Review of Transmission-Line Theory

The term transmission-line in electromagnetics is commonly reserved for those structures which are capable of guiding TEM waves. Transmission-lines are a special class of the more general electromagnetic waveguide. TEM waves can only exist in structures which contain two or more separate conductors. Coaxial lines, parallel plates, and two-wire lines are examples of practical transmission-lines.

Transmission-Line Equations

The most important property of TEM waves on transmission-lines is that the electromagnetic fields can be uniquely related to a voltage and current via

$$V = - \int \overline{E} \cdot d\ell \qquad I = \oint \overline{H} \cdot d\ell$$

For this reason, we can analyze the structures using circuit theory concepts, provided we break the problem into small parts so that the circuit element dimensions will be much smaller than a wavelength. To do this, we describe the transmission-line by a series resistance per unit length R, series inductance per unit length L, shunt conductance per unit length G, and shunt capacitance per unit length C. A small section of transmission-line with length dz thus has the following equivalent circuit:
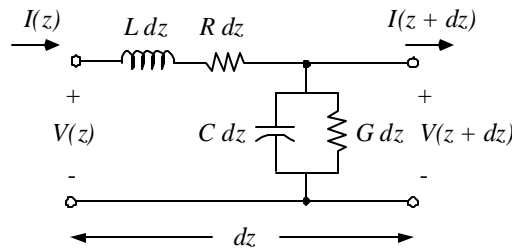
Analysis of this circuit for time-harmonic signals gives the wave equations

$$\frac{\partial V(z)}{\partial z} - \gamma^2 V(z) = 0 \qquad \frac{\partial I(z)}{\partial z} - \gamma^2 I(z) = 0$$

where γ is the complex propagation constant given by

$$\gamma = \alpha + \jmath\beta = \sqrt{(R + \jmath\omega L)(G + \jmath\omega C)}$$

Case	Propagation Constant $\gamma = \alpha + \jmath\beta$	Characteristic Impedance Z_0
General	$\sqrt{(R + \jmath\omega L)(G + \jmath\omega C)}$	$\sqrt{(R + \jmath\omega L)/(G + \jmath\omega C)}$
Lossless ($R = G = 0$)	$0 + \jmath\omega\sqrt{LC}$	$\sqrt{L/C}$
Distortionless ($R/L = G/C$)	$\sqrt{RG} + \jmath\omega\sqrt{LC}$	$\sqrt{L/C}$

The solutions to the wave equations are superpositions of forward and reverse waves,

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \qquad I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

Just like with plane waves, we define a characteristic impedance as the ratio of voltage to current (for positive travelling waves)
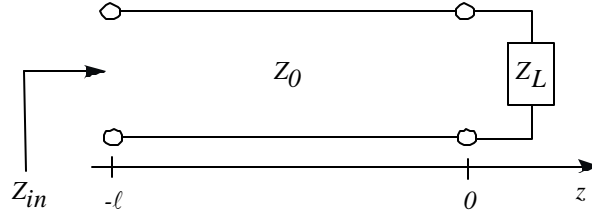
$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-}$$

and the wave velocity (or phase velocity) and wavelength are given by

$$\lambda = \frac{2\pi}{\beta} \qquad v_p = \frac{\omega}{\beta} = f\lambda$$

TEM waves always propagate with velocity $v = 1/\sqrt{\mu\epsilon}$. For lossless transmission lines, we find

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{LC}} \quad \Rightarrow \quad LC = \mu\epsilon$$

Which is true regardless of the transmission-line geometry.

For the transmission-line circuit above, we define a voltage reflection coefficient (at the load) as the ratio of reflected voltage to incident voltage, which can in general be complex:

$$\Gamma_L = \frac{V_0^-}{V_0^+} = |\Gamma_L|e^{\jmath\theta_\Gamma} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Some special cases to remember are

$$\text{Short: } Z_L = 0 \rightarrow \Gamma_L = -1 \quad \text{Open: } Z_L = \infty \rightarrow \Gamma_L = +1 \qquad \text{Match: } Z_L = Z_0 \rightarrow \Gamma_L = 0$$

Using the load reflection coefficient gives

$$V(z) = V_0^+ e^{-\jmath\beta z}\left(1 + |\Gamma_L|e^{\jmath(2\beta z + \theta_\Gamma)}\right)$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\jmath\beta z}\left(1 - |\Gamma_L|e^{\jmath(2\beta z + \theta_\Gamma)}\right)$$

In the general case, the input impedance $Z_{in} = V(-l)/I(-l)$ becomes

$$Z_{in} = Z_0 \frac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l} \qquad \text{(lossy)}$$

and for the lossless case we get

$$Z_{in} = Z_0 \frac{Z_L + \jmath Z_0 \tan\beta l}{Z_0 + \jmath Z_L \tan\beta l} \qquad \text{(lossless)}$$

The term βl is called the electrical length of the transmission-line. Some special cases to consider:

$$\text{Shorted line:} \quad Z_L = 0 \quad Z_{in} = \jmath Z_0 \tan\beta l$$
$$\text{Open Line:} \quad Z_L = \infty \quad Z_{in} = -\jmath Z_0 \cot\beta l$$
$$\text{Quarter-Wave Line:} \quad l = \lambda/4 \quad Z_{in} = \frac{Z_0{}^2}{Z_L}$$
$$\text{Half-Wavelength Line:} \quad l = \lambda/2 \quad Z_{in} = Z_L$$

The most important thing to remember is that the input impedance is a function of the line length. For that reason, the reflection coefficient also be a function of the line length. Note that the reflection coefficient defined above is the reflection coefficient at the load. We can define a generalized reflection coefficient along the line as

$$\Gamma(-l) = \frac{\text{reflected wave}}{\text{incident wave}} = \frac{V_0^- e^{-\jmath\beta l}}{V_0^+ e^{\jmath\beta l}} = \Gamma_L e^{-\jmath 2\beta l} = |\Gamma_L|e^{\jmath(\theta_\Gamma - 2\beta l)}$$

So the magnitude of Γ does not change as we move along the line; only the phase changes.

Standing Waves

The combination of forward and reverse travelling waves produce a standing-wave, which is so-called because the positions of maximum and minimum signal do not vary with time. The actual shape of this standing wave is a function of the load impedance. We define a standing wave ratio as

$$S = \frac{|V_{max}|}{|V_{min}|} = \frac{|I_{max}|}{|I_{min}|} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

The Smith Chart

The expression for the load reflection coefficeint Γ_L in terms of the load impedance Z_L can be manipulated to show that points of constant resistance (conductance) and reactance (susceptance) define circles in the complex Γ plane. A plot of these circles is called a Smith chart, and is diagramed below.

Some important aspects of the Smith chart are summarized:

- The Smith chart can be used for both impedances and admittances.
- The Smith chart is defined for normalized impedances or admittances.
- To find the reflection coefficient: first normalize the load impedance, and plot on the Smith chart. The distance between the center of the chart and this point gives the magnitude, and the phase can be found from a scale on the periphery.
- Circles of constant-Γ are centered on the Smith chart.
- Moving clockwise around the Smith chart corresponds to moving away from the load, or toward the generator.
- To find admittance from impedance: plot the normalized impedance on the chart. The point diametrically opposite this point through the center gives the admittance.
- The resistance at the intersection of the constant-Γ circle with the positive real axis is equal to the standing-wave ratio.
- The positions of maximum and minimum voltage on the line are found by determining the intersections of the constant-Γ circle with the positive and negative real axes, respectively.