

Integrability in sigma-models

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Preface

The following topics are covered in this chapter: (1) homogeneous spaces, (2) classical integrability of sigma-models in two dimensions, (3) topological terms, (4) background-field method and beta-function, (5) S-matrix bootstrap in the $O(N)$ model, (6) super-symmetric cosets and strings on $AdS_d \times X$.

5.1 Introduction

A sigma-model is a field theory wherein fields take values in a curved manifold \mathcal{M} . In other words, a field configuration is a map¹

$$X^M(\sigma) : \Sigma \rightarrow \mathcal{M}. \quad (5.1)$$

Sigma-models arise as effective field theories in many physical contexts that range from low-energy QCD [1] to condensed-matter systems [2].

Another vast area of applications of sigma-models is string theory, where they govern string propagation in curved backgrounds. The ultimate goal of this chapter is to introduce string sigma-models relevant for holographic duality whose key feature is complete integrability. Quantization of integrable sigma-models will then be exemplified by the S-matrix bootstrap in the $O(N)$ model [3]. Although sigma-models that arise in the holographic duality are in many respects different, methods for their exact solution are conceptually the same.

The canonical example of holography, the AdS/CFT correspondence, deals with string theory on $AdS_5 \times S^5$, a non-linear manifold whose curvature is controlled by the 't Hooft coupling of the dual gauge theory. The sigma-model on $AdS_5 \times S^5$ [4] is completely integrable [5], a remarkable property not only of this particular model, but of a whole class of holographic string backgrounds, which underlies proliferation of integrability methods in the gauge–string duality. The goal of this chapter is a step-by-step introduction to integrable sigma-models.

Given local coordinates X^M on \mathcal{M} , the most general two-derivative Lagrangian of a sigma-model is

$$\mathcal{L} = \frac{1}{2} \left(\sqrt{|h|} h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N + \varepsilon^{ab} B_{MN}(X) \partial_a X^M \partial_b X^N \right). \quad (5.2)$$

Transformations that leave the metric and B-field invariant translate into global symmetries of the sigma-model. Symmetries are neither necessary nor sufficient for integrability, but they allow one to build a large class of integrable models. The sigma-models arising in the holographic duality are precisely of this type. For this reason we concentrate on the cases when the target space \mathcal{M} admits an action of a (simple) Lie group (or supergroup) G , and in Section 5.2 recollect basic facts about action of Lie groups on manifolds,

¹ The space-time manifold Σ , for the purpose of this chapter, will almost exclusively be the 2d Minkowski space with $(+ -)$ signature or the Euclidean \mathbb{R}^2 .

following [6]. We then describe classical integrability of sigma-models on such manifolds, following [7].

5.2 Geometry

We start by collecting basic geometric facts about homogeneous spaces, closely following [6].

Def 1 *A map*

$$T_g(x) : G \times \mathcal{M} \rightarrow \mathcal{M}$$

defines left (right) action of group G on manifold \mathcal{M} , if

$$T_g T_h = T_{gh} \quad (T_g T_h = T_{hg})$$

and $T_1 = \text{id}$.

The stability group—or the little group—of a point $x \in \mathcal{M}$ is defined as the set of all elements in G that leave x intact:

$$H_x = \{g \in G \mid T_g(x) = x\}.$$

Def 2 *\mathcal{M} is a left (right) homogeneous space of group G , if the action of G on \mathcal{M} is transitive, namely if*

$$\forall x, y \in \mathcal{M} \quad \exists g \in G : T_g(x) = y.$$

Stability groups of any two points in a homogeneous space are isomorphic to one another. To see this we notice that any two points in \mathcal{M} , x and y , are related by the group action: $T_g(x) = y$ and $T_{g^{-1}}(y) = x$. Consider now an element $h \in H_x$ ($T_h(x) = x$), then

$$y = T_g(x) = T_g T_h(x) = T_g T_h T_{g^{-1}}(y) = T_{ghg^{-1}}(y).$$

Therefore $ghg^{-1} \in H_y$ and the stability groups H_y and H_x are related by conjugation:

$$H_y = gH_x g^{-1}.$$

In particular, they have the same multiplication table.

Given a subgroup $H \subset G$, one can define a (right) coset G/H as a set of equivalence classes with respect to right multiplication by H :

$$G/H = \{g \sim gh \mid g \in G, h \in H\}.$$

A set gH , obtained by multiplying all elements of H by g , constitutes one point in G/H (Figure 5.1). One can show that for a closed Lie subgroup $H \subset G$, the coset space G/H is a smooth manifold. The left coset $H \backslash G$ is defined in a similar way.

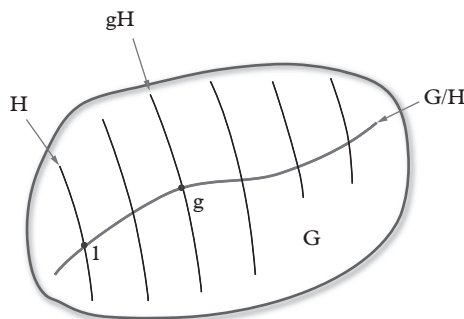


Figure 5.1 A group manifold foliated over equivalence classes gH . In the coset G/H , each equivalence class is projected to one point.

The coset G/H is a left homogeneous space of G , with the group action defined by left multiplication:

$$T_k(gH) = kgH. \quad (5.3)$$

Interestingly, the converse is also true, by virtue of the following theorem.

Th 1. Homogeneous space \mathcal{M} is isomorphic to the coset of its symmetry group G by the stability group H_{x_0} of any point $x_0 \in \mathcal{M}$: $\mathcal{M} = G/H_{x_0}$.

Proof. Define a map $f : G/H_{x_0} \rightarrow \mathcal{M}$ by

$$f(gH) = T_g(x_0).$$

This map

- does not depend on the representative in the equivalence class gH :

$$T_{gh}(x_0) = T_g T_h(x_0) = T_g(x_0),$$

and thus maps the whole equivalence class to a single point in \mathcal{M} .

- covers all \mathcal{M} . This follows from transitivity of the group action.
- is one-to-one. Indeed, assuming that $f(g_1H) = f(g_2H)$, we have $T_{g_1}(x_0) = T_{g_2}(x_0)$ and $T_{g_2^{-1}g_1}(x_0) = x_0$. Consequently, $g_2^{-1}g_1 \in H_{x_0}$. Denoting $g_2^{-1}g_1 = h$ we can thus write $g_1 = g_2h$ with $h \in H_{x_0}$, which proves that g_1, g_2 belong to the same equivalence class. Different equivalence classes, therefore, map to different points on \mathcal{M} .

The construction does not really depend on the base point x_0 , because the stability groups H_x for different $x \in \mathcal{M}$ are all isomorphic. \square

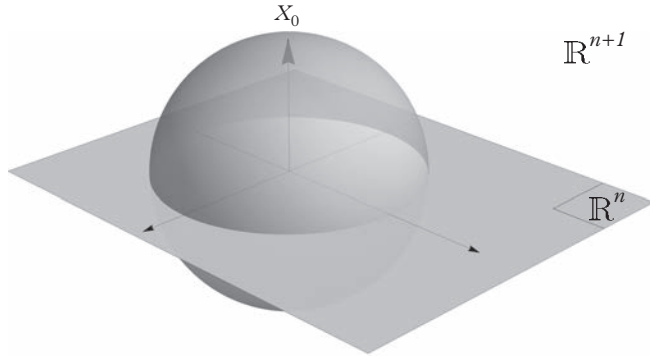


Figure 5.2 n -dimensional sphere is $SO(n+1)/SO(n)$.

An algebraic characterization of homogeneous spaces as cosets of a group by its subgroup is very convenient for field-theory applications. Since the construction may look rather abstract in the beginning, we illustrate it on a number of examples.

Ex 1. *Group manifold itself is both left and right homogeneous space.*

Proof. The group action is defined by left or right multiplication:

$$\begin{aligned} T_g^L(x) &= gx \\ T_g^R(x) &= xg. \end{aligned} \quad (5.4)$$

Ex 2. n -dimensional sphere S^n □

Proof. The $SO(n+1)$ rotation group acts transitively on the unit sphere in \mathbb{R}^{n+1} . The stability group of the north pole is the $SO(n)$ rotation group of the lateral hyperplane (Figure 5.2). Consequently,

$$S^n = SO(n+1)/SO(n). \quad (5.5)$$

□

Ex 3. *Complex projective space* $\mathbb{CP}^n = \{\mathbf{z} \sim \lambda \mathbf{z} | \lambda \in \mathbb{C}^*, \mathbf{z} \in \mathbb{C}^{n+1}\}$

Proof. $SU(n+1)$ acts transitively on \mathbb{CP}^n . The invariance subgroup of the point $(1, \mathbf{0}) \sim (\lambda, \mathbf{0})$ is

$$\left(\begin{array}{c|c} e^{i\varphi} & \\ \hline & SU(n) \times e^{-\frac{i\varphi}{n}} \end{array} \right) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Consequently, $\mathbb{CP}^n = SU(n+1)/SU(n) \times U(1)$. □

Ex 4. Anti-de-Sitter space

Proof. The $(d+1)$ -dimensional anti-de-Sitter space has coordinates (x^μ, z) and the line element

$$ds^2 = \frac{dx_\mu dx^\mu + dz^2}{z^2}, \quad (5.6)$$

where $\mu = 0 \dots d-1$ and the indices are contracted with the $- + \dots +$ Minkowski metric (for AdS_{d+1}) or with a Euclidean metric (for $EAdS_{d+1}$). The AdS space can be realized as a hypersurface in $\mathbb{R}^{d,2}$ (or $\mathbb{R}^{d+1,1}$) with embedding coordinates,

$$\begin{aligned} X^\mu &= \frac{x^\mu}{z} \\ X^{-1} &= \frac{x^2 + z^2 + 1}{2z} \\ X^d &= \frac{x^2 + z^2 - 1}{2z}, \end{aligned} \quad (5.7)$$

that satisfy

$$\eta_{MN} X^M X^N + 1 = 0, \quad (5.8)$$

where $M, N = -1 \dots d$ and η_{MN} is the pseudo-Euclidean metric with signature $- + \dots +$ (the upper sign is for Minkowskian AdS).

Euclidean anti-de-Sitter (Lobachevsky) space $EAdS_{d+1}$ is the upper sheet of the two-sheeted hyperboloid imbedded into the $(d+2)$ -dimensional Minkowski space, Figure 5.3(a). The hyperboloid asymptotes to the future light-cone, and the boundary of Euclidean AdS, the sphere S^d at infinity shown in Figure 5.3(a) as a circle, can be thought of as a set of all light rays emitted from the origin.

Minkowski AdS, as defined by equation (5.8), is a one-sheeted hyperboloid, Figure 5.3(b). The relationship between the space with the Poincaré metric (5.6) and the surface (5.8) is more intricate in this case. The coordinates (x^μ, z) cover only part of the hyperboloid, the Poincaré patch with $X^{-1} > X^d$ (which corresponds to

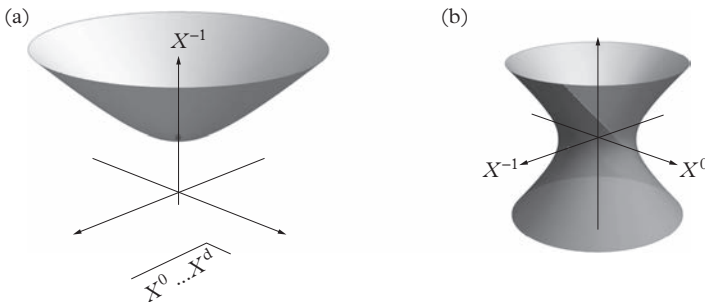


Figure 5.3 (a) $EAdS_{d+1}$ (b) AdS_{d+1} .

$z > 0$). The two patches, shaded differently in Figure 5.3(b), are glued together along the horizon at $z = \infty$. The lateral sections of the hyperboloid are actually time-like, and on the boundary that has the $S^{d-1} \times S^1$ geometry, the time direction is cyclic. The space which is usually called global AdS_{d+1} is the universal cover of the hyperboloid (5.8) obtained by unwinding the time direction.

The embedding (5.8) defines a transitive action of $SO(d, 2)$ on AdS_{d+1} and of $SO(d+1, 1)$ on $EAdS_{d+1}$. The stability group of the point shown as a dot in Figure 5.3 is $SO(d+1)$ for $EAdS_{d+1}$ and $SO(d, 1)$ for AdS_{d+1} . Hence,

$$EAdS_{d+1} = SO(d+1, 1)/SO(d+1), \quad AdS_{d+1} = SO(d, 2)/SO(d, 1). \quad (5.9)$$

□

5.3 Principal chiral field

The principal chiral field is a non-linear field that takes values in a group manifold:

$$g(x) : \Sigma \rightarrow G.$$

One can define a Lie-algebra-valued current

$$j_a = g^{-1} \partial_a g \in \mathfrak{g}, \quad (5.10)$$

and construct the Lagrangian as

$$\mathcal{L} = -\text{tr } j_a j^a = -\text{tr } j \wedge *j. \quad (5.11)$$

Here ‘tr’ denotes the invariant quadratic form on \mathfrak{g} . The overall normalization and sign are chosen so as to give a conventional kinetic term to X^A in the parameterization $g = e^{iX^A T_A}$, where T_A are Lie algebra generators normalized such that $\text{tr } T_A T_B = \delta_{AB}/2$. Another commonly used normalization includes an extra factor of $1/4$.

The Lagrangian is invariant under global $G_L \times G_R$ transformations

$$g(x) \rightarrow g_L g(x), \quad g(x) \rightarrow g(x) g_R. \quad (5.12)$$

The current is left-invariant and transforms in the adjoint under right multiplications: $j_a \rightarrow g_R^{-1} j_a g_R$. The invariance of the Lagrangian in the latter case follows from the invariance of the quadratic form, denoted by tr, under the group action.

The equations of motion of the principal chiral field follow from an infinitesimal variation

$$\delta g = g \xi, \quad \xi \in \mathfrak{g}.$$

The variation of the current is

$$\delta j_a = \partial_a \xi + [j_a, \xi]. \quad (5.13)$$

The commutator term does not contribute to the variation of the Lagrangian, because of the invariance of the quadratic form under the adjoint action of the Lie algebra. Partial integration in the remaining derivative term yields

$$\delta S = 2 \int d^2 x \operatorname{tr} \xi \partial_a j^a.$$

The equations of motion therefore are

$$\partial_a j^a = 0, \quad (5.14)$$

and are equivalent to the conservation of current j_a . The current j_a is nothing but the Noether current of the right group multiplication and the equations of motion can be alternatively derived from the Noether theorem.

When written in terms of $g(x)$, the conservation of current is a second-order non-linear differential equation. Boundary conditions, for instance,

$$g(x^0, \pm\infty) = 1, \quad (5.15)$$

and initial conditions at some fixed time slice lead to a well-defined Cauchy problem.

However, it is far more convenient to regard the current itself as a dynamical variable. To this end, we notice that (5.10) defines a pure-gauge potential, a flat connection, whose curvature is equal to zero:

$$\partial_a j_b - \partial_b j_a + [j_a, j_b] = 0. \quad (5.16)$$

Unlike (5.14), this equation is an identity, a consequence of definition (5.10). Nevertheless we can treat equations (5.14) and (5.16) on equal footing, as two equations of motion for the two components of the current. Once the current is known, $g(x)$ can be reconstructed from (5.10) after imposing the boundary conditions (5.15).

In the index-free notations,

$$d * j = 0 \quad (5.17)$$

$$dj + j \wedge j = 0. \quad (5.18)$$

These two equations can be combined into one by the following simple trick. Let us multiply the first equation by z and add to the second equation: $z(5.17) + (5.18)$. We do not lose any information by doing so, because the two expressions are null as soon as their arbitrary linear combination is null. Explicitly,

$$d(j + z*j) + j \wedge j = 0. \quad (5.19)$$

Here z is just a dummy variable, an arbitrary complex number for instance.

The last equation suggest redefining the current to $j + z*j$. And indeed, using the identities

$$*a \wedge b = -a \wedge *b, \quad *^2 = 1,$$

valid for one-forms in two dimensions, the last term in the equation can be expressed through the new current as well, up to a rescaling factor

$$(j + z*j) \wedge (j + z*j) = (1 - z^2)j \wedge j.$$

A simple rescaling factor

$$L = \frac{j + z*j}{1 - z^2}, \quad (5.20)$$

makes the current so defined flat:

$$dL + L \wedge L = 0. \quad (5.21)$$

The current L is called the Lax connection. Explicitly,

$$L_a = \frac{j_a + z\varepsilon_{ab}j^b}{1 - z^2}. \quad (5.22)$$

The equations of motion for the principal chiral field are equivalent to the condition that the Lax connection is flat [8]:

$$\partial_a L_b - \partial_b L_a + [L_a, L_b] = 0 \quad \forall z. \quad (5.23)$$

The zero curvature, or Lax representation of the equations of motion is a hallmark of integrability.

By virtue of the zero curvature condition, the holonomy (the Wilson line) of L is invariant under continuous deformations of the contour. The holonomy along a constant time slice defines the monodromy matrix

$$M(t; z) = P \exp \int_{-\infty}^{+\infty} dx L_1(t, x; z), \quad (5.24)$$

which for generic complex z is an element of the complexified group $G^{\mathbb{C}}$. Translation of the contour along the time direction is a continuous deformation and therefore leaves the monodromy matrix intact. Hence, the monodromy matrix is time-independent:

$$\partial_t M(t; z) = 0. \quad (5.25)$$

A quantity that does not change with time, once the equations of motion are imposed, is called the conserved charge. The monodromy matrix defines an infinite number of those because it contains an auxiliary parameter z (called the spectral parameter). The individual conserved charges are generated by expanding the monodromy matrix in z , for instance by the Laurent expansion at infinity:

$$M(z) = \exp \left(\sum_{n=1}^{\infty} \frac{Q_n}{z^n} \right). \quad (5.26)$$

Written explicitly in terms of currents, the monodromy matrix is

$$M(z) = P \exp \int_{-\infty}^{+\infty} dx \frac{j_1 - z j_0}{1 - z^2}, \quad (5.27)$$

and to the first order in $1/z$,

$$M(z) = 1 + \frac{1}{z} \int_{-\infty}^{+\infty} dx j_0 + \mathcal{O} \left(\frac{1}{z^2} \right). \quad (5.28)$$

The first charge in the hierarchy, Q_1 , therefore, is just the Noether charge of right group multiplications.

Exercise 1 Compute the second (Yangian) charge Q_2 by expanding the monodromy matrix to the second order in $1/z$.

Exercise 2 Derive the Noether current of the left group multiplications. Show that the corresponding charge appears at the first order in the Taylor expansion of the monodromy matrix at $z = 0$.

Higher conserved charges obtained by expanding the monodromy matrix at $z = \infty$ are non-local, they cannot be represented by integrals of local densities. Local conserved quantities are generated by a Taylor expansion at $z = \pm 1$. The conventional energy and momentum of the sigma-model, however, cannot be obtained from the monodromy matrix. Integrability of the principal chiral field is more consistent with an alternative canonical structure that does not follow from the Lagrangian (5.11), but whose Hamiltonian and momentum appear as first terms in the expansion of the monodromy matrix at $z = \pm 1$ [9].

The zero-curvature representation has far-reaching consequences. It can be used to explicitly construct solutions of the equations of motion, or to find the action-angle variables, in which the dynamics is trivial. The counterpart of the Lax representation in quantum theory is the Algebraic Bethe Ansatz [10], which defines the building blocks of

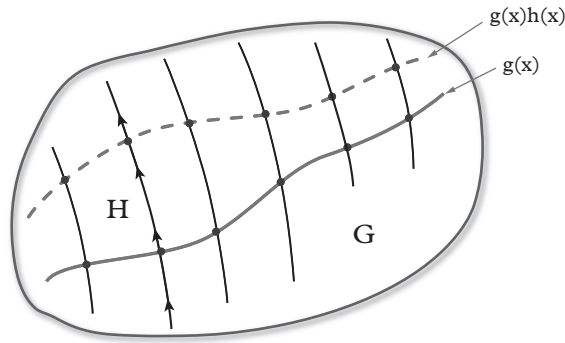


Figure 5.4 Locally, a coset can be viewed as a section of the group manifold transverse to the orbits of H . Two sections that differ by multiplication with elements in H are equivalent.

the quantum spectrum. The usual issues of UV regularization and definition of the true ground state are very non-trivial in integrable QFTs, and it is often easier to reconstruct the final answer from the constraints imposed by integrability on the dynamics, rather than to solve the model directly. Unfortunately, many steps of this bootstrap procedure are model dependent and may also depend on the boundary conditions. We exemplify the integrability bootstrap in Section 5.8 by the exact solution of the $O(N)$ sigma-model.

5.4 Symmetric cosets

Sigma-models on homogeneous spaces are also integrable, provided that the target space is *symmetric*, an algebraic property described later. The action of a sigma-model can be always written as (5.2) in an explicit coordinate system, but for homogeneous spaces an elegant invariant construction based on the coset representation [11] is in many ways more convenient. Points in a coset are orbits of H . Picking one representative per orbit introduces an explicit coordinate system. This defines a section of the group manifold that is transverse to the orbits of H at any point.² A field configuration then is a map from space-time to the group manifold which is restricted to this section (Figure 5.4). But any other choice differing by right multiplication with elements in H is equivalent, and it is desirable to have a formulation covariant with respect to the action of H . In other words, any map $g(x)$ should be allowed, but the field configurations $g(x)$ and $g(x)h(x)$ with $h(x) \in H$ should be physically indistinguishable. This can be achieved by realizing the right action of H as a gauge symmetry:

$$g(x) \rightarrow g(x)h(x), \quad h(x) \in H. \quad (5.29)$$

For this construction to work the action of the sigma-model has to be gauge invariant.

² Such a section may not be globally defined.

The Lagrangian (5.11) fails to satisfy this condition. Indeed, the current (5.10) transforms as

$$j_a \rightarrow h^{-1} j_a h + h^{-1} \partial_a h, \quad (5.30)$$

and the inhomogeneous term does not cancel in the gauge variation of the action.

The inhomogeneous term in the variation of the current can be projected out by decomposing the Lie algebra \mathfrak{g} of G into the denominator subalgebra \mathfrak{h} and its orthogonal complement \mathfrak{f} with respect to the invariant quadratic form:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}. \quad (5.31)$$

The current then expands in two components,

$$j_a = A_a + K_a, \quad A_a \in \mathfrak{h}, \quad K_a \in \mathfrak{f}. \quad (5.32)$$

Since $h \in H$, and $h^{-1} \partial_a h \in \mathfrak{h}$, it is A_a that transforms as a gauge field,

$$A_a \rightarrow h^{-1} A_a h + h^{-1} \partial_a h, \quad (5.33)$$

while the coset component transforms as a matter field in the adjoint,

$$K_a \rightarrow h^{-1} K_a h. \quad (5.34)$$

The simplest gauge-invariant Lagrangian built from these fields is

$$\mathcal{L} = -\text{tr} K_a K^a. \quad (5.35)$$

It is manifestly invariant under the local gauge transformations (5.29) and is also symmetric under global left multiplications $g(x) \rightarrow g_L g(x)$, $g_L \in G$, because the current is left invariant from the very beginning. The model possesses all the requisite symmetries of the coset space G/H .

In order to make the discussion less abstract let us illustrate the coset construction by a simple example of the Hopf fibration $S^3 \rightarrow S^2$:

Ex 5. $S^2 = SU(2)/U(1)$.

Proof. The generators of $\mathfrak{su}(2)$ (the Pauli matrices) are split into the coset generators and the denominator subalgebra as

$$\mathfrak{h} = \langle \sigma_1 \rangle, \quad \mathfrak{f} = \langle \sigma_2, \sigma_3 \rangle. \quad (5.36)$$

The generic element of $SU(2)$ can be parameterized by three Euler angles:

$$g = e^{\frac{i\varphi\sigma_1}{2}} e^{\frac{i\theta\sigma_3}{2}} e^{\frac{i\psi\sigma_1}{2}}$$

The last factor belongs to the coset denominator and can be removed by a gauge transformation. We can just drop it altogether, which is equivalent to fixing a gauge. For the current we then have

$$g^{-1}dg = e^{-\frac{i\theta\sigma_3}{2}} \left(d + \frac{i}{2}d\varphi\sigma_1 \right) e^{\frac{i\theta\sigma_3}{2}} = \frac{i}{2} \left(\underbrace{d\theta\sigma_3 + d\varphi\sin\theta\sigma_2}_K + \underbrace{d\varphi\cos\theta\sigma_1}_A \right).$$

The resulting line element is

$$ds^2 = -2\operatorname{tr} K^2 = \frac{1}{2}\operatorname{tr} (d\theta\sigma_3 + d\varphi\sin\theta\sigma_2)^2 = d\theta^2 + \sin^2\theta d\varphi^2,$$

which is the familiar metric of the sphere. \square

Returning to the general case, we can re-write the variation of the current (5.13) as

$$\delta j_a = D_a \xi + [K_a, \xi], \quad (5.37)$$

where D_a is the standard covariant derivative,

$$D_a = \partial_a + [A_a, \cdot]. \quad (5.38)$$

The ensuing variation of the Lagrangian is

$$\delta \mathcal{L} = -2\operatorname{tr} \delta K_a K^a = -2\operatorname{tr} \delta j_a K^a = -2\operatorname{tr} D_a \xi K^a,$$

where the first equality follows from orthogonality of $\delta A_a \in \mathfrak{h}$ and $K^a \in \mathfrak{k}$, and the commutator term in (5.37) was dropped because of the invariance of the quadratic form. The resulting equations of motion are

$$D_a K^a = 0. \quad (5.39)$$

Together with the flatness condition (5.16), they form a complete set of equations for the current components. The flatness condition, when expressed in terms of K_a and A_a , takes the form

$$F_{ab} + D_a K_b - D_b K_a + [K_a, K_b] = 0, \quad (5.40)$$

where $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$ is the field strength of the gauge connection.

Before proceeding further let us pause for a few general remarks on the coset decomposition (5.31). It is orthogonal with respect to the invariant scalar product on \mathfrak{g} , but what about the commutation relations? How are they affected by the coset decomposition? Since \mathfrak{h} is a subalgebra, the commutator of two elements of \mathfrak{h} is again an element of \mathfrak{h} ,

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}. \quad (5.41)$$

In terms of the structure constraints this means that $f_{\mathfrak{h}\mathfrak{h}\mathfrak{f}} = 0$. Antisymmetry of the structure constants (a very general property that follows from the existence of a non-degenerate invariant quadratic form) implies that $f_{\mathfrak{h}\mathfrak{f}\mathfrak{h}} = 0$. Hence,

$$[\mathfrak{h}, \mathfrak{f}] \subset \mathfrak{f}. \quad (5.42)$$

What can be said about $[\mathfrak{f}, \mathfrak{f}]$? Since $f_{\mathfrak{f}\mathfrak{f}\mathfrak{h}}$ are related to $f_{\mathfrak{h}\mathfrak{f}\mathfrak{f}}$, they are different from zero. The structure constants $f_{\mathfrak{f}\mathfrak{f}\mathfrak{f}}$ are not restricted by any general principles. An interesting special cases arises if they also vanish. Then,

$$[\mathfrak{f}, \mathfrak{f}] \subset \mathfrak{h}. \quad (5.43)$$

If the Lie algebra decomposition obeys this constraint, the coset space is called symmetric. The commutation relations (5.41), (5.42), and (5.43) follow a systematic pattern in this case. One can formalize it by introducing a \mathbb{Z}_2 transformation acting on the Lie algebra \mathfrak{g} as

$$\Omega(\mathfrak{h}) = \mathfrak{h}, \quad \Omega(\mathfrak{f}) = -\mathfrak{f}. \quad (5.44)$$

This operation constitutes a symmetry of \mathfrak{g} once (5.43) holds. Compatibility of the \mathbb{Z}_2 symmetry (5.44) with the commutation relations of \mathfrak{g} can be taken as a definition of symmetric homogeneous space.

One can invert the logic and think of a symmetric space as being defined by the symmetry group G and a \mathbb{Z}_2 automorphism of its Lie algebra (a linear map $\Omega : \mathfrak{g} \rightarrow \mathfrak{g}$ that preserves the commutation relations and squares to the identity: $\Omega^2 = \text{id}$). The denominator of the coset \mathfrak{h} is then identified with the invariant subalgebra of Ω . The \mathbb{Z}_2 automorphism of the Lie algebra translates into a geometric reflection symmetry of the homogeneous space, as illustrated by the following example.

Ex 6. $S^2 = SU(2)/U(1)$ is a symmetric space.

Proof. The coset decomposition (5.36) is \mathbb{Z}_2 invariant because $[\sigma_2, \sigma_3] = 2i\sigma_1 \in \mathfrak{h}$. Since σ^3 pairs with θ in the Euler parameterization, the \mathbb{Z}_2 symmetry of the algebra translates to the invariance of the sphere under $\theta \rightarrow -\theta$. \square

Exercise 3 Check that S^n , \mathbb{CP}^n , and $(E)AdS_{d+1}$ are symmetric spaces.

Exercise 4 The conformal algebra $\mathfrak{so}(d, 2)$ is generated by Lorentz transformations $L_{\mu\nu}$, translations P_μ , dilatation D , and special conformal transformations K_μ . Its commutation relations are

$$\begin{aligned} [L_{\mu\nu}, L_{\lambda\rho}] &= \eta_{\mu\lambda}L_{\nu\rho} + \eta_{\nu\rho}L_{\mu\lambda} - \eta_{\mu\rho}L_{\nu\lambda} - \eta_{\nu\lambda}L_{\mu\rho}, \\ [L_{\mu\nu}, P_\lambda] &= \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu, \quad [L_{\mu\nu}, K_\lambda] = \eta_{\mu\lambda}K_\nu - \eta_{\nu\lambda}K_\mu, \\ [D, P_\mu] &= P_\mu, \quad [D, K_\mu] = -K_\mu, \quad [P_\mu, K_\nu] = 2L_{\mu\nu} - 2\eta_{\mu\nu}D. \end{aligned} \quad (5.45)$$

The Killing metric on $\mathfrak{so}(d, 2)$ is

$$\mathrm{tr} L_{\mu\nu} L_{\lambda\rho} = \eta_{\mu[\lambda} \eta_{\nu\rho]}, \quad \mathrm{tr} D^2 = -1, \quad \mathrm{tr} P_\mu K_\nu = -2\eta_{\mu\nu}. \quad (5.46)$$

Consider a \mathbb{Z}_2 transformation,

$$\Omega(L_{\mu\nu}) = L_{\mu\nu}, \quad \Omega(D) = -D, \quad \Omega(K_\mu) = P_\mu, \quad \Omega(P_\mu) = K_\mu. \quad (5.47)$$

Check that Ω is consistent with the commutation relations of $\mathfrak{so}(4, 2)$, and therefore constitutes an automorphism of the algebra. Show that the invariant subalgebra of Ω is $\mathfrak{so}(d, 1)$, and the symmetric coset defined by Ω is AdS_{d+1} .

Take the coset representative to be

$$g(x, z) = e^{iP_\mu x^\mu} z^{-D}, \quad (5.48)$$

and derive the metric in the (x^μ, z) coordinates from the coset construction.

A remarkable fact about symmetric-space sigma-models is their complete integrability [12]. While coset sigma-models, which are not symmetric, can still be integrable in exceptional cases [13], integrability of symmetric cosets can be demonstrated in a uniform, essentially algebraic way.

Here is the key point, that singles out symmetric spaces. The first two terms in (5.40) belong to \mathfrak{h} and \mathfrak{f} , respectively, because of (5.41) and (5.42), while the third term in general has both \mathfrak{f} and \mathfrak{h} components. But for symmetric cosets all four terms in the flatness condition (5.40) have definite \mathbb{Z}_2 grading and the equation can therefore be neatly projected onto \mathfrak{h} and \mathfrak{f} subspaces of the coset decomposition:

$$\begin{aligned} F_{ab} + [K_a, K_b] &= 0 \\ D_a K_b - D_b K_a &= 0. \end{aligned} \quad (5.49)$$

These equations, together with the equation of motion (5.39) follow from the flatness condition for the Lax connection:

$$L_a = A_a + \frac{z^2 + 1}{z^2 - 1} K_a - \frac{2z}{z^2 - 1} \varepsilon_{ab} K^b. \quad (5.50)$$

Exercise 5 Show that the zero-curvature condition (5.23) for (5.50) is equivalent to the equations of motion (5.39), (5.49).

5.5 B-field and topology

So far we have discussed the metric term in the sigma-model Lagrangian (5.2), and showed how to construct it for manifolds that admit an action of a symmetry group G . Is it also possible to construct an invariant B-field? To better formulate the question, it is convenient to switch to the form notations:

$$\frac{1}{2} \int_{\Sigma} d^2 x \varepsilon^{ab} B_{MN} \partial_a X^M \partial_b X^N = \int_{\Sigma} B = \int_D H, \quad (5.51)$$

where

$$B = B_{MN} dX^M \wedge dX^N, \quad H = dB = H_{MNL} dX^M \wedge dX^N \wedge dX^L, \quad H_{MNL} = \partial_{[M} B_{NL]}, \quad (5.52)$$

and D is a fiducial three-dimensional space whose boundary is Σ : $\partial D = \Sigma$.

Re-writing a 2d integral in a 3d form may look superficial, but it helps to uncover interesting possibilities that would be overlooked otherwise. The three-form H , as defined in (5.52), is exact. Interestingly, this requirement can be relaxed and replaced by a weaker condition that the form is closed ($dH = 0$). The three-dimensional form of the B-field coupling cannot be then transformed back to a two-dimensional integral of an invariant two-form. Yet it defines a consistent local coupling in two-dimensional field theory [14, 15].

If $\Sigma = S^2$, then D is the three-dimensional ball. Field configurations on S^2 can be unfolded to \mathbb{R}^2 by stereographic projection and, conversely, field configurations on \mathbb{R}^2 sufficiently well behaved at infinity can be compactified on S^2 , so the ensuing construction applies to conventional field theories on the flat 2d space under assumption of regularity conditions at infinity.

The B-field coupling in the sigma-model action can thus be constructed from an invariant two-form on the target space or from an invariant closed three-form. Both cases lead to topological terms in the sigma-model action. We consider them in turn. A more comprehensive introduction to topological terms in QFT can be found in [16].

5.5.1 Theta-term

If the coset denominator contains an Abelian factor, $H \ni U(1)$, the corresponding field strength F_{ab} is a gauge-invariant two-form. The coset sigma-model on G/H then admits a theta-term:

$$S_{\theta} = \frac{\theta}{4\pi} \int d^2 x \varepsilon^{ab} F_{ab} = \frac{\theta}{2\pi} \int F. \quad (5.53)$$

The theta-term is topological, it only depends on the values of the fields at infinity,

$$\frac{1}{2\pi} \int F = \frac{1}{2\pi} \oint_{\infty} A.$$

Since the field must approach pure gauge at $x \rightarrow \infty$: $A = d\varphi$,

$$\frac{1}{2\pi} \oint_{\infty} A = \frac{1}{2\pi} \oint_{\infty} d\varphi = \frac{\Delta\varphi}{2\pi} = n.$$

The theta-term is non-zero only on topologically non-trivial field configurations (instantons) and measures their topological charge. As such it does not contribute to the equations of motion. In the path integral the theta-term produces a phase factor $e^{iS\theta} = e^{i\theta n}$. Since n is an integer, the theta-angle is a periodic variable: the path integral does not change under $\theta \rightarrow \theta + 2\pi$.

Examples of sigma-models that admit a theta-term, and have instantons, are the \mathbb{CP}^n models, for any n . These include the $O(3)$ model (because $S^2 = \mathbb{CP}^1$), while $O(n)$ models with $n > 3$ do not admit a theta-term and do not have instantons.

The theta-term measures the winding of pure gauge field at infinity and is associated with the homotopy group³ $\pi_1(H) = \mathbb{Z}$. If $\pi_2(G/H) = \mathbb{Z}$, the degree of mapping $\Sigma \rightarrow G/H$ also defines a topological charge, but one can show on very general grounds that $\pi_2(G/H) = \pi_1(H)$ and the degree of mapping actually coincides with the winding of the pure gauge at infinity, so we do not get anything new. It is instructive to explicitly work out the equivalence of the two definitions of the topological charge for the $O(3)$ model.

Exercise 6 The $O(3)$ sigma model can be realized as the $SU(2)/U(1) = S^2$ coset, or as a field theory of three-dimensional unit vector \mathbf{n} , $\mathbf{n}^2 = 1$. The degree of mapping defined by \mathbf{n} (the number of times the target-space S^2 wraps Σ) is given by the integral

$$n = \frac{1}{8\pi} \int d^2x \epsilon^{ab} \mathbf{n} \cdot \partial_a \mathbf{n} \times \partial_b \mathbf{n}.$$

Prove that this number coincides with the instanton charge in the coset formulation, perhaps up to a sign. (Hint: use $F = -K \wedge K$, which is an identity, not an equation of motion).

Instanton solutions of arbitrary topological charge in the $O(3)$ model have been constructed in [17]. An interplay between instantons and integrability in this and other two-dimensional models remains an interesting open problem.

5.5.2 Wess–Zumino term

An invariant three-form on a group manifold can be obtained by taking the wedge product of three currents. This defines the Wess–Zumino (WZ) term that can be added to the action of the principal chiral field sigma-model [18]:

$$S_{\text{WZ}} = \frac{1}{3} \int_D \text{tr} j \wedge j \wedge j. \quad (5.54)$$

³ The definition and properties of homotopy groups can be found, for instance, in [6].

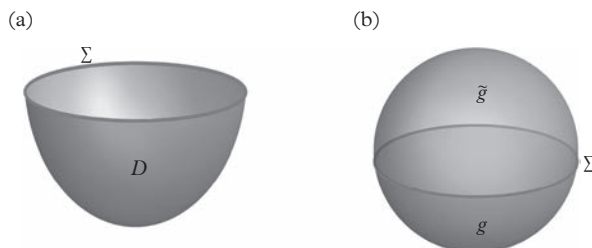


Figure 5.5 (a) The space-time manifold Σ is the boundary of the integration domain D in the WZ term. (b) Two different continuations of the field variable in the interior of D can be combined into a map $\hat{g}: S^3 \rightarrow G$.

The boundary of the integration domain is the physical 2d space-time: $\partial D = \Sigma$, Figure 5.5(a). The definition tacitly assumes that the field variable $g(x)$ is continued into the interior of D in some way. This is always possible to do, since $\pi_2(G) = 0$ for any Lie group G and consequently any map $g: S^2 \rightarrow G$ is contractible. The process of contraction defines a map $D \rightarrow G$.

If considered as part of the action functional in a 2d field theory, the WZ term should be independent of the way the field variable is continued inside D . Let us see if this is indeed the case [15, 18], by considering the WZ term evaluated on two different maps, $g: D \rightarrow G$ and $\tilde{g}: D \rightarrow G$, which coincide on the boundary, $g|_{\Sigma} = \tilde{g}|_{\Sigma}$ and thus correspond to one and the same field configuration in space-time. Consider first infinitesimally close maps: $\tilde{g} = g(1 + \xi)$, where $\xi \in \mathfrak{g}$ is small. Substituting the variation of the current from (5.13) in the WZ functional, we find

$$\delta S_{\text{WZ}} = \int_D d\xi \wedge j \wedge j = - \int_D \xi d(j \wedge j). \quad (5.55)$$

The boundary term can be omitted upon integration by parts because ξ vanishes on Σ . The flatness condition (5.18) then gives

$$d(j \wedge j) = dj \wedge j - j \wedge dj = -j \wedge j \wedge j + j \wedge j \wedge j = 0,$$

so the variation of the WZ action indeed vanishes.

Continuous deformations of the map g do not change the WZ term, but can any two maps from D to the group manifold be continuously connected? Consider two such maps g and \tilde{g} and combine them into a map $\hat{g}: S^3 \rightarrow G$ defined as shown in Figure 5.5(b):

$$\hat{g}(x) = \begin{cases} \tilde{g}(x) & x \in \text{upper hemisphere} \\ g(x) & x \in \text{lower hemisphere.} \end{cases}$$

Then,

$$S_{\text{WZ}}[\hat{g}] - S_{\text{WZ}}[g] = \frac{1}{3} \int_{S^3} \text{tr} \hat{j} \wedge \hat{j} \wedge \hat{j} = 16\pi^2 \text{in}. \quad (5.56)$$

The number n does not change under continuous deformations of the map \hat{g} and therefore is a topological invariant. Under appropriate normalization of the quadratic form on the Lie algebra this number is an integer that characterizes an element of $\pi_3(G) = \mathbb{Z}$.

Exercise 7 For $G = SU(2) = S^3$, \hat{g} defines a map $S^3 \rightarrow S^3$. Show that n given by the integral (5.56) is the degree of this map, the number of times $\hat{g}(x)$ wraps the target S^3 while x goes around the sphere.

The WZ action thus is a multi-valued functional. Depending on how the field variable g is continued inside the three-dimensional domain of integration, the result may change by an integer:

$$S_{WZ} \sim S_{WZ} + 16\pi^2 in. \quad (5.57)$$

This is not a problem in classical field theory, because the equations of motion only depend on the variation of the action, and the variation of the WZ action is single-valued. But in quantum theory the action enters the path integral through the phase factor,

$$e^{\frac{k}{8\pi} S_{WZ}} \in \text{path integral},$$

where k is the coupling constant (called level). This factor will be single valued provided that the coupling is quantized:

$$k \in \mathbb{Z}. \quad (5.58)$$

The principal chiral field with the WZ term, the Wess–Zumino–Witten (WZW) model, is defined by the action [18]

$$S_{WZW} = -\frac{1}{\kappa^2} \int_{\Sigma} \text{tr} j \wedge *j + \frac{ik}{24\pi^2} \int_D \text{tr} j \wedge j \wedge j. \quad (5.59)$$

Exercise 8 Derive the equations of motion for the WZW model and show that they admit a Lax representation [19]. What happens at $\kappa^2 = 2\pi/k$?

5.6 Quantum sigma-models

A quantum coset sigma-model is defined by the path integral (in Euclidean signature)

$$Z = \int \mathcal{D}g e^{\frac{1}{\kappa^2} \int d^2x \text{tr} K_a K^a}. \quad (5.60)$$

This is a highly non-linear quantum field theory with a single dimensionless coupling. To get a first glance at the properties of this QFT we consider its response to a macroscopic external force, assuming momentarily that residual quantum effects can be treated perturbatively.

To this end we separate the integration variable in the path integral into the classical background field \bar{g} , which models the external force and which we assume to be a solution of the classical equations of motion, and quantum field X whose fluctuations are assumed to be small:

$$g = \bar{g}e^X. \quad (5.61)$$

The field X belongs to the Lie algebra of G . An appropriate gauge transformation (5.29) can eliminate the \mathfrak{h} component of X :

$$X \in \mathfrak{f}. \quad (5.62)$$

The restriction to the coset subspace is a convenient gauge-fixing condition.

To develop perturbation theory we need to expand the current (5.10) in powers of X . We start by deriving a general formula for such an expansion. Substituting (5.61) into the definition of the current, we get

$$j_a = e^{-X} (\partial_a + \bar{j}_a) e^X = \bar{j}_a + \int_0^1 ds e^{-sX} \mathcal{D}_a X e^{sX}, \quad (5.63)$$

where $\bar{j}_a = \bar{g}^{-1} \partial_a \bar{g}$ is the background current and $\mathcal{D}_a = \partial_a + [\bar{j}_a, \cdot]$. The last term expands in multiple commutators of X . To write the series in a compact form, we commute both sides of (5.63) with X , which eliminates the integral over s ,

$$[X, j_a - \bar{j}_a] = - \int_0^1 ds \frac{d}{ds} e^{-sX} \mathcal{D}_a X e^{sX} = \mathcal{D}_a X - e^{-X} \mathcal{D}_a X e^X. \quad (5.64)$$

Introducing the notation

$$\text{ad } X \cdot Y = [X, Y], \quad (5.65)$$

and taking into account that

$$e^{-X} Y e^X = e^{-\text{ad } X} Y,$$

we find

$$j_a = \bar{j}_a + \frac{1 - e^{-\text{ad } X}}{\text{ad } X} \mathcal{D}_a X. \quad (5.66)$$

Using this formula the current can be expanded to any desired order in X .

After rescaling $X \rightarrow \kappa X$, the first two orders become

$$\begin{aligned} j_a &\simeq \bar{j}_a + \kappa \mathcal{D}_a X - \frac{\kappa^2}{2} [X, \mathcal{D}_a X] \\ &= \bar{A}_a + \bar{K}_a + \kappa (D_a X + [\bar{K}_a, X]) - \frac{\kappa^2}{2} ([X, D_a X] + [X, [\bar{K}_a, X]]), \end{aligned} \quad (5.67)$$

where D_a is the covariant derivative with respect to the background gauge field. We can use \mathbb{Z}_2 grading to assign commutators to either \mathfrak{h} or \mathfrak{f} , and thus already here assume that the coset is a symmetric space.

The action of the sigma-model, to the leading order in κ , is

$$S = S_{\text{cl}} - \int d^2x \text{tr} (D_a X D^a X - [\bar{K}_a, X] [\bar{K}^a, X]) + \mathcal{O}(\kappa). \quad (5.68)$$

Gaussian integration over X then yields

$$Z \simeq e^{-S_{\text{cl}}} \det^{-\frac{1}{2}} (-D^2 + \text{ad } \bar{K}^2). \quad (5.69)$$

The result of these manipulations is a one-loop effective action for the background field,

$$S_{\text{eff}} = S_{\text{cl}} + \frac{1}{2} \ln \det (-D^2 + \text{ad } \bar{K}^2). \quad (5.70)$$

The one-loop determinant diverges and requires regularization. Since in two dimensions the only scalar diagram that diverges is the bubble, the divergent part of the effective action is

$$\text{Bubble Diagram} = \frac{1}{2} \int d^2x \text{tr}_{\mathfrak{f}} \text{ad } \bar{K}^2 \times \int^{\Lambda} \frac{d^2p}{(2\pi)^2} \frac{1}{p^2} = \frac{1}{4\pi} \ln \frac{\Lambda}{\mu} \int d^2x \text{tr}_{\mathfrak{f}} \text{ad } \bar{K}^2. \quad (5.71)$$

As a matrix in the adjoint representation, $\text{ad } \bar{K}_a$ has \mathbb{Z}_2 grading -1 and thus the following matrix structure:

$$\text{ad } \bar{K}_a : \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \begin{pmatrix} \mathfrak{h} \\ \mathfrak{f} \end{pmatrix}. \quad (5.72)$$

So,

$$\text{tr}_{\mathfrak{f}} \text{ad } \bar{K}^2 = \langle \mathfrak{f} | \text{ad } \bar{K}_a | \mathfrak{h} \rangle \langle \mathfrak{h} | \text{ad } \bar{K}^a | \mathfrak{f} \rangle = \text{tr}_{\mathfrak{h}} \text{ad } \bar{K}^2 = \frac{1}{2} \text{tr}_{\mathfrak{g}} \text{ad } \bar{K}^2.$$

The divergence renormalizes the sigma-model coupling constant,

$$S_{\text{eff}} = - \int d^2x \left(\frac{1}{\kappa^2} \text{tr} - \frac{1}{8\pi} \ln \frac{\Lambda}{\mu} \text{tr}_{\text{adj}} \right) \bar{K}_a \bar{K}^a + \text{finite}. \quad (5.73)$$

Taking into account that

$$\text{tr}_{\text{adj}} XY = 2C_G \text{tr} XY, \quad (5.74)$$

where C_G is the dual Coxeter number⁴ of the group G , we find that the coupling renormalized at scale μ is related to the bare coupling by

$$\frac{1}{\kappa^2(\mu)} = \frac{1}{\kappa^2(\Lambda)} - \frac{C_G}{4\pi} \ln \frac{\Lambda}{\mu}. \quad (5.75)$$

The running of sigma-model coupling with the energy scale is governed by the beta-function, which can be read off from the above calculation:

$$\beta(\kappa) = \frac{d\kappa(\mu)}{d\ln\mu} = -\frac{C_G}{8\pi} \kappa^3. \quad (5.76)$$

The one-loop beta-function is negative, which means that two-dimensional coset sigma-models are asymptotically free quantum field theories [20].

Exercise 9 Calculate the beta-function of the WZW model (5.59) [15].

The RG flow in two-dimensional sigma-models is often interpreted geometrically as shrinking of the target-space in the IR [21]. The coupling constant rescales the metric in the sigma-model action (5.2). Its inverse thus plays the role of an overall radius of the target space. The theory flows to stronger coupling in the IR, with the radius of the target space becoming smaller and smaller along the flow, Figure 5.6.

The RG flow will eventually stop and generate a dynamical mass scale through dimensional transmutation:⁵

$$M = \Lambda e^{-\frac{4\pi}{C_G \kappa^2}}. \quad (5.77)$$

This means that coset sigma-models are massive field theories, with particles transforming under some representation of the symmetry group G . The dynamics of these particles is non-perturbative at low energies, but is highly constrained by integrability, of course under the assumption that integrability is not broken by quantization.

⁴ $C_{SU(N)} = N$, $C_{SO(N)} = N - 2$.

⁵ In the presence of a theta-term or a WZ term this is not necessarily true. The flow may then end in a non-trivial CFT in the IR.

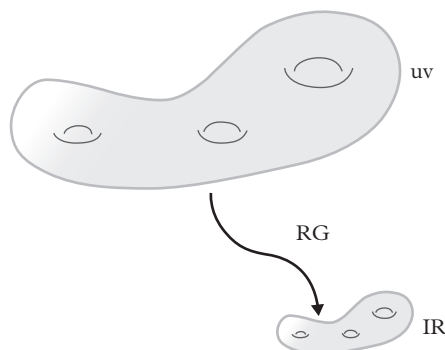


Figure 5.6 The RG flow in a sigma-model.

Table 5.1 Examples of classically integrable sigma-models, which can be either integrable or not at the quantum level.

QFT	Coset	Integrability
Principal chiral field	G	✓
$O(N)$ model	$SO(N)/SO(N-1)$	✓
\mathbb{CP}^N model	$SU(N+1)/SU(N) \times U(1)$	×
Superstring on $AdS_5 \times S^5$	$PSU(2,2 4)/SO(4,1) \times SO(5)$	✓
$O(3)$ model with $\theta \neq 0, \pi$	$SO(3)/SO(2)$	×

Conservation laws are often affected by quantum anomalies, and integrable QFTs are no exception. Whether or not higher conserved charges are anomalous is a dynamical question, with a model-dependent answer. There are cases where integrability is preserved at the quantum level, and cases where integrability is broken by anomalies. Some representative examples are listed in Table 5.1.

The dynamical information of a massive QFT is contained in its S-matrix. The S-matrix of an integrable QFT is so constrained by symmetries and analyticity that it can be reconstructed from the kinematical requirements alone, with the help of a minimal amount of explicit perturbative data. We illustrate how such integrable bootstrap works, taking the $O(N)$ model as an example. The departure point for constructing the exact scattering theory for the $O(N)$ model is its solution at large- N , which can be obtained by elementary manipulations of the path integral described in Section 5.7.

5.7 $O(N)$ model: large N

The $O(N)$ model is a quantum field theory of an N -component vector field \mathbf{n} subject to the constraint $\mathbf{n}^2 = 1$. The constraint can be imposed by a Lagrange multiplier in the path integral:

$$Z = \int D\mathbf{n} D\sigma e^{\frac{i}{\kappa^2} \int d^2x [(\partial\mathbf{n})^2 - \sigma(\mathbf{n}^2 - 1)]}. \quad (5.78)$$

Integration over σ yields the requisite delta-function, so $D\mathbf{n}$ is the linear, unconstrained measure in the field space. Integrating over \mathbf{n} first we get an effective action for σ :

$$S_{\text{eff}} = \frac{1}{\kappa^2} \int d^2x \sigma + \frac{iN}{2} \ln \det(-\partial^2 - \sigma). \quad (5.79)$$

The two terms are of the same order of magnitude in the 't Hooft limit:

$$N \rightarrow \infty, \quad \lambda = \frac{\kappa^2 N}{2} - \text{fixed}. \quad (5.80)$$

In the 't Hooft limit the effective action is of an overall order N . The semiclassical approximation becomes exact at $N \rightarrow \infty$, and it suffices to solve the classical equations of motion that follow from (5.79). Denoting

$$\langle \sigma \rangle = m^2, \quad (5.81)$$

and taking the variation of (5.79), we get

$$\frac{1}{\lambda} = i \left\langle x \left| \frac{1}{-\partial^2 - m^2} \right| x \right\rangle \iff \frac{1}{\lambda} = \int \frac{d^2p}{(2\pi)^2} \frac{i}{p^2 - m^2}. \quad (5.82)$$

This is known as the gap equation⁶, which after the Wick rotation, $p_0 \rightarrow ip_0$, becomes

$$\frac{1}{\lambda} = \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + m^2} = \frac{1}{2\pi} \ln \frac{\Lambda}{m}. \quad (5.83)$$

The UV cut-off Λ is necessary for regulating the divergent momentum integral. The gap equation determines the mass scale of the theory:

$$m = \Lambda e^{-\frac{2\pi}{\lambda}}. \quad (5.84)$$

This is consistent with (5.77) at large N , when the difference between N and $C_{SO(N)} = N - 2$ is immaterial.

If we return to (5.78), the expectation value of the Lagrange multiplier $\langle \sigma \rangle = m^2$ gives mass to the field \mathbf{n} , which thus describes N massive particles in the vector representation of $SO(N)$. The $N - 1$ Goldstone modes of the naive perturbation theory rearrange

⁶ The ground state of the system is of course homogeneous, but the gap equation also has space-time dependent soliton solutions that describe excited states; see [22] for an example.

themselves into N massive states at low energies. This rearrangement is obviously non-perturbative.

Exercise 10 Calculate the $2 \rightarrow 2$ scattering amplitude in the $O(N)$ model to the first non-vanishing order in $1/N$ [3].

5.8 $O(N)$ model: exact solution

Integrability strongly constrains the scattering processes. The constraints follow from no more than simple kinematics. Consider, to begin with, the scattering of two identical particles $A(p_1^{\text{in}}) + A(p_2^{\text{in}}) \rightarrow A(p_1^{\text{out}}) + A(p_2^{\text{out}})$ in two dimensions, where $p_i^{\text{in}}, p_i^{\text{out}}$ are incoming and outgoing spacial momenta. If the initial momenta are given, energy and momentum conservation constitute two equations for two final-state momenta:

$$\begin{aligned} p_1^{\text{out}} + p_2^{\text{out}} &= p_1^{\text{in}} + p_2^{\text{in}} \\ E(p_1^{\text{out}}) + E(p_2^{\text{out}}) &= E(p_1^{\text{in}}) + E(p_2^{\text{in}}). \end{aligned}$$

Two equations for two unknowns have a discrete set of solutions—two-body scattering has no phase space in two dimensions.

It is easy to convince oneself that the conservation laws admit but two obvious solutions: $p_1^{\text{out}} = p_1^{\text{in}}, p_2^{\text{out}} = p_2^{\text{in}}$ (transmission) and $p_1^{\text{out}} = p_2^{\text{in}}, p_2^{\text{out}} = p_1^{\text{in}}$ (reflection). This is true in any translationally invariant theory in two dimensions, but imagine now that there is an infinite number of additional conservation laws. In any interaction process, even if it involves more than two particles, exchange of momenta would still be possible, but any other outcome of the scattering process would be kinematically forbidden, because conservation laws constitute a hugely overconstrained system of equations. In integrable theories, which feature an infinite number of conservation laws, particle production is forbidden, scattering is diagonal in particle number, and the final-state momenta of any $n \rightarrow n$ process constitute a permutation of the initial-state momenta: $p_i^{\text{out}} = p_{\sigma(i)}^{\text{in}}$. Any permutation can be written as a product of transpositions, likewise $n \rightarrow n$ scattering in an integrable theory factorizes into a sequence of elementary $2 \rightarrow 2$ processes. The dynamics of an integrable QFT is thus encoded in the two-body scattering S-matrix. This is a huge simplification that puts integrable models in between trivial (free) theories and generic QFTs, where the intricate phase space of intermediate states makes the analytic structure of the S-matrix enormously complicated.

Taking into account that the $O(N)$ model is integrable, and drawing intuition from the large- N approximation, we conclude that the model is characterized by:

- N massive particles transforming in the vector representation of $SO(N)$
- no particle production
- factorized scattering.

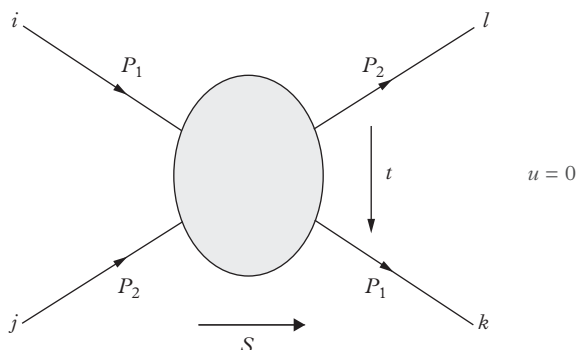


Figure 5.7 The $2 \rightarrow 2$ scattering amplitude.

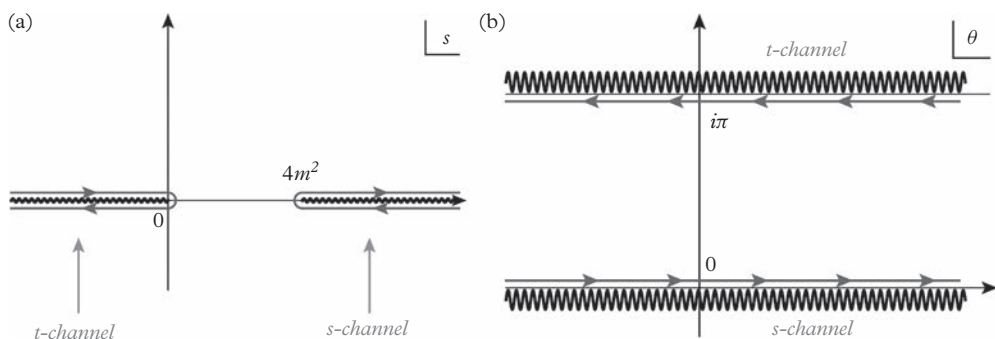


Figure 5.8 Analytic structure of the S -matrix: (a) in the s -plane; (b) in the rapidity plane.

These basic data are sufficient to completely reconstruct the S -matrix of the theory. Our exposition of this bootstrap solution of the model closely follows the original derivation [3].

Consider $2 \rightarrow 2$ scattering, illustrated in Figure 5.7. The Mandelstam invariant $u = 0$ in two dimensions, while s and t are related by

$$s + t = 4m^2. \quad (5.85)$$

Hence the $2 \rightarrow 2$ S -matrix is a function of only one kinematic variable: $S_{ij}^{kl}(s)$. Moreover, the S -matrix is analytic in s almost everywhere on the complex plane. All the singularities of the S -matrix are associated with intermediate virtual states going on-shell. And here integrability brings in enormous simplifications. The absence of particle production implies that the only allowed singularities are two-particle cuts in the s - and t -channels, Figure 5.8(a). The paucity of analytic functions on the complex plane with two cuts is the ultimate reason for the success of integrability bootstrap.

Two-dimensional relativistic kinematics drastically simplifies in the rapidity variables:

$$\begin{aligned} p_{1,2} &= m \sinh \theta_{1,2} \\ E_{1,2} &= m \cosh \theta_{1,2}. \end{aligned} \quad (5.86)$$

The centre-of-mass energy squared is then

$$s = 4m^2 \cosh^2 \frac{\theta_1 - \theta_2}{2}. \quad (5.87)$$

The S-matrix is thus a function of the rapidity difference between the two colliding particles, $S = S(\theta_1 - \theta_2)$. The entire s -plane maps onto the strip of the theta-plane between the lines $\text{Im} \theta = 0$ and $\text{Im} \theta = \pi$ (called the physical strip), as shown in Figure 5.8.

In spite of what the name might suggest, most of the physical strip corresponds to unphysical kinematics. The S-matrix is a probability amplitude of a physical scattering process only for real positive $s > 4m^2$, cf. (5.87). The rest is analytic continuation. More precisely, the scattering amplitude is defined for s real positive $+i\epsilon$, according to the usual Feynman prescription. The upper side of the s -channel cut maps to the positive real semi-axis on the θ -plane (Figure 5.8). Analytic continuation to the entire s -plane leads to a set of S-matrix axioms [3, 23]:

- Unitarity (+ real analyticity):

$$S(-\theta)S(\theta) = 1 \quad (5.88)$$

- Crossing symmetry:⁷

$$S_{ij}^{kl}(\theta) = S_{it}^{kj}(i\pi - \theta) \quad (5.89)$$

- Yang–Baxter equation (YBE), Figure 5.9:

$$S_{12}(\theta_1 - \theta_2)S_{13}(\theta_1 - \theta_3)S_{23}(\theta_2 - \theta_3) = S_{23}(\theta_2 - \theta_3)S_{13}(\theta_1 - \theta_3)S_{12}(\theta_1 - \theta_2) \quad (5.90)$$

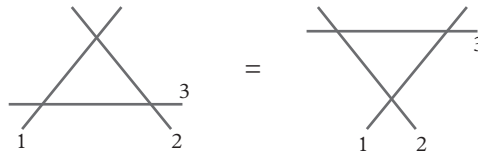


Figure 5.9 The Yang–Baxter equation.

⁷ Physical particles in the $O(N)$ model are their own anti-particles. More generally, crossing involves charge conjugation.

The YBE is a consistency condition on the factorization of the $3 \rightarrow 3$ scattering into elemental $2 \rightarrow 2$ processes. The outcome should not depend on the order in which the three particles collide pairwise. The labels on S_{ab} indicate that it acts on the $SO(N)$ indices of particles a and b .

Exercise 11 Write down the YBE in the explicit index notations.

Symmetries and the YBE usually produce a skeleton of the S-matrix, a fixed matrix structure without the detailed dependence on the rapidity. Indeed, if $S_{ij}^{kl}(\theta)$ solves the YBE, then $f(\theta)S_{ij}^{kl}(\lambda\theta)$ does so also. To bootstrap the remaining ambiguity one needs to invoke crossing symmetry, unitarity, and analyticity. Here we show how this program works for the $O(N)$ model.

The general form of the S-matrix consistent with the $O(N)$ symmetry is

$$S_{ij}^{kl}(\theta) = \sigma(\theta) \left(\delta_i^k \delta_j^l + \frac{1}{p(\theta)} \delta_i^l \delta_j^k + \frac{1}{k(\theta)} \delta_{ij} \delta^{kl} \right). \quad (5.91)$$

Firstly, we impose the YBE. The overall ‘dressing’ factor $\sigma(\theta)$ then drops out. It is in principle straightforward to determine what the constraints on the two remaining functions are, albeit the calculations are rather lengthy. The algebra can be facilitated by introducing permutation and trace operators, that act on the tensor product of two $SO(N)$ vectors as

$$Pa \otimes b = b \otimes a, \quad Ka \otimes b = (a \cdot b)1. \quad (5.92)$$

In these notations,

$$S = \sigma(\theta) \left(1 + \frac{1}{p(\theta)} P + \frac{1}{k(\theta)} K \right). \quad (5.93)$$

The permutation and trace operators satisfy the following braiding relations:

$$\begin{aligned} P_{ab}P_{bc} &= P_{ac}P_{ab}, & K_{ab}^2 &= NK_{ab}, & K_{ab}P_{ab} &= P_{ab}K_{ab} = K_{ab}, \\ K_{ab}P_{cb} &= P_{cb}K_{ac}, & K_{ab}K_{cb} &= P_{bc}P_{ab}K_{bc}, & P_{ab}P_{bc}K_{ab} &= P_{ac}K_{ab}. \end{aligned} \quad (5.94)$$

Algebraic manipulations based on (5.94)—the details of which we omit⁸—reduce the YBE to a set of three independent equations on functions $k(\theta)$ and $p(\theta)$:

$$\begin{aligned} p(u) - p(v) &= p(u - v) \\ k(u) - k(v) &= -p(u - v) \\ k(u)(k(u - v) + k(v) + p(u) + N) &= k(v) + k(u - v) - p(u). \end{aligned} \quad (5.95)$$

⁸ When dealing with the YBE, one really appreciates the power of *Mathematica*.

The first two equations simply tell us that $k(\theta)$ and $p(\theta)$ are linear functions,

$$p(\theta) = -\lambda\theta, \quad k(\theta) = \lambda\theta - \frac{1}{2\Delta}, \quad (5.96)$$

where λ and Δ are arbitrary constants. The last equation is then also satisfied, provided

$$\Delta = \frac{1}{N-2}. \quad (5.97)$$

As anticipated, the YBE fixes the functional structure of the S-matrix up to rescaling $\theta \rightarrow \lambda\theta$, and up to an overall dressing factor. The permutation and trace terms in the S-matrix map to one another under the crossing transformation, which interchanges the $SO(N)$ indices j and l in (5.91). The crossing-symmetry condition (5.89) then requires that

$$p(i\pi - \theta) = k(\theta), \quad (5.98)$$

which fixes

$$\lambda = \frac{1}{2\pi i\Delta}. \quad (5.99)$$

This determines the S-matrix up to the dressing factor:

$$S_{ij}^{kl}(\theta) = \sigma(\theta) \left(\delta_i^k \delta_j^l - \frac{2\pi i\Delta}{\theta} \delta_i^l \delta_j^k + \frac{2\pi i\Delta}{\theta - i\pi} \delta_{ij} \delta^{kl} \right). \quad (5.100)$$

The dressing factor is constrained by the unitarity and crossing conditions (5.88) and (5.89). Substitution of (5.93) into (5.88), and the use of the identities $P^2 = 1$, $K^2 = NK$, $PK = K = KP$, gives

$$\begin{aligned} S(-\theta)S(\theta) = \sigma(-\theta)\sigma(\theta) & \left[1 + \frac{1}{p(-\theta)p(\theta)} + \left(\frac{1}{p(-\theta)} + \frac{1}{p(\theta)} \right) P \right. \\ & \left. + \left(\frac{1}{k(-\theta)} + \frac{1}{k(\theta)} + \frac{1}{k(-\theta)p(\theta)} + \frac{1}{k(\theta)p(\theta)} + \frac{N}{k(-\theta)k(\theta)} \right) K \right]. \end{aligned}$$

The coefficients of P and K in this equation identically vanish, and the unitarity condition boils down to

$$\sigma(-\theta)\sigma(\theta) = \frac{\theta^2}{\theta^2 + 4\pi^2\Delta^2}. \quad (5.101)$$

The crossing symmetry simply requires that

$$\sigma(i\pi - \theta) = \sigma(\theta). \quad (5.102)$$

We need to solve these two functional equations on $\sigma(\theta)$.

The dressing factor $\sigma(\theta)$ by itself is not an amplitude of any physical process. It is much more instructive to reformulate the crossing/unitarity equations in terms of a physical S-matrix element for some particular scattering channel. General features of integrability bootstrap, which are important, for example, in the context of thermodynamic Bethe ansatz (TBA), become more transparent if the equations are written in terms of a physical amplitude.

To construct these we concentrate on a $U(1)$ subgroup of $SO(N)$ generated by rotations in the (12) plane. An $SO(N)$ vector has two components with $U(1)$ charges $+1$ and -1 and $N-2$ neutral components. In terms of the fields in the path integral (5.78), the charge $+1$ state corresponds to

$$\phi = n_1 + in_2.$$

Because of the charge conservation and the absence of particle production the $\phi\phi \rightarrow \phi\phi$ amplitude is purely elastic, and the S-matrix element

$$s(\theta) \equiv S_{\phi\phi \rightarrow \phi\phi}(\theta) \quad (5.103)$$

is a pure phase,

$$s(\theta) = e^{i\Phi(\theta)}. \quad (5.104)$$

From (5.100) we find

$$s(\theta) = \sigma(\theta) \frac{\theta - 2\pi i\Delta}{\theta}. \quad (5.105)$$

The crossing/unitarity equations for $s(\theta)$ are

$$\begin{aligned} s(i\pi - \theta) &= s(\theta) \frac{\theta(\theta + 2\pi i\Delta - i\pi)}{(\theta - 2\pi i\Delta)(\theta - i\pi)} \\ s(-\theta)s(\theta) &= 1. \end{aligned} \quad (5.106)$$

From these we infer that

$$s\left(\theta + \frac{i\pi}{2}\right)s\left(\theta - \frac{i\pi}{2}\right) = \frac{\left(\theta - \frac{i\pi}{2}\right)\left(\theta + \frac{i\pi}{2} - 2\pi i\Delta\right)}{\left(\theta + \frac{i\pi}{2}\right)\left(\theta - \frac{i\pi}{2} + 2\pi i\Delta\right)}. \quad (5.107)$$

Written in terms of the phase, the equation becomes

$$\Phi\left(\theta + \frac{i\pi}{2}\right) + \Phi\left(\theta - \frac{i\pi}{2}\right) = 2 \arctan \frac{2\theta}{\pi} - 2 \arctan \frac{2\theta}{\pi(1-4\Delta)} + \text{p.a.} \quad (5.108)$$

The last term (to be specified shortly) reflects an ambiguity in choosing the branch of the logarithm when reformulating the problem in terms of the scattering phase, which is defined up to shifts by integer multiples of 2π . The branch of the logarithm can be chosen differently for different θ , and we can parameterize the phase ambiguity by writing

$$\text{p.a.} = 2\pi \sum_k (\vartheta(\theta - i\alpha_k) + \vartheta(\theta + i\alpha_k)), \quad (5.109)$$

where $\vartheta(x)$ is the step function, and $\{\alpha_k\}$ is a potentially arbitrary set of parameters.

Another function, that plays an important role in all types of Bethe ansatz equations, is the derivative of the scattering phase,

$$K(\theta) = \frac{d\Phi(\theta)}{d\theta}. \quad (5.110)$$

For this function we get a difference equation

$$\begin{aligned} K\left(\theta + \frac{i\pi}{2}\right) + K\left(\theta - \frac{i\pi}{2}\right) &= \frac{\pi}{\theta^2 + \frac{\pi^2}{4}} - \frac{\pi(1-4\Delta)}{\theta^2 + \frac{\pi^2}{4}(1-4\Delta)^2} \\ &\quad + 2\pi \sum_k (\delta(\theta - i\alpha_k) + \delta(\theta + i\alpha_k)), \end{aligned} \quad (5.111)$$

which can be also written as

$$\begin{aligned} 2 \cos\left(\frac{\pi}{2} \frac{d}{d\theta}\right) K(\theta) &= \frac{\pi}{\theta^2 + \frac{\pi^2}{4}} - \frac{\pi(1-4\Delta)}{\theta^2 + \frac{\pi^2}{4}(1-4\Delta)^2} \\ &\quad + 2\pi \sum_k (\delta(\theta - i\alpha_k) + \delta(\theta + i\alpha_k)). \end{aligned} \quad (5.112)$$

The equation becomes algebraic in the Fourier space,

$$\cosh \frac{\pi\omega}{2} K(\omega) = \pi e^{-\frac{\pi}{2}|\omega|} - \pi e^{-\frac{\pi}{2}(1-4\Delta)|\omega|} + 2\pi \sum_k \cosh \alpha_k \omega. \quad (5.113)$$

Fourier transforming back to the θ -representation, we get

$$\begin{aligned} 2\pi K(\theta) = & \psi\left(1 + \frac{i\theta}{2\pi}\right) + \psi\left(1 - \frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{2} + \frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right) \\ & - \psi\left(1 - \Delta + \frac{i\theta}{2\pi}\right) - \psi\left(1 - \Delta - \frac{i\theta}{2\pi}\right) \\ & + \psi\left(\frac{1}{2} - \Delta + \frac{i\theta}{2\pi}\right) + \psi\left(\frac{1}{2} - \Delta - \frac{i\theta}{2\pi}\right) + \sum_k \frac{4\pi \cos \alpha_k \cosh \theta}{\sinh^2 \theta + \cos^2 \alpha_k}. \end{aligned} \quad (5.114)$$

Integrating and exponentiating the result, we find

$$s(\theta) = S_{\text{CDD}}(\theta) \frac{\Gamma\left(1 + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right) \Gamma\left(1 - \Delta - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} - \Delta + \frac{i\theta}{2\pi}\right)}{\Gamma\left(1 - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} + \frac{i\theta}{2\pi}\right) \Gamma\left(1 - \Delta + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} - \Delta - \frac{i\theta}{2\pi}\right)}, \quad (5.115)$$

where

$$S_{\text{CDD}}(\theta) = \prod_k \frac{\sinh \theta - i \cos \alpha_k}{\sinh \theta + i \cos \alpha_k} \quad (5.116)$$

is known as the CDD (Castillejo–Dalitz–Dyson) factor.

The CDD factor parameterizes fundamental ambiguity in the crossing + unitarity conditions, which do not fix the S-matrix completely. However, each CDD factor introduces extra poles in the rapidity plane. Any pole on the physical strip is associated with a bound state, and no new poles can be added by hand without changing the particle content of the theory. Usually this means that the minimal solution of the crossing equation is the physical one, but sometimes the minimal solution has spurious poles that have to be cancelled by CDD factors. The $O(N)$ model is a good example of this phenomenon.

The minimal solution without CDD factors would have a pole on the physical strip, at $\theta = i\pi(1 - 2\Delta)$. The pole has to be cancelled by a single CDD factor with

$$\alpha = \frac{\pi}{2} (1 - 4\Delta).$$

After which (5.115) becomes

$$s(\theta) = - \frac{\Gamma\left(1 + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right) \Gamma\left(\Delta - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} + \Delta + \frac{i\theta}{2\pi}\right)}{\Gamma\left(1 - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} + \frac{i\theta}{2\pi}\right) \Gamma\left(\Delta + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} + \Delta - \frac{i\theta}{2\pi}\right)}. \quad (5.117)$$

This is the exact scattering amplitude of the ϕ quanta.

The dressing factor can be extracted from (5.105):

$$\sigma(\theta) = \underline{Q}(\theta)\underline{Q}(i\pi - \theta), \quad \underline{Q}(\theta) = \frac{\Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right)\Gamma\left(\Delta - \frac{i\theta}{2\pi}\right)}{\Gamma\left(-\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} + \Delta - \frac{i\theta}{2\pi}\right)}. \quad (5.118)$$

Crossing symmetry is manifest in (5.118), while unitarity is manifest in (5.117).

Exercise 12 Expand the exact S-matrix of the $O(N)$ model to the first non-vanishing order in $1/N$, and compare with the result of the explicit calculation from Exercise 10.

5.9 Crash course in superalgebras

String theory requires supersymmetry for internal consistency. If a sigma-model is to describe critical ten-dimensional string it has to be supersymmetric. Supersymmetric counterparts of the coset sigma-models that we have studied so far are ubiquitous in holographic duality, and many of them are integrable, just like in the purely bosonic case. Fields in these models take values in a supermanifold, and are invariant under the action of a supergroup. Any detailed introduction to supergroups and superalgebras (which can be found in [24]) would occupy too large a space-time volume, but the most common examples can be introduced very easily, as algebras of matrices of a certain type, just like most common examples of ordinary Lie algebras are algebras of matrices constrained to be traceless, anti-symmetric, and so on.

In addition to bosonic generators T_M , a superalgebra contains supercharges Q_α . An element of a superalgebra is a linear combination,

$$X = X^M T_M + \theta^\alpha Q_\alpha,$$

where θ^α are Grassmann-odd variables. The commutator on a superalgebra must satisfy the usual requirements of linearity, anti-symmetry, and obeying the Jacobi identity. A group element is defined by an exponential map $g = e^X$, which is actually polynomial in supercharges because a sufficiently large power of Grassmann variables vanishes identically.

In any finite-dimensional representation the generators T_M and Q_α are ordinary numeric matrices. An element of the Lie superalgebra X then forms a supermatrix—a matrix some of whose elements are Grassmann-odd numbers.

Complex supermatrices can be defined as linear transformations of $\mathbb{C}^{(n|m)}$, a space of vectors whose first n components are Grassmann-even and the last m of which are Grassmann-odd:

$$\begin{pmatrix} B & | & \Theta \\ \hline \bar{\Theta} & | & \bar{B} \end{pmatrix} \begin{pmatrix} b \\ \hline f \end{pmatrix}.$$

To preserve the Grassmann-parity structure, the diagonal blocks of a supermatrix must be even and the off-diagonal blocks must be odd.

Supermatrices can be multiplied and commuted, with the commutator obeying all the usual axioms of the Lie bracket. The algebra of all $(n|m) \times (n|m)$ supermatrices is denoted by $\mathfrak{gl}(n|m)$. This superalgebra is not simple. An obvious way to get a simple algebra is to impose the traceless condition, and here the first deviation from the usual Lie algebras occurs.

Consider this simple example:

$$\left[\begin{pmatrix} 0 & \theta_1 \\ \psi_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \theta_2 \\ \psi_2 & 0 \end{pmatrix} \right] = \begin{pmatrix} \theta_1\psi_2 - \theta_2\psi_1 & 0 \\ 0 & \psi_1\theta_2 - \psi_2\theta_1 \end{pmatrix}.$$

When θ_i, ψ_i are ordinary numbers the sum of diagonal matrix elements on the right-hand side vanishes; the trace of a commutator is always zero. But if θ 's and ψ 's are Grassmann numbers, the diagonal matrix elements differ by a sign and the trace does not vanish. Since the trace of a commutator of supermatrices is not zero, we conclude that traceless supermatrices do not form a subalgebra in $\mathfrak{gl}(1|1)$.

This can be fixed in the following way. In the above example the difference, rather than the sum, of the diagonal matrix elements is zero. This is known as supertrace, in general defined as

$$\text{Str} \left(\begin{array}{c|c} B & \Theta \\ \hline \Theta' & B' \end{array} \right) = \text{tr} B - \text{tr} B'. \quad (5.119)$$

Exercise 13 Show that for any supermatrices,

$$\text{Str}[M_1, M_2] = 0. \quad (5.120)$$

Because of the last property, restriction to *supertraceless* supermatrices gives a closed algebra, known as the special linear superalgebra $\mathfrak{sl}(n|m)$. The $\mathfrak{sl}(n|m)$ superalgebras are simple if $m \neq n$, but the case of $m = n$ is special. Since

$$\text{Str} \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right) = n - m$$

for $m = n$ the trace condition does not eliminate the unit matrix! The unit matrix commutes with anything and thus represents a central element of the algebra. The factor-algebra, where this central element is set to zero is denoted by

$$\mathfrak{psl}(n|n) = \mathfrak{sl}(n|n)/C. \quad (5.121)$$

In any irreducible representation the central element takes a constant numeric value, and $\mathfrak{psl}(n|n)$ is defined by keeping only those representations of $\mathfrak{sl}(n|n)$ in which the central

element is equal to zero. It is interesting to notice that the defining $(n|n)$ -dimensional representation of $\mathfrak{sl}(n|n)$ is not a representation of $\mathfrak{psl}(n|n)$.

The $\mathfrak{psl}(n|n)$ superalgebras are remarkable in many respects. Being finite-dimensional they admit a central extension (never happens for ordinary Lie algebras). They also have zero dual Coxeter number, where the dual Coxeter number is defined as in the bosonic case through equation (5.74). To see this, consider a diagonal supermatrix $K = \text{diag}(a_1 \dots a_n | b_1 \dots b_n)$, which represents an element of $\mathfrak{psl}(n|n)$ if the sums of a_i 's and b_I 's are separately zero. As for its action in the adjoint representation, the eigenvalues of $\text{ad } K$ are $a_i - a_j$, $b_I - b_J$, $a_i - b_J$, and $b_I - a_j$ for all unordered pairs of indices ij , IJ , iJ , and Ij . Therefore,

$$\text{Str}_{\text{adj}} K^2 = \sum_{ij} (a_i - a_j)^2 + \sum_{IJ} (b_I - b_J)^2 - \sum_{iJ} (a_i - b_J)^2 - \sum_{Ij} (b_I - a_j)^2 = 0.$$

The unitary superalgebras $(\mathfrak{p})\mathfrak{su}(p, q|r, s)$ are real forms of $(\mathfrak{p})\mathfrak{sl}(n|m)$ singled out by the Hermiticity condition

$$M^\dagger = -B^{-1}MB, \quad B = \text{diag}(+^p -^q | +^r -^s). \quad (5.122)$$

Another important class of superalgebras is the supersymmetric counterpart of $\mathfrak{so}(n)$. Ordinary orthogonal algebra is singled out by the anti-symmetry condition imposed on elements of $\mathfrak{gl}(n)$. For supermatrices this does not work because transposition is not compatible with commutators. In general, $(M_1 M_2)^t \neq M_2^t M_1^t$ for supermatrices, since their off-diagonal Grassmann entries do not commute but anti-commute.

The supersymmetric generalization of transposition, called supertransposition, is defined as

$$\begin{pmatrix} A & \Theta \\ \Psi & B \end{pmatrix}^{st} = \begin{pmatrix} A^t & -\Psi^t \\ \Theta^t & B^t \end{pmatrix}. \quad (5.123)$$

It follows that

$$(M_1 M_2)^{st} = M_2^{st} M_1^{st}, \quad (5.124)$$

and consequently supertransposition preserves commutators up to a sign.

An orthosymplectic algebra $\mathfrak{osp}(n|2m)$ is defined as a subset of $\mathfrak{gl}(n|2m)$ singled out by the condition

$$M^{st} = -H^{-1}MH, \quad (5.125)$$

where

$$H = \begin{pmatrix} 1_{n \times n} & 0 \\ 0 & \mathcal{J} \end{pmatrix}, \quad (5.126)$$

and \mathcal{J} is the $2m \times 2m$ symplectic matrix,

$$\mathcal{J} = \begin{pmatrix} 0 & 1_{m \times m} \\ -1_{m \times m} & 0 \end{pmatrix}. \quad (5.127)$$

Explicitly,

$$\mathfrak{osp}(n|2m) = \left\{ \begin{pmatrix} A & \Theta \\ \Psi & B \end{pmatrix} \in \mathfrak{gl}(n|2m) \left| \begin{array}{l} A^t = -A, B^t = \mathcal{J}B\mathcal{J}, \\ \Theta^t = \mathcal{J}\Psi^t \end{array} \right. \right\}. \quad (5.128)$$

The bosonic subalgebra of $\mathfrak{osp}(n|2m)$ is $\mathfrak{so}(n) \oplus \mathfrak{sp}(2m)$.

Exercise 14 Check that superalgebras $\mathfrak{osp}(2n+2|2n)$ have zero dual Coxeter number.

5.10 Supercoset sigma-models

A natural supersymmetric generalization of a symmetric space is based on an automorphism $\Omega : \mathfrak{g} \rightarrow \mathfrak{g}$ that squares to the fermion parity,

$$\Omega^2 = (-1)^F. \quad (5.129)$$

Such automorphism has order four: $\Omega^4 = \text{id}$, and the superalgebra has a \mathbb{Z}_4 grading,

$$\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3, \quad (5.130)$$

where

$$\Omega(\mathfrak{h}_n) = i^n \mathfrak{h}_n. \quad (5.131)$$

The \mathbb{Z}_4 decomposition is consistent with the Grassmann parity: $\mathfrak{h}_0 \oplus \mathfrak{h}_2$ form the bosonic subalgebra of \mathfrak{g} and all the supercharges belong to $\mathfrak{h}_1 \oplus \mathfrak{h}_3$.

If H_0 is the subgroup of G whose Lie algebra \mathfrak{h}_0 is \mathbb{Z}_4 -invariant, the coset G/H_0 is called semi-symmetric superspace [25]. Anti-de-Sitter geometries in various dimensions arise as bosonic sections of \mathbb{Z}_4 cosets, which play an important role in the holographic duality. The key example is string theory on $AdS_5 \times S^5$ whose sigma-model [4] is a semi-symmetric coset [26]. Just as for ordinary symmetric-space sigma-models, integrability of semi-symmetric cosets can be established at the algebraic level, using constraints imposed on the equations of motion by the \mathbb{Z}_4 -symmetry.

The current (5.10) decomposes into four components according to their \mathbb{Z}_4 grading,

$$j = j_0 + j_1 + j_2 + j_3. \quad (5.132)$$

The current components j_0, j_2 expand in bosonic generators of the superalgebra, while the components j_1, j_3 are fermionic—they are linear combinations of supercharges with Grassmann-odd coefficients.

The action of the \mathbb{Z}_4 coset sigma-model is

$$S = \int d^2x \text{Str} \left(\sqrt{-h} h^{ab} j_{2a} j_{2b} + \varepsilon^{ab} j_{1a} j_{3b} \right). \quad (5.133)$$

The first term is the usual sigma-model Lagrangian, while the second term descends from the WZ action, which for supercosets is not topological and can be written in a manifestly 2d form.

Exercise 15 *Derive the equations of motion for a \mathbb{Z}_4 coset and check that the combined system of equations of motion and the flatness condition of the left-invariant current admits zero-curvature representation with the Lax connection [5]*

$$L = j_0 + \frac{z^2 + 1}{z^2 - 1} j_2 - \frac{2z}{z^2 - 1} * j_2 + \sqrt{\frac{z + 1}{z - 1}} j_1 + \sqrt{\frac{z - 1}{z + 1}} j_3. \quad (5.134)$$

Therefore the model is completely integrable.

The action (5.133), taken at face value, is quadratic in derivatives, which is fine for bosonic degrees of freedom, but a fermion Lagrangian is normally of Dirac type and is expected to be linear in derivatives. It is instructive to see how this apparent contradiction is resolved in \mathbb{Z}_4 cosets.

To get some intuition about the structure of the sigma-model action we can take the coset representative in the standard exponential form,

$$g = e^X, \quad X = X_1 + X_2 + X_3 = \theta_1^\alpha Q_{1\alpha} + X^M T_M + \theta_3^{\dot{\alpha}} Q_{3\dot{\alpha}}, \quad (5.135)$$

and expand the currents in X to the linear order. Then $j_{na} \simeq \partial_a X_n$. The sigma-model term becomes $(\partial X^M)^2$, while the WZ term gives $\varepsilon^{ab} \partial_a X_1 \partial_b X_3$, which looks like a second-order kinetic term for fermions, but in fact is a total derivative that integrates to zero,

$$\varepsilon^{ab} \partial_a X_1 \partial_b X_3 = \partial_a \left(\varepsilon^{ab} X_1 \partial_b X_3 \right).$$

The quadratic term for fermions appears only at the next order in the expansion.

The cubic part of the action, one can show, contains⁹

$$\begin{aligned}\mathcal{L}_F^{(3)} &= \text{Str} \left(\Pi_-^{ab} X_1 [\partial_a X_2, \partial_b X_1] + \Pi_+^{ab} X_3 [\partial_a X_2, \partial_b X_3] \right) \\ &= \Pi_-^{ab} \theta_{1\alpha} f_{M\beta}^\alpha \partial_a X^M \partial_b \theta_1^\beta + \Pi_+^{ab} \theta_{3\dot{\alpha}} f_{M\dot{\beta}}^{\dot{\alpha}} \partial_a X^M \partial_b \theta_3^{\dot{\beta}},\end{aligned}\quad (5.136)$$

where f_{AB}^C are structure constants of the superalgebra \mathfrak{g} , and Π_\pm are the world-sheet chirality projectors,

$$\Pi_\pm^{ab} = \eta^{ab} \pm \varepsilon^{ab}. \quad (5.137)$$

This action is exactly the same as the quadratic part of the Green–Schwarz (GS) action for the superstring [27], with structure constants $f_{M\alpha}^\beta$ replacing the Dirac matrices.

The GS action is notoriously difficult to quantize covariantly, one manifestation of which is absence of kinetic terms for fermions. Those can be induced if bosonic currents have a non-zero expectation value: $\langle \partial X^M \rangle \neq 0$. The most familiar instance when this happens is the light-cone gauge:

$$X^+ = \tau. \quad (5.138)$$

Taking into account that

$$\Pi_\pm^{0b} \partial_b = \partial_\pm, \quad (5.139)$$

where ∂_\pm are the world-sheet light-cone derivatives, we see that the GS action in the light-cone gauge describes the standard left- and right-moving Majorana fermions,

$$\mathcal{L}_{F,\text{lc}}^{(3)} = \theta_1 f_+ \partial_- \theta_1 + \theta_3 f_+ \partial_+ \theta_3. \quad (5.140)$$

Exercise 16 Consider a slightly more general setup. As in Section 5.6, take the coset representative in the form (5.61) where \bar{g} is the background field that belongs to the bosonic subgroup of G and satisfies the equations of motion, and expand the action to the quadratic order in fluctuations. Reproduce the trilinear terms in the action (5.136) from this more general result. Show that the light-cone gauge-fixed action in addition to (5.140) contains a mass term that couples θ_1 and θ_3 .

Exercise 17 Use the results from Exercise 16 to compute the one-loop beta-function for generic semi-symmetric coset [28]. Show that the beta-function vanishes as soon as the superalgebra \mathfrak{g} has zero dual Coxeter number ($\text{Str}_{\text{adj}} K^2 = 0 \forall K \in \mathfrak{h}_2$).

⁹ To simplify the notations we consider a flat Minkowski metric on the world-sheet from now on.

Semi-symmetric superspaces are all classified [25]. As we have seen each of them gives rise to an integrable GS-type sigma-model. Not all of these sigma-models are consistent as string backgrounds. There are a number of additional constraints that need to be satisfied. The sigma-model must be conformally invariant and must have zero beta-function, to begin with. The one-loop beta-function of semi-symmetric cosets is proportional to the supertrace of the background current in the adjoint, and only those cosets whose dual Coxeter number vanishes have a chance to be consistent as string sigma-models. Two infinite series of superalgebras with vanishing adjoint supertrace were identified in Section 5.9: $\mathfrak{psl}(n|n)$ and $\mathfrak{osp}(2n+2|2n)$.

To define a critical superstring in ten dimensions a sigma-model should also have central charge 26, which is more or less equivalent to the condition that the physical degrees of freedom in the light-cone gauge are $8_b + 8_f$ transverse fluctuation modes of the string. We are not going to present a comprehensive analysis of all possible cases, but restrict exposition to two representative examples.

The $\mathfrak{psl}(m|m)$ superalgebras admit a number of \mathbb{Z}_4 automorphisms. One of them, defined for even $m = 2n$, is a combination of supertransposition (5.123) and a conjugation:

$$\Omega = -st \circ \text{Ad} \, \text{diag}(\mathcal{J}, \mathcal{J}), \quad (5.141)$$

where \mathcal{J} is the symplectic matrix (5.127). The invariant subalgebra of Ω is

$$\mathfrak{h}_0 = \{\text{diag}(A, B) \mid A = \mathcal{J}A^t\mathcal{J}, B = \mathcal{J}B^t\mathcal{J}\}, \quad (5.142)$$

namely,

$$\mathfrak{h}_0 = \mathfrak{sp}(2n) \oplus \mathfrak{sp}(2n). \quad (5.143)$$

The bosonic dimension of the coset (the number of generators in \mathfrak{h}_2) is $2[(4n^2 - 1) - (2n^2 + n)] = 4n^2 - 2n - 2$. For $n = 2$, this equals 10.

Choosing the real form to be $\mathfrak{psu}(2, 2|4)$, and taking into account that $SU(4) = Spin(6)$, $SU(2, 2) = Spin(4, 2)$, $USp(4) = Spin(5)$ and $USp(2, 2) = Spin(4, 1)$, we get the coset $PSU(2, 2|4)/Spin(4, 1) \times Spin(5)$ whose bosonic section is

$$Spin(4, 2)/Spin(4, 1) \times Spin(6)/Spin(5) = AdS_5 \times S^5.$$

The coset therefore supersymmetrizes $AdS_5 \times S^5$, the background that plays so prominent a rôle in the AdS/CFT correspondence.

The off-diagonal blocks of a $\mathfrak{psu}(2, 2|4)$ element supply $2 \times 4^2 = 32$ real supercharges. In the coset parameterization (5.135) those give rise to 32 fermion fields in the sigma-model. However, only half of them correspond to physical, propagating degrees of freedom. This is readily seen in the light-cone gauge associated with $T_+ = (+ + - - \mid + + - -) \sim D - \mathcal{J}$, where D is the dilatation generator of the conformal algebra and \mathcal{J} is the $\mathfrak{so}(6)$ angular momentum. So chosen T_+ is indeed a null element of $\mathfrak{psu}(2, 2|4)$:

$\text{Str } T_+^2 = 0$. The light-cone coordinate X^+ is a combination of global AdS time and an angle on S^5 , so the classical solution (5.138) describes a point-like string moving along the big circle of S^5 at the speed of light [29]. The kinetic terms in the gauge-fixed fermion action contain $f_+ = \text{ad } T_+$. When restricted to the fermionic subspace of $\mathfrak{psu}(2, 2|4)$, $\text{ad } T_+$ has eight eigenvalues $+2$, eight eigenvalues -2 , and sixteen zero eigenvalues. Half of the fermions (those corresponding to the zero modes of $\text{ad } T_+$) do not appear in the action at all. This is a manifestation of the kappa-symmetry, a Grassmann-odd gauge symmetry that at the linearized level acts as a shift symmetry: $\theta \rightarrow \theta_\alpha + \varepsilon \kappa_\alpha$, where κ_α are zero eigenvectors of $\text{ad } T_+$. Gauge-fixing the kappa-symmetry leaves sixteen Majorana fermions, eight of which are left- and eight are right-moving. This is the spectrum of critical superstring in ten dimensions.

Another example is the sigma-model based on the $\mathfrak{osp}(2n + 2|2n)$ series of superalgebras. These superalgebras admit a \mathbb{Z}_4 automorphism

$$\Omega = \text{Ad diag}(\mathcal{I} | +^p -^{n-p} +^p -^{n-p}). \quad (5.144)$$

The invariant subalgebra is $\mathfrak{h}_0 = \mathfrak{u}(n+1) \oplus \mathfrak{sp}(2p) \oplus \mathfrak{sp}(2n-2p)$, and the bosonic dimension of the coset is $(n+1)(2n+1) - (n+1)^2 + (2n^2+n) - (2p^2+p) - [2(n-p)^2 + n-p] = n^2 + n + 4np - 4p^2$, which gives ten for $n=2$ and $p=1$ (for larger n the dimension is too big). Taking into account that the appropriate real form of $\mathfrak{sp}(4)$ is $\mathfrak{so}(3, 2)$, that $\mathfrak{so}(6) = \mathfrak{su}(3)$, and that the denominator subalgebra is $\mathfrak{u}(3) \oplus \mathfrak{so}(3, 1)$, we get the coset $OSp(6|4)/U(3) \times SO(3, 1)$ whose bosonic section is

$$SU(4)/U(3) \times SO(3, 2)/SO(3, 1) = \mathbb{CP}^3 \times AdS_4, \quad (5.145)$$

another representative holographic background.

Exercise 18 *Show that the number of kappa symmetries in the $OSp(6|4)/U(3) \times SO(3, 1)$ coset is such that the light-cone spectrum contains $8_L + 8_R$ Majorana fermions.*

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