VII
The Algebraic Bethe Ansatz

Introduction

The algebraic Bethe Ansatz is presented in this chapter. This is an important generalization of the coordinate Bethe Ansatz presented in Part I, and is one of the essential achievements of QISM. The algebraic Bethe Ansatz is based on the idea of constructing eigenfunctions of the Hamiltonian via creation and annihilation operators acting on a pseudovacuum. The matrix elements of the monodromy matrix play the role of these operators. The transfer matrix (the sum of the diagonal elements of the monodromy matrix) commutes with the Hamiltonian; thus constructing eigenfunctions of $\tau(\mu)$ determines the eigenfunctions of the Hamiltonian.

The basis of the algebraic Bethe Ansatz is stated in section 1. The commutation relations between matrix elements of the monodromy matrix are specified by the R-matrix. The explicit form of the commutation relations allows the construction of eigenfunctions of the transfer matrix (the trace of the monodromy matrix). (Recall that the Hamiltonian may also be obtained from the transfer matrix via the trace identities.) Further developments of the algebraic Bethe Ansatz necessary for the computation of correlation functions are given in section 2. The general scheme is illustrated with some examples in section 3. The NS model, the sine-Gordon model and spin models are considered in detail. The Pauli principle for interacting one-dimensional bosons plays an important role in constructing the ground state of the system and is discussed in section 4. The eigenvalues of the shift operator acting on the monodromy matrix are calculated in section 5. The classification of monodromy matrices possessing a given R-matrix is given in sections 6 and 7. It is shown that they are parametrized by two arbitrary functions. The important concept of the monodromy matrix determinant in the quantum case is introduced in section 8. The properties of the partition function Z_N of the XXX model with domain wall boundary conditions (introduced in section 6 of Chapter VI), are examined in section 9. This partition function which can be explicitly calculated is presented in section 10 where the solution is given as a determinant. We want to emphasize the importance of this determinant representation, because this is the beginning of evaluation of correlation functions, which will be continued in Part III.

Let us mention that only models having the R-matrices of the XXX and XXZ models (VI.3.17), (VI.3.18) and (VI.4.3), (VI.4.4) are considered in this chapter. However, models with different R-matrices may be treated in a similar fashion.

VII.1 The algebraic Bethe Ansatz

Models connected with the simplest R-matrices, those for the XXX and XXZ models (VI.3.17), (VI.4.3) will be considered in this section. In these cases, the monodromy matrix is a 2×2-matrix:

$$\mathsf{T}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{1.1}$$

The commutation relations between its matrix elements are given by the bilinear relation

$$\mathsf{R}(\lambda,\mu)\left(\mathsf{T}(\lambda)\bigotimes\mathsf{T}(\mu)\right) = \left(\mathsf{T}(\mu)\bigotimes\mathsf{T}(\lambda)\right)\mathsf{R}(\lambda,\mu) \tag{1.2}$$

with

$$\mathsf{R}(\lambda,\mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix} \tag{1.3}$$

where, for the XXX case,

$$f(\mu,\lambda) = 1 + \frac{ic}{\mu-\lambda}, \qquad g(\mu,\lambda) = \frac{ic}{\mu-\lambda}, \tag{1.4}$$

and for the XXZ case

$$f(\mu,\lambda) = \frac{\sinh(\mu-\lambda+2i\eta)}{\sinh(\mu-\lambda)}, \qquad g(\mu,\lambda) = i\frac{\sin 2\eta}{\sinh(\mu-\lambda)} \tag{1.5}$$

(we put $\eta = -\gamma/2$ in (VI.4.3), (VI.4.4)). The Hamiltonian and the transfer matrix have the same eigenfunctions; this follows from the trace identities (see sections 3 and 5 of Chapter VI) and the commutativity of the transfer matrix at different values of the spectral parameter. To construct eigenfunctions of $\tau(\lambda)$, we should have a generating vector (pseudovacuum $|0\rangle$), which must satisfy the following requirements:

$$\begin{aligned} &A(\lambda)|0\rangle = a(\lambda)|0\rangle; \qquad D(\lambda)|0\rangle = d(\lambda)|0\rangle; \\ &C(\lambda)|0\rangle = 0. \end{aligned} \tag{1.6}$$

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Here $a(\lambda)$ and $d(\lambda)$ are complex-valued functions called the "vacuum eigenvalues." The pseudovacuum $|0\rangle$ is similar to the the highest-weight vector in the theory of representations of Lie algebras. The action of A, B, C and D on the vacuum can be understood by looking at the L-operator for the NS model (VI.3.14). For A and D the vacuum $|0\rangle$ is an eigenvector and C annihilates it. B acts like a creation operator and is treated as such below.

It should be noted that the existence of an R-matrix does not automatically guarantee the existence of a pseudovacuum. The latter property should be established for each given model independently. However, the pseudovacuum vector does indeed exist for a large class of integrable models (see examples in section 3). When constructing the pseudovacuum for a model, the following remark is useful. Let the monodromy matrix be the matrix product of two factors:

$$\mathsf{T}(\lambda) = \mathsf{T}(2|\lambda)\mathsf{T}(1|\lambda) \tag{1.7}$$

where the commutation relations between matrix elements of each matrix $\mathsf{T}(1|\lambda)$, $\mathsf{T}(2|\lambda)$ are given by (1.2) and matrix elements of different matrices commute: $[\mathsf{T}^{ab}(1|\lambda), \mathsf{T}^{cd}(2|\lambda)] = 0$, for any a, b, c, d. (Thus, $\mathsf{T}(\lambda)$ consists of two "commuting" matrix factors.) If $|0\rangle_1$ is the pseudovacuum for $\mathsf{T}(1|\lambda)$, and $|0\rangle_2$ is that for $\mathsf{T}(2|\lambda)$, with eigenvalues $a_1(\lambda)$, $d_1(\lambda)$, and $a_2(\lambda)$, $d_2(\lambda)$, respectively, then the pseudovacuum $|0\rangle$ exists and is given by

$$|0\rangle = |0\rangle_2 \bigotimes |0\rangle_1$$

and

$$a(\lambda) = a_1(\lambda)a_2(\lambda), \qquad d(\lambda) = d_1(\lambda)d_2(\lambda). \tag{1.8}$$

To prove this statement, it is sufficient to note that the product of triangular matrices is a triangular matrix. This statement is easily generalized to the case when there are more than two commuting factors in the above product. The vacuum eigenvalues $a(\lambda)$ and $d(\lambda)$ of the monodromy matrix $\mathsf{T}(\lambda) = \mathsf{L}(\mathsf{M}|\lambda)\cdots\mathsf{L}(1|\lambda)$ (VI.1.1) are

$$a(\lambda) = \prod_{j=1}^{M} a_j(\lambda), \qquad d(\lambda) = \prod_{j=1}^{M} d_j(\lambda) \tag{1.9}$$

where $a_j(\lambda)$, $d_j(\lambda)$ are the vacuum eigenvalues of the L-operator $\mathsf{L}(j|\lambda)$.

Later we will see that the eigenfunctions of $\tau(\lambda) = A(\lambda) + D(\lambda)$ are of the form

$$|\Psi_N(\{\lambda_j\})\rangle = \prod_{j=1}^{N} B(\lambda_j)|0\rangle \tag{1.10}$$

where the λ_j satisfy a system of Bethe equations.

We begin by explicitly writing down the commutation relations (1.2) of the matrix elements of the monodromy matrix:

$$[B(\lambda),\, B(\mu)] = 0; \qquad [C(\lambda),\, C(\mu)] = 0; \tag{1.11}$$

$$A(\mu)B(\lambda) = f(\mu,\lambda)B(\lambda)A(\mu) + g(\lambda,\mu)B(\mu)A(\lambda); \tag{1.12}$$

$$D(\mu)B(\lambda) = f(\lambda,\mu)B(\lambda)D(\mu) + g(\mu,\lambda)B(\mu)D(\lambda); \tag{1.13}$$

$$C(\lambda)A(\mu) = f(\mu,\lambda)A(\mu)C(\lambda) + g(\lambda,\mu)A(\lambda)C(\mu); \tag{1.14}$$

$$C(\lambda)D(\mu) = f(\lambda,\mu)D(\mu)C(\lambda) + g(\mu,\lambda)D(\lambda)C(\mu); \tag{1.15}$$

$$[C(\lambda),\, B(\mu)] = g(\lambda,\mu)\left\{A(\lambda)D(\mu) - A(\mu)D(\lambda)\right\}; \tag{1.16}$$

$$[A(\lambda),\, A(\mu)] = 0; \qquad [D(\lambda),\, D(\mu)] = 0; \tag{1.17}$$

$$B(\mu)A(\lambda) = f(\mu,\lambda)A(\lambda)B(\mu) + g(\lambda,\mu)A(\mu)B(\lambda); \tag{1.18}$$

$$D(\mu)C(\lambda) = f(\mu,\lambda)C(\lambda)D(\mu) + g(\lambda,\mu)C(\mu)D(\lambda); \tag{1.19}$$

$$A(\lambda)C(\mu) = f(\mu,\lambda)C(\mu)A(\lambda) + g(\lambda,\mu)C(\lambda)A(\mu); \tag{1.20}$$

$$B(\lambda)D(\mu) = f(\mu,\lambda)D(\mu)B(\lambda) + g(\lambda,\mu)D(\lambda)B(\mu); \tag{1.21}$$

$$[D(\lambda),\, A(\mu)] = g(\lambda,\mu)\left\{B(\lambda)C(\mu) - B(\mu)C(\lambda)\right\}; \tag{1.22}$$

$$[A(\lambda),\, D(\mu)] = g(\lambda,\mu)\left\{C(\lambda)B(\mu) - C(\mu)B(\lambda)\right\}; \tag{1.23}$$

$$[B(\lambda),\, C(\mu)] = g(\lambda,\mu)\left\{D(\lambda)A(\mu) - D(\mu)A(\lambda)\right\}. \tag{1.24}$$

Starting with equations (1.11)–(1.13) we can calculate the result of the action of operators $A(\mu)$ and $D(\mu)$ on the state $\prod B(\lambda_j)|0\rangle$:

$$A(\mu)\prod_{j=1}^{N} B(\lambda_j)|0\rangle = \Lambda \prod_{j=1}^{N} B(\lambda_j)|0\rangle + \sum_{n=1}^{N} \Lambda_n B(\mu) \prod_{\substack{j=1\\ j\neq n}}^{N} B(\lambda_j)|0\rangle; \tag{1.25}$$

$$D(\mu)\prod_{j=1}^{N} B(\lambda_j)|0\rangle = \widetilde{\Lambda} \prod_{j=1}^{N} B(\lambda_j)|0\rangle + \sum_{n=1}^{N} \widetilde{\Lambda}_n B(\mu) \prod_{\substack{j=1\\ j\neq n}}^{N} B(\lambda_j)|0\rangle. \tag{1.26}$$

The coefficients Λ and $\widetilde{\Lambda}$ are

$$\Lambda = a(\mu)\prod_{j=1}^{N} f(\mu,\lambda_j); \quad \Lambda_n = a(\lambda_n)g(\lambda_n,\mu)\prod_{\substack{j=1\\ j\neq n}}^{N} f(\lambda_n,\lambda_j); \tag{1.27}$$

$$\widetilde{\Lambda} = d(\mu)\prod_{j=1}^{N} f(\lambda_j,\mu); \quad \widetilde{\Lambda}_n = d(\lambda_n)g(\mu,\lambda_n)\prod_{\substack{j=1\\ j\neq n}}^{N} f(\lambda_j,\lambda_n). \tag{1.28}$$

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We shall prove these formulæ following the arguments of paper [2].

The commutation relation (1.12) allows us to "move" the operator A from the left to the right through the operators B in (1.10). The first term on the right hand side of (1.12) corresponds to preserving the arguments of the operators, and the second term corresponds to an exchange of the arguments. To compute Λ in (1.25), one always has to use the first term on the right hand side of (1.12) when moving $A(\mu)$. If one applies the second term on the right hand side of (1.12), then $B(\mu)$ will appear (giving a contribution to the next term of (1.25)) and never vanish. So the second term in (1.12) does not contribute to the coefficient Λ. Repeating this procedure, one can move A to the right of all the B (1.10) and then use the relation (1.6) to obtain (1.27) for Λ.

Next we compute Λ_n. Using (1.11) we can rewrite (1.10) as

$$|\Psi_N(\{\lambda_j\})\rangle = B(\lambda_n) \prod_{\substack{j=1 \\ j\neq n}}^{N} B(\lambda_j)|0\rangle. \tag{1.29}$$

It is clear that for the first step, moving $A(\mu)$ past $B(\lambda_n)$, one has only to take into account the second term in (1.12), since this term does not contain $B(\lambda_n)$. After this first step, we obtain:

$$-g(\mu, \lambda_n)B(\mu)A(\lambda_n) \prod_{j\neq n} B(\lambda_j)|0\rangle. \tag{1.30}$$

The next step is the commuting of $A(\lambda_n)$ with $B(\lambda_j)$. Only the first term in (1.12) must be used, as the second one contains $B(\lambda_n)$, which should not be present. Therefore, it is clear that the final result will be given by (1.27). The formulæ (1.26), (1.28) can be obtained similarly. Due to the commutativity of the operators $B(\lambda)$ (1.11), the function $|\Psi_N\rangle$ is symmetric in the set $\{\lambda_j\}$. The linearly independent vectors $|\Psi_N\rangle$ correspond to different sets $\{\lambda_j\}$. Hence, $|\Psi_N\rangle$ is an eigenfunction of $\tau(\mu)$ if and only if $\Lambda_n + \widetilde{\Lambda}_n = 0$. This requirement leads to the Bethe equations

$$r(\lambda_n) \prod_{\substack{j=1 \\ j\neq n}}^{N} \frac{f(\lambda_n, \lambda_j)}{f(\lambda_j, \lambda_n)} = 1, \qquad n = 1, \ldots, N \tag{1.31}$$

where

$$r(\lambda) \equiv \frac{a(\lambda)}{d(\lambda)}. \tag{1.32}$$

The logarithmic form of the Bethe equations is often used:

$$\varphi_k = 2\pi n_k, \qquad k = 1, \ldots, N. \tag{1.33}$$

Here n_k is an arbitrary set of N integers and

$$\varphi_k = i \ln r(\lambda_k) + i \sum_{\substack{j=1 \\ j \neq n}}^{N} \ln \left[\frac{f(\lambda_k, \lambda_j)}{f(\lambda_j, \lambda_k)} \right]. \tag{1.34}$$

The eigenvalues, $\theta(\mu)$, of the transfer matrix $\tau(\mu)$ are

$$\begin{aligned} &\theta(\mu, \{\lambda_j\}) = a(\mu) \prod_{j=1}^{N} f(\mu, \lambda_j) + d(\mu) \prod_{j=1}^{N} f(\lambda_j, \mu), \\ &\tau(\mu)|\Psi_N(\{\lambda_j\})\rangle = \theta(\mu, \{\lambda_j\})|\Psi_N(\{\lambda_j\})\rangle. \end{aligned} \tag{1.35}$$

The eigenvalues of the Hamiltonian are computed by means of trace identities.

We have supposed that all $\{\lambda_j\}$ $(j = 1, \dots, N)$ are different. The case where some of them are equal will be considered in section 4.

VII.2 Comments on the algebraic Bethe Ansatz

In the previous section, the result of the action of the operators $A(\mu)$, $D(\mu)$ and $B(\mu)$ on the state (Bethe vector)

$$\prod_{j=1}^{N} B(\lambda_j)|0\rangle \tag{2.1}$$

was calculated.

To investigate correlation functions, we must also compute the action of $C(\mu)$ on (2.1):

$$\begin{aligned} C(\mu) \prod_{j=1}^{N} B(\lambda_j)|0\rangle = \sum_{n=1}^{N} M_n \prod_{\substack{j=1 \\ j \neq n}}^{N} B(\lambda_j)|0\rangle \\ + \sum_{k>n} M_{kn} B(\mu) \prod_{\substack{j=1 \\ j \neq k,n}}^{N} B(\lambda_j)|0\rangle. \end{aligned} \tag{2.2}$$

The coefficients M_n, M_{kn} are given by

$$\begin{aligned} M_n = {} & g(\mu, \lambda_n) a(\mu) d(\lambda_n) \prod_{j \neq n} f(\lambda_j, \lambda_n) f(\mu, \lambda_j) \\ & + g(\lambda_n, \mu) a(\lambda_n) d(\mu) \prod_{j \neq n} f(\lambda_j, \mu) f(\lambda_n, \lambda_j) \end{aligned} \tag{2.3}$$

$$\begin{aligned} M_{kn} = {} & d(\lambda_k) a(\lambda_n) g(\mu, \lambda_k) g(\lambda_n, \mu) f(\lambda_n, \lambda_k) \prod_{j \neq k,n} f(\lambda_j, \lambda_k) f(\lambda_n, \lambda_j) \\ & + d(\lambda_n) a(\lambda_k) g(\mu, \lambda_n) g(\lambda_k, \mu) f(\lambda_k, \lambda_n) \prod_{j \neq k,n} f(\lambda_j, \lambda_n) f(\lambda_k, \lambda_j). \end{aligned} \tag{2.4}$$

These formulæ can be obtained analogously to (1.25)–(1.28). In addition to the relations (1.11)–(1.13), one uses (1.16) and $C(\lambda)|0\rangle = 0$.

Let us now consider the dual pseudovacuum $\langle 0|$: $\langle 0| = |0\rangle^\dagger$, $\langle 0|0\rangle = 1$. It is easy to verify that

$$\langle 0|B(\mu) = 0, \quad \langle 0|A(\mu) = a(\mu)\langle 0|, \quad \langle 0|D(\mu) = d(\mu)\langle 0|. \tag{2.5}$$

In complete analogy with the results of section 1, the action of the operators $A(\mu)$, $D(\mu)$ and $B(\mu)$ on the state conjugate to (1.10),

$$\langle \widetilde{\Psi}_N(\{\lambda_j\})| = \langle 0| \prod_{j=1}^{N} C(\lambda_j), \tag{2.6}$$

can be calculated:

$$\begin{aligned}\langle 0| \prod_{j=1}^{N} C(\lambda_j) A(\mu) = \Lambda \langle 0| \prod_{j=1}^{N} C(\lambda_j)& \\ + \sum_{n=1}^{N} \Lambda_n \langle 0|C(\mu) \prod_{\substack{j=1\\ j\neq n}}^{N} C(\lambda_j)&;\end{aligned} \tag{2.7}$$

$$\begin{aligned}\langle 0| \prod_{j=1}^{N} C(\lambda_j) D(\mu) = \widetilde{\Lambda} \langle 0| \prod_{j=1}^{N} C(\lambda_j)& \\ + \sum_{n=1}^{N} \widetilde{\Lambda}_n \langle 0|C(\mu) \prod_{\substack{j=1\\ j\neq n}}^{N} C(\lambda_j)&.\end{aligned} \tag{2.8}$$

$$\begin{aligned}\langle 0| \prod_{j=1}^{N} C(\lambda_j) B(\mu) = \sum_{n=1}^{N} M_n \langle 0| \prod_{\substack{j=1\\ j\neq n}}^{N} C(\lambda_j)& \\ + \sum_{k>n} M_{kn} \langle 0|C(\mu) \prod_{\substack{j=1\\ j\neq k,n}}^{N} C(\lambda_j)&.\end{aligned} \tag{2.9}$$

The coefficients here are the same as in (1.25)–(1.28), (2.2)–(2.4). If the Bethe equations are valid, then $\langle \widetilde{\Psi}_N|$ is also an eigenfunction of $\tau(\mu)$:

$$\langle \widetilde{\Psi}_N|\tau(\mu) = \theta(\mu)\langle \widetilde{\Psi}_N| \tag{2.10}$$

with the previous eigenvalue (1.35).

The eigenfunctions (2.1), (2.6) are orthogonal:

$$\begin{aligned}&\langle \Psi_N(\{\lambda_j^C\})|\Psi_N(\{\lambda_j^B\})\rangle = \langle 0| \prod_{j=1}^{N} C(\lambda_j^C) \prod_{k=1}^{N} B(\lambda_k^B)|0\rangle = 0, \\ &(\{\lambda_j^C\} \neq \{\lambda_j^B\}),\end{aligned} \tag{2.11}$$

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if $\{\lambda_j^C\}$ and $\{\lambda_j^B\}$ are different sets of solutions of the Bethe equations. To prove this statement, one considers the matrix elements

$$\langle 0|\prod_{j=1}^{N} C(\lambda_j^C)\tau(\mu)\prod_{k=1}^{N} B(\lambda_k^B)|0\rangle. \tag{2.12}$$

Taking into account that

$$\theta(\mu,\{\lambda_j^B\}) \neq \theta(\mu,\{\lambda_j^C\}), \qquad (\{\lambda_j^C\} \neq \{\lambda_j^B\}) \tag{2.13}$$

and then comparing (1.35) and (2.10) one obtains (2.11).

As a concluding remark, let us mention that the R-matrix commutes with the matrix $\widehat{\varepsilon}\bigotimes\widehat{\varepsilon}$, where

$$\widehat{\varepsilon} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \tag{2.14}$$

and ε is a c-number. Hence, if the commutation relations of the matrix elements of the monodromy matrix $\mathsf{T}(\lambda)$ are given by the R-matrix (1.3), then the commutation relations of the elements of the matrix

$$\mathsf{T}_\varepsilon(\lambda) = \begin{pmatrix} \varepsilon A(\lambda) & \varepsilon B(\lambda) \\ \varepsilon^{-1}C(\lambda) & \varepsilon^{-1}D(\lambda) \end{pmatrix} \tag{2.15}$$

are given by the same R-matrix. The matrix T_ε possesses the same pseudovacuum $|0\rangle$ and its vacuum eigenfunctions are

$$a_\varepsilon(\lambda) = \varepsilon a(\lambda), \qquad d_\varepsilon(\lambda) = \varepsilon^{-1}d(\lambda). \tag{2.16}$$

VII.3 Examples

Now we shall calculate eigenfunctions for some important models of two-dimensional quantum field theory and statistical mechanics.

(1) The nonlinear Schrödinger (NS) model. The pseudovacuum is just the Fock vacuum $|0\rangle$ ($\Psi(x)|0\rangle = 0$, $\forall x$). The vacuum eigenvalues (1.6) of the L-operator (VI.3.14) are

$$a_j(\lambda) = 1 - i\frac{\lambda\Delta}{2}; \qquad d_j(\lambda) = 1 + i\frac{\lambda\Delta}{2}. \tag{3.1}$$

Using

$$\lim_{M\to\infty}\left(1 + i\frac{\lambda\Delta}{2}\right)^M = \exp\{i\lambda L/2\}, \qquad L = M\Delta, \tag{3.2}$$

and (1.9), one obtains the vacuum eigenvalues of the monodromy matrix for the continuum model:

$$a(\lambda) = \exp\left\{-i\frac{\lambda L}{2}\right\}; \qquad d(\lambda) = \exp\left\{i\frac{\lambda L}{2}\right\}. \tag{3.3}$$

The Bethe equations (1.31)

$$\exp\{i\lambda_n L\} = \prod_{\substack{j=1\\ j\neq n}}^{N} \left(\frac{\lambda_n - \lambda_j + ic}{\lambda_n - \lambda_j - ic}\right); \qquad n = 1, \dots, N \tag{3.4}$$

coincide with (I.2.2). The eigenvalues of the transfer matrix (1.35) are

$$\theta(\mu, \{\lambda_j\}) = \exp\left\{-i\frac{\mu L}{2}\right\} \prod_{j=1}^{N} f(\mu, \lambda_j) + \exp\left\{i\frac{\mu L}{2}\right\} \prod_{j=1}^{N} f(\lambda_j, \mu). \tag{3.5}$$

To calculate the eigenvalues of H, P and Q one has to use the trace identities (VI.3.6) which give:

$$\ln\left[\exp\left\{i\frac{\mu L}{2}\right\}\theta_N(\mu, \{\lambda_j\})\right] \underset{\mu\to i\infty}{=} \frac{ic}{\mu}\Big\{Q_N + \mu^{-1}\left[P_N - (ic/2)Q_N\right] + \mu^{-2}\left[E_N - icP_N - (c^2/3)Q_N\right] + O\left(\mu^{-3}\right)\Big\}. \tag{3.6}$$

The second term in (3.5) is negligibly small as $\mu \to i\infty$, and one obtains

$$Q_N = N; \qquad P_N = \sum_{j=1}^{N} \lambda_j; \qquad E_N = \sum_{j=1}^{N} \lambda_j^2 \tag{3.7}$$

which coincide with (I.1.29).

We have thus reproduced all the corresponding results of Chapter I within the context of QISM. The coefficients of the expansion (3.6) in inverse powers of μ can be expressed in terms of

$$I_n = \sum_{j=1}^{N} \lambda_j^n. \tag{3.8}$$

Conservation of these quantities in scattering processes prevents multiparticle production.

(2) XXX Heisenberg model. The monodromy matrix $\mathsf{T}(\lambda)$ of the XXX model (VI.5.15) is constructed in a standard way by means of the L-operator (VI.5.11). The R-matrix (1.3), (1.4) coincides with that of the NS model. Now we have two pseudovacua $|0\rangle$, $|0'\rangle$:

$$|0\rangle = \prod_{j=1}^{M} (\uparrow)_j, \qquad C(\lambda)|0\rangle = 0; \tag{3.9}$$

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$$|0'\rangle = \prod_{j=1}^{M} (\downarrow)_j\,, \qquad B(\lambda)|0'\rangle = 0. \tag{3.10}$$

The vacuum eigenvalues for the first pseudovacuum are

$$a(\lambda) = \left(\lambda - i\frac{c}{2}\right)^M, \qquad d(\lambda) = \left(\lambda + i\frac{c}{2}\right)^M. \tag{3.11}$$

Below we shall consider the lattice of even length M.

The eigenfunctions and eigenvalues of the transfer matrix are constructed as for the NS model. It follows from (VI.5.13) that

$$a^*(\lambda^*) = d(\lambda), \qquad B^\dagger(\lambda^*) = -C(\lambda). \tag{3.12}$$

The Bethe equations are:

$$\left(\frac{\lambda_j + ic/2}{\lambda_j - ic/2}\right)^M = \prod_{\substack{k=1\\k\neq j}}^{N} \left(\frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}\right), \qquad j = 1, \ldots, N. \tag{3.13}$$

Without loss of generality we can put $c = 1$ everywhere, since this corresponds to a rescaling of the spectral parameter $\lambda \to c\lambda$.

(3) Inhomogeneous XXX Heisenberg model. The inhomogeneous generalization of the XXX monodromy matrix (VI.5.26) is a 2×2 matrix similar to (1.1). The vacuum is (3.9) (as is (3.10)), with eigenvalues

$$a(\lambda) = \prod_{j=1}^{M} \left(\lambda - \nu_j - i\frac{c}{2}\right); \qquad d(\lambda) = \prod_{j=1}^{M} \left(\lambda - \nu_j + i\frac{c}{2}\right). \tag{3.14}$$

The operator for the number of particles is essentially the operator for the third component of total spin,

$$S_z = \frac{1}{2} \sum_{k=1}^{N} \sigma_z^{(k)}, \tag{3.15}$$

with the two pseudovacua (3.9), (3.10) as eigenfunctions:

$$S_z|0\rangle = \frac{M}{2}\,|0\rangle, \qquad S_z|0'\rangle = -\frac{M}{2}\,|0'\rangle. \tag{3.16}$$

The function (1.10) with arbitrary $\{\lambda_j\}$ is also an eigenfunction of S_z:

$$S_z \prod_{j=1}^{N} B(\lambda_j)|0\rangle = \left(\frac{M}{2} - N\right) \prod_{j=1}^{N} B(\lambda_j)|0\rangle \tag{3.17}$$

where the largest and smallest eigenvalues are $+M/2$ and $-M/2$, respectively.

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Considering $M = N$, we obtain

$$\prod_{j=1}^{N} B(\lambda_j)|0\rangle = Z_N|0'\rangle. \tag{3.18}$$

The numerical coefficient

$$Z_N = \langle 0'| \prod_{j=1}^{N} B(\lambda_j)|0\rangle \tag{3.19}$$

coincides with the partition function with domain wall boundary conditions introduced in section 6, Chapter VI. To see this it is sufficient to compare the representations (VI.6.12) and (3.19).

(4) The XXZ Heisenberg model (VI.5.18), (VI.5.21). The analysis is similar to that of the XXX model in sections 1 and 2. We need only replace the XXX R-matrix by the XXZ R-matrix.

The vacuum eigenvalues are

$$a(\lambda) = (-i)^M \cosh^M(\lambda - i\eta), \qquad d(\lambda) = (-i)^M \cosh^M(\lambda + i\eta) \tag{3.20}$$

with pseudovacuum (3.9).

The Bethe equations are

$$\left(\frac{\cosh(\lambda_n - i\eta)}{\cosh(\lambda_n + i\eta)}\right)^M \prod_{\substack{j=1 \\ j\neq n}}^{N} \frac{\sinh(\lambda_n - \lambda_j + 2i\eta)}{\sinh(\lambda_n - \lambda_j - 2i\eta)} = 1. \tag{3.21}$$

With the help of the trace identities (VI.5.19), the one-particle energy and momentum are calculated:

$$\begin{aligned} \varepsilon_0(\lambda) &= -2\sin^2 2\eta\, [\cosh(\lambda + i\eta)\cosh(\lambda - i\eta)]^{-1} + 2h; \\ p_0(\lambda) &= i \ln\left[\frac{\cosh(\lambda - i\eta)}{\cosh(\lambda + i\eta)}\right]. \end{aligned} \tag{3.22}$$

Results (3.21) and (3.22) are the same as (II.1.26) and (II.1.13). In this way, all the results of Chapter II are reproduced within QISM.

(5) The sine-Gordon model (VI.4.2). We shall consider the product of two L-operators on neighboring sites in order to construct the pseudovaccum

$$\mathsf{L}_{(2)}(n|\lambda) = \mathsf{L}(2n|\lambda)\mathsf{L}(2n-1|\lambda). \tag{3.23}$$

For this L-operator, the pseudovacuum was found in [22]:

$$|0\rangle_n = \delta\left(u_{2n-1} - u_{2n} + (\beta/4) - (2\pi/\beta)\right). \tag{3.24}$$

The vacuum for $\mathsf{T}(\lambda)$ is the product of the $|0\rangle_n$:

$$|0\rangle = \bigotimes_{n=1}^{M/2} |0\rangle_n.$$

For rational values of γ/π, the quantum operators $\exp\{i\beta u_n/2\}$ and $\exp\{i\beta p_n/2\}$ can be represented as finite matrices (see Appendix 1). The normalized vacuum can be constructed and the vacuum eigenvalues of the monodromy matrix found:

$$\begin{aligned} a(\lambda) &= \exp\{Ms\cosh(2\lambda - i\gamma)\}; \\ d(\lambda) &= \exp\{Ms\cosh(2\lambda + i\gamma)\}; \qquad s = \left(\frac{m\Delta}{4}\right)^2. \end{aligned} \tag{3.25}$$

The Bethe equations are of the form

$$\exp\{-2iMs\sin\gamma\,\sinh 2\lambda_n\}\prod_{\substack{j=1\\ j\neq n}}^{N}\frac{\sinh(\lambda_n - \lambda_j - i\gamma)}{\sinh(\lambda_n - \lambda_j + i\gamma)} = 1, \tag{3.26}$$

$$n = 1, \ldots, N.$$

The eigenfunctions are calculated by means of (1.10). The energy and momentum are given by

$$\begin{aligned} \varepsilon_0(\lambda) &= m_0\cosh 2\lambda, \\ p_0(\lambda) &= m_0\sinh 2\lambda. \end{aligned} \tag{3.27}$$

Now one can compare the Bethe equations for the massive Thirring (III.1.19) and sine-Gordon models. Examination of (3.26) shows the two models to be equivalent. For more details on their similarity, see [22].

VII.4 The Pauli principle for one-dimensional interacting bosons

The distinctive feature of one-dimensional quantum systems is that the Pauli principle is fulfilled not only for fermions, but also for interacting bosons (they may not possess equal momenta). Let us consider, for example, the NS model. The eigenfunctions of the transfer matrix (VI.1.3) are constructed by (1.10). The momenta $\{\lambda_j\}$ satisfy the Bethe equations (1.31). When deriving these equations, it was essential to assume that all λ_j are different. Let us now relax this condition and consider the simplest example: λ_1 occurs twice, with $\lambda_1 \equiv \lambda_2$, and the remaining momenta, λ_j ($j = 3, \ldots, N$) are all different.

The corresponding eigenfunction is

$$|\Psi_N\rangle = B^2(\lambda_1)\prod_{j=3}^{N} B(\lambda_j)|0\rangle. \tag{4.1}$$

Theorem 1. *Eigenfunctions of the form (4.1) do not exist for the NS model.*

Proof: Let $|\Psi_N\rangle$ (4.1) be an eigenfunction of the transfer matrix $\tau(\mu) = A(\mu) + D(\mu)$. This demand results in restrictions on the $\{\lambda_j\}$. One can show that the Bethe equations for (4.1) are

$$\frac{a(\lambda_n)}{d(\lambda_n)}\left[\frac{f(\lambda_n,\lambda_1)}{f(\lambda_1,\lambda_n)}\right]^2 \prod_{\substack{j=3\\ j\neq n}}^{N} \frac{f(\lambda_n,\lambda_j)}{f(\lambda_j,\lambda_n)} = 1, \qquad n = 3,\dots,N; \tag{4.2}$$

$$\frac{a(\lambda_1)}{d(\lambda_1)} \prod_{j=3}^{N} \frac{f(\lambda_1,\lambda_j)}{f(\lambda_j,\lambda_1)} = -1; \tag{4.3}$$

$$\frac{d}{d\lambda_1}\varphi_1 = i\frac{d}{d\lambda_1}\ln\frac{a(\lambda_1)}{d(\lambda_1)} + i\sum_{j=3}^{N}\frac{d}{d\lambda_1}\ln\frac{f(\lambda_1,\lambda_j)}{f(\lambda_j,\lambda_1)} + \frac{4}{c} = 0. \tag{4.4}$$

To prove this, let us take $\lambda_2 \to \lambda_1$ in (1.25)–(1.28); then we find:

$$\begin{aligned} A(\mu)B^2(\lambda_1)\prod_{j=3}^{N} B(\lambda_j)|0\rangle = {}& \Lambda B^2(\lambda_1)\prod_{j=3}^{N} B(\lambda_j)|0\rangle \\ & + B(\mu)B^2(\lambda_1)\sum_{l=3}^{N}\Lambda_l^{(1)}\prod_{\substack{j=3\\ j\neq l}}^{N} B(\lambda_j)|0\rangle \\ & + \Lambda^{(2)}B(\mu)B(\lambda_1)\prod_{j=3}^{N} B(\lambda_j)|0\rangle \\ & + \Lambda^{(3)}B(\mu)B'(\lambda_1)\prod_{j=3}^{N} B(\lambda_j)|0\rangle; \end{aligned} \tag{4.5}$$

$$\begin{aligned} D(\mu)B^2(\lambda_1)\prod_{j=3}^{N} B(\lambda_j)|0\rangle = {}& \widetilde{\Lambda} B^2(\lambda_1)\prod_{j=3}^{N} B(\lambda_j)|0\rangle \\ & + B(\mu)B^2(\lambda_1)\sum_{l=3}^{N}\widetilde{\Lambda}_l^{(1)}\prod_{\substack{j=3\\ j\neq l}}^{N} B(\lambda_j)|0\rangle \\ & + \widetilde{\Lambda}^{(2)}B(\mu)B(\lambda_1)\prod_{j=3}^{N} B(\lambda_j)|0\rangle \\ & + \widetilde{\Lambda}^{(3)}B(\mu)B'(\lambda_1)\prod_{j=3}^{N} B(\lambda_j)|0\rangle. \end{aligned} \tag{4.6}$$

The coefficients are given by

$$\begin{aligned}\Lambda &= a(\mu) f^2(\mu, \lambda_1) \prod_{j=3}^{N} f(\mu, \lambda_j);\\ \widetilde{\Lambda} &= d(\mu) f^2(\lambda_1, \mu) \prod_{j=3}^{N} f(\lambda_j, \mu);\end{aligned} \tag{4.7}$$

$$\Lambda_l^{(1)} = -\, a(\lambda_l) g(\mu, \lambda_l) f^2(\lambda_l, \lambda_1) \prod_{\substack{j=3\\ j\neq l}}^{N} f(\lambda_l, \lambda_j); \tag{4.8}$$

$$\widetilde{\Lambda}_l^{(1)} = d(\lambda_l) g(\mu, \lambda_l) f^2(\lambda_1, \lambda_l) \prod_{\substack{j=3\\ j\neq l}}^{N} f(\lambda_j, \lambda_l); \tag{4.9}$$

$$\begin{aligned}\Lambda^{(2)} = &-\, a(\lambda_1) g(\mu, \lambda_1) \left[1 + f(\mu, \lambda_1)\right] \prod_{j=3}^{N} f(\lambda_1, \lambda_j)\\ &- icg(\mu, \lambda_1) \frac{\partial}{\partial \lambda_1} \left[a(\lambda_1) \prod_{j=3}^{N} f(\lambda_1, \lambda_j) \right];\end{aligned} \tag{4.10}$$

$$\begin{aligned}\widetilde{\Lambda}^{(2)} = &\, d(\lambda_1) g(\mu, \lambda_1) \left[1 + f(\lambda_1, \mu)\right] \prod_{j=3}^{N} f(\lambda_j, \lambda_1)\\ &- icg(\mu, \lambda_1) \frac{\partial}{\partial \lambda_1} \left[d(\lambda_1) \prod_{j=3}^{N} f(\lambda_j, \lambda_1) \right];\end{aligned} \tag{4.11}$$

$$\begin{aligned}\Lambda^{(3)} &= ica(\lambda_1) g(\mu, \lambda_1) \prod_{j=3}^{N} f(\lambda_1, \lambda_j);\\ \widetilde{\Lambda}^{(3)} &= icd(\lambda_1) g(\mu, \lambda_1) \prod_{j=3}^{N} f(\lambda_j, \lambda_1).\end{aligned} \tag{4.12}$$

Taking into account that on the right hand sides of (4.5) and (4.6) all the vectors are linearly independent, we obtain that $|\Psi_N\rangle$ is an eigenvector of $\tau(\mu)$ with

$$\tau(\mu)|\Psi_N\rangle = \left(\Lambda + \widetilde{\Lambda}\right)|\Psi_N\rangle \tag{4.13}$$

if the following system of equations is satisfied:

$$\Lambda_l^{(1)} + \widetilde{\Lambda}_l^{(1)} = 0; \qquad \Lambda^{(2)} + \widetilde{\Lambda}^{(2)} = 0; \qquad \Lambda^{(3)} + \widetilde{\Lambda}^{(3)} = 0. \tag{4.14}$$

This system coincides with (4.2)–(4.4). So we have proved that the Bethe equations for $|\Psi_N\rangle$ can be written in the form (4.2)–(4.4).

We shall now prove that these equations for the NS model are unsolvable. Equations (4.2) and (4.3) for this model are

$$e^{-i\lambda_n L}\left(\frac{\lambda_n-\lambda_1+ic}{\lambda_n-\lambda_1-ic}\right)^2\prod_{\substack{j=3\\ j\neq n}}^{N}\left(\frac{\lambda_n-\lambda_j+ic}{\lambda_n-\lambda_j-ic}\right)=1,$$

$$n=3,\ldots,N \tag{4.15}$$

$$e^{-i\lambda_1 L}\prod_{j=3}^{N}\left(\frac{\lambda_1-\lambda_j+ic}{\lambda_1-\lambda_j-ic}\right)=-1, \tag{4.16}$$

and possess solutions for real $\{\lambda_j\}$ only (the proof is similar to that of Theorem 1 in section 2, Chapter I). On the other hand, equation (4.4) looks like:

$$\frac{4}{c}+L+\sum_{l=3}^{N}\frac{2c}{c^2+(\lambda_1-\lambda_l)^2}=0 \tag{4.17}$$

and has no solutions for real $\{\lambda_j\}$. The theorem is proved.

We have thus proved that two momenta cannot coincide. The case when several momenta coincide is treated in a similar way [5]. Thus we see that in the set $\{\lambda_j\}$ (1.10), there are no repeated momenta if $|\Psi_N\rangle$ is an eigenfunction of $\tau(\mu)$. This theorem plays an important role since, as was shown in Chapter I, the state of the NS Hamiltonian with minimal energy could be constructed as a Dirac sea. The Pauli principle ensures the stability of the Dirac sea. One may show that the unsolvability of equations (4.17) is connected with the convexity of Yang's action, which was discussed in Chapter I. This leads to the conjecture that in all integrable models possessing a convex Yang-Yang action, the Pauli principle holds. If the Yang-Yang action is not convex, the Pauli principle may not be valid.

VII.5 The shift operator

Let us consider the monodromy matrix consisting of two "commuting" factors (1.7)

$$\mathsf{T}(\lambda)=\mathsf{T}(2|\lambda)\mathsf{T}(1|\lambda)=\begin{pmatrix}A(\lambda) & B(\lambda)\\ C(\lambda) & D(\lambda)\end{pmatrix}. \tag{5.1}$$

Each factor $\mathsf{T}(\mathsf{j}|\lambda)$ $(j=1,2)$,

$$\mathsf{T}(\mathsf{j}|\lambda)=\begin{pmatrix}A_j(\lambda) & B_j(\lambda)\\ C_j(\lambda) & D_j(\lambda)\end{pmatrix}, \tag{5.2}$$

is intertwined by the R-matrix (1.3) and has vacuum $|0\rangle_j$:

$$\begin{aligned} &A_j(\lambda)|0\rangle_j = a_j(\lambda)|0\rangle_j; \qquad D_j(\lambda)|0\rangle_j = d_j(\lambda)|0\rangle_j; \\ &C_j(\lambda)|0\rangle_j = 0. \end{aligned} \tag{5.3}$$

(The case of more than two factors is considered in Appendix 2.) Let us define the shift operator O by the following:

$$O\mathsf{T}(\lambda)O^{-1} = \widetilde{\mathsf{T}}(\lambda) = \mathsf{T}(1|\lambda)\mathsf{T}(2|\lambda) = \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix}. \tag{5.4}$$

(Note that $\operatorname{tr} \mathsf{T}(\lambda) = \operatorname{tr} \widetilde{\mathsf{T}}(\lambda)$.) The commutation relations for the matrix elements of $\widetilde{\mathsf{T}}(\lambda)$ are given by the same R-matrix (1.3). The vacuum is $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$. The vacuum eigenvalues of $\widetilde{\mathsf{T}}$ and T coincide (1.8): $a(\lambda) = \widetilde{a}(\lambda) = a_1(\lambda)a_2(\lambda)$, $d(\lambda) = \widetilde{d}(\lambda) = d_1(\lambda)d_2(\lambda)$.

Consider now the wave functions

$$|\Psi_N\rangle = \prod_{j=1}^{N} B(\lambda_j)|0\rangle, \tag{5.5}$$

$$|\widetilde{\Psi}_N\rangle = \prod_{j=1}^{N} \widetilde{B}(\lambda_j)|0\rangle. \tag{5.6}$$

Theorem 2. *The wave functions (5.5) and (5.6) are proportional to each other if and only if the Bethe equations are satisfied:*

$$\frac{a(\lambda_n)}{d(\lambda_n)} \prod_{\substack{j=1 \\ j\neq n}}^{N} \frac{f(\lambda_n, \lambda_j)}{f(\lambda_j, \lambda_n)} = 1, \qquad n = 1, \ldots, N. \tag{5.7}$$

Proof: Expressing the wave function (5.5) in terms of the wave functions (5.2) of the individual sites $\prod B_1(\lambda_j)|0\rangle_1$ and $\prod B_2(\lambda_j)|0\rangle_2$, we have for $B(\lambda)$ from (5.1) and (5.2):

$$B(\lambda) = A_2(\lambda)B_1(\lambda) + D_1(\lambda)B_2(\lambda). \tag{5.8}$$

Applying the logic of the algebraic Bethe Ansatz we get

$$\begin{aligned} \prod_{j=1}^{N} B(\lambda_j)|0\rangle = &\sum_{\{\lambda\}=\{\lambda^I\}\bigcup\{\lambda^{II}\}} \prod_{j\in I}^{n_1} \prod_{k\in II}^{n_2} a_1(\lambda_j^I)d_2(\lambda_k^{II}) \\ &\times f(\lambda_j^I, \lambda_k^{II}) \left(B_2(\lambda_k^{II})|0\rangle_2\right) \left(B_1(\lambda_j^I)|0\rangle_1\right) \end{aligned} \tag{5.9}$$

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where summation is over all decompositions of the set $\{\lambda_j\}$ into two disjoint subsets $\{\lambda^I\}$ and $\{\lambda^{II}\}$, for which

$$\begin{aligned} &\{\lambda^I\}\bigcap\{\lambda^{II}\} = \emptyset; \qquad \{\lambda^I\}\bigcup\{\lambda^{II}\} = \{\lambda\}; \\ &\mathrm{card}\{\lambda^I\} = n_1, \qquad \mathrm{card}\{\lambda^{II}\} = n_2. \end{aligned} \tag{5.10}$$

Each term in (5.9) is a double product; the index j belongs to the first set and the index k to the second. The generalization of the formula (5.9) for many commuting factors instead of just two (5.1) is given in Appendix 2. The wave function (5.6) is expressed as

$$\begin{aligned} \prod_{j=1}^{N} \widetilde{B}(\lambda_j)|0\rangle = &\sum_{\{\lambda\}=\{\lambda^I\}\bigcup\{\lambda^{II}\}} \prod_{j\in I}^{n_1}\prod_{k\in II}^{n_2} a_2(\lambda_k^{II})d_1(\lambda_j^I) \\ &\times f(\lambda_k^{II},\lambda_j^I)\left(B_1(\lambda_j^I)|0\rangle_1\right)\left(B_2(\lambda_k^{II})|0\rangle_2\right). \end{aligned} \tag{5.11}$$

All the terms on the right hand side of (5.9) are linearly independent. Calculation shows that the right hand sides of (5.9) and (5.11) are proportional if and only if the system (5.7) is fulfilled. In this context,

$$\begin{aligned} &O|\Psi_N\rangle = |\widetilde{\Psi}_N\rangle = \nu|\Psi_N\rangle; \\ &\nu = \prod_{j=1}^{N}\frac{a_1(\lambda_j)}{d_1(\lambda_j)} = \prod_{j=1}^{N}\left(\frac{a_2(\lambda_j)}{d_2(\lambda_j)}\right)^{-1} \end{aligned} \tag{5.12}$$

where ν is the eigenvalue of the shift operator. The theorem is proved.

Theorem 2 means exactly that the Bethe Ansatz equations (5.7) guarantee the periodicity.

The dual wavefunctions are also proportional to each other if and only if the system (5.7) is fulfilled. The proof is based on the representation

$$\begin{aligned} \langle 0|\prod_{j=1}^{N} C(\lambda_j) = &\sum_{\{\lambda\}=\{\lambda^I\}\bigcup\{\lambda^{II}\}} \prod_{j\in I}^{n_1}\prod_{k\in II}^{n_2} \left({}_1\langle 0|C_1(\lambda_j^I)\right)\bigotimes\left({}_2\langle 0|C_2(\lambda_k^{II})\right) \\ &\times f(\lambda_k^{II},\lambda_j^I)a_1(\lambda_k^{II})d_2(\lambda_j^I). \end{aligned} \tag{5.13}$$

In addition,

$$\langle 0|\prod_{j=1}^{N}\widetilde{C}(\lambda_j) = \langle 0|\prod_{j=1}^{N} C(\lambda_j)O^{-1} = \nu^{-1}\langle 0|\prod_{j=1}^{N} C(\lambda_j). \tag{5.14}$$

Thus, if the wave function (5.5) is an eigenfunction of the shift operator, it is necessarily an eigenfunction of the transfer matrix.

Theorem 2 may be applied to the calculation of momentum eigenvalues. Let us consider the NS model. One represents the NS monodromy

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matrix as the product of two "commuting terms": $\mathsf{T}(\lambda) = \mathsf{T}(2|\lambda)\mathsf{T}(1|\lambda)$, where $\mathsf{T}(1|\lambda)$ is the monodromy matrix on the interval $[0, x]$ with vacuum eigenvalues equal to

$$a_1(\lambda) = \exp\left\{-i\frac{\lambda x}{2}\right\}; \qquad d_1(\lambda) = \exp\left\{i\frac{\lambda x}{2}\right\}, \tag{5.15}$$

and matrix $\mathsf{T}(2|\lambda)$ is the monodromy matrix on the interval $[x, L]$, with vacuum eigenvalues equal to

$$a_2(\lambda) = \exp\left\{-i\frac{\lambda(L-x)}{2}\right\}; \qquad d_2(\lambda) = \exp\left\{i\frac{\lambda(L-x)}{2}\right\}. \tag{5.16}$$

The operator $O = \exp\{i(L-x)\widehat{P}\}$ is the shift operator on $L - x$ and $\widehat{P}$ is the momentum operator. The eigenvalue of the shift operator (5.12) is in this case

$$\nu = \exp\left\{i(L-x)\sum_{j=1}^{N}\lambda_j\right\}.$$

This means that the eigenvalue of the momentum operator is $P_N = \sum_{j=1}^{N}\lambda_j$, which coincides with (3.7).

Let us consider the homogeneous lattice model where the vacuum eigenfunctions of each L-operator are similar and equal to $a_{\mathsf{L}}(\lambda)$ and $d_{\mathsf{L}}(\lambda)$. In such a model, the eigenvalue of the momentum operator is

$$P_N = i\sum_{j=1}^{N}\ln\left(\frac{a_{\mathsf{L}}(\lambda_j)}{d_{\mathsf{L}}(\lambda_j)}\right). \tag{5.17}$$

For $N = 1$, we have reproduced formula (3.22) for the XXZ magnet.

The theorem proved in this section plays an important role when calculating correlation functions. It is also useful when constructing quantum local Hamiltonians on the lattice ([29] and [30]).

VII.6 Classification of monodromy matrices

Let us consider the monodromy matrix

$$\mathsf{T}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{6.1}$$

intertwined by the R-matrix (1.3), (1.4):

$$\mathsf{R}(\lambda,\mu)\left(\mathsf{T}(\lambda)\bigotimes\mathsf{T}(\mu)\right) = \left(\mathsf{T}(\mu)\bigotimes\mathsf{T}(\lambda)\right)\mathsf{R}(\lambda,\mu) \tag{6.2}$$

and assume the existence of a pseudovacuum $|0\rangle$:

$$A(\lambda)|0\rangle = a(\lambda)|0\rangle; \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle; \quad C(\lambda)|0\rangle = 0. \tag{6.3}$$

Theorem 3. *The monodromy matrix $\mathsf{T}(\lambda)$ (6.1) exists for any given functions $a(\lambda)$ and $d(\lambda)$ if conditions (6.2) and (6.3) are fulfilled.*

Proof: To prove this statement, one has to represent four linear operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ and the space they act in. The basis vectors of this space are N particle states with "momenta" $\lambda_1, \ldots, \lambda_N$ (the momenta are arbitrary complex numbers):

$$|\lambda_1, \ldots, \lambda_N\rangle_N, \quad \lambda_j \in \mathbb{C}, \quad N = 0, 1, 2, \ldots \tag{6.4}$$

These states are symmetric functions of all λ_j. The $N = 0$ state is identified with the pseudovacuum. The action of the operators A, B, C and D on the basis vectors is defined by the formulæ for the algebraic Bethe Ansatz. The operator $B(\mu)$ adds a particle with momentum μ to the basis vector:

$$B(\mu)|\lambda_1, \ldots, \lambda_N\rangle_N = |\mu, \lambda_1, \ldots, \lambda_N\rangle_{N+1}. \tag{6.5}$$

The action of operators $A(\mu)$ and $D(\mu)$ preserves the number of particles:

$$A(\mu)|\{\lambda_j\}\rangle_N = \Lambda|\{\lambda_j\}\rangle_N + \sum_{n=1}^{N} \Lambda_n|\mu, \{\lambda_{j\neq n}\}\rangle_N; \tag{6.6}$$

$$D(\mu)|\{\lambda_j\}\rangle_N = \widetilde{\Lambda}|\{\lambda_j\}\rangle_N + \sum_{n=1}^{N} \widetilde{\Lambda}_n|\mu, \{\lambda_{j\neq n}\}\rangle_N. \tag{6.7}$$

The coefficients Λ, $\widetilde{\Lambda}$ are given here by (1.27), (1.28). The operator $C(\mu)$ removes a momentum from the basis vector:

$$\begin{aligned} C(\mu)|\{\lambda_j\}\rangle_N = &\sum_{n=1}^{N} M_n|\{\lambda_{j\neq n}\}\rangle_{N-1} \\ &+ \sum_{k>n} M_{kn}|\mu, \{\lambda_{j\neq k,n}\}\rangle_{N-1}. \end{aligned} \tag{6.8}$$

The coefficients M are given by (2.3), (2.4). Thus, the monodromy matrix is built up. It should be emphasized that equations (6.6) and (6.7) are now definitions.

Let us now prove that the monodromy matrix thus obtained satisfies (6.2). Writing down (6.2) as the 16 scalar relations (1.11)–(1.24), we can act with the right and left hand sides of these equalities on some basis vector (6.4).

The fact that these operators standing on both sides of each equality act equally on arbitrary basis vectors is obvious from direct calculation. When calculating identities, the functions $f(\mu, \lambda)$ and $g(\mu, \lambda)$ should be used. For illustration we shall prove that

$$A(\mu_1)A(\mu_2) = A(\mu_2)A(\mu_1). \tag{6.9}$$

Acting with $A(\mu_2)A(\mu_1)$ on $|\{\lambda_j\}\rangle_N$, we obtain

$$\begin{aligned}
A(\mu_2)A(\mu_1)|\{\lambda_j\}\rangle_N =& a(\mu_1)a(\mu_2)\prod_{j=1}^{N} f(\mu_1,\lambda_j)f(\mu_2,\lambda_j)|\{\lambda_j\}\rangle_N \\
&+ \sum_{n=1}^{N} a(\mu_1)\prod_{j=1}^{N} f(\mu_1,\lambda_j)a(\lambda_n)g(\lambda_n,\mu_2) \\
&\quad\times \prod_{j\neq n} f(\lambda_n,\lambda_j)|\mu_2,\{\lambda_{j\neq n}\}\rangle_N \\
&+ \sum_{n=1}^{N} a(\mu_2)\prod_{j\neq n} f(\mu_2,\lambda_j)f(\mu_2,\mu_1)a(\lambda_n)g(\lambda_n,\mu_1) \\
&\quad\times \prod_{j\neq n} f(\lambda_n,\lambda_j)|\mu_1,\{\lambda_{j\neq n}\}\rangle_N \\
&+ \sum_{n=1}^{N} a(\lambda_n)g(\lambda_n,\mu_1)\prod_{j\neq n} f(\lambda_n,\lambda_j)a(\mu_1)g(\mu_1,\mu_2) \\
&\quad\times \prod_{j\neq n} f(\mu_1,\lambda_j)|\mu_2,\{\lambda_{j\neq n}\}\rangle_N \\
&+ \sum_{n=1}^{N}\sum_{k\neq n} a(\lambda_n)g(\lambda_n,\mu_1)\prod_{j\neq n} f(\lambda_n,\lambda_j)a(\lambda_k)g(\lambda_k,\mu_2) \\
&\quad\times \prod_{j\neq n,k} f(\lambda_k,\lambda_j)f(\lambda_k,\mu_1)|\mu_1,\mu_2,\{\lambda_{j\neq n,k}\}\rangle_N
\end{aligned} \tag{6.10}$$

assuming that the momenta μ_1, μ_2 and $\{\lambda_j\}$ are different. If (6.9) is correct, then the right hand side of (6.10) must not change when $\mu_1 \leftrightarrow \mu_2$. Let us check all coefficients separately. The symmetry of $a(\mu_1)a(\mu_2)|\{\lambda_j\}\rangle_N$ is obvious. Next, the coefficient of $a(\mu_1)a(\lambda_n)|\mu_2, \{\lambda_{j\neq n}\}\rangle_N$ transfers into the coefficient of $a(\mu_2)a(\lambda_n)|\mu_1,\{\lambda_{j\neq n}\}\rangle_N$. Canceling the common factor $(\prod_{j\neq n} f(\mu_2,\lambda_j)f(\lambda_n,\lambda_j))$, we get the equality

$$g(\lambda_n,\mu_1)f(\mu_2,\lambda_n) + g(\lambda_n,\mu_2)g(\mu_2,\mu_1) = f(\mu_2,\mu_1)g(\lambda_n,\mu_1) \tag{6.11}$$

which is identically true for both the R-matrices (1.4) and (1.5). The coefficient of $a(\lambda_n)a(\lambda_k)|\mu_1,\mu_2,\{\lambda_{j\neq n,k}\}\rangle_N$ should not change when

$\mu_1 \leftrightarrow \mu_2$. Testing this statement leads to the identity:

$$\begin{aligned} & g(\lambda_n, \mu_1) g(\lambda_k, \mu_2) f(\lambda_k, \mu_1) f(\lambda_n, \lambda_k) \\ & \qquad + g(\lambda_k, \mu_1) g(\lambda_n, \mu_2) f(\lambda_n, \mu_1) f(\lambda_k, \lambda_n) \\ & = g(\lambda_n, \mu_2) g(\lambda_k, \mu_1) f(\lambda_k, \mu_2) f(\lambda_n, \lambda_k) \\ & \qquad + g(\lambda_k, \mu_2) g(\lambda_n, \mu_1) f(\lambda_n, \mu_2) f(\lambda_k, \lambda_n) \end{aligned} \tag{6.12}$$

which is indeed valid. Thus, both sides of (6.9) act equally on the basis vector (6.4). We assumed at the start that in the set $\{\mu_1, \mu_2, \lambda_j\}$ all momenta were different. One may continue the equality

$$\Big(\mathsf{R}(\lambda, \mu)\, \mathsf{T}(\lambda) \bigotimes \mathsf{T}(\mu) - \mathsf{T}(\mu) \bigotimes \mathsf{T}(\lambda)\, \mathsf{R}(\lambda, \mu)\Big) |\lambda_1, \ldots, \lambda_N\rangle_N = 0 \tag{6.13}$$

for coinciding momenta and verify that it is always valid.

Thus, the commutation relations are proved. The relations (6.3) are also fulfilled. The theorem is proved. Similar statements are also true for the XXZ R-matrix (1.5) (see [8]).

VII.7 Comments on the classification of monodromy matrices

(1) The following equality can be proved by applying the techniques of the previous section:

$$\begin{aligned} A\left(\lambda - i\frac{c}{2}\right) D\left(\lambda + i\frac{c}{2}\right) - B\left(\lambda - i\frac{c}{2}\right) C\left(\lambda + i\frac{c}{2}\right) & \\ = a\left(\lambda - i\frac{c}{2}\right) d\left(\lambda + i\frac{c}{2}\right) & . \end{aligned} \tag{7.1}$$

The left hand side is a quadratic combination of operators, while the right hand side is a complex-valued function (a c-number). Acting with both parts of this equality on an arbitrary basis vector, one obtains the same result. The essential notion of the determinant of the monodromy matrix in the quantum case will be introduced in the next section with the help of relation (7.1).

(2) The following identification may be performed:

$$|\lambda_1, \ldots, \lambda_N\rangle_N = \prod_{j=1}^{N} B(\lambda_j)|0\rangle. \tag{7.2}$$

Due to the linearity of the intertwining relations, there is a trivial arbitrariness connected with the multiplication of $\mathsf{T}(\mu)$ by an arbitrary complex-valued function. Thus, essentially different monodromy matrices, $\mathsf{T}(\mu)$, can be parametrized by one arbitrary function $r(\lambda) = a(\lambda)/d(\lambda)$. The action of the operators A, B, C and D on the basis

(7.2) is defined by the formulæ (1.25)–(1.28), (2.2)–(2.4) for the algebraic Bethe Ansatz and depends only on the vacuum eigenvalues $a(\lambda)$ and $d(\lambda)$. This leads to the uniqueness theorem which states that for a given R-matrix, the functions $a(\lambda)$ and $d(\lambda)$ uniquely determine the model.

(3) Linear functionals in the space considered are of interest. Let us define $\langle 0|$ (the dual pseudovacuum) by the relations

$$\langle 0|0\rangle = 1, \qquad \langle 0|\{\lambda_j\}_N\rangle = 0, \quad N = 1, 2, \ldots. \tag{7.3}$$

It is easy to evaluate the result of the action of certain operators on a dual pseudovacuum:

$$\begin{aligned} &\langle 0|A(\mu) = a(\mu)\langle 0|; \qquad \langle 0|D(\mu) = d(\mu)\langle 0|; \\ &\langle 0|B(\mu) = 0. \end{aligned} \tag{7.4}$$

Let us consider the linear functionals

$$\langle 0|C(\lambda_1)\cdots C(\lambda_N). \tag{7.5}$$

We shall call them *basis* functionals. These linear functionals (7.5) are symmetric functions of the $\{\lambda_j\}$ (see (1.11)). The action of the operators $A(\mu)$, $B(\mu)$, $C(\mu)$ and $D(\mu)$ on the basis functionals (7.5) is easily calculated by means of (1.11)–(1.24); the answers coincide with (2.7)–(2.9).

(4) Let us introduce the particle number operator, Q, in the constructed space. On the basis vectors, it acts as follows:

$$\begin{aligned} &Q|0\rangle = 0; \qquad Q|\{\lambda_j\}\rangle_N = N|\{\lambda_j\}\rangle_N; \\ &\langle 0|\left(\prod_{j=1}^{N} C(\lambda_j)\right) Q = N\langle 0|\left(\prod_{j=1}^{N} C(\lambda_j)\right); \\ &\langle 0|Q = 0. \end{aligned} \tag{7.6}$$

The momenta $\{\lambda_j\}$ are arbitrary (and not subject to the Bethe equations). The commutation relations are easily calculated:

$$\begin{aligned} &[Q,\, B(\lambda)] = B(\lambda); \qquad [Q,\, C(\lambda)] = -C(\lambda); \\ &[Q,\, A(\lambda)] = [Q,\, D(\lambda)] = 0, \end{aligned} \tag{7.7}$$

or, in matrix form, $2[Q,\, \mathsf{T}(\lambda)] = [\sigma_z,\, \mathsf{T}(\lambda)]$. The action of the operator $\exp\{\alpha Q\}$ (where α is an arbitrary constant) on the basis vector is

$$\exp\{\alpha Q\}\,|\{\lambda_j\}\rangle_N = \exp\{\alpha N\}\,|\{\lambda_j\}\rangle_N. \tag{7.8}$$

(5) The properties of the basis (7.2) are similar to those of the basis (7.5). This allows the introduction of Hermitian conjugation into the space as follows. Let us demand that

$$B^\dagger(\lambda^*) = \pm C(\lambda); \qquad \langle 0| = |0\rangle^\dagger \tag{7.9}$$

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which is possible if

$$a^*(\lambda^*) = d(\lambda) \tag{7.10}$$

and the bases (7.2) and (7.5) are connected by Hermitian conjugation, and

$$A^\dagger(\lambda^*) = D(\lambda). \tag{7.11}$$

In this case, the value of the linear functional on the basis vector with a similar set of $\{\lambda_j\}$,

$$\langle 0| \prod_{j=1}^{N} C(\lambda_j) \prod_{k=1}^{N} B(\lambda_k)|0\rangle, \tag{7.12}$$

is real for real $\{\lambda_j\}$.

(6) It is shown in [8] that all the results of the previous section are also true for the XXZ R-matrix (1.3), (1.5). In this case, it is natural to consider only the arbitrary periodic function $a(\lambda)/d(\lambda)$ [29]:

$$\frac{a(\lambda)}{d(\lambda)} = \frac{a(\lambda + i\pi)}{d(\lambda + i\pi)}. \tag{7.13}$$

This is connected with the problem of representing $\mathsf{T}(\lambda)$ as a product of L-operators (see section 4, Chapter VIII).

Thus, we have classified all the monodromy matrices for a given R-matrix by the function $r(\lambda)$. Later (section 4, Chapter VIII) we shall classify all the L-operators for the same R-matrix. The connection between $\mathsf{T}(\lambda)$ and $\mathsf{L}(\lambda)$ is given by (VI.1.1). We shall be able to construct an L-operator depending only on four complex parameters (the matrix $\mathsf{T}(\lambda)$ is a functional of $r(\lambda)$). Matrix elements of the L-operator will act in a rather narrow subspace of the Fock space constructed above; this L-operator generates the most general monodromy matrix of the form (VI.1.1). In regard to this relation, the parallel between the theory of Lie group representations and QISM mentioned in (VI.2) should be emphasized.

(7) Let us discuss the consequences of the theorem proved in the previous section. Consider the Bethe equations (1.31):

$$r(\lambda_j) \prod_{\substack{k=1 \\ k\neq j}}^{N} \frac{f(\lambda_j, \lambda_k)}{f(\lambda_k, \lambda_j)} = 1, \qquad j = 1, \ldots, N \tag{7.14}$$

and fix an arbitrary set of N different numbers $\{\lambda_j\}$.

Are there any integrable models for which this set is the solution of the Bethe equations? The answer is positive; many such models are available. They are parametrized by a function $r(\lambda)$ with a (rather loose)

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restriction: for the points λ_j, $r(\lambda)$ must take the fixed values

$$r(\lambda_j) = \prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{f(\lambda_k, \lambda_j)}{f(\lambda_j, \lambda_k)}, \qquad j = 1, \ldots, N. \tag{7.15}$$

This makes it possible to consider the solutions $\{\lambda_j\}$ of the Bethe equations as free independent variables; this is used frequently in Part III.

(8) Finally, let us discuss the scattering matrix for the case of arbitrary functions $a(\lambda)$ and $d(\lambda)$. Consider the Bethe equations (1.31), (7.14). The factors $f(\lambda_j, \lambda_k)/f(\lambda_k, \lambda_j)$ on the left hand side have the meaning of the bare scattering matrices for a particle with spectral parameter λ_j and a particle with spectral parameter λ_k. Thus, the bare scattering matrix does not depend on the arbitrary functions $a(\lambda)$ and $d(\lambda)$, but is defined entirely by the R-matrix. The dressed scattering phase is obtained from the bare one by means of the dressing equations considered in section 4 of Chapter I. The kernel of these integral equations is defined by the R-matrix:

$$K(\lambda, \mu) = i \frac{\partial}{\partial \lambda} \ln \frac{f(\lambda, \mu)}{f(\mu, \lambda)};$$

the same is true for the right hand side of the dressing equations (I.4.39) for the phase. Thus, the dressed scattering matrix is also entirely defined by the R-matrix. To be more precise, let us consider two models with the same R-matrix but different functions $a(\lambda)$ and $d(\lambda)$. If the values of the spectral parameters on the Fermi boundary in these models coincide, the dressed scattering matrices also coincide.

It should be emphasized that in the above we have considered the S-matrices as functions of the spectral parameter λ. As functions of the dressed (physical) momenta they are, in general, different.

VII.8 The quantum determinant

The generalization of the determinant of the monodromy matrix in the quantum case is introduced here. Let us consider the monodromy matrix $\mathsf{T}(\lambda)$ (1.1) intertwined by the XXX R-matrix (1.3), (1.4) and possessing the vacuum (1.6). When $\lambda = \mu + ic$, the commutation relations (1.11)–(1.24) between the matrix elements of $\mathsf{T}(\lambda)$ and $\mathsf{T}(\mu)$ are essentially simplified due to the fact that $f(\mu, \lambda) = 0$, $g(\mu, \lambda) = -1$, and $\mathsf{R}(\lambda, \mu)$ is proportional to the one-dimensional projector

$$\mathsf{R}(\mu + ic, \mu) = \Pi - I. \tag{8.1}$$

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Examples of these commutation relations are:

$$\begin{aligned} C(\mu)D(\mu+ic) &= D(\mu)C(\mu+ic); \\ A(\mu)B(\mu+ic) &= B(\mu)A(\mu+ic). \end{aligned} \tag{8.2}$$

The determinant of the monodromy matrix in the quantum case is defined as

$$\det_q \mathsf{T}(\lambda) = A\left(\lambda - i\frac{c}{2}\right) D\left(\lambda + i\frac{c}{2}\right) - B\left(\lambda - i\frac{c}{2}\right) C\left(\lambda + i\frac{c}{2}\right). \tag{8.3}$$

Using the commutation relations between $\mathsf{T}(\lambda)$ and $\mathsf{T}(\lambda + ic)$, this can also be rewritten in the form

$$\det_q \mathsf{T}(\lambda) = D\left(\lambda - i\frac{c}{2}\right) A\left(\lambda + i\frac{c}{2}\right) - C\left(\lambda - i\frac{c}{2}\right) B\left(\lambda + i\frac{c}{2}\right). \tag{8.4}$$

The quantum determinant commutes with the operators $A(\mu)$, $B(\mu)$, $C(\mu)$ and $D(\mu)$, i.e., with any matrix element of the monodromy matrix:

$$[\det_q \mathsf{T}(\lambda),\ \mathsf{T}(\mu)] = 0 \tag{8.5}$$

(the commutator here is in the quantum space).

It follows from (7.1) that the quantum determinant is a complex-valued function, but not a quantum operator (more exactly, it is proportional to the unit operator in the quantum space, the coefficient being a c-number)

$$\det_q \mathsf{T}(\lambda) = a\left(\lambda - i\frac{c}{2}\right) d\left(\lambda + i\frac{c}{2}\right). \tag{8.6}$$

In matrix form, the commutation relations (8.2)–(8.5) are

$$\begin{aligned} &\mathsf{T}\left(\lambda - i\frac{c}{2}\right) \sigma_y \mathsf{T}^T\left(\lambda + i\frac{c}{2}\right) \sigma_y \\ &\quad = \det_q \mathsf{T}(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a\left(\lambda - i\frac{c}{2}\right) d\left(\lambda + i\frac{c}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{8.7}$$

or,

$$\mathsf{T}^{-1}(\lambda) = \sigma_y \mathsf{T}^T(\lambda + ic)\sigma_y \left[\det_q \mathsf{T}\left(\lambda + i\frac{c}{2}\right)\right]^{-1}. \tag{8.8}$$

This formula is the natural generalization of the classical Cramer's formula for the inverse of a 2×2 matrix. Using this, we can prove that the determinant of the "commuting" matrix product $\mathsf{T}(\lambda) = \mathsf{T}(2|\lambda)\mathsf{T}(1|\lambda)$ (1.7) is the product of determinants:

$$\det_q \mathsf{T}(\lambda) = \det_q \mathsf{T}(1|\lambda) \det_q \mathsf{T}(2|\lambda) \tag{8.9}$$

which means that the determinant of the monodromy matrix is equal to the product of the determinants of the L-operators:

$$\det_q \mathsf{T}(\lambda) = \prod_{j=1}^{M} \det_q \mathsf{L}(j|\lambda). \tag{8.10}$$

It is also possible to define the quantum determinant for the XXZ R-matrix:

$$\begin{aligned}\det_q \mathsf{T}(\lambda) &= A(\lambda - i\eta)D(\lambda + i\eta) - B(\lambda - i\eta)C(\lambda + i\eta) \\ &= a(\lambda - i\eta)d(\lambda + i\eta)\end{aligned} \tag{8.11}$$

which possesses all the properties mentioned above.

Let us list the quantum determinants $\det_q \mathsf{T}(\lambda)$ for the models considered above:

(1) NS model (see (3.3)):

$$\det_q \mathsf{T}(\lambda) = \exp\{-cL/2\} \tag{8.12}$$

(2) XXX chain (see (3.11)):

$$\begin{aligned}\det_q \mathsf{L}(\lambda) &= (c^2 + \lambda^2); \\ \det_q \mathsf{T}(\lambda) &= (c^2 + \lambda^2)^M\end{aligned} \tag{8.13}$$

(3) XXZ chain (see (3.20)):

$$\begin{aligned}\det_q \mathsf{L}(\lambda) &= -\cosh(\lambda + 2i\eta)\cosh(\lambda - 2i\eta); \\ \det_q \mathsf{T}(\lambda) &= [\det_q \mathsf{L}(\lambda)]^M\end{aligned} \tag{8.14}$$

(4) sine-Gordon model (see (3.25)):

$$\det_q \mathsf{T}(\lambda) = \exp\{2Ms\cosh 2\lambda\}, \qquad s = (m\Delta/4)^2 \tag{8.15}$$

VII.9 Recursion properties of the partition function

The partition function Z_N (VI.6.6) for the XXX model with domain wall boundary conditions is examined in more detail in this section. An interesting quantity in itself, Z_N is also of importance when studying scalar products. It is a function of $2N$ variables, $\{\lambda_\alpha\}$, $\{\nu_j\}$.

Lemma 1. The partition function $Z_N(\{\lambda_\alpha\}, \{\nu_j\})$ is symmetric in all variables $\{\lambda_\alpha\}$ and $\{\nu_j\}$ separately, and is a polynomial of degree $(N-1)$ in both λ_α and ν_j, with all other variables fixed:

$$\frac{\partial^N Z_N}{\partial \lambda_\alpha^N} = \frac{\partial^N Z_N}{\partial \nu_j^N} = 0. \tag{9.1}$$

The proof follows from the representation (VI.6.12). Since (1.11)

$$[B(\lambda),\, B(\mu)] = 0,$$

(this follows because $B(\lambda)$ is an element of the monodromy matrix (VI.6.7)), Z_N is a symmetric matrix of all λ_α. The relations (VI.6.4) and (VI.6.7) imply that:

$$\frac{\partial^N \mathsf{T}_\alpha(\lambda)}{\partial \lambda^N} = N!\, I \equiv N! \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \tag{9.2}$$

then $\partial^N B(\lambda)/\partial\lambda^N = 0$ and hence $B(\lambda)$ is also a polynomial of degree $(N-1)$ in each λ_α. The representation (VI.6.15) results in similar properties for each ν_j. Lemma 1 is thus proved.

For further investigation of Z_N, we list some properties of the L-operator (VI.6.4):

(1) $\mathsf{L}_{\alpha k}(\nu_k - \lambda_\alpha = ic/2) = -ic\Pi_{\alpha k}$. (9.3)

Here $\Pi_{\alpha k}$ is the permutation matrix

$$\Pi_{\alpha k} v_k w_\alpha = w_k v_\alpha \tag{9.4}$$

with subscripts α and k denoting the spaces to which vectors v and w belong.

(2) The L-operator possesses two simple eigenvectors $(\uparrow_\alpha\uparrow_k)$ and $(\downarrow_\alpha\downarrow_k)$:

$$\mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k)\,(\uparrow_\alpha\uparrow_k) = \left(\lambda_\alpha - \nu_k - i\frac{c}{2}\right)(\uparrow_\alpha\uparrow_k)\,; \tag{9.5}$$

$$\mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k)\,(\downarrow_\alpha\downarrow_k) = \left(\lambda_\alpha - \nu_k - i\frac{c}{2}\right)(\downarrow_\alpha\downarrow_k)\,. \tag{9.6}$$

(3) $[\mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k),\, \sigma_x^{(k)}\sigma_x^{(\alpha)}] = 0$. (9.7)

Let us introduce the operator W defined by

$$W = \prod_{\alpha=1}^{N} \sigma_x^{(\alpha)} \prod_{k=1}^{N} \sigma_x^{(k)}.$$

This operator has the following properties:

$$W^2 = 1;$$

and

$$\left[W, \prod_{\alpha=1}^{N}\prod_{k=1}^{N} \mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k)\right] = 0, \tag{9.8}$$

which follow from property (3) of the L-operator. Here the double product indicates space ordering as in section VI.6:

$$\prod_{\alpha=1}^{N}\prod_{k=1}^{N} \mathsf{L}_{\alpha k} \equiv (\cdots \mathsf{L}_{23}\mathsf{L}_{22}\mathsf{L}_{21})\,(\cdots \mathsf{L}_{13}\mathsf{L}_{12}\mathsf{L}_{11})\,.$$

Lemma 2. The function Z_N can be represented in the following form:

$$Z_N(\{\lambda_\alpha\},\{\nu_j\}) = \langle 0| \prod_{j=1}^{N} C(\lambda_j)|0'\rangle. \tag{9.9}$$

The proof is as follows. From (VI.6.5) we have

$$\begin{aligned} Z_N &= \left\{\prod_{\beta=1}^N \uparrow_\beta\right\}\left\{\prod_{j=1}^N \downarrow_j\right\} W \\ &\qquad \times \left\{\prod_{\alpha=1}^N \prod_{k=1}^N \mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k)\right\} W \left\{\prod_{\beta=1}^N \downarrow_\beta\right\}\left\{\prod_{j=1}^N \uparrow_j\right\} \\ &= \left\{\prod_{\beta=1}^N \downarrow_\beta\right\}\left\{\prod_{j=1}^N \uparrow_j\right\} \\ &\qquad \times \left\{\prod_{\alpha=1}^N \prod_{k=1}^N \mathsf{L}_{\alpha k}(\lambda_\alpha - \nu_k)\right\}\left\{\prod_{\beta=1}^N \uparrow_\beta\right\}\left\{\prod_{j=1}^N \downarrow_j\right\}. \end{aligned} \tag{9.10}$$

Repeating the arguments of section VI.6 ((VI.6.11)–(VI.6.12)), we obtain (9.9). The lemma is proved.

Lemma 3. The function Z_N satisfies the following recursion relation:

$$\begin{aligned} Z_N\Big|_{\lambda_\beta=\nu_l-ic/2} &= -ic \prod_{\substack{k=1\\k\neq l}}^N \left(\lambda_\beta - \nu_k - i\frac{c}{2}\right) \prod_{\substack{\alpha=1\\ \alpha\neq\beta}}^N \left(\lambda_\alpha - \nu_l - i\frac{c}{2}\right) \\ &\qquad \times Z_{N-1}\left(\{\lambda_{\alpha\neq\beta}\}, \{\nu_{j\neq l}\}\right). \end{aligned} \tag{9.11}$$

It is sufficient to prove (9.11) for $\beta = l = 1$ since Z_N is symmetric in all the λ_α and in all the ν_j. Notice that the operator L_{11} is the furthest to the right in the product chain of L-operators in the representation (VI.6.5), and at $\lambda_1 = \nu_1 - ic/2$, it is equal to the permutation matrix:

$$\mathsf{L}_{11}(\nu_1 - \lambda_1 = ic/2) = -ic\Pi_{11}. \tag{9.12}$$

Hence we can calculate the following equality

$$\begin{aligned} \mathsf{L}_{11} &\left\{\prod_{j=1}^N \uparrow_j\right\}\left\{\prod_{\alpha=1}^N \downarrow_\alpha\right\} \\ &= -ic\left(\downarrow_1 \left\{\prod_{j=2}^N \uparrow_j\right\}\right)\left(\uparrow_1 \left\{\prod_{\alpha=2}^N \downarrow_\alpha\right\}\right). \end{aligned} \tag{9.13}$$

The vector on the right hand side here is the eigenstate of all the $\mathsf{L}_{1j}(\lambda_1 - \nu_j)$, $(j = 2, \ldots, N)$ (9.5) and of all the $\mathsf{L}_{\alpha 1}(\lambda_\alpha - \nu_1)$ $(\alpha = 2, \ldots, N)$ (see (9.6)). After application of these L-operators to the vector (9.13), we

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have

$$Z_N\Big|_{\lambda_1=\nu_1-ic/2} = -ic\prod_{k=2}^{N}\left(\lambda_1-\nu_k-i\frac{c}{2}\right)\prod_{\alpha=2}^{N}\left(\lambda_\alpha-\nu_1-i\frac{c}{2}\right) \times Z_{N-1}\left(\{\lambda_{\alpha\neq1}\},\{\nu_{j\neq1}\}\right). \tag{9.14}$$

The lemma is proved.

Lemma 4. The properties which uniquely determine the partition function are:

(a) $Z_1 = -ic$.
(b) Z_N is a symmetric function of $\{\lambda_\alpha\}$ and $\{\nu_j\}$ separately.
(c) Z_N is a polynomial of degree $(N-1)$ in each variable λ_α or ν_j when all others are assumed fixed.
(d) Z_N satisfies the recursion relation (9.11).

This statement is proved by induction. For

$$Z_1 = -ic \tag{9.15}$$

it is obvious. If the function Z_{N-1} is known, then the function Z_N is fixed uniquely. Z_N is an $(N-1)$th-degree polynomial in λ_N. The values of this polynomial at $\lambda_N = \nu_j$ $(j = 1, \dots, N)$ are known, (9.11). Thus, it is fixed uniquely. The lemma is proved.

It will be suitable to use functions $G_N(\{\lambda_\alpha^B\},\{\lambda_j^C\})$:

$$G_N\left(\{\lambda_\alpha^B\},\{\lambda_j^C\}\right) \equiv Z_N\left(\{\lambda_\alpha^B\},\{\lambda_j^C+i\frac{c}{2}\}\right). \tag{9.16}$$

These functions are easily calculated for small N starting from the recursive properties of Z_N:

$$\begin{aligned} &G_1 = -ic;\\ &G_2(\lambda_1^B,\lambda_2^B|\lambda_1^C,\lambda_2^C) = c^4 + ic^3(\lambda_1^B+\lambda_2^B-\lambda_1^C-\lambda_2^C)\\ &\qquad - c^2(\lambda_1^B-\lambda_2^C)(\lambda_2^B-\lambda_1^C) - c^2(\lambda_1^B-\lambda_1^C)(\lambda_2^B-\lambda_2^C). \end{aligned} \tag{9.17}$$

VII.10 Z_N as a determinant

The determinant representation obtained in this section initiates the calculation of correlation functions which will be continued in Part III. It is of interest that the recursion relations of the previous section can be solved explicitly. The answer is given by the determinant of an $N \times N$

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matrix:

$$Z_N\left(\{\lambda_\alpha\},\{\nu_j\}\right) = (-1)^N \frac{\prod_j^N \prod_\alpha^N \left(\nu_j - \lambda_\alpha - i\frac{c}{2}\right)\left(\nu_j - \lambda_\alpha + i\frac{c}{2}\right)}{\prod_{N\geq j>k\geq 1}^N (\nu_k - \nu_j) \prod_{N\geq \alpha>\beta\geq 1}^N (\lambda_\alpha - \lambda_\beta)} \det \mathcal{M} \tag{10.1}$$

with

$$\mathcal{M}_{j\alpha} = \frac{ic}{(\nu_j - \lambda_\alpha + ic/2)(\nu_j - \lambda_\alpha - ic/2)}. \tag{10.2}$$

One can prove a similar formula for the six-vertex (ice) model [6] which provides a trigonometric version of formulae (10.1) and (10.2). We shall use this expression when constructing the main coefficient of the scalar product in Part III.

We will now show that this expression for Z_N satisfies the properties stated in Lemma 4. This will prove that (10.1) is a unique expression for the partition function.

(a) This result follows trivially from (10.1) by setting $N = 1$. The double product in the numerator contains only one term. $\mathcal{M}$ consists of only one element which cancels the double product leaving a factor of ic. A factor of -1 comes from $(-1)^N$. Hence $Z_1 = -ic$.

(b) Exchanging two λ's ($\lambda_\gamma \leftrightarrow \lambda_\delta$) in (10.2) is the same as interchanging two columns in $\mathcal{M}$ which gives rise to a factor of -1 in $\det \mathcal{M}$. This factor is compensated by a factor of -1 from the exchange of two λ's in

$$\prod_{N\geq \alpha>\beta\geq 1} (\lambda_\alpha - \lambda_\beta),$$

which leaves Z_N unchanged. A similar argument holds for the exchange of two ν's. Thus, Z_N is a symmetric function of $\{l_\alpha\}$ and $\{\nu_j\}$ separately.

(c) This item of Lemma 4 involves proving two statements. First we will show that Z_N is a polynomial rather than a rational function. Then, we will show that Z_N is a polynomial of degree $(N-1)$.

To prove Z_N is a polynomial rather than a rational function, we need to demonstrate that Z_N has zero residue at its poles. An analysis of (10.1) shows that there are two sources of poles. The first source is in $\mathcal{M}$. The poles here are given by

$$\nu_k - \lambda_\gamma = \pm i\frac{c}{2}.$$

However, these poles are also zeros of the double product in the numerator of (10.1). Thus the residue of these poles is zero. The other source of poles is when two λ's coincide in the denominator. Again this pole has zero residue since $\mathcal{M}$ is degenerate when two λ's coincide (making the determinant of $\mathcal{M}$ zero). A similar argument holds for coinciding ν's. Thus Z_N is a polynomial.

To find the degree of Z_N we should use the symmetry proved in (b) which allows us to check the degree of λ_1 only. The double product in the numerator of (10.1)

$$\prod_j^N \prod_\alpha^N \left(\nu_j - \lambda_\alpha - i\frac{c}{2}\right)\left(\nu_j - \lambda_\alpha + i\frac{c}{2}\right) \tag{10.3}$$

is of order λ_1^{2N} while the product in the denominator of (10.1)

$$\prod_{N\geq\alpha>\beta\geq 1} (\lambda_\alpha - \lambda_\beta) \tag{10.4}$$

is of order λ_1^{N-1}. In $\mathcal{M}$, λ_1 only occurs in the first column which shows that $\det\mathcal{M}$ is of order λ_1^{-2}. Combining these results we see that Z_N is of degree $N-1$ in λ_1. The above can also be shown to be true for ν_1 since Z_N is also symmetric in $\{\nu_j\}$. Hence, Z_N is a polynomial of degree $N-1$ in each variable λ_α or ν_j when all others are assumed fixed.
(d) To show that Z_N satisfies the recursion relation (9.11), we shall use formula (10.1) and the symmetry property proved above. Thus it is sufficient to prove (9.11) for $\beta = l = 1$ which is given by

$$\begin{aligned} Z_N\Big|_{\lambda_1=\nu_1-ic/2} &= -ic\prod_{k=2}^N\left(\lambda_1-\nu_k-i\frac{c}{2}\right)\prod_{\alpha=2}^N\left(\lambda_\alpha-\nu_1-i\frac{c}{2}\right)\\ &\quad\times Z_{N-1}\left(\{\lambda_{\alpha\neq 1}\},\{\nu_{j\neq 1}\}\right). \end{aligned} \tag{10.5}$$

Since we are interested in the parameters λ_1 and ν_1, we can isolate their contribution to Z_N as follows:

$$\begin{aligned} Z_N &= (-1)^N\left[(\nu_1-\lambda_1-i\frac{c}{2})(\nu_1-\lambda_1+i\frac{c}{2})\right]\\ &\times\frac{\prod_{j=2}^N(\nu_j-\lambda_1-i\frac{c}{2})(\nu_j-\lambda_1+i\frac{c}{2})\prod_{\alpha=2}^N(\nu_1-\lambda_a-i\frac{c}{2})(\nu_1-\lambda_a+i\frac{c}{2})}{\prod_{j=2}^N(\nu_1-\nu_j)\prod_{\alpha=2}^N(\lambda_\alpha-\lambda_1)}\\ &\times\frac{\prod_{j=2}^N\prod_{\alpha=2}^N\left(\nu_j-\lambda_\alpha-i\frac{c}{2}\right)\left(\nu_j-\lambda_\alpha+i\frac{c}{2}\right)}{\prod_{N\geq j>k\geq 2}^N(\nu_k-\nu_j)\prod_{N\geq\alpha>\beta\geq 2}^N(\lambda_\alpha-\lambda_\beta)}\det\mathcal{M}. \end{aligned} \tag{10.6}$$

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To further isolate the contribution from the parameters λ_1 and ν_1, we need to understand how $\det \mathcal{M}$ behaves when $\lambda_1 \to \nu_1 - ic/2$. By examining (10.2), it is easily seen that $\mathcal{M}$ has a pole in this limit. Thus the determinant will be dominated by this contribution and we can write

$$\det \mathcal{M} = \det \mathcal{M}_{N-1} \left(\frac{1}{\nu_1 - \lambda_1 - i\frac{c}{2}} \right) \Bigg|_{\lambda_1 = \nu_1 - ic/2} . \tag{10.7}$$

where $\mathcal{M}_{N-1}$, the $(N-1)\times(N-1)$ minor of $\mathcal{M}$, is independent of λ_1 and ν_1. This pole is cancelled by a zero in (10.6). Using (10.1) for Z_{N-1}, it easily follows that (10.6) reduces to (10.5) which implies that Z_N satisfies the recursion relation (9.11).

We have shown that the determinant formula for Z_N satisfies the requirements of Lemma 4 from section 9 which uniquely specify the partition function. Hence (10.1) is the unique expression of the partition function Z_N.

Conclusion

In this chapter we have shown how to reproduce the results of the Bethe Ansatz starting from the quantum inverse scattering method. The algebraic Bethe Ansatz provides new opportunities for investigation of the various models. It allows for the classification of exactly solvable models and construction of the quantum determinant. The partition function with domain wall boundary conditions Z_N (evaluated in section 10) will be used later for the evaluation of correlation functions, norms of eigenfunctions and scalar products. The partition function for the six-vertex model with periodic boundary conditions was found in [15], [16], [17], [26].

The algebraic Bethe Ansatz was constructed for the first time in [2] and [27] where the Bethe equations and the spectrum of eigenvalues of the energy and momentum operators over the Fock vacuum were derived for the Heisenberg magnet, sine-Gordon model and Bose gas. The coincidence of the formulæ for the eigenfunctions of the Hamiltonian obtained within QISM with the coordinate wave functions of Part I may be proved by generalizing formula (5.9) for the case where the monodromy matrix consists of an arbitrary number of "commuting" factors (see [7] and Appendix 2). It is shown in a similar way that all observable values in the quantum sine-Gordon [22] and massive Thirring models are the same as in Part I. Let us mention that the normalized vacuum of the sine-Gordon model was constructed in [5].

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The action of the operator $C(\mu)$ on the Bethe vector (2.1) was investigated in [9] and [10]. The Pauli principle for one-dimensional interacting bosons was formulated in [5].

The algebraic Bethe Ansatz allows us to go further in the investigation of integrable models. It is of particular interest in that it reveals the structure of the monodromy matrix to be of the general form connected with a given R-matrix. The theorem on the classification of monodromy matrices (section 6) in the XXX and XXZ cases was proved in [8]. The concept of the quantum determinant was introduced in [3], [8]. It plays an important role in QISM and was used by many authors. Formula (8.8) was used in [25] when deriving the quantum Gel'fand-Levitan equation. The concept of the quantum determinant was extended for matrices of greater dimensions in [13]. It was used for the generation of new R-matrices in [11] and for the construction of the Casimir operator in quadratic algebras in [23] and [24]. It is related to the antipode in the theory of quantum groups [1].

The recursive properties of the partition function Z_N were investigated in [9]. Its representation in the determinant form was given in [6].

We should also mention important issues (which are not in the book but related) such as the analytic Bethe Ansatz ([18]) and the Bethe Ansatz hierarchy ([12] and [27]). The ideas of the algebraic Bethe Ansatz were applied to the Hubbard model by S. Shastry in [19], [20] and [21].

Appendix VII.1: Matrix representation of quantum operators

Let us consider the rational coupling constant in the quantum sine-Gordon model

$$\frac{\gamma}{\pi} = \frac{Q}{P} \tag{A.1.1}$$

with coprime integers Q and P, $Q < P$.

The relations

$$\pi\chi = \chi\pi \exp(i\gamma) \tag{A.1.2}$$

$$\chi = \exp(i\beta u/2); \qquad \pi = \exp(i\beta p/4) \tag{A.1.3}$$

give

$$\chi^{2P} = \pi^{2P} = 1. \tag{A.1.4}$$

Thus χ and π may be represented as $2P \times 2P$ matrices

$$\begin{aligned} \chi_{ab} &= \delta_{ab} \exp\{i\pi(a-1)/P\}, \\ \pi_{ab} &= \delta_{a+Q,b} \qquad (a, b = 1, \ldots, 2P;\ a + 2P \equiv a). \end{aligned} \tag{A.1.5}$$

This representation may be used in the sine-Gordon model since χ and π enter the monodromy matrix via integer powers. For the pseudovacuum (3.24), we then have the Kronecker δ-symbol:

$$\delta_{a_{2n}+P, a_{2n-1}+Q}. \tag{A.1.6}$$

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Appendix VII.2: Multisite model

Let us consider a multisite model. In other words, the monodromy matrix $\mathsf{T}(\lambda)$ in this case can be represented as a product of L factors:

$$\mathsf{T}(\lambda) = \mathsf{T}(\mathsf{L}|\lambda)\cdots\mathsf{T}(2|\lambda)\mathsf{T}(1|\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{A.2.1}$$

This is a generalization of section 5. Each factor has the same R-matrix and matrix elements of different factors commute as quantum operators. Each factor can be represented in the form (5.2)

$$\mathsf{T}(\mathsf{j}|\lambda) = \begin{pmatrix} A_j(\lambda) & B_j(\lambda) \\ C_j(\lambda) & D_j(\lambda) \end{pmatrix}. \tag{A.2.2}$$

and has its own pseudovacuum (5.3)

$$\begin{aligned} &A_j(\lambda)|0\rangle_j = a_j(\lambda)|0\rangle_j; \qquad D_j(\lambda)|0\rangle_j = d_j(\lambda)|0\rangle_j; \\ &C_j(\lambda)|0\rangle_j = 0. \end{aligned} \tag{A.2.3}$$

The multisite generalization of formula (5.9) is

$$\begin{aligned} \prod_{\alpha=1}^{N} B(\lambda_\alpha)|0\rangle = &\sum_{\{\lambda\}=\bigcup_{j=1}^{L}\{\lambda_j\}} \prod_{j=1}^{L} B_j(\lambda_j)|0\rangle_j \\ &\times \prod_{1\le j<k\le L} a_k(\lambda_j)d_j(\lambda_k)f(\lambda_j,\lambda_k). \end{aligned} \tag{A.2.4}$$

Here the summation is with respect to the partition of the set of all λ into L subsets; j and k label the subsets. On the right hand side of (A.2.4) we have a product with respect to numbers of the subsets. Each factor (corresponding to one subset) means the product with respect to all the λ's entering this subset. To illustrate this one should compare (A.2.4) for $L = 2$ with (5.9). These two formulæ coincide.

The development of this idea helped to identify the coordinate and algebraic Bethe Ansatzes for the Heisenberg model.