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The Integrable Structure of Wess-Zumino-Witten Models

Master Thesis

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Abstract

In this thesis, we investigate the integrable structure of Wess-Zumino-Witten models. We look for local integrable hierarchies, which are infinite towers of commuting conserved local charges. We go beyond the simplest example of a quantum KdV hierarchy and take a brute force approach to obtain a new integrable hierarchy built from the Kac-Moody currents of the model. We explictly construct the densities of this hierarchy up to conformal dimension 8. This introduces new operators which commute with the Hamiltonian of the model and amongst themselves, and opens the way for a finer diagonalisation of the Hilbert space of the theory.
As well, we recast these results in the over arching framework of Affine Gaudin Models, and use the previous results as explicit checks of the conjecture on the integrable hierarchy of these models. We continue working with this conjecture, and present a rational reparametrisation of the hierarchy.

Contents

Chapter 1

Introduction

Non-perturbativity is undoubtedly one of the most important and hardest open questions of quantum field theory, as non-perturbative regimes of physical theories remain hard to understand. However, there exist techniques allowing one to analytically obtain exact all-orders results for very specific theories: we call these integrable theories or exactly solvable models, and the associated techniques are usually referred to under the umbrella term of integrability. Methods of integrability have allowed for tremendous progress in understanding many aspects of theoretical physics: they have found wide ranging application, with consequences in statistical physics, string theory, condensed matter, quantum field theory and many others.

All of these methods heavily rely on the quantum integrability of the models they are applied on, which is generally understood for field theories to be the existence of an infinite tower of conserved commuting quantities. We usually call these towers an integrable hierarchy:

$$[Q_i, Q_j] = 0, \qquad [Q_i, H] = 0 \Leftrightarrow \partial_0 Q_i = 0 \tag{1.1}$$

where Q_i are the conserved commuting charges and H is the Hamiltonian, usually amongst the Q_is and generates time evolution. As well, we will look for local hierarchies in 2 dimensional field theories, which means that the conserved charges are obtained as

$$Q_i = \int q_i^0(x)dx \tag{1.2}$$

where x is the spacial direction and $q_i^\mu(x)$ is some quantum current satisfying conservation equations

$$\partial_\mu q_i^\mu = 0 \quad . \tag{1.3}$$

These charges typically generate a very large number of symmetries, and act as constraints on the dynamics of the system, which then allows one

to use the various aforementioned methods. One goal of constructing an integrable hierarchy is the diagonalisation of all the conserved charges in a single basis, which allows one to obtain the spectrum of the theory.

However, in many cases, quantum integrability is very difficult to formally establish, as there is no general way of finding arguments proving the existence and commutation of infinitely many conserved charges. In particular, formally establishing integrablity for quantum field theories is a very delicate affair, as fields themselves and their commutation relations are not well understood. It is therefore very tricky to build the conserved commuting charges from the fields. The few cases in which integrability is formally established are therefore extremely important, but they describe a quite narrow range of theories, and in practice, many methods simply assume integrability as an hypothesis and carry through. While there are background checks and heuristic arguments for the well-foundedness of this hypothesis, it nonetheless remains unsatisfying to not have access to the hierarchies of the theories.

This will be the main motivation of this thesis: to strengthen the hypothesis of quantum integrability through explicit calculations of conserved charges, as we look for integrable hierarchies. Following [1], the one point where fields and their commutation relations are well understood in quantum field theories is their conformal limit. In this very specific context, the conservation equations become chirality equations

$$\partial_\mu q^\mu = 0 \to \partial_z q^z = 0 \quad \text{or} \quad \partial_{\bar{z}} q^{\bar{z}} = 0 \tag{1.4}$$

such that the densities of the hierarchy become chiral or holomorphic/antiholomorphic fields. These chiral fields then satisfy chiral algebras, which formally describe the commutation relations of the fields. This is where it is formally doable to to build commuting conserved charges, and where we shall look for conserved charges in this thesis. The first example of such a hierarchy is the so called quantum KdV hierarchy [2][3][4][5], which is built from the Virasoro algebra, the simplest chiral algebra.

As indicated by the title of this thesis, the focus of thesis will be to build a local hierarchy for the Wess-Zumino-Witten (WZW) model. This is motivated by several reasons: they are models of extreme importance in theoretical physics in general ever since their introduction by [6][7][8][9], with repercussions ranging from holography to quantum Hall effect, and deepening the understanding of their integrable structure can never hurt. As well, they constitute the conformal limit of several integrable field theories, such as the Principal Chiral Model + Wess Zumino term and the Lambda model [10]. As such, WZW models are the one energy scale where these theories

are well understood at quantum level: understanding the integrable structure of these theories in the conformal limit is then a first step to verifying integrability at all scales.
From the chiral algebra point of view, studying WZW models simply amounts to studying a specific chiral algebra: the so-called affine Kac-Moody or Current algebra, built of holomorphic current fields. They are an infinite dimensional generalisation of the familiar Lie algebras, and are a widespread example of chiral algebras in conformal field theory. We will then focus on building a local integrable hierarchy within this Kac-Moody algebra. We will also be guided by the classical construction of such a hierarchy, due to [11][12] (see [13] for a review).

As well, in a second phase, we will study the same problem in the context of Affine Gaudin Models (AGMs). We shall follow [14][15][16] and the presentation of [13] and [17]. These models constitute an over arching formalism for the study of integrable sigma models: many of the familiar integrable sigma models can be recast in terms of AGMs, and reciprocally, one can generate many sigma models from these AGMs, including previously unexplored ones. As such, there is great value in understanding the integrable structure of these models, as this would allow one to automatically understand the structure of many sigma models at once.
Classically, AGMs are very similar to sigma models, and their integrable structure is known and formally well established. However, at the quantum level, these theories are not conformal invariant, and as such suffer from the usual problems of quantum field theory: the fields and their commutation relations are not well understood. We again follow [1] and place ourselves at the conformal point of the theory, in which the chiral algebra is simply N Kac-Moody algebras taken together. This is then a straightforward generalisation of the chiral algebra of WZW models.

As well, there exists connections between AGMs and the geometric Langlands conjecture, which led to a powerful conjecture on the integrable structure of these models [18]. This gives another reason to study WZW and other sigma models in terms of AGMs, as this conjecture provides a guide to resolve their integrable structure. Then, we shall focus the second part of this thesis on building the local integrable hierarchy of AGMs, using the conjecture as a guide. Explicit verification of this conjecture is then again a valuable strengthening of the hypothesis of quantum integrability, and serves as a check to the structure of WZW models, which the AGMs encompass.

This thesis is organised in three chapters, meant to reflect the progression of the last six months. Chapter two introduces basics of two dimensional CFT, starting from conformal invariance, in order to define the chiral alge-

bra and local integrable hierarchies, which will be the main ingredient of the thesis. As well, it gives an introduction to the quantum KdV hierarchy as a motivating example. Chapter three introduces WZW models, focusing on their chiral algebra, and presents the methods and the results obtained when looking for conserved charges. Chapter four follows analogously by presenting AGMs and the results derived from the conjecture.

Chapter 2

Two Dimensional CFT Prerequisites

This chapter is meant to introduce the theoretical prerequisites for the rest of the thesis, that is, elements of two dimensional conformal field theory (CFT). After a basic description of conformal invariance, we focus on the methods of Operator Product Expansion and Normal Ordering, which are the essential ingredients used throughout the thesis.
With these ingredients at hand, we define the Chiral Algebra of fields, which is the central structural element of CFTs we will concern ourselves with and that we will use as a starting point to look for conserved charges.
We finally introduce elements of integrable 2d CFT, defining local integrable hierarchies, which we present with the example of the quantum KdV construction, acting as the motivating example of the thesis.

2.1 Conformal Invariance

We begin by discussing the conformal group and global conformal invariance in arbitrary dimension. We then follow by discussing local conformal invariance, which we quickly specialise to two dimensions. This section roughly follows chapters 4,5 and 6 of [19].

Conformal Transformations in $d \neq 2$ dimensions

Consider a d dimensional spacetime equipped with metric $g_{\mu\nu}$. Then, conformal transformations are transformations of the spacetime coordinates $\mathbf{x} \to \mathbf{x}'$ such that

$$g'_{\mu\nu}(\mathbf{x}') = \Lambda(\mathbf{x}) g_{\mu\nu}(\mathbf{x}) \quad . \tag{2.1}$$

That is, they are the coordinate transformations that leave the metric invariant up to a scaling. These transformations form the conformal group, and one notices that the Poincaré group is the $\Lambda = 1$ subgroup. One therefore

expects to find translations, rotations and dilations in the connected component to the identity. We derive the conformal group from infinitesimal transformations in appendix A.1. The conformal group in $d \neq 2$ dimensions is then given by

$$\text{Translation:} \qquad x'^{\mu} = x^{\mu} + a^{\mu} \tag{2.2}$$

$$\text{Dilation:} \qquad x'^{\mu} = \alpha x^{\mu} \tag{2.3}$$

$$\text{Rotation (+boost):} \qquad x'^{\mu} = M^{\mu}_{\nu} x^{\nu} \tag{2.4}$$

$$\text{SCT:} \qquad x'^{\mu} = \frac{x^{\mu} - b^{\mu}\mathbf{x}^2}{1 - 2\mathbf{b}\cdot\mathbf{x} + b^2\mathbf{x}^2} \tag{2.5}$$

where SCT stands for Special Conformal Transformation. Together, they form the component connected to the identity of the conformal group in $d \neq 2$ dimensions. The SCT can be written as

$$\frac{x'^{\mu}}{\mathbf{x}'^2} = \frac{x^{\mu}}{\mathbf{x}^2} - b^{\mu} \tag{2.6}$$

which is simply an inversion followed by a translation and another inversion (inversions are part of the conformal group, but not of its component connected to the identity).

Conformal Transformations in $d = 2$ dimensions

As can be seen in appendix A.1, the derivation of the conformal transformation that led to equations (2.2)-(2.5) is not valid in the $d = 2$ case, and we need to proceed differently to characterise the two dimensional conformal group. We start again from equation (2.1), and impose the definition of the conformal group. This gives the following conditions

$$\left(\frac{\partial x'^0}{\partial x^0}\right)^2 + \left(\frac{\partial x'^0}{\partial x^1}\right)^2 = \left(\frac{\partial x'^1}{\partial x^0}\right)^2 + \left(\frac{\partial x'^1}{\partial x^1}\right)^2 \tag{2.7}$$

$$\frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^0} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^1} = 0 \quad . \tag{2.8}$$

They are equivalent to either

$$\frac{\partial x'^1}{\partial x^0} = \frac{\partial x'^0}{\partial x^1} \quad \text{and} \quad \frac{\partial x'^0}{\partial x^0} = -\frac{\partial x'^1}{\partial x^1} \tag{2.9}$$

or

$$\frac{\partial x'^1}{\partial x^0} = -\frac{\partial x'^0}{\partial x^1} \quad \text{and} \quad \frac{\partial x'^0}{\partial x^0} = \frac{\partial x'^1}{\partial x^1} \quad . \tag{2.10}$$

Equations (2.9) are the Cauchy-Riemann equations defining holomorphic functions, while equations (2.10) define anti-holomorphic functions. There-

fore, we define complex coordinates (similar to light-cone coordinates) as

$$z = x^0 + ix^1 \tag{2.11}$$
$$\bar{z} = x^0 - ix^1 \tag{2.12}$$
$$\partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \tag{2.13}$$
$$\partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1) \quad . \tag{2.14}$$

The metric in these coordinates is then:

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad . \tag{2.15}$$

Then, with these holomorphic coordinates, equation (2.9) becomes:

$$\partial_{\bar{z}} z'(z, \bar{z}) = 0 \tag{2.16}$$

which is solved by any holomorphic function f

$$z \to z'(z) = f(z) \quad . \tag{2.17}$$

Analogously, equation (2.10) is solved by any anti-holomorphic transformation. Therefore, we have that conformal transformations in two dimensions are simply holomorphic or anti-holomorphic transformations of the complex coordinates $(z, \bar{z})$.
However, it is important to note that equations (2.9) and (2.10) are purely local - they do not impose that the holomorphic and anti-holomorphic transformations are defined and invertible everywhere on the complex plane. As such, strictly speaking, the set of all **local** conformal transformations in 2d is given by any holomorphic or anti-holomorphic transformations of the coordinates, and we need to impose additional constraints to obtain **global** conformal transformations.

For a conformal transformation to be global, one must have that the transformation maps the whole Riemann sphere (complex plane + point at infinity) back to itself, and be invertible. For the mapping f to be invertible, it must not have any branch points nor essential singularity, so it can be written as a ratio of polynomials. As well, for the mapping to be one-to-one, there must be a single point that maps to infinity and to zero, and the neighbourhood of these points must be mapped a single time on itself. As such, the only acceptable form for f is a ratio of 1st order polynomials with different zeros

$$f(z) = \frac{az + b}{cz + d} \quad \text{with} \quad ad - bc \neq 0 \quad . \tag{2.18}$$

These are the so called Moebius transformations, which are the global conformal transformations of dimension 2. They can also be written in a matrix form:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det(A) \neq 0 \tag{2.19}$$

which makes the isomorphism of these transformations with $PSL(2,\mathbb{C})$ explicit [1]. One in fact can see that for specific choices of the parameters a, b, c, d, one recovers the usual translations, rotations, dilations and SCT of the conformal group: one gets the translations for $a = d = 1$, $c = 0$, dilations for $b = c = 0$, $a/d \in \mathbb{R}$, rotations for $b = c = 0$, $a/d \in U(1)$ and SCTs can be written as a product matrix of an inversion $a = d = 0$, $b = c = 1$, a translation and another inversion.

In conclusion, the global conformal transformations are described by the Poincaré group, supplemented with dilations and SCTs in any dimension. Local conformal transformations follow the same classification in $d \neq 2$, while in $d = 2$ they consists of any holomorphic or anti-holomorphic transformation of the complex coordinates.

Primary Fields

We have characterised the group of conformal transformations, and now investigate how these transformations affect the QFT. This begins by considering how fields transform.
Consider a conformal map $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$, and a field $\varphi(z, \bar{z})$. We call φ quasi-primary if it transforms under conformal mapping as

$$\varphi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \varphi(z, \bar{z}) \tag{2.20}$$

with $h, \bar{h}$ the holomorphic and anti-holomorphic conformal dimensions. As well, we call φ primary if it transforms as equation (2.20) for any local conformal transformations (and not only the global ones).

Ward Identities

To any continuous global transformation, one can associate a Ward identity. Recall the definition of the the stress-energy tensor:

$$T^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi \tag{2.21}$$

[1] and not $SL(2)$ as one can rescale all parameters together without changing the transformation

with $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$ the Lagrangian density of the theory. Using the stress-energy tensor, we can write the Ward identities associated to the global conformal group, which will relate correlation functions $\langle \_ \rangle$ to the symmetries of the theory:

$$\frac{\partial}{\partial x^\mu}\langle T^\mu_\nu(\mathbf{x})X\rangle = -\sum_{i=1}^{n}\delta(\mathbf{x}-\mathbf{x}_i)\frac{\partial}{\partial x_i^\nu}\langle X\rangle \tag{2.22}$$

$$\varepsilon_{\mu\nu}\langle T^{\mu\nu}(\mathbf{x})X\rangle = -i\sum_{i=1}^{n} s_i\delta(\mathbf{x}-\mathbf{x}_i)\langle X\rangle \tag{2.23}$$

$$\langle T^\mu_\mu(\mathbf{x})X\rangle = -\sum_{i=1}^{n}\delta(\mathbf{x}-\mathbf{x}_i)\Delta_i\langle X\rangle \quad . \tag{2.24}$$

Here, X is a string of n primary fields $\varphi(\mathbf{x}_1)...\varphi(\mathbf{x}_n)$, each with spin s_i and scaling dimension Δ_i, which are related to the conformal dimension h as $h_i = \frac{1}{2}(s_i + \Delta_i)$ and $\bar{h}_i = \frac{1}{2}(s_i - \Delta_i)$, and $\varepsilon_{\mu\nu}$ is the Levi-Civita symbol in two dimensions. Equation (2.22) is associated to translations, (2.23) to rotations and (2.24) to dilations. We will want to convert these Ward identities in terms of complex coordinates and components. This is done in appendix A.2, and yields the following Ward identity:

$$\langle T(z)X\rangle = \sum_{i=1}^{n}\left[\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right] + \text{reg} \tag{2.25}$$

where $T(z) = T^{z,z}(z)$ is the z,z component of the stress energy tensor, and reg stands for a regular holomorphic function of z as $z = w_i$. That $T^{z,z}$ is a holomorphic function of z is a consequence of conformal invariance and is the first example of holomorphic splitting, which we will detail later.

Conformal Generators

We now find the algebra of the two dimensional local conformal group. Consider the following infinitesimal conformal transformation:

$$z' = z + \varepsilon(z), \qquad \varepsilon(z) = \sum_{-\infty}^{\infty} c_n z^n \tag{2.26}$$

Together with its anti holomorphic counter part $\bar{\varepsilon}(\bar{z})$. Consider a spinless, dimensionless field $\varphi(z,\bar{z})$. Under the previous transformation, the field transforms as:

$$\varphi'(z',\bar{z}') = \varphi(z,\bar{z}) \tag{2.27}$$

$$= \varphi(z',\bar{z}') - \varepsilon(z')\partial'_z\varphi(z',\bar{z}') - \bar{\varepsilon}(\bar{z}')\partial'_{\bar{z}}\varphi(z',\bar{z}') \quad . \tag{2.28}$$

or equivalently:

$$\delta\varphi = -\varepsilon(z)\partial_z\varphi - \bar{\varepsilon}(\bar{z})\partial_{\bar{z}}\varphi \tag{2.29}$$

$$= \sum_n \{c_n l_n \varphi(z,\bar{z}) + \bar{c}_n \bar{l}_n \varphi(z,\bar{z})\} \tag{2.30}$$

with the generators

$$l_n = -z^{n+1}\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{2.31}$$

which satistfy two independant copies of the Witt algebra

$$[l_n, l_m] = (n-m)l_{n+m} \tag{2.32}$$
$$[\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m} \tag{2.33}$$
$$[\bar{l}_n, l_m] = 0 \quad . \tag{2.34}$$

One can see that the Witt algebra has a subalgebra made of l_{-1}, l_0 and l_1 which corresponds to the Lie algebra of the global conformal group: $l_{-1} = -\partial_z$ is associated to translations, $l_0 = -z\partial_z$ generates dilations and rotations, and $l_1 = -z^2\partial_z$ generates SCTs.

The splitting of the structure of the theory into two independent copies of the Witt algebra hints at a very important property of CFTs: in many CFTs (such as the ones we will study here), the algebraic structure of the theory will split into a holomoprhic and an anti-holomorphic part, which will not talk to each other. This gives us a motivation to study the so-called chiral fields $\varphi_L(z)$,$\varphi_R(\bar{z})$, defined by their property $\partial_{\bar{z}}\varphi_L(z) = 0 = \partial_z\varphi_R(\bar{z})$, and as such transform only in one copy of the algebra of the theory. In particularly nice cases, such as the free bosonic field CFT $\mathcal{L} = \partial^\mu\varphi\partial_\mu\varphi$, one can straightforwardly build holomorphic and anti holomorphic fields by taking derivatives of the fundamental field: $\partial_z\varphi$ and $\partial_{\bar{z}}\varphi$. From these fields, one is then able to reconstruct the whole theory, and it is then enough to consider only a single copy of the algebra, that is, considering only the holomorphic fields. Note that this is not true in general, even in the case that the algebra splits.

This pushes us to shift the focus on studying chiral fields of the theories. We shall then reduce the notation $\varphi_L(z) \to \varphi(z)$, where we identify the chirality of a field simply by its $z, \bar{z}$ dependence.

A slight comment on the semantics: holomorphic/anti-holomorphic is a name originating in Euclidean CFT, as we consider all directions of the plane equivalently, while chiral fields refer to left/right movers, and have origin in Minkowskian CFT. We shall use the two interchangeably, but we shall keep the name of chiral algebra.

2.2 Operator Product Expansion and Operator Formalism

In this subsection, we introduce the basics of the Operator Product Expansion and the Operator Formalism, as well as Mode Expansions of Fields.

We then derive the Virasoro Algebra as our first and simplest example of a Chiral Algebra, which we will define later.

The Operator Product Expansion

It is typical of correlation functions to have singularities when the position of fields coincide, which reflects the infinite fluctuations of a quantum field taken at an exactly known precision. The Operator Product Expansion (OPE) is the representation of a product of operators by a sum of single well behaved operator terms multiplied by a possibly diverging function of the positions of the operators. The first example of such an OPE is given by the Holomorphic Ward identity (equation (2.25)): it describes the divergent behaviour of a correlator of the stress energy tensor with a string of primary fields. Then, for a chiral primary field φ of conformal dimensions h, the OPE reads:

$$T(z)\varphi(w) = \frac{h}{(z-w)^2}\varphi(w) + \frac{1}{z-w}\partial_w\varphi(w) + reg \tag{2.35}$$

where *reg* is an infinite sum of regular terms as $w \to z$.
For two chiral fields $a(z), b(w)$, one writes the OPE of a and b as

$$a(z)b(w) = \sum_{n=-\infty}^{N} \frac{\{ab\}_n(w)}{(z-w)^n} \tag{2.36}$$

with $\{ab\}_n(w)$ regular fields at $z = w$. The details of OPEs are model specific; however, one OPE of particular importance is the one of the stress energy tensor, which always take the following form for conformally invariant theories:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg \tag{2.37}$$

where c is a model specific constant named central charge: 1 for free bosons, $1/2$ for free fermions, etc... . While we just state this OPE here, its form can be derived from conformal invariance, and the values of c can be straightforwardly computed from the actions of free bosons, fermions, etc... . See [19] page 128-135.

Mode Expansions

Consider a chiral field $a(z)$ of conformal dimension h. It can be mode expanded as follows

$$a(z) = \sum_{n\in\mathbb{Z}} z^{-n-h} a_n \tag{2.38}$$

$$a_n = \frac{1}{2\pi i}\oint dz z^{n+h-1} a(z) \quad . \tag{2.39}$$

This will allow us to study the modes corresponding to the fields and how they act on the Hilbert space of the theory, as well as allowing us to express conserved charges in terms of modes.

Commutation relations

As always when working in QFT, the fields inside a correlation function $\langle\varphi_1(z_1)...\varphi_n(z_n)\rangle$ have to be time ordered for the correlation function to reflect reality. However, this is typically done when working with QFTs defined on the plane. In string theory, we are typically interested in CFTs defined on the cylinder, which we can map on the plane with $z = e^{iu}$, where u is the coordinate on the cylinder. On the cylinder, time is defined along the non-periodic direction, which gets mapped to a radial direction on the plane. As such the time ordering that appears inside correlation functions is converted to a radial ordering, formally defined by:

$$\mathcal{R}a(z)b(w) = \begin{cases} a(z)b(w) & \text{if} \quad |z| > |w| \\ b(w)a(z) & \text{if} \quad |z| < |w| \end{cases} \tag{2.40}$$

when the two fields a, b are bosonic (a minus has to be added when dealing with fermions). This radial ordering is actually in every correlation function, and therefore also in every OPE, but it will be left implicit. See appendix A.3 for more details on radial quantisation.

Using this radial ordering, one can relate OPEs to commutation relations. Let $a(z)$ and $b(z)$ be holomorphic fields and consider the integral

$$\oint_w dz a(z)b(w) \tag{2.41}$$

where the w in the integration contour means counterclockwise integration around w but not the origin. As the integrand must be radial ordered, we can split the integral into two contours C_1 and C_2 which are respectively $|w| + \varepsilon$ and $|w| - \varepsilon$ circles centered on the origin:

$$\oint_w dz a(z)b(w) = \oint_{C_1} dz a(z)b(w) - \oint_{C_2} dz b(w)a(z) \tag{2.42}$$

$$= [A, b(w)] \tag{2.43}$$

with

$$A = \oint a(z)dz \quad . \tag{2.44}$$

We therefore see that contour integration together with radial ordering defines an equal "time" commutator of operators. In practice, these integrals are evaluated by taking the OPE of a and b, in which only the pole contributes. Defining B analogously to A, one has the commutator of A and B as

$$[A, B] = \oint_0 dw \oint_w dz a(z)b(w) \quad . \tag{2.45}$$

Equation (2.45) allows us to translate into operator language the dynamical statements that are the OPEs. They will be particularly relevant in the context of integrability, as they allow to translate the commutation of conserved charges into an OPE condition. This equation allows to compute the commutation of any modes of any fields: recalling the mode expansion equation (2.39), we can take $A = a_n$ and $B = b_m$

$$A = a_n = \frac{1}{2\pi i}\oint dz z^{n+h_a-1}a(z) \qquad B = b_m = \frac{1}{2\pi i}\oint dw w^{m+h_b-1}b(w) \quad (2.46)$$

with h_a and h_b the conformal dimensions of a and b. Therefore, one has

$$[a_n, b_m] = \frac{1}{(2\pi i)^2}\oint_0 dw \oint_w dz w^{m+h_b-1}z^{n+h_a-1}a(z)b(w) \quad . \quad (2.47)$$

We therefore clearly see that the OPE contains the same information as commutator of modes.

The Virasoro Algebra

As a crucial example, we consider the OPE of the stress energy tensor $T(z)$ and derive its corresponding algebra. As in equation (2.38), we expand the stress-energy tensor:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \qquad L_n = \frac{1}{2\pi i}\oint dz z^{n+1}T(z) \quad (2.48)$$

with $h_T = 2$. The modes L_n of the stress-energy tensor are the generators of the local conformal transformations on the Hilbert space, as l_n are the generators of conformal transformations on the space of fields. We therefore expect the modes L_n (and $\bar{L}_n$) to satisfy the Witt algebra: we can now use equation (2.47) to compute this commutator. We do this in appendix A.4. Altogether, one has

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{1}{12}cn(n^2-1)\delta_{n+m,0} \quad (2.49)$$

$$[L_n, \bar{L}_m] = 0 \quad (2.50)$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{m+n} + \frac{1}{12}cn(n^2-1)\delta_{n+m,0} \quad (2.51)$$

which are two independent copies of the so-called Virasoro algebra equation (2.49). Notice that this is identical to the Witt algebra of the generators l_n on the space of fields, except for a central extension term proportional to the central charge c. Once again, one sees the factorisation property appearing, as the algebra splits into a holomorphic and anti-holomorphic part. An important point to notice is that these commutation relations contain the same information as the OPE (2.37), and define the structure of the theory. This consists the first example of a chiral algebra, which we will define in section 2.4.

2.3 Normal Ordering

In quantum field theory, in general, composite fields will have divergences when the positions of the component fields coincide, as illustrated sharply by the OPE. However, in quantum field theory, it is important that fields have a vanishing vacuum expectation value, so it becomes necessary to remove these divergences. This is typically done by using an ordering freedom, which comes from the different choices of promoting classical modes to quantum ones. The most common prescription of what is known as Normal Ordering (NO) is then to put all annihilation operators/positive modes to the right of creation ones: this results in normal ordered terms having vanishing vacuum expectation values. We illustrate with a simple example:

Free Boson

The classical action for a free boson is

$$S = \frac{1}{2} g \int d^2x \partial_\mu \varphi \partial^\mu \varphi \tag{2.52}$$

and the two-point function/propagator is

$$\langle \varphi(\mathbf{x}) \varphi(\mathbf{y}) \rangle = -\frac{1}{4\pi g} \ln |\mathbf{x} - \mathbf{y}|^2 + const \tag{2.53}$$

with g some normalisation parameter. Converting to complex coordinates gives

$$\langle \varphi(z, \bar{z}) \varphi(w, \bar{w}) \rangle = -\frac{1}{4\pi g} \left(\ln |z - w| + \ln |\bar{z} - \bar{w}| \right) \tag{2.54}$$

and taking derivatives w.r.t. z and w gives the holomorphic components

$$\langle \partial\varphi(z) \partial\varphi(w) \rangle = -\frac{1}{4\pi g} \frac{1}{(z - w)^2} \tag{2.55}$$

which is also the main OPE for the free boson.
Using the definition of the stress-energy tensor equation (2.21), the classical stress energy tensor is

$$T(z) = -2\pi g \partial\varphi(z) \partial\varphi(z) \quad . \tag{2.56}$$

However, when going at quantum level, it becomes clear by equation (2.55) that this stress-energy tensor is divergent: we need therefore a prescription to get rid of these infinities.
We then define normal ordering for the free boson as follows:

$$: \partial\varphi \partial\varphi : (z) = \lim_{w \to z} \left(\partial\varphi(z) \partial\varphi(w) - \langle \partial\varphi(z) \partial\varphi(w) \rangle \right) \quad . \tag{2.57}$$

One can see that using this definition, $T = -2\pi g : \partial\varphi\partial\varphi :$ is no longer divergent. As well, one can show by mode expansion that this is equivalent to rearranging the modes such that the positive modes stand to the right of the left ones, such that in the classical limit, when modes commute, one recovers the classical definition.

Generalised Normal Ordering

The previous definition in which we subtract from the composite field its vacuum expectation value works only for a special class of fields, which we call free fields: they are defined as fields whose OPE with themselves contains a single divergent term with a constant coefficient.
As it turns out, normal ordering needs to be generalised for non free fields. Consider the stress energy tensor T, and consider the attempt at a normal ordered product $:TT:$ following the definition for $\partial\varphi$ equation (2.57). Then, using the OPE (2.37) of T with itself, one has

$$\lim_{w\to z}\left(T(z)T(w) - \langle T(z)T(w)\rangle\right) \tag{2.58}$$

$$= \lim_{w\to z}\left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg - \frac{c/2}{(z-w)^4}\right) \tag{2.59}$$

$$= \lim_{w\to z}\left(\frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg\right) \tag{2.60}$$

which is still very much divergent. It is therefore necessary to generalise the definition of normal ordering to general fields, and the procedure is quite straightforward. Considering the general OPE of two chiral fields a and b

$$a(z)b(w) = \sum_{n=-\infty}^{N} \frac{\{ab\}_n(w)}{(z-w)^n} \tag{2.61}$$

then we define the normal ordered product $:ab:(w)$ and the contraction $\overbracket{a(z)b}(w)$ as

$$:ab:(w) = \{ab\}_0(w) \qquad \overbracket{a(z)b}(w) = \sum_{n=1}^{N} \frac{\{ab\}_n(w)}{(z-w)^n} \quad . \tag{2.62}$$

As well, a very useful formula for normal ordering comes from contour integration, as normal ordering is essentially extracting the constant term from the OPE:

$$:ab:(w) = \frac{1}{2\pi i}\oint_w \frac{dz}{z-w} a(z)b(w) \quad . \tag{2.63}$$

Finally, as to show that this normal ordering is really just a particular choice of picking the order of modes when passing from classical to quantum field theory, we give the mode equivalent of equation (2.63)

$$:ab:_m = \sum_{n\leq -h_a} a_n b_{m-n} + \sum_{n\geq -h_a} b_{m-n}a_n \tag{2.64}$$

where h_a is the conformal dimension of a. Notice in particular that this definition is not symmetric, unlike the usual normal ordering of free fields. One can also show it is also not associative: $:(:ab:)c:\neq:a(:bc:):$.

Generalised Wick Theorem

Introducing normal ordering and contractions suggests a Wick theorem, and it is indeed necessary to use a Wick theorem when computing correlation functions of composite fields. In the free case, the Wick theorem takes the following familiar form:

$$\overbracket{\varphi_1(x):\varphi_2}\varphi_3:(y) = \overbracket{\varphi_1(x)\varphi_2}(y):\varphi_3(y):+\overbracket{\varphi_1(x)\varphi_3}(y):\varphi_2(y): \quad . \tag{2.65}$$

The generalisation of this theorem to the generalised normal ordering is

$$\overbracket{a(z):bc}:(w) = \frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\left\{\overbracket{a(z)b}(x)c(w) + b(x)\overbracket{a(z)c}(w)\right\} \quad . \tag{2.66}$$

That this expression is the correct generalisation can be shown by expanding the contractions into their power series, and observing that only negative powers of $z-w$ survive. A subtlety lies in that, after expanding the contractions, one is left with full OPEs to expand, which include all regular terms in the OPE, which can and will contribute in general.
This technical result is in fact of key importance to the structure of the theory, as it allows one to extend the meaning of an OPE to composite fields. As we will show in the next section, this is needed to define a consistent closed algebra on the field space of the theory.

2.4 Chiral Algebra

We have now introduced all the elements to define the so-called Chiral Algebra of a CFT, which is the core structure that will define the different types of CFTs. The essential ingredients are then the chiral/holomorphic factorisation property, the OPE (2.36), the normal ordering equation (2.62) and the associated generalised Wick theorem (2.66). That the OPE closes in the space of holomorphic fields - i.e. for any two holomorphic fields $a(z)$ and $b(w)$, their OPE $a(z)b(w)$ will be a power series involving only other holomorphic fields - allows us to define the chiral algebra $\mathcal{A}$ of the theory. We take it to be the vector space of holomorphic fields

$$\mathcal{A} := \{a \text{ s.t. } \partial_{\bar{z}} a(z,\bar{z}) = 0\} \tag{2.67}$$

equipped with the bilinear products

$$\{\_,\_\}_n : \mathcal{A}\times\mathcal{A} \mapsto \mathcal{A} \tag{2.68}$$

$$a, b \mapsto \{ab\}_n \quad . \tag{2.69}$$

That is, we consider the space of holomorphic fields $\partial_{\bar{z}} a = 0$, on which we define infinitely many bilinear products $\{\_,\_\}_n$ indexed by $n \in \mathbb{Z}$, and which we take to be the coefficients of the OPE defined in equation (2.36). As well, considering $\{ab\}_0 =: ab :$ and $\{Ta\}_1 = \partial a$ make it clear that normal ordering and derivation are stable in the chiral algebra (T is in the algebra as we consider conformal invariant theories) .
We already encountered the simplest example of a chiral algebra, which is in fact the Virasoro algebra (or its equivalent mode formulation): it is generated by the single field T in the sense that $\mathcal{A}_T$ is built of all normal ordered products of T and its derivatives: $\mathcal{A}_T := Span(T, : TT :, \partial T, : \partial TT :, ...)$. Its chiral algebra structure is then defined by the OPE

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg \quad . \tag{2.70}$$

This example also highlights the importance of the Generalised Wick theorem, as it is the tool that allows one to extend the algebraic structure of the OPE to composite fields of a generator. Note as well that the chiral algebra $\mathcal{A}_T$ contains the same information as the homolorphic copy of the Virasoro algebra (2.49).

2.5 Integrable CFT and Quantum KdV construction

2.5.1 Integrable CFT

When considering field theories, one can deem the theory as integrable if it is possible to find an infinite set of pairwise commuting conserved charges. Typically, they would correspond to the generators of some continuous symmetries, by virtue of Noether's theorem. Recalling that conformal invariance in two dimensions comes equipped with infinitely many generators of symmetry, one would then expect to find infinitely many conserved charges. This is trivial when considering the chiral algebra: $A = \oint a(z)dz$ is conserved for any chiral field $a(z)$. However conformal invariance brings no guarantee on the commutation of these charges.

The chiral algebra provides an especially convenient framework to deal with commuting charges. Referring to subsection 2.2, specifically equation (2.45), we can see that the commutator of two charges is sensitive only to the pole term of the OPE of the densities of the charges: taking equation (2.45) for

the commutator and inserting the general OPE (2.36), one has

$$[A,B] = \oint_0 dw \oint_w dz a(z)b(w) \tag{2.71}$$

$$= \oint_0 dw \oint_w dz \sum_{n=-\infty}^{N} \frac{\{ab\}_n(w)}{(z-w)^n} \tag{2.72}$$

$$= \oint_0 dw \{ab\}_1(w) \quad . \tag{2.73}$$

$\{ab\}_1(w)$ is in general some operator, and the integral does not vanish. However, if $\{ab\}_1$ is a total derivative of some operator, then the integral vanishes immediately, as no total derivative admits a single pole in its Laurent expansion. As such, the condition for two charges to be commuting, is that the singular term of their OPE be a total derivative:

$$[A,B] = 0 \Leftrightarrow \{ab\}_1 = \partial(...) \quad . \tag{2.74}$$

This key condition will be at the center of this thesis.

2.5.2 Quantum KdV

As stated in the previous section, CFTs are essentially defined by the choice of their chiral algebra $\mathcal{A}$. As well, every field theory that is conformally invariant at quantum level will admit a Virasoro algebra: that is, the Virasoro algebra must exist amongst the sub-algebras of the chiral algebra of the theory (that is a subspace closed under OPE). Going back to the question of finding local conserved charges, establishing integrability then lies in showing that the chiral algebra of the theory contains infinitely many commuting conserved charges. One then must wonder if it is possible to find such charges for the simplest chiral algebra, that is $\mathcal{A}_T$.
The answer turns out to be yes, as was shown in the streak of papers [2][3][4][5], and we recall their results as an example, which is known as the quantum KdV (Korteweg-De-Vries) hierarchy. Using only the Virasoro algebra, one can obtain infinitely many conserved charges of the form

$$I_{2k-1} = \oint T_{2k}(z)dz \tag{2.75}$$

with $T_{2k}(z) =: T^k(z) : +...$. With ... some fields involving normal ordered products and derivatives of T such that the total conformal dimension of T_{2k} is $2k$. Explicitly, the first three densities are

$$T_2(z) = T(z), \quad T_4(z) =: TT : (z), \quad T_6(z) =: TTT : (z) + \frac{c+2}{12} : (\partial T(z))^2 : \tag{2.76}$$

and [5] shows that infinitely many of such charges exist and are uniquely fixed up to total derivatives by the commutation condition $[I_{2k-1}, I_{2l-1}] = 0$ and by their conformal dimensions. This quantum KdV hierarchy is the minimal example in which one can discuss integrable hierarchies.

Brute Force Construction

We shall really use this hierarchy as the motivating example of this thesis, and we present in some details the procedure used to obtain the first few densities of the hierarchy, as we will use similar methods later for WZW models. We then proceed order by order in the conformal dimension:

- At dimension 2, finding T_2 is trivial, as the only field is T.
- At dimension 4, the chiral algebra contains two fields: $:TT:$ and $\partial^2 T$. Since the latter one is a total derivative, it does not contribute to the charge and we ignore it. T_4 is then simply $:TT:$.
- The first non-trivial case appears at dimension 6, where the algebra contains $\partial^4 T$, $\partial^2(:TT:)$, $:TTT:$ and $:(\partial T)^2:$. Removing total derivatives and scaling freedom, one would then have

  $$T_6 =: TTT : +\alpha : (\partial T)^2 :$$

  with the α an a priori undetermined constant. It is then imposing the commutation of I_5 with I_3, that is, imposing $\{T_4 T_6\}_1 = \partial(...)$, that will fix this constant to $\alpha = \frac{c+2}{12}$, as reported in equation (2.76).

One would then in principle proceed analogously, building order by order the conserved charges. [5], however, was able to prove from the Virasoro algebra that this construction carries on indefinitely.

We will follow a conceptually identical method in chapter 3 to build order by order the hierarchy of WZW models, but we shall not attempt to prove the hierarchy at all order.

Comment on Sphere/Cylinder Charges

We mention here a subtlety that appears in equations (2.75) and (2.76), namely the use of z as a variable. Here, the conserved charges were defined on the complex plane/ Riemann sphere. However, as we mentioned with equation (2.40), one could have as well wished to build conserved commuting charges on the cylinder, then parametrised by u. When passing from the sphere to the cylinder, the fields themselves will change with the change of coordinates, for example a primary field will change as equation (2.20). It is therefore not trivial that the densities on the sphere will give densities on the cylinder. As it turns out, the commutation condition on the densities will work the same, and as such, the densities found on the sphere carry straightforwardly to densities on the cylinder. For this, one must use the conformal mapping $z = e^{iu}$. However, this will in the end change the expression of the conserved charges in terms of modes. This distinction will play an important role in the next subsection, where we will consider the integrated densities on the cylinder. We shall go into more details in appendix

D, but as a quick example:

$$I_1^{sphere} = \frac{1}{2\pi i}\oint T(z)dz = L_{-1}$$
$$I_1^{cyl} = \frac{1}{2\pi i}\oint T^{cyl}(u)du = -\left(L_0 - \frac{c}{24}\right) \tag{2.77}$$

where $T^{cyl}(u)$ is the stress energy tensor on the cylinder after the conformal mapping $z = e^{iu}$. In both cases, I_1 will serve as the generator of holomorphic translations: $L_{-1} = \partial_z$ while $L_0 = -z\partial_z = i\partial_u$. As the underlying spacetime is different, they take up different expressions in terms of modes.

Zero Mode of the Densities

As mentioned in the introduction, a main goal of integrability is to resolve the spectrum of the theory; that is, to completely characterise the Hilbert space of the theory as eigenstates of the theory's conserved charges. Then, obtaining a tower of commuting conserved charges allows one to refine the spectrum of the theory, as the charges can be diagonalised in a common basis.
The first step of this procedure is then to determine how the commuting conserved charge act on the Hilbert space of the theory: this means expressing them in terms of modes. One can use this expression in terms of modes to compute the action of the charges on the Hilbert space.
We give the example of these expressions for the KdV charges. As we mentioned previously, we will be interested in the charges defined on the cylinder, and we notice that we have in fact already given the expression for the first charge in equation (2.77):

$$I_1^{cyl} = -\left(L_0 - \frac{c}{24}\right) \quad . \tag{2.78}$$

We give here the expression in terms of modes for the second charge, as presented in [5] and [20]

$$I_3^{cyl} = \frac{1}{2\pi i}\oint :TT:^{cyl}(u)du = 2\sum_{n=1}^{+\infty} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880} \quad . \tag{2.79}$$

We give more details on how to compute these expressions in appendix D. From these expressions, one can then figure out how the charges act on the Hilbert space of the theory, and diagonalise them. We do not pursue this here.

Chapter 3

Local Conserved Charges of Wess-Zumino-Witten Models

In this section, we introduce the first main player of this thesis, the Wess-Zumino-Witten Models (WZW models) [6][7][8][9]. These paradigmatic models have become indispensable tools in modern theoretical physics, as they play a role in numerous areas. Perhaps their best-known use comes from string theory, where they describe the propagation of strings on group manifolds and play a crucial role in the AdS/CFT correspondence. They also play a role in condensed matter, where they can be used to describe the quantum Hall effect and some spin chains. However, in this thesis, we approach WZW models from the integrable CFT point of view, and we will focus on building their local commuting conserved charges.
This section introduces the WZW models, starting from their action up to deriving their chiral algebra from the Ward identities, following the steps of section 2. For subsections 3.1.1 to 3.1.4, we follow again [19], chapter 15. After this introduction, we present the classical conserved charges, due to [11][12], and present a brute force approach for obtaining quantum conserved commuting local charges. We finally report the results of this method in the $\mathfrak{g} = \mathfrak{su}(2)$ case, that is, the expression for the densities of local commuting conserved charges.

3.1 Wess-Zumino-Witten Models

We here introduce WZW models from their action, and then proceed through the machinery of section 2 to finally derive their chiral algebra.

3.1.1 Principal Chiral Model

Before introducing WZW models, we first introduce a simpler but similar theory: the Principal Chiral Model (PCM). Consider the following action

$$S_0 = \frac{1}{4a^2}\int d^2x Tr(\partial^\mu g^{-1}\partial_\mu g) \tag{3.1}$$

with a^2 a positive, dimensionless coupling constant, and $g(x)$ a matrix bosonic field valued in a semisimple compact group manifold G, associated with a Lie algebra $\mathfrak{g}$. This action defines the so called PCM. For this action to be real valued, g must be valued in a unitary representation of G. The trace Tr is then taken over the generators of $\mathfrak{g}$, which we define with the following normalisation

$$Tr(T^a T^b) = 2\delta^{ab}, \qquad [T^a, T^b] = if^{ab}_{\ c}T^c \tag{3.2}$$

where T^a stands for the generator of $\mathfrak{g}$ in any unitary matrix representation. This action has two symmetries by right and left global group action $g \to Lg$ and $g \to gR^{-1}$, where L, R are in the aforementioned unitary representation of G. We say the PCM has $G \times G$ symmetry. Then, associated to both symmetries are conserved currents J^μ and $\tilde{J}^\mu$ for the right and left action respectively. They are given by

$$J_\mu = g^{-1}\partial_\mu g \qquad \tilde{J}_\mu = \partial_\mu g g^{-1} \tag{3.3}$$

and they satisfy the following conservation equations

$$\partial_\mu J^\mu = \partial_\mu \tilde{J}^\mu = 0 \quad . \tag{3.4}$$

Expanding in complex coordinates, this is then

$$\partial_{\bar{z}} J_z + \partial_z J_{\bar{z}} = \partial_{\bar{z}} \tilde{J}_z + \partial_z \tilde{J}_{\bar{z}} = 0 \tag{3.5}$$

where we have lowered the coordinate index using the metric of equation (2.15): $J^z = g^{z\mu}J_\mu = 2J_{\bar{z}}$. However, neither J nor $\tilde{J}$ are holomorphic/anti-holomorphic currents, in the sense that $\partial_\mu J^\mu \neq \partial_{\bar{z}} J_z$, such that they are not chiral currents. This is a sign that this theory is not conformal invariant at quantum level.
Indeed, it is known that, while this theory is classically conformal invariant, renormalisation grants a scale dependancy to the coupling parameter, which breaks conformal invariance at quantum level.

3.1.2 Wess-Zumino Term and WZW models

We want to obtain a quantum conformal invariant theory, in which the currents become chiral. For this, one considers the following Wess-Zumino term

$$\Gamma = \frac{i}{24\pi}\int_B d^3y \varepsilon_{\alpha\beta\gamma} Tr'(\tilde{g}^{-1}\partial^\alpha \tilde{g}\tilde{g}^{-1}\partial^\beta \tilde{g}\tilde{g}^{-1}\partial^\gamma \tilde{g}) \tag{3.6}$$

which is defined on a three dimensional manifold B whose boundary is the original two dimensional worldsheet on which we took the PCM action (3.1), and $\tilde{g}$ is the extension of the field g to this three dimensional manifold. One then considers the action

$$S = S_0 + k\Gamma \tag{3.7}$$

where k is an integer, called the level of the model. That k is an integer is non-trivial, and comes from topological considerations, which we present in appendix B.1.

We now turn to the currents. Again, the theory is invariant by left and right group action, which gives rise to two conserved currents:

$$J_\mu = \left(\left(1 + \frac{a^2 k}{4\pi}\right)(g^{-1}\partial_z g), \left(1 - \frac{a^2 k}{4\pi}\right)(g^{-1}\partial_{\bar{z}} g)\right) \tag{3.8}$$

$$\tilde{J}_\mu = \left(\left(1 - \frac{a^2 k}{4\pi}\right)(\partial_z g g^{-1}), \left(1 + \frac{a^2 k}{4\pi}\right)(\partial_{\bar{z}} g g^{-1})\right) \tag{3.9}$$

which again satisfy conservation equations:

$$\partial_{\bar{z}} J_z + \partial_z J_{\bar{z}} = \partial_{\bar{z}} \tilde{J}_z + \partial_z \tilde{J}_{\bar{z}} = 0 \quad . \tag{3.10}$$

We see that picking $a^2 = 4\pi/k$ kills $J_{\bar{z}}$ and $\tilde{J}_z$. This then reduces the conservation equations to

$$\partial_{\bar{z}} J_z = \partial_z \tilde{J}_{\bar{z}} = 0 \tag{3.11}$$

which is now the conservation law of a holomorphic current J_z and an anti-holomorphic current $\tilde{J}_{\bar{z}}$. This choice of a implies k to be a positive integer; one could also have chosen $a^2 = -4\pi/k$, which would require a negative integer and lead to the conservation of the $\tilde{J}_z$ as holomorphic and $J_{\bar{z}}$ as anti holomorphic, which is equivalent and irrelevant for the algebra of the theory.

Having the conserved currents associated to the right and left group actions be holomorphic and anti-holomorphic allows us to promote the previous global group action to a chiral one, as any local holomorphic (resp. anti.) transformation will preserve the equations of motion (3.11). One then has invariance under

$$g(z, \bar{z}) \to h_L(\bar{z}) g(z, \bar{z}) h_R^{-1}(z) \tag{3.12}$$

with h_L and h_R two independent matrices valued in the unitary representation of G. The global $G \times G$ invariance of the PCM has been extended to a local $G(z) \times G(\bar{z})$ invariance.

3.1.3 Affine Lie Algebras

We now consider the holomorphic current $J = J_z = J(z)$, and we wish to study its operator properties; its OPEs, mode expansion and commutation relations. As in section 2.4, the goal is to derive the Chiral Algebra of the theory, which will be equivalent to its algebra in terms of commutation relations. We will simply state the final results here, with the more detailed derivation kept for appendix B.2.
The OPE of the currents is then

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^c(w)}{z-w} + reg \tag{3.13}$$

with J^a the component of J expanded in the generators of $\mathfrak{g}$[1]:

$$J = \sum_{a=1}^{\dim \mathfrak{g}} J^a T_a \quad . \tag{3.14}$$

This OPE defines the chiral algebra of the theory, which we will call current algebra or affine Kac-Moody algebra, for reasons that will become clear immediately. We call this algebra $\mathcal{A}_J$ and keep in mind that it is defined by the Lie algebra $\mathfrak{g}$ of the group G and the level of the theory k.
As in the Virasoro case, equation (2.49), we can convert this chiral algebra into an algebra in terms of modes: we first expand J as in equation (2.38)

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1} \tag{3.15}$$

and using equation (2.47) we obtain

$$[J^a_n, J^b_m] = if^{ab}{}_c J^c_{n+m} + kn\delta^{ab}\delta_{n+m,0} \quad . \tag{3.16}$$

These commutation relations define what is known as an affine Lie algebra $\hat{\mathfrak{g}}_k$, or affine Kac-Moody algebra, which is an infinite dimensional symmetry algebra. This gives the name of Kac-Moody current to J^a, and explains the previous name for the chiral algebra. One gets a second independent copy of this algebra for the anti-holomorphic current, illustrating again the splitting of the structure.

3.1.4 Sugawara Construction

We now wonder about the quantum conformal invariance. We showed in section 3.1.3 that the fields of the WZW models fit into an affine Kac-Moody algebra symmetry. However, as argued in section 2.4, quantum conformal invariance imposes that the stress-energy tensor - the generator of conformal

[1] formally the generators of $\mathfrak{g}^*$, but we will take an orthogonal basis

transformations - fits into a Virasoro algebra, which is $\mathcal{A}_T \subset \mathcal{A}_J$. The classical stress-energy tensor for WZW models is $(1/2k)\sum_a J_a J^a$, and we look for a quantum version of this expression as a normal ordered product

$$T(z) = \gamma \sum_a : J_a J^a : (z) \quad . \tag{3.17}$$

γ is a so-far undetermined constant: it cannot be determined from the classical theory, as it is subject to quantum corrections. To fix it, we have two conditions: first, that the OPE of T with itself is of the correct form of equation (2.37)

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg \tag{3.18}$$

and the second one being the requirement that J^a transforms as a primary field of conformal dimension 1

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} + reg \quad . \tag{3.19}$$

To compute equation (3.19), one needs to compute contractions of the form $\overbrace{J^a(z):J_b J^b:}(w)$. This requires the use of the generalised Wick theorem (2.66), in which one needs to perform stacked OPEs (3.13). We start with

$$\begin{aligned}
\overbrace{J^a(z):J_b J^b:}(w) &= \frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\left[\overbrace{J^a(z)J_b}(x)J^b(w) + J_b(x)\overbrace{J^a(z)J^b}(w)\right] \\
&= \frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\left\{\left[\frac{k\delta^a_b}{(z-x)^2} + \sum_c i f^a_{\ bc}\frac{J^c(x)}{z-x}\right]J^b(w)\right. \\
&\quad \left. + J_b(x)\left[\frac{k\delta^{ab}}{(z-w)^2} + \sum_c i f^{ab}_{\ \ c}\frac{J^c(w)}{z-w}\right]\right\}
\end{aligned}$$

where in this equation the b index is not summed over, and we used the OPE (3.13) in the contractions. However, we have to once again use this OPE in its full form, including the normal ordered terms. This yields

$$\begin{aligned}
\overbrace{J^a(z):J_b J^b:}(w) =& \frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\left\{\frac{k\delta^a_b J^b(w)}{(z-x)^2}\right. \\
&+ \sum_c \frac{i f^a_{\ bc}}{z-x}\left[i f^{cb}_{\ \ d}\frac{J^d(w)}{x-w} + \frac{k\delta^{cb}}{(x-w)^2} + :J^c J^b:(w)\right] \\
&\left. + \frac{k\delta^a_b J^b(w)}{(z-w)^2} + \sum_c i f^a_{\ bc}\frac{:J^b J^c:(w)}{z-w}\right\}
\end{aligned}$$

The $f^a_{\ bc}\delta^{bc}$ term dies by antisymmetry. We now add the sum over b and use the following identity/definition

$$-\sum_{b,c} f^a_{\ bc} f^{cbd} = 2\check{h}\delta^{ad}$$

with $\check{h}$ the dual Coxeter number. In the end, we get

$$\overbrace{J^a(z)\sum_b : J_b J^b :}(w) = 2(k+\check{h})\frac{J^a(w)}{(z-w)^2} \quad . \tag{3.20}$$

Now, we swap the two contracted fields, multiply by γ and expand around w:

$$\overbrace{T(z)J^a}(w) = \gamma 2(k+\check{h})\frac{J^a(z)}{(z-w)^2} \tag{3.21}$$

$$= \gamma 2(k+\check{h})\left\{\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)}\right\} \quad . \tag{3.22}$$

Comparing with equation (3.19), one sees that for T to be the stress-energy tensor, one must impose

$$\gamma = \frac{1}{2(k+\check{h})} \tag{3.23}$$

that is

$$T(z) = \frac{1}{2(k+\check{h})}\sum_a : J_a J^a : (z) \quad . \tag{3.24}$$

This expression is known as the Sugawara construction [21], and T is then the Sugawara stress energy tensor.
From this expression, one can work out the central charge (equation (3.18)) by taking the OPE of T with itself, and one finds

$$c = \frac{k\dim\mathfrak{g}}{k+\check{h}} \quad . \tag{3.25}$$

One can evidently perform an exactly analog construction for the anti -holomorphic field $\bar{T}$ expressed in terms of $\bar{J}^a$.
The Sugawara construction is an explicit embedding of a Virasoro algebra in an affine Kac-Moody one: every theory with an affine Kac-Moody symmetry will admit an operator construction of a stress-energy tensor, and therefore, be conformally invariant.
Finally, one can expand as usual T in modes L_n, which will then satisfy the Virasoro algebra. Their expression in terms of current modes J^a_n is

$$L_n = \frac{1}{2(k+\check{h})}\sum_a\sum_m : J_{a,m}J^a_{n-m} : \tag{3.26}$$

and they satisfy

$$[L_n, J^a_m] = -mJ^a_{n+m} \quad . \tag{3.27}$$

3.2 Brute Force Approach to Conserved Charges

Referring to section 2.5, it is known that any two dimensional conformal field theory will admit an infinite tower of commuting charges, generated by the KdV charges inside the theory's Virasoro algebra. However in WZW models, the fundamental fields of the chiral algebra [2]are the (holomorphic) Kac-Moody currents $J^a(z)$, which satisfy the OPE (3.13), and while the Virasoro algebra can be embedded in the affine Kac-Moody algebra (see 3.1.4), the KdV charges of [5] only make reference to the Virasoro algebra, and completely ignore the finer structure of the currents. As such, one can wonder if there exists another such hierarchy which picks up this finer structure. Classically, this question has been solved by [11][12] which were able to construct an infinite hierarchy of Poisson-commuting charges.

Our main goal is then to find a quantum version of these charges. We shall look for local densities inspired by the classical version. However, we do not attempt a general proof of the existence of this hierarchy, and instead focus on obtaining explicit expressions for the charges at low conformal dimension.
The strategy is to take a general ansatz inspired by the classical charges which takes the form of classical charge + all possible corrections. Then, we will solve for the coefficients of the different fields by imposing the key constraint (2.74). This ansatz is very general, and in practice, taking as many coefficients as fields in the sub-space of conformal dimension h is a very expensive process. However, global group invariance comes in to restrict the ansatz.

3.2.1 Classical charges and Group Invariance

Classical Charges

We first introduce the conserved commuting charges of the classical theory, by which we mean the theory which classical currents satisfy a classical Poisson-Kac-Moody algebra. This is simply equation (3.16) with commutators replaced by Poisson brackets. These charges are due to [11][12] and shall serve as our guide for looking at quantum level.
The classical Poisson commuting charges are then given by

$$I_p^{class} = \frac{1}{2\pi i}\oint T_{p+1}^{class}(z)dz, \qquad T_{p+1}^{class} = \kappa^{EHMM}_{a_1\dots a_{p+1}} J^{a_1}\dots J^{a_{p+1}} \tag{3.28}$$

where $\kappa^{EHMM}_{a_1\dots a_{p+1}}$ are the so-called EHMM tensors, in honor of Evans, Hassan, Mountain, McKay to whom these charges are due [11][12], see [13] for a

[2]the actual fundamental fields are the group valued fields g, but they are not chiral

review. These tensors satisfy very specific properties (see [11],[13]) which ensure that $\{I_p, I_{p'}\} = 0$.[3]
These properties have as a consequence that the rank $p+1$ of these tensors must be such that p is an affine exponent of $\hat{\mathfrak{g}}$. These exponents are a subset of positive integers which depend on the choice of $\mathfrak{g}$; for example, they are $\mathbb{N}$ modulo N for $\mathfrak{su}(N)$. This then fixes the conformal dimensions of the conserved charges I_p to be the affine exponents of $\hat{\mathfrak{g}}$, as the conformal dimension of the density will have to coincide with the rank of the tensor, as can be seen in the expression for T^{class}_{p+1}.

One property that these tensors must satisfy is that these tensors belong to the class of invariant tensors of the group G: this guarantees that the conserved charges I_p are invariant under the global (right) action of the group. As these tensors will be relevant at quantum level later on, we now detail the construction of these invariant tensors.

Group Invariance and Invariant Tensors

We clarify here what we mean by group action and group invariance. As shown with equation (3.12), the theory is invariant under left and right global group action. As well, the holomorphic current $J^a(z)$ transforms only under the adjoint right group action as

$$J \to hJh^{-1} \text{ for } h \in G \tag{3.29}$$

where we, referring to equation (3.12), we have replace h_R with h.
There exist invariant tensors of the group G, which were already relevant with the classical charges. We now define them.
One starts by considering a function κ

$$\begin{aligned}&\kappa : \mathfrak{g}^{\otimes n} \mapsto \mathbb{C}\\ &X_1, ..., X_n \mapsto \kappa_{a_1 ... a_n} X_1^{a_1} ... X_n^{a_n}\end{aligned}$$

with $X_i = X_i^a T_a$, and T_a the generators of $\mathfrak{g}$. We consider these Xs as they transform as J under group action. Then, demanding that $\kappa(X_1, ..., X_n)$ be invariant under adjoint group action is the condition

$$0 = \kappa(X_1, ..., X_n) - \kappa(hX_1h^{-1}, ..., hX_nh^{-1}) \quad . \tag{3.30}$$

To see how this constrains $\kappa_{a_1 ... a_n}$, we consider infinitesimal group action on X_j

$$\begin{aligned}X_j \to& hX_jh^{-1}\\ =& X_j + i[\alpha, X_j]\\ =& (X_j^a + if^a_{\ bc}\alpha^b X_j^c)T_a \quad \text{with} \quad h = e^{i\alpha} = e^{i\alpha^a T_a}\end{aligned}$$

[3] $\{\_,\_\}$ is here the Poisson bracket, not the OPE

then, the invariance of $\kappa(X_1, ..., X_n)$ amounts to

$$\begin{aligned} 0 &= \kappa(X_1, ..., X_n) - \kappa(hX_1h^{-1}, ..., hX_nh^{-1}) \\ &= -i\alpha^b(f^c_{ba_1}\kappa_{ca_2...a_n} + f^c_{ba_2}\kappa_{a_1ca_3...a_n} + ... + f^c_{ba_n}\kappa_{a_1...a_{n-1}c})X_1^{a_1}...X_n^{a_n} \quad . \end{aligned}$$

As such, the equation defining the invariant tensors at any order is

$$f^c_{ba_1}\kappa_{ca_2...a_n} + f^c_{ba_2}\kappa_{a_1ca_3...a_n} + ... + f^c_{ba_n}\kappa_{a_1...a_{n-1}c} = 0 \quad . \tag{3.31}$$

This equation can be solved numerically and gives the desired tensors. However, numerically solving this equation does not give traceable solutions, in the sense that it does not give a formula for each solution.

A class of traceable solutions that satisfy condition (3.31) is given by

$$\kappa_{a_1...a_n} = Tr(T_{a_1}...T_{a_{k_1}})Tr(T_{a_{k_1+1}}...T_{a_{k_1+k_2}})...Tr(T_{a_{\sum_i^{l-1}k_i+1}}...T_{a_n}) \tag{3.32}$$

with $\sum_{i=1}^{l} k_i = n$. These solutions are guaranteed to satisfy condition (3.31) as

$$\begin{aligned} \kappa(X_1, ..., X_n) =& \kappa_{a_1...a_n}X_1^{a_1}...X_n^{a_n} \\ =& Tr(X_1...X_{k_1})...Tr(X_{\sum_i^{l-1}k_i+1}...X_n) \end{aligned} \tag{3.33}$$

is immediately group invariant.
Amongst the solution to equation (3.31) are the EHMM tensors of the classical densities (equation (3.28)), which also happen to be fully symmetric. However, in general, there are many more invariant tensors, and they will become important when considering the quantum densities, as we shall see immediately.
As well, note that equation (3.33) will produce in general redundant tensors, as simply permuting traces will give identical tensors with a different choice of $\{k_i\}$s; as well, cyclicity of the trace will bring in internal symmetries which will increase again the redundancy.

3.2.2 Casimir Algebra

Casimir Algebra

When looking at the quantum theory, one can see that simply promoting the densities of equation (3.28) to a normal ordered version will prove insufficient. As such, these charges will receive corrections at quantum level. As we still want the conserved charges to be invariant under the global group action, we will have to take specific combinations of the fields; they are built from the invariant tensors of the group, as defined by equation (3.31).

We call the space of group invariant local fields the Casimir (sub) algebra of the theory, which we write C. The first element in this sub algebra is simply

$$\kappa_{ab} : J^a J^b : \tag{3.34}$$

where κ is the Killing form of $\mathfrak{g}$ in its fundamental representation. With the normalisation (3.2), $\kappa_{ab} = 2\delta_{ab}$. This field is known as the Quadratic Casimir field of $\hat{\mathfrak{g}}$, and is actually proportional to T. We see that the stress energy tensor is always included in the Casimir algebra of the theory, and in fact, T will always be the first conserved charge density that we find, as predicted by the classical construction.

We call the subspace of the Casimir algebra at conformal dimension h C_h. It will be important to be able to obtain an explicit basis of the fields of C_h for every h. Using the invariant tensors, the fields of C_h will then be linear combinations of fields of the form

$$\kappa_{a_1 \ldots a_h} : J^{a_1} \ldots . J^{a_h} : \tag{3.35}$$

as well as all possible combinations of the form

$$\kappa_{a_1 .. a_n} : \partial^{k_1} J^{a_1} \ldots \partial^{k_n} J^{a_n} :, \quad \text{with} \quad n + \sum_{i=1}^{n} k_i = h \quad . \tag{3.36}$$

In these two expressions, one can use in principle all possible invariant tensors for κ. However, as already mentioned with equation (3.33), many of the tensors are redundant. As well, because of normal ordering and rearrangement of fields within normal ordering, many different invariant tensors will give linearly dependant fields once contracted with the normal ordered products of currents and their derivatives.
One must therefore be careful when building an explicit basis of the Casimir algebra. While it is most convenient to use equation (3.33) to obtain the tensors, one must always check that the fields built from these tensors are linearly independent of the previously obtained fields.

Two considerations come in to help us build the basis of C_h. First is knowledge of the character of the Casimir algebra, which we present in the next section. As well, having available a lexicographic basis in which to project our fields is essential for checking linear independence of the fields. We present this in appendix C.1.

Character of the Casimir Algebra

Group invariance helps us restrict further the search for an invariant basis of fields, through the knowledge of the character of the Casimir Algebra χ.

This character acts as a generating functional for the number N_h of independant fields in C_h:

$$\chi(q) = \sum_{h=0}^{\infty} N_h q^h \quad . \tag{3.37}$$

Therefore, knowledge of the character greatly simplifies the task of building a basis of the Casimir algebra: one can stop looking for fields as soon as the number of independent fields found reaches N_h. We detail how we built a basis of the Casimir algebra in appendix C.2. As an example, the character for the Casimir algebra of $\mathfrak{su}(2)$ WZW is

$$\chi(q) = \frac{1}{\prod_{n=1}(1-q^n)^3}\left(1 + 3\sum_{n=1}(-1)^n q^{n(n+1)/2} + \sum_{n=2}(-1)^n q^{n(n+1)/2-1}\right) \tag{3.38}$$

which first few orders are

$$\chi(q) = 1 + q^2 + q^3 + 3q^4 + 3q^5 + 8q^6 + ... \quad . \tag{3.39}$$

The expression for this character is due to [22].

3.2.3 Imposing the Commutation

We now have available a basis of C_h at every conformal dimension. The idea is then as follows: we start with the classical construction, and promote the classical densities (3.28) to a normal ordered version. Evidently, this is not enough, and we need to include corrections. We then take a general ansatz, using a basis of the Casimir algebra, conformal dimension by conformal dimension, and fix the coefficients of these fields such that the commutation condition is satisfied.
Denoting the fields in a basis of C_h as $\{c_i^h\}$, we write a general ansatz for the conserved charge density at dimension h

$$T^h = \sum_{i}^{N_h} \alpha_i^h c_i^h \tag{3.40}$$

and we explicitly compute

$$\{T^h T^{h'}\}_1 \quad . \tag{3.41}$$

Then, for $T^h, T^{h'}$ to give commuting charges, we need $\{T^h T^{h'}\}_1$ to be a total derivative, as in condition (2.74). To check whether this is a total derivative, one has to identify a sub-basis of total derivatives in the Casimir algebra at dimension $h + h' - 1$, and check for linear dependence of $\{T^h T^{h'}\}_1$ in this sub-basis. However, a necessary but not sufficient condition for a field Ψ to be a total derivative is

$$\Psi = \partial Y \Rightarrow \{\Psi J^a\}_1 = 0 \tag{3.42}$$

as

$$\Upsilon(z)J^a(w) = ... + \frac{\{\Upsilon J^a\}_1(w)}{z-w} + :\Upsilon J^a: + ...$$
$$\Rightarrow \Psi(z)J^a(w) = \partial_z\Upsilon(z)J^a(w) = ... + \frac{-\{\Upsilon J^a\}_1(w)}{(z-w)^2} + :\partial\Upsilon J^a: + ... \quad .$$

Condition (3.42) is easier to solve in practice as it does not require a higher conformal dimension basis. Therefore, we solve

$$\{\{T^h T^{h'}\}_1 J^a\}_1 = 0 \tag{3.43}$$

for the coefficients α_i^h, and later check that the solutions satisfy condition (2.74).

3.3 $\mathfrak{su}(2)$ Results

We now present the results of the previously described procedure, considering the $\mathfrak{su}(2)$ WZW model at arbitrary level. In this specific case, the structure constants f^{abc} reduce to Levi-Civita symbols ε^{abc}, the dimension of the algebra is $\dim \mathfrak{g} = 3$, and the dual Coxeter number is $\check{h} = 2$. As well, the affine exponents of $\hat{\mathfrak{su}}(2)$ are every odd number. Referring to section 3.2.1, we then expect to find densities at every even conformal dimension.

We present then the consecutive steps that were taken to obtain these charges, which follow a similar logic to the brute force method presented for the quantum KdV hierarchy in section 2.5.

Dimension 4

As we mentioned in section 3.2.1, the conserved charges occur at every odd integer, which means the densities will occur at every even integer conformal dimension. At dimension $h = 2$, finding the density is trivial as, as in the KdV case, the only density is T.

Then, the first non-trivial step occurs at $h = 4$. According to the character formula (3.39), a basis of C_4 will contain three fields. One choice of basis is then

$$C_4 = \left\{\kappa^4_{abcd} : J^a J^b J^c J^d :, \quad \delta_{ab}(: J^a J^b :)'', \quad \delta_{ab} : J^a J^{b\prime\prime} : \right\} \tag{3.44}$$

where $\kappa^4_{abcd} = \delta_{(ab}\delta_{cd)}$ is the unique EHMM tensor of rank 4 for $\mathfrak{su}(2)$ (and $g_{(ab)}$ is the symmetrisation over the indices of the tensor). Notice that the second field in C_4 is proportional to T'', which is a total derivative, and as

such will never contribute to the charges. As such, before imposing any constraint of commutation, the most general density we will consider is

$$T_4 = \kappa^4_{abcd} : J^a J^b J^c J^d : -\Delta\delta_{ab} : J^a J^{b\prime\prime} : \quad . \tag{3.45}$$

We now want to constrain Δ such that the integral of T_4 commutes with all the other charges, that is condition (2.74), or condition (3.43) in practice. So far, the only available density is $T_2 = T$. However, as it turns out, imposing $\{\{T_2 T_4\}_1 J^a\}_1 = 0$ does not bring any constraint on Δ, as T acts simply as the derivation operator on T_4. Therefore, to constrain Δ, we need to look at the next conformal dimension, that is $h = 6$.

Dimension 6

We now have to build the Casimir algebra C_6. According to the character (3.39), its basis should contain 8 fields. Immediately, one has that 3 of these fields are total derivatives, and are simply the second derivatives of the fields at $h = 4$: $C_4'' \subset C_6$. As for the other fields, one has the generalisation of the classical charge T_6^{class} involving an EHMM tensor, as well as 4 other fields which shall involve other invariant tensors as well as derivatives of the currents. See appendix C.2 for a detailed construction of the basis. Explicitly, one choice of basis of the Casimir algebra at dimension 6 is

$$\begin{aligned} C_6 = \Big\{ & \delta_{ab} : J^{a,(4)} J^b :, \quad f_{abc} : J^{a\prime\prime} J^{b\prime} J^c, \quad s^4_{abcd} : J^{a\prime\prime} J^b J^c J^d :, \\ & s^4_{abcd} : J^{a\prime} J^{b\prime} J^c J^d :, \quad s^6_{abcdef} : J^a J^b J^c J^d J^e J^f :, \quad C4'' \Big\} \end{aligned}$$

where $s^4_{abcd} = Tr(t_a t_b t_c t_d)$, and $s^6_{abcdef} = Tr(t_a t_b t_c t_d t_e t_f)$. We here have chosen to include the field in s^6 instead of the field with the EHMM tensor κ^6 for computational reasons [4]. As we shall see, there are many choices of basis for the Casimir algebra. Again, the last 3 fields are total derivatives, such that we can ignore them. Then, we can take a general ansatz for the 6-th density as

$$T_6 = \sum_{i=1}^{5} \beta_i c_i^6 \tag{3.46}$$

where c_i^6 is just the i-th field of C_6, and β_i are a priori free coefficients.

The commutation

We can now impose condition (3.43) [5], that is $\{\{T_4 T_6\}_1 J^a\}_1 = 0$. As it turns out, this completely fixes the freedom of Δ and of all the β_is up to one overall

[4] it would have been more natural to take κ^6, but this changes nothing to the final charge.

[5] This was in practice done with the Mathematica package [23].

free scaling. It turns out there are only two possible solutions. They are

$$\begin{aligned} \Delta = k+5, \quad & \beta_1 = 1176 + 628k + 59k^2, \beta_2 = 984 + 420k, \\ & \beta_3 = 480 + 276k, \beta_4 = -168 + 60k, \beta_5 = 72 \end{aligned} \tag{3.47}$$

$$\begin{aligned} \Delta = \frac{19+5k}{3}, \quad & \beta_1 = 66168 + 41524k + 5845k^2, \beta_2 = 60(758 + 329k) \\ & \beta_3 = 300(88 + 49k), \beta_4 = 300(-26 + 7k), \beta_5 = 3000 \quad . \end{aligned} \tag{3.48}$$

These solutions are not particularly nice, and it seems weird that there are two of them, as we started by looking for corrections to a single solution.

Interpreting the results

We now interpret the previously found results. The first thing to recall is that the densities we found are defined up to a total derivative and a scaling. Because of this, some seemingly unobvious equalities can happen. In fact, if we consider the first solution (3.47), one can find that

$$-\frac{1}{(2+k)^2} T^4_{sol1} - \frac{1}{2(2+k)} T'' =: TT : \tag{3.49}$$

with

$$T^4_{sol1} = \kappa^4_{abcd} : J^a J^b J^c J^d : -(k+5)\delta_{ab} : J^a J^{b\prime\prime} : \quad . \tag{3.50}$$

This is just the second KdV density (equation (2.76))! This makes sense; if we recall 3.1.4, it was argued that one must have $\mathcal{A}_T \subset \mathcal{A}_J$, that is, one must recover the Virasoro algebra as a sub-algebra of the Kac-Moody algebra. Then, as shown in section 2.5, any theory with a Virasoro algebra will admit a quantum KdV construction. In fact, it turns out that T_6 with the values of β_i of equation (3.47) is also equal up to a scaling and total derivative to the KdV density $T^{KdV}_6 =: TTT : -\frac{c+2}{12} : \partial T \partial T :$. That we obtain the KdV hierarchy as a solution after applying this method is a very important sanity check on the method, and shows that it is a valid approach to obtaining the charges.

Now onto the second solution (3.48). We applied the same "change of basis" as in equation (3.49). This gave the following result:

$$-\frac{1}{(2+k)^2} T^4_{sol2} - \frac{1}{2(2+k)} T'' =: TT : + \frac{2}{3} \frac{\delta_{ab} : J^{a\prime\prime} J^b :}{2+k} \tag{3.51}$$

with

$$T^4_{sol2} = \kappa^4_{abcd} : J^a J^b J^c J^d : -\frac{19+5k}{3} \delta_{ab} : J^a J^{b\prime\prime} : \quad . \tag{3.52}$$

This is now different from KdV! As well, applying again a "change of basis" on the T_6 associated to this solution does not give back the KdV density,

(although we shall not report its explicit form, as it does not bring any insights). As well, we report that we were able to determine a fourth non-KdV density at dimension $h = 8$, which expression we do not report, and that its expression was completely fixed (up to scaling and total derivatives) by imposing condition (3.43) for T_4 and T_8.

In conclusion, the reported result is then the existence of at least four commuting conserved charges, which differ from the quantum KdV (Virasoro) ones. Their first two densities are given as

$$T_2(z) = T(z), \qquad T_4(z) =: TT : (z) + \frac{2}{3}\frac{: J^{a\prime\prime}J^a : (z)}{2+k} \tag{3.53}$$

up to total derivatives and overall scaling.
As well, the sixth and eighth charges are of the form $T_{2k} = T_{2k,KdV} + ...$, where $T_{2k,KdV}$ refers to the charges from [5] (equation (2.76)), and ... is some normal ordered product of the affine Kac-Moody currents, which is not writable as terms of the form $\partial^k T^n$. However, we do not report the explicit form of T_8, as it is not particularly instructive, and takes up much space. For computational reasons, the analysis was not pushed further than conformal dimension eight, but we expect the method to be in principle identical at higher dimensions, and to carry through.

Additionally, we report that applying the method of section 3.2 also yielded as solutions to the key condition (2.74) the quantum KdV charges, which constitutes an important check of the method.

Zero Mode of the Densities

Again, as in section 2.5.2, we give the expression of the conserved charges in terms of modes of the fields. The expression is then

$$\begin{aligned} I_3 =& \frac{1}{2\pi i}\oint T_4^{cyl}(u)du \\ =& 2\sum_{n=1}^{+\infty} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 - \frac{2}{3}\frac{1}{2+k}\sum_{n=1}^{\infty} n^2 J^a_{-n}J^a_n + \frac{c(c+6)}{576} \end{aligned} \tag{3.54}$$

where we derive this expression in appendix 4. In this expression $T_4^{cyl}(u)$ is the cylinder equivalent field to $T_4(z)$ on the plane, which we also explain how to obtain in appendix D.

The next step would be to use this expression to work out how I_3 acts on the Hilbert space of the theory and then to diagonalise it on this Hilbert space. As for section 2.5, we do not pursue this endeavour in the main text, but we present some preliminary calculations in appendix D.

Chapter 4

Local Charges of Affine Gaudin Models

In this section, we introduce Affine Gaudin Models (AGMs), a N current generalisation of WZW models. These models constitute an important tool in the study of quantum integrability, as the conjecture on their local conserved charges provides a hint of first principle quantisation.

We first introduce AGMs in more detail and present the conjecture on local conserved quantum charges due to [18]. We will follow the review [13] of [14][15][16]. We then present some explicit calculations of the conserved charge densities, in the one and two currents cases, and we investigate their dependency on the model's external parameters. Finally, we present an interesting result on a rational reparametrisation of these densities.

4.1 Affine Gaudin Models

In this section, we define the quantum Affine Gaudin Models by first discussing the algebraic structure of the model. Additionally, we present their formulations in terms of Lax-Gaudin matrix and twist functions, and we briefly mention their non-local conserved charges. The general references for this section are [13],[17] and [24].

4.1.1 Chiral and Casimir Algebras of Affine Gaudin Models

We first address a slight naming ambiguity: Formally, AGMs, are a non-conformally invariant theory with $2N$, non-holomorphic currents (similar to the PCM currents (3.3)). For certain configurations, these AGMs display a conformal point in their renormalisation group flow, at which the $2N$ currents will split into N holomorphic and N anti-holomorphic currents. Then, what we shall call AGMs in this thesis are the chiral AGMs of N holomorphic Kac-Moody currents at the conformal point. As in the WZW case, one also obtains an anti-holomorphic version, which is completely analogous.

From the point of view of the algebra, AGMs are a straightforward extension of the WZW models: where WZW models have a single (holomorphic) Kac-Moody current J^a at level k, AGMs consists of N independent currents J^a_r each with level k_r $r = 1, ..., N$, all with the same associated Lie algebra $\mathfrak{g}$. Then, the OPE of these currents is directly

$$J^a_r(z)J^b_p(w) = \delta_{rp}\left(\frac{k_r\delta^{ab}}{(z-w)^2} + \frac{if^{ab}_{\ c}J^c_r(w)}{z-w} + reg\right) \tag{4.1}$$

which is simply then N independent copies of the OPE (3.13).
This OPE defines the chiral algebra of the theory, which is $\mathcal{A}_{AGM} = \bigotimes_{i=1}^{N}\mathcal{A}_{J_i}$[1].
As well, one can easily build a Virasoro sub-algebra $\mathcal{A}_T$, generated by the stress-energy tensor

$$T = \sum_{i=1}^{N} T_i, \qquad T_i = \frac{:J_{a,i}J^a_i:}{2(k_i + \check{h})} \tag{4.2}$$

which is simply the sum of the Sugawara stress-energy tensors T_i associated to each current (as in 3.1.4) and generates the conformal transformations of the theory. One can see that it is the correct expression for the stress-energy tensor by writing its OPE with the currents

$$T(z)J^a_r(w) = \frac{J^a_r(w)}{(z-w)^2} + \frac{\partial J^a_r(w)}{z-w} + reg \tag{4.3}$$

which matches the OPE (2.35). As well, one straightforwardly has that the central charge of the system c is

$$c = \sum_{r=1}^{N} c_r = \sum_{r=1}^{N} \frac{k_r \dim \mathfrak{g}}{k_r + \check{h}} \quad .$$

As it turns out, AGMs have an additional global symmetry, called diagonal symmetry generated by the integral of $J_{diag} = \sum_{r=1}^{N} J_r$ and which acts as

$$J_r \to h^{-1}J_r h \qquad \forall r \in \{1, ..., N\} \quad . \tag{4.4}$$

This generalises the global group (right) invariance of WZW models. Then, as was the case for WZW models, we shall look for conserved charges invariant under this diagonal action, which means taking densities in a a diagonal invariant sub-algebra of the Chiral algebra $\mathcal{A}_{inv} := \{\varphi \in \mathcal{A}_{AGM} s.t. \{\varphi J_{diag}\}_1 = 0\}$, which is the sub-algebra of "commuting" fields with the generator of the symmetry. This is the "Casimir" algebra of the AGM, in the sense that it's the natural generalisation of the WZW's Casimir algebra.

[1] Here, the tensor product is understood as normal ordered tensor products of the fields, i.e. $:J^a_1J^b_2:$.

4.1.2 Lax-Gaudin Matrix and Twist Function

As it turns out, AGMs come equipped with an additional N parameters. They are written $\{\lambda_r\}_{r=1}^N$, and we call them the punctures associated to the currents: one defines each Kac-Moody current J_r^a to be attached to a corresponding puncture $\lambda_r \in \mathbb{CP}^1$. We call this $\mathbb{CP}^1$ the auxiliary or spectral space, and we shall explore it using an auxiliary or spectral parameter λ. This is very typical of integrability, and this is typically motivated by allowing one to use methods of complex analysis to study the different components of the theory, as we shall immediately illustrate.
One can see the use of the spectral parameter by defining the quantum Lax-Gaudin matrix Γ^a and the quantum Twist function φ:[2]

$$\Gamma^a(\lambda, z) = \sum_{r=1}^{N} \frac{J_r^a(z)}{\lambda - \lambda_r} \qquad \varphi(\lambda) = \sum_{r=1}^{N} \frac{k_r}{\lambda - \lambda_r} - 1 \quad . \tag{4.5}$$

Then, using the spectral parameter, one sees that the Lax-Gaudin and twist functions are readily interpreted as the generating functionals of the currents and of the levels: one has $\text{Res}_{\lambda=\lambda_r} \Gamma^a(\lambda, z) = J_r^a(z)$ and $\text{Res}_{\lambda=\lambda_r} \varphi(\lambda) = k_r$. This also justifies saying the currents are "attached" to the punctures, and shows the use of the spectral parameter.

However, the Lax-Gaudin and twist functions are not just convenient generating functionals of the quantities of the theory: we show here that they are a meaningful way of repackaging the OPE (4.1) of the theory.
The OPE of two Lax-Gaudin matrices is then

$$\Gamma^a(\lambda, z)\Gamma^b(\mu, w) = -\frac{if^{ab}_{\ c}}{z - w} \frac{\Gamma^c(\lambda, w) - \Gamma^c(\mu, w)}{\lambda - \mu} - \frac{\delta^{ab}}{(z - w)^2} \frac{\varphi(\lambda) - \varphi(\mu)}{\lambda - \mu} + reg \quad . \tag{4.6}$$

The key point is that this involves only Γ^a and φ, which themselves contain all the information on the punctures and the levels. As such, this single OPE encodes the full chiral algebra of the theory $\mathcal{A}_{AGM}$, regardless of the number of currents.

Lax Formulation and Standard Approach

We here very briefly mention that AGMs are a classically integrable class of model, which integrable structure involves both local and non-local conserved charges. We present these non-local charges in appendix E, while the local charges shall be the focus of the main text. These non-local charges come from the standard monodromy trace approach to integrability, which

[2] The -1 is actually an additional constant, named k_∞ which we are free to set to -1 because of a dilation freedom in the spectral space.

generates an infinite tower of Poisson commuting non local charges. This monodromy trace is

$$\mathcal{T}(\lambda) = Tr\left[\overleftarrow{Pexp}\left(-\int_0^{2\pi}\mathcal{L}(\lambda,z)dz\right)\right] \tag{4.7}$$

where $\mathcal{L}(\lambda,z)$ is the Lax matrix of the theory, defined by $\mathcal{L}^a = \Gamma^a/\varphi$ [16]. These non local charges however suffer from the so-called problem of non ultra-locality, which prevents the classical non-local charges to be straightforwardly quantised. This provides an additional motivation to study local charges. See appendix E.

4.2 Conjecture on Local Conserved Charges

We now turn to the central guide of this work, that is, the conjecture on the local commuting charges of Affine Gaudin Models, due to [25] and [26] [3]. While this conjecture initially was inspired by the study of local charges in the classical algebra, and can be motivated by the ODE/IM correspondence, we shall ignore all of this to stay within the scope of this thesis, and we simply state the conjecture without worrying about its origin.

As for WZW models, the charges will only occur at specific conformal dimensions, given by the exponents p of the affine algebra $\hat{\mathfrak{g}}$. We call the conserved charges $Q_{i,p}$ where $i \in 1,...,N$ indexes the different conserved charges at each level. Then the conjecture is that these conserved commuting charges take the following form:

$$Q_{i,p} = \int W_{i,p+1}(z)dz \qquad W_{i,p+1}(z) = \oint_{\mathcal{H}_i}\mathcal{P}(\lambda)^{-p/\check{h}}S_{p+1}(\lambda,z)d\lambda \quad . \tag{4.8}$$

We now need to explain the various elements of the conjecture:

- $\mathcal{P}(\lambda)$ is defined from the twist function by

$$\partial_\lambda \ln(\mathcal{P}(\lambda)) = \varphi(\lambda) \tag{4.9}$$

  which explicitly gives

$$\mathcal{P}(\lambda) = e^{-\lambda}\prod_{r=1}^{N}(\lambda-\lambda_r)^{k_r} \quad . \tag{4.10}$$

- The contours $\mathcal{H}_i$ are N Hankel's loop-type contours in the spectral space: they are contours starting and ending at $\lambda = -\infty$ which circle only the i-th puncture λ_i, while excluding all the others, see fig. 4.1.

[3] In these references, the conjecture is presented for AGMs with a local diagonal symmetry, while we work only with global invariance; as such, the expressions will vary from this thesis to theses references, but the conjecture presented here is self contained.

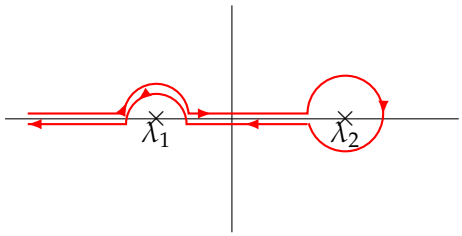

Figure 4.1: A Hankel's loop, $\mathcal{H}_2$ in a $N = 2$ current case

- $S_{p+1}(\lambda, z)$ is a local "spectral" density built of normal ordered products of the Lax-Gaudin matrix and both their spectral and spatial derivatives. Although there is so far no general expression at all orders for these densities, it is expected that they are of the form

$$S_{p+1}(\lambda, z) = \kappa^{a_1 \dots a_{p+1}}_{EHMM} : \Gamma_{a_1}(\lambda, z) \dots \Gamma_{a_{p+1}}(\lambda, z) : + \dots \tag{4.11}$$

  which involves the invariant EHMM tensors, supplemented by some corrections ... of the form

$$\partial^{\alpha_1}_\lambda \varphi(\lambda) \dots \partial^{\alpha_n}_\lambda \varphi(\lambda) : \partial^{\beta_1}_\lambda \partial^{\gamma_1}_z \Gamma_{a_1}(\lambda, z) \dots \partial^{\beta_m}_\lambda \partial^{\gamma_m}_z \Gamma_{a_m}(\lambda, z) : \kappa^{a_1 \dots a_m} \tag{4.12}$$

  with $m, n, \alpha_i, \beta_j, \gamma_k$ positive integers, and $\kappa^{a_1 \dots a_m}$ some invariant tensor of rank m. For S_{p+1} to have a well defined conformal dimension, one must then have that $m + \sum_{k=1}^{m} \gamma_k = p + 1$. As well, there is another constraint on the powers of λ, which is $\sum_{k=1}^{n} \alpha_k + \sum_{k=1}^{m} \beta_k + n + m = p + 1$ [4]. This will be relevant when considering the so-called "fixed sum" sub-lattice later on.

For this conjecture to hold and give commuting charges, it is now needed to impose some very specific constraints on the spectral densities. To explain them, we first need to introduce twisted derivatives.

Twisted Derivatives

We need to introduce a technical element, the so called twisted derivative $D_{\lambda,p}$. They are defined by

$$D_{\lambda,p} f(\lambda) = \partial_\lambda f(\lambda) - \frac{p}{\check{h}} \varphi(\lambda) f(\lambda) \quad . \tag{4.13}$$

The relevant property of this derivative is that, for any of the contours $\mathcal{H}_i$, then

$$\oint_{\mathcal{H}_i} \mathcal{P}(\lambda)^{-p/\check{h}} D_{\lambda,p} f(\lambda) d\lambda = 0 \tag{4.14}$$

[4] We call this "powers of λ" as in the $N = 1$ current case, it actually is a constraint on the lowest power of λ that appears in the spectral density. In the more general case, it becomes a grading analog to a "conformal dimension" in λ, with Γ^a, φ and ∂ counting for 1.

which, using the definition of $\mathcal{P}$ in terms of φ, comes from

$$\oint_{\mathcal{H}_i} \mathcal{P}(\lambda)^{-p/\check{h}} D_{\lambda,p} f(\lambda) d\lambda = \oint_{\mathcal{H}_i} \partial_\lambda \left(\mathcal{P}(\lambda)^{-p/\check{h}} f(\lambda) \right) d\lambda = 0 \tag{4.15}$$

where the final $= 0$ is a consequence of the properties of the contours $\mathcal{H}_i$: this integral vanishes as the exponential $e^{p\lambda/\check{h}}$ in $\mathcal{P}^{-p/\check{h}}$ vanishes at the start and end points of the contours. In general one could use other contours in the conjecture, with the required properties that the integrand of equation (4.15) is single valued along the contours, and that the contours close, such that equation (4.14) holds. As we shall see later, there exist other contours than $\mathcal{H}_i$ which satisfy these requirements, although they will not produce independent densities to the ones presented here. As well, this shows that the spectral densities will be defined up to twisted derivatives as well as total derivatives.

Commutation Property

For the conjecture to hold, it must be that the conserved charges $Q_{i,p}$ must be commuting: $[Q_{i,p}, Q_{j,l}] = 0$. Then, the claim of the conjecture is that it is possible to find a set of spectral densities S_{p+1}, included in the diagonal invariant sub-chiral algebra, such that the singular part of the OPE of $S_{p+1}(\lambda, z)$ and $S_{p'+1}(\mu, w)$ is given by a sum of a total derivative, twisted derivative in λ and twisted derivative in μ.

$$\{S_{p+1}(\lambda, z) S_{p'+1}(\mu, w)\}_1 = \partial_w(...) + D_{\lambda,p}(...) + D_{\mu,p'}(...) \quad . \tag{4.16}$$

Then, the key property (4.14) of the twisted derivatives guarantees that condition (4.16) implies commutation $[Q_{i.p}, Q_{j,p'}] = 0$. As such, one does produce commuting conserved charges, so long as one is able to build spectral densities satisfying condition (4.16). The conjecture is then on the existence of the spectral densities, which we expect to be always possible to build; one takes a very general ansatz for the spectral densities, then computes condition (4.16) using the OPE (4.6) of the Lax-Gaudin matrix. This then constrains the values of the coefficients in the ansatz.

As a first check of the conjecture, we give the example of the quadratic (in the currents) density S_2

$$S_2(\lambda, z) = \delta^{ab} : \Gamma_a(\lambda, z) \Gamma_b(\lambda, z) : \tag{4.17}$$

which has no corrections to the leading term. As well as the expression of the quartic density S_4 in the specific $\mathfrak{g} = \mathfrak{su}(2)$ case, from [17]

$$\begin{aligned} S_4(\lambda,z) = \kappa^{abcd}_{EHMM} &: \Gamma_a(\lambda,z)\Gamma_b(\lambda,z)\Gamma_c(\lambda,z)\Gamma_d(\lambda,z) : \\ &+ \frac{5i}{4} f^{abc} : \partial_z\Gamma_a(\lambda,z)\partial_\lambda\Gamma_b(\lambda,z)\Gamma_c(\lambda,z) : \\ &+ \frac{\delta^{ab}}{48}\left(45\varphi^2(\lambda) : \partial_z\Gamma_a(\lambda,z)\partial_z\Gamma_b(\lambda,z) : -140 : \partial^2_\lambda\partial_z\Gamma_a(\lambda,z)\partial_z\Gamma_b(\lambda,z) : \right. \\ &\left. -30 : \partial_\lambda\partial_z\Gamma_a(\lambda,z)\partial_\lambda\partial_z\Gamma_b(\lambda,z)\right) + \frac{5\delta^{ab}}{12}\left(3 : \partial_\lambda\partial^2_z\Gamma_a(\lambda,z)\Gamma_b(\lambda,z) : \right. \\ &\left. + : \partial^2_z\partial^2_\lambda\Gamma_a(\lambda,z)\Gamma_b(\lambda,z) :\right) \end{aligned} \tag{4.18}$$

in which one would need to use $f^{abc} = \varepsilon^{abc}$ and $\check{h} = 2$.
That these densities satisfy condition (4.16) is not trivial to see, and we will not show it. It was shown in [27].

4.3 Analysis of the Conjecture and Explicit Checks

In this section, we first attempt to predict results of the conjecture, and state what we can reasonably expect of the integrable structure of the AGMs. We then make use of the conjecture to explicitly compute the first few densities of the integrable hierarchies as to both check our expectations and to investigate the non-trivial aspects of the results. We will do so first for the $N = 1$ current AGM, that is, WZW models, then for a $N = 2$ currents AGM.

4.3.1 Expected Integrable Structure

We now discuss what one can reasonably expect to observe when looking at the integrable hierarchy and at the results of the conjecture.

First, we consider the parameters. By construction, AGMs come equipped with N external parameters which define its chiral algebra, which are the levels k_r. Then, as in the WZW case, the integrable hierarchy will explicitly depend on these. However, AGMs have another N parameters, the punctures λ_r. Given the form of the conjecture, which has λ_r dependence in every density through $\mathcal{P}$ and S_{p+1}, it is reasonable to expect that the conserved charges will be dependent on the punctures. One will see then that, even at fixed levels, one will obtain a family of hierarchies parametrised by the punctures. One sees that the algebraical data of the model is not enough to characterise its integrable structure.

When one considers a single current $N = 1$ AGM, one expects to land back on a WZW model, and as such to obtain the conserved charges as in equation (3.53). These charges however do not have any dependence on an ex-

ternal parameter. One then realises that, by a change of spectral parameter $\lambda \to \lambda - \lambda_1$ together with redefinitions $\lambda_r \to \lambda_r - \lambda_1$, one gets completely rid of any λ_1 dependence. As such, we are free to send λ_1 to 0. This should then match our expectation that at $N = 1$, there is no free parameter; we should check explicitly that one lands back on the WZW densities.

Now, we consider $N > 1$ currents. We will then obtain N towers of densities, and expect them to be parametrised by $N - 1$ punctures. It is hard to predict the form of these densities from the conjecture. However, considering that the chiral algebra is simply a few WZW chiral algebras taken together, one should expect that, there must be a point in the parameter space such that the N towers of densities are simply N independent copies of WZW hierarchies, each with their own level k_r.
One can expect that, in the limit where the punctures are all infinitely far from each other, one should exactly obtain a direct sum of WZW hierarchies. This can be seen directly from the conjecture, as each each density W_i will be associated a Hankel's loop circling only λ_i; intuitively, if the other punctures are infinitely far, the contour will not pick up any information about them and give a single WZW density. We shall check this expectation explicitly in the $N = 2$ current case.

While the limit of infinitely distant punctures admits a straightforward geometric interpretation, at this point, the limit in which punctures collide is not obvious, simply by the definition of our Hankel's loop: they circle only a single puncture. We shall then also investigate this limit with explicit calculations in the $N = 2$ case.

4.3.2 WZW Densities

The WZW model is simply a $N = 1$ current AGM. We then explicitly compute the densities of the integrable hierarchy from the conjecture, and we expect to be able to find the brute-force result equation (3.53) of section 3.
In this section, the Lax-Gaudin and twist functions become

$$\Gamma^a(\lambda, z) = \frac{J^a}{\lambda} \qquad \varphi(\lambda) = \frac{k}{\lambda} - 1 \tag{4.19}$$

in which we have already set $\lambda_1 = 0$, as described in section 4.3.1. As well, to reduce notation, we removed the 1 index for this section: $J_1^a \to J^a$, $k_1 \to k$. Similarly, $\mathcal{P}$ is

$$\mathcal{P}(\lambda) = e^{-\lambda}\lambda^k \quad . \tag{4.20}$$

WZW quadratic

We start from the spectral density S_2 given by equation (4.17). The spectral density is

$$S_2(\lambda, z) = \frac{1}{\lambda^2}\delta_{ab} : J^a J^b : (z) \quad . \tag{4.21}$$

Then, we consider the Hankel loop $\mathcal{H}$ circling the origin $\lambda = 0$ such that, from the conjecture, the density is

$$W_2(z) = \delta_{ab} : J^a J^b : (z) \oint_{\mathcal{H}} e^{\lambda/\check{h}} \lambda^{-k/\check{h}-2} d\lambda \quad . \tag{4.22}$$

To evaluate this density, we need the following identity[5]

$$\frac{1}{\Gamma_E(a)} = \frac{1}{2\pi i} \oint_{\mathcal{H}} e^t t^{-a} dt \tag{4.23}$$

where $\Gamma_E(a)$ is the gamma-Euler function (not the Lax-Gaudin matrix). One can bring equation (4.22) into the form of the right-hand-side of this identity by a rescaling of variables, and it becomes straightforward to evaluate the integral. We then find

$$W_2(z) = \delta_{ab} \frac{: J^a J^b : (z)}{k + \check{h}} \cdot \left(-1(\check{h})^{k/\check{h}} \Gamma_E(-\frac{k}{\check{h}}) 2i \sin(\frac{\pi k}{\check{h}}) \right) \tag{4.24}$$

and one recognizes directly that the first density of the hierarchy is simply the stress-energy tensor T given by the Sugawara construction, up to a proportionality constant.

$$W_2(z) \propto T(z) \quad . \tag{4.25}$$

WZW quartic

We now turn to the quartic charge. For this, one needs to start with $S_4(\lambda, z)$ as given by equation (4.18) (which we recall is valid only for $\mathfrak{su}(2)$).
To proceed, one can see that in the one puncture (one current) case, S_4 is simply an expansion in powers of $1/\lambda$: terms without φ are proportional to $1/\lambda^4$, while φ^2 will generate terms in $1/\lambda^3$ and $1/\lambda^2$. Then, one has to repeatedly use the identity (4.23), which, because of the different powers in λ, will yield gamma functions evaluated at different points. As such, one needs to use the recursion identity of the gamma function to bring all the terms to the same prefactors, leaving out only an operator part. Finally, one

[5]Note that actually, we could stop at equation (4.22), as we do not care about the proportionality factor. It is still useful to introduce the following identity, as the more complicated densities will follow a similar pattern

obtains:

$$W_4/N = \kappa^{abcd}_{EHMM} : J_a J_b J_c J_d : -\frac{5i}{4} f^{abc} : J'_a J_b J_c : +\delta^{ab} : J''_a J_b : \frac{5}{12}$$
$$+ \delta^{ab} : J'_a J'_b : \left[\frac{45}{48}k^2 - \frac{45}{24}\check{h}k(1+\frac{k}{\check{h}}) + \frac{15}{48}\check{h}^2(1+\frac{k}{\check{h}})(2+3\frac{k}{\check{h}}) - \frac{310}{48}\right] \quad (4.26)$$

with

$$N = (\check{h}/3)^{-3k\check{h}-3} 2i \sin(3k\pi/\check{h})\Gamma_E(-3-3k/\check{h}) \quad (4.27)$$

some proportionality constant we will forget about. Expression (4.26) might be correct, but it is not exactly instructive, nor does it match the result of section 3.3, equation (3.53). This is because the conserved densities are defined up to a scaling factor and up to total derivative. Using Mathematica [23], it was possible to find out that in fact

$$W_4(z)/N = \frac{3}{4}(k+2)^2 T_4(z) - \frac{1}{24}(26+15k)T''(z) \quad (4.28)$$

and so we see that in fact, the conjecture does predict the result T_4 we found through brute force in section 3.3, which was equation (3.53).

4.3.3 N=2 AGM Densities

We now attempt the next easiest scenario, that is, the AGM with $N = 2$ currents. The goal is then to compute the densities from the conjecture, and check if one obtains what we were expecting to observe. That is, do we obtain two towers of densities parametrised by the puncture λ_2, and that in the $\lambda_2 \to \infty$ limit, the towers decouple.
These charges will prove much more challenging to compute than in the WZW case due to the appearance of confluent hypergeometric functions.
In the $N = 2$ case, the Lax-Gaudin matrix, twist function and $\mathcal{P}$ are then

$$\Gamma^a(\lambda, z) = \frac{J_1^a(z)}{\lambda} + \frac{J_2^a(z)}{\lambda - \lambda_2} \quad (4.29)$$

$$\varphi(\lambda) = \frac{k_1}{\lambda} + \frac{k_2}{\lambda - \lambda_2} - 1 \quad (4.30)$$

$$\mathcal{P}(\lambda) = e^{-\lambda}\lambda^{k_1}(\lambda - \lambda_2)^{k_2} \quad . \quad (4.31)$$

N=2 AGM Quadratic Density

According to the conjecture, the quadratic density $S_2(\lambda, z)$, equation (4.17) is valid for any number of currents, as it only involves the Lax-Gaudin matrix and twist function, independently of the number of currents. However, because they involve all the currents, the resulting integral will be more complicated. One has that the quadratic density is then

$$W_{2,i}(z) = \oint_{\mathcal{H}_i} e^{\lambda/\check{h}}\lambda^{-k_1/\check{h}}(\lambda - \lambda_2)^{-k_2/\check{h}}\delta_{ab} : \Gamma^a(\lambda, z)\Gamma^b(\lambda, z) : d\lambda \quad (4.32)$$

where i denotes the two possible choices of contour, two Hankel's loop $\mathcal{H}_i$ circling 0 and λ_2 respectively. To proceed, one needs an identity analogous to (4.23) for both choices of contours. It turns out that these exists, and they involve the independent solutions to the Kummer differential equation, the confluent hypergeometric functions of the second kind, or Tricomi functions:

$$U(a,b,\eta) = e^{-a\pi i}\frac{\Gamma_E(1-a)}{2\pi i}\oint_{\tilde{\mathcal{H}}} e^{-\eta t}t^{a-1}(1+t)^{b-a-1}dt \tag{4.33}$$

where $U(a,b,\eta)$ is the Tricomi function, and $\tilde{\mathcal{H}}$ is a Hankel's loop type contour which starts and ends at $+\infty$. This identity can be found in [28].
Starting from this identity, one does a scaling of t by $-\lambda_2$, which transforms $\tilde{\mathcal{H}}$ into $\mathcal{H}_1$ and then matches the form of equation (4.32). This gives then a way of computing $W_{2,1}$. As well, starting from the identity, one can also start by shifting $t \to t-1$, then perform the scaling by $-\lambda_2$, which then gives an identity involving $\mathcal{H}_2$, and allows to compute $W_{2,2}$.
One can then repeat the business of section 4.3.2, that is expanding the product $\Gamma^a\Gamma^b$ into a sum of terms proportional to $1/\lambda^a \quad 1/(\lambda-\lambda_2)^b$ and apply the previous identity (4.33) on every term of the sum. One then gets the following results:

$$\begin{aligned} W_{2,1}(z) \propto \quad & \frac{1}{(a-1)}U(a-1,b,\eta)\delta_{cd} : J_1^c J_1^d : (z) \\ & + aU(a+1,b,\eta)\delta_{cd} : J_2^c J_2^d : (z) \\ & + 2U(a,b,\eta)\delta_{cd} : J_1^c J_2^d : (z) \end{aligned} \tag{4.34}$$

$$\begin{aligned} W_{2,2}(z) \propto \quad & (b-a)U(b-a+1,b,-\eta)\delta_{cd} : J_1^c J_1^d : (z) \\ & + \frac{1}{b-a-1}U(b-a-1,b,-\eta)\delta_{cd} : J_2^c J_2^d : (z) \\ & + 2U(b-a,b,-\eta)\delta_{cd} : J_1^c J_2^d : (z) \end{aligned} \tag{4.35}$$

with

$$a = -\frac{k_1}{\check{h}}, \qquad b = -\frac{k_1+k_2}{\check{h}}, \qquad \eta = \frac{\lambda_2}{\check{h}} \quad .$$

One sees that, as expected, we do get two conserved charges, parametrised by λ_2.

N=2 AGM Quartic and Higher Order charges

Here, we simply mention that one can in principle continue this game to any order, and we have computed explicitly the quartic densities, although we do not report their expression as they are not particularly instructive and take up much space. Its calculation was aided by Mathematica [23] to keep track of the power counting of the expansion in $1/\lambda^\alpha 1/(\lambda-\lambda_2)^\beta$.

These charges have been also checked to commute with the quadratic charges using Mathematica. In principle, one can continue this game indefinitely, so long as 1) one is able to find the densities S_{p+1} and 2) one is able to perform the power counting. These two tasks are in principle straight forward, and generalise to any order, but they are very, very computationally heavy, hence why we did not pursue them.

λ_2 Limits

Decoupling Limit We now check the $\lambda_2 \to \infty$ limit. As it turns out, $U(a,b,\eta) \propto \eta^{-a}$ as $\eta \to \infty$. Then, looking at equation (4.34), one sees that the three terms scale as

$$\begin{aligned}\lim_{\lambda_2\to\infty} W_{2,1}(z) \propto &\frac{1}{(a-1)}(\lambda_2/\check{h})^{-a+1}\delta_{cd} : J_1^c J_1^d : (z) \\ &+ a(\lambda_2/\check{h})^{-a-1}\delta_{cd} : J_2^c J_2^d : (z) \\ &+ 2(\lambda_2/\check{h})^{-a}\delta_{cd} : J_1^c J_2^d : (z) \end{aligned} \tag{4.36}$$

such that the leading power λ_2^{-a+1} is in the first term. Analogously, for the second solution, one has

$$\begin{aligned}\lim_{\lambda_2\to\infty} W_{2,2}(z) \propto &(b-a)(\lambda_2/\check{h})^{-b+a-1}\delta_{cd} : J_1^c J_1^d : (z) \\ &+ \frac{1}{b-a-1}(\lambda_2/\check{h})^{-b+a+1}\delta_{cd} : J_2^c J_2^d : (z) \\ &+ 2(\lambda_2/\check{h})^{-b+a}\delta_{cd} : J_1^c J_2^d : (z) \end{aligned} \tag{4.37}$$

such that the leading power λ_2^{-b+a+1} is in the second term. Therefore, by scaling out the non-leading powers, one has

$$\lim_{\lambda_2\to\infty} W_{2,1}(z) \propto T_1(z) \qquad \lim_{\lambda_2\to\infty} W_{2,2}(z) \propto T_2(z) \quad . \tag{4.38}$$

One then really has that the hierarchies decouple in the $\lambda_2 \to \infty$ limit.
This matches our expectation from section 4.3.1: the Hankel's loops which appear in the conjecture will only pick up information about the puncture they circle when the other punctures become infinitely distant. This then causes the integrable hierarchies to decouple, and one obtains 2 independent copies of the WZW hierarchies.
We also computed this limit for the quartic densities, and checked that they indeed decouple and produce two copies of T_4 given by equation (3.53), which is the second density of the WZW hierarchy. This supports the claim that the whole hierarchies decouple in the $\lambda_2 \to \infty$ limit.

Colliding Limit We now want to investigate the $\lambda_2 \to 0$ limit, that is, when the two punctures collide. It turns out $U(a,b,\eta)$ is ill-defined as $\eta \to 0$,

which is not so surprising considering our Hankel's loops must include only a single puncture each.
U was called the Tricomi, or confluent hypergeometric function of the second kind. As the name suggests, there exists a confluent hypergeometric function of the first kind, written M and also called Kummer's function. These are well defined as $\lambda_2 = 0$; is there a way to reformulate the densities in terms of these functions? As it turns out, they satisfy analogous identities to (4.33), see [28]:

$$M(a,b,\eta) = e^{-bi\pi}\frac{\Gamma_E(1-a)\Gamma_E(1-b+a)}{\Gamma_E(b)}\frac{1}{4\pi^2}\oint_\gamma e^{\eta t}t^{a-1}(1-t)^{b-a-1}dt \quad (4.39)$$

$$\eta^{1-b}M(a-b+1,2-b,\eta) = \frac{1}{2\pi i}\frac{1}{\Gamma_E(2-b)}\oint_{\mathcal{H}} e^{\eta t}t^{a-2b+1}(t-1)^{a-b+1}dt \quad (4.40)$$

where $M(a,b,\eta)$ is the so-called confluent hypergeometric function, or Kummer's function. γ is a Pochhammer contour, which is a contour that circles both punctures twice, once in every direction (see fig. 4.2), and $\mathcal{H}$ is a big Hankel's loop, which circles both punctures. The arguments and prefactor in identity (4.40) look arbitrary, but they were in fact precisely chosen as $M(a,b,\eta)$ and $\eta^{1-b}M(a-b+1,2-b,\eta)$ are the two independent solutions to Kummer's equation which are well-behaved as $\eta \to 0$: $M(a,b,\eta) = \eta^{1-b}M(a-b+1,2-b,\eta) \to 1$ as $\eta = 0$ (see [28]).

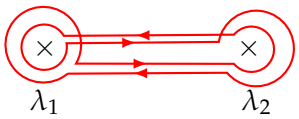

Figure 4.2: A Pochhammer contour, γ circling λ_1 and λ_2

The conjecture used the Hankel's loops $\mathcal{H}_i$, but as we explained, any contour such that the integrand of equation (4.14) is single valued and equation (4.14) vanishes is a valid choice. The big Hankel's loop obviously satisfies these conditions, but it turns out that the Pochhammer contour also works: it is precisely defined so that it stays on a single sheet of $\mathcal{P}^\alpha$, and it is closed so that the integral vanishes. As such, these contours are also valid choices to plug in the conjecture. We therefore use these identities in the conjecture, and one gets the following results:

$$\begin{aligned} W_{2,\gamma}(z) \propto \quad & \frac{b-a}{a-1}M(a-1,b,\eta)\delta_{cd} : J_1^c J_1^d : (z) \\ & + \frac{a}{b-a-1}M(a+1,b,\eta)\delta_{cd} : J_2^c J_2^d : (z) \\ & - 2M(a,b,\eta)\delta_{cd} : J_1^c J_2^d : (z) \end{aligned} \quad (4.41)$$

$$\begin{aligned} W_{2,\mathcal{H}}(z) \propto \quad & M(a-b,2-b,\eta)\delta_{cd} : J_1^c J_1^d : (z) \\ & + M(a-b+2,2-b,\eta)\delta_{cd} : J_2^c J_2^d : (z) \\ & + 2M(a-b+1,2-b,\eta)\delta_{cd} : J_1^c J_2^d : (z) \quad . \end{aligned} \tag{4.42}$$

One must keep in mind that the conjecture still predicts only 2 conserved charges per exponent. As such, while using $\mathcal{H}$ and γ as contours in the conjecture is a valid choice, the densities produced using these contours will not be independent from the densities obtained with $\mathcal{H}_1$, $\mathcal{H}_2$. One can think of the choice of contours to use in the conjecture as a choice of basis for the 2 densities of the hierarchy. Therefore, one has that the densities given by equations (4.41) and (4.42) are not independent of the densities of equations (4.34) and (4.35), but they are another basis well defined at $\lambda_2 = 0$. As such, we can study the $\lambda_2 = 0$ limit, which corresponds to the punctures colliding. We define the following "stress-energy tensors" by analogy to the $N = 1$ current case, although they are not the actual stress-energy tensor of theory.

$$T_i(z) = \delta_{ab} \frac{: J_i^a J_i^b : (z)}{\check{h} + k_i}, i = 1,2 \qquad T_{diag}(z) = \delta_{ab} \frac{: J_{diag}^a J_{diag}^b : (z)}{\check{h} + k_1 + k_2} \quad . \tag{4.43}$$

Then, when taking the $\lambda_2 \to 0$ limit, one gets:

$$\lim_{\lambda_2 \to 0} W_{2,\gamma}(z) \propto T_{GKO}(z) = T_1(z) + T_2(z) - T_{diag}(z) \tag{4.44}$$

$$\lim_{\lambda_2 \to 0} W_{2,\mathcal{H}}(z) \propto T_{diag}(z) \tag{4.45}$$

where T_{GKO} is the so called Goddard-Kent-Olive stress energy tensor [29].

It is not immediately clear which interpretation to give to these charges, but $W_{2,\mathcal{H}}$ can guide us. When one considers the big Hankel's loop which circle both punctures, one can straightfowardly collide the two punctures without crossing any contour line. Then, recalling the expressions of Γ^a and φ, one sees that colliding the punctures is analytically equivalent to a single current AGM with J_{diag} and $k_1 + k_2$ as a current. Therefore, we expect it to be possible to build another WZW hierarchy out of this J_{diag} current. As such, seeing T_{diag} show up is not surprising.
We then look at the chiral algebra, to see if one can express this result algebraically. It is clear that $\mathcal{A}_{AGM} = \mathcal{A}_{J_1} \otimes \mathcal{A}_{J_2}$ contains a sub algebra $\mathcal{A}_{diag}$, which is simply the Kac-Moody algebra generated by $J_{diag} = J_1 + J_2$. This gives an easy interpretation to the $W_{2,\mathcal{H}} \to T_{diag}$ limit, as simply the first charge in the WZW hierarchy of the $\mathcal{A}_{diag}$ sub-algebra.

This still doesn't explain T_{GKO}.
Within $\mathcal{A}_{AGM}$, we can build another sub algebra

$$\mathcal{A}_{GKO} := \{\varphi \in \mathcal{A}_{AGM}, \{\varphi J_{diag}\}_n = 0\} \quad .$$

This is simply the sub algebra of "commuting" fields with J_{diag}. Then, within the commuting sub algebra, one finds T_{GKO} as the first density of a conserved charge.

As it turns out, $\mathcal{A}_{GKO}$ is actually the chiral algebra of a gauged AGM, that is an AGM in which the global diagonal symmetry has been promoted to a local one. Referring to [17], one sees the GKO stress-energy tensor is indeed the quadratic density of the hierarchy of gauged AGM. This hierarchy commutes by definition with the WZW hierarchy generated by J_{diag}, and as such, we do obtain two full commuting hierarchies in the colliding limit.

To conclude, we recall that the statements about both $\lambda_2 \to 0$ and $\lambda_2 \to \infty$ limits rest on arguments concerning the whole chiral algebra. As such, we expect that at both limits, the whole hierarchy splits in the described ways, and not only the quadratic and quartic charges.

4.4 Rational Reparametrisation

While looking for a local hierarchy of AGMs, it would also be possible to proceed as in section 3, by doing a brute-force procedure. This would then amount to, as for WZW models, build a general ansatz of all the fields in the (diagonal invariant) chiral algebra, and then impose the commutation condition (2.74). This condition will then translate into quadratic equations on the coefficients.
One would not expect to find highly transcendent functions, such as the confluent hypergeometric functions, to appear as solutions of quadratic equations. This then motivates us to look for some way of reparametrising the results of the conjecture into a simpler form.

We present in this section an interesting result on a reparametrisation of the conserved charges of Affine Gaudin Models which allows one to get rid of the special functions that appear as part of the conjecture.

We first give the motivating example of the quadratic charges for $N = 2$ currents, then present a more general setup for a rational parametrisation of the densities through a change of basis. That this specific choice of basis is rational will rely on some recurrence identities, and we conclude this section by presenting elements of a proof of the existence of these identities in the most general case.

4.4.1 First example: Quadratic Charge at $N = 2$ Currents

We start with the quadratic densities at two currents. As mentioned above, one could proceed brute force. One possible expression of the densities is of

the form

$$\tilde{W}_{2,1} = T_1 + \delta_1 O_{12} \tag{4.46}$$

$$\tilde{W}_{2,2} = T_2 - \delta_2 O_{12} \tag{4.47}$$

where, from here on $O_{12}(z) = \delta_{ab} : J_1^a J_2^b : (z)$, and δ_i are some scalar functions to be determined. In principle one would fix the values of δ_1 and δ_2 as in the brute force approach of section 3.2.3.
This form of the densities must still be just another choice of basis than the densities of the conjecture, which involve the confluent hypergeometric functions. Therefore we look for a change of basis by a matrix $\mathbf{M}$ and we take for convenience the $M(a,b,\eta)$ expressions of equations (4.41) and (4.42)

$$\begin{pmatrix} T_1 + \delta_1 O_{12} \\ T_2 - \delta_2 O_{12} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} W_{2,\gamma} \\ W_{2,\mathcal{H}} \end{pmatrix} \quad . \tag{4.48}$$

Imposing the coefficients proportional to T_1 and T_2 to be $1,0$ in the first line, and $0,1$ in the second line completely fixes the matrix $\mathbf{M}$, from which we can then extract the expression for δ_1 and δ_2. Very explicitly, one gets

$$\begin{aligned}
\delta_1 = \frac{2}{\check{h}} \Bigg[& M(a,b,\eta) M(a-b+2,-b+2,\eta) \\
& + \frac{a}{b-a-1} M(a+1,b,\eta) M(a-b+1,-b+2,\eta) \Bigg] \\
& \cdot \Bigg[(b-a) M(a-1,b,\eta) M(a-b+2,-b+2,\eta) \\
& - a \frac{a-1}{b-a-1} M(a+1,b,\eta) M(a-b,-b+2,\eta) \Bigg]^{-1}
\end{aligned}$$

$$\begin{aligned}
\delta_2 = \frac{2}{\check{h}} \Bigg[& M(a,b,\eta) M(a-b,-b+2,\eta) \\
& + \frac{b-a}{a-1} M(a-1,b,\eta) M(a-b+1,-b+2,\eta) \Bigg] \\
& \cdot \Bigg[(b-a) M(a-1,b,\eta) M(a-b+2,-b+2,\eta) \\
& - a \frac{a-1}{b-a-1} M(a+1,b,\eta) M(a-b,-b+2,\eta) \Bigg]^{-1} \quad .
\end{aligned}$$

This is not so nice. However, it turns out that the confluent hypergeometric functions satisfy very specific recurrence relations, as follows:

$$(b-a)M(a-1,b,\eta)+(2a-b+\eta)M(a,b,\eta)-aM(a+1,b,\eta)=0 \quad . \tag{4.49}$$

As it turns out, one can use this relation in the expression for δ_1 and δ_2, and one ends up with

$$\delta_1=\delta_2=\delta=\frac{2}{k_1-k_2-\lambda_2} \tag{4.50}$$

giving the conserved charges as

$$\tilde{W}_{2,1}=T_1+\frac{2O_{12}}{k_1-k_2-\lambda_2} \tag{4.51}$$

$$\tilde{W}_{2,2}=T_2-\frac{2O_{12}}{k_1-k_2-\lambda_2} \quad . \tag{4.52}$$

This is way nicer! Not only are the δs equal, they are rational! This holds because of the very specfic recurrence relations satisfied by the confluent hypergeometric function, and gives back a rational solution for the conserved charges, well defined as both $\lambda_2=0$ and $\lambda_2=\infty$. As well, this matches our expectation from the brute force approach, where all the equations are at most quadratic, and it clearly highlights that T_1+T_2, which is the stress-energy tensor of the theory, is in the hierarchy. This result also brings in the next question: is this a low dimension/low current accident, or does this generalise ? This is the topic of the rest of the section.

4.4.2 General Setup

We now consider a more general setup, as to be able to deal with more currents.

Definition of The Functions

The first step is to simplify the setup by defining our own functions, instead of using unnamed generalisations of the confluent hypergeometric functions.
The start point of the conjecture is the expression for the spectral densities S_{p+1}. One notices that S_{p+1} can in general be expressed as

$$S_{p+1}(\lambda,z)=\sum_{a_1=0}^{p+1}\dots\sum_{a_N=0}^{p+1}\rho_{a_1\dots a_N}(z)\prod_{j=1}^{N}(\lambda-\lambda_j)^{-a_j} \tag{4.53}$$

where we have introduced $\rho_{a_1\dots a_N}$ as a general expression containing combinations of operators such as the currents together with the levels. That this

expression holds is a consequence of the grading on the powers of λ that was introduced with equation (4.12).
This fact pushes us to define a lattice Λ, and we shall think of the densities from the conjecture as expanded on this lattice. We take the lattice to be $\Lambda = \{(a_1, .., a_N), a_k \in (0, 1, ..., p+1)\}$, where each point of the lattice corresponds to one given term of the form $\prod_{j=1}^{N}(\lambda - \lambda_j)^{-a_j}$. Then, we will now define our own functions on the lattice as to simplify the expression of the densities.
Plugging the previous expression for S_{p+1} in the conjecture, one has that

$$W_{p+1,i}(z) = \sum_{a_1=0}^{p+1} ... \sum_{a_N=0}^{p+1} \rho_{a_1...a_N}(z) \oint_{\mathcal{H}_i} e^{p\lambda/\check{h}} \prod_{j=1}^{N} (\lambda - \lambda_j)^{-pk_j/\check{h}-a_j} d\lambda \quad . \tag{4.54}$$

Then, we define our own functions $F_i(a_1, ..., a_N)$ by

$$F_i(a_1, ..., a_N) = \oint_{\mathcal{H}_i} e^{p\lambda/\check{h}} \prod_{j=1}^{N} (\lambda - \lambda_j)^{-pk_j/\check{h}-a_j} d\lambda \quad . \tag{4.55}$$

We here suppress the dependence on the k_is, on the λ_is and on p as we will work with all of these fixed for this section.
Formally, these functions are related by some scaling of the variables to the confluent limit of the $N+1$ degree Lauricella function [28], but we shall not care about them here. These functions are then defined on every site $\vec{a} = (a_1, ..., a_N)$ of the lattice Λ. Using these functions allows for a compact rewriting of the densities

$$W_{p+1,i}(z) = \sum_{a_1=0}^{p+1} ... \sum_{a_N=0}^{p+1} \rho_{a_1...a_N}(z) F_i(a_1, ..., a_N) \quad . \tag{4.56}$$

As an example to this notation, the now familiar quadratic charges of the 2 current AGM are then expressed as

$$W_{2,\mathcal{H}_i}(z) = O_{11}(z)F_i(2,0) + O_{22}(z)F_i(0,2) + 2O_{12}(z)F_i(1,1) \tag{4.57}$$

where we have generalised the notation $O_{ij} = \eta_{ab} : J_i^a J_j^b :$. Introducing this notation and functions has the advantages of getting rid of prefactors, of being symmetric in the levels of the current and to treat in a unified way the different choices of contour, simply by changing the index i of F_i.
As well, making use of the lattice notation, we shall condense the expressions in the cases of more than $N = 2$ currents as

$$W_{p+1,i}(z) = \sum_{\vec{a} \in \Lambda} \rho_{\vec{a}}(z) F_i(\vec{a}) \tag{4.58}$$

where we simply mean $\rho_{\vec{a}} = \rho_{a_1...a_N}$ and analogously for F_i.

Fixed Sum Sub-Lattice

We now study where on the lattice does the conjecture predict coefficients to land. We see that in the quadratic $N = 2$ case, all the coefficients land in a 1-dimensional sub-lattice defined by $a_1 + a_2 = 2$. As well, looking at the quartic spectral density S_4 (4.18), we see that all the terms land again on a 1-dimensional sub-lattice $a_1 + a_2 = 4$, as the power expansion reads $\lambda^{a_1}(\lambda - \lambda_2)^{4-a_1}$, except for the terms involving the twist function φ, in which the constant term reduces the total power of λ.
As the power expansion of the density is controlled by the spectral density S_{p+1}, which is itself a differential polynomial of Γ and φ, this argument stands at any order and any number of currents: the coefficients will land on a fixed sum sub-lattice $\Lambda_* := \{\vec{a} \in \Lambda, \sum_{j=1}^{N} a_j = p + 1\} \subset \Lambda$, except for these involving a twist function. That this holds is again a consequence of the λ power grading that appears in equation (4.12).
We now show that we have the freedom to restrict ourselves to dealing solely with the fixed sum sub-lattice.
As mentioned previously, S_{p+1} will be expanded completely on Λ_*, except for the terms involving non derived twist functions, which we write $\varphi^k A$, with A some term of the form of equation (4.12) in which no twist function without derivative appear. Then, one can use the definition of the twisted derivative equation (4.14), to write

$$\varphi^k A = \left[D_{\lambda,p}(\varphi^{k-1} A) - \partial_\lambda(\varphi^{k-1} A) \right] \frac{\check{h}}{p} \quad . \tag{4.59}$$

Since the density is defined up to a twisted derivative, we can neglect the first term, and we are left with only the second term, which involves one power less of φ than what we started with. As such, we can iterate this procedure until only $\partial_\lambda^k \varphi$ terms remain in the density, in which the constant term vanishes. Therefore, we can systematically restrict ourselves to work with expansions on this fixed sum sub-lattice.

4.4.3 Rational Basis

Recalling the $N = 2$ currents quadratic example, the goal is then clear: we want to find a new basis of the densities in which only rational functions of the punctures remain. Because this is just a change of basis, we will be looking for this new basis by a rotation matrix

$$\tilde{W}_i = \sum_{j=1}^{N} M_{ij} W_j \tag{4.60}$$

where $\tilde{W}_i$ is the i-th charge in the rational basis, W_j is the the j-th charge in the basis predicted by the conjecture and M_{ij} is the change of matrix basis.

We shall then first define this new basis in an a priori unrelated way, and then focus on proving that this basis is in fact rational, which will require to use recurrence identities of the functions on the lattice. We first give the $N = 2$ current AGM as an example to illustrate the change of basis, then proceed with the general case. We then derive the recurrence identities of our functions and finish by attempting a proof.

Change of Basis: N=2 Example

Again, we give the example of the 2 current AGM at any conformal dimension first to illustrate the idea, and we will present the most general setup later. At any conformal dimension, there will be two conserved charges, expanded on a one dimensional sub-lattice $a_1 + a_2 = p + 1$, such that we only need one index to characterise the sub-lattice, and we pick a_1. We will define the new basis by

$$\begin{pmatrix} \tilde{W}_1 = \sum_{a_1=0}^{p+1} \delta_1(a_1, p+1-a_1)\rho_{a_1,p+1-a_1} \\ \tilde{W}_2 = \sum_{a_1=0}^{p+1} \delta_2(a_1, p+1-a_1)\rho_{a_1,p+1-a_1} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} W_{p+1,1} \\ W_{p+1,2} \end{pmatrix} \tag{4.61}$$

where we introduce $\delta_{1,2}(a_1, p+1-a_1)$ as the coefficients associated to each point of the fixed sum sub-lattice, and $W_{p+1,1}, W_{p+1,2}$ are given by equation (4.56). This is so far just a general change of basis and nothing is fixing the δ_is.

To fix the basis, we then impose $\delta_i(\tilde{e}_j) = \delta_{ij}$, where $\tilde{e}_j$ are the corners or extremities of the lattice: $\tilde{e}_1 = (p+1, 0)$, $\tilde{e}_2 = (0, p+1)$. The idea is to associate a corner of the lattice to each density in the new basis.
One can realise that this is what we were doing in disguise with equation (4.48): T_1 and T_2 corresponded to the operators at the reference points $\rho_{2,0}$ and $\rho_{0,2}$ with $p = 1$, and we defined the new densities $\tilde{W}_i$ to have coefficients $\delta_{ij}T_j+$ a bulk term.

Imposing $\delta_i(\tilde{e}_j) = \delta_{ij}$ then gives 4 conditions, which will fix the 2×2 rotation matrix uniquely, and therefore fix the δ_is. We can solve this change of basis: starting from equation (4.60), we have

$$\begin{gathered} \tilde{W}_i = M_{ij}W_j \\ \sum_{a_1=0}^{p+1} \delta_i(a_1, p+1-a_1)\rho_{a_1,p+1-a_1} = \sum_{a_1=0}^{p+1} M_{ij}F_j(a_1, p+1-a_1)\rho_{a_1,p+1-a_1} \end{gathered} \tag{4.62}$$

which gives

$$\delta_i = M_{ij}F_j \quad . \tag{4.63}$$

Then the condition $\delta_i(\tilde{e}_j) = \delta_{ij}$ implies that the change of basis matrix M is given by

$$M = C^{-1} \qquad C := (C_{ij}) = (F_i(\tilde{e}_j)) \quad . \tag{4.64}$$

This then gives for the δ_is

$$\delta_1(\vec{a}) = \frac{F_2(\tilde{e}_2)F_1(\vec{a}) - F_1(\tilde{e}_2)F_2(\vec{a})}{F_2(\tilde{e}_2)F_1(\tilde{e}_1) - F_1(\tilde{e}_2)F_2(\tilde{e}_1)} \tag{4.65}$$

$$\delta_2(\vec{a}) = \frac{F_1(\tilde{e}_1)F_2(\vec{a}) - F_2(\tilde{e}_1)F_1(\vec{a})}{F_2(\tilde{e}_2)F_1(\tilde{e}_1) - F_1(\tilde{e}_2)F_2(\tilde{e}_1)} \quad . \tag{4.66}$$

From equations (4.65), (4.66), one gets $\delta_1(\tilde{e}_1) = 1, \delta_1(\tilde{e}_2) = 0$ and $\delta_2(\tilde{e}_1) = 0, \delta_2(\tilde{e}_2) = 1$, which is how we defined the new basis.

Change of Basis: General case

We now present the definition of the new basis in the N current case. One will get N commuting charges at every conformal dimension, indexed by i the choice of contour. All of these densities will have the same expression as an expansion on the lattice equation (4.56), only with different index i corresponding to the contours. As well, we can restrict the expansion to land on the fixed sum sub-lattice. This sub-lattice will in general have N corners, defined as the points $\{\tilde{e}_k = (p+1)\hat{e}_k, k = (1, ..., N)\}$ where $\hat{e}_k$ is the unit vector in the direction k. We will call these points the reference points of the sub-lattice, and they will be essential in defining the change of basis.

We define the new basis of charges at spin $p+1$ by $\tilde{W}_i, \quad i = (1, ..., N)$

$$\tilde{W}_i = \sum_{\vec{a} \in \Lambda_*} \delta_i(\vec{a}) \rho_{\vec{a}} \tag{4.67}$$

together with the condition

$$\delta_i(\tilde{e}_j) = \delta_{ij} \tag{4.68}$$

that is to say, the k-th conserved charge of the new basis is defined to have coefficients landing only on the k-th reference point, plus coefficients in the bulk of the sub-lattice.
As in the previous section, this completely fixes the change of basis matrix to be

$$M = C^{-1} \qquad C_{ij} = F_i(\tilde{e}_j) \tag{4.69}$$

where we have to assume that C is invertible.
The expression for the δ_is at any number of currents is then

$$\delta_i(\vec{a}) = \sum_{j=1}^{N} F_j(\vec{a}) M_{ij} \tag{4.70}$$

where $\vec{a}$ is a point on the fixed sum sub-lattice Λ_*.

Recurrence Relations

As was shown in the $N = 2$ quadratic densities example, special recurrence relations are needed in order to express the δ_is in terms of rational functions. In the example, these recurrence relations (4.49) involved the confluent hypergeometric functions M, which we have since then traded for our own functions. In this section, we derive/present these recurrence relations from the definition of the functions F_i, (4.55).
The first operation one can do on the these integrals is an Integration By Parts (IBP) . It will give the following relation:

$$\sum_{j=1}^{N}(p\frac{k_j}{\check{h}} - a_j)F_i(a_1, ..., a_j - 1, ..., a_N) + \frac{p}{\check{h}}F_i(a_1, ..., a_N) = 0 \quad . \tag{4.71}$$

It is a relation that involves a lattice point and N of its neighbours in the negative direction.
As well, one can do a so called Partial Fraction Expansion (PFE), which is simply noticing

$$\frac{1}{\lambda - \lambda_k}\frac{1}{\lambda - \lambda_h} = \frac{1}{\lambda_k - \lambda_h}\left(\frac{1}{\lambda - \lambda_k} - \frac{1}{\lambda - \lambda_h}\right) \quad . \tag{4.72}$$

One can do this in the integrand of (4.55) for any pair of punctures. This gives $N(N-1)/2$ relations of the form

$$F_i(a_1, ..., a_N) = \frac{1}{\lambda_k - \lambda_h}\Big(F_i(a_1, ..., a_k, ..., a_h + 1, ...a_N) - F_i(a_1, ..., a_k + 1, ..., a_h, ...a_N)\Big) \quad . \tag{4.73}$$

These relations always link a lattice point to two of its neighbours in the positive direction.
For convenience, we will from here on rewrite these relations in the vector form of the lattice points

$$(IBP) \rightarrow F_i(\vec{a}) + \sum_{j=1}^{N} b_j F_i(\vec{a} - \hat{e}_j) = 0$$
$$(PFE) \rightarrow F_i(\vec{a}) + c_{kh}(F_i(\vec{a} + \hat{e}_k) - F_i(\vec{a} + \hat{e}_h)) = 0$$

where $b_j = pk_j/\check{h} - a_j, c_{kh} = 1/(\lambda_h - \lambda_k)$, and $\hat{e}_k$ are some unit vectors of the lattice.

One can see that both the IBP (4.71) and PFE (4.73) relations do not respect the fixed sum sub-lattice. However, as we shall show now, it is possible to combine them into recurrence relations on the sub-lattice.

Consider a point $\vec{a} \in \Lambda_*$ and its immediate first neighbour $\vec{a} - \hat{e}_1$. We write the IBP as

$$F_i(\vec{a}) + b_1 F_i(\vec{a} - \hat{e}_1) + \sum_{k=2}^{N} b_k F_i(\vec{a} - \hat{e}_k) = 0 \quad . \tag{4.74}$$

We will focus on trading $\vec{a} - \hat{e}_1$ for a point on the sub-lattice. For this, we consider a PFE centered on $\vec{a} - \hat{e}_1$

$$F_i(\vec{a} - \hat{e}_1) + c_{1h} \left(F_i(\vec{a}) - F_i(\vec{a} - \hat{e}_1 + \hat{e}_h) \right) = 0 \quad . \tag{4.75}$$

Then, one sees that by taking (4.74)-b_1(4.75), one obtains

$$F_i(\vec{a})(1 - b_1 c_{1h}) + b_1 c_{1h} F_i(\vec{a} - \hat{e}_1 + \hat{e}_h) + \sum_{k=2}^{N} b_k F_i(\vec{a} - \hat{e}_k) = 0 \tag{4.76}$$

such that we have traded the direct neighbour $\vec{a} - \hat{e}_1$ for a sub-lattice neighbour $\vec{a} - \hat{e}_1 + \hat{e}_h$. One can then generically repeat this procedure for all of the other direct neighbours $\vec{a} - \hat{e}_k$ by taking PFEs centered on them. Then, one obtains generically

$$F_i(\vec{a}) \left(1 - \sum_{k=1}^{N} b_k c_{kh_k} \right) + \sum_{k=1}^{N} b_k c_{kh_k} F_i(\vec{a} - \hat{e}_k + \hat{e}_{h_k}) = 0 \tag{4.77}$$

which is a relation involving N possible sub-lattice 1-neighbours of $\vec{a}$. Since every choice of $\{h_k \neq k\}$ is possible, one has $(N-1)^N$ relations, although not all independent of one another.

4.4.4 Idea of the Proof

We have now in hand all the elements to state the idea of the proof.
We assume that there exists a recurrence relation of the form

$$F_i(\vec{a}) - \sum_{k=1}^{N} \beta_k(\vec{a}) F_i(\tilde{e}_k) = 0 \tag{4.78}$$

with $\beta(\vec{a})$ a rational function for all $\vec{a} \in \Lambda_*$. That is, we assume there exists a rational recurrence relation linking any point on the sub-lattice to the N reference points of Λ_*.
It then directly follows

$$\begin{aligned} \delta_i(\vec{a}) &= \sum_{j=1}^{N} M_{ij} F_j(\vec{a}) && (4.79) \\ &= \sum_{j=1}^{N} M_{ij} \sum_{k=1}^{N} \beta_k(\vec{a}) F_j(\tilde{e}_k) && (4.80) \\ &= \sum_{k=1}^{N} \beta_k(\vec{a}) \delta_i(\tilde{e}_k) && (4.81) \\ &= \beta_i(\vec{a}) && (4.82) \end{aligned}$$

where we have used $\delta_i(\tilde{e}_k) = \delta_{ik}$ in the last line. Therefore, assuming (4.78) allows us to conclude that our basis is a rational parametrisation.
We only have at our disposition relations (4.77), which link a point to its lattice nearest neighbours.

Conjecture:
One can always combine several copies of equations (4.77) to obtain a recurrence relation

$$F_i(\vec{a}) - \sum_{k=1}^{N} \beta_k(\vec{a}) F_i(\tilde{e}_k) = 0 \tag{4.83}$$

linking any point $\vec{a} \in \Lambda_*$ to the reference points $\tilde{e}_k$ of the sub-lattice. As relations (4.77) have rational coefficients, their combinations will also only have rational coefficients, such that (4.82) holds.

We will attempt to prove this conjecture by iterating on the distance between points (which we define below) in a relation, as if it is possible to build relations for points at any distance, then it is possible to build (4.83). The relations (4.77) satisfy then the initialisation of the iteration.

Notice that in the $N = 2, p = 1$ case (which was the example case), relation (4.77) is already the target relation (4.83), as the only non reference point of the lattice is a first neighbour of both reference points.

4.4.5 Elements of Proof

We now attempt a proof that it is always possible to find a relation as (4.83) with $\beta_k(\vec{a})$ rational. We first define the distance of a relation.

Distance between points

We take the distance $||\_||$ between two points on the lattice $\vec{a}_1$ and $\vec{a}_2$ to be half of the $||\_||_1$ norm, that is, the total number of unit vectors $\hat{e}_k$ one needs to add to $\vec{a}_1$ to obtain $\vec{a}_2$, divided by 2.
As an example: consider $\vec{a}_1 = (a_1, ..., a_N)$ and $\vec{a}_2 = (a_1, ..., a_7 + 11, ..., a_5 - 3, ..., a_N)$. Then, $||\vec{a}_1 - \vec{a}_2||_1 = 14$ and $||\vec{a}_1 - \vec{a}_2|| = 7$. We take this half because we will always be working with points on Λ_*, on which nearest neighbours are always 2 unit vectors apart.

Then, we call "n-neighbours" two points $\vec{a}$ and $\vec{b} \in \Lambda_*$ s.t. $||\vec{a} - \vec{b}|| = n$. As well, we call "up-to-n-neighbours" two points s.t. $||\vec{a} - \vec{b}|| \leq n$.

Element of Iterated Proof

The iteration hypothesis is then as such:

Assume that, $\forall \vec{a}_0 \in \Lambda_*, \quad \forall \{\vec{a}_j\}_{j=1,...N} \in \Lambda_*$ *s.t.* $||\vec{a}_j - \vec{a}_0|| \leq n$,
then, $\exists A_n := F(\vec{a}_0) + \sum_{j=1}^{N} \beta_j F(\vec{a}_j) = 0$ *s.t.* β_j *rational.*

That is, we assume that given any starting point on the sub-lattice, there is a rational relation linking this point to any N up-to-n-neighbours on the sub-lattice.
We show now that this hypothesis allows one to build relations at distance $n+1$:

Pick a $\vec{a}_0$ *and* $\{\vec{a}_k\}_{k=1...N}$ *s.t.* $||\vec{a}_0 - \vec{a}_k|| \leq n$. *Consider* $\vec{a}'_0$ *a 1-neighbour of* $\vec{a}_0$.
$\rightarrow$ *By iteration hypothesis,* $\exists A_n := F(\vec{a}_0) + \sum_{j=1}^{N} \beta_j F(\vec{a}_j) = 0$.
$\rightarrow$ *Also by iteration hypothesis,* $\exists \tilde{A}_n := F(\vec{a}_0) + \sum_{j=1}^{N-1} \beta'_j F(\vec{a}_j) + \gamma F(\vec{a}'_0) = 0$.

Therefore, if one considers $B_{n+1} := (\tilde{A}_n - A_n)/\gamma$, *one has that*

$$B_{n+1} = F(\vec{a}'_0) + \sum_{j=1}^{N-1} \frac{\beta_j - \beta'_j}{\gamma} F(\vec{a}_j) + \frac{\beta_N}{\gamma} F(\vec{a}_N) = 0 \tag{4.84}$$

s.t. $||\vec{a}'_0 - \vec{a}_j|| \leq n+1 \forall j$, *and all the coefficients in* B_{n+1} *are rational. Therefore, we have obtained a relation between a point* $\vec{a}'_0$ *and* N *of its up-to-*$(n+1)$*-neighbours.*

Caveat

We have called section 4.4.5 an element of proof for the following reason: while relation (4.84) does link $\vec{a}'_0$ to N of its up-to-$n+1$-neighbours $\vec{a}_j$ on Λ_*, the assumption is that all of these are up-to-n-neighbours of $\vec{a}_0$. However, in the most general case, not all $n+1$-neighbours of $\vec{a}'_0$ are $n-$neighbours of the same 1-neighbour $\vec{a}_0$ of $\vec{a}'_0$. As such, this element of proof is insufficient to validate the conjecture that one can always link any point on the sub-lattice to any N other points.
However, it is true that all $n+1$-neighbours of a point are n neighbours of some 1-neighbour of the initial point. As such we believe that it is still possible to find relations of the form (4.83) and that it could be proved possibly by considering a detailed study of the 1-neighbours of the initial point and applying similar arguments as the element of proof to each 1-neighbour separately.
As well, we report that, through Mathematica, we were able to verify that it is possible to find such a reparametrisation in the $N = 3$ and $N = 4$ case up to $p = 5$, which further supports the claim that this reparametrisation is

always possible.
Finally we can state that in the 2 current case, the previously mentioned caveat vanishes as in this case, the sub-lattice is a single line. One can then define a direction, say left to right, and pick the relations A_n and $\tilde{A}_n$ to be in this direction: we pick $\vec{a}_0$ to be the "leftmost" point in the relation $\tilde{A}_n$, and $\vec{a}'_0$ to be the left neighbour of $\vec{a}_0$. Then, all the $n+1$-neighbours of $\vec{a}'_0$ will be n-neighbours of $\vec{a}_0$, and the caveat vanishes. One can then iterate on the distance, and we obtain a then rational relation linking any 3 points on the 1 dimensional sub-lattice.
As such, the proof holds for $N = 2$ currents, and there is always a rational reparametrisation at $N = 2$ currents.

Chapter 5

Conclusion

The first goal of this thesis was to investigate the integrable structure of Wess-Zumino-Witten models and to characterise a hypothetical integrable hierarchy built of the Kac-Moody currents of the theory. To answer this question, we took a brute force approach to building the charges, and we were able to obtain expressions for the first four densities of the commuting charges, for the $\mathfrak{g} = \mathfrak{su}(2)$ WZW model. As well, these charges were explicitly different from both their "classical counterparts" [1] and from the quantum KdV hierarchy, which was expected to be obtainable. This hints towards a new, previously unknown integrable hierarchy built from Kac-Moody algebras.

The second goal of this thesis was to extend this study to the over arching framework of Affine Gaudin Models, in which the conjecture on local conserved charges predicted the expressions of these charges. We were able to explicitly match our brute force result on the WZW charges to the expressions predicted by the conjecture. As well, using the conjecture, we obtained expressions for the commuting conserved charge densities for the $N = 2$ currents AGMs, and characterised their λ_2 dependency through a geometric interpretation in terms of contours. Finally, matching these expressions to results from a brute force approach allowed us to get a hint for a rational parametrisation, which we proved in the $N = 2$ current case, and setup to be proven in the most general case.

An immediate next step, which we already started in appendix D, is then to use the expressions of the WZW commuting charges to completely organise the Hilbert space of the theory into eigenstates of the conserved charges.
When considering WZW models as a CFT, the Hilbert space of the theory organises as eigenstates of two conserved charges: the generator of holomorphic translations $I_1 = L_{-1}$ and the generator of the global symmetry J_0^a. However, as it turns out, the spectrum of these two charges is highly

[1] understood as the normal ordered version of the classical charges

degenerate, and the whole Hilbert space of the theory cannot be uniquely characterised by the eigenvalues of these charges.
Considering WZW models as an integrable CFT allowed us to obtain an integrable hierarchy, that is, the expressions of the "higher" commuting conserved charges $I_3, I_5, \ldots$. As such, these higher conserved charges from the new hierarchy are expected to lift degeneracies in the spectrum of L_{-1} and J_0^a, and it would be the next step to check to which extent these degeneracies are lifted. In particular, one very interesting question to investigate would be whether the charge I_3 alone is enough to totally lift the degeneracies of the spectrum. While in general, one would not expect this to be the case, the fact that the expressions of I_5 and I_7 were completely fixed by imposing commutation with I_3 (and not amongst themselves) suggests that all the information of the hierarchy could be contained in I_3. If this pattern continues for all of the higher charges I_{2n+1}, then I_3 would contain all the information of the hierarchy (it would "generate" it in a sense), and one would expect I_3 to lift all the degeneracy of the spectrum.

The next next step would be then to prove the conjecture on local charges of AGMs. This is of course not a straightforward question, but it can be taken in steps. One could explicitly check the conjecture for other Lie algebras. While we expect the construction to continue analogously for $\mathfrak{su}(N)$, it might be interesting to test the conjecture on exotic Lie algebras, such as the exceptional Lie algebras. One could also continue explicit checks at higher order, with more currents, and checks of commutation with non-local charges [30][31]. Finally, formally proving the existence of the rational parametrisation at all currents could provide some hints in the structure of the spectral densities.
Proving the conjecture, even in specific case would be a great step forward in formally establishing integrability for sigma models, as one would have immediate access to the local integrable hierarchy of a wide class of sigma models at their conformal point. Amongst these sigma models are many with direct applications to various areas of physics. As such, solving this conjecture is clearly a worthwhile endeavour, with wide ranging consequences.

A (maybe very) long term next step is then to investigate integrable deformations of the new integrable hierarchy we obtained. As mentioned in the introduction, WZW models are the conformal limit of several models, for instance lambda models [10]. These are integrable deformations of WZW models: they are obtained by adding terms to the CFT action that recouple chiralities such that conformal invariance is broken. As such, the new theory becomes massive, but integrability is preserved. These deformations can, at least classically, be recast in terms of AGMs [13]. It would therefore be very interesting to study how the hierarchy gets deformed by these deformations, and how it relates to the dependence on spectral parameters.

Appendix A

Proofs of Section 2

A.1 Conformal Transformations

We now show how to obtain equations (2.2)-(2.5).
To derive the conformal group, one considers an infinitesimal transformation $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(\mathbf{x})$. Then the metric changes as

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \tag{A.1}$$

$$= g_{\mu\nu} - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) \quad . \tag{A.2}$$

Then, equation (2.1) implies

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(\mathbf{x}) g_{\mu\nu} \tag{A.3}$$

with $f(\mathbf{x}) = \frac{2}{d}\partial_\rho \varepsilon^\rho$ by taking the trace of both sides.

From this equation, one can derive the explicit form of the conformal transformations.
By permuting, taking traces and renaming indices (and one can assume for simplicity a flat background metric $g_{\mu\nu} = \eta_{\mu\nu}$), one can obtain the following series of constraints (see [19] page 96-97):

$$2\partial_\mu\partial_\nu\varepsilon_\rho = \eta_{\nu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f \tag{A.4}$$

$$2\partial^2\varepsilon_\mu = (2-d)\partial_\mu f \tag{A.5}$$

$$(2-d)\partial_\mu\partial_\nu f = \eta_{\mu\nu}\partial^2 f \tag{A.6}$$

$$(d-1)\partial^2 f = 0 \quad . \tag{A.7}$$

One can use the following ansatz $f(\mathbf{x}) = A + B_\mu x^\mu$, which translates into

$$\varepsilon_\mu = a_\mu + b_{\mu\nu}x^\mu + = c_{\mu\nu\rho}x^\nu x^\rho, \qquad c_{\mu\nu\rho} = c_{\mu\rho\nu} \tag{A.8}$$

and then treat every power of x separately. The constraints then translate into constraints on the coefficients (we treat $d = 2$ separately later): a_μ is free and therefore gives a translation, $b_{\mu\nu}$ is constrained by equation (A.3) to

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu} \qquad m_{\mu\nu} = -m_{\nu\mu} \tag{A.9}$$

and therefore leads to a dilation and a rotation; $c_{\mu\nu\rho}$ is related to $b_{\mu\nu}$ by the constraints equations (A.4)-(A.7)

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu \qquad b_\mu = \frac{1}{d}c^\sigma_{\sigma\mu} \tag{A.10}$$

and leads to the following infinitesimal transformation, known as Special Conformal Transformation (SCT):

$$x'^\mu = x^\mu + 2(\mathbf{x}\cdot\mathbf{b})x^\mu - b^\mu\mathbf{x}^2 \quad . \tag{A.11}$$

One can then take the infinitesimal transformations and exponentiate them. This leads to the following coordinate transformations:

$$\text{Translation:} \qquad x'^\mu = x^\mu + a^\mu \tag{A.12}$$

$$\text{Dilation:} \qquad x'^\mu = \alpha x^\mu \tag{A.13}$$

$$\text{Rotation (+boosts):} \qquad x'^\mu = M^\mu_\nu x^\nu \tag{A.14}$$

$$\text{SCT:} \qquad x'^\mu = \frac{x^\mu - b^\mu\mathbf{x}^2}{1 - 2\mathbf{b}\cdot\mathbf{x} + b^2\mathbf{x}^2} \quad . \tag{A.15}$$

A.2 Derivation of Ward Identities

In this section, the goal is to derive the Ward identity (2.25) from the Ward identities of the conformal group (2.22)-(2.24).

For this, the first necessary ingredient is the following expression for the delta distributions:

$$\delta(\mathbf{x}) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}} \tag{A.16}$$

which is justified when integrating over some region M of the complex plane bounded by some contour ∂M: Gauss's theorem gives the expected result straightforwardly.

The Ward identities (2.22)-(2.24) are then:

$$2\pi\partial_z\langle T_{\bar{z}z}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{zz}X\rangle = -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle \tag{A.17}$$

$$2\pi\partial_z\langle T_{\bar{z}\bar{z}}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle \tag{A.18}$$

$$2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle = -\sum\delta(\mathbf{x}-\mathbf{x}_i)\Delta_i\langle X\rangle \tag{A.19}$$

$$-2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle = -\sum\delta(\mathbf{x}-\mathbf{x}_i)s_i\langle X\rangle \quad . \tag{A.20}$$

Summing and subtracting the last two equations, we obtain:

$$2\pi\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}h_i\langle X\rangle \tag{A.21}$$

$$2\pi\langle T_{z\bar{z}}X\rangle = -\sum_{i=1}^{n}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\bar{h}_i\langle X\rangle \tag{A.22}$$

$$\tag{A.23}$$

which justifies the relation between the conformal dimensions, scaling dimensions and planar spin. We can then plug these equations into the first two Ward identities to obtain

$$\partial_{\bar{z}}\left[\langle T(z,\bar{z})X\rangle - \sum_{i=1}^{n}\left[\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right]\right] = 0 \tag{A.24}$$

and its complex conjugate, with $T = -2\pi T_{zz}$, $\bar{T} = -2\pi T_{\bar{z}\bar{z}}$. This expression tells us that the expression between brackets is holomorphic (resp. anti-holomorphic). As such, one can write

$$\langle T(z)X\rangle = \sum_{i=1}^{n}\left[\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right] + \text{reg} \tag{A.25}$$

where reg stands for a regular holomorphic function of z as $z = w_i$. This is equation (2.25).

One can derive the other form of the Ward identities starting from an infinitesimal variation of the coordinates, which will turn out essential later when deriving commutation relations. One starts with an infinitesimal coordinate transformations $\delta x^\mu = \varepsilon^\mu$. Then one can write:

$$\partial_\mu(\varepsilon_\nu T^{\mu\nu}) = \varepsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu)T^{\mu\nu} + \frac{1}{2}(\partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu)T^{\mu\nu} \tag{A.26}$$

$$= \varepsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\rho\varepsilon^\rho)\eta_{\mu\nu}T^{\mu\nu} + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta\varepsilon_{\mu\nu}T^{\mu\nu} \quad . \tag{A.27}$$

One can see in the last line that the three T terms correspond to the l.h.s of equations (2.22),(2.23) and (2.24). Therefore, one can see that integrating both sides of equation (A.27), one can write the three Ward identities in a unified way

$$\delta_\varepsilon \langle X \rangle = \int_M d^2x \partial_\mu \langle T^{\mu\nu} \varepsilon_\nu X \rangle \tag{A.28}$$

with $\delta_\varepsilon X$ is the variation of X under a local conformal transformation, and M is an integration domain containing all the positions of the fields in X. As the integrand is a total derivative, one can use Gauss's theorem to convert this into a contour integral over a boundary $C = \partial M$; as well taking the conventions of equation (2.25), one obtains:

$$\delta_\varepsilon \langle X \rangle = -\frac{1}{2\pi i} \oint_C dz \varepsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi i} \oint_C d\bar{z} \bar{\varepsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle \quad . \tag{A.29}$$

This equation is known as the Conformal Ward Identity.

A.3 Radial Quantisation

In Euclidean field theory (which comes from statistical physics) it is unnatural to pick out an arbitrary direction to call "time": it amounts to picking directions in the lattice and call them "space" which defines the "time" direction as the one orthogonal to the "space" ones. However, in the continuum limit, there is a continuum of directions to pick as "space": a symmetric choice is then to pick the "time" direction as the radial directions, with the "space" directions as concentric circles centred on the origin.

In the Minkowski case, this choice can also be motivated by string theory, more specifically by considering the CFT on a closed string worldsheet, which for a free string, is a cylinder with the "space" direction along the compactified dimension of length L and the "time" direction along the length of the cylinder. Then, by "exploding" the cylinder onto the plane, with $t = -\infty$ at the origin, one obtains the same setup as in the statistical physics case. Explicitly, we convert Minkowski coordinates t, x into holomorphic coordinates by the map $z = e^{2\pi(t+ix)/L}$. Then, when considering correlation functions, time ordering becomes radial ordering: the fields inside correlators are ordered such that the smallest radius (=earlier time) acts always first.

Using this "radial time" also means we have to revise our definition of Hermitian conjugation: in Minkowski space, time is left unchanged by Hermitian conjugation. This means that when considering holomorphic coordinates, one must map z to $1/z*$ as to not change the Minkowski time t. Considering a primary field φ of conformal dimensions $h, \bar{h}$. Then, we de-

fine the Hermitian conjugate of this field as

$$[\varphi(z,\bar{z})]^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \varphi(1/\bar{z}, 1/z) \tag{A.30}$$

where the right hand side prefactors can be justified by imposing the inner product of asymptotic "in" and "out" states be well defined:

$$\begin{aligned}
\langle \varphi_{out} | \varphi_{in} \rangle &= \lim_{z,\bar{z},w,\bar{w}\to 0} \langle 0 | \varphi(z,\bar{z})^\dagger \varphi(w,\bar{w}) | 0 \rangle && \text{(A.31)} \\
&= \lim_{z,\bar{z},w,\bar{w}\to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0 | \varphi(1/\bar{z}, z)^\dagger \varphi(w,\bar{w}) | 0 \rangle && \text{(A.32)} \\
&= \lim_{\xi,\bar{\xi}\to\infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \langle 0 | \varphi(\bar{\xi}, \xi) \varphi(0,0) | 0 \rangle && \text{(A.33)} \\
&= Constant && \text{(A.34)}
\end{aligned}$$

where, to conclude that this expression is constant, one needs to use conformal invariance to constrain the two point function of primary fields:

$$\langle \varphi_1(\mathbf{x}_1) \varphi_2(\mathbf{x}_2) \rangle = \frac{C_{1,2} \delta_{h_1,h_2} \delta_{\bar{h}_1,\bar{h}_2}}{|\mathbf{x}_1 - \mathbf{x}_2|^{2h_1 + 2\bar{h}_1}} \quad . \tag{A.35}$$

Equation (A.35) is the only non trivial conformally invariant two point function of two primary fields. This justifies the prefactors of the Hermitian conjugation.

A.4 Derivation of Virasoro Algebra

We here prove equation (2.49).

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{(2\pi i)^2} \oint_0 dw w^{n+1} \oint_w dz z^{m+1} \left\{ \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg \right\} \\
&= \frac{1}{2\pi i} \oint_0 dw w^{m+1} \left\{ \frac{1}{12} c(n+1)n(n-1) w^{n-2} + \right. \\
&\qquad \left. 2(n+1) w^n T(w) + w^{n+1} \partial T(w) \right\} \\
&= \frac{1}{12} cn(n^2-1)\delta_{n+m,0} + 2(n+1) L_{m+n} - \frac{1}{2\pi i} \oint_0 dw (n+m+2) w^{n+m+1} T(w) \\
&= \frac{1}{12} cn(n^2-1)\delta_{n+m,0} + (n-m) L_{m+n}
\end{aligned}$$

and the computation for the anti-holomorphic mode is identical. Altogether, one has

$$\begin{aligned}
[L_n, L_m] &= (n-m) L_{m+n} + \frac{1}{12} cn(n^2-1)\delta_{n+m,0} && \text{(A.36)} \\
[L_n, \bar{L}_m] &= 0 && \text{(A.37)} \\
[\bar{L}_n, \bar{L}_m] &= (n-m) \bar{L}_{m+n} + \frac{1}{12} cn(n^2-1)\delta_{n+m,0} && \text{(A.38)}
\end{aligned}$$

which is the claim of equation (2.49).

Appendix B

Details on WZW models

B.1 Topology of WZW models

We show here the topological considerations that lead to requiring k to be an integer. The starting point is equation (3.6), which we recall here

$$\Gamma = \frac{-i}{24\pi}\int_B d^3y\varepsilon_{\alpha\beta\gamma}Tr'(\tilde{g}^{-1}\partial^\alpha\tilde{g}\tilde{g}^{-1}\partial^\beta\tilde{g}\tilde{g}^{-1}\partial^\gamma\tilde{g}) \quad . \tag{B.1}$$

The extension $\tilde{g}$ is guaranteed to exist if one requires G to be compact. However, this extension is not unique, as there is an ambiguity in the choice of B: two possible Bs exist for a given G, we call them B_1 and B_2. The cost of this ambiguity is:

$$\begin{aligned}\Delta\Gamma =& \frac{-i}{24\pi}\int_{B_1} d^3y\varepsilon_{\alpha\beta\gamma}Tr'(\tilde{g}^{-1}\partial^\alpha\tilde{g}\tilde{g}^{-1}\partial^\beta\tilde{g}\tilde{g}^{-1}\partial^\gamma\tilde{g}) &\text{(B.2)}\\ & \frac{+i}{24\pi}\int_{B_2} d^3y\varepsilon_{\alpha\beta\gamma}Tr'(\tilde{g}^{-1}\partial^\alpha\tilde{g}\tilde{g}^{-1}\partial^\beta\tilde{g}\tilde{g}^{-1}\partial^\gamma\tilde{g}) &\text{(B.3)}\\ =& \frac{-i}{24\pi}\int_{S^3} d^3y\varepsilon_{\alpha\beta\gamma}Tr'(\tilde{g}^{-1}\partial^\alpha\tilde{g}\tilde{g}^{-1}\partial^\beta\tilde{g}\tilde{g}^{-1}\partial^\gamma\tilde{g}) &\text{(B.4)}\end{aligned}$$

where the last line stands because the two extensions can be glued together at their boundaries M, and therefore the resulting three dimensional manifold is topologically equivalent to the 3-sphere S^3.

To evaluate this integral, we can restrict $\tilde{g}$ to be valued in an $SU(2)$ subgroup of G as one can always continously deform a mapping of $S^3 \to G$ into one of $S^3 \to SU(2) \in G$. We can parametrise the 3-sphere by 4 coordinates $y^i, i = 0,1,2,3$ with the condition $(y^0)^2 + y^i y^i = 1$. Then, within the $SU(2)$ subgroup, we can write

$$\tilde{g}(y) = y^0 - iy^k\sigma_k \tag{B.5}$$

with σ_k the usual Pauli matrices. Then, using $Tr'\sigma_i\sigma_j = 4\delta_{ij}$ and $\int_{S^3} d^3y = 2\pi^2$ it becomes straightforward to evaluate the integral. This gives

$$\Delta\Gamma = 2\pi i \quad . \tag{B.6}$$

As such, one can see that this ambiguity in the choice of the extension domain becomes irrelevant in an exponential (therefore in a partition function) if one multiplies Γ by an integer. As such, one considers the action

$$S = S_0 + k\Gamma \tag{B.7}$$

where k is an integer, called the level of the model.

B.2 Derivation of Affine Kac-Moody OPE and Commutation Relations

We here show how to obtain the OPE and commutation relations for the chiral currents. We will then follow an analogous procedure to section 2.2, starting by deriving its OPE from Ward identities.
We consider an infinitesimal transformation $g \to g + \omega g$. Then, the transformation of the action is

$$\delta S = -\frac{1}{2\pi}\int d^2x \left\{\partial_{\bar{z}}(Tr'[\omega(z)J(z)]) + \partial_z(Tr'[\bar{\omega}(\bar{z})\bar{J}(\bar{z})])\right\} \quad . \tag{B.8}$$

Where J and $\bar{J}$ are rescaled conserved chiral current defined by

$$J(z) := -k\partial_z g g^{-1} \tag{B.9}$$

$$\bar{J}(\bar{z}) := k g^{-1}\partial_{\bar{z}} g \quad . \tag{B.10}$$

We then proceed as for equation (A.29), by replacing d^2x by $(-i/2)dzd\bar{z}$, and using Gauss's theorem to replace two dimensional integrals by holomorphic and anti-holomorphic contours. This gives the following Ward identity:

$$\delta_{\omega,\bar{\omega}} S = \frac{i}{4\pi}\oint dz Tr'[\omega(z)J(z)] - \frac{i}{4\pi}\oint d\bar{z} Tr'[\bar{\omega}(\bar{z})\bar{J}(\bar{z})] \quad . \tag{B.11}$$

According to their definitions, the currents J and $\bar{J}$ take values in $\mathfrak{g}$, as does ω. We therefore expand J and ω in the generators T^a of $\mathfrak{g}$:

$$J = \sum_a J^a T_a \qquad \omega = \sum_a \omega^a T_a \quad . \tag{B.12}$$

Plugging this in (B.11) and evaluating the traces, we get

$$\delta_{\omega,\bar{\omega}} S = -\frac{1}{2\pi i}\oint dz \sum_a \omega_a J^a + \frac{1}{2\pi i}\oint d\bar{z}\sum_a \bar{\omega}_a \bar{J}^a \quad . \tag{B.13}$$

Finally, we show that $\delta\langle X\rangle = \langle(\delta S)X\rangle$ with X some string of fields.

$$\langle X\rangle = \frac{1}{Z}\int \mathcal{D}\Phi X e^{-S[\Phi]} \tag{B.14}$$

$$= \frac{1}{Z}\int \mathcal{D}\Phi (X+\delta X)\Phi e^{-S[\Phi]-\delta S} \tag{B.15}$$

$$= \frac{1}{Z}\int \mathcal{D}\Phi \left(Xe^{-S[\Phi]} + \delta X e^{-S[\Phi]} - X\delta S e^{-S[\Phi]} + O(\delta^2)\right) \tag{B.16}$$

$$= \langle X\rangle + \delta\langle X\rangle - \langle \delta S\rangle \quad . \tag{B.17}$$

From which one concludes. Using this on (B.13), one gets the Ward identity

$$\delta_{\omega,\bar{\omega}}\langle X\rangle = -\frac{1}{2\pi i}\oint dz \sum_a \omega_a \langle J^a X\rangle + \frac{1}{2\pi i}\oint d\bar{z}\sum_a \bar{\omega}_a\langle \bar{J}^a X\rangle \quad . \tag{B.18}$$

We can use this Ward identity with $X = J^a$ to compute the OPE of two currents. One can compute $\delta_\omega J$ from the definition and finds

$$\delta_\omega J = [\omega, J] - k\partial_z \omega \tag{B.19}$$

which expanded gives

$$\delta_\omega J^a = i f^a{}_{bc}\omega^b J^c - k\partial_z \omega^a \quad . \tag{B.20}$$

Plugging this transformation into equation (B.18) gives the OPE

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c(w)}{z-w} + reg \tag{B.21}$$

which is the OPE given in equation (3.13).
As in the Virasoro case (2.49), we can convert this chiral algebra into an algebra in terms of modes: we expand J^a into modes

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J^a_n \tag{B.22}$$

and proceed as for the Virasoro case (equation (2.49)) to obtain the following commutation relations

$$[J^a_n, J^b_m] = i f^{ab}{}_c J^c_{n+m} + kn\delta^{ab}\delta_{n+m,0} \tag{B.23}$$

giving the commutation relations (3.16).

Appendix C

Implementation of the Bases

In this section, we present in more detail the implementation of the lexicographic and invariant basis that was used in the brute force method to obtain local charges. We present "recipes" that were formally implemented in Mathematica, using the package [23] to account for fields.

C.1 Lexicographic Basis

The lexicographic basis of the Hilbert space (or more correctly the field space) is defined as normal ordered products of the form

$$: \partial^{k_1} J^1 \partial^{k_2} J^1 ... \partial^{k_{l_1}} J^1 \partial^{k'_1} J^2 ... \partial^{k'_{l_2}} J^2 ... : \tag{C.1}$$

with

$$k_1 \geq k_2 \geq ... \geq k_{l_1}, \qquad k'_1 \geq ... \geq k'_{l_2} \quad ... \tag{C.2}$$

and the conformal dimension

$$h = \sum_{i=1}^{dim\mathfrak{g}} \sum_{j=1}^{l_i} (1 + k_j^{(i)}) \quad . \tag{C.3}$$

This basis spans the entire space of fields. It is not so relevant conceptually to the thesis, but it is extremely important in practice, as we use this basis to check for independence of the more complicated fields that appear in the invariant basis.
The question is then how to implement such a basis? We show how to do this in the $\mathfrak{g} = \mathfrak{su}(2)$ case, in which one has three currents, but the generalisation to any $\mathfrak{g}$ is straightforward.

Consider a fixed conformal dimension h,

- Find all 3 integer partitions of h.

- Take all possible permutations of these 3 integers within one partition: this gives a set of 3 integers which are the total powers given to each current J^a.
- For a given set of integers $\{I_1, I_2, I_3\}$, we take for each of these integers all its possible integer partitions: call k_1 the set of all non zero integer partitions of I_1.
- Now, to every partition in k_a, we associate the field $\prod_{j=1}^{|k_a|} \partial^{k_a^j - 1} J^a$.
- We then join together the fields associated to k_1 k_2 and k_3, and take a normal ordering of this product of fields.
- Repeat this procedure for every set $\{I_1, I_2, I_3\}$ and for every of its set of partitions $\{k_1, k_2, k_3\}$.

This then gives us the lexicographic basis at every conformal dimension for $\mathfrak{g} = \mathfrak{su}(2)$. One can generalise this to any $\mathfrak{g}$ simply by replacing 3 by $\dim \mathfrak{g}$.

C.2 Basis of the Casimir Algebra

We now present how we build the basis of fields of the Casimir algebra C_h at conformal dimension h. As explained in section 3.2.1, the start point is the knowledge of the invariant tensors of the group. We then first need to obtain a basis of all the invariant tensors of rank h:

- We start by numerically solving equation (3.31) for a fixed rank h. From this numerical solve we only keep the number of independent tensors in this space. We call it N_T.
- We then use equation (3.33) to generate invariant tensors with known expressions:
  - We start with either the EHMM tensor or a fully symmetric trace $Tr(T^{(a_1}....T^{a_h)})$, which we know belongs to this space.
  - We then take integer partitions of h [1]. Then, writing a partition as $\{k_i\}$, with $\sum_i^n k_i = h$, one has an invariant tensor of the form

$$Tr(T_{a_1}...T_{a_{k_1}})...Tr(T_{a_{k_{n-1}}+1}...T_{a_{k_n}}) \quad . \tag{C.4}$$

  - For each tensor we produce in this way, we check if it is linearly independent from the previous ones; if it is, we add it to the basis.
  - We continue until N_T independent tensors are obtained.

[1] with the smallest integer being 2, as $Tr(T_a) = 0$ for $\mathfrak{su}(2)$

We now have a basis of invariant tensors of rank h, and we know their explicit expressions. As well, notice that we also have a basis of all invariant tensors of dimensions $\leq h$, which will be relevant when considering fields with derivatives of J. We now use these to build the basis of fields. We also have available the number of fields in this basis from the character equation (3.38).

- We start with the "EHMM" field, which we can always take in the basis. It is simply $\kappa^{EHMM}_{a_1...a_h} : J^{a_1}...J^{a_h} :$.
- Then, we proceed starting from the lowest rank tensors. We start at rank 2 by taking all 2 integer partitions $h-2$: this gives for example fields like $\delta_{ab} : J^a \partial^{h-2} J^b :$, $\delta_{ab} : \partial J^a \partial^{h-3} J^b :$, etc...
- For every of these fields, we check for linear independence to the previous fields in the basis by projecting these fields in the lexicographic basis: if independent, we add it to the basis.
- We proceed then increasing the ranks of the tensors until the expected number of fields is reached.

This then gives a basis which will always include the "quantum EHMM" field, together with fields involving increasing tensor rank. This choice of basis was made for computational reasons: having simpler tensors will reduce calculation times.
As well, we note that another choice of basis we used for example in C_6 was to replace the EHMM tensor field by a simpler trace tensor $s^6_{abcdef} = Tr(T_a T_b T_c T_d T_e T_f)$. This is again because it reduces computation time.

Appendix D

Charges on the Hilbert Space

This appendix serves as an introduction to the next steps one could follow after chapter 3. The main goal of this appendix is to present how one would proceed to diagonalise the conserved commuting charges, such that one could completely determine the spectrum of the theory in terms of eigenvalues under the commuting charges.
The first step is then to introduce how to compute conserved charges on the cylinder from their densities on the plane.
Then, we will introduce the Hilbert space of WZW models and define how the modes act on the states, following [32]. Then, finally, we shall compute how the conserved charges act explicitly on some simple states of the Hilbert space.

D.1 Fields and Normal Ordering on the Cylinder

As we hinted in section 2.5, with equation (2.76), we are actually interested in computing the conserved charges of the WZW theory defined on the cylinder, and not on the complex plane/sphere. As such, we need to convert our fields defined on the sphere to fields on the cylinder. For this, we use the mapping $z = e^{iu}$, with z defined on the sphere and u defined on the cylinder. For the Kac-Moody current J^a, converting from the sphere to the cylinder is very straightforward, as J^a is a primary. Therefore, under $z \to u = -i\ln(z)$, J^a changes as

$$J^a_{cyl}(u) = \left(\frac{du}{dz}\right)^{-1} J^a_{sphere}(z)|_{z=e^{iu}} \tag{D.1}$$

$$= izJ^a_{sphere}(z)|_{z=e^{iu}} \tag{D.2}$$

$$= \sum_{i\in\mathbb{Z}} (iJ^a_n)e^{-inu} \tag{D.3}$$

as such, we see that the modes $\tilde{J}^a_n$ of J^a_{cyl} are equal iJ^a_n.
This solves the situation for J. However, what about T? T is not a primary field, so its transformation under a change of coordinates is non trivial: we have to figure it out from the conformal Ward Identity (A.29). Then, from the conformal Ward Identity and an infinitesimal transformation $z \to z + \varepsilon(z)$, one has

$$\delta_\varepsilon T(u) = -\frac{1}{2\pi i}\oint dz \varepsilon(z) T(z) T(u) \tag{D.4}$$

$$= -\frac{1}{12} c \partial_u^3 \varepsilon(u) - 2T(u)\partial_u \varepsilon(u) - \varepsilon(u)\partial_u T(u) \quad . \tag{D.5}$$

One then has to non trivially exponentiate this infinitesimal transformation to find

$$T'(u) = \left(\frac{du}{dz}\right)^{-2} \left[T(z) - \frac{c}{12}\{u;z\}\right]_{z=z(u)} \tag{D.6}$$

with $\{u;z\}$ the Schwarzian derivative:

$$\{u;z\} = \frac{\partial_z^3 u}{\partial_z u} - \frac{3}{2}\left(\frac{\partial_z^2 u}{\partial_z u}\right)^2 \quad . \tag{D.7}$$

Then, using $z = e^{iu}$, one finds that

$$T_{cyl}(u) = -1\left(\sum_{n\in\mathbb{Z}} L_n e^{-inu} - \frac{c}{24}\right) \quad . \tag{D.8}$$

One then sees that the modes are offset by the conformal dimension compared to the mode expansion on the sphere.
Recalling the Sugawara construction (3.1.4), one has that T is proportional to $: J^a J^b : \delta_{ab}$. As such, one should be able to recover the expression above from the normal ordering of the currents: but sraightforwardly applying our definition of normal ordering will never yield the $\frac{c}{24}$ term that appears in (D.8) As such, we need to adapt our definition of normal ordering to the cylinder. We then refer ourselves to [20], which tells us that the correct definition of normal ordering on the cylinder is given by

$$: AB : (u) = A_-(u)B(u) + B(u)A_+(u) + \oint \frac{dz}{2\pi}\sum_{k=0}^{\infty} c_k \left(e^{i(z-u)} - 1\right)^k [A(z), B(u)] \tag{D.9}$$

where:

$$A_-(u) = \sum_{n=1}^{\infty} A_{-n} e^{inu} \qquad A_+(u) = \sum_{n=0}^{\infty} A_n e^{-inu} \tag{D.10}$$

and c_k are defined as the coefficients of the Taylor expansion of $1/\ln(u)$. We shall not show that this is the correct definition of normal ordering, and

instead trust [20]. Then, using this definition of Normal Odering on the cylinder equation (D.9) and the mode expansion of the currents on the cylinder equation (D.3), one can compute the Sugawara construction of T, and see that it matches the one given by the Schwarzian derivative equation (D.8). As well, we note that it would have been anyways necessary to define the normal ordering on the cylinder, as higher charges will anyways involve normal ordered products of the currents and of the stress-energy tensor.

D.2 Representations of WZW models

We now know how to deal with fields on the cylinder, and we wish to figure out the spectrum of the conserved charges of the theory. This means we will apply these operators on the Hilbert space of the theory: as such, we first need to characterise the Hilbert space of the theory, and how the various operators act on it.
The states of the Hilbert space must organise themselves in representations of the algebra. As such, there must be a highest weight state $|\lambda\rangle$ of the Lie algebra $\mathfrak{g}$ such that

$$J_n^a|\lambda\rangle = 0 \forall n > 0 \quad . \tag{D.11}$$

As well, the $0-$modes are simply the generators of the Lie algebra. We then go into the $\mathfrak{su}(2)$ case, in which we label the highest weight with $|l\rangle$.We go in the Cartan basis of the algebra, with $J_n^{\pm} = J_n^1 \pm iJ_n^2$ and J_n^3 as the basis.
They satisfy the following commutation relations

$$[J_n^3, J_m^3] = \frac{k}{2} m\delta_{m,-n} \qquad [J_n^3, J_m^{\pm}] = \pm J_{m+n}^{\pm} \qquad [J_m^+, J_n^-] = km\delta_{m,-n} + 2J_{m+n}^3 \quad . \tag{D.12}$$

They act on the highest weight state as

$$J_0^3|l\rangle = l|l\rangle \qquad J_0^+|l\rangle = 0 \tag{D.13}$$

and one can raise/lower the indices as

$$J_{\pm,n} = J_n^{\mp} \qquad J_{3,n} = J_n^3 \quad . \tag{D.14}$$

As well, there must exist a lowest weight state, given by $(J_0^-)^{2l}|l\rangle$.
We can figure out the conformal dimension of the highest weight state by applying the conformal generator L_0

$$L_0|l\rangle = \frac{1}{2(2+k)} \left(\sum_{n\leq -1} J_n^a J_{-n}^a + \sum_{n\geq 0} J_{-n}^a J_n^a \right) |l\rangle \tag{D.15}$$

$$= \frac{1}{2(2+k)} J_0^a J_0^a |l\rangle \tag{D.16}$$

$$= \frac{l(l+1)}{2+k} |l\rangle \tag{D.17}$$

$$= h_l |l\rangle \quad . \tag{D.18}$$

As well, from the commutation relation

$$[L_0, J^a_n] = -nJ^a_n \tag{D.19}$$

we can figure out that the conformal dimension of any state of the form $J^{a_1}_{-n_1}...J^{a_m}_{-n_m}|l\rangle$ with $n_i > 0\ \forall i$ is $h_l + \sum_{i=1}^m n_i$.
Finally, we consider the Verma Module of the representation (defined by the highest weight l) by

$$\mathcal{V} = \{...(J^3_{-2})^{n^3_2}(J^+_{-2})^{n^+_2}(J^-_{-1})^{n^-_1}(J^3_{-1})^{n^3_1}(J^+_{-1})^{n^+_1}(J^-_0)^{n^-_0}|l\rangle, ns > 0\} \tag{D.20}$$

this space will contain all the states of the representation, but is not irreducible, due to the presence of null vectors. We shall however not care for them, as we will restrain ourselves to the first few excitations of the ground state.

D.3 Diagonalisation of Charges

D.3.1 Zero-mode v of the I_3 Charge

We first compute the 0-modes of the conserved charges. We already computed $T(u)$ in equation (D.8), and so its 0-mode is simply

$$I_1 = (-)T_0 = L_0 - \frac{c}{24} \tag{D.21}$$

where the $-$ in front is a matter of convention.
The next charge to consider is the 0-mode of the quartic density from (3.53):

$$: TT :_0 + \frac{2}{3}\frac{: J^{a\prime\prime}J^a :_0}{2+k} \tag{D.22}$$

this computation is quite involved, but it is essentially applying equation (D.9) on both terms, and working out all the details. We present here the computation of $: J^{a\prime\prime}J^a :_0$ and refer to [20] for the computation of $: TT :_0$, which we will use for our final result.
According to equation (D.9),

$$: J^{a\prime\prime}J^a : (u) = J^{a\prime\prime}_-(u)J^a(u) + J^a(u)J^{a\prime\prime}_+(u) \tag{D.23}$$

$$+ \int_0^{2\pi}\frac{dz}{2\pi}\sum_{k=0}^{\infty} c_k \left(e^{i(z-u)} - 1\right)^k [J^{a\prime\prime}(z), J^a(u)] \quad . \tag{D.24}$$

We work out all the components step by step.
First, the mode expansions:

$$J^a(u) = \sum_{n\in\mathbb{Z}} (iJ^a_n)e^{-inu} \tag{D.25}$$

$$J^{a\prime\prime}(u) = \partial^2_u J^a(u) \tag{D.26}$$

$$= -\sum_{n\in\mathbb{Z}} n^2 (iJ^a_n)e^{-inu} \quad . \tag{D.27}$$

Then the partial fields $J^{a\prime\prime}_{\pm}$:

$$J^{a\prime\prime}_{-}(u) = -\sum_{n=1}^{\infty} n^2 (iJ^a_{-n}) e^{inu} \qquad J^{a\prime\prime}_{+}(u) = -\sum_{n=0}^{\infty} n^2 (iJ^a_n) e^{-inu} \quad . \tag{D.28}$$

Altogether, the terms except the integral give:

$$J^{a\prime\prime}_{-}(u) J^a(u) + J^a(u) J^{a\prime\prime}_{+}(u) \tag{D.29}$$

$$= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \left((m-n)^2 : (iJ^a_{m-n})(iJ^a_n) \right) e^{-inu} \quad . \tag{D.30}$$

Then, to compute the integral term, we first need the commutator:

$$[J^{a\prime\prime}(z), J^a(u)] = \sum_{n,m\in Z} e^{-inz} e^{-imu} n^2 [iJ^a_n, iJ^a_m] \tag{D.31}$$

$$= \sum_{n\in\mathbb{Z}} e^{-inz} e^{imu} n^3 k \dim(\mathfrak{g})(-1) \tag{D.32}$$

$$= k \dim(\mathfrak{g})(-1) 2\pi (i\partial_z)^3 \delta(z-u) \quad . \tag{D.33}$$

We plug this in the integral term

$$\int_0^{2\pi} \frac{dz}{2\pi} \sum_{p=0}^{\infty} c_p \left(e^{i(z-u)} - 1 \right)^p [J^{a\prime\prime}(z), J^a(u)] \tag{D.34}$$

$$= (-1) \int_0^{2\pi} dz 3k \sum_{p=0}^{\infty} c_p \left(e^{i(z-u)} - 1 \right)^p (i\partial_z)^3 \delta(z-u) \tag{D.35}$$

$$= \int_0^{2\pi} dz 3k \sum_{p=0}^{\infty} c_p (i\partial_z)^3 \left(e^{i(z-u)} - 1 \right)^p \delta(z-u) \tag{D.36}$$

$$= 3k \sum_{p=0}^{\infty} c_p (i\partial_z)^3 \left(e^{i(z-u)} - 1 \right)^p |_{z=u} \tag{D.37}$$

$$= 3k \left[\sum_{p=0} c_p \left(p(e^{i(z-u)-1})^{p-1} + 3p(p-1)(e^{i(z-u)-1})^{p-2} \right. \right. \tag{D.38}$$

$$\left. \left. + p(p-1))p-2)(e^{i(z-u)-1})^{p-3} \right) \right] |_{z=u} \tag{D.39}$$

$$= 3k(c_1 + 6c_2 + 6c_3) \tag{D.40}$$

$$= 3k(-\frac{1}{12} + \frac{6}{24} - 6\frac{19}{720}) \tag{D.41}$$

$$= \frac{k}{40} \quad . \tag{D.42}$$

Such that, altogether:

$$: J^{a\prime\prime} J^a : (u) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \left((m-n)^2 : (iJ^a_{m-n})(iJ^a_n) \right) e^{-inu} + \frac{k}{40} \quad . \tag{D.43}$$

This was very painful, so we shall skip the analogous calculation for $:TT:$, we simply take its expression from [20], also in [5].
Altogether, the zero mode of our conserved density - that is, our conserved charge, is

$$I_3 = 2\sum_{m=1}^{\infty} L_{-m}L_m + L_0^2 - \frac{c+2}{12}L_0 - \frac{2}{3}\frac{1}{2+k}\sum_{m=1}^{\infty} m^2 J^a_{-m}J^a_m + \frac{c(c+6)}{576} \quad . \tag{D.44}$$

D.3.2 Eigenvalues and Eigenstates of the I_3 Charge

We now have the mode expansion of our first non KdV conserved charge, and we know how each operator acts on the highest weight state. We will therefore now figure out the eigenvalues and eigenvectors of I_3 acting on excitations of a highest weight state $|l\rangle$.
It is no work to compute $I_3|l\rangle$:

$$I_3|l\rangle = \left(h_l^2 - \frac{c+2}{12}h_l + \frac{c(c+6)}{576}\right)|l\rangle \tag{D.45}$$

$$= I_3^{(h_l)}|l\rangle \tag{D.46}$$

where one has h_l from equation (D.18) and c the central charge from equation (3.25).
The first state in the Verma module is $J_0^-|l\rangle$. We figure out its charge by commuting the non trivial terms to the ground state:

$$I_3 J_0^-|l\rangle = I_3^{(h_l)}J_0^-|l\rangle - \frac{2}{3}\frac{1}{k+2}\sum_{m=1}^{\infty} m^2 J_{a,-m}[J^a_m J_0^-]|l\rangle \tag{D.47}$$

$$= I_3^{(h_l)}J_0^-|l\rangle - \frac{2}{3}\frac{1}{k+2}\sum_{m=1}^{\infty} m^2 J_{a,-m}[-\delta^a_3 J^-_m \tag{D.48}$$

$$+ \delta^a_+(km\delta_{m,0} + 2J^3_m)]|l\rangle \tag{D.49}$$

$$= I_3^{(h_l)}J_0^-|l\rangle \tag{D.50}$$

and, using this result, we can obtain

$$I_3(J_0^-)^k|l\rangle = I_3^{(h_l)}(J_0^-)^k|l\rangle \tag{D.51}$$

for $k \leq 2l$, which are all eigenstates of I_3.
We then proceed analogously for higher excitation states, commuting through more and more operators.
At conformal dimension $h_l + 1$, weight l, we find for instance

$$I_3 J^3_{-1}|l\rangle = \left(I_3^{(h_l+1)} - \frac{1}{3}\frac{k}{k+2}\right)J^3_-1|l\rangle - \frac{2}{3}\frac{1}{k+2}J^+_{-1}J^-_0|l\rangle \tag{D.52}$$

$$I_3 J^+_{-1}J^-_0|l\rangle = \left(I_3^{(h_l+1)} - \frac{2}{3}\frac{k+2-2l}{k+2}\right)J^+_{-1}J^-_0|l\rangle - \frac{2}{3}\frac{2l}{k+2}J^3_{-1}|l\rangle \tag{D.53}$$

we see that the states of the Verma module are not eigenstates anymore, and we need to diagonalise the matrix.
We stop here, as we have to stop somewhere. However, the next steps are clear: diagonalise the matrix to find the eigenvectors, and repeat this procedure systematically on the whole Hilbert space. One expects then that the existence of I_3 will then lift degeneracies in the spectrum, that is, states with identical I_1, J_3 and $\vec{J}^2$ eigenvalues.

Appendix E

Integrable structure of classical AGMs

We here present the classical integrable structure of Affine Gaudin Models and the standard monodromy trace approach to its conserved charges.

E.1 Integrable Structure

It might be instructive to present the standard approach to conserved charges through the monodromy traces and the R-matrix, which justifies the name "Lax" in "Lax-Gaudin matrix". We thus present the standard approach to conserved charges for the affine Gaudin model.

A very important point to keep in mind is that, while this approach is guaranteed to work classically, the so-called problem of "non-ultralocality" (which we introduce later) makes it that it is not expected that the standard approach carries on to quantum level. Therefore, we only present the standard approach classically.

The classical limit of the Lax-Gaudin matrix and the twist function is virtually identical to its quantum definition, with the exception that the quantum Kac-Moody current J_r^a are replaced by classical Kac-Moody currents $\mathcal{J}_r^a$ currents satisfying a Poisson-Kac-Moody algebra:

$$\{\mathcal{J}_r^a(x), \mathcal{J}_s^b(y)\} = \delta_{r,s}\left(f_c^{ab}\mathcal{J}_r^c(x)\delta(x-y) - k_r\kappa_{ab}\partial_x\delta(x-y)\right) \tag{E.1}$$

where κ_{ab} is the quadratic Casimir of the algebra. One can see this Poisson bracket as a classical version of the affine Kac-Moody algebra equation (3.16). We now define the proper Lax matrix of the theory [16]

$$\mathcal{L}(z,x) = \frac{\Gamma(z,x)}{\varphi(z)} \tag{E.2}$$

which one can check that, using the Poisson bracket (E.1) and some properties of the quadratic Casimir, satisfies a Maillet bracket [33][34]:

$$\{\mathcal{L}_{\mathbf{1}}(z,x),\mathcal{L}_{\mathbf{2}}(w,y)\} = [\mathcal{R}(z,w)_{\mathbf{12}},\mathcal{L}_{\mathbf{1}}(z,x)]\delta(x-y) \tag{E.3}$$
$$- [\mathcal{R}(w,z)_{\mathbf{21}},\mathcal{L}_{\mathbf{2}}(w,y)]\delta(x-y) \tag{E.4}$$
$$- (\mathcal{R}(z,w)_{\mathbf{12}} + \mathcal{R}(w,z)_{\mathbf{21}})\,\partial_x\delta(x-y) \tag{E.5}$$

where we have introduced tensorial notation $\mathcal{L}_{\mathbf{1}} = \mathcal{L}\otimes Id$, and $\mathcal{R}$ is the so-called R-matrix of the theory, given by:

$$\mathcal{R}_{\mathbf{12}}(z,w) = \frac{1}{\varphi(w)}\frac{C_{\mathbf{12}}}{z-w} \tag{E.6}$$

where $C_{\mathbf{12}} = \kappa_{ab}T^a\otimes T^b$ is the so-called split Casimir. One can check that $\mathcal{R}$ satisfies the classical Yang-Baxter equation:

$$[\mathcal{R}_{12},\mathcal{R}_{13}] + [\mathcal{R}_{12},\mathcal{R}_{23}] + [\mathcal{R}_{32},\mathcal{R}_{13}] = 0 \quad . \tag{E.7}$$

E.2 Conserved Non-Local Charges, Non-ultralocality

From the Lax matrix, we can define the monodromy trace

$$\mathcal{T}_V(z) = Tr_V\left[\overleftarrow{Pexp}\left(-\int_0^{2\pi}\mathcal{L}(z,x)dx\right)\right] \tag{E.8}$$

with V a representation of the group G, and $\overleftarrow{Pexp}$ the path ordered exponential. By virtue of the Maillet bracket equation (E.5), the monodromy trace is conserved, and commutes with itself at different values of the spectral parameter. This guarantees that the monodromy trace contains an infinite tower of commuting conserved charges as its power series in λ. These charges are then going to be non local, and will take the form of linear combinations of nested integrals of the currents J_r^a

$$\int_0^{2\pi}dx_1\int_0^{x_1}dx_2...\int_0^{x_{k-1}}dx_k Tr_V[J_{r_1}(x_1)...J_{r_k}(x_k)] \quad . \tag{E.9}$$

As the currents in the previous expression are evaluated at generically different point, these charges are called non-local, and they form half of the classical integral structure.

Here is why this section focuses on classical AGMs: in equation (E.5), a term involving derivatives of Dirac delta distribution is present, called the ultralocal term. The presence of this terms makes it that the first principle quantisation of the local conserved charges included in $\mathcal{T}$ is very very convoluted (see [30][31] for a quantisation). As such, the standard approach to

integrability is not great at providing any conserved charges when considering the quantum level. This is the main motivation of studying local charges, as they do not suffer of the problem of non-ultralocality, which lets us hope that conisdering local charges will allow for obtaining a quantum integrable hierarchy.

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