VI
The Quantum Inverse Scattering Method

Introduction

The quantum inverse scattering method (QSIM) appears as the quantized form of the classical inverse scattering method. It allows us to reproduce the results of the Bethe Ansatz and to move ahead. QISM is now a well developed branch of mathematical physics. In this chapter the fundamentals of QISM are given and illustrated by concrete examples.

In section 1 the general scheme of QISM, which allows the calculation of commutation relations between elements of the transfer matrix (necessary to construct eigenfunctions of the Hamiltonian in Chapter VII) is presented, and the quantum R-matrix is introduced. As in the classical case, the existence of an R-matrix and trace identities ensures that a Lax representation for the model exists. Thus, there are infinitely many conservation laws.

The Yang-Baxter equation, which is satisfied by the R-matrix, is discussed in section 2. Some important features of the R-matrix are also mentioned. The trace identities for the quantum nonlinear Schrödinger equation are proved in section 3. The general scheme of QISM is applied to the quantum sine-Gordon and Zhiber-Shabat-Mikhailov models in section 4. Spin models of quantum statistical physics are discussed in section 5. It is shown that a fundamental spin model can be constructed with the help of any given R-matrix.

The connection between classical statistical models on a two-dimensional lattice and QISM is established in section 6. QISM is the generalization of the classical inverse scattering method. In section 6 the partition function of the six-vertex model with domain wall boundary conditions is introduced. Later it will be studied extensively, and finally it will be represented as the determinant of a matrix. This idea will be developed in Part III. Finally this will permit us to write down a determinant representation for quantum correlation functions, and in Part IV this will lead to the complete solution of the problem of correlation functions. Hence, the main formalism of Chapter V will be used here again, but in the quantum version.

VI.1 General scheme

The Hamiltonian H and the matrix elements of the matrices $\mathsf{L}(n|\lambda)$ and $V(n|\lambda)$ (see section 1 of Chapter V) for a given model are now quantum operators depending on the dynamical variables of the system. For relativistic systems, regularization is necessary due to ultraviolet divergences. Thus, we shall consider the lattice formulation where the inverse lattice spacing, Δ^{-1}, has the meaning of an ultraviolet cut-off. Continuous models can also be considered within QISM (see section 3, where the quantum continuous nonlinear Schrödinger equation is considered). We shall consider models on a periodic lattice with M sites.

As in the classical case, the transition matrix T is defined by the following relation:

$$\mathsf{T}(n,m|\lambda) = \mathsf{L}(n|\lambda)\cdots\mathsf{L}(m|\lambda), \qquad (n \geq m). \tag{1.1}$$

Here, $\mathsf{L}(m|\lambda)$ can be obtained by direct quantization of the corresponding classical expression. This definition coincides with (V.1.8), but the matrix elements of T and L are now quantum operators which, generally speaking, do not commute with each other. The monodromy matrix, $\mathsf{T}(\lambda)$, is the transition matrix through the entire lattice:

$$\mathsf{T}(\lambda) \equiv \mathsf{T}(M,1|\lambda). \tag{1.2}$$

The transfer matrix, $\tau(\lambda)$, defined as the trace of the monodromy matrix (exactly as in (V.1.9)),

$$\tau(\lambda) = \mathrm{tr}\ \mathsf{T}(\lambda), \tag{1.3}$$

plays an important role. Eigenfunctions of the quantum Hamiltonian will be constructed with the help of the monodromy matrix in Chapter VII. Therefore, it is important to find the commutators between all its matrix elements. This problem is solved within QISM by means of the R-matrix method, which is applicable if the commutation relations between the elements of the L-operator can be represented in the form

$$\mathsf{R}(\lambda,\mu)\left(\mathsf{L}(k|\lambda)\bigotimes\mathsf{L}(k|\mu)\right) = \left(\mathsf{L}(k|\mu)\bigotimes\mathsf{L}(k|\lambda)\right)\mathsf{R}(\lambda,\mu) \tag{1.4}$$

where the matrix elements of the L-operator commute at different sites:

$$[\mathsf{L}_{ij}(p|\lambda),\ \mathsf{L}_{kl}(q|\mu)] = 0, \qquad \text{when } p \neq q. \tag{1.5}$$

The relation (1.5) is usually referred to as the ultralocality property.

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Theorem 1. *If the relations (1.4) and (1.5) are valid, then the commutation relations between matrix elements of the transition matrix* $\mathsf{T}(n,m|\lambda)$ *(1.1) are given by a formula similar to (1.4):*

$$\mathsf{R}(\lambda,\mu)\left(\mathsf{T}(n,m|\lambda)\bigotimes\mathsf{T}(n,m|\mu)\right)=\left(\mathsf{T}(n,m|\mu)\bigotimes\mathsf{T}(n,m|\lambda)\right)\mathsf{R}(\lambda,\mu). \tag{1.6}$$

Here $\mathsf{R}(\lambda,\mu)$ *is a c-number* $k^2\times k^2$ *matrix depending on the spectral parameters* λ *and* μ *only.*

Proof: The proof can be easily obtained by induction. The relation (1.4) is the basis of the induction. Fixing m in (1.6) and assuming that this relation is valid for $n=j$, one then easily proves that (1.6) is valid for $n=j+1$. Indeed,

$$\begin{aligned}
&\mathsf{R}(\lambda,\mu)\left(\mathsf{T}(j+1,m|\lambda)\bigotimes\mathsf{T}(j+1,m|\mu)\right)\\
&\quad=\mathsf{R}(\lambda,\mu)\left(\mathsf{L}(j+1|\lambda)\bigotimes\mathsf{L}(j+1|\mu)\right)\\
&\qquad\qquad\times\ \mathsf{R}^{-1}(\lambda,\mu)\mathsf{R}(\lambda,\mu)\left(\mathsf{T}(j,m|\lambda)\bigotimes\mathsf{T}(j,m|\mu)\right)\mathsf{R}^{-1}(\lambda,\mu)\mathsf{R}(\lambda,\mu)\\
&\quad=\left(\mathsf{L}(j+1|\mu)\bigotimes\mathsf{L}(j+1|\lambda)\right)\left(\mathsf{T}(j,m|\mu)\bigotimes\mathsf{T}(j,m|\lambda)\right)\mathsf{R}(\lambda,\mu)\\
&\quad=\left(\mathsf{T}(j+1,m|\mu)\bigotimes\mathsf{T}(j+1,m|\lambda)\right)\mathsf{R}(\lambda,\mu).
\end{aligned} \tag{1.7}$$

So Theorem 1 is proved.

The relations (1.4) and (1.6) are usually called bilinear relations. Relation (1.4) means that the L-operator is "intertwined" by the R-matrix. The transition matrix is "intertwined" by the same R-matrix (1.6).

Sometimes it is preferable to rewrite the bilinear relation in a different notation. Multiplying (1.6) from the left by the permutation matrix Π (V.2.2), one obtains

$$\widetilde{\mathsf{R}}(\lambda,\mu)\left(\mathsf{T}(\lambda)\bigotimes I\right)\left(I\bigotimes\mathsf{T}(\mu)\right)=\left(I\bigotimes\mathsf{T}(\mu)\right)\left(\mathsf{T}(\lambda)\bigotimes I\right)\widetilde{\mathsf{R}}(\lambda,\mu) \tag{1.8}$$

where

$$\widetilde{\mathsf{R}}(\lambda,\mu)=\Pi\mathsf{R}(\lambda,\mu). \tag{1.9}$$

In (1.8) we have suppressed the space coordinates of $\mathsf{T}(n,m|\lambda)$. Equation (1.8) is also valid for the monodromy matrix (1.2). The $k\times k$ matrix $\mathsf{T}(\lambda)$ acts only on the first vector space, and the matrix $\mathsf{T}(\mu)$ acts only on the second vector space. Thus we can rewrite (1.8) in the form

$$\widetilde{\mathsf{R}}_{12}(\lambda,\mu)\mathsf{T}_1(\lambda)\mathsf{T}_2(\mu)=\mathsf{T}_2(\mu)\mathsf{T}_1(\lambda)\widetilde{\mathsf{R}}_{12}(\lambda,\mu), \tag{1.10}$$

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where the subindices denote the vector space(s) on which the matrix acts. The matrix $\widetilde{\mathsf{R}}$ acts on the tensor product of the two spaces 1 and 2. (Tensor notation is explained in the Appendix to Chapter V.)

To construct integrable lattice models, we have to consider tensor products of many vector spaces. Then expression (1.10) will be generalized as

$$\widetilde{\mathsf{R}}_{\alpha\beta}(\lambda,\mu)\mathsf{T}_{\alpha}(\lambda)\mathsf{T}_{\beta}(\mu)=\mathsf{T}_{\beta}(\mu)\mathsf{T}_{\alpha}(\lambda)\widetilde{\mathsf{R}}_{\alpha\beta}(\lambda,\mu) \tag{1.11}$$

where α and β denote (different) vector spaces.

A consequence of (1.6) is the existence of a commuting family of transfer matrices $\tau(\lambda)$, for which

$$[\tau(\lambda),\,\tau(\mu)]=0. \tag{1.12}$$

To prove this, one rewrites the relation (1.6) as

$$\mathsf{R}(\lambda,\mu)\,\mathsf{T}(\lambda)\bigotimes\mathsf{T}(\mu)\,\mathsf{R}^{-1}(\lambda,\mu)=\mathsf{T}(\mu)\bigotimes\mathsf{T}(\lambda) \tag{1.13}$$

and takes the trace in the space of $k^2\times k^2$ matrices to obtain $\tau(\lambda)\tau(\mu)=\tau(\mu)\tau(\lambda)$ (using the fact that the trace of the tensor product of matrices is equal to the product of their traces).

In the quantum case, as in the classical one, the Hamiltonian of a model can be represented as a linear combination of logarithmic derivatives of the transfer matrix $\tau(\lambda)$ (1.3) at some points ν_a:

$$H=\sum_k\sum_a c_{ka}\frac{d^k}{d\lambda^k}\ln\tau(\lambda)\bigg|_{\lambda=\nu_a} \tag{1.14}$$

(here the c_{ka} are coefficients). The points ν_a are chosen as to make the Hamiltonian local. These formulæ are known as the trace identities. Examples are given in sections 3 and 5. It follows from (1.12) that (as in the classical case) an infinite number of conservation laws exist in the infinite lattice limit $M\to\infty$. Indeed, from (1.12) it follows immediately that $[\ln\tau(\lambda),\,\tau(\mu)]=0$. Differentiating this with respect to λ, and using (1.14), we get $[H,\,\tau(\mu)]=0$. Now we can expand this in a Taylor series in μ and obtain an infinite set of conservation laws, similar to the classical case (section 2, Chapter V).

The Lax representation for the nonlinear equation also follows from the existence of the R-matrix. First, let us construct the generating functional $U(n|\lambda,\mu)$ of the time evolution operators (in a way similar to that of the classical case (V.2.19)–(V.2.20)).

Theorem 2. *The following Lax representation exists in the quantum case:*

$$i\left[\frac{d}{d\mu}\ln\tau(\mu), \mathsf{L}(n|\lambda)\right] = U(n+1|\lambda,\mu)\mathsf{L}(n|\lambda) - \mathsf{L}(n|\lambda)U(n|\lambda,\mu). \quad (1.15)$$

The generating functional of the time evolution operators is equal to

$$U(n|\lambda,\mu) = i\frac{d}{d\mu}\ln\tau(\mu)I - iq^{-1}(n|\lambda,\mu)\frac{d}{d\mu}q(n|\lambda,\mu) \quad (1.16)$$

where $q(n|\lambda,\mu)$ is the $k\times k$ matrix

$$q_2(n|\lambda,\mu) = \mathrm{tr}_1\left[\mathsf{T}_1(M,n|\mu)\widetilde{\mathsf{R}}_{12}(\mu,\lambda)\mathsf{T}_1(n-1,1|\mu)\right]. \quad (1.17)$$

The subscripts 1, 2 *label different vector spaces. (Note that the subindex* 2 *is only the label of the auxiliary vector space and can be dropped.)*

Proof: The commutation relation between $\mathsf{L}(n|\lambda)$ and $q(n|\lambda,\mu)$ is easily obtained from (1.4), (1.6) and (1.10):

$$q(n+1|\lambda,\mu)\mathsf{L}(n|\lambda) = \mathsf{L}(n|\lambda)q(n|\lambda,\mu) \quad (1.18)$$

where these are matrix products of L and q on both sides. Differentiating this expression with respect to μ and multiplying it by $q^{-1}(n+1|\lambda,\mu)$ on the left gives

$$q^{-1}(n+1|\lambda,\mu)\frac{d}{d\mu}q(n+1|\lambda,\mu)\mathsf{L}(n|\lambda) =$$
$$q^{-1}(n+1|\lambda,\mu)\mathsf{L}(n|\lambda)\frac{d}{d\mu}q(n|\lambda,\mu). \quad (1.19)$$

To transform the right hand side, let us rewrite the commutation relation (1.18) between q and L as

$$q^{-1}(n+1|\lambda,\mu)\mathsf{L}(n|\lambda) = \mathsf{L}(n|\lambda)q^{-1}(n|\lambda,\mu) \quad (1.20)$$

so that

$$q^{-1}(n+1|\lambda,\mu)\frac{d}{d\mu}q(n+1|\lambda,\mu)\mathsf{L}(n|\lambda)$$
$$= \mathsf{L}(n|\lambda)q^{-1}(n|\lambda,\mu)\frac{d}{d\mu}q(n|\lambda,\mu). \quad (1.21)$$

Now we see that (1.15) follows from (1.16) and (1.21). The theorem is thus proved. The first proof was published in [30].

With the help of the trace identities, it is possible to change the left hand side in (1.15) to $\partial_t\mathsf{L}(n|\lambda) = i[H, \mathsf{L}(n|\lambda)]$ and the time evolution operator $U(n|\lambda)$ for the Hamiltonian H is then equal to a linear combination of $d^lU(n|\lambda,\mu)/d\mu^l$ at $\mu = \nu_a$. For the Hamiltonian (1.14), the zero curvature condition exists:

$$\partial_t \mathsf{L}(n|\lambda) = i[H,\, \mathsf{L}(n|\lambda)] = U(n+1|\lambda)\mathsf{L}(n|\lambda) - \mathsf{L}(n|\lambda)U(n|\lambda). \tag{1.22}$$

Moreover, the evolution operator is constructed from the generating functional as follows:

$$U(n|\lambda) = \sum_k \sum_a c_{ka} \frac{d^{(k-1)}}{d\mu^{(k-1)}} U(n|\lambda,\mu)\Big|_{\mu=\nu_a}. \tag{1.23}$$

This operator possesses the correct quasiclassical limit and is local (for Hamiltonians with local interactions).

All considerations in this section also remain valid for quantum continuum models (they can be obtained, for example, as a formal continuous limit of lattice models). The relation between lattice and continuous models was shown in Chapter V (quantization does not change this).

VI.2 The Yang-Baxter relation

Not any $k^2 \times k^2$ matrix $\mathsf{R}(\lambda,\mu)$ can be an R-matrix for some integrable model. There are some restrictions which can be derived as the compatibility conditions for the relations (1.6). Namely, consider the tensor product $\mathsf{T}(\lambda) \otimes \mathsf{T}(\mu) \otimes \mathsf{T}(\nu)$. This expression can be reduced to $\mathsf{T}(\nu) \otimes \mathsf{T}(\mu) \otimes \mathsf{T}(\lambda)$ in two different ways. Using (1.6) one has

$$\begin{aligned}
&\mathsf{T}(\lambda) \bigotimes \mathsf{T}(\mu) \bigotimes \mathsf{T}(\nu) \\
&\quad = \left(\mathsf{R}^{-1}(\lambda,\mu) \bigotimes I\right) \left(I \bigotimes \mathsf{R}^{-1}(\lambda,\nu)\right) \left(\mathsf{R}^{-1}(\mu,\nu) \bigotimes I\right) \\
&\qquad \times \left(\mathsf{T}(\nu) \bigotimes \mathsf{T}(\mu) \bigotimes \mathsf{T}(\lambda)\right) \\
&\qquad \times \left(\mathsf{R}(\mu,\nu) \bigotimes I\right) \left(I \bigotimes \mathsf{R}(\lambda,\nu)\right) \left(\mathsf{R}(\lambda,\mu) \bigotimes I\right) \\
&\quad = \left(I \bigotimes \mathsf{R}^{-1}(\mu,\nu)\right) \left(\mathsf{R}^{-1}(\lambda,\nu) \bigotimes I\right) \left(I \bigotimes \mathsf{R}^{-1}(\lambda,\mu)\right) \\
&\qquad \times \left(\mathsf{T}(\nu) \bigotimes \mathsf{T}(\mu) \bigotimes \mathsf{T}(\lambda)\right) \\
&\qquad \times \left(I \bigotimes \mathsf{R}(\lambda,\mu)\right) \left(\mathsf{R}(\lambda,\nu) \bigotimes I\right) \left(I \bigotimes \mathsf{R}(\mu,\nu)\right).
\end{aligned} \tag{2.1}$$

The sufficient condition for the validity of this equality is the famous Yang-Baxter relation

$$\begin{aligned}
&\left(I \bigotimes \mathsf{R}(\lambda,\mu)\right) \left(\mathsf{R}(\lambda,\nu) \bigotimes I\right) \left(I \bigotimes \mathsf{R}(\mu,\nu)\right) = \\
&\quad = \left(\mathsf{R}(\mu,\nu) \bigotimes I\right) \left(I \bigotimes \mathsf{R}(\lambda,\nu)\right) \left(\mathsf{R}(\lambda,\mu) \bigotimes I\right).
\end{aligned} \tag{2.2}$$

Later we shall discover that exactly solvable models which have the same R-matrix but different L-operators are closely related to each other. One can compare this with different representations of the same Lie al-

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gebra. The R-matrix should be compared with the set of structure constants and the Yang-Baxter equation can be viewed as the analogue of the Jacobi identity. In section 6 of Chapter VII, we shall enumerate all the monodromy matrices whose commutation relations are given by a fixed R-matrix. In section 4 of Chapter VIII we shall enumerate all the L-operators which give rise to the same R-matrix. This is similar to the enumeration of all representations of a fixed Lie algebra.

The simplest example of an R-matrix is the rational R-matrix for the XXX model:

$$\mathsf{R}(\lambda,\mu) = i\hbar E + (\mu-\lambda)\sqcap. \tag{2.3}$$

An example of a nontrivial R-matrix is presented in section 4. Before continuing, a few comments must be made about some properties of the R-matrix. Usually the spectral parameter is a complex number which can be chosen so that the R-matrix depends on the difference of its arguments:

$$\mathsf{R}(\lambda,\mu) \equiv \mathsf{R}(\lambda-\mu). \tag{2.4}$$

Thus, we can shift the spectral parameter of the L-operator. If some operator $\mathsf{L}(\lambda)$ satisfies the bilinear relation (1.4), then $\mathsf{L}(\lambda-\nu)$ satisfies this relation also:

$$\mathsf{R}(\lambda-\mu)\left(\mathsf{L}(\lambda-\nu)\bigotimes \mathsf{L}(\mu-\nu)\right) = \left(\mathsf{L}(\mu-\nu)\bigotimes \mathsf{L}(\lambda-\nu)\right)\mathsf{R}(\lambda-\mu) \tag{2.5}$$

for any complex number ν.

It follows from (1.6) that the R-matrix can be multiplied by an arbitrary complex-valued function, which can be chosen such that

$$\mathsf{R}(\lambda,\mu) = \mathsf{R}^{-1}(\mu,\lambda), \qquad \mathsf{R}(\lambda,\lambda) = E. \tag{2.6}$$

The quasiclassical limit for such R-matrices is especially simple:

$$\mathsf{R} \underset{\hbar\to 0}{\longrightarrow} \sqcap(E - i\hbar \mathsf{r})$$

where r is the corresponding classical r-matrix. As a rule, this normalization condition (2.6) will be used below.

At the end of section 4 we include, as an example, an R-matrix which depends on a three-dimensional spectral parameter $\widetilde{\lambda}$ (actually, $\widetilde{\lambda}$ can be realized as a 2×2 matrix belonging to $SL(2,\mathbb{C})$).

VI.3 Trace identities

We now discuss the trace identities for the quantum nonlinear Schrödinger equation which describes a one-dimensional Bose gas. The Hamiltonian is:

$$H = \int dx\, \left[\partial_x\Psi^\dagger\,\partial_x\Psi + c\Psi^\dagger\Psi^\dagger\Psi\Psi\right]. \tag{3.1}$$

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Here Ψ, $\Psi^\dagger$ are Hermitian-conjugate bosonic operators with the following canonical commutation relations:

$$\begin{aligned}&[\Psi(x),\ \Psi^\dagger(y)] = \delta(x-y);\\ &[\Psi(x),\ \Psi(y)] = [\Psi^\dagger(x),\ \Psi^\dagger(y)] = 0.\end{aligned} \tag{3.2}$$

The operators for momentum, P, and number of particles, Q, are given by (V.1.13). The quantum transition matrix is constructed from the classical one by normal ordering (the operators $\Psi^\dagger$ must stand to the left of the operators Ψ):

$$\mathsf{T}_{\mathrm{q}}(x,y|\lambda) = \ :\mathsf{T}_{\mathrm{cl}}(x,y|\lambda):$$

In the quantum case, one can act almost as in the classical case. Let us consider the quantum operator $\partial_x + V(x|\lambda)$ (see (V.1.14)). One obtains for the transfer matrix analogously to (V.1.6):

$$\left(\partial_x + i\frac{\lambda}{2}\sigma_z + \Omega(x)\right)\mathsf{T}(x,y|\lambda) = 0. \tag{3.3}$$

Periodic boundary conditions in x are imposed on $\Psi(x)$ (the period being equal to L). It should be mentioned that the matrix elements of the 2×2 matrix T are now functionals of the quantum field operators $\Psi(z)$, $\Psi^\dagger(z)$ $(x \geq z \geq y)$. The transition matrix possesses the following property:

$$\sigma_x \mathsf{T}^*(x,y|\lambda^*)\sigma_x = \mathsf{T}(x,y|\lambda). \tag{3.4}$$

Here the asterisk following T means Hermitian conjugation of its matrix elements (but no transposition is performed). The monodromy matrix is defined as in the classical case:

$$\mathsf{T}(\lambda) = \mathsf{T}(L,0|\lambda). \tag{3.5}$$

As before, the matrix trace of the monodromy matrix, $\tau(\lambda) = \operatorname{tr}\,\mathsf{T}(\lambda)$, is the transfer matrix. One can express the integrals of motion H, P and Q with the help of the trace identities:

$$\begin{aligned}\ln\left(e^{i\lambda L/2}\tau(\lambda)\right) \underset{\lambda\to i\infty}{\longrightarrow} ic\Big\{&\lambda^{-1}Q + \lambda^{-2}[P-(ic/2)Q]\\ &+\lambda^{-3}[H - icP - (c^2/3)Q] + O\left(\lambda^{-4}\right)\Big\}.\end{aligned} \tag{3.6}$$

The proof of this formula is similar to that in the classical case (see section 1 of Chapter V). Making a gauge transformation (V.1.22)–(V.1.28) which diagonalizes $\mathsf{T}(x,y|\lambda)$, one looks for $D(L,0|\lambda)$ of the following form:

$$D(L,0|\lambda) = \exp\left\{-i\frac{\lambda L}{2}\sigma_z\right\}\left[1 + A_1\lambda^{-1} + A_2\lambda^{-2} + A_3\lambda^{-3} + O\left(\lambda^{-4}\right)\right] \tag{3.7}$$

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and, using (V.1.27) one finds that

$$A_1 = -:\int_0^L W_1(z)\,dz: \tag{3.8}$$

$$A_2 = -:\int_0^L W_2(z)\,dz: + :\int_0^L W_1(z)\,dz \int_0^z W_1(y)\,dy: \tag{3.9}$$

$$\begin{aligned} A_3 = &-:\int_0^L W_3(z)\,dz: + :\int_0^L W_2(z)\,dz \int_0^z W_1(y)\,dy: \\ &+ :\int_0^L W_1(z)\,dz \int_0^z W_2(y)\,dy: \\ &- :\int_0^L W_1(z)\,dz \int_0^z W_1(y)\,dy \int_0^y W_1(t)\,dt: \end{aligned} \tag{3.10}$$

Here, the W_i are the same as in the classical case (see V.1.28). Due to periodic boundary conditions, we have $G(0) = G(L)$ and $\tau(\lambda) = \operatorname{tr} D(L,0|\lambda)$. As $D_{22} \ll D_{11}$ for $\lambda \to i\infty$, one has that $\tau(\lambda) = D_{11}(L,0|\lambda)$ $(\lambda \to i\infty)$. Taking the logarithm one gets:

$$\begin{aligned} \ln\left(e^{i\lambda L/2}\tau(\lambda)\right) \underset{\lambda\to i\infty}{\longrightarrow} &\ln\left(1 + a_1\lambda^{-1} + a_2\lambda^{-2} + a_3\lambda^{-3}\right) \\ &= \lambda^{-1}b_1 + \lambda^{-2}b_2 + \lambda^{-3}b_3. \end{aligned} \tag{3.11}$$

Here, by a_i we denote the element in position 11 of the corresponding matrix A_i (see (3.7)–(3.10)), and

$$\begin{aligned} &b_1 = a_1; \qquad b_2 = a_2 - \frac{a_1^2}{2}; \\ &b_3 = a_3 - \frac{a_1 a_2 + a_2 a_1}{2} + \frac{a_1^3}{3}. \end{aligned} \tag{3.12}$$

The quantities a_1, a_2, a_3 and b_1 are normal ordered, but the quantities b_2 and b_3 are not. Reducing b_2 and b_3 to the normal ordered form, one obtains "quantum corrections" (differences between the quantum and the classical trace identities):

$$\begin{aligned} &b_1 = icQ; \qquad b_2 = icP + \frac{c^2}{2}Q; \\ &b_3 = icH + c^2 P - i\frac{c^3}{3}Q. \end{aligned} \tag{3.13}$$

Thus we obtain the quantum trace identities (3.6). When constructing

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the Hamiltonian of the lattice nonlinear Schrödinger model (see Chapter VIII, section 3), we shall return to the quantum trace identities.

Let us now calculate the R-matrix, which gives the commutation relations between matrix elements of the monodromy matrix. The calculation will be obvious if we use the infinitesimal lattice formulation instead of the continuous one. The L-operator is given by the same classical formula ((V.1.19) and (V.1.20)):

$$\mathsf{L}(n|\lambda) = \begin{pmatrix} 1 - i\frac{\lambda\Delta}{2} & -i\sqrt{c}\Psi_n^\dagger\Delta \\ i\sqrt{c}\Psi_n\Delta & 1 + i\frac{\lambda\Delta}{2} \end{pmatrix} + O\left(\Delta^2\right) \tag{3.14}$$

where the quantum operators Ψ_n, $\Psi_n^\dagger$ satisfy

$$[\Psi_n,\, \Psi_m^\dagger] = \frac{1}{\Delta}\delta_{nm}. \tag{3.15}$$

Elementary calculations show that the bilinear relation (1.4) is valid and the R-matrix is the same as (2.3):

$$\mathsf{R}(\lambda, \mu) = \Pi - i\frac{c}{\lambda - \mu}E \tag{3.16}$$

or, in explicit form,

$$\mathsf{R}(\lambda, \mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix} \tag{3.17}$$

where

$$f(\mu,\lambda) = 1 + \frac{ic}{\mu - \lambda}, \qquad g(\mu,\lambda) = \frac{ic}{\mu - \lambda}. \tag{3.18}$$

This R-matrix will be called the R-matrix of the XXX model (the XXX R-matrix).

VI.4 Quantum field theory models

In this section some examples of quantum field theory models solvable by QISM will be presented.

(1) The quantum sine-Gordon (SG) model is given by the same Hamiltonian (V.3.2) as in the classical case (see section 3 of Chapter V). The fields $\pi(x)$ and $u(x)$ are now quantum bosonic operators with canonical commutation relations $[u(x),\, \pi(y)] = i\delta(x-y)$. The lattice quantities u_n and p_n are also canonical operators:

$$[u_n,\, p_m] = i\delta_{mn}. \tag{4.1}$$

The correct version of the infinitesimal quantum L-operator of the model,

$$\mathsf{L}(n|\lambda) = \begin{pmatrix} \exp(-i\beta p_n/4) & (m\Delta/2)\sinh(\lambda - i\beta u_n/2) \\ (-m\Delta/2)\sinh(\lambda + i\beta u_n/2) & \exp(i\beta p_n/4) \end{pmatrix} + O\left(\Delta^2\right) \tag{4.2}$$

was introduced in [54]. This formula will be clarified in Chapter VIII, section 2. It is interesting to note that in the classical case this L-operator can also be used. The difference between the quantum and classical cases is the following: in the classical case p_n can be considered as bounded and $\exp\{i\beta p_n/4\}$ can be replaced by $1 + i\beta p_n/4$. This is impossible in the quantum case, because p_n is an unbounded operator.

The quantum R-matrix is

$$\mathsf{R}(\lambda,\mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix} \tag{4.3}$$

where the functions f and g are

$$f(\mu,\lambda) = \frac{\sinh(\mu - \lambda - i\gamma)}{\sinh(\mu - \lambda)}; \qquad g(\mu,\lambda) = -i\frac{\sin\gamma}{\sinh(\mu - \lambda)} \tag{4.4}$$

and

$$0 < \gamma \equiv \beta^2/8 < \pi. \tag{4.5}$$

This R-matrix will be called the R-matrix of the XXZ model (XXZ R-matrix).

The monodromy matrix possesses the involution

$$\sigma_y \mathsf{T}^*(\lambda^*)\sigma_y = \mathsf{T}(\lambda). \tag{4.6}$$

Later on (see Chapter VIII), we shall put the model on a lattice while preserving the property of complete integrability; that is, we will construct the exact L-operator intertwined by the R-matrix (4.3), (4.4). This will be the direct solution of the ultraviolet divergence problem.

(2) The Zhiber-Shabat-Mikhailov model in the quantum case is given by the Hamiltonian (V.3.11) with

$$[u(y),\, \pi(x)] = i\delta(x - y). \tag{4.7}$$

The nonzero elements of the 9×9 R-matrix are given by (setting $\varphi = \gamma/8$, $\beta = \lambda - \mu$ and $s = e^{-3\beta+5i\varphi} - e^{3\beta-5i\varphi} + e^{-i\varphi} - e^{i\varphi}$):

$$s\mathsf{R}^{11}_{11} = -e^{3\beta-3i\varphi} + e^{-3\beta+3i\varphi} + e^{5i\varphi} - e^{-5i\varphi} - e^{3i\varphi} + e^{-3i\varphi} - e^{i\varphi} + e^{-i\varphi};$$

$$s\mathsf{R}^{11}_{22} = s\mathsf{R}^{33}_{11} = e^{-2\beta+5i\varphi} - e^{-2\beta+i\varphi} + e^{\beta-i\varphi} - e^{\beta-5i\varphi};$$

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$$
\begin{aligned}
&\mathsf{R}_{22}^{22} = \mathsf{R}_{33}^{33} = 1;\\
&s\mathsf{R}_{33}^{11} = s\mathsf{R}_{11}^{22} = e^{2\beta - i\varphi} - e^{2\beta - 5i\varphi} + e^{-\beta + 5i\varphi} - e^{-\beta + i\varphi};\\
&s\mathsf{R}_{33}^{22} = e^{-2\beta + 5i\varphi} - e^{-2\beta - 3i\varphi} - e^{\beta + i\varphi} + e^{\beta - i\varphi} + e^{\beta - 3i\varphi} - e^{\beta - 5i\varphi};\\
&s\mathsf{R}_{22}^{33} = e^{2\beta + 3i\varphi} - e^{2\beta - 5i\varphi} + e^{-\beta + 5i\varphi} - e^{-\beta + 3i\varphi} + e^{-\beta - i\varphi} - e^{-\beta + i\varphi};\\
&s\mathsf{R}_{21}^{12} = s\mathsf{R}_{12}^{21} = s\mathsf{R}_{31}^{13} = s\mathsf{R}_{13}^{31} = -e^{3\beta - 3i\varphi} + e^{-3\beta + 3i\varphi} - e^{3i\varphi} + e^{-3i\varphi};\\
&s\mathsf{R}_{13}^{12} = s\mathsf{R}_{31}^{21} = e^{2\beta} - e^{2\beta - 4i\varphi} - e^{-\beta} + e^{-\beta - 4i\varphi};\\
&s\mathsf{R}_{12}^{13} = s\mathsf{R}_{21}^{31} = e^{-2\beta + 4i\varphi} - e^{-2\beta} - e^{\beta + 4i\varphi} + e^{\beta};\\
&s\mathsf{R}_{32}^{23} = s\mathsf{R}_{23}^{32} = -e^{3\beta - i\varphi} + e^{-3\beta + i\varphi} + e^{-i\varphi} - e^{i\varphi}.
\end{aligned}
\tag{4.8}
$$

This R-matrix is an example of a nontrivial solution of the Yang-Baxter equation.

Let us also mention another curious solution of the Yang-Baxter equation ([8] and [31]). This is an R-matrix which depends on a three-dimensional spectral parameter. In this case, we can represent the spectral parameter as a 2×2 matrix with unit determinant

$$
\widehat{\lambda} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}, \qquad \det \widehat{\lambda} = 1.
$$

In other words, $\widehat{\lambda} \in SL(2, \mathbb{C})$. The R-matrix depends on the ordered ratio of two spectral parameters:

$$
\mathsf{R}(\widehat{\lambda}, \widehat{\mu}) = \mathsf{R}(\widehat{\lambda}\, \widehat{\mu}^{-1}).
$$

Let us define $\widehat{g} = \widehat{\lambda}\, \widehat{\mu}^{-1}$. The nonzero matrix elements of R can now be written in terms of the matrix elements of

$$
\widehat{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} :
$$

$$
\begin{aligned}
&\mathsf{R}_{11}^{11} = g_{11}; &\qquad &\mathsf{R}_{22}^{22} = g_{22};\\
&\mathsf{R}_{21}^{12} = i g_{12}; &\qquad &\mathsf{R}_{12}^{21} = i g_{21};\\
&\mathsf{R}_{22}^{11} = \mathsf{R}_{11}^{22} = 1 :
\end{aligned}
$$

The most interesting example is quantization of the nonabelian Toda lattice, see [40].

VI.5 Fundamental spin models

Spin models present another set of examples solved by QISM. Their L-operators are constructed directly from R-matrices satisfying the Yang-Baxter equation. First, we shall construct the L-operator and then the monodromy and transfer matrices. Then, by means of trace identities, we shall construct the Hamiltonian.

Let us consider a one-dimensional lattice with M sites, and let n number the sites ($1 \leq n \leq M$). One begins the construction of the spin model connected with a given $k^2 \times k^2$ R-matrix by constructing the corresponding L-operator. This L-operator is a $k \times k$ matrix with matrix elements $\mathsf{L}_{\alpha\beta}(n|\lambda)$ (α, $\beta = 1, \ldots, k$) which are themselves $k \times k$ matrices (quantum operators):

$$\begin{aligned} &\mathsf{L}^{ij}_{\alpha\beta}(n|\lambda) = \mathsf{R}^{i\beta}_{\alpha j}(\lambda,\nu)\varphi(\lambda,\nu) = \widetilde{\mathsf{R}}^{\alpha\beta}_{ij}(\lambda,\nu)\varphi(\lambda,\nu),\\ &(\alpha,\beta,i,j = 1,\ldots,k). \end{aligned} \tag{5.1}$$

Here $\varphi(\lambda,\nu)$ is an arbitrary c-number function. It is possible to include φ because all the relations are linear. Sometimes it is convenient to choose φ in a nontrivial way. This L-operator is essentially the R-matrix (1.9) at some fixed point $\mu = \nu$. The indices α, β are called matrix indices, while i, j are quantum indices. The matrix indices originate from the classical L-operator. The quantum space number is labeled by the index n. The Yang-Baxter equation can be rewritten with the help of (5.1) as a bilinear relation:

$$\mathsf{R}(\lambda,\mu)\left(\mathsf{L}(n|\lambda)\bigotimes\mathsf{L}(n|\mu)\right) = \left(\mathsf{L}(n|\mu)\bigotimes\mathsf{L}(n|\lambda)\right)\mathsf{R}(\lambda,\mu). \tag{5.2}$$

Thus, the L-operator is constructed directly from the R-matrix. The equality of dimensions of the quantum space and the matrix space is a distinctive property of these models. The models defined by these L-operators are called fundamental. Relation (5.2) can be generalized by multiplying the R-matrix by an arbitrary complex-valued function.

The further investigation of spin models is analogous to that of section 1. The monodromy matrix $\mathsf{T}(\lambda)$ and the transfer matrix $\tau(\lambda)$ are constructed as explained there. In the present case, $\tau(\lambda)$ is a quantum operator acting in the tensor product of M different quantum spaces. The Hamiltonian is defined by the trace identities:

$$H = \text{const}\, \frac{d}{d\lambda} \ln \tau(\lambda)\Big|_{\lambda=\nu}. \tag{5.3}$$

The point $\lambda = \nu$ is chosen in such a way that $\mathsf{R}(\nu,\nu) = E$; thus, the Hamiltonian describes the interaction of nearest neighbors on a lattice and is easily calculated:

$$H = \text{const} \sum_{n=1}^{M} H_{n-1,n}. \tag{5.4}$$

The operator $H_{n-1,n}$ acts nontrivially only at the two neighboring sites (n and $n-1$):

$$(H_{n-1,n})^{\{i\}}_{\{k\}} = \left(\frac{d}{d\lambda}\mathsf{R}^{i_{n-1}k_{n-1}}_{i_n k_n}(\lambda,\nu)\right)\Big|_{\lambda=\nu} + \frac{d}{d\lambda}\ln\varphi(\lambda,\nu)\Big|_{\lambda=\nu}, \tag{5.5}$$

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while at sites $j \neq n-1$, n it acts trivially. The second term in (5.5) is evident. Let us explain that the first term in (5.5) is presented correctly. The proof of these formulæ is based on the fact that $\tau(\nu)$ is the shift operator on one lattice site (provided that $\mathsf{R}(\nu,\nu)=E$):

$$(\tau(\nu))^{\{i\}}_{\{k\}} = \prod_{j=1}^{M} \delta_{i_j,k_{j-1}}, \qquad (M+1\equiv 1). \tag{5.6}$$

Let us discuss the Lax representation of the equations of motion which are generated by the above Hamiltonian. This representation exists and the explicit form of the time evolution operator can be obtained from the general formulæ (1.15), (1.16), (1.17) and (1.23):

$$\begin{aligned}[U_{\alpha\beta}(n|\lambda,\nu)]^{\{i\}}_{\{k\}} = i\frac{d}{d\mu}\Big\{&\delta_{\alpha\beta}\,[\mathsf{R}(\mu,\nu)]^{i_{n-1}k_{n-1}}_{i_n k_n} \\ &-\left[\mathsf{R}^{-1}(\nu,\lambda)\right]^{i_{n-1}a}_{\alpha b}[\mathsf{R}(\mu,\nu)]^{bc}_{i_n k_n}\,[\mathsf{R}(\mu,\lambda)]^{ak_{n-1}}_{c\beta}\Big\}\Big|_{\mu=\nu}. \end{aligned} \tag{5.7}$$

As a quantum operator, it acts nontrivially on two neighboring sites.

The function $\varphi(\lambda,\nu)$ controls the dependence of the above construction on the normalisation of the R-matrix. The starting point above was $\mathsf{R}(\nu,\nu)=E$. We shall use different normalizations, φ, in the examples which follow.

Let us consider the R-matrix of the XXX model (3.16)–(3.18) as an example. It is convenient, instead, to use the matrix $\widetilde{\mathsf{R}}$ (1.9):

$$\widetilde{\mathsf{R}}_{12}(\lambda,\mu) = E_{12} - \frac{ic}{\lambda-\mu}\Pi_{12} \tag{5.8}$$

(the subscripts 1, 2 denote the vector spaces). We can rewrite this R-matrix as

$$\widetilde{\mathsf{R}}_{12}(\lambda,\mu) = \left[1-\frac{ic}{2(\lambda-\mu)}\right]E_{12} - \frac{ic}{2(\lambda-\mu)}\sum_{i=x,y,z}\sigma_i^1\sigma_i^2 \tag{5.9}$$

where we have used the identity

$$2\Pi_{12} = I^1 I^2 + \sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \sigma_z^1\sigma_z^2. \tag{5.10}$$

(Here σ are the Pauli matrices, with superscripts denoting the space upon which they act. In both (5.9) and (5.10), we have a tensor product of σ matrices.)

The corresponding L-operator (5.1) is

$$\mathsf{L}(n|\lambda) = \lambda - \frac{ic}{2}\begin{pmatrix}\sigma_z^n & 2\sigma_-^n \\ 2\sigma_+^n & -\sigma_z^n\end{pmatrix} \tag{5.11}$$

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where $2\sigma_\pm = \sigma_x \pm i\sigma_y$. This can be rewritten as

$$\mathsf{L}(n|\lambda) = \lambda - \frac{ic}{2}\sigma_z\sigma_z^n - ic\left(\sigma_-\sigma_+^n + \sigma_+\sigma_-^n\right). \tag{5.12}$$

Usually, when discussing magnets, we shall assume that $c = 1$, which can always be achieved by rescaling the spectral parameter.

To obtain this L-operator, we must multiply the R-matrix (5.8) by $\varphi(\lambda, \nu) = (\lambda - \mu)$, use (5.1) and put $\nu = -i/2$. It should be noted that if $\mathsf{R}(\lambda, \mu)$ satisfies the Yang-Baxter equation, then $(\lambda - \mu)\mathsf{R}(\lambda, \mu)$ does also. The monodromy matrix T generated by this L-operator possesses the property

$$\mathsf{T}^*(\lambda^*) = \sigma_y \mathsf{T}(\lambda)\sigma_y \tag{5.13}$$

where the asterisk denotes Hermitian conjugation of quantum operators (without matrix transposition) and complex conjugation of c-numbers. The spin at the k-th site is described by the matrices σ^k (the quantum operators). The trace identities are

$$H = \pm 2i\frac{d}{d\mu}\ln\tau(\mu)\bigg|_{\mu=-i/2} \pm 2M - 2hS_z. \tag{5.14}$$

Thus, the Hamiltonian of the isotropic XXX model considered in Chapter II is obtained:

$$H = \mp\sum_{k=1}^{M}\left(\sigma_x^k\sigma_x^{k+1} + \sigma_y^k\sigma_y^{k+1} + \sigma_z^k\sigma_z^{k+1} - 1\right) - 2hS_z. \tag{5.15}$$

The upper sign corresponds to the ferromagnetic case and the lower one to the antiferromagnetic case. S_z is the operator of the third component of the total spin:

$$S_z = \frac{1}{2}\sum_{n=1}^{M}\sigma_z^n \tag{5.16}$$

which commutes with the Hamiltonian and the transfer matrix:

$$[H, S_z] = [\tau(\lambda), S_z] = 0. \tag{5.17}$$

Let us now consider the anisotropic XXZ magnet. The Hamiltonian (see Chapter II)

$$H = -\sum_{k=1}^{M}\left[\sigma_x^k\sigma_x^{k+1} + \sigma_y^k\sigma_y^{k+1} + \Delta\left(\sigma_z^k\sigma_z^{k+1} - 1\right) + h\sigma_z^k\right] \tag{5.18}$$

is generated by the monodromy matrix of the fundamental spin model, which is constructed from the XXZ R-matrix (4.3). The trace identities are

$$H = 2i(\sin 2\eta)\frac{\partial}{\partial\mu}\ln\tau(\mu)\bigg|_{\mu=i\pi/2-i\eta} + 2M\cos 2\eta - 2hS_z. \tag{5.19}$$

Here the coupling constant η, related to γ in the R-matrix (4.3), (4.4), is defined as

$$\gamma = -2\eta \tag{5.20}$$

where the anisotropy parameter is $\Delta = \cos 2\eta$. The operator S_z commutes with the Hamiltonian and the transfer matrix as before. The L-operator is constructed by means of the R-matrix and is given by

$$\mathsf{L}(n|\lambda) = -i\begin{pmatrix} \cosh(\lambda - i\eta\sigma_z^n) & \sigma_-^n \sin 2\eta \\ \sigma_+^n \sin 2\eta & \cosh(\lambda + i\eta\sigma_z^n) \end{pmatrix}. \tag{5.21}$$

The corresponding monodromy matrix satisfies the involution

$$\sigma_y \mathsf{T}(\lambda)\sigma_y = \mathsf{T}^*(\lambda^* + i\pi). \tag{5.22}$$

The isotropic XXX magnet may be considered as the limiting case of the XXZ magnet. In the ferromagnetic case, the Hamiltonian (5.15), L-operator (5.11) and R-matrix (3.17), (3.18) are obtained from the corresponding quantities in the XXZ case (5.18), (5.21), (4.3), (4.4) in the limit (for even M)

$$2\eta = \kappa \to 0; \qquad \lambda_{XXZ} = i\frac{\pi}{2} + \kappa\lambda_{XXX} \tag{5.23}$$

(λ_{XXX} finite, $c = 1$). In this case, the trace identities (5.19) reproduce formula (5.14).

The following comment must be made in conclusion. We can shift the spectral parameter in the L-operator (5.11) by a complex number ν_n which may depend on the number n of the site, passing to the operator $\lambda(n|\lambda - \nu_n)$ (see (2.5)). This shift corresponds to the inhomogeneous XXX model since the L's differ at each site of the lattice. This L-operator also satisfies the bilinear relation (5.2). The corresponding monodromy matrix is

$$\mathsf{T}(\lambda) = \mathsf{L}(M|\lambda - \nu_M)\cdots\mathsf{L}(1|\lambda - \nu_1). \tag{5.24}$$

VI.6 Fundamental models of classical statistical physics

This section is an introduction to classical statistical physics on a lattice. The coverage will be brief and will only touch on the construction of the partition function. For the interested reader, we recommend the excellent book by Baxter [5]. Here we shall introduce the extremely important

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partition function of the six-vertex model with domain wall boundary conditions. In section VII.10 it will be represented as a determinant. Finally this will lead to the solution of the problem of correlation functions.

Any R-matrix satisfying the Yang-Baxter equation generates an exactly solvable model of classical statistical physics on a two-dimensional lattice. The partition function of this model can be calculated by means of QISM. Let us consider the $N \times M$ square lattice (see Figure VI.1). Vertical lines will be enumerated by Latin indices ($k = 1, \dots, M$) while horizontal lines will be labeled by Greek indices ($\alpha = 1, \dots, N$). Each spin variable is situated on a lattice edge (link) and takes k different values (this k should not be confused with the index of the vertical lines). The spin variables will be denoted a and b, with subscripts indicating the number of the line carrying them: a_k, b_k, a_α, b_α.

The statistical weight $\mathsf{L}_{\alpha k}$ is associated with the intersection of the α-th and k-th lines and depends on the values of the spin variables situated on the adjoining edges of the vertex:

$$(\mathsf{L}_{\alpha k}(\lambda_\alpha, \nu_k))_{a_\alpha b_\alpha}^{a_k b_k} \tag{6.1}$$

Here λ_α is the spectral parameter (or rapidity) associated with the horizontal line α, and ν_k is the other spectral parameter, associated with the vertical line k (see Figure VI.2). Let us associate the weight with the spin L-operator (6.1) (see also (5.1)):

$$(\mathsf{L}_{\alpha k}(\lambda_\alpha, \nu_k))_{a_\alpha b_\alpha}^{a_k b_k} = \mathsf{R}_{a_\alpha b_k}^{a_k b_\alpha}(\lambda_\alpha, \nu_k)\varphi(\lambda_\alpha, \nu_k) = \widetilde{\mathsf{R}}_{a_k b_k}^{a_\alpha b_\alpha}(\lambda_\alpha, \nu_k)\varphi(\lambda_\alpha, \nu_k) \tag{6.2}$$

where α and k denote the spaces where $\widetilde{\mathsf{R}}$ acts nontrivially. The function $\varphi(\lambda, \nu)$ controls the normalization of the R-matrix. The partition function is equal to

$$Z = \sum_{\text{spins}} \prod_{\text{vertices}} (\mathsf{L}_{\alpha k}(\lambda_\alpha, \nu_k))_{a_\alpha b_\alpha}^{a_k b_k} \tag{6.3}$$

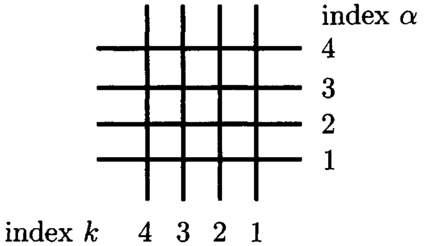

Figure VI.1

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where the product is taken over all vertices of the lattice and the summation is over all values of the spins, taken independently. For a finite lattice, the partition function depends on the values of the spins on the boundary of the lattice. Usually periodic boundary conditions are imposed. We shall also consider some different types of boundary conditions (domain wall boundary conditions).

The R-matrix (4.3), (4.4) generates the well known ice model. The model associated with the R-matrix (3.17), (3.18) is a special case of the ice model (isotropic in spin space). Solutions for many similar models have been found. We shall not discuss them in detail, but only mention that Baxter's model on the honeycomb lattice [6] is a special case of a model generated by the R-matrix (4.8).

Let us construct a special partition function which plays an important role in the calculation of correlation functions. The statistical weight is given by the L-operator, (5.11), (5.12) of the XXX model (5.9):

$$\mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k) = \lambda_\alpha - \nu_k - i\frac{c}{2}\sigma_z^\alpha \sigma_z^k - ic\left(\sigma_-^\alpha \sigma_+^k + \sigma_+^\alpha \sigma_-^k\right) \tag{6.4}$$

where α and k denote the corresponding spaces. In this case, each spin variable takes only two values: ± 1. Following tradition, we shall denote them by up and down arrows $\uparrow$ $(+1)$ or $\downarrow$ (-1). Consider the square $N \times N$ lattice. Each spectral parameter λ_α corresponds to a horizontal line, and each ν_k to a vertical one (see Figure VI.3).

The statistical weight $\mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k)$ (6.4) corresponds to the vertex at the intersection of these lines. The partition function is given by (6.3) with special boundary conditions, which we shall call domain wall bound-

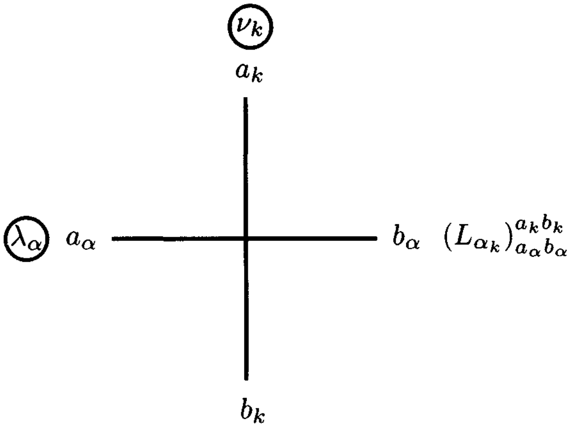

Figure VI.2

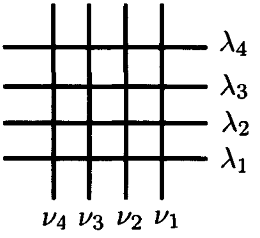

Figure VI.3

ary conditions (see Figure VI.4). On the upper and right boundaries, the spins are down ($\downarrow_k$ and $\downarrow_\alpha$), while on the lower and left boundaries the spins are up ($\uparrow_k$ and $\uparrow_\alpha$) (see Figure VI.4). Thus, the partition function $Z_N(\{\lambda_\alpha\}, \{\nu_k\})$ depends on $2N$ arguments:

$$Z_N(\{\lambda_\alpha\}, \{\nu_k\}) = \left\{ \prod_{\beta=1}^{N} \uparrow_\beta \right\} \left\{ \prod_{j=1}^{N} \downarrow_j \right\} \\ \times \left\{ \prod_{\alpha=1}^{N} \prod_{k=1}^{N} \mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k) \right\} \left\{ \prod_{\beta=1}^{N} \downarrow_\beta \right\} \left\{ \prod_{j=1}^{N} \uparrow_j \right\}. \quad (6.5)$$

It must be mentioned that the L-operator (6.2) is a matrix in the "vertical" space (a_k, b_k) as well as in the "horizontal" space (a_α, b_α); hence, the summation in (6.3) can be interpreted as matrix multiplication. Here the double product indicates space-ordering:

$$\prod_{\alpha=1}^{N} \prod_{k=1}^{N} \mathsf{L}_{\alpha k} \equiv (\cdots \mathsf{L}_{23}\mathsf{L}_{22}\mathsf{L}_{21})(\cdots \mathsf{L}_{13}\mathsf{L}_{12}\mathsf{L}_{11}).$$

Thus defined, Z_N is called the partition function on the inhomogeneous lattice [4]. (This quantity is also called the partition function of the isotropic 6-vertex model.)

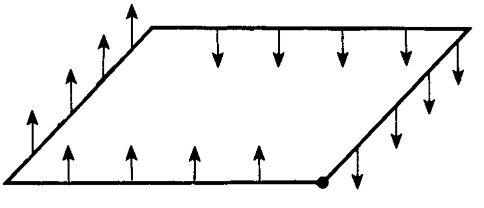

Figure VI.4

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The product of L-operators in (6.5) may be rearranged as

$$Z_N = \left\{ \prod_{j=1}^{N} \downarrow_j \right\} \left\{ \prod_{\beta=1}^{N} \uparrow_\beta \right\} \\ \times \left\{ \prod_{k=1}^{N} \prod_{\alpha=1}^{N} \mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k) \right\} \left\{ \prod_{j=1}^{N} \uparrow_j \right\} \left\{ \prod_{\beta=1}^{N} \downarrow_\beta \right\}. \tag{6.6}$$

Let us express the partition function using the monodromy matrix (5.26):

$$\mathsf{T}_\alpha(\lambda) = \prod_{k=1}^{N} \mathsf{L}_{\alpha k}(\lambda - \nu_k) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{6.7}$$

where the product is taken as space-ordered as above. Here $\mathsf{T}_\alpha(\lambda)$ is a 2×2 matrix in the α-th space; $\mathsf{T}_\alpha(\lambda)$ is also the quantum operator in the spaces $k = 1, \ldots, N$. The operator $\mathsf{L}_{\alpha k}$ is intertwined by $\widetilde{\mathsf{R}}$ (5.8), (5.9) in each space (α or k):

$$\widetilde{\mathsf{R}}_{\alpha\beta}(\lambda_\alpha, \lambda_\beta) \mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k) \mathsf{L}_{\beta k}(\lambda_\beta - \nu_k) \\ = \mathsf{L}_{\beta k}(\lambda_\beta - \nu_k) \mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k) \widetilde{\mathsf{R}}_{\alpha\beta}(\lambda_\alpha, \lambda_\beta). \tag{6.8}$$

$$\widetilde{\mathsf{R}}_{kl}(\nu_l, \nu_k) \mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k) \mathsf{L}_{\alpha l}(\lambda_\alpha - \nu_l) \\ = \mathsf{L}_{\alpha l}(\lambda_\alpha - \nu_l) \mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k) \widetilde{\mathsf{R}}_{kl}(\nu_l, \nu_k). \tag{6.9}$$

Hence, for $\mathsf{T}_\alpha(\lambda)$ we have

$$\widetilde{\mathsf{R}}_{\alpha\beta}(\lambda, \mu) \mathsf{T}_\alpha(\lambda) \mathsf{T}_\beta(\mu) = \mathsf{T}_\beta(\mu) \mathsf{T}_\alpha(\lambda) \widetilde{\mathsf{R}}_{\alpha\beta}(\lambda, \mu) \tag{6.10}$$

with the same R-matrix. Here α and β denote vector spaces. Now we can rewrite Z_N in the form

$$Z_N = \left\{ \prod_{j=1}^{N} \downarrow_j \right\} \left\{ \prod_{\alpha=1}^{N} (\uparrow_\alpha) \, \mathsf{T}_\alpha(\lambda_\alpha) \, (\downarrow_\alpha) \right\} \left\{ \prod_{j=1}^{N} \uparrow_j \right\}. \tag{6.11}$$

Using (6.7) we see that

$$(\uparrow_\alpha) \, \mathsf{T}_\alpha(\lambda) \, (\downarrow_\alpha) = B(\lambda)$$

and, therefore,

$$Z_N = \left\{ \prod_{j=1}^{N} \downarrow_j \right\} \left\{ \prod_{\alpha=1}^{N} B(\lambda_\alpha) \right\} \left\{ \prod_{j=1}^{N} \uparrow_j \right\}. \tag{6.12}$$

Let us rewrite Z_N with the help of the monodromy matrix in the "orthogonal" direction. Introducing the "vertical" monodromy matrix t_j

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one has (as a space ordered product):

$$\mathsf{t}_j(\nu_j) = \prod_{\alpha=1}^{N} \mathsf{L}_{\alpha j}(\lambda_\alpha - \nu_j) = \begin{pmatrix} A(\nu_j) & B(\nu_j) \\ C(\nu_j) & D(\nu_j) \end{pmatrix} \tag{6.13}$$

which is a 2×2 matrix in the j-th space and quantum operator in the spaces $\alpha = 1, \ldots, N$. We can express Z_N, starting from (6.6), as

$$Z_N = \left\{ \prod_{\beta=1}^{N} \uparrow_\beta \right\} \left\{ \prod_{j=1}^{N} (\downarrow_j)\, \mathsf{t}_j(\nu_j)\, (\uparrow_j) \right\} \left\{ \prod_{\beta=1}^{N} \downarrow_\beta \right\}. \tag{6.14}$$

Due to $(\downarrow_j)\, \mathsf{t}_j(\nu)\, (\uparrow_j) = C(\nu)$ we have

$$Z_N = \left\{ \prod_{\beta=1}^{N} \uparrow_\beta \right\} \left\{ \prod_{j=1}^{N} C(\nu_j) \right\} \left\{ \prod_{\beta=1}^{N} \downarrow_\beta \right\}. \tag{6.15}$$

For the monodromy matrix, we also have

$$\widetilde{\mathsf{R}}_{kl}(\nu_l, \nu_k)\mathsf{t}_k(\nu_k)\mathsf{t}_l(\nu_l) = \mathsf{t}_l(\nu_l)\mathsf{t}_k(\nu_k)\widetilde{\mathsf{R}}_{kl}(\nu_l, \nu_k) \tag{6.16}$$

as a consequence of (6.9). Thus, different representations are obtained for Z_N. Later we shall analyze Z_N as a function of $\{\lambda_\alpha\}$ and $\{\nu_k\}$. This partition function will play an important role in the investigation of correlation functions.

We would like to emphasize that a similar partition function Z_N can be constructed for the XXZ model (5.21), see for example [38].

Conclusion

In this chapter we have seen that the main concepts of the quantum inverse scattering method (QISM) can be obtained by direct quantization of the classical case (see Chapter V). The quantum R-matrix is extremely important. It is easy to evaluate it because one needs only to know the explicit expression for the L-operator. Nevertheless, it is very powerful since it gives the commutation relations with the monodromy matrix (which is highly nontrivial). In the next chapter, we shall use it to construct the algebraic Bethe Ansatz and to reproduce the results of Part I.

We have also explained how to apply the ideas of QISM to different types of models—quantum field theory, spin models, and lattice models of classical statistical physics. In the next chapter, we shall explicitly evaluate the partition function with domain wall boundary conditions (introduced in section 6 of this chapter). It will play an important role later when we study correlation functions, norms and scalar products.

We have stated the main ideas of QISM ([23], [24], [54], [55] and [63]). We also recommend the following reviews and papers: [15], [26], [31], [43], [46], [58], [60] and [61].

The Yang-Baxter relation plays a central role in modern mathematical physics. It is important in the theory of factorized S-matrices ([14], [65]–[70]). The study of the Bethe wave function in [48] (see also [16], [12] and [13]) led to the discovery of the factorization of a many-body scattering matrix into the product of two-body ones. In papers [65], [66] the one-dimensional quantum mechanics of particles with arbitrary statistics interacting via a δ potential was solved and the "nested Bethe Ansatz" was discovered. The Yang-Baxter relation appears there in its present form as the consistency condition for factorization. The Yang-Baxter relation is important also in statistical physics (see [1]–[9], [27], [35], [49], [50], [51] and [64], especially the book by Baxter [5]). The role of the Yang-Baxter relation in QISM can be understood from [10], [18], [19], [23]–[31], [33]–[35], [37]–[47], [50]–[59] and [63]. One can find a splendid collection of pioneering works and references on the subject in the book [36]. Extensive literature is devoted to the search for solutions of the Yang-Baxter equation ([9]–[11], [18], [19], [34], [35], [43]–[47], [49], [52], [68] and [70]). In the paper [53] it was shown that the Yang-Baxter relation for the scattering matrix follows from an infinite set of conservation laws.

QISM was first applied to the nonlinear Schrödinger (NS) model ([23], [55] and [56]). The comparison of classical and quantum results was performed in [47]. For the trace identities, see references [15], [17], [20], [21], [22], [32], [57], [62] and [63].

The sine-Gordon model was treated by QISM in [54]. The quantum L-operator and R-matrix of the Zhiber-Shabat-Mikhailov model were constructed in [29].

Let us mention that the models considered in this chapter by no means exhaust the number of quantum field models solvable by QISM. Due to lack of space, we refer the reader to [5], [7]–[9], [19], [25]–[27], [34], [35], [37], [43], [44], [49]–[52], [59] and [64].

Fundamental spin models are examined in [5] and [58]. An interesting spin Hamiltonian is generated by the R-matrix (4.8) of the Zhiber-Shabat-Mikhailov model. This Hamiltonian describes the interaction of spins in the presence of an external magnetic field. Its explicit form is given in [29]. Later, this model was solved in [51] and [64]. It is connected with a classical statistical physics model on a honeycomb lattice [6].