Probability and Statistics: MA6.101

Homework 4

Topics Covered: Continuous Random Variables, PDF, CDF, Joint random variables

Q1: You have two friends, Alice and Bob and they both promised to call you "sometime after 7pm on Saturday". So you are sitting at home at 7pm on Saturday and you decide to go out with either Alice or Bob, whomever calls you first. Let A be the amount of time before Alice calls you, and B the amount of time before Bob calls you. Assume A and B are independent exponential random variables; the expectation of A is 60 minutes and the expectation of B is 30 minutes. What is the probability that you will go out with Alice?

A: The question asks for $P(B \geq A)$. Note that the joint density function of (A, B) is

$$f(x,y) = \frac{1}{60} \cdot \frac{1}{30} \exp\left(-\frac{x}{60} - \frac{y}{30}\right)$$

if $x, y > 0$ and 0 otherwise. So the answer is

$$P(B \geq A) = \int_0^\infty \int_x^\infty \frac{1}{60} \cdot \frac{1}{30} \exp\left(-\frac{x}{60} - \frac{y}{30}\right) dy\, dx.$$

Switching the order of integration,

$$P(B \geq A) = \int_0^\infty \int_0^y \frac{1}{60} \cdot \frac{1}{30} \exp\left(-\frac{x}{60} - \frac{y}{30}\right) dx\, dy.$$

Integrating with respect to x,

$$P(B \geq A) = \int_0^\infty \left[-\frac{1}{30} \exp\left(-\frac{x}{60} - \frac{y}{30}\right)\right]_{x=0}^{x=y} dy.$$

$$P(B \geq A) = \int_0^\infty \frac{1}{30} \left[\exp\left(-\frac{y}{30}\right) - \exp\left(-\frac{y}{60} - \frac{y}{30}\right)\right] dy.$$

$$P(B \geq A) = \int_0^\infty \frac{1}{30} \left[\exp\left(-\frac{y}{30}\right) - \exp\left(-\frac{y}{20}\right)\right] dy.$$

Evaluating the integrals,

$$\int_0^\infty \frac{1}{30} \exp\left(-\frac{y}{30}\right) dy = 1,$$

$$\int_0^\infty \frac{1}{30} \exp\left(-\frac{y}{20}\right) dy = \frac{2}{3}.$$

Thus,

$$P(B \geq A) = 1 - \frac{2}{3} = \frac{1}{3}.$$

Q2: X is a random variable distributed uniformly in the interval $[0, 1]$. Let $Y = \min(X, 1-X)$. Calculate the probability density function and expected value of Y.

A: First, find the cumulative distribution function (CDF) of Y:

$$F_Y(y) = \mathbb{P}(Y \le y).$$

For $Y = \min(X, 1-X)$ to be less than or equal to y:

$$\mathbb{P}(\min(X, 1-X) \le y) = 1 - \mathbb{P}(X \ge y \text{ and } 1 - X \ge y).$$

The condition $X \ge y$ and $1 - X \ge y$ implies:

$$\mathbb{P}(y \le X \le 1-y) = (1-y) - y = 1 - 2y.$$

Thus, the CDF is:

$$F_Y(y) = 1 - (1-2y) = 2y.$$

The probability density function (PDF) $f_Y(y)$ is:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = 2 \quad \text{for } 0 \le y \le \frac{1}{2}.$$

Thus, the expected value of Y is:

$$\mathbb{E}[Y] = \int_0^{\frac{1}{2}} 2y\, dy = \frac{1}{4}.$$

Q3: Suppose that buses arrive are scheduled to arrive at a bus stop at noon but are always X minutes late, where X is an exponential random variable with probability density function $f_X(x) = \lambda e^{-\lambda x}$. Suppose that you arrive at the bus stop precisely at noon.

(a) Compute the probability that you have to wait for more than five minutes for the bus to arrive.

(b) Suppose that you have already waiting for 10 minutes. Compute the probability that you have to wait an additional five minutes or more.

A:

(a) We compute

$$P(X \ge 5) = 1 - P(X < 5) = 1 - \int_0^5 \lambda e^{-\lambda x} dx = 1 - \left(1 - e^{-5\lambda}\right) = e^{-5\lambda}.$$

(b) We want $P(X \ge 15 \mid X \ge 10)$. First observe that $P(X \ge 15, X \ge 10) = P(X \ge 15)$. From similar computations in (a), we know

$$P(X \ge 15) = e^{-15\lambda} \quad \text{and} \quad P(X \ge 10) = e^{-10\lambda}.$$

From the definition of conditional probability,

$$P(X \ge 15 \mid X \ge 10) = \frac{P(X \ge 15, X \ge 10)}{P(X \ge 10)} = \frac{P(X \ge 15)}{P(X \ge 10)} = e^{-5\lambda}.$$

Note: This is an illustration of the memorylessness property of the exponential distribution.

Q4: Let X and Y be two continuous random variables with joint pdf

$$f(x, y) = \begin{cases} cx^2y(1+y) & \text{for } 0 \le x, y \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of c.

(b) Find the probability $P(1 \le X \le 2, 0 \le Y \le 1)$.

(c) Determine the joint cumulative distribution function (CDF), $F(a, b)$, of X and Y for a and b between 0 and 3.

(d) Find the marginal CDF $F_X(a)$ for a between 0 and 3.

(e) Find the marginal probability density function (PDF) $f_X(x)$ directly from $f(x, y)$ and check that it is the derivative of $F_X(x)$.

(f) Are X and Y independent?

A: (a) Total probability must be 1, so

$$1 = \int_0^3 \int_0^3 f(x, y) dy\ dx = \int_0^3 \int_0^3 c(x^2y + x^2y^2) dy\ dx = c \cdot \frac{243}{2}$$

(Here we skipped showing the arithmetic of the integration) Therefore,

$$c = \frac{2}{243}$$

(b)

$$\begin{aligned} P(1 \le X \le 2, 0 \le Y \le 1) &= \int_1^2 \int_0^1 f(x, y) dy\ dx \\ &= \int_1^2 \int_0^1 c(x^2y + x^2y^2) dy\ dx \\ &= c \cdot \frac{35}{18} \\ &= \frac{70}{4374} \approx 0.016 \end{aligned}$$

(c) For $0 \le a \le 3$ and $0 \le b \le 3$ we have

$$F(a, b) = \int_0^a \int_0^b f(x, y) dy\ dx = c(\frac{a^3b^2}{6} + \frac{a^3b^3}{9})$$

(d) Since $y = 3$ is the maximum value for Y, we have

$$F_X(a) = F(a, 3) = c(\frac{9a^3}{6} + 3a^3) = \frac{9}{2}c\ a^3 = \frac{a^3}{27}$$

(e) For $0 \le x \le 3$ we have, by integrating over the entire range for y,

$$f_X(x) = \int_0^3 f(x, y) dy = cx^2(\frac{3^2}{2} + \frac{3^3}{3}) = c\frac{27}{2}x^2 = \frac{1}{9}x^2.$$

This is consistent with (c) because $\frac{d}{dx}(x^3/27) = x^2/9$

(f) Since $f(x, y)$ separates into a product as a function of a times a function of y we know X and Y are independent.

Q5: Let X and W be random variables. Let $Y = X + W$. Suppose that the joint probability density of X and Y is

$$f_{X,Y}(x, y) = \lambda^2 e^{-\lambda y}, \quad 0 < x < y < \infty$$

(a) Find the density of X.

(b) Find the density of Y.

(c) Find the joint density of X and W.

(d) Find the density of W.

A:

(a) $f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y} dy = -\lambda^2 \left[\frac{e^{-\lambda y}}{-\lambda}\right]_x^\infty = \lambda e^{-\lambda x}$

(b) $f_Y(y) = \lambda^2 e^{-\lambda y} \int_0^y dx = \lambda^2 y e^{-\lambda y}$

(c) $f_{X,W}(x, w) = f(X = x, W = w) = f(X = x, Y = x + w)$
$= \lambda^2 e^{-\lambda(x+w)} = \lambda^2 e^{-\lambda w} e^{-\lambda x}$

(d) $f_W(w) = \int_0^\infty f_{X,W}(x, w) dx = \lambda^2 e^{-\lambda w} \int_0^\infty e^{-\lambda x} dx = \lambda e^{-\lambda w}$

Q6: Let X be an exponential random variable with parameter λ. Suppose random variable $Y = -\ln X$

(a) Find $F_Y(a)$.

(b) Find $f_Y(y)$.

A:

(a)

$$P(Y \leq a) = P(-\ln X \leq a) = P(X \geq e^{-a})$$
$$P(X \geq e^{-a}) = 1 - F_X(e^{-a}) = \lambda e^{-\lambda e^{-a}}$$

(b) Let $Y = g(X)$. Let $h(Y) = g^{-1}(Y) = e^{-Y}$.
As X increases, Y decreases. So,

$$F_Y(y) = P(g(X) \leq y) = P(X \geq h(y)) = 1 - F_X(h(y))$$

Using chain rule,

$$f_y(Y) = \frac{d}{dy}(1 - F_X(h(y)) = e^{-y} f_x(e^{-y}) = \lambda e^{-y} e^{-\lambda e^{-y}}$$