Probability and Statistics: MA6.101

Homework 1

Topics Covered: Sigma Algebra, Set Theory, Probability Axioms, Conditional Probability, Permutations and Combinations

Q1: Let $\mathcal{F}$ be a σ-algebra of subsets of Ω. Show that $\mathcal{F}$ is closed under countable intersections $\bigcap_n A_n$, under set differences $(A \setminus B)$, under symmetric differences $(A\Delta B)$. Will it be closed under countable union $\bigcup_n A_n$?

> A: By the definition of σ algebra, it is closed under countable union ;-;
> (just asked here to confuse them a little)
> For countable intersection, refer to this proof; other proof could be to take $A_i \in \mathcal{F}$ $\forall i \in \mathbb{N}$, and then combine them together as a basic set theory proof.
> For set difference, express $(A \setminus B)$ as $A \cap B^C$ and for symmetric difference, express $(A\Delta B)$ as $(A \cap B^C) \cup (A^C \cap B)$.

Q2: Player X has \$1 and Player Y has \$2. They play a game in which the loser gives \$1 to the winner. Player X is enough better than player Y that he wins $\frac{2}{3}$ of the time. They play until one of them gets bankrupt. What is the probability that Player x wins ?

A: The game ends when one of the players has \$3 and the other one has \$0. At any time, Player X can have either \$0, \$1, \$2 or \$3.

Let p_n be probability of winning if Player X has \$n. Then,

$$p_0 = 0$$
$$p_3 = 1$$

When player X has 1, they have a probability p=$\frac{2}{3}$ of winning and $\frac{1}{3}$ probability of losing.

$$p_1 = \frac{1}{3}p_0 + \frac{2}{3}p_2$$

Similarly,

$$p_2 = \frac{1}{3}p_1 + \frac{2}{3}p_3$$

Solving, we get $p_1 = \frac{4}{7}$

Q3: A wireless sensor grid consists of $21 \times 11 = 231$ sensor nodes that are located at points (i, j) in the plane such that $i \in \{0, 1, \dots, 20\}$ and $j \in \{0, 1, \dots, 10\}$ as shown in the figure below. The sensor node located at point $(0, 0)$ needs to send a message to a node located at $(20, 10)$. The messages are sent to the destination by going from each sensor to a neighboring sensor located above or to the right. That is, we assume that each node located at point (i, j) will only send messages to the nodes located at $(i + 1, j)$ or $(i, j + 1)$. How many different paths do exist for sending the message from node $(0, 0)$ to node $(20, 10)$?

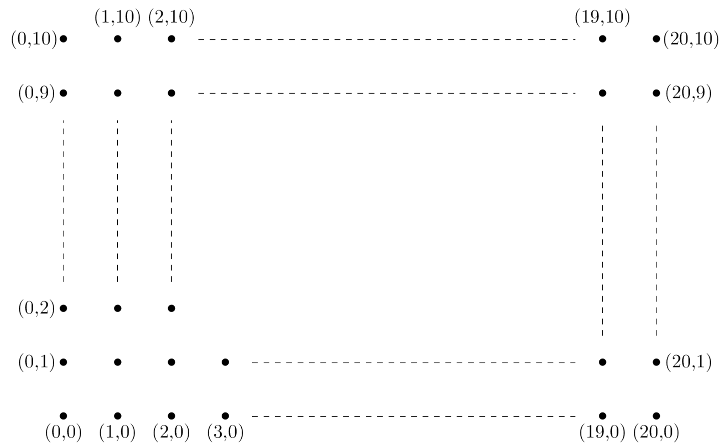

Figure 1: Wireless Sensor Grid

A: Each path from $(0,0)$ to $(20,10)$ consists of 20 steps to the right and 10 steps up. This is a classic problem of arranging 20 right moves (R) and 10 up moves (U).
Now it's just about finding how many ways are there to arrange these 30 letters. Which is given by :

$$\binom{30}{10} = \frac{30!}{10! \cdot 20!}$$

BONUS: Give a recursive solution.
A: Number of ways to reach $(20,10)$ is the sum of number of ways to reach $(19,10)$ and $(20,9)$
$w(20,10) = w(20,9) + w(19,10)$
for the node at point $w(i,j)$:
$w(i,j) = w(i,j-1) + w(i-1,j)$
base case: $w(i,j) = 1$, if i or j is 0

Q4: In Problem 3, assume that all the appropriate paths are equally likely. What is the probability that the sensor located at point (10,5) receives the message? That is, what is the probability that a randomly chosen path from (0,0) to (20,10) goes through the point (10,5)?

A: We need to count the number of paths that pass through the point $(10,5)$. So, let's break the problem into 2 variants of Problem 12. First one being: starting from point $(0,0)$ and reaching $(10,5)$ & second one: starting from point $(10,5)$ and reaching $(20,10)$. Solving both of these in the same way as Q.12,
1. Number of paths from $(0,0)$ to $(10,5)$:

$$\binom{15}{5} = \frac{15!}{5! \cdot 10!}$$

2. Number of paths from $(10,5)$ to $(20,10)$:

$$\binom{10}{5} = \frac{10!}{5! \cdot 5!}$$

3. Total number of paths from $(0,0)$ to $(20,10)$:

$$\binom{30}{10} = \frac{30!}{10! \cdot 20!}$$

4. Probability that the path goes through $(10,5)$: Will be the total no. of paths through $(10,5)$ (which is a product of 1 and 2) divided by the total paths from $(0,0)$ to $(20,10)$

$$P(\text{through } (10,5)) = \frac{\binom{15}{5} \cdot \binom{10}{5}}{\binom{30}{10}}$$

Q5: In Problem 3, given that the message has reached the node at $(10,5)$, find the probability of the message reaching the top-right node passing through the node at $(14,8)$.

A: To find the probability that the message reaches the node at $(20,10)$ passing through the node at $(14,8)$, given that it has already reached $(10,5)$, we use conditional probability.

Let A be the event that the message passes through $(14,8)$ and let B be the event that the message has reached $(10,5)$.

We need to find $P(A \mid B)$, which is the probability of A given B.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

To Calculate $P(A \cap B)$: This is the probability that the message passes through both $(10,5)$ and $(14,8)$.

- Number of paths from $(0,0)$ to $(10,5)$ and then from $(10,5)$ to $(14,8)$ and then from $(14,8)$ to $(20,10)$ is:

$$\binom{15}{5} \times \binom{7}{3} \times \binom{8}{2}$$

- Total number of paths from $(0,0)$ to $(20,10)$:

$$\binom{30}{10}$$

So,

$$P(A \cap B) = \frac{\binom{15}{5} \times \binom{7}{3} \times \binom{8}{2}}{\binom{30}{10}}$$

To Calculate $P(B)$: This is the probability that the message reaches $(10,5)$ from $(0,0)$ and then continues to $(20,10)$.

$$P(B) = \frac{\binom{15}{5} \times \binom{15}{5}}{\binom{30}{10}}$$

Therefore, the conditional probability is:

$$P(A \mid B) = \frac{\frac{\binom{15}{5}\times\binom{7}{3}\times\binom{8}{2}}{\binom{30}{10}}}{\frac{\binom{15}{5}\times\binom{15}{5}}{\binom{30}{10}}} = \frac{\binom{7}{3}\times\binom{8}{2}}{\binom{15}{5}}$$

Note: This can also be thought of as follows: since the signal has already reached $(10, 5)$, the problem reduces to finding the probability that the remaining path from $(10, 5)$ to $(20, 10)$ passes through $(14, 8)$. Thus, we calculate the number of ways to travel from $(10, 5)$ to $(20, 10)$ via $(14, 8)$ and divide it by the total number of paths from $(10, 5)$ to $(20, 10)$.

Q6: 4 people are standing in a line, numbered 1,2,3,4 from left to right. A ball is initially given to 3. Each person passes the ball to their left and right neighbours with equal probability, and a person at the end always passes the ball back to their neighbour. A person wins if when they receive the ball for the first time, every other person has already received the ball atleast once. Find probability of winning for every person.

A: Clearly, a person can win only if they are standing at one of the ends.

$$\therefore P(2 \text{ win}) = P(3 \text{ win}) = 0$$

Now let $P(i|j)$ denote conditional probability of i winning, given j has the ball. By symmetry,

$$P(1|3) = P(4|2) \tag{1}$$
$$P(4|3) = P(1|2) \tag{2}$$

Also, since only 1 or 4 can win, we have

$$P(1|3) + P(4|3) = 1 \tag{3}$$
$$P(1|2) + P(4|2) = 1 \tag{4}$$

Since 3 initially has the ball, and can pass it to 2 or 4 with equal probability, we have

$$\begin{aligned} P(1|3) &= \frac{1}{2}P(1|2) + \frac{1}{2} \\ &= \frac{1}{2}P(4|3) + \frac{1}{2} \quad \text{(from (1))} \\ &= \frac{1}{2}(1 - P(1|3)) + \frac{1}{2} \quad \text{(from (3))} \end{aligned}$$

Solving, we get

$$P(1|3) = \frac{2}{3}$$

$$P(4|3) = \frac{1}{3}$$