# Boosting

Boosting is part of ensemble techniques to improve prediction result and can be applied th any machine learning model , It tackle the high bias unlike Bagging which focuses on averaging multiple high variance models.

**Weak Learners** : a model is consider a weak learner if it's error rate is slightly less than randomly guessing (very high bias/underfit)

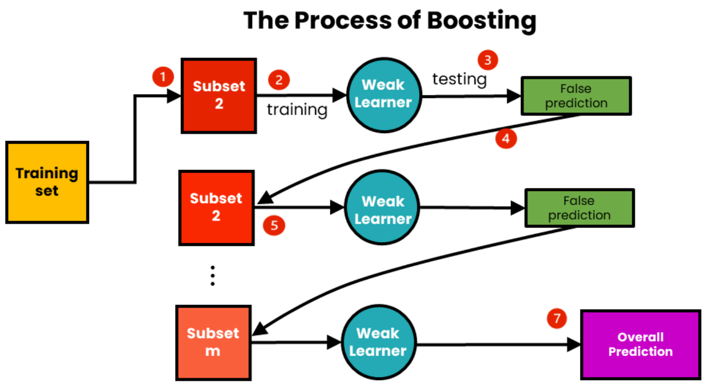
$$\text{error rate} \approx 0.45 - 4.9$$

Where :

- An $\text{error rate} \leq 0.3$ is a **Strong Learner**
- An $\text{error rate} = 0.5$ is a **Useless Learner** (randomly guessing)

## Boosting idea :

Before Showing Formulas, The main idea of **Boosting** is **Sequentially** using those weak learners to get a **Strong Learner** by :




Now we Focus on **misclassified data** of the weak learner by:

- Increase the weights of misclassified data $\rightarrow$ Weighted Training
- Update the weights
- Train the next **Weak Learner** on the updated weights data
  - So now the rows that the first weak learner got wrong got their weight increase which means the next weak learner will focus on them

**Important note** : Weights in this context means importance of **Samples/rows** so a weight $w_1$ is the importance score of a single observation(row) $x_1$, Don't get it confused with coefficients and weights in Linear Regression , Neural Nets...

## AdaBoost (Adaptive Boosting)

One of the most widely used **boosting** techniques, let's represent the algorithm steps and discuss it :

1. Initialize the observation weights $w_i = 1/N,\ i = 1, 2, \ldots, N$.
2. For $m = 1$ to $M$ :

   $(a)$ Fit a classifier $G_m(x)$ to the training data using weights $w_i$.

   $(b)$ Compute
   $$\text{err}_m = \frac{\sum_{i=1}^{N} w_i I(y_i \neq G_m(x_i))}{\sum_{i=1}^{N} w_i}.$$
   $(c)$ Compute $\alpha_m = \log\big((1 - \text{err}_m)/\text{err}_m\big)$.

   $$(d)\ \text{Set } w_i \leftarrow \frac{w_i \cdot \exp\big[\alpha_m \cdot I(y_i \neq G_m(x_i))\big]}{Z_m},\ i = 1, 2, \ldots, N$$
3. Output $G(x) = \text{sign}\Big[\sum_{m=1}^{M} \alpha_m G_m(x)\Big]$.

So After fitting the **Binary Classifier** $G_m(x)$, we compute the $\text{error rate}$ associated with that model.

$$\text{err}_m = \frac{\sum_{i=1}^{N} w_i I(y_i \neq G_m(x_i))}{\sum_{i=1}^{N} w_i}$$

- Here **misclassified** data $I(y_i \neq G_m(x))$ get multiplied by their weight so they contribute more to the $\text{error rate}$

$$\alpha_m = \log\big((1 - \text{err}_m)/\text{err}_m\big).$$

- After it we compute $\alpha_m$ which is the **classifier** $G_m$ weight also can think of it as **voting power
  - Models with **error rate** has higher $\alpha$ and their votes are more of **importance**, which we need in the last step

$$(d) \text{ Set } w_i \leftarrow \frac{w_i \cdot \exp\big[\alpha_m \cdot I(y_i \neq G_m(x_i))\big]}{Z_m},\ i = 1, 2, \ldots, N$$

- Update the weights(importance) for each misclassified observation(row), the misclassified rows weights are multiplied by $\exp[\alpha_m]$

$$G(x) = \text{sign}\left[\sum_{m=1}^{M} \alpha_m G_m(x)\right]$$

- The final step is **Majority Vote**, Since it's a **binary classification** setting it's either $[-1, 1]$ each weak learner $G_m(x)$ is multiplied by "how important they are" **Classifier weight** $\alpha_m$

## Forward Stage-wise additive modeling

It's the framework behind boosting and here we will proof starting from the base algorithm to **Ada boost** , The main idea is instead of training the full **complex** model , we sequentially add smaller models .

$$\begin{aligned}
&\text{Forward Stagewise additive modeling algorithm}\\
&1. \text{ Initialize } f_0(x) = 0\\
&2. \text{ for } m = 1 \ldots M\\
&\quad (a).\ \text{ Compute}(\beta_m, h_m(x)) = \text{argmin} \sum_i L(y_i, f_{m-1}(x_i) + \beta_m h_m(x))\\
&\quad (b).\ \text{ Set }\ f_m(x) = f_{m-1}(x) + \beta_m h_m(x)
\end{aligned}$$

- With $h(x)$ being a basis function, see Basis Functions and $\beta$ the coefficient for the model
- $L()$ is the loss function

This algorithm can be used in any model from linear regression to Tress and neural nets, in the case of **Ada boost** we use the **exponential loss** as a loss function (reason will be stated later) and basis function $h(x) = G(x)$ which are weak learners defined above.

The exponential loss function is :

$$L(y, f(x)) = e^{-yf(x)}$$

Applying the **forward stage-wise** $(a)$:

$$(\beta_m, G_m) = \text{argmin} \sum_i^n \exp\Big[-y_i(f_{m-1}(x) + \beta_m G_m(x_i)\Big]$$

Sharing the $y_i$ on the terms we get :

$$\text{argmin} \sum_i^n \exp[-y_i f_{m-1}(x_i)]\ \ \exp[-y_i \beta_m G_m(x_i)]$$

- Where $w_i = \exp[y_i f_{m-1}(x_i)]$ from exponential loss.

$$\text{argmin} \sum_i^n w_i^{(m)} \exp[-y_i \beta_m G_m(x_i)]$$

Since $G_m(x) = \{-1, 1\}$ and $y = \{-1, 1\}$ we get the following :

$$\text{argmin } e^{-\beta} \sum_{i:y_i=G(x_i)} w_i + e^{\beta} \sum_{i:y_i \neq G(x_i)} w_i$$

Given that :

$$\sum_{i:y_i=G(x_i)} w_i = W - \sum_{i:y_i \neq G(x_i)}$$

- With $W$ being the **sum of all weights** for classified and classified samples

By substituting we get :

$$\text{argmin } e^{-\beta}(W - \sum_{i:y_i!=G(x_i)} w_i) + e^{\beta} \sum_{i:y_i \neq G(x_i)} w_i$$

$$\text{argmin } e^{-\beta}W - e^{-\beta} \sum_{i:y_i!=G(x_i)} w_i + e^{\beta} \sum_{i:y_i \neq G(x_i)} w_i$$

Factoring $\sum_{y \neq G} w_i$:

$$\text{argmin } e^{-\beta}W + (e^{\beta} - e^{-\beta}) \sum_{i:y_i \neq G(x_i)} w_i$$

Remember that we are minimizing for $(\beta_m, G_m)$, first let's minimize for $G_m$

$$\text{argmin } \sum_{i:y_i \neq G(x_i)} w_i$$

- Since $e^{-\beta}W$ is a constant
- $(e^{\beta} - e^{-\beta})$ can be neglected when minimizing

Can also be written :

$$\text{argmin } \sum_{i}^{N} w_i^{(m)} \ I(y_i \neq G(x_i))$$

Thus :

$$G_m = \text{argmin } \sum_{i}^{N} w_i^{(m)} \ I(y_i \neq G(x_i))$$

Now the weighted error will be : (notice that is the same as in the Ada boost algorithm above)

$$\text{err}_m = \frac{\sum_i^N w_i^{(m)} \ I(y_i \neq G(x_i))}{W}$$

Which allow us to write the weights for the classified and misclassified samples as :

$$\sum_{i:y_i \neq G(x_i)} w_i = \text{err}_m W$$

$$\sum_{i:y_i=G(x_i)} w_i = (\text{1-err}_m)W$$

Now plugging those into this expression to minimize $\beta_m$

$$\text{argmin } e^{-\beta} \sum_{i:y_i=G(x_i)} w_i + e^{\beta} \sum_{i:y_i \neq G(x_i)} w_i$$

We get :

$$\text{argmin } e^{-\beta}(\text{1-err}_m)W + e^{\beta}\text{err}_m W$$

Taking the derivative w.r.t. $\beta$ :

$$\frac{d}{d\beta} = -e^{-\beta}(\text{1-err}_m)W + e^{\beta}\text{err}_m W = 0$$

$$\beta = \frac{1}{2}\log\left(\frac{1-\text{err}_m}{\text{err}_m}\right)$$

Which is the same as the **Classifier weight** in Ada boost

Now we got at the last step in the forward stage-wise algorithm :

$$(b).\ \ f_m(x) = f_{m-1}(x) + \beta_m G_m(x)$$

With the weight at a step $m$ is :

$$w_i^{(m)} = e^{-y_i f_{m-1}(x_i)}$$

and the weight for the next step is :

$$w_i^{(m+1)} = e^{-y_i(f_{m-1}(x)+\beta_m G_m(x_i))} = e^{-y_i f_{m-1}(x)} \cdot e^{-y_i\beta_m G_m(x_i)}$$

$$w_i^{(m+1)} = w_i^{(m)} \cdot e^{-y_i\beta_m G_m(x_i)}$$

Knowing that :

- When $y_i = G_m(x_i) \to y_i G_m(x_i) = 1$ and $I(y_i \neq G_m(x_i)) = 0$ (correct classification)
- When $y_i \neq G_m(x_i) \to y_i G_m(x_i) = -1$ and $I(y_i \neq G_m(x_i)) = 1$ (wrong classification)

Now w want a formula that gives:

- $-1$ when $I(y \neq G_m) = 0$ (we want $-1$ cause $-y_i G(x_i)_m$ so it result in $1$)
- $1$ when $I(y \neq G_m) = 1$

$$-y_i G(x_i)_m = 2.I(y_i \neq G(x_i)_m) - 1$$

> **Note** : It might be a bit confusing but try write it and check

Now we plug the formula into the weight formula above :

$$w_i^{(m+1)} = w_i^{(m)} \cdot e^{\beta_m(-y_i G_m(x_i))}$$

$$w_i^{(m+1)} = w_i^{(m)} \cdot e^{2\beta_m I(y_i \neq G_m(x_i)-1)}$$

$$w_i^{(m+1)} = w_i^{(m)} \cdot e^{2\beta_m I(y_i \neq G_m(x_i)} \cdot e^{-\beta_m}$$

With $\alpha = 2\beta$

$$w_i^{(m+1)} = w_i^{(m)} \cdot e^{\alpha I(y_i \neq G_m(x_i)} \cdot e^{-\beta_m}$$

Now we minimize for $\beta$ we get :

$$w_i^{(m+1)} = w_i^{(m)} \cdot e^{\alpha I(y_i \neq G_m(x_i)}$$

- $e^{\beta_m}$ is dropped since this value is multiplied by all the weights

Resulting in the same expression used in the **Ada Boost**:

$$(d)\ \text{Set } w_i \leftarrow \frac{w_i \cdot \exp\left[\alpha_m \cdot I(y_i \neq G_m(x_i))\right]}{Z_m},\ i = 1, 2, \ldots, N$$

# Why exponential Loss

The reason why the original **Ada Boost** uses exponential loss is for computational reasons and later they discover it got a statistical meaning since it's **estimates the log odds** of the cross entropy loss which is statistical right but expensive and doesn't have a close form solution.

In modern day's using cross entropy is the standard due to improvements of computational power.

## Exponential loss estimates the log odds ?

It should be clear now that Ada Boost minimize the exponential loss function via **forward stage-wise additive modeling** , but what makes exponential loss desirable here :

Let's start by minimizing the expected value of the **exponential loss** :

$$f(x)^* = \text{argmin}_{f(x)} \mathbb{E}[e^{-Yf(x)}]$$

Since $Y = \{-1, 1\}$ :

$$P(Y = 1|x) = p$$

$$P(Y = -1|x) = (1 - p)$$

We can expand it into :

$$f(x)^* = \text{argmin}_{f(x)}\, p \cdot e^{-(1)f(x)} + (1 - p) \cdot e^{-(-1)f(x)}$$

Minimizing w.r.t. $f(x)$ results in :

$$f(x)^* = \frac{1}{2} \log \frac{p}{1 - p} = \frac{1}{2} \log \frac{P(Y = 1|X)}{P(Y = -1|X)}$$

- Which is half log of odds, check Logistic Regression for log of odds

equivalently :

$$P(Y = 1|x) = \frac{1}{1 + e^{-2f(x)}}$$

## Cross Entropy Loss comparison

Starting from the question if we can show that the **exponential loss** estimates the log odds of the **cross entropy loss** which make it valid for classification :

First transforming $Y = \{-1, 1\}$ to $Y^{'} = \{0, 1\}$:

$$Y^{'} = \frac{Y + 1}{2}$$

The cross entropy loss is given :

$$-l(Y^{'}, p(x)) = Y^{'} \log(p(x)) + (1 - Y^{'}) \log(1 - p(x))$$

**Now let's simplify :**

- When $Y = 1$ $(Y' = 1)$: $-l = -\log p(x)$
- When $Y = -1$ $(Y' = 0)$: $-l = -\log(1 - p(x))$

Using $p(x) = \frac{1}{1+e^{-2f(x)}}$ and $1 - p(x) = \frac{1}{1+e^{2f(x)}}$:

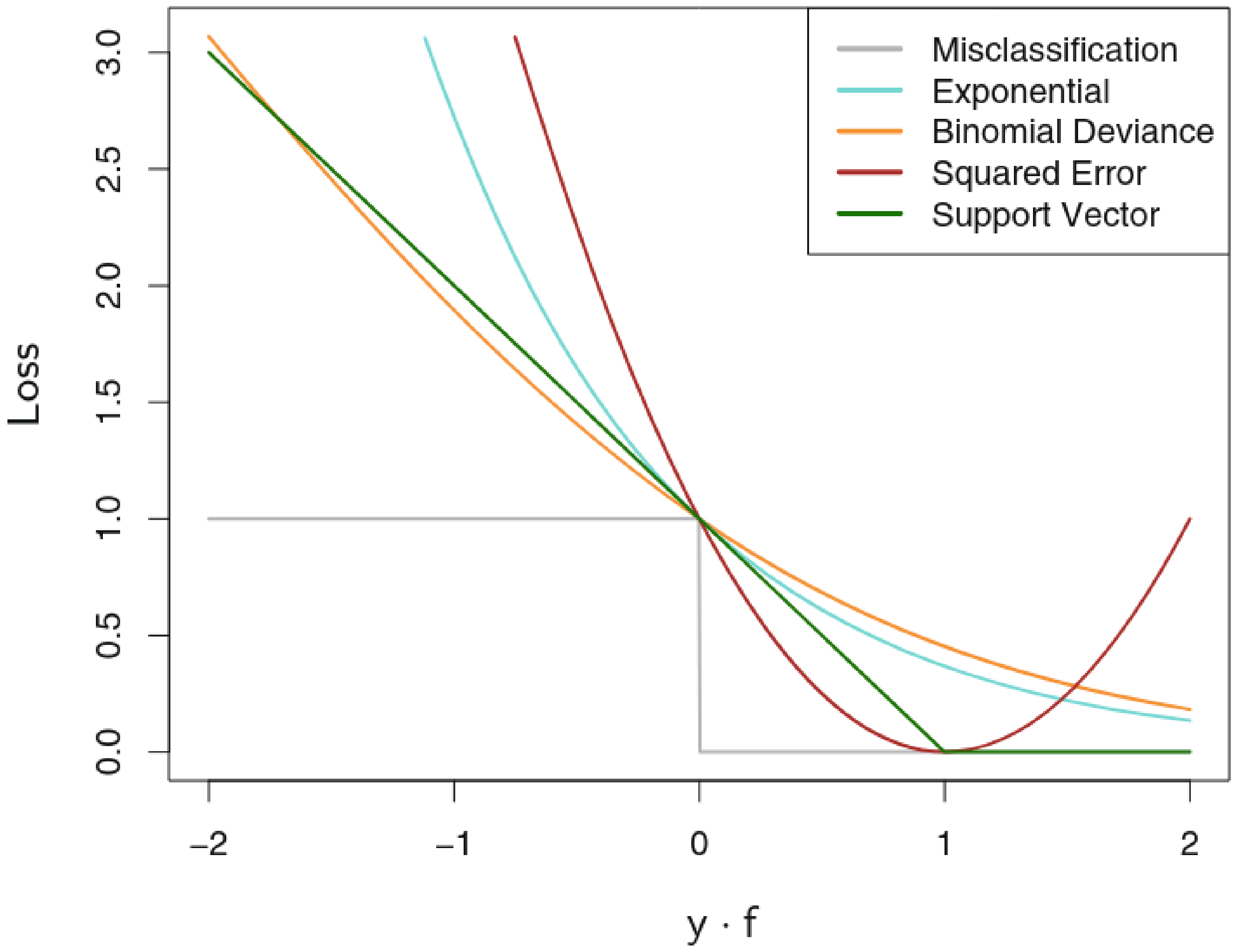
- For $Y = 1$: $-\log p(x) = \log(1 + e^{-2f(x)})$
- For $Y = -1$: $-\log(1 - p(x)) = \log(1 + e^{2f(x)})$

Combining both cases :

$$-l(Y, f(x)) = log(1 + e^{-2Yf(x)})$$

**Check:**

- If $Y = 1$: $\log(1 + e^{-2f})$
- If $Y = -1$: $\log(1 + e^{-2(-1)f}) = \log(1 + e^{2f})$




Proving that the **exponential loss** estimates half of log odds that the cross entropy loss do without iterative optimization algorithms like Gradient Descent, its a close simple formula that's why the original Ada boost used it.