# V
# The Classical r-matrix

## Introduction

The modern way to solve partial differential equations is called the classical inverse scattering method. (One can think of it as a nonlinear generalization of the Fourier transform.)

Nowadays, the classical inverse scattering method (CISM) is a well developed branch of mathematical physics (see Preface references [1], [9], [10], [11], [18], [21]–[24], [29], [37], [45]). In this chapter, we shall give only the information necessary for the quantization which will be performed in the next chapter. The concepts of the Lax representation, the transition matrix and the trace identities are stated in section 1. Classical completely integrable partial differential equations will appear once more in this book. In Chapters XIV and XV we shall derive them for quantum correlation functions. In those chapters we shall study completely integrable differential equations from a different point of view. We shall apply the Riemann-Hilbert problem in order to evaluate the asymptotics. The classical r-matrix, which enables calculation of the Poisson brackets between matrix elements of the transition matrix (and also construction of the action-angle variables) is introduced in section 2. As explained there, the existence of the r-matrix guarantees the existence of the Lax representation. The r-matrix satisfies a certain bilinear relation (the classical Yang-Baxter relation). The existence of the r-matrix also guarantees the existence of an infinite number of conservation laws which restrict in an essential way the dynamics of the system. In the next chapter, the notion of the r-matrix will be generalized to the quantum case. In the first two sections of this chapter, general statements are demonstrated by example using the nonlinear Schrödinger equation which is the simplest dynamical model (it should be mentioned that in the classical case this name is more natural than the one-dimensional Bose gas). Other mod-

els (the sine-Gordon equation, the Mikhailov-Shabat-Zhiber model) are considered in section 3. Tensor notation, the application of which substantially simplifies the calculations both in the classical and quantum cases, is discussed in the appendix to this chapter.

If the reader finds this chapter too brief, we recommend the excellent book by L.D. Faddeev and L.A. Takhtajan (see [18] in the Preface references), which describes CISM in explicit detail.

## V.1 The Lax representation

Let us consider a classical nonlinear Hamiltonian evolutionary equation in two-dimensional space-time. The corresponding Hamiltonian will be denoted by $H$. We study the system on a periodic interval of length $L$ ($0 \leq x \leq L$). The traditional base for application of the inverse scattering method (ISM) to this equation is that it can be represented in the Lax form:

$$[\partial_t - U(x|\lambda),\ \partial_x + V(x|\lambda)] = 0 \tag{1.1}$$

which is valid for all $\lambda$. Here $U$ and $V$ are $k \times k$ matrices (the integer $k$ depends on the equation under consideration) which depend on a complex spectral parameter $\lambda$ and on the dynamical variables of the problem. The matrix $V(x|\lambda)$ is called the potential and $U(x|\lambda)$ is the time evolution operator. Condition (1.1) must be valid at any $\lambda$ and can be regarded as the consistency condition for the following differential equations:

$$\begin{aligned} \partial_t \Phi(x,t) &= U(x|\lambda)\Phi(x,t), \\ \partial_x \Phi(x,t) &= -V(x|\lambda)\Phi(x,t). \end{aligned} \tag{1.2}$$

Here $\Phi(x,t)$ is an unknown vector function which also depends on $\lambda$. In the book [5], the similarity to the Yang-Mills fields is explained. Condition (1.1) plays the role of the zero-curvature condition, with $U$ and $V$ playing the role of gauge fields. In Chapters XIV and XV we shall derive nonlinear partial differential equations for quantum correlation functions, starting from the Lax representation. These will give interesting examples of completely integrable differential equations.

It is useful to consider the translation of the solution of the system (1.2), $\Phi$, along the $x$-direction (at fixed time $t$):

$$\Phi(x) = \mathsf{T}(x,y|\lambda)\Phi(y). \tag{1.3}$$

(Here we have suppressed the time argument.) The matrix $\mathsf{T}(x,y|\lambda)$ is called the transition matrix. Below we shall discuss this $k \times k$ matrix in detail. But first, let us discuss the Lax representation for lattice models (with continuous time). We shall use lattice versions of quantum field theory models to solve the problem of ultraviolet divergences. For quantization, we also need the Lax representation on the periodic lattice with

$M$ sites and lattice spacing $\Delta$:

$$\partial_t \mathsf{L}(n|\lambda) = U(n+1|\lambda)\mathsf{L}(n|\lambda) - \mathsf{L}(n|\lambda)U(n|\lambda). \tag{1.4}$$

Here $\mathsf{L}$ and $U$ are $k \times k$ matrices which depend on the spectral parameter and dynamical variables. The equality (1.4) is a consequence of the consistency condition for the following problem on the lattice:

$$\partial_t \Phi(n,t) = U(n|\lambda)\Phi(n,t),$$
$$\Phi(n+1,t) = \mathsf{L}(n,\lambda)\Phi(n,t),$$

where $n$ is the lattice site number. To study continuous models, it is convenient to consider the infinitesimal lattice ($\Delta \to 0$). The coordinate of the $n$-th site of the lattice thus obtained is $x_n = n\Delta$ ($n = 1, \ldots, M$, and $M = L/\Delta$). For such a lattice

$$\mathsf{L}(n|\lambda) = I - V(x_n|\lambda)\Delta + O\left(\Delta^2\right), \tag{1.5}$$

where $I$ is the $k \times k$ unit matrix.

Let us now study the transition matrix $\mathsf{T}(x,y|\lambda)$ (1.3) which plays an important role in the ISM. For the continuous case, this $k \times k$ matrix is defined on the interval $[y,x]$ ($x \geq y$), by the following requirements:

$$\begin{aligned} &[\partial_x + V(x|\lambda)]\,\mathsf{T}(x,y|\lambda) = 0; \\ &\mathsf{T}(y,y|\lambda) = I. \end{aligned} \tag{1.6}$$

Sometimes it is useful to write down a formal solution of this equation:

$$\mathsf{T}(x,y|\lambda) = \mathsf{P}\exp\left\{-\int_y^x V(z|\lambda)\,dz\right\}, \tag{1.7}$$

where $\mathsf{P}$ denotes path ordering of noncommutative factors.

The transition matrix possesses the following group-like property: if $z$ is any interior point in the interval $[y,x]$ then

$$\mathsf{T}(x,z|\lambda)\mathsf{T}(z,y|\lambda) = \mathsf{T}(x,y|\lambda) \qquad (x \geq z \geq y).$$

The left hand side here is the product of two $k \times k$ matrices. The transition matrix for the whole periodic interval $[0,L]$ is called the monodromy matrix $\mathsf{T}(L,0|\lambda)$.

The lattice transition matrix from the $m$-th site to the $(n+1)$-th site can be represented as the product of $(n-m+1)$ matrices:

$$\mathsf{T}(n,m|\lambda) = \mathsf{L}(n|\lambda)\mathsf{L}(n-1|\lambda)\cdots\mathsf{L}(m|\lambda), \quad n \geq m \tag{1.8}$$

where $\mathsf{L}(k,\lambda) \equiv \mathsf{T}(k,k|\lambda)$ is the elementary transition matrix for one lattice site. The transition matrix for the whole lattice length, $\mathsf{T}(M,1|\lambda)$, is called the monodromy matrix. The matrix $\mathsf{L}(k|\lambda)$ is called the $\mathsf{L}$-operator.

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The trace of the monodromy matrix in both the continuous and lattice cases plays a particularly important role:

$$\tau(\lambda) = \operatorname{tr} \mathsf{T}(L, 0|\lambda); \quad \tau(\lambda) = \operatorname{tr} \mathsf{T}(M, 1|\lambda). \tag{1.9}$$

In the next section we shall see that $\tau(\lambda)$ is time-independent. The Hamiltonian of the initial evolutionary equation is expressed in terms of logarithmic derivatives of $\tau(\lambda)$ by means of trace identities.

As an example, we shall consider the nonlinear Schrödinger equation

$$i\partial_t \Psi = -\partial_x{}^2 \Psi + 2c\Psi^*\Psi\Psi \tag{1.10}$$

with the Hamiltonian

$$H = \int_0^L dx \, (\partial_x \Psi^* \, \partial_x \Psi + c\Psi^*\Psi^*\Psi\Psi) \tag{1.11}$$

and the Poisson brackets of the fields $\Psi$, $\Psi^*$ given as

$$\{\Psi(x), \, \Psi^*(y)\} = i\delta(x-y). \tag{1.12}$$

The charge (number of particles) $Q$ and momentum $P$

$$Q = \int_0^L \Psi^*\Psi \, dx; \qquad P = -i \int_0^L \Psi^* \partial_x \Psi \, dx \tag{1.13}$$

commute with $H$: $\{H, Q\} = \{H, P\} = 0$. This model is the classical limit of the quantum nonlinear Schrödinger equation studied in detail in Chapter I. The nonlinear Schrödinger equation can be represented in the Lax form (1.1); the $2 \times 2$ matrices $V$ and $U$ are given by

$$V(x|\lambda) = i\frac{\lambda}{2}\sigma_z + \Omega(x), \tag{1.14}$$

$$U(x|\lambda) = i\frac{\lambda^2}{2}\sigma_z + \lambda\Omega(x) + i\sigma_z \left(\partial_x \Omega + c\Psi^*\Psi\right), \tag{1.15}$$

where $\sigma_z$ is the Pauli matrix ($\sigma_z = \operatorname{diag}(1, -1)$) and the matrix $\Omega$ is

$$\Omega(x) = \begin{pmatrix} 0 & i\sqrt{c}\Psi^*(x) \\ -i\sqrt{c}\Psi(x) & 0 \end{pmatrix}. \tag{1.16}$$

The transition matrix has the following properties:

$$\det \mathsf{T}(x, y|\lambda) = 1; \tag{1.17}$$

$$\sigma_x \mathsf{T}^*(x, y|\lambda^*)\sigma_x = \mathsf{T}(x, y|\lambda); \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.18}$$

The corresponding L-operator on the infinitesimal lattice is:

$$\mathsf{L}(n|\lambda) = \begin{pmatrix} 1 - i\frac{\lambda\Delta}{2} & -i\sqrt{c}\Psi_n^*\Delta \\ i\sqrt{c}\Psi_n\Delta & 1 + i\frac{\lambda\Delta}{2} \end{pmatrix} + O(\Delta^2), \tag{1.19}$$

$$\Psi_n = \frac{1}{\Delta}\int_{x_{n-1}}^{x_n} \Psi(x)\,dx; \qquad \{\Psi_n,\, \Psi_m^*\} = \frac{i}{\Delta}\delta_{nm}. \tag{1.20}$$

The trace identities for this model are as follows (see (1.9), (1.11), (1.13)):

$$\ln\left[e^{i\lambda L/2}\tau(\lambda)\right] \underset{\lambda\to i\infty}{\longrightarrow} ic\left[\lambda^{-1}Q + \lambda^{-2}P + \lambda^{-3}H + O\left(\lambda^{-4}\right)\right]. \tag{1.21}$$

Let us derive this formula. Taking $\lambda \to i\infty$ the potential $V$ in (1.14) becomes close to diagonal and one can represent the transition matrix as follows:

$$\mathsf{T}(x, y|\lambda) = G(x|\lambda)D(x, y|\lambda)G^{-1}(y|\lambda). \tag{1.22}$$

Here $D$ is a diagonal matrix, and the matrix $G$ is chosen have the following form:

$$G(x|\lambda) = I + \sum_{k=1}^{\infty} \lambda^{-k} G_k(x), \tag{1.23}$$

where $I$ is the unit matrix and the $G_k$ are antidiagonal matrices. The meaning of representation (1.22) is that the transition matrix can be diagonalized by means of a gauge transformation. The differential equation (1.6) results in the following equation for $D$:

$$[\partial_x + W(x|\lambda)]\, D(x, y|\lambda) = 0; \qquad D(y, y|\lambda) = I \tag{1.24}$$

where the potential $W$ is equal to

$$W(x|\lambda) = G^{-1}(x)\,\partial_x G(x) + i\frac{\lambda}{2}G^{-1}(x)\sigma_z G(x) + G^{-1}(x)\Omega(x)G(x). \tag{1.25}$$

The matrices $G_k$ are defined by the requirement that the potential $W$ be a diagonal matrix. It is easy to show that

$$G_1 = i\sigma_z\Omega; \quad G_2 = -\partial_x\Omega; \quad G_3 = i\sigma_z\left(-\partial_x^2\Omega + \Omega^3\right). \tag{1.26}$$

Thus, the potential $W$ is

$$W = i\frac{\lambda}{2}\sigma_z + \lambda^{-1}W_1 + \lambda^{-2}W_2 + \lambda^{-3}W_3 + O\left(\lambda^{-4}\right) \tag{1.27}$$

where

$$\begin{aligned} &W_1 = -i\sigma_z\Omega^2; \qquad W_2 = -\Omega\partial_x\Omega; \\ &W_3 = i\sigma_z\left[\Omega\,\partial_x^2\Omega - \Omega^4\right]. \end{aligned} \tag{1.28}$$

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Due to the diagonality of the matrix $W$, equation (1.24) can be solved explicitly:

$$D(x,y|\lambda) = \exp\left\{-\int_y^x W(z|\lambda)\,dz\right\}. \tag{1.29}$$

Now take $y = 0$ and $x = L$. The periodic boundary conditions imply that $G(L) = G(0)$. Using (1.17), (1.22) one has that $\det D(L,0) = 1$. Hence

$$D(L,0|\lambda) = \exp\{\sigma_z Z(\lambda)\} \tag{1.30}$$

with a scalar function $Z(\lambda)$. It is easily shown from (1.27)–(1.29) that

$$Z(\lambda) = -i\frac{\lambda L}{2} + ic\left[\lambda^{-1}Q + \lambda^{-2}P + \lambda^{-3}H + O\left(\lambda^{-4}\right)\right]. \tag{1.31}$$

Due to the periodic boundary conditions, we have

$$\tau(\lambda) = \operatorname{tr} \mathsf{T}(L,0|\lambda) = \operatorname{tr} D(L,0|\lambda) \tag{1.32}$$

and, for $\lambda \to i\infty$, $D_{11}(L,0|\lambda) \gg D_{22}(L,0|\lambda)$. Using (1.30)–(1.32) we can calculate $\ln\tau(\lambda)$ and obtain (1.21). Actually, higher order terms in (1.21) are also interesting:

$$\ln\left[e^{i\lambda L/2}\tau(\lambda)\right] \underset{\lambda\to i\infty}{\longrightarrow} ic\sum_{n=1}^{\infty}\lambda^{-n}I_n. \tag{1.33}$$

In the next section, we shall see that each $I_n$ is time-independent. Thus, the $\{I_n\}$ comprise the infinite set of conservation laws that exactly solvable models possess.

Let us write down the first nontrivial conservation law:

$$\int dx\left\{\Psi^*\Psi_{xxx} - \frac{3c}{2}\Psi^{*2}\left(\Psi^2\right)_x\right\}. \tag{1.34}$$

## V.2 The classical r-matrix

To construct the action-angle variables, it is necessary to calculate the Poisson brackets (PB) between matrix elements of the transition matrix (see [5]). There exists an effective method for performing such calculations, based on the classical r-matrix. We shall use the following notation for tensor products. The tensor product of two $k \times k$ matrices $A$ and $B$ will be denoted by $A \otimes B$ (a $k^2 \times k^2$ matrix). The $k^2 \times k^2$ permutation matrix $\Pi$ has the following property:

$$\Pi(A \otimes B)\Pi = B \otimes A. \tag{2.1}$$

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This equality is valid for all numerical matrices $A$ and $B$. The minimum dimension of $\Pi$ is $4 \times 4$; in this case it can be written as

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

Definition: The Poisson brackets of the tensor product $\{A \overset{\otimes}{,} B\}$ is a $k^2 \times k^2$ matrix, its matrix elements being equal to the PB of some matrix element of $A$ with some matrix element of $B$. The labeling of the elements of the matrix $\{A \overset{\otimes}{,} B\}$ is the same as for the matrix $A \otimes B$. (Tensor notation is discussed in detail in Appendix 1.)

**Theorem 1**. *If the PB between matrix elements of $V(x|\lambda)$ can be represented in the form*

$$\{V(x|\lambda) \overset{\otimes}{,} V(y|\mu)\} = \delta(x-y)\,[\mathsf{r}(\lambda,\mu),\, V(x|\lambda) \otimes I + I \otimes V(x|\mu)] \tag{2.3}$$

*then the PB between matrix elements of the transition matrix is given by*

$$\{\mathsf{T}(x,y|\lambda) \overset{\otimes}{,} \mathsf{T}(x,y|\mu)\} = [\mathsf{T}(x,y|\lambda) \otimes \mathsf{T}(x,y|\mu),\, \mathsf{r}(\lambda,\mu)]. \tag{2.4}$$

*The square brackets on the right hand side denote the matrix commutator of two $k^2 \times k^2$ matrices. The matrix* r, *a $k^2 \times k^2$ matrix with elements depending on $\lambda$ and $\mu$, acts in the tensor product of the two spaces (*r *does not depend on space variables).*

Proof: The transition matrix $\mathsf{T}(x,y|\lambda)$ depends on the dynamical variables only through the potential $V(x|\lambda)$. Thus the PB of its matrix elements can be expressed in the form

$$\{\mathsf{T}_{ij}(x,y|\lambda),\, \mathsf{T}_{kl}(x,y|\mu)\} = \int_y^x dz_\lambda \int_y^x dz_\mu \left(\frac{\delta \mathsf{T}_{ij}(x,y|\lambda)}{\delta V_{ab}(z_\lambda|\lambda)}\right)\left(\frac{\delta \mathsf{T}_{kl}(x,y|\mu)}{\delta V_{cd}(z_\mu|\mu)}\right) \times \{V_{ab}(z_\lambda|\lambda),\, V_{cd}(z_\mu|\mu)\}. \tag{2.5}$$

The variational derivative of $\mathsf{T}(x,y|\lambda)$ with respect to $V(z|\lambda)$ is obtained from

$$\delta \mathsf{T}(x,y|\lambda) = -\int_y^x \mathsf{T}(x,z|\lambda)\,\delta V(z|\lambda) \mathsf{T}(z,y|\lambda)\,dz. \tag{2.6}$$

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Substituting this expression into (2.5), one obtains in tensor notation

$$\{\mathsf{T}(x,y|\lambda) \overset{\otimes}{,} \mathsf{T}(x,y|\mu)\} = \int_y^x dz_\lambda \int_y^x dz_\mu \, (\mathsf{T}(x,z_\lambda|\lambda) \otimes \mathsf{T}(x,z_\mu|\mu)) \times \{V(z_\lambda|\lambda) \overset{\otimes}{,} V(z_\mu|\mu)\} \, (\mathsf{T}(z_\lambda,y|\lambda) \otimes \mathsf{T}(z_\mu,y|\mu)) \,. \quad (2.7)$$

Using (2.3) one writes

$$\{\mathsf{T}(x,y|\lambda) \overset{\otimes}{,} \mathsf{T}(x,y|\mu)\} = \int_y^x dz \, (\mathsf{T}(x,z|\lambda) \otimes \mathsf{T}(x,z|\mu)) \times [\mathsf{r}(\lambda,\mu), \, V(z|\lambda) \otimes I + I \otimes V(z|\mu)] \, (\mathsf{T}(z,y|\lambda) \otimes \mathsf{T}(z,y|\mu)) \quad (2.8)$$

It follows from (1.7) that

$$\partial_y \mathsf{T}(x,y|\lambda) = \mathsf{T}(x,y|\lambda) V(y|\lambda) \quad (2.9)$$

which removes $V(z)$ from (2.8) to give

$$\{\mathsf{T}(x,y|\lambda) \overset{\otimes}{,} \mathsf{T}(x,y|\mu)\} = -\int_y^x dz \, \frac{d}{dz} \Big[ (\mathsf{T}(x,z|\lambda) \otimes \mathsf{T}(x,z|\mu)) \times \mathsf{r}(\lambda,\mu) \, (\mathsf{T}(z,y|\lambda) \otimes \mathsf{T}(z,y|\mu)) \Big] . \quad (2.10)$$

Evaluating the integral over the total derivative and using $\mathsf{T}(x,x|\lambda) = I$ one arrives at formula (2.4). The theorem is proved. (In [6] and [7], this theorem was generalized to the case where r depends on $x$.)

With the help of the r-matrix, the PB of the transition matrix in the discrete case can also be calculated.

**Theorem 2.** *If the PB between the elements of* $\mathsf{L}(\lambda)$ *can be expressed in the form*

$$\{\mathsf{L}(k|\lambda) \overset{\otimes}{,} \mathsf{L}(l|\mu)\} = \delta_{kl} [\mathsf{L}(k|\lambda) \otimes \mathsf{L}(l|\mu), \, \mathsf{r}(\lambda,\mu)] \quad (2.11)$$

*then the PB of* $\mathsf{T}(\lambda)$ *is given by*

$$\{\mathsf{T}(n,m|\lambda) \overset{\otimes}{,} \mathsf{T}(n,m|\mu)\} = [\mathsf{T}(n,m|\lambda) \otimes \mathsf{T}(n,m|\mu), \, \mathsf{r}(\lambda,\mu)] . \quad (2.12)$$

Proof: The theorem is proved by induction using expression (2.11) as the basis of the induction. Using (1.8), now consider

$$\mathsf{T}(n+1,m|\lambda) = \mathsf{L}(n+1,m|\lambda) \mathsf{T}(n,m|\lambda) . \quad (2.13)$$

It follows from (2.11) that $\{\mathsf{L}(n+1|\lambda) \overset{\otimes}{,} \mathsf{T}(n,m|\mu)\} = 0$. Supposing that (2.12) is valid, one can easily obtain

$$\{\mathsf{T}(n+1,m|\lambda) \overset{\otimes}{,} \mathsf{T}(n+1,m|\mu)\} = [\mathsf{T}(n+1,m|\lambda) \otimes \mathsf{T}(n+1,m|\mu),\ \mathsf{r}(\lambda,\mu)]. \tag{2.14}$$

The proof is thus finished.

The r-matrix in (2.4), (2.12) must satisfy some identities. It acts on the tensor product of two $k$-dimensional vector spaces. Consider the triple tensor product of $k$-dimensional vector spaces. Then

$$[\mathsf{r}_{13}(\lambda,\nu),\ \mathsf{r}_{23}(\mu,\nu)] + [\mathsf{r}_{12}(\lambda,\mu),\ \mathsf{r}_{13}(\lambda,\nu) + \mathsf{r}_{23}(\mu,\nu)] = 0. \tag{2.15}$$

The spaces on which the r-matrix acts non-trivially are denoted by subscripts (see Appendix 1). The square brackets denote the commutator, in (2.15).

As follows from (2.4), the r-matrix is defined up to a scalar term $a(\lambda,\mu)E$ ($E$ is the $k^2 \times k^2$ unit matrix, $a$ is an arbitrary c-number function). The r-matrix can be chosen as an antisymmetric matrix: $\mathsf{r}(\lambda,\mu) = -\Pi\mathsf{r}(\mu,\lambda)\Pi$ ($\Pi$ is the permutation matrix). The relation (2.15) is called the classical Yang-Baxter relation. It is merely a restatement of the Jacobi identity for the PB (2.12). The point $\lambda = \mu = \nu$ is a pole of the r-matrix; thus the relation (2.15) should be defined more precisely at this point [12].

The simplest r-matrix is

$$\mathsf{r}(\lambda,\mu) = \frac{c}{\lambda-\mu}\Pi, \tag{2.16}$$

which is the r-matrix for the nonlinear Schrödinger equation.

The existence of the r-matrix for a given potential $V$ (2.3) or L-operator (2.11) is not obvious in advance. However, r-matrices exist for the majority of models solved by ISM.

The existence of the r-matrix results in the presence of infinitely many conservation laws in the model. This follows from the fact that the traces of the monodromy matrix, defined in (1.9), commute for different values of the spectral parameter:

$$\{\tau(\lambda),\ \tau(\mu)\} = 0. \tag{2.17}$$

To obtain this relation, one calculates the traces of both sides of (2.4) (or (2.12)) in the $k^2 \times k^2$-matrix space and then uses the fact that the trace of the tensor product of two matrices is equal to the product of the traces of each matrix. From (2.17) it follows that $\{\ln\tau(\lambda),\ \tau(\mu)\} = 0$. If we expand this in inverse powers of $\lambda$ and use the trace identities (1.21), we shall arrive at $\{H,\ \tau(\mu)\} = 0$. This means that the trace of the monodromy matrix, $\tau(\mu)$, is time-independent: $\partial_t\tau(\mu) = 0$. Thus, all the $I_n$ in (1.33)

give rise to conservation laws. This means that $\tau(\mu)$ is the generating functional for the integrals of motion. It should be mentioned that the r-matrix replaces, in a sense, the time evolution operator $U(x|\lambda)$ (1.2). It can be proved that if the Hamiltonian and Poisson brackets are given, and if the potential $V$ (or L-operator) and the r-matrix exist and the trace identities hold, then the initial nonlinear equation can be expressed in the Lax form (1.1).

Let us explain the construction of the corresponding time evolution operator $U(x|\lambda)$ in the continuous case. Changing $\partial_t V$ in (1.1) to $\partial_t V = \{H, V\}$ and using the fact that the Hamiltonian is expressed by means of the trace identities (1.21) in terms of $\tau(\lambda)$, one concludes that it is sufficient to consider the Poisson brackets between the monodromy matrix and the potential.

**Theorem 3**. *The following Lax representation is valid:*

$$\{\tau(\mu),\, V(x|\lambda)\} = \partial_x U(x|\lambda,\mu) + [V(x|\lambda),\, U(x|\lambda,\mu)]. \tag{2.18}$$

The proof is analogous to that of Theorem 1. Note that on the left hand side of (2.18) $\tau$ is a scalar and $V$ is a matrix. Let us indicate by a subscript 2 the space in which the matrix $V$ acts and introduce an auxiliary space with subscript 1. Rewriting (2.18) in the form

$$\{\tau(\mu),\, V_2(x|\lambda)\} = \partial_x U_2(x|\lambda,\mu) + [V_2(x|\lambda),\, U_2(x|\lambda,\mu)], \tag{2.19}$$

we can introduce $U_2$ as a trace in the first space:

$$U_2(x|\lambda,\mu) = \mathrm{tr}_1\left(\mathsf{T}_1(L,x|\mu)\mathsf{r}_{12}(\mu,\lambda)\mathsf{T}_1(x,0|\mu)\right). \tag{2.20}$$

The matrices act in the spaces indicated by the subscripts, and $\mathsf{T}$ is the transition matrix. It is natural to call $U_2(x|\lambda,\mu)$ (2.20) the generating functional of time evolution operators. An analogous theorem is valid for lattice models. Due to Theorem 3, we shall not (as a rule) use the time evolution operator further.

To get (1.1) from (2.18) one should first rewrite (2.18) in the form

$$\{\ln\tau(\mu),\, V(x|\lambda)\} = \partial_x\left(\tau^{-1}(\mu)U(x|\lambda,\mu)\right) + [V(x|\lambda),\, \left(\tau^{-1}(\mu)\right)U(x|\lambda,\mu)]. \tag{2.21}$$

Then one should use the trace identities. In the case of the nonlinear Schrödinger equation, one should expand both sides of equation (2.21) in inverse powers of $\mu$ (see (1.21)). Then one should consider the coefficient of $\mu^{-3}$ in (2.21). On the left hand side, one will obtain $ic\{H, V(x|\lambda)\}$. This is proportional to $\partial_t V(x|\lambda)$. This means that $\tau^{-1}(\mu)U(x|\lambda,\mu)$ when

expanded in inverse powers of $\mu$ gives $U(x|\lambda)$ (from (1.1)) as the coefficient of $\mu^{-3}$. This example shows the relation between (2.18) and (1.1). In other words, $\tau^{-1}(\mu)U(x|\lambda,\mu)$ is the generating functional for time evolution operators $U(x|\lambda)$.

The method of constructing action-angle variables for completely integrable Hamiltonian equations is explained in [5]. Basically, one considers the scattering of plane-waves on the potential in the equation $\partial_x\Phi + V(x|\lambda)\Phi = 0$. The scattering matrix provides an opportunity for the explicit construction of action-angle variables.

The classical r-matrix fixes the structure of the action-angle variables. This suggests the construction of lattice variants of continuous field theory models. The explicit form of the r-matrix should be unchanged when changing the continuous Lax operator $(\partial_x + V)$ to the discrete operator $\mathsf{L}(n|\lambda)$. This will guarantee the conservation of the structure of the action-angle variables in the transition from the continuum to the lattice. Thus, the continuum limit $\Delta \to 0$ (condensation of the lattice) will proceed very smoothly.

## V.3 Examples

In this section examples of completely integrable Lorentz-invariant equations are given.

(1) The sine-Gordon (SG) model. The equation of motion is given by

$$\left(\partial_t^2 - \partial_x^2\right) u + \frac{m^2}{\beta} \sin \beta u = 0. \tag{3.1}$$

The Hamiltonian, momentum and charge are

$$H = \int dx \left( \frac{1}{2}\pi^2 + \frac{1}{2}\left(\partial_x u\right)^2 + \frac{m^2}{\beta^2}(1 - \cos \beta u) \right); \tag{3.2}$$

$$P = -\int dx\, \pi\, \partial_x u; \qquad Q = \frac{\beta}{2\pi} \int dx\, \partial_x u, \tag{3.3}$$

where $\pi(x) = \partial_t u(x)$, $\{\pi(x), u(y)\} = \delta(x - y)$. This model has many physical applications. It is of interest as a nontrivial model of a relativistic scalar field. The equation of motion possesses a zero curvature representation with the potential

$$V(x|\lambda) = \begin{pmatrix} i\beta\pi(x)/4 & (m/2)\, \sinh\{-\lambda + [i\beta u(x)/2]\} \\ (m/2)\, \sinh\{\lambda + [i\beta u(x)/2]\} & -i\beta\pi(x)/4 \end{pmatrix}. \tag{3.4}$$

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The L-operator on the infinitesimal lattice is

$$\mathsf{L}(n|\lambda) = \begin{pmatrix} 1 - i\beta p_n/4 & (m\Delta/2)\,\sinh\left[\lambda - (i\beta u_n/2)\right] \\ -(m\Delta/2)\,\sinh[\lambda + (i\beta u_n/2)] & 1 + i\beta p_n/4 \end{pmatrix} \tag{3.5}$$

where

$$u_n = \frac{1}{\Delta}\int_{x_{n-1}}^{x_n} u(x)\,dx; \qquad p_n = \int_{x_{n-1}}^{x_n} \pi(x)\,dx; \qquad \{p_n,\, u_m\} = \delta_{nm}. \tag{3.6}$$

The classical r-matrix (2.3) is

$$\mathsf{r}(\lambda,\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathsf{r}_{22}^{11} & \mathsf{r}_{21}^{12} & 0 \\ 0 & \mathsf{r}_{12}^{21} & \mathsf{r}_{11}^{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

with the nonzero elements ($\gamma \equiv \beta^2/8$)

$$\begin{aligned} \mathsf{r}_{22}^{11} &= \mathsf{r}_{11}^{22} = \gamma\coth(\lambda - \mu); \\ \mathsf{r}_{21}^{12} &= \mathsf{r}_{12}^{21} = -\frac{\gamma}{\sinh(\lambda - \mu)}. \end{aligned} \tag{3.8}$$

The trace identities can be found in [5]. For quantization, we must put the model on the lattice. The trace identities for the lattice model, which differ from the continuous case, are constructed in section 2 of Chapter VIII. The transition matrix possesses the following symmetry properties:

$$\sigma_y \mathsf{T}^*(x, y|\lambda^*)\sigma_y = \mathsf{T}(x, y|\lambda); \tag{3.9}$$

$$\det \mathsf{T}(x, y|\lambda) = 1. \tag{3.10}$$

(2) The Zhiber-Shabat-Mikhailov model. The Hamiltonian and the PB of this model are

$$\begin{aligned} H = \int dx \Bigg\{ &\frac{\gamma}{2}\pi^2 + \frac{1}{2\gamma}\,(\partial_x u)^2 \\ &+ \frac{m^2}{\gamma}\left[\exp(u) + \frac{1}{2}\exp(-2u) - \frac{3}{2}\right]\Bigg\}; \\ \pi(x) = \frac{1}{\gamma}\partial_t u(x), \qquad &\{\pi(x),\, u(y)\} = \delta(x - y). \end{aligned} \tag{3.11}$$

The equation of motion

$$\left(\partial_t^2 - \partial_x^2\right) u + m^2\left[\exp(u) - \exp(-2u)\right] = 0 \tag{3.12}$$

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can be expressed in the Lax form (1.1) with potential

$$V(x|\lambda)=\frac{1}{2}\begin{pmatrix} 0 & im\exp(u/2-\lambda) & -im\exp(u/2+\lambda) \\ -im\exp(u/2+\lambda) & -\partial_t u & im\exp(-u-\lambda) \\ im\exp(u/2-\lambda) & -im\exp(-u+\lambda) & \partial_t u \end{pmatrix} \tag{3.13}$$

The classical r-matrix has the following nonzero elements ($\kappa \equiv \exp(\lambda-\mu)$):

$$\begin{aligned}
&\mathrm{r}_{11}^{11}=\gamma\frac{2-\kappa^3-\kappa^{-3}}{4(\kappa^3-\kappa^{-3})};\\
&\mathrm{r}_{22}^{11}=\mathrm{r}_{33}^{11}=\mathrm{r}_{11}^{22}=\mathrm{r}_{11}^{33}=-\gamma\frac{2+\kappa^3+\kappa^{-3}}{4(\kappa^3-\kappa^{-3})};\\
&\mathrm{r}_{33}^{22}=\mathrm{r}_{22}^{33}=-\gamma\frac{\kappa^3+\kappa^{-3}}{2(\kappa^3-\kappa^{-3})};\\
&\mathrm{r}_{21}^{12}=\mathrm{r}_{13}^{31}=\gamma\frac{\kappa^2+\kappa^{-1}}{2(\kappa^3-\kappa^{-3})};\\
&\mathrm{r}_{13}^{12}=\mathrm{r}_{21}^{31}=\gamma\frac{\kappa^2-\kappa^{-1}}{2(\kappa^3-\kappa^{-3})};\\
&\mathrm{r}_{31}^{13}=\mathrm{r}_{12}^{21}=\gamma\frac{\kappa+\kappa^{-2}}{2(\kappa^3-\kappa^{-3})};\\
&\mathrm{r}_{12}^{13}=\mathrm{r}_{31}^{21}=-\gamma\frac{\kappa-\kappa^{-2}}{2(\kappa^3-\kappa^{-3})};\\
&\mathrm{r}_{32}^{23}=\gamma\frac{\kappa^2}{(\kappa^3-\kappa^{-3})};\\
&\mathrm{r}_{23}^{32}=\gamma\frac{\kappa^{-2}}{(\kappa^3-\kappa^{-3})}.
\end{aligned} \tag{3.14}$$

One should mention finally that there are very few completely integrable systems where one relativistic scalar (Hermitian) field, $u$, satisfies an equation of the form

$$\partial_t^2 u-\partial_x^2 u+f(u)=0.$$

Here $f(u)$ is some arbitrary function of the field. Besides the sine-Gordon (sinh-Gordon) and Zhiber-Shabat-Mikhailov equations, the Liouville ($f(u)=e^u$) and Klein-Gordon ($f(u)=m^2u$) equations are the only other forms that are completely integrable.

More examples of classical completely integrable differential equations can be found in Chapter XIV. They will appear in relation to quantum correlation functions.

## Conclusion

In this chapter we have learned that completely integrable evolution equations are very special. They have an infinite set of conservation laws. The trace of the monodromy matrix $\tau(\lambda)$ is the generating functional of these conservation laws. The Lax representation (1.1) introduces an auxiliary matrix structure (and spectral parameter $\lambda$), which helps in solving the model. Theorem 3 in section 2 shows that the existence of an r-matrix guarantees the existence of a Lax representation for the given equation. This is the reason why we shall mainly pay attention to the spatial shift operator $(\partial_x + V)$ rather than the time-evolution operator $(\partial_t - U)$.

The Hamiltonian aspects of the classical inverse scattering method which we have presented above will be used for quantization in the next chapter. The method of calculating the PB based on the classical r-matrix was introduced in [13]. The proof of Theorem 1 which we have given appeared in [6], and the proof of its discrete analogue (Theorem 2) in [9]. Solutions of equation (2.15) are completely investigated in [2], [4], [12] and [10]. Let us mention that the nonlinear Schrödinger equation was solved with the help of CISM in [19] and its r-matrix (2.16) found in [13]. The trace identities were obtained in [18]. The sine-Gordon model was solved in [1], [8], [15], [16], [17] and [20] and the r-matrix was found in [14]. The trace identities can be found, for example, in [5]. The Zhiber-Shabat-Mikhailov model was introduced in [3] and solved in [11] and [21]. Its r-matrix was computed in [6]. It should be noted that all integrable equations for one interacting relativistic scalar field in two dimensions were enumerated in [11] and [21]. These are the sine-Gordon model, the hyperbolic sine-Gordon model (obtained by analytic continuation in the coupling constant) and the Zhiber-Shabat-Mikhailov model (the Liouville model is a limiting case of the latter).

In order to find out how the Lax representation helps in actually solving differential equations one should look through chapters XIV–XVI.

# Appendix V.1: Tensor notation

We use the following notation. The tensor product of two $k \times k$ matrices $A$ and $B$ (which is a $k^2 \times k^2$ matrix $A \otimes B$) is defined as

$$(A \otimes B)^{ij}_{kl} = A_{ij} B_{kl}, \qquad (i, j, k, l = 1, \ldots, k). \tag{A.1.1}$$

The matrix elements of the $k^2 \times k^2$ matrices are labeled by the block indices $i, j$ ($i$ is the number of the block row and $j$ that of the block column) and by the intrinsic indices $k$, $l$ ($k$ is the number of the row and $l$ that of the column). As an example, we shall write out a $4 \times 4$ matrix $R$ which acts in the tensor product of two $2 \times 2$ spaces:

$$R = \begin{pmatrix} R^{11}_{11} & R^{11}_{12} & R^{12}_{11} & R^{12}_{12} \\ R^{11}_{21} & R^{11}_{22} & R^{12}_{21} & R^{12}_{22} \\ R^{21}_{11} & R^{21}_{12} & R^{22}_{11} & R^{22}_{12} \\ R^{21}_{21} & R^{21}_{22} & R^{22}_{21} & R^{22}_{22} \end{pmatrix}. \tag{A.1.2}$$

In this notation, the matrix product of two $k^2 \times k^2$ matrices is written in the following way:

$$(RS)^{ij}_{kl} = R^{im}_{kn} S^{mj}_{nl} \tag{A.1.3}$$

(summation over repeated indices is assumed). The unit matrix $E$ and permutation matrix $\Pi$ (both $k^2 \times k^2$ matrices) are:

$$E = I \otimes I, \qquad E^{ij}_{kl} = \delta_{ij} \delta_{kl}; \tag{A.1.4}$$

$$\Pi^{ij}_{kl} = \delta_{il} \delta_{kj}, \qquad \Pi^2 = E. \tag{A.1.5}$$

Let us define the PB tensor product of two matrices $A$ and $B$ (both of which depend on the dynamical variables) by

$$\{A \overset{\otimes}{,} B\}^{ij}_{kl} = \{A_{ij}, B_{kl}\}. \tag{A.1.6}$$

The right hand side is the usual PB of the corresponding matrix elements of the matrices $A$ and $B$.

Sometimes a different notation is used. Denoting the space where a matrix acts by a subscript, we can rewrite the relation

$$\{\mathsf{T}(\lambda) \overset{\otimes}{,} \mathsf{T}(\mu)\} = [\mathsf{T}(\lambda) \otimes \mathsf{T}(\mu),\ \mathsf{r}(\lambda, \mu)] \tag{A.1.7}$$

in the form

$$\{\mathsf{T}_1(\lambda),\ \mathsf{T}_2(\mu)\} = [\mathsf{T}_1(\lambda)\mathsf{T}_2(\mu),\ \mathsf{r}_{12}(\lambda, \mu)]. \tag{A.1.8}$$

This notation is used in (2.15), (2.19).

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