Eigenvectors & Eigenvalues

Before talking about Eigenvectors, let's establish that any matrix can be interpreted as a **Linear Transformation**, for example :
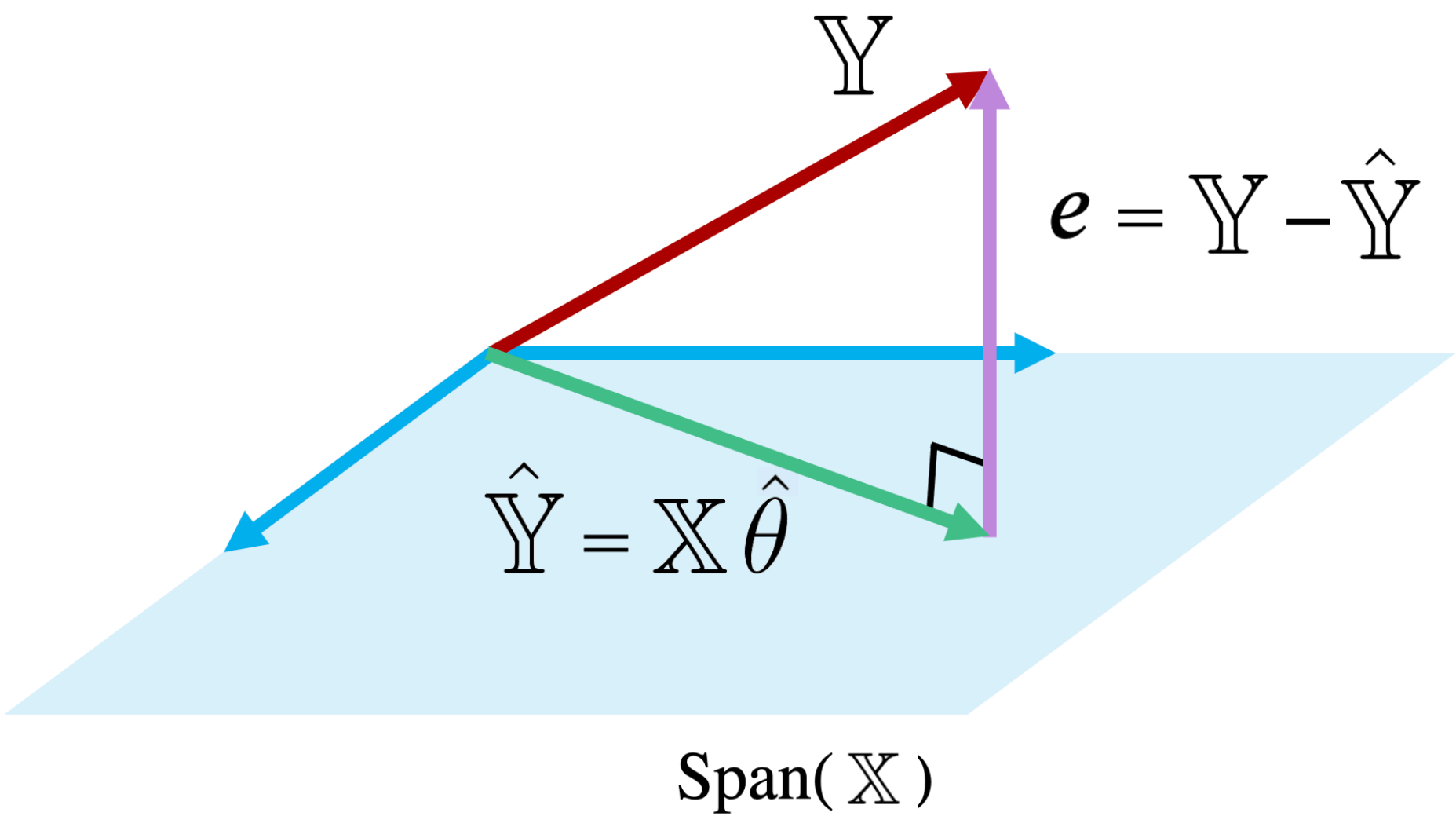
$$y = Ax$$

- The **matrix** A is the Transformation matrix for the **vector** x

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

- This matrix scales the x axis by 1 and the y axis by 2
- The columns of the matrix A tells you what happen to the standard basis

The Geometry of Linear Transformations makes easier to understand as scaling , rotating ,projection :

- Same as in the [Hat Matrix](#) which Transform Y into $\hat{Y}$

$$\hat{Y} = HY$$

Formally any matrix defines a **Linear Transformation**

$$T_A(x) = Ax$$

- Linearity check : $T_A(\alpha x + \beta = \alpha T_A(x) + \beta T_A(y)$

Eigen vectors & values

Let A be a square matrix $\in \mathbb{R}^{n \times n}$ and

- λ a scalar
- x a non-zero column vector $\in \mathbb{R}^n \{0\}$

$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

\begin{align} Ax-\lambda Ix=0 \\ \\

(A-\lambda I)x=0
\end{align}$$

- x is called the **Eigenvector** and can not be the zero vector
- λ is the **Eigenvalue**
- The Transformation matrix A can have multiple **Eigenvectors** (no more than n)
- $(A - \lambda I)$ can't be invertible cause :
  if $(A - \lambda I)$ is invertible

$$(A - \lambda I)^{-1}(A - \lambda I)x = (A - \lambda I)^{-1}\vec{0}$$
$$x = (A - \lambda I)^{-1}\vec{0}$$
$$x = \vec{0}$$

- Which break the rule of x not being the zero vector

Since $(A - \lambda I)$ isn't invertible :

- $\det(A - \lambda I) = 0$
- $\text{rk}(A - \lambda I) < n$

Solving $|A - \lambda I| = 0$ will results in **Eigenvalue** λ

A vectorized version is written :

$$AT = TD$$

- T the columns of this matrix are **Eigenvectors**
- D is a diagonal matrix with each entry as a **Eigenvalue**

Geometrically **Eigenvectors** remain on their own span when transformed by the matrix A , that's what makes them special since most of the vectors will get knocked their own span.

- The **Linear Transformation** A applied to the **Eigenvector** x will only scale it by an **Eigenvalue** λ
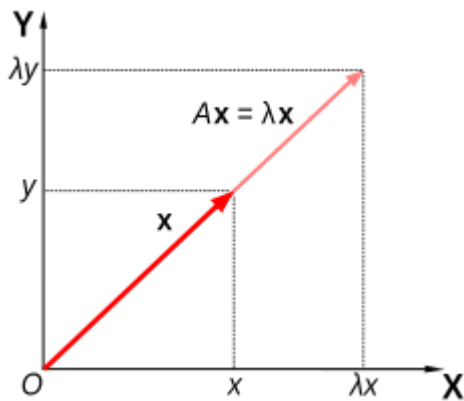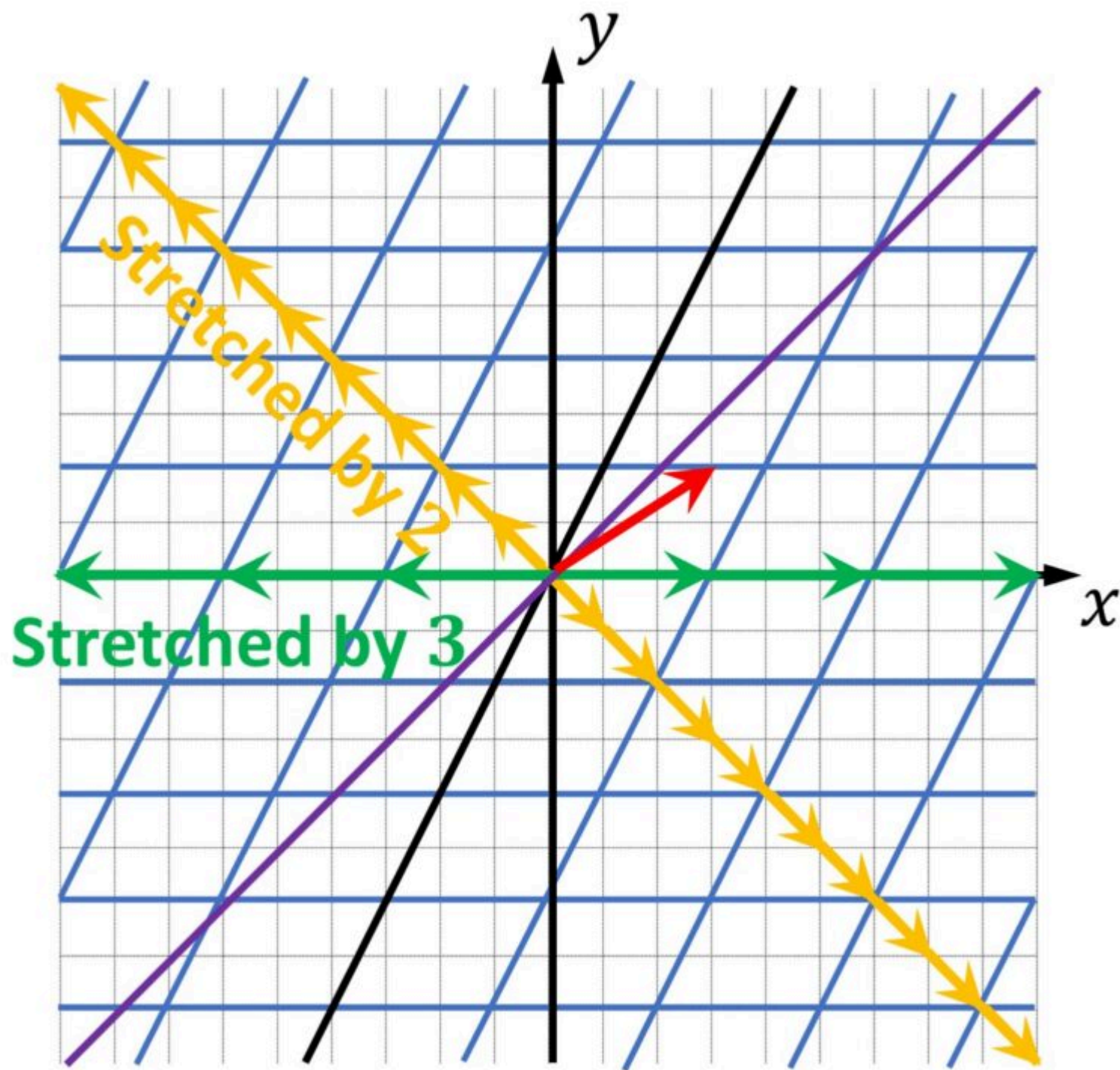
- The green and yellow vectors stays on their spans(origin) after the **Transformation** these are the **Eigenvectors**
- As for the purple vector gets knocked off it's span shown in the red vector