1 Drift-kinetic classical transport

These notes derive the classical transport under the same set of assumptions as in the linearized drift-kinetic equation solved by the δf-code neoclassical code SFINCS (and probably most other neoclassical codes).

1.1 Motivation

For a mass-ratio expanded ion-impurity collision operator, we found that the classical transport of collisional impurities in W7-X can be comparable to the neoclassical transport. Thus, it would be of interest to have a general numerical tool to calculate the classical transport alongside the neoclassical transport from a drift-kinetic solver such as SFINCS.

1.2 Gyrophase dependent part of f

In many ways, calculating the classical transport is simpler than calculating the neoclassical transport, as the gyrophase dependent part of the distribution function is smaller than the gyrophase-independent in the expansion parameter ρ/L, and the gyrophase dependent part to required order is given entirely by the zeroth order gyrophase independent distribution. When f_{a0} is a Maxwellian f_{aM}, the gyrophase dependent part is given by

$$\tilde{f}_{a1} = -\vec{\rho}_a(\gamma) \cdot \nabla f_{Ma} = -\vec{\rho}(\gamma) \cdot \nabla \psi \frac{\partial f_{Ma}}{\partial \psi}, \tag{1}$$

where γ is the gyrophase, ψ a flux-label and $\vec{\rho}$ the gyrophase vector

$$\vec{\rho}_a = \frac{1}{\Omega_a} \vec{b} \times \vec{v}_\perp, \tag{2}$$

with $\vec{v}_\perp$ the velocity perpendicular to the magnetic field, $\Omega_a = Z_a eB/m_a$ the gyrofrequency, $\vec{b}$ the unit vector in the direction of the magnetic field $\vec{B}$; and ψ a flux-label, which acts as our radial coordinate.

Equation 1 can be massaged a bit to yield (we drop the tilde since we will only consider the gyrophase dependent part of f_{a1} here)

$$f_{a1} = \frac{1}{\Omega_a} (\vec{b} \times \nabla \psi) \cdot \vec{v}_\perp \frac{\partial f_{Ma}}{\partial \psi}, \tag{3}$$

and we introduce the local coordinate system $\vec{v}_\perp = v_2 \hat{e}_2 + v_3 \hat{e}_3$, where $\hat{e}_2$, $\hat{e}_3$ form an orthonormal triplet with $\vec{b}$. There is some freedom in how we align our $\hat{e}_2$, $\hat{e}_3$ coordinates, and we choose these vectors so that $\hat{e}_2$ is entirely in the $(\vec{b} \times \nabla\psi)$ direction. Thus

$$f_{a1} = \frac{1}{\Omega_a}|\vec{b} \times \nabla\psi| v_2 \frac{\partial f_{Ma}}{\partial \psi}. \tag{4}$$

Finally, we evaluate $\frac{\partial f_{Ma}}{\partial \psi}$. For heavy impurities, the lowest order distribution may vary on the flux-surface in response to a flux-surface variation of the electrostatic potential, in which case, we write

$$f_{Ma} = \eta_a(\psi)\left(\frac{m_a}{2\pi T_a}\right)^{3/2} \exp\left(-\frac{m_a v^2}{2T_a} - \frac{Z_a e\tilde{\Phi}}{T_a}\right), \tag{5}$$

where the *pseudo-density* η_a is related to the ordinary density by

$$\eta_a = n_a \mathrm{e}^{Z_a e\tilde{\Phi}/T_a}. \tag{6}$$

The gradient thus becomes

$$\frac{\partial f_{Ma}}{\partial \psi} = f_{Ma}\left(\frac{\mathrm{d}\ln \eta_a}{\mathrm{d}\psi} + \frac{Z_a e}{T_a}\frac{\partial \tilde{\Phi}}{\partial \psi} + \frac{Z_a e\tilde{\Phi}}{T_a}\frac{\mathrm{d}\ln T_a}{\mathrm{d}\psi} + \left[\frac{m_a v^2}{2T_a} - \frac{3}{2}\right]\frac{\mathrm{d}\ln T_a}{\mathrm{d}\psi}\right). \tag{7}$$

To simplify notation, we will introduce the quantities

$$\alpha_{1a} \equiv \frac{\mathrm{d}\ln \eta_a}{\mathrm{d}\psi} + \frac{Z_a e}{T_a}\frac{\partial \tilde{\Phi}}{\partial \psi} + \frac{Z_a e\tilde{\Phi}}{T_a}\frac{\mathrm{d}\ln T_a}{\mathrm{d}\psi} - \frac{3}{2}\frac{\mathrm{d}\ln T_a}{\mathrm{d}\psi} \tag{8}$$

$$\alpha_{2a} \equiv \frac{\mathrm{d}\ln T_a}{\mathrm{d}\psi}, \tag{9}$$

so that $\frac{\partial f_{Ma}}{\partial \psi} = f_{Ma}(\alpha_{a1} + \alpha_{2a}x_a^2)$, where

$$x_a^2 \equiv \frac{m_a v^2}{2T_a} \equiv \frac{v^2}{v_{Ta}^2}. \tag{10}$$

Thus

$$f_{a1} = \frac{|\vec{b} \times \nabla\psi|}{\Omega_a} v_2 f_{Ma}(\alpha_{1a} + \alpha_{2a}x_a^2). \tag{11}$$

1.3 Radial fluxes

The classical radial flux of particle and heat is given by

$$\Gamma_a \equiv \left\langle \vec{\Gamma}_a \cdot \nabla\psi \right\rangle^{\mathrm{C}} \equiv \left\langle \frac{\vec{b} \times \nabla\psi}{Z_a e B} \cdot \vec{R}_a \right\rangle, \tag{12}$$

$$Q_a \equiv \left\langle \vec{Q}_a \cdot \nabla\psi \right\rangle^{\mathrm{C}} \equiv \left\langle \frac{\vec{b} \times \nabla\psi}{Z_a e B} \cdot \vec{G}_a \right\rangle, \tag{13}$$

where we have introduced the *friction force* and *energy-weighted friction force*

$$\vec{R}_a = \int \mathrm{d}^3 v m_a \vec{v} C[f_a], \tag{14}$$

$$\vec{G}_a = \int \mathrm{d}^3 v \frac{m_a v^2}{2} m_a \vec{v} C[f_a]. \tag{15}$$

Here, $C[f_a]$ is the Fokker-Planck collision operator accounting for the effects of all species acting on f_a

$$C[f_a] = \sum_b C_{ab}\left[f_a, f_b\right]. \tag{16}$$

An important property of the Fokker-Planck operator is that it preserves the gyrophase dependence of f_a. Thus, only the gyrophase dependent part of f_a will contribute to the friction force in the $\vec{b}\times\nabla\psi$ direction. As the gyrophase dependent part of f_a acts as a small correction to f_{Ma}, we can linearize the Fokker-Planck operator around f_{Ma} and write

$$\begin{aligned} C[f_a] &= \sum_b C_{ab}\left[f_{Ma} + f_{1a}, f_{Mb} + f_{1b}\right] \\ &\approx \sum_b C_{ab}[f_{Ma}, f_{Mb}] + C_{ab}[f_{Ma}, f_{b1}] + C_{ab}[f_{a1}, f_{Mb}], \end{aligned} \tag{17}$$

where the gyrophase-independent $C_{ab}[f_{Ma}, f_{Mb}]$ part will not contribute to the classical transport. With this result, we write the relevant components of $\vec{R}_a$ and $\vec{G}_a$ as

$$\vec{b}\times\nabla\psi\cdot\vec{R}_a = m_a|\vec{b}\times\nabla\psi|\sum_b\int \mathrm{d}^3 v\, v_2\left(C_{ab}[f_{a1}, f_{Mb}] + C_{ab}[f_{Ma}, f_{b1}]\right), \tag{18}$$

$$\vec{b}\times\nabla\psi\cdot\vec{G}_a = m_a T_a|\vec{b}\times\nabla\psi|\sum_b\int \mathrm{d}^3 v\, v_2 x_a^2\left(C_{ab}[f_{a1}, f_{Mb}] + C_{ab}[f_{Ma}, f_{b1}]\right). \tag{19}$$

Inserting our expressions for f_1, we thus get

$$\begin{aligned} (\vec{b}\times\nabla\psi)\cdot\vec{R}_a = m_a|\vec{b}\times\nabla\psi|^2\sum_b\int \mathrm{d}^3 v\, v_2 \Bigg(&\frac{\alpha_{1a}}{\Omega_a}C_{ab}[v_2 f_{Ma}, f_{Mb}] + \frac{\alpha_{2a}}{\Omega_a}C_{ab}[v_2 x_a^2 f_{Ma}, f_{Mb}] \\ &+\frac{\alpha_{1b}}{\Omega_b}C_{ab}[f_{Ma}, v_2 f_{Mb}] + \frac{\alpha_{2b}}{\Omega_b}C_{ab}[f_{Ma}, v_2 x_b^2 f_{Mb}]\Bigg), \end{aligned} \tag{20}$$

$$\begin{aligned} (\vec{b}\times\nabla\psi)\cdot\vec{G}_a = m_a T_a|\vec{b}\times\nabla\psi|^2\sum_b\int \mathrm{d}^3 v\, v_2 x_a^2 \Bigg(&\frac{\alpha_{1a}}{\Omega_a}C_{ab}[v_2 f_{Ma}, f_{Mb}] + \frac{\alpha_{2a}}{\Omega_a}C_{ab}[v_2 x_a^2 f_{Ma}, f_{Mb}] \\ &+\frac{\alpha_{1b}}{\Omega_b}C_{ab}[f_{Ma}, v_2 f_{Ma}] + \frac{\alpha_{2b}}{\Omega_b}C_{ab}[f_{Ma}, v_2 x_b^2 f_{Ma}]\Bigg). \end{aligned} \tag{21}$$

These friction-force projections can be conveniently expressed in terms of Braginskii matrix elements

$$M_{ab}^{jk} = \frac{\tau_{ab}}{n_a}\int v_2 L_j^{(3/2)}(x_a^2) C_{ab}\left[\frac{m_a v_2}{T_a} L_k^{(3/2)}(x_a^2) f_{a0}, f_{b0}\right] \tag{22}$$

$$N_{ab}^{jk} = \frac{\tau_{ab}}{n_a}\int v_2 L_j^{(3/2)}(x_a^2) C_{ab}\left[f_{a0}, \frac{m_b v_2}{T_b} L_k^{(3/2)}(x_b^2) f_{b0}\right], \tag{23}$$

where

$$\tau_{ab} = \frac{12\pi^{3/2}\epsilon_0^2}{\sqrt{2}e^4 \ln\Lambda}\frac{m_a^{1/2}T_a^{3/2}}{Z_a^2 Z_b^2 n_b}, \tag{24}$$

and the relevant Sonine polynomials are

$$L_0^{(3/2)}(x_a^2) = 1 \tag{25}$$

$$L_1^{(3/2)}(x_a^2) = \frac{5}{2} - x_a^2, \tag{26}$$

so that

$$x_a^2 - \frac{5}{2} = - L_1^{(3/2)}(x_a^2) \tag{27}$$

$$x_a^2 = \frac{5}{2}L_0^{(3/2)}(x_a^2) - L_1^{(3/2)}(x_a^2) \tag{28}$$

$$1 = L_0^{(3/2)}(x_a^2). \tag{29}$$

Equation 27 makes it convenient to calculate the conductive heat flux $q_a = Q_a - \frac{5}{2}T_a\Gamma_a$ rather than directly calculating Q_a. With this in mind, we write

$$\begin{aligned}(\vec{b}\times\nabla\psi)\cdot\vec{R}_a = m_a|\vec{b}\times\nabla\psi|^2\sum_b \frac{n_a}{\tau_{ab}}\Bigg[&\frac{\alpha_{1a}}{\Omega_a}\frac{T_a}{m_a}M_{ab}^{00} + \frac{\alpha_{2a}}{\Omega_a}\frac{T_a}{m_a}\left(\frac{5}{2}M_{ab}^{00} - M_{ab}^{01}\right)\\ &+\frac{\alpha_{1b}}{\Omega_b}\frac{T_b}{m_b}N_{ab}^{00} + \frac{\alpha_{2b}}{\Omega_b}\frac{T_b}{m_b}\left(\frac{5}{2}N_{ab}^{00} - N_{ab}^{01}\right)\Bigg],\end{aligned} \tag{30}$$

$$\begin{aligned}(\vec{b}\times\nabla\psi)\cdot\left(\vec{G}_a - T_a\frac{5}{2}\vec{R}_a\right) = -m_a T_a|\vec{b}\times\nabla\psi|^2\sum_b \frac{n_a}{\tau_{ab}}\Bigg[&\frac{\alpha_{1a}}{\Omega_a}\frac{T_a}{m_a}M_{ab}^{10} + \frac{\alpha_{2a}}{\Omega_a}\frac{T_a}{m_a}\left(\frac{5}{2}M_{ab}^{10} - M_{ab}^{11}\right)\\ &+\frac{\alpha_{1b}}{\Omega_b}\frac{T_b}{m_b}N_{ab}^{10} + \frac{\alpha_{2b}}{\Omega_b}\frac{T_b}{m_b}\left(\frac{5}{2}N_{ab}^{10} - N_{ab}^{11}\right)\Bigg].\end{aligned} \tag{31}$$

The classical particle-flux and heat-flux thus become

$$\Gamma_a = \frac{m_a}{Z_a e^2}\left\langle |\nabla\psi|^2 \sum_b \frac{n_a n_b}{B^2 \tau_{ab} n_b}\left[\frac{\alpha_{1a}T_a}{Z_a}M_{ab}^{00} + \frac{\alpha_{2a}T_a}{Z_a}\left(\frac{5}{2}M_{ab}^{00} - M_{ab}^{01}\right) + \frac{\alpha_{1b}T_b}{Z_b}N_{ab}^{00} + \frac{\alpha_{2b}T_b}{Z_b}\left(\frac{5}{2}N_{ab}^{00} - N_{ab}^{01}\right)\right]\right\rangle \tag{32}$$

$$q_a = -\frac{T_a m_a}{Z_a e^2}\left\langle |\nabla\psi|^2 \sum_b \frac{n_a n_b}{B^2 \tau_{ab} n_b}\left[\frac{\alpha_{1a}T_a}{Z_a}M_{ab}^{10} + \frac{\alpha_{2a}T_a}{Z_a}\left(\frac{5}{2}M_{ab}^{10} - M_{ab}^{11}\right) + \frac{\alpha_{1b}T_b}{Z_b}N_{ab}^{10} + \frac{\alpha_{2b}T_b}{Z_b}\left(\frac{5}{2}N_{ab}^{10} - N_{ab}^{11}\right)\right]\right\rangle. \tag{33}$$

The values for the matrix-components can be found in the appendix of Helander+Sigmar. We can simplify the above expressions slightly by noting the symmetry properties

$$M_{ab}^{jk} = M_{ab}^{kj}, \tag{34}$$

$$N_{ab}^{jk} = \frac{T_a v_{Ta}}{T_b v_{Tb}} N_{ba}^{kj}, \tag{35}$$

$$N_{ab}^{j0} = -M_{ab}^{j0} = -M_{ab}^{0j}, \tag{36}$$

$$N_{ab}^{0j} = \frac{T_a v_{Ta}}{T_b v_{Tb}} N_{ba}^{j0} = -\frac{T_a v_{Ta}}{T_b v_{Tb}} M_{ba}^{j0} = -\frac{T_a^{3/2} m_b^{1/2}}{T_b^{3/2} m_a^{1/2}} M_{ba}^{0j}, \tag{37}$$

$$\tag{38}$$

where the last property follows from applying the other 3. Thus, we only need

$$M_{ab}^{00} = -\frac{1+\frac{m_a}{m_b}}{\left(1+\frac{m_a T_b}{m_b T_a}\right)^{3/2}} = -\frac{\left(1+\frac{m_a}{m_b}\right)\left(1+\frac{m_a T_b}{m_b T_a}\right)}{\left(1+\frac{m_a T_b}{m_b T_a}\right)^{5/2}} \tag{39}$$

$$M_{ab}^{01} = -\frac{3}{2}\frac{1+\frac{m_a}{m_b}}{\left(1+\frac{m_a T_b}{m_b T_a}\right)^{5/2}} \tag{40}$$

$$M_{ba}^{01} = -\frac{3}{2}\frac{1+\frac{m_b}{m_a}}{\left(1+\frac{m_b T_a}{m_a T_b}\right)^{5/2}} = -\frac{3}{2}\frac{\left(1+\frac{m_a}{m_b}\right)\frac{m_b}{m_a}\left(\frac{m_a T_b}{m_b T_a}\right)^{5/2}}{\left(1+\frac{m_a T_b}{m_b T_a}\right)^{5/2}} \tag{41}$$

$$M_{ab}^{11} = -\frac{\frac{13}{4} + 4\frac{m_a T_b}{m_b T_a} + \frac{15}{2}\left(\frac{m_a T_b}{m_b T_a}\right)^2}{\left(1+\frac{m_a T_b}{m_b T_a}\right)^{5/2}} \tag{42}$$

$$N_{ab}^{11} = \frac{27}{4}\frac{\frac{m_a}{m_b}}{\left(1+\frac{m_a T_b}{m_b T_a}\right)^{5/2}}. \tag{43}$$

$$\tag{44}$$

The property $N_{ab}^{j0} = -M_{ab}^{j0} = -M_{ab}^{0j}$ also means that the radial electric field does not contribute to the classical heat or particle flux.

Writing out the α_1 and α_2 and employing the relevant symmetry properties, we obtain the particle flux

$$\begin{aligned}\Gamma_a = \frac{m_a}{Z_a e^2}\sum_b \frac{1}{\tau_{ab} n_b}\Bigg[&\left\langle n_a n_b \frac{|\nabla\psi|^2}{B^2}\right\rangle M_{ab}^{00}\left(\frac{T_a}{Z_a}\frac{\mathrm{d}\ln\eta_a}{\mathrm{d}\psi} - \frac{T_b}{Z_b}\frac{\mathrm{d}\ln\eta_b}{\mathrm{d}\psi}\right)\\ &+\left\langle n_a n_b \frac{|\nabla\psi|^2}{B^2} e\tilde{\Phi}\right\rangle M_{ab}^{00}\left(\frac{\mathrm{d}\ln T_a}{\mathrm{d}\psi} - \frac{\mathrm{d}\ln T_b}{\mathrm{d}\psi}\right)\\ &+\left\langle n_a n_b \frac{|\nabla\psi|^2}{B^2}\right\rangle\left(\left(M_{ab}^{00} - M_{ab}^{01}\right)\frac{T_a}{Z_a}\frac{\mathrm{d}\ln T_a}{\mathrm{d}\psi} - \left(M_{ab}^{00} - \frac{m_a T_b}{m_b T_a}M_{ab}^{01}\right)\frac{T_b}{Z_b}\frac{\mathrm{d}\ln T_b}{\mathrm{d}\psi}\right)\Bigg]\end{aligned} \tag{45}$$

and the heat-flux

$$\begin{aligned}q_a = -\frac{T_a m_a}{Z_a e^2}\sum_b \frac{1}{\tau_{ab} n_b}\Bigg[&\left\langle n_a n_b \frac{|\nabla\psi|^2}{B^2}\right\rangle M_{ab}^{01}\left(\frac{T_a}{Z_a}\frac{\mathrm{d}\ln\eta_a}{\mathrm{d}\psi} - \frac{T_b}{Z_b}\frac{\mathrm{d}\ln\eta_b}{\mathrm{d}\psi}\right)\\ &+\left\langle n_a n_b \frac{|\nabla\psi|^2}{B^2} e\tilde{\Phi}\right\rangle M_{ab}^{01}\left(\frac{\mathrm{d}\ln T_a}{\mathrm{d}\psi} - \frac{\mathrm{d}\ln T_b}{\mathrm{d}\psi}\right)\\ &+\left\langle n_a n_b \frac{|\nabla\psi|^2}{B^2}\right\rangle\left(\left(M_{ab}^{01} - M_{ab}^{11}\right)\frac{T_a}{Z_a}\frac{\mathrm{d}\ln T_a}{\mathrm{d}\psi} - \left(M_{ab}^{01} + N_{ab}^{11}\right)\frac{T_b}{Z_b}\frac{\mathrm{d}\ln T_b}{\mathrm{d}\psi}\right)\Bigg]\end{aligned} \tag{46}$$

2 SFINCS implementation

SFINCS uses normalized quantities (denoted with a hat), where the normalized radial fluxes are defined in terms of the physical fluxes as

$$\hat{\Gamma}_a = \frac{\bar{R}}{\bar{n}\bar{v}}\frac{1}{Z_a e}\left\langle B^{-2}\left(\vec{B}\times\nabla\hat{\psi}\right)\cdot\vec{R}_a\right\rangle, \tag{47}$$

$$\hat{Q}_a = \frac{\bar{R}}{\bar{n}\bar{v}\bar{T}}\frac{1}{Z_a e}\left\langle B^{-2}\left(\vec{B}\times\nabla\hat{\psi}\right)\cdot\vec{G}_a\right\rangle, \tag{48}$$

where $\hat{\psi} = \psi/(\bar{B}\hat{R}^2)$ is just a new choice of radial coordinate rather than a normalization – see the SFINCS manual for more details. We also define the dimensionless parameters $\alpha = e\bar{\Phi}/\bar{T}$ and $\Delta = \frac{\sqrt{2\bar{m}\bar{T}}}{e\bar{R}\bar{B}}$, and the dimensionless collisionality ν_n

$$\bar{\nu} \equiv \frac{\sqrt{2}}{12\pi^{3/2}}\frac{\bar{n}e^4\ln\Lambda}{\epsilon_0^2\bar{m}^{1/2}\bar{T}^{3/2}} \tag{49}$$

$$\nu_n \equiv \bar{\nu}\frac{\bar{R}\sqrt{\bar{m}}}{\sqrt{2\bar{T}}} = \frac{1}{12\pi^{3/2}}\frac{\bar{n}\bar{R}e^4\ln\Lambda}{\epsilon_0^2\bar{T}^2}, \tag{50}$$

so that

$$\frac{1}{\tau_{ab} n_b} = \frac{Z_a^2 Z_b^2}{\hat{m}_a^{1/2}\hat{T}_a^{3/2}} \frac{\sqrt{2}\bar{T}^{1/2}}{\bar{n}\bar{R}\bar{m}^{1/2}}\nu_n = \frac{Z_a^2 Z_b^2}{\hat{m}_a^{1/2}\hat{T}_a^{3/2}} \frac{\sqrt{2}\bar{T}^{1/2}\bar{m}^{1/2}\bar{B}}{\bar{n}\bar{R}\bar{B}\bar{m}}\nu_n = \frac{Z_a^2 Z_b^2}{\hat{m}_a^{1/2}\hat{T}_a^{3/2}} \frac{e\bar{B}}{\bar{n}\bar{m}}\Delta\nu_n. \tag{51}$$

With this, the normalized particle and conductive heat-flux becomes

$$\begin{aligned}\hat{\Gamma}_a = \frac{\Delta^2\nu_n}{2}\frac{Z_a\hat{m}_a^{1/2}}{\hat{T}_a^{3/2}}\sum_b Z_b^2 \Bigg[&\left\langle \hat{n}_a\hat{n}_b\frac{|\bar{R}\nabla\hat{\psi}|^2}{\hat{B}^2}\right\rangle M_{ab}^{00}\left(\frac{\hat{T}_a}{Z_a}\frac{\mathrm{d}\ln\eta_a}{\mathrm{d}\hat{\psi}} - \frac{\hat{T}_b}{Z_b}\frac{\mathrm{d}\ln\eta_b}{\mathrm{d}\hat{\psi}}\right) \\ &+ \left\langle \hat{n}_a\hat{n}_b\frac{|\bar{R}\nabla\hat{\psi}|^2}{\hat{B}^2}\hat{\Phi}_1\right\rangle \alpha M_{ab}^{00}\left(\frac{\mathrm{d}\ln T_a}{\mathrm{d}\hat{\psi}} - \frac{\mathrm{d}\ln T_b}{\mathrm{d}\hat{\psi}}\right) \\ &+ \left\langle \hat{n}_a\hat{n}_b\frac{|\bar{R}\nabla\hat{\psi}|^2}{\hat{B}^2}\right\rangle \left(\left(M_{ab}^{00} - M_{ab}^{01}\right)\frac{\hat{T}_a}{Z_a}\frac{\mathrm{d}\ln T_a}{\mathrm{d}\hat{\psi}} - \left(M_{ab}^{00} - \frac{\hat{m}_a\hat{T}_b}{\hat{m}_b\hat{T}_a}M_{ab}^{01}\right)\frac{\hat{T}_b}{Z_b}\frac{\mathrm{d}\ln T_b}{\mathrm{d}\hat{\psi}}\right)\Bigg]\end{aligned} \tag{52}$$

$$\begin{aligned}\hat{q}_a = -\frac{\Delta^2\nu_n}{2}\frac{Z_a\hat{m}_a^{1/2}}{\hat{T}_a^{1/2}}\sum_b Z_b^2 \Bigg[&\left\langle \hat{n}_a\hat{n}_b\frac{|\bar{R}\nabla\hat{\psi}|^2}{\hat{B}^2}\right\rangle M_{ab}^{01}\left(\frac{\hat{T}_a}{Z_a}\frac{\mathrm{d}\ln\eta_a}{\mathrm{d}\hat{\psi}} - \frac{\hat{T}_b}{Z_b}\frac{\mathrm{d}\ln\eta_b}{\mathrm{d}\hat{\psi}}\right) \\ &+ \left\langle \hat{n}_a\hat{n}_b\frac{|\bar{R}\nabla\hat{\psi}|^2}{\hat{B}^2}\hat{\Phi}_1\right\rangle \alpha M_{ab}^{01}\left(\frac{\mathrm{d}\ln T_a}{\mathrm{d}\hat{\psi}} - \frac{\mathrm{d}\ln T_b}{\mathrm{d}\hat{\psi}}\right) \\ &+ \left\langle \hat{n}_a\hat{n}_b\frac{|\bar{R}\nabla\hat{\psi}|^2}{\hat{B}^2}\right\rangle \left(\left(M_{ab}^{01} - M_{ab}^{11}\right)\frac{\hat{T}_a}{Z_a}\frac{\mathrm{d}\ln T_a}{\mathrm{d}\hat{\psi}} - \left(M_{ab}^{01} + N_{ab}^{11}\right)\frac{\hat{T}_b}{Z_b}\frac{\mathrm{d}\ln T_b}{\mathrm{d}\hat{\psi}}\right)\Bigg].\end{aligned} \tag{53}$$

Finally, using $\hat{n}_a = \hat{\eta}_a \mathrm{e}^{-Z_a\alpha\hat{\Phi}_1/\hat{T}_a}$, we get

$$\hat{n}_a\hat{n}_b = \hat{\eta}_a\hat{\eta}_b \exp\left(-\alpha\hat{\Phi}_1\left[\frac{Z_a}{\hat{T}_a} + \frac{Z_b}{\hat{T}_b}\right]\right), \tag{54}$$

where $\hat{\eta}$ is the "density" in SFINCS, which is a flux-function. In the code, we define

$$G_{ab}^{(1)} = \left\langle \exp\left(-\alpha\hat{\Phi}_1\left[\frac{Z_a}{\hat{T}_a} + \frac{Z_b}{\hat{T}_b}\right]\right)\frac{|\bar{R}\nabla\hat{\psi}|^2}{\hat{B}^2}\right\rangle \tag{55}$$

$$G_{ab}^{(2)} = \left\langle \exp\left(-\alpha\hat{\Phi}_1\left[\frac{Z_a}{\hat{T}_a} + \frac{Z_b}{\hat{T}_b}\right]\right)\frac{|\bar{R}\nabla\hat{\psi}|^2}{\hat{B}^2}\hat{\Phi}_1\right\rangle, \tag{56}$$

so that

$$\hat{\Gamma}_a = \frac{\Delta^2 \nu_n}{2} \frac{Z_a \hat{m}_a^{1/2} \hat{\eta}_a}{\hat{T}_a^{3/2}} \sum_b Z_b^2 \hat{\eta}_b \left[G_{ab}^{(1)} M_{ab}^{00} \left(\frac{\hat{T}_a}{Z_a} \frac{\mathrm{d} \ln \eta_a}{\mathrm{d}\hat{\psi}} - \frac{\hat{T}_b}{Z_b} \frac{\mathrm{d} \ln \eta_b}{\mathrm{d}\hat{\psi}} \right) \right.$$
$$+ G_{ab}^{(2)} \alpha M_{ab}^{00} \left(\frac{\mathrm{d} \ln T_a}{\mathrm{d}\hat{\psi}} - \frac{\mathrm{d} \ln T_b}{\mathrm{d}\hat{\psi}} \right)$$
$$\left. + G_{ab}^{(1)} \left(\left(M_{ab}^{00} - M_{ab}^{01} \right) \frac{\hat{T}_a}{Z_a} \frac{\mathrm{d} \ln T_a}{\mathrm{d}\hat{\psi}} - \left(M_{ab}^{00} - \frac{\hat{m}_a \hat{T}_b}{\hat{m}_b \hat{T}_a} M_{ab}^{01} \right) \frac{\hat{T}_b}{Z_b} \frac{\mathrm{d} \ln T_b}{\mathrm{d}\hat{\psi}} \right) \right], \tag{57}$$

$$\hat{q}_a = -\frac{\Delta^2 \nu_n}{2} \frac{Z_a \hat{m}_a^{1/2} \hat{\eta}_a}{\hat{T}_a^{1/2}} \sum_b Z_b^2 \hat{\eta}_b \left[G_{ab}^{(1)} M_{ab}^{01} \left(\frac{\hat{T}_a}{Z_a} \frac{\mathrm{d} \ln \eta_a}{\mathrm{d}\hat{\psi}} - \frac{\hat{T}_b}{Z_b} \frac{\mathrm{d} \ln \eta_b}{\mathrm{d}\hat{\psi}} \right) \right.$$
$$+ G_{ab}^{(2)} \alpha M_{ab}^{01} \left(\frac{\mathrm{d} \ln T_a}{\mathrm{d}\hat{\psi}} - \frac{\mathrm{d} \ln T_b}{\mathrm{d}\hat{\psi}} \right)$$
$$\left. + G_{ab}^{(1)} \left(\left(M_{ab}^{01} - M_{ab}^{11} \right) \frac{\hat{T}_a}{Z_a} \frac{\mathrm{d} \ln T_a}{\mathrm{d}\hat{\psi}} - \left(M_{ab}^{01} + N_{ab}^{11} \right) \frac{\hat{T}_b}{Z_b} \frac{\mathrm{d} \ln T_b}{\mathrm{d}\hat{\psi}} \right) \right], \tag{58}$$

and

$$\hat{Q}_a = \hat{q}_a + \frac{5}{2} \hat{T}_a \hat{\Gamma}_a. \tag{59}$$

Thanks to Sarah Newton for bringing to my attention that the classical transport can be calculated using the Braginskii matrices.