Boundary value problems

Consider problems of the type:

$$\frac{d^2T}{dx^2} + h(T_a - T) = 0$$

Where we want to find $T(x)$

Given $T(0) = \mathrm{T}_0$ and $T(L) = T_1$

We don not have two initial conditions. Instead, two conditions at different locations (positions of the independent variable x)

Shooting Method

Convert the BVP into an equivalent initial value problem.

But we need to guess the initial conditions,

A trial and error approach is used

Example: in the problem

$$\frac{d^2T}{dx^2} + h(T_a - T) = 0$$

Take, $h = 0.01\ m^{-2}$, $T_a = 20$ and the boundary conditions $T(0) = 40$ and $T(10) = 200$.

Find $T(x)$ for x in the range 0 to 10.

This problem is from Chapra And Canale . 27.1

Step 1- Convert into first order ODEs. We will have two dependent variables.

$$\frac{dT}{dx} = z \qquad \text{Eqn 1}$$

$$\frac{dz}{dx} = -h(T - T_a) \qquad \text{Eqn 2}$$

Step 2.

We need an initial value for z

Guess value 1. Take $z(0) = 10$ (Note- this is purely a guess)

Step 3.

Solve Eqn 1 and 2, using the conditions, $T(0) = 40, z(0) = 10$

Using a suitable technique- eg. RK4.

This gives, $T(10) = 168.3797$ (calculated from first guess)

Step 4. But given problem says $T(10) = 200$.

So take a second guess- $z(0) = 20$

Repeat Step 3 using

$T(0) = 40, z(0) = 20$ and RK4 method.

Now we get $T(10) = 285.8980$

Step 5.Use the two previous solutions to improve the guess for $z(0)$.

$z(0) = 10$ gives $T(10) = 168.3797$ (point 1)

$z(0) = 20$ gives $T(10) = 285.8980$ (point 2)

$z(0) =?$ Will give $T(10) = 200$

Try linear interpolation formula by finding a straight line that passes through points 1 and 2.

$$z(0) = 10 + \frac{(20 - 10)}{(285.8980 - 168.3797)}(200 - 168.3797)$$

Gives new estimate: $z(0) = 12.6907$

Step 6. Try once more with $z(0) = 12.6907$

and check if the final solutions $T(10) = 200$ or is close (within a tolerance).

Shooting method for nonlinear problems

The solution to the BVP is a function of the initial condition on z

$f(z_0) = 200$ (only that we do not know the functional form of $f(z_0)$).

You can "shoot" multiple times and then fit a function/interpolate to get the optimum z_0.

Finite Difference Method

Solve $y'' = f(x, y, y')$ with $y(a) = \alpha$ and $y(b) = \beta$

This will be very useful when we do Partial Differential Equations.

In the finite difference method, we divide the range of integration (a, b) into m equal sub-intervals.

The denoted by $y_i, i = 0,1,2, \dots . m$

1. Derivatives of y are replaced by finite difference expression.

It is common to use central difference formula.

But with end-points- use higher difference formula, with error of the same order. Alternately, introduce extra spill-over points outside the boundaries to continue using central difference formula.

$y' = \frac{y_{i+1} - y_{i-1}}{2h}$ and $y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$ for the middle points

2. Differential equation is enforced at the mesh points:

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = f(\mathrm{x_i}, \mathrm{y_i}, \frac{y_{i+1} - y_{i-1}}{2h}) \text{ , } i = 0,1,2, \dots . m$$

$y_0 = \alpha$ or $\frac{y_1 - y_{-1}}{2h} = \alpha$, in case $y'(a)$ is given

And $y_m = \beta$ or $\frac{y_{m-1} - y_{m+1}}{2h} = \beta$

Here there is a spill over- y_{-1} and y_{m+1} are introduced

$$y_{-1} - 2y_0 + y_1 - h^2 f\left(x_0, y_0, \frac{y_1 - y_{-1}}{2h}\right) = 0$$ Equation A

$$y_{i-1} - 2y_i + y_{i+1} - h^2 f\left(\mathrm{x_i}, \mathrm{y_i}, \frac{y_{i+1} - y_{i-1}}{2h}\right) = 0$$ for $i = 1,2, \dots m - 1$

Equation B

And

$$y_{m-1} - 2y_m + y_{m+1} - h^2 f\left(x_0, y_0, \frac{y_1 - y_{-1}}{2h}\right) = 0$$ Equation C

To introduce Boundary conditions

$y_0 - \alpha = 0$ to replace Equation A if $y(a) = \alpha$ is given.
Or $-2y_0 + 2y_1 - h^2 f(x_0, y_0, \alpha) - 2h\alpha = 0$ is used to replace Equation A

For the other boundary
$y_m - \beta = 0$ to replace Equation C if $y(b) = \beta$ is supplied.
Or $2y_{m-1} - 2y_m - h^2 f(x_m, y_m, \beta) + 2h\beta = 0$ to replace Equation C

Finally,

Solve for $y_0, y_1, y_2, , y_m$ using Gauss Elimination or standard techniques. In case of nonlinear methods, you can use a variation of Newton Raphson for many variables.

Note that when we introduce the BC, we eliminate y_{-1} and y_{m+1} terms.

Also note that you can be given mixed Boundary conditions.