Probability and Statistics: MA6.101

Homework 2 Solutions

Topics Covered: Conditional Probability, Mutual Independence and Exclusion, Principles of Counting (Permutations and Combinations)

Q1: A person has n boxes, where $n \geq 2$. We assume that either a box can have one or zero items (both cases have equal probability). What is the probability that all of the boxes contain an item for the following cases:

(a) We pick one box at random and find that it contains an item.

(b) We ask the person, "Do you have at least one box with a special item?" They respond, "Yes!" (Here, assume that if a box contains an item, it has a special item with probability $\alpha \ll 1$, independently of other boxes. If the box is empty, it does not have a special item.)

A:

$E_1 \Rightarrow$ Event that the random box we pick has an item
$E_2 \Rightarrow$ Event that atleast one of the boxes has a special item
$G_i \Rightarrow$ Event that exactly i boxes have an item
To find: $P(G_n \mid E_1)$ $P(G_n \mid E_2)$

(a)

$$P(G_n \mid E_1) = \frac{P(G_n \cap E_1)}{P(E_1)}$$

$$P(E_1) = \sum_{i=0}^{n} P(E_1 \cap G_i)$$

$$P(E_1 \cap G_i) = P(E_1 \mid G_i)\, P(G_i)$$

$$P(E_1 \cap G_0) = P(E_1 \mid G_0)\, P(G_0) = 0$$

Then,

$$P(E_1 \cap G_i) = P(E_1 \mid G_i)\, P(G_i)$$

As we pick the box randomly and it happens to have an item:

$$P(E_1 \mid G_i) = \frac{i}{n}$$

Probability of having exactly i boxes with an item inside them:

$$P(G_i) = \frac{1}{2^n} * {}^{n}\mathrm{C}_i$$

So,

$$P(E_1 \cap G_i) = \frac{1}{2^n} * \frac{i}{n} * {}^{n}\mathrm{C}_i$$

So,

$$P(E_1) = \frac{1}{2^n} * \frac{1}{n} * \sum_{i=1}^{n} i * {}^{n}\mathrm{C}_i = \frac{1}{2^n} * \frac{1}{n} * (n * 2^{n-1}) = \frac{1}{2}$$

(Using binomial expansion for $(1+x)^n$)

$$P(G_n \mid E_1) = \frac{\frac{1}{2^n} * \frac{n}{n} * {}^n\mathrm{C}_n}{\frac{1}{2}}$$

$$\boxed{P(G_n \mid E_1) = \frac{1}{2^{n-1}}}$$

(b) Here, we need to find:

$$P(G_n \mid E_2)$$

Similar to part (a),

$$P(E_2 \cap G_i) = P(E_2 \mid G_i)\, P(G_i)$$

To calculate $P(E_2 \mid G_i)$ we know that the probability that atleast one box has a special item is the same as the complement of the probability that no box has a special item.

$$P(E_2 \mid G_i) = 1 - P(E_2^c \mid G_i)$$
$$P(E_2^c \mid G_i) = (1-\alpha)^i$$
$$P(E_2 \mid G_i) = 1 - (1-\alpha)^i$$

Probability of having a excatly i boxes with an item inside them:

$$P(G_i) = \frac{1}{2^n} * {}^n\mathrm{C}_i$$

$$P(E_2 \cap G_i) = \left[1 - (1-\alpha)^i\right] * \frac{1}{2^n} * {}^n\mathrm{C}_i$$

So,

$$P(E_2) = \frac{1}{2^n} * \sum_{i=1}^{n} \left[1 - (1-\alpha)^i\right] * {}^n\mathrm{C}_i$$

$$P(E_2) = \frac{1}{2^n} * \left[\sum_{i=1}^{n} {}^n\mathrm{C}_i - \sum_{i=1}^{n} (1-\alpha)^i * {}^n\mathrm{C}_i\right]$$

Using binomial expansion for $(1+x)^n$ we find that:

$$\sum_{i=1}^{n} {}^n\mathrm{C}_i = 2^n - 1$$

$$P(E_2) = \frac{1}{2^n} * [2^n - 1 - ((2-\alpha)^n - 1)] = \frac{1}{2^n} * [2^n - (2-\alpha)^n]$$

$$P(G_n \mid E_2) = \frac{P(G_n \cap E_2)}{P(E_2)}$$

$$\boxed{P(G_n \mid E_2) = \frac{1 - (1-\alpha)^n}{2^n - (2-\alpha)^n}}$$

Q2: A coin is tossed n times, what is the probability of getting an odd number of heads if the probability of getting heads is p.

A: Let $P(n)$ denote the probability of getting an odd number of heads when a coin is tossed n times.

Base Case:

For $n = 1$, the probability of getting an odd number of heads is simply the probability of getting a head:

$$P(1) = p$$

Recursive Case:

For $n > 1$, the probability P_n can be expressed in terms of $P(n-1)$ as follows:

$$P(n) = p \cdot (1 - P(n-1) + (1-p) \cdot P(n-1)$$

Simplifying, we get:

$$P(n) = p + (1-2p) \cdot P(n-1)$$

Solving the above recursion:

$$P(2) = p + (1-2p) \cdot P(1)$$
$$P(3) = p + (1-2p) \cdot P(2)$$

Substituting,

$$P(3) = p + (1-2p) \cdot (p + (1-2p) \cdot P(1))$$

Simplifying,

$$P(3) = p + (1-2p) \cdot p + (1-2p)^2 \cdot P(1)$$

By observing the pattern an putting $P(1) = p$,

$$P(n) = p + p \cdot (1-2p) + p \cdot (1-2p)^2 + \cdots + p \cdot (1-2p)^{n-2} + (1-2p)^{n-1} \cdot p$$

Solving, using sum of GP,

$$P(n) = \frac{p \cdot (1 - (1-2p)^n)}{1 - (1-2p)}$$

$$P(n) = \frac{1 - (1-2p)^n}{2}$$

Q3: Harry Potter's closet contains 12 numbered brooms, of which 8 are Comet Two Sixty's (numbered 1 - 8) and 4 are Nimbus Two Thousand's (Numbered 9-12). Harry, Ron, George and Fred want to sneak out for a game of Quidditch in the middle of the night. They don't want to turn on the light in case Snape catches them. They reach in the closet and pull out a sample of 4 brooms. Give answers for both unordered and ordered samples.

(a) How many different samples are possible?

(b) How many samples have exactly one Comet Two Sixty in them?

(c) How many samples have at least 3 Comet Two Sixty's?

(d) Now, Ginny and Demelza wanna join so they grab 6 brooms, but Demelza believes that the both teams should have equal number of each type of broom for the game to be balanced. So, for any broom for a team we would need another broom of the same type for the other team. How many samples will keep Demelza happy?

A:

(a)

$$^{12}\mathrm{C}_4 = 495$$

$$^{12}\mathrm{P}_4 = 11880$$

(b)

$$^{8}\mathrm{C}_1.^{4}\mathrm{C}_3 = 32$$

$$^{8}\mathrm{C}_1.^{4}\mathrm{C}_3.4! = 768$$

(c)

$$^{8}\mathrm{C}_3.^{4}\mathrm{C}_1 + ^{8}\mathrm{C}_4.^{4}\mathrm{C}_0 = 294$$

$$(^{8}\mathrm{C}_3.^{4}\mathrm{C}_1 + ^{8}\mathrm{C}_4.^{4}\mathrm{C}_0).4! = 7056$$

(d) So, we would need to look at three cases: A = (4Nimbus, 2 Comet), B = (2 Nimbus, 4 Comet) , C = (6 Comet), since these are mutually exclusive,

$$Ans = P(A) + P(B) + P(C)$$

$$A = ^{4}\mathrm{C}_4.^{8}\mathrm{C}_2 = 28$$

$$B = ^{4}\mathrm{C}_2.^{8}\mathrm{C}_4 = 420$$

$$B = ^{4}\mathrm{C}_0.^{8}\mathrm{C}_6 = 28$$

$$Total = A + B + C = 28 + 420 + 28 = 476$$

For ordered since there are 6 elements, and each is numbered, multiplying total by 6! accounts for all samples

$$Total = 6!(476) = 342720$$

Q4: The Celtics and the Lakers are set to play a playoff series of n basketball games, where n is odd. The Celtics have a probability p of winning any one game, independent of other games.

(a) Find the values of p for which $n = 5$ is better for the Celtics than $n = 3$.

(b) Generalize part (a). For any $k > 0$, find the values for p for which $n = 2k + 1$ is better for the Celtics than $n = 2k - 1$.

A: a For $n = 5$ and $n = 3$, we compare the probabilities of the Celtics winning the series.
Let $P(n)$ denote the probability of the Celtics winning the series when there are n games.
For $n = 3$: The Celtics need to win 2 out of 3 games. The probability is:

$$P(3) = \binom{2}{1} p^2(1-p)^1 + p^3 = 2p^2(1-p) + p^3$$

For $n = 5$: The Celtics need to win 3 out of 5 games. The probability is:

$$P(5) = \binom{4}{2} p^3(1-p)^2 + \binom{4}{1} p^4(1-p)^1 + p^5 = 6p^3(1-p)^2 + 4p^4(1-p) + p^5$$

To find when $n = 5$ is better than $n = 3$, solve:

$$P(5) > P(3)$$

$$6p^3(1-p)^2 + 4p^4(1-p) + p^5 > 2p^2(1-p) + p^3$$

b
Generalizing the problem to $n = 2k+1$ and $n = 2k-1$:
Let P_A be the probability of winning a series of $2k+1$ games and P_B be the probability of winning a series of $2k-1$ games. The Celtics need to win $k+1$ out of $2k+1$ games and k out of $2k-1$ games.

$$P_A = \sum_{i=k+1}^{2k+1} \binom{2k}{i} p^i(1-p)^{2k+1-i}$$

$$P_B = \sum_{i=k}^{2k-1} \binom{2k-1}{i} p^i(1-p)^{2k-1-i}$$

To determine when $P_A > P_B$, we need:

$$P_A - P_B = \binom{2k}{k+1} p^{k+1}(1-p)^k - \binom{2k-1}{k} p^k(1-p)^{k+1}$$

Simplify:

$$P_A - P_B = \frac{(2k)!}{(k+1)!(k-1)!} p^{k+1}(1-p)^k - \frac{(2k-1)!}{k!(k-1)!} p^k(1-p)^{k+1}$$

$$= \frac{(2k)!}{(k+1)k!} p^{k+1}(1-p)^k - \frac{(2k-1)!}{k!(k-1)!} p^k(1-p)^{k+1}$$

$$= \frac{(2k)!}{k!(k+1)!} p^k(1-p)^k(2p-1)$$

Thus, $n = 2k+1$ is better than $n = 2k-1$ if and only if $p > \frac{1}{2}$.

Q5: In a company, there are three teams: Team A, Team B, and Team C. Each team has different success probabilities for their projects:

- Team A has a 70% chance of completing a project successfully.
- Team B has a 60% chance of completing a project successfully.
- Team C has a 50% chance of completing a project successfully.

Projects are assigned to these teams with the following probabilities:

- 40% of projects are assigned to Team A
- 35% of projects are assigned to Team B
- 25% of projects are assigned to Team C

Consider a project is completed successfully.

(a) What is the probability that the project was assigned to Team A given that it was completed successfully?

(b) If you have two projects, what is the probability that both projects are completed successfully?

(c) Calculate the probability that a project assigned to Team B is not completed successfully.

(d) Suppose the company has 12 projects to assign, and you want to assign exactly 5 projects to Team A, 4 to Team B, and 3 to Team C. How many different ways can you assign these projects?

(e) Out of these 12 projects, how many ways can you select 6 projects such that exactly 3 are from Team A and 3 are from Team B?

A:

(a) To find the probability that the project was assigned to Team A given that it was completed successfully, use Bayes' Theorem:

$$P(A \mid \text{Success}) = \frac{P(\text{Success} \mid A) \cdot P(A)}{P(\text{Success})}$$

where:

$P(\text{Success}) = P(\text{Success} \mid A) \cdot P(A) + P(\text{Success} \mid B) \cdot P(B) + P(\text{Success} \mid C) \cdot P(C)$

Substituting the values:

$$P(\text{Success}) = 0.70 \cdot 0.40 + 0.60 \cdot 0.35 + 0.50 \cdot 0.25$$

$$P(\text{Success}) = 0.28 + 0.21 + 0.125 = 0.615$$

$$P(A \mid \text{Success}) = \frac{0.70 \cdot 0.40}{0.615} = \frac{0.28}{0.615} \approx 0.456$$

(b) If the success of each project is independent, the probability that both projects are completed successfully is:

$$P(\text{Both Success}) = P(\text{Success}) \times P(\text{Success})$$

Since the probabilities of success for two randomly chosen projects depend on their respective probabilities:

$$P(\text{Both Success}) = 0.615 \times 0.615 \approx 0.378$$

(c) The probability that a project assigned to Team B is not completed successfully is:

$$P(\text{Not Success} \mid B) = 1 - P(\text{Success} \mid B)$$

$$P(\text{Not Success} \mid B) = 1 - 0.60 = 0.40$$

(d) To assign 12 projects with 5 to Team A, 4 to Team B, and 3 to Team C, use the multinomial coefficient:

$$\frac{12!}{5!4!3!}$$

Calculating this:

$$\frac{12!}{5!4!3!} = \frac{479001600}{120 \cdot 24 \cdot 6} = 27720$$

(e) To select 6 projects such that exactly 3 are from Team A and 3 are from Team B:

$$\binom{5}{3} \times \binom{4}{3}$$

Calculating this:

$$\binom{5}{3} = 10$$

$$\binom{4}{3} = 4$$

$$\text{Total Ways} = 10 \times 4 = 40$$