VIII
Lattice Integrable Models of Quantum Field Theory

Introduction

Lattice variants of integrable models, both classical and quantum, are formulated in the present chapter. The nonlinear Schrödinger (NS) equation and the sine-Gordon (SG) model are considered. QISM makes it possible to put continuous models of field theory on a lattice while preserving the property of integrability. In addition, the explicit form of the R-matrix is kept unchanged; this means that the structure of the action-angle variables for the classical models is unchanged. For quantum models, the analogue is the preservation of the S-matrix (see the end of section VII.7). The critical exponents, which characterize the power-law decay of correlators for large distances, are also preserved. For relativistic models of quantum field theory, lattice models may be used to rigorously solve the problem of ultraviolet divergences. The construction of local Hamiltonians for lattice models in quantum field theory is given much attention in this chapter. It is of interest to note that the L-operators of lattice models depend on some additional parameter Δ (which is absent in the R-matrix). This is the lattice spacing. However, the L-operator can be continued in Δ to the whole complex plane. Based on this fact, the most general L-operator may be constructed which is intertwined by a given R-matrix. This solves the problem of enumerating all the integrable models connected with a given R-matrix.

The preservation of the R-matrix when going to the lattice is the most important feature of the constructed lattice models. Let us also note that the lattice models constructed in this chapter are formulated in terms of initial Bose fields. This makes our approach different from other discretizations ([1], [8], [9] and [21]). Our approach is from the point of view of group theory. Going from the continuum to the lattice, we keep the group (R-matrix) but change the representation (L-operator).

The classical lattice models are constructed in sections 1 and 2. The explicit form of the L-operator on the lattice is obtained. The local Hamiltonians on the lattice are constructed by means of trace identities. The construction is based on the fact that the L-operator becomes a one-dimensional projector at some value of the spectral parameter.

The classical lattice NS equation is constructed in section 1, while the sine-Gordon equation is constructed in section 2.

The quantum discrete NS model is constructed in section 3. The local Hamiltonian on the lattice can be constructed using (as in the classical case) the degeneracy of the L-operator at values of the spectral parameter. The Hamiltonian is an elementary function of local Bose fields.

The L-operator of the quantum discrete NS model is considered in section 4. It depends on the continuous parameter Δ. At special values of this parameter, it is converted into the L-operator of the XXX Heisenberg model of arbitrary spin and is the most general form of the XXX R-matrix. This L-operator exhausts all possible L-operators related to the XXX R-matrix. The connection of the L-operator with representations of the $SU(2)$ algebra is also discussed. The discrete quantum sine-Gordon model is constructed in section 5.

VIII.1 Classical lattice nonlinear Schrödinger equation

Let us start with the NS model. The continuous variant of this model was discussed in section V.1. The lattice model describes the interaction of fields on a lattice Ψ_n, Ψ_n^* (n being the site number), with Poisson brackets $\{\Psi_n, \Psi_m^*\} = i\delta_{nm}/\Delta$, $\{\Psi_n, \Psi_m\} = \{\Psi_n^*, \Psi_m^*\} = 0$. This model is defined by the L-operator

$$\mathsf{L}(n|\lambda) = \begin{pmatrix} 1 - \frac{i\lambda\Delta}{2} + \frac{c\Delta^2}{2}\Psi_n^*\Psi_n & -i\sqrt{c}\Delta\Psi_n^*\rho_n^+ \\ i\sqrt{c}\Delta\rho_n^-\Psi_n & 1 + \frac{i\lambda\Delta}{2} + \frac{c\Delta^2}{2}\Psi_n^*\Psi_n \end{pmatrix}. \tag{1.1}$$

The quantities $\rho^\pm$ are functions of the products of fields Ψ_n^*, Ψ_n only and

$$\rho_n^+\rho_n^- = 1 + \frac{c\Delta^2}{4}\Psi_n^*\Psi_n. \tag{1.2}$$

The Poisson brackets of this L-operator are given by the classical r-matrix (V.2.11), which coincides with that for the continuous case (V.2.16), and the bilinear relation (V.2.11) is now exact. This can be checked by simple calculations.

Though $\rho^\pm$ are not uniquely defined by (1.2), it will prove convenient to use the following form:

$$\rho_n^+ = \rho_n^- = \rho_n = \sqrt{1 + \frac{c\Delta^2}{4}\Psi_n^*\Psi_n}. \tag{1.3}$$

The L-operator (1.1) differs from the continuous one (V.1.19) only at second order in Δ as $\Delta \to 0$. However, the symmetry properties are the same:

$$\sigma_x \mathsf{L}^*(\lambda^*)\sigma_x = \mathsf{L}(\lambda); \qquad \mathsf{L}^T(-\lambda) = \sigma_x \mathsf{L}(\lambda)\sigma_x; \tag{1.4}$$

$$\det \mathsf{L}(n|\lambda) = \frac{\Delta^2}{4}(\lambda-\nu)(\lambda-\nu^*), \quad \nu = -\frac{2i}{\Delta}. \tag{1.5}$$

The monodromy matrix is constructed as usual (V.1.8) and possesses the same symmetry properties: $\sigma_x \mathsf{T}^*(\lambda^*)\sigma_x = \mathsf{T}(\lambda)$, $\det \mathsf{T}(\lambda) = (\det \mathsf{L})^M$ (V.1.18). In section 4 we shall use another representation of $\rho^{\pm}$:

$$\rho_n^+ = 1; \qquad \rho_n^- = 1 + \frac{c\Delta^2}{4}\Psi_n^*\Psi_n. \tag{1.6}$$

Let us now construct the local Hamiltonian using the trace identities. We shall follow the papers [7] and [15]–[17]. Consider the Hamiltonian of the lattice model

$$H = \sum_{n=1}^{M} H_n \tag{1.7}$$

with the Hamiltonian density H_n depending only on the dynamical variables Ψ_j, Ψ_j^* in a certain neighborhood of the n-th site $(n-m \le j \le n+l)$. If, in the limit $M \to \infty$ (interesting for physical applications), $m+l$ remains finite, $m+l \le i$, the Hamiltonian is local and describes the interaction of $i+1$ neighbors (for i the minimal possible value should be taken).

The Hamiltonian can be obtained as a linear combination of logarithmic derivatives of the transfer matrix $\tau(\lambda) = \operatorname{tr} \mathsf{T}(\lambda) = A(\lambda)+D(\lambda)$. Locality is achieved by the following method. At points $\lambda = \nu$, ν^*, $\det \mathsf{L}(n|\lambda) = \det \mathsf{T}(\lambda) = 0$, i.e., the matrices $\mathsf{L}(n|\lambda)$ and $\mathsf{T}(\lambda)$ are proportional to one-dimensional projectors. Namely,

$$\mathsf{L}_{ik}(n|\nu) = \alpha_i(n)\alpha_k^*(n) \tag{1.8}$$

where α is a two-component vector, with components

$$\begin{aligned} \alpha_1(n) &= -i\Delta\sqrt{\frac{c}{2}}\Psi_n^*, \\ \alpha_2(n) &= \sqrt{2\left(1+\frac{c\Delta^2}{4}\Psi_n^*\Psi_n\right)} = \sqrt{2}\rho_n. \end{aligned} \tag{1.9}$$

At $\lambda = \nu^*$, the L-operator is also a one-dimensional projector:

$$\mathsf{L}_{ik}(n|\nu^*) = (\sigma_x\alpha^*(n))_i(\sigma_x\alpha(n))_k. \tag{1.10}$$

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Using this representation, one can easily put the transfer matrix (V.1.9) into factorized form at $\lambda = \nu$, ν^*:

$$\begin{aligned} \tau(\lambda) &= \prod_{n=1}^{M} \tau_n, \\ \tau_n &= (\alpha^*(n+1)\,\alpha(n)), \qquad \alpha(M+1) = \alpha(1), \end{aligned} \tag{1.11}$$

where the parentheses denote the scalar product of two one-dimensional vectors:

$$(\alpha^*\,\alpha) = \alpha_1^*\alpha_1 + \alpha_2^*\alpha_2. \tag{1.12}$$

Thus, the conservation law is local,

$$\ln\tau(\lambda) = \sum_{n=1}^{M} \ln\tau_n, \tag{1.13}$$

and describes the interactions of nearest neighbors. The logarithmic derivatives $d^m \ln\tau(\lambda)/d\lambda^m$ at $\lambda = \nu$, ν^* are also local [15]. The first derivative is

$$\frac{\partial}{\partial\lambda}\ln\tau(\lambda)\bigg|_{\lambda=\nu} = -i\frac{\Delta}{2}\sum_{n=1}^{M}\frac{(\alpha^*(n+1)\sigma_z\alpha(n-1))}{(\alpha^*(n+1)\alpha(n))(\alpha^*(n)\alpha(n-1))}. \tag{1.14}$$

We define the Hamiltonian of the model as follows:

$$\begin{aligned} H = &-\frac{8i}{3c\Delta^4}\frac{\partial}{\partial\lambda}\ln\left[\tau(\lambda)\tau_0^{-1}(\lambda)\right]\bigg|_{\lambda=\nu} \\ &+\frac{8i}{3c\Delta^4}\frac{\partial}{\partial\lambda}\ln\left[\tau(\lambda)\tau_0^{-1}(\lambda)\right]\bigg|_{\lambda=\nu^*} + \frac{4}{3\Delta^2}Q. \end{aligned} \tag{1.15}$$

Here Q is the operator of the number of particles

$$Q = \Delta\sum_{n=1}^{M}\Psi_n^*\Psi_n \tag{1.16}$$

which commutes with $\tau(\lambda)$. The value of $\tau(\lambda)$ at zero field ($\Psi = 0$) is $\tau_0(\lambda)$:

$$\tau_0(\lambda) = \left(1 - i\frac{\lambda\Delta}{2}\right)^M + \left(1 + i\frac{\lambda\Delta}{2}\right)^M. \tag{1.17}$$

However, one can take as the Hamiltonian $\ln\tau(\lambda)$ itself, or some other combination of logarithmic derivatives at ν, ν^*. The demand for the proper continuous limit is a rather weak restriction.

Direct calculation gives the following explicit expression for the Hamiltonian (1.15):

$$H = -\frac{4}{3c\Delta^3}\sum_{n=1}^{M}\Bigg[\frac{(\alpha^*(n+1)\sigma_z\alpha(n-1))}{(\alpha^*(n+1)\alpha(n))(\alpha^*(n)\alpha(n-1))} + \frac{(\alpha(n+1)\sigma_z\alpha^*(n-1))}{(\alpha(n+1)\alpha^*(n))(\alpha(n)\alpha^*(n-1))} + 1 - c\Delta^2\Psi_n^*\Psi_n\Bigg]. \quad (1.18)$$

This Hamiltonian is real and invariant under spatial reflection. It has the same structure of action-angle variables as the corresponding continuous model considered in section V.1. One can easily prove by direct calculation that, as $\Delta \to 0$, (1.18) turns into the continuous Hamiltonian (V.1.11), while the charge (1.16) becomes (V.1.13). This transition is smooth. It is also remarkable that the Hamiltonian is an elementary function of local Bose fields.

Having the Hamiltonian and Poisson brackets, we can easily obtain the nonlinear completely integrable equations of motion (the nonlinearity is of rational nature). The r-matrix and the trace identities are obtained above, the time evolution operator $U(n|\lambda)$ is constructed by the generating functional (V.2.20) and is local. Thus the classical equations of motion can be represented in the Lax form.

VIII.2 Classical lattice sine-Gordon model

The lattice sine-Gordon model is defined by local lattice fields p_n and u_n with Poisson brackets $\{p_n, u_n\} = \delta_{nm}$ and by the L-operator

$$\mathsf{L}(n|\lambda) = \begin{pmatrix} e^{-i\beta p_n/8}\rho_n e^{-i\beta p_n/8} & \frac{m\Delta}{2}\sinh(\lambda - i\beta u_n/2) \\ \frac{-m\Delta}{2}\sinh(\lambda + i\beta u_n/2) & e^{i\beta p_n/8}\rho_n e^{i\beta p_n/8} \end{pmatrix} \quad (2.1)$$

where $\rho_n = \sqrt{1 + 2s\cos\beta u_n}$, $s = (m\Delta/4)^2$, with $0 < s < 1/2$. In the continuum limit ($p_n \to \Delta\pi(x_n)$, $u_n \to u(x_n)$), the L-operator turns into the infinitesimal L-operator (V.3.5). The Poisson brackets of matrix elements of the L-operator (2.1) are given by exactly the same r-matrix (V.3.7) as in the continuous case.

The properties of the monodromy matrix $\mathsf{T}(\lambda)$ are the same as in the continuous case (V.3.9). Considering an even number of sites, we have:

$$\sigma_y \mathsf{T}^*(\lambda^*)\sigma_y = \mathsf{T}(\lambda), \quad (2.2)$$

$$\det\mathsf{T}(\lambda) = \det{}^M\mathsf{L}(\lambda); \qquad \det\mathsf{L}(\lambda) = 1 + 2s\cosh 2\lambda. \quad (2.3)$$

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The Hamiltonian is defined by the trace identities

$$\begin{aligned} H - P &= \frac{m^2\Delta}{4\gamma}\frac{\partial}{\partial e^{2\lambda}}\ln\left[\tau(\lambda)\tau_0^{-1}(\lambda)\right]\Big|_{e^{2\lambda}=-b}; \\ H + P &= \frac{m^2\Delta}{4\gamma}\frac{\partial}{\partial e^{-2\lambda}}\ln\left[\tau(\lambda)\tau_0^{-1}(\lambda)\right]\Big|_{e^{2\lambda}=-b^{-1}}; \end{aligned} \tag{2.4}$$

Here $\tau_0(\lambda)$ is the trace of $\mathsf{T}(\lambda)$ at $p_n = u_n = 0$. The reality of H and P is due to the involution (2.2); b is given by

$$b = 2s\left(1 + \sqrt{1 - 4s^2}\right)^{-1}; \tag{2.5}$$

$\tau(\lambda)$ is a meromorphic function of $\exp(2\lambda)$. At $\exp(2\lambda) = -b^{\pm 1}$, the determinant of $\mathsf{T}(\lambda)$ (2.3) is equal to zero; hence H and P are local. The trace identities (2.4) have the correct continuous limit ($\Delta \to 0$, $s \to 0$, $b \to 0$) (see, for example, reference [2] from the Introduction to Part II). The Hamiltonian (2.4) describes the interaction of three nearest neighbors. It is convenient to write out the explicit form of H and P using the variables

$$\begin{aligned} \chi_n^{\pm} &= \sqrt{\frac{s}{b}}\left(\frac{e^{\pm i\beta u_n/2} + b e^{\mp i\beta u_n/2}}{\rho_n}\right) e^{-i\beta p_n/4}, \\ |\chi_n^{\pm}|^2 &= 1. \end{aligned} \tag{2.6}$$

Then,

$$H \mp P = \frac{m^2\Delta}{4\gamma b(1-b^2)}\sum_{k=1}^{M}\left[\frac{(\chi_{k\mp 1}^{+} - b\chi_{k\pm 1}^{-})(\chi_k^{-} - b\chi_k^{+})}{(1 + \chi_{k\pm 1}^{-}\chi_k^{+})(1 + \chi_{k\mp 1}^{+}\chi_k^{-})} - \frac{(1-b)^2}{4}\right]. \tag{2.7}$$

The phase space of the constructed model is the direct product of toruses. The Hamiltonian and momentum are periodic functions of p_n and u_n with periods $2\pi/\beta$ and $4\pi/\beta$, respectively. Let us define the charge Q by

$$Q = \frac{\beta}{\pi}\sum_{k=1}^{M/2}(u_{2k} - u_{2k-1}). \tag{2.8}$$

The Poisson brackets of Q and $\mathsf{T}(\lambda)$ are easily calculated:

$$\{Q,\, \mathsf{T}(\lambda)\} = i\frac{\gamma}{\pi}[\sigma_z,\, \mathsf{T}(\lambda)]. \tag{2.9}$$

Hence, Q is an integral of motion: $\{Q,\, \tau(\lambda)\} = \{Q,\, H\} = 0$. The continuum limit of the L-operator (2.1) and the Hamiltonian (2.7) of the lattice sine-Gordon model is correct. When $\Delta \to 0$, we obtain (V.3.3). The lattice and continuous models have the same structure of action-angle variables.

VIII.3 Quantum lattice nonlinear Schrödinger equation

The quantum NS model on a lattice is defined by the same L-operator as the classical NS model on a lattice (1.1). However, Ψ_n and $\Psi_n^\dagger$ are now quantum operators with commutation relations $[\Psi_n, \Psi_m^\dagger] = \delta_{nm}/\Delta$. The commutation relations of the lattice L-operator of the NS model (1.1) are given by the same matrix

$$\mathsf{R}(\lambda, \mu) = \Pi - \frac{ic}{\lambda - \mu} I$$

as in the continuous case. Thus, the relation

$$\mathsf{R}(\lambda,\mu)\left(\mathsf{L}(\lambda)\bigotimes \mathsf{L}(\mu)\right) = \left(\mathsf{L}(\mu)\bigotimes \mathsf{L}(\lambda)\right)\mathsf{R}(\lambda,\mu)$$

remains exactly true on the lattice.

Using the arbitrariness in $\rho_n^\pm$ (see (1.2)), we shall fix $\rho_n = \rho_n^+ = \rho_n^-$ as in (1.3). The quantum L-operator possesses symmetry properties which are similar to the classical ones:

$$\sigma_x \mathsf{L}^*(\lambda^*)\sigma_x = \mathsf{L}(\lambda). \tag{3.1}$$

The asterisk here denotes Hermitian conjugation of the matrix elements, without matrix transposition, and complex conjugation of c-numbers. The quantum determinant is

$$\begin{aligned} \det{}_q \mathsf{L}(\lambda) &= \frac{\Delta^2}{4}\left(\lambda - \nu - i\frac{c}{2}\right)\left(\lambda + \nu + i\frac{c}{2}\right), \\ \nu &= -\frac{2i}{\Delta}. \end{aligned} \tag{3.2}$$

The operator of the number of particles is given by

$$Q = \sum_{n=1}^{M} \Delta\, \Psi_n^\dagger \Psi_n, \tag{3.3}$$

and is the same as for the classical model.

Now one has to define the Hamiltonian by means of trace identities. To do this, let us modify the L-operator (1.1) by making it different at even and odd sites of the lattice:

$$\mathsf{L}(n|\lambda) = \begin{pmatrix} \begin{matrix} 1 + (-1)^n\left(\frac{c\Delta}{4}\right) - \frac{i\lambda\Delta}{2} \\ +\frac{c\Delta^2}{2}\Psi_n^\dagger\Psi_n \end{matrix} & -i\sqrt{c}\,\Delta\Psi_n^\dagger\rho_n \\ i\sqrt{c}\,\Delta\rho_n\Psi_n & \begin{matrix} 1 + (-1)^n\left(\frac{c\Delta}{4}\right) + \frac{i\lambda\Delta}{2} \\ +\frac{c\Delta^2}{2}\Psi_n^\dagger\Psi_n \end{matrix} \end{pmatrix}, \tag{3.4}$$

$$\rho_n = \sqrt{1 + (-1)^n \frac{c\Delta}{4} + \frac{c\Delta^2}{4} \Psi_n^\dagger \Psi_n}. \tag{3.5}$$

This L-operator is intertwined by the XXX R-matrix (VI.3.17). The transfer matrix constructed from these L-operators has the correct continuous limit. By multiplying the L-operators from two adjacent sites, we see that when $\Delta \to 0$, the resulting L-operator is a product of two of the usual infinitesimal L-operators (VI.3.14) up to terms of order Δ^2. The following properties of $\mathsf{L}(n|\lambda)$ can be easily established:

$$\sigma_x \mathsf{L}^*(n|\lambda^*)\sigma_x = \mathsf{L}(n|\lambda), \tag{3.6}$$

$$\mathsf{L}^T(n|-\lambda) = \sigma_x \mathsf{L}(n|\lambda)\sigma_x. \tag{3.7}$$

We can invert the quantum L-operator using the quantum determinant (see (VII.8.8)):

$$\mathsf{L}(n|\lambda)\sigma_y \mathsf{L}^T(n|\lambda + ic)\sigma_y = \frac{\Delta^2}{4}(\lambda - \nu_1^{(n)})(\lambda - \nu_2^{(n)}), \tag{3.8}$$

$$\nu_1^{(n)} = -\frac{2i}{\Delta}\left[1 + (-1)^n \frac{c\Delta}{4}\right]; \quad \nu_2^{(n)} = -ic + \frac{2i}{\Delta}\left[1 + (-1)^n \frac{c\Delta}{4}\right] \tag{3.9}$$

The L-operator can be represented (as in the classical case) as a projector (1.8) at the points $\nu_{1,2}^{(n)}$. However, the components of α in this case are noncommuting quantum operators; hence, we must distinguish the "direct" projector

$$\mathsf{L}_{ik} = \alpha_i \beta_k \tag{3.10}$$

from the "inverse" one

$$\mathsf{L}_{ik} = \delta_k \gamma_i. \tag{3.11}$$

To be more precise, let us introduce the two-component vector α (its components being quantum operators). In the quantum case, α looks like the classical version (1.9), but now a dependence on the parity of the site operator appears. At even sites ($n = 0 \pmod 2$)

$$\begin{aligned} \alpha_1(n) &= -i\sqrt{\frac{c}{2}}\Delta \Psi_n^\dagger, \\ \alpha_2(n) &= \sqrt{2\left(1 + \frac{c\Delta^2}{4}\Psi_n^\dagger \Psi_n\right)}, \end{aligned} \tag{3.12}$$

while at odd sites ($n = 1 \pmod 2$)

$$\begin{aligned} \alpha_1(n) &= -i\sqrt{\frac{c}{2}}\Delta\Psi_n^\dagger, \\ \alpha_2(n) &= \sqrt{2\left(1 - \frac{c\Delta}{4} + \frac{c\Delta^2}{4}\Psi_n^\dagger\Psi_n\right)}. \end{aligned} \tag{3.13}$$

Taking λ at the point where the quantum determinant (3.8) vanishes,

$$\nu = -\frac{2i}{\Delta} + \frac{ic}{2}, \tag{3.14}$$

we see that at odd sites, the L-operator is the "direct" projector

$$\mathsf{L}_{ik}(n|\nu) = \alpha_i(n)\alpha_k^\dagger(n) \qquad (n \text{ odd}), \tag{3.15}$$

while at even sites, it is the "inverse" one:

$$\mathsf{L}_{ik}(n|\nu) = \alpha_k^\dagger(n)\alpha_i(n) \qquad (n \text{ even}). \tag{3.16}$$

Taking the conjugated zero of the quantum determinant, $\lambda = \nu^*$, where

$$\nu^* = -\nu = \frac{2i}{\Delta} - \frac{ic}{2}, \tag{3.17}$$

and using (3.6), we have the L-operator as the "inverse" projector at odd sites:

$$\mathsf{L}_{ik}(n|\nu^*) = (\sigma_x\alpha(n))_k(\sigma_x\alpha^\dagger(n))_i \qquad (n \text{ odd}), \tag{3.18}$$

and as the "direct" projector at even sites:

$$\mathsf{L}_{ik}(n|\nu^*) = (\sigma_x\alpha^\dagger(n))_i(\sigma_x\alpha(n))_k \qquad (n \text{ even}). \tag{3.19}$$

We shall consider a lattice with an even number of sites, M. The operator Q (3.3) commutes with the transfer matrix:

$$[\tau(\lambda), Q] = 0. \tag{3.20}$$

The first logarithmic derivative at $\lambda = \nu$ is local. Using the representation

$$\begin{aligned} \tau(\nu) = {} & (\alpha^\dagger(M)\,\alpha(M-1)) \\ & \times \left\{\prod_{n=1}^{M/2-1} (\alpha^\dagger(2n)\,\alpha(2n-1))(\alpha^\dagger(2n+1)\,\alpha(2n))\right\} (\alpha^\dagger(1)\,\alpha(M)). \end{aligned} \tag{3.21}$$

Here the product is ordered from left to right. We can calculate the

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explicit form of the first logarithmic derivative:

$$\frac{\partial}{\partial\lambda}\ln\tau(\lambda)\Big|_{\lambda=\nu} = -i\frac{\Delta}{2}\sum_{n=1}^{M} t_n. \tag{3.22}$$

The local density, t_n, depends on the parity of the site; for odd sites

$$\begin{aligned} t_n = {} & \left(\alpha^\dagger(n+2)\,\alpha(n+1)\right)^{-1} \\ & \times \left\{\left(\alpha^\dagger(n)\,\alpha(n-1)\right)^{-1}\left(\alpha^\dagger(n+1)\,\alpha(n)\right)^{-1}\left(\alpha^\dagger(n+1)\sigma_z\alpha(n-1)\right)\right\} \\ & \times \left(\alpha^\dagger(n+2)\,\alpha(n+1)\right), \end{aligned} \tag{3.23}$$

while for even sites

$$\begin{aligned} t_n = {} & \left(\alpha^\dagger(n-1)\,\alpha(n-2)\right)^{-1} \\ & \times \left\{\left(\alpha^\dagger(n+1)\,\alpha(n)\right)^{-1}\left(\alpha^\dagger(n)\,\alpha(n-1)\right)^{-1}\left(\alpha^\dagger(n+1)\sigma_z\alpha(n-1)\right)\right\} \\ & \times \left(\alpha^\dagger(n-1)\,\alpha(n-2)\right). \end{aligned} \tag{3.24}$$

(We note that the quantum t_n differ from the classical ones (1.14) only by a similarity transformation.) Similar expressions for $\partial\ln\tau(\lambda)/\partial\lambda$ are valid for $\lambda = \nu^*$. It can be proved that

$$\left[\frac{\partial\ln\tau(\lambda)}{\partial\lambda}\Big|_{\lambda=\nu}\right]^\dagger = \frac{\partial\ln\tau(\lambda)}{\partial\lambda}\Big|_{\lambda=\nu^*}, \tag{3.25}$$

$$\widetilde{P}\,\frac{\partial\ln\tau(\lambda)}{\partial\lambda}\Big|_{\lambda=\nu}\widetilde{P} = -\frac{\partial\ln\tau(\lambda)}{\partial\lambda}\Big|_{\lambda=\nu^*}. \tag{3.26}$$

(Here $\widetilde{P}$ is the operator of spatial reflection.) The Hamiltonian of the lattice NS model is defined similarly to the classical one:

$$\begin{aligned} H = {} & -\frac{8i}{3c\Delta^4}\frac{\partial}{\partial\lambda}\ln\left[\tau(\lambda)\tau_0^{-1}(\lambda)\right]\Big|_{\lambda=\nu} + \frac{8i}{3c\Delta^4}\frac{\partial}{\partial\lambda}\ln\left[\tau(\lambda)\tau_0^{-1}(\lambda)\right]\Big|_{\lambda=\nu^*} \\ & + \left(\frac{4}{3\Delta^2(1-\frac{\Delta^2c^2}{16})}\right)Q. \end{aligned} \tag{3.27}$$

Here Q is the operator of the number of particles (3.3), and $\tau_0(\lambda)$ is the transfer matrix at zero field:

$$\begin{aligned} \tau_0(\lambda) = {} & \left(1+\frac{c\Delta}{4}-\frac{i\lambda\Delta}{2}\right)^{M/2}\left(1-\frac{c\Delta}{4}-\frac{i\lambda\Delta}{2}\right)^{M/2} \\ & + \left(1+\frac{c\Delta}{4}+\frac{i\lambda\Delta}{2}\right)^{M/2}\left(1-\frac{c\Delta}{4}+\frac{i\lambda\Delta}{2}\right)^{M/2}. \end{aligned} \tag{3.28}$$

It is interesting to mention that in the continuous limit, the contribution of $c\Delta/4$ can be dropped.

We have thus constructed the local Hamiltonian in terms of initial Bose fields:

$$
\begin{aligned}
H = & -\frac{4}{3c\Delta^3} \sum_{n=1}^{M} \left(t_n + t_n^\dagger + \frac{8 - c\Delta}{8 - 2c\Delta} \right) \\
& + \left(\frac{4}{3\Delta^2 (1 - \frac{\Delta^2 c^2}{16})} \right) \sum_{n=1}^{M} \Delta \Psi_n^\dagger \Psi_n,
\end{aligned} \tag{3.29}
$$

where t_n is given by (3.23), (3.24). The Hamiltonian is Hermitian and possesses the proper continuous limit.

It is clear that the corresponding model is integrable. The Fock vacuum is the pseudovacuum $|0\rangle$ $(\Psi_n|0\rangle = 0)$:

$$
\begin{aligned}
& A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \\
& C(\lambda)|0\rangle = 0;
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
& a(\lambda) = \left(1 + \frac{c\Delta}{4} - i\frac{\lambda\Delta}{2}\right)^{M/2} \left(1 - \frac{c\Delta}{4} - i\frac{\lambda\Delta}{2}\right)^{M/2}; \\
& d(\lambda) = \left(1 + \frac{c\Delta}{4} + i\frac{\lambda\Delta}{2}\right)^{M/2} \left(1 - \frac{c\Delta}{4} + i\frac{\lambda\Delta}{2}\right)^{M/2}.
\end{aligned} \tag{3.31}
$$

The Bethe equations are

$$
\left[\frac{\left(1 + \frac{c\Delta}{4} - i\frac{\lambda_j\Delta}{2}\right)\left(1 - \frac{c\Delta}{4} - i\frac{\lambda_j\Delta}{2}\right)}{\left(1 + \frac{c\Delta}{4} + i\frac{\lambda_j\Delta}{2}\right)\left(1 - \frac{c\Delta}{4} + i\frac{\lambda_j\Delta}{2}\right)} \right]^{M/2}
= \prod_{\substack{k \neq j \\ k=1}}^{N} \left(\frac{\lambda_j - \lambda_k - ic}{\lambda_j - \lambda_k + ic} \right). \tag{3.32}
$$

The eigenvalues of the Hamiltonian can be interpreted in terms of particles:

$$
\begin{aligned}
& E_N = \sum_{k=1}^{N} \varepsilon_0(\lambda_k), \\
& \varepsilon_0(\lambda) = \left(\frac{4}{3\Delta^2 (1 - \frac{\Delta^2 c^2}{16})} \right) - \frac{4}{3\Delta^2} \left[h(\lambda) + \frac{\lambda^2 \Delta^2}{h(\lambda)} \right]^{-1}, \\
& h(\lambda) = 1 - \frac{\Delta^2}{16} \left(4\lambda^2 + c^2 \right).
\end{aligned} \tag{3.33}
$$

The eigenvalue of momentum is

$$p_0(\lambda) = \frac{i}{2\Delta} \ln \left[\frac{\left(1 + \frac{c\Delta}{4} - i\frac{\lambda\Delta}{2}\right)\left(1 - \frac{c\Delta}{4} - i\frac{\lambda\Delta}{2}\right)}{\left(1 + \frac{c\Delta}{4} + i\frac{\lambda\Delta}{2}\right)\left(1 - \frac{c\Delta}{4} + i\frac{\lambda\Delta}{2}\right)} \right]. \tag{3.34}$$

Thus, we have constructed the quantum lattice NS model with a local Hamiltonian. Its distinctive feature is that it is formulated in terms of initial local bose fields. We see that the main idea behind the construction of local quantum Hamiltonians is the same as in the classical case (see section 1).

It must be mentioned that with the help of trace identities (generalizing those for the fundamental spin models), local Hamiltonians of different types may be constructed. This was done for the lattice NS model and for the sine-Gordon model in [27].

VIII.4 Classification of quantum L-operators

It appears that the most general L-operator for the XXX R-matrix can be obtained from the L-operator of the lattice NS model. Let us fix $\rho_n^{\pm}$ in (1.1) as follows: $\rho_n^+ = 1$, $\rho_n^- = 1 + (c\Delta^2/4)\Psi_n^\dagger\Psi_n$ (see (1.6)). This L-operator depends on an arbitrary complex variable Δ. It is easy to introduce three more complex variables. Shifting the spectral parameter in (1.1), we shall pass to the operator $\mathsf{L}(\lambda - \nu)$ (VI.2.5). Multiplying the L-operator by an arbitrary complex number and also multiplying on the left by the matrix $\exp\{\varepsilon\sigma_z\}$ (VII.2.14), (VII.2.15) (ε is an arbitrary parameter), we obtain the following L-operator:

$$\mathsf{L}(n|\lambda) = \begin{pmatrix} a_n^{(1)}(\lambda + ic\Psi_n^\dagger\Psi_n) + a_n^{(0)} & \Psi_n^\dagger \\ c\rho_n\Psi_n & d_n^{(1)}(\lambda - ic\Psi_n^\dagger\Psi_n) + d_n^{(0)} \end{pmatrix} \tag{4.1}$$

where

$$\rho_n = i\left[a_n^{(1)}d_n^{(0)} - a_n^{(0)}d_n^{(1)}\right] + cd_n^{(1)}a_n^{(1)}\Psi_n^\dagger\Psi_n,$$
$$[\Psi_n, \Psi_m^\dagger] = \delta_{nm}, \quad [\Psi_n, \Psi_m] = [\Psi_n^\dagger, \Psi_m^\dagger] = 0.$$

This L-operator possesses the vacuum ($\Psi_n|0\rangle = 0$) and is intertwined by the XXX R-matrix. The vacuum eigenvalues are

$$\begin{aligned} a_n(\lambda) &= a_n^{(1)}\lambda + a_n^{(0)}, \\ d_n(\lambda) &= d_n^{(1)}\lambda + d_n^{(0)}, \end{aligned} \tag{4.2}$$

where $a_n^{(0)}$, $d_n^{(0)}$, $a_n^{(1)}$, $d_n^{(1)}$ are four arbitrary complex parameters.

Theorem 1. *A monodromy matrix with arbitrary rational function $a(\lambda)/d(\lambda)$ is generated by the* L*-operator (4.1).*

Proof: Let us construct the monodromy matrix

$$\mathsf{T}(\mathsf{L}) = \mathsf{L}(M|\lambda)\cdots\mathsf{L}(1|\lambda). \tag{4.3}$$

The numbers a_n and d_n are in general different at each lattice site. Hence, $\mathsf{T}(\lambda)$ is a function of $4M$ arbitrary complex parameters. For $\mathsf{T}(\lambda)$ the ratio $a(\lambda)/d(\lambda)$ (VII.1.9) is equal to

$$\frac{a(\lambda)}{d(\lambda)} = \prod_{n=1}^{M} \frac{a_n^{(1)}\lambda + a_n^{(0)}}{d_n^{(1)}\lambda + d_n^{(0)}}. \tag{4.4}$$

The right hand side can be made equal to any arbitrary rational function since M is arbitrary. The theorem is proved.

It should be noted that monodromy matrices are parametrized by the function $a(\lambda)/d(\lambda)$ (see section VII.6). Using the L-operator (4.1), the monodromy matrix $\mathsf{T}(\lambda)$ with arbitrary analytic function $a(\lambda)/d(\lambda)$ may be constructed. Doing this, one must approximate the analytic functions by rational ones. A function with cuts may also be obtained; this can be dealt with by approximating the cuts by convergent sequences of zeros and poles.

The L-operator of the lattice NS model is the most general one and contains the L-operator of the XXX chain (VI.5.11) as a special case (as well as its generalization to higher spins [20]). This is quite natural since the classical continuous Heisenberg magnet and the NS equation are gauge-invariant (see reference [2] in the Introduction to Part II). To illustrate this, let us use the L-operator (1.1), with

$$\rho_n^+ = \rho_n^- = \sqrt{1 + \frac{c\Delta^2}{4}\Psi_n^\dagger\Psi_n}. \tag{4.5}$$

Or, after a simple transformation:

$$\mathsf{L}(n|\lambda) = -\frac{2}{\Delta}\sigma_z\mathsf{L}(n|\lambda) = i\lambda + c\sum_{l=x,y,z}\sigma_l t_l^{(n)}. \tag{4.6}$$

This L-operator is intertwined by the same XXX R-matrix (since the XXX R-matrix commutes with the matrix $\sigma_z \otimes \sigma_z$). We shall call this L-operator the L-operator of the generalized XXX model. In (4.6) σ_l are

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the Pauli matrices, and the t_l are quantum operators:

$$\begin{aligned} t_1^{(n)} &= \frac{i}{\sqrt{c}}\left(\Psi_n^\dagger \rho_n + \rho_n \Psi_n\right), \\ t_2^{(n)} &= \frac{1}{\sqrt{c}}\left(\rho_n \Psi_n - \Psi_n^\dagger \rho_n\right), \\ t_3^{(n)} &= -\frac{2}{c}\left(\frac{1}{\Delta} + \frac{\Delta c}{2}\Psi_n^\dagger \Psi_n\right). \end{aligned} \tag{4.7}$$

These operators generate a representation of the $SU(2)$ algebra:

$$[t_i,\, t_k] = i\epsilon_{ikl} t_l \tag{4.8}$$

with spin $t^2 = s(s+1)$, $s = -2/c\Delta$. This is the well known Holstein-Primakoff representation [11]. In general, this representation is infinite-dimensional, but for special values of Δ it is finite-dimensional. Namely, s integer or half-integer corresponds to XXX models; higher spins were introduced in [20]. For $s = 1/2$, the operator (4.6) becomes the usual XXX L-operator (VI.5.11).

VIII.5 Quantum lattice sine-Gordon model

The quantum sine-Gordon (SG) model on a lattice allows us to solve the problem of ultraviolet divergences in a rigorous way. The L-operator is given by (2.1):

$$\begin{aligned} &\mathsf{L}(n|\lambda) = \begin{pmatrix} e^{-i\beta p_n/8}\rho_n e^{-i\beta p_n/8} & \frac{m\Delta}{2}\sinh(\lambda - i\beta u_n/2) \\ \frac{-m\Delta}{2}\sinh(\lambda + i\beta u_n/2) & e^{i\beta p_n/8}\rho_n e^{i\beta p_n/8} \end{pmatrix} \\ &\rho_n = \sqrt{1 + 2s\cos\beta u_n}, \quad \beta = \sqrt{8\gamma}, \quad s = \left(\frac{m\Delta}{4}\right)^2, \end{aligned} \tag{5.1}$$

with quantum operators u_n and p_n: $[u_n,\, p_m] = i\delta_{nm}$. This L-operator is intertwined by the XXZ R-matrix exactly; its symmetry properties are analogous to the classical ones (2.2):

$$\sigma_y \mathsf{L}^*(\lambda^*)\sigma_y = \mathsf{L}(\lambda). \tag{5.2}$$

The asterisk here means Hermitian conjugation of the quantum operators. The quantum determinant is

$$\det{}_q \mathsf{L}(\lambda) = 1 + 2s\cosh 2\lambda, \tag{5.3}$$

and is equal to the classical determinant (2.3) (this should be compared with the quantum continuous case (VII.8.15)). The continuous limit of the L-operator is proper (VI.4.2). The construction of different Hamiltonians for the lattice SG model ([2], [3], [5], [6], [17] and [27]) with a proper continuous ($\Delta \to 0$) quasiclassical limit is based on this L-operator.

The model considered above is solved by means of the algebraic Bethe Ansatz. To construct the pseudovacuum, let us use the two-site monodromy matrix $\mathsf{L}_2(n|\lambda) = \mathsf{L}(2n|\lambda)\mathsf{L}(2n-1|\lambda)$. The pseudovacuum is constructed in a similar manner to the continuous case (VII.3.24):

$$|0\rangle_n = \left\{1 - 2s\cos\left[\frac{\beta}{2}(u_{2n}+u_{2n-1})\right]\right\}^{-1/2} \times\delta\left(u_{2n}-u_{2n-1}-\frac{\beta}{4}+\frac{2\pi}{\beta}\right) \tag{5.4}$$

(in addition to the pseudovacuum $|0\rangle_n$, the operator L_2 also possesses another pseudovacuum). For rational γ/π, the vacuum may be normalized with quantum operators represented as finite matrices (see Appendix 1 in the previous chapter). The vacuum eigenvalues of L_2 are:

$$a_n(\lambda) = 1 + 2s\cosh(2\lambda - i\gamma), \quad d_n(\lambda) = 1 + 2s\cosh(2\lambda + i\gamma). \tag{5.5}$$

The Bethe equations are of the form

$$\left(\frac{1+2s\cosh(2\lambda_l - i\gamma)}{1+2s\cosh(2\lambda_l + i\gamma)}\right)^{M/2} = -\prod_{k=1}^{N}\frac{\sinh(\lambda_l-\lambda_k+i\gamma)}{\sinh(\lambda_l-\lambda_k-i\gamma)}. \tag{5.6}$$

The operator for the number of particles may be introduced for the lattice model as

$$Q = \frac{4}{\beta}\sum_{n=1}^{M/2}(u_{2n}-u_{2n-1}) + M\frac{(\pi-\gamma)}{2\gamma}. \tag{5.7}$$

It is easy to check that Q commutes with the transfer matrix and hence with the Hamiltonian. The Bethe wave function Ψ_N (VII.1.10) is an eigenfunction of the operator Q with eigenvalue N. The continuous limit of the physical quantities that have been introduced is proper.

The problem of ultraviolet divergences was solved in [2], [3], [5], [6], [12], [14], [16] and [17] by means of the lattice SG model. The bound states in the lattice model are described by the same inequalities as in the continuous case (see (III.1.15)). The ground state of the lattice model is a rather complicated combination of bound states, the latter being dependent on the arithmetic structure of the coupling constant. In the continuous limit, the picture is simplified, and is in agreement with that of the massive Thirring model obtained there with the help of standard perturbation theory (see section III.2).

The dependence of the L-operator of the lattice SG model on a continuous parameter allows one to construct the most general L-operator intertwined by the XXZ R-matrix as was done in section 4 for the lattice NS model.

Conclusion

Continuous models of quantum field theory have been put on a lattice without spoiling their dynamical structure. Our approach (based on the preservation of the R-matrix) guarantees that the continuous limit of the lattice model will be smooth (no phase transitions as $\Delta \to 0$). The most important dynamical characteristics of lattice models will coincide with those of the continuous model.

The construction of local Hamiltonians on a lattice made use of the trace identities, which are similar to the continuous models. The Hamiltonian was expressed at points where the L-operator was degenerate. This guaranteed that interactions were limited to nearest neighbors even in the limit of an infinite lattice.

The lattice models considered in this chapter, playing an important role in the development of QISM, were introduced in [12], [15], [16] and [17], and studied in [2]–[7], [13], [14], [18], [22], [25]–[28]. There exist different approaches to the construction of lattice models starting from continuous models ([1], [8]–[10] and [21]). The distinctive feature of our approach is the coincidence of the Hamiltonian structures (the R-matrices for continuous and lattice cases are the same.)

The general method of constructing local Hamiltonians stated here was worked out in papers [14], [15] and [17]. The Hamiltonian (1.15) is suggested in [7].

The explicit form of the time evolution operator, the local equations of motion, the Gel'fand-Levitan-Marchenko equations and soliton solutions for the lattice SG model are given in [25].

The solution of the quantum lattice NS model is given in [7]. Let us note that other local lattice Hamiltonians may be constructed by means of trace identities from the quantum L-operator (1.1) (generalizing trace identities for the fundamental spin models). This was done in [27]. On the basis of the same L-operator, the quantum Gel'fand-Levitan-Marchenko equation for the NS model was constructed in [24], and [20] is devoted to the generalization of the multicomponent NS model. The thermodynamics of the lattice model was investigated in [4].

Theorem 1 in section 4 was proved in [18]; in [26] it was generalized to the case of the XXZ R-matrix. The quadratic algebras [22], [23] closely connected with the L-operators we have considered should be mentioned here.

The problem of ultraviolet divergences for the SG model was solved in [2], [3], [5], [6], [12], [14], [16] and [17] with the help of the lattice SG model. The ground state of the lattice SG model was constructed in the same papers and the equations describing its thermodynamics were derived.

In the limiting case, the L-operator of the lattice SG model generates the lattice L-operator of the Liouville equation, which plays an important role when investigating conformal quantum field theory [28].

The lattice L-operator of the sine-Gordon equation is the most general L-operator related to R-matrices of the XXZ model ([25] and [26]).