > 7) = (1 - 0.25)^7 = 0.75^7 \approx 0.1335$$

Solution for $P(3 < X \leq 8)$

We have:

$$P(3 < X \leq 8) = P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8)$$

This is computed as:

$$P(X = k) = (0.75)^{k-1} \times 0.25$$

Thus:

$$P(3 < X \leq 8) = 0.75^3 \times 0.25 + 0.75^4 \times 0.25 + \cdots + 0.75^7 \times 0.25$$

Summing these terms, we get:

$$P(3 < X \leq 8) \approx 0.246$$

PART (b)

Given $X \sim \text{Binomial}(12, 0.4)$, we want to find:

$$P(X > 7) \quad \text{and} \quad P(3 < X \leq 8)$$

The probability mass function for a Binomial distribution is:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

where $n = 12$ and $p = 0.4$.

Solution for $P(X > 7)$

$$P(X > 7) = 1 - P(X \leq 7) = 1 - \sum_{k=0}^{7} P(X = k)$$

Using the binomial formula, we calculate each term and sum to find $P(X \leq 7)$, then subtract from 1.

Solution for $P(3 < X \leq 8)$

$P(3 < X \leq 8) = P(X = 4)+P(X = 5)+P(X = 6)+P(X = 7)+P(X = 8)$

Using the binomial formula:

$$P(X = k) = \binom{12}{k}(0.4)^k(0.6)^{12-k}$$

We calculate these probabilities and sum them to find $P(3 < X \leq 8)$.

Q5: Suppose you participate in a quiz competition that consists of 15 multiple-choice questions. Each question has 5 possible options. You know the correct answer to 7 questions, but for the remaining 8 questions, you guess randomly. Let your total score Z on the quiz be the number of correct answers you get.

(a) Find the PMF of Z.

(b) What is $P(Z > 10)$?

A: Part (a): Finding the PMF of Z

Let's define the random variable W as the number of correct answers out of the 8 questions you guess randomly. Then your total score will be $Z = W + 7$.

For each of the 8 questions that you guess randomly, the probability of getting the correct answer is $\frac{1}{5}$. Since the answers to these 8 questions are independent, W follows a binomial distribution: $W \sim \text{Binomial}(8, \frac{1}{5})$. Thus, the PMF of W is:

$$p_W(w) = \begin{cases} \binom{8}{w}\left(\frac{1}{5}\right)^w\left(\frac{4}{5}\right)^{8-w} & \text{for } w = 0, 1, 2, \ldots, 8, \\ 0 & \text{otherwise.} \end{cases}$$

The range of Z is $R_Z = \{7, 8, 9, \ldots, 15\}$. The PMF of Z can be written as:

$$p_Z(7) = P(Z = 7) = P(W+7 = 7) = P(W = 0) = \binom{8}{0}\left(\frac{4}{5}\right)^8 = \left(\frac{4}{5}\right)^8,$$

$$p_Z(8) = P(Z = 8) = P(W + 7 = 8) = P(W = 1) = \binom{8}{1}\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^7,$$

In general, for $k \in R_Z = \{7, 8, 9, \ldots, 15\}$,

$$p_Z(k) = P(Z = k) = P(W+7 = k) = P(W = k-7) = \binom{8}{k-7}\left(\frac{1}{5}\right)^{k-7}\left(\frac{4}{5}\right)^{15-k}.$$

To summarize:

$$p_Z(k) = \begin{cases} \binom{8}{k-7}\left(\frac{1}{5}\right)^{k-7}\left(\frac{4}{5}\right)^{15-k} & \text{for } k = 7, 8, 9, \ldots, 15, \\ 0 & \text{otherwise.} \end{cases}$$

Part (b): Calculating $P(Z > 10)$

To find $P(Z > 10)$, we need to consider the values k where $Z > 10$. This corresponds to $W = 4, 5, 6, 7, 8$:

$$P(Z > 10) = \sum_{w=4}^{8} \binom{8}{w} \left(\frac{1}{5}\right)^{w} \left(\frac{4}{5}\right)^{8-w},$$

Specifically:

$$p_Z(11) = \binom{8}{4}\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^4,$$

$$p_Z(12) = \binom{8}{5}\left(\frac{1}{5}\right)^5\left(\frac{4}{5}\right)^3,$$

$$p_Z(13) = \binom{8}{6}\left(\frac{1}{5}\right)^6\left(\frac{4}{5}\right)^2,$$

$$p_Z(14) = \binom{8}{7}\left(\frac{1}{5}\right)^7\left(\frac{4}{5}\right)^1,$$

$$p_Z(15) = \binom{8}{8}\left(\frac{1}{5}\right)^8\left(\frac{4}{5}\right)^0.$$

Thus,

$$P(Z > 10) = p_Z(11) + p_Z(12) + p_Z(13) + p_Z(14) + p_Z(15).$$

Q6: Let X denote a discrete random variable that can take the values -2, -1, M and 2. Given that X has probability distribution function $f(X) = (X + 4)/16$, find the variance of X.

A: Calculate the value of M using property of summation of probabilies to be 1 and then get the corresponding PMF for X

$$f(x) = \begin{cases} \frac{1}{8}, & x = -2 \\ \frac{3}{16}, & x = -1 \\ \frac{5}{16}, & x = 1 \\ \frac{3}{8}, & x = 2 \end{cases}$$

We want to find the variance of X. To do this, we first need to calculate $E[X]$ (the expected value) and $E[X^2]$ (the expected value of X^2).

- Calculate $E[X]$
  The expected value $E[X]$ is given by:

$$E[X] = \sum_x x \cdot f(x)$$

  Substitute the values:

$$E[X] = (-2) \cdot \frac{1}{8} + (-1) \cdot \frac{3}{16} + 1 \cdot \frac{5}{16} + 2 \cdot \frac{3}{8}$$

  Convert all terms to a common denominator of 16:

$$E[X] = -\frac{2}{8} - \frac{3}{16} + \frac{5}{16} + \frac{6}{8}$$

$$E[X] = -\frac{4}{16} - \frac{3}{16} + \frac{5}{16} + \frac{12}{16}$$

$$E[X] = \frac{-4 - 3 + 5 + 12}{16}$$

$$E[X] = \frac{10}{16} = \frac{5}{8}$$

- Calculate $E[X^2]$
  The expected value $E[X^2]$ is given by:

$$E[X^2] = \sum_x x^2 \cdot f(x)$$

  Substitute the values:

$$E[X^2] = (-2)^2 \cdot \frac{1}{8} + (-1)^2 \cdot \frac{3}{16} + 1^2 \cdot \frac{5}{16} + 2^2 \cdot \frac{3}{8}$$

  Calculate each term:

$$E[X^2] = 4 \cdot \frac{1}{8} + 1 \cdot \frac{3}{16} + 1 \cdot \frac{5}{16} + 4 \cdot \frac{3}{8}$$

$$E[X^2] = \frac{4}{8} + \frac{3}{16} + \frac{5}{16} + \frac{12}{8}$$

  Convert all terms to a common denominator of 16:

$$\frac{4}{8} = \frac{8}{16}$$

$$\frac{12}{8} = \frac{24}{16}$$

$$E[X^2] = \frac{8}{16} + \frac{3}{16} + \frac{5}{16} + \frac{24}{16}$$

$$E[X^2] = \frac{8 + 3 + 5 + 24}{16}$$

$$E[X^2] = \frac{40}{16} = \frac{5}{2}$$

- Calculate the Variance $\text{Var}(X)$
  The variance $\text{Var}(X)$ is given by:

  $$\text{Var}(X) = E[X^2] - (E[X])^2$$

  Substitute the values:

  $$(E[X])^2 = \left(\frac{5}{8}\right)^2 = \frac{25}{64}$$

  $$E[X^2] = \frac{5}{2} = \frac{160}{64}$$

  Calculate the variance:

  $$\text{Var}(X) = \frac{160}{64} - \frac{25}{64}$$

  $$\text{Var}(X) = \frac{135}{64}$$

  Thus, the variance of X is $\frac{135}{64}$.