ÁLGEBRA LINEAR EE

Exercícios - Determinantes

1. Considere a matriz

$$A = \begin{bmatrix} 2 & 2 & 0 & -2 \\ 3 & 1 & 1 & 1 \\ -1 & -2 & 0 & 3 \\ -2 & 0 & 2 & -1 \end{bmatrix}$$

(a) Calcule $|A|$ utilizando eliminação de Gauss.

$$|A| = \begin{vmatrix} 2 & 2 & 0 & -2 \\ 3 & 1 & 1 & 1 \\ -1 & -2 & 0 & 3 \\ -2 & 0 & 2 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 0 & -1 \\ 3 & 1 & 1 & 1 \\ -1 & -2 & 0 & 3 \\ -2 & 0 & 2 & -1 \end{vmatrix} \overset{-3L_1+L_2, L_1+L_3, 2L_1+L_4}{=} 2\begin{vmatrix} 1 & 1 & 0 & -1 \\ 0 & -2 & 1 & 4 \\ 0 & -1 & 0 & 2 \\ 0 & 2 & 2 & -3 \end{vmatrix} \overset{L_2 \leftrightarrow L_3}{=}$$

$$-2\begin{vmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 2 \\ 0 & -2 & 1 & 4 \\ 0 & 2 & 2 & -3 \end{vmatrix} \overset{-2L_1+L_3, 2L_2+L_4}{=} -2\begin{vmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{vmatrix} \overset{-2L_3+L_4}{=} -2\begin{vmatrix} 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 2.$$

(b) Calcule $|A|$ utilizando o Teorema de Laplace.

$$|A| = \begin{vmatrix} 2 & 2 & 0 & -2 \\ 3 & 1 & 1 & 1 \\ -1 & -2 & 0 & 3 \\ -2 & 0 & 2 & -1 \end{vmatrix} \overset{L1}{=} 2\begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 3 \\ 0 & 2 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 & 1 \\ -1 & 0 & 3 \\ -2 & 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 3 & 1 & 1 \\ -1 & -2 & 3 \\ -2 & 0 & -1 \end{vmatrix} -$$

$$(-2)\begin{vmatrix} 3 & 1 & 1 \\ -1 & -2 & 0 \\ -2 & 0 & 2 \end{vmatrix} = \underbrace{2\left(\begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}\right)}_{\text{(por c1)}} \underbrace{-2\left(\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix}\right)}_{\text{(por l2)}} \underbrace{+2\left(\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix}\right)}_{\text{(por l2)}}$$

$$= 2((-6) + 2(-1-2)) - 2((-1-2) - 3(6+2)) + 2(2 - 2(6+2)) = 2$$

2. Sabendo que $\begin{vmatrix} a & b & c & x \\ 0 & 0 & 0 & 2 \\ d & e & f & y \\ g & h & i & z \end{vmatrix} = 6$, determine:

(a) $\begin{vmatrix} b & c & a \\ e & f & d \\ h & i & g \end{vmatrix}$; (b) $\begin{vmatrix} b & h & e \\ c & i & f \\ a & g & d \end{vmatrix}$; (c) $\begin{vmatrix} 6b & 2c & 2a \\ 3e & f & d \\ 3h & i & g \end{vmatrix}$.

$$\begin{vmatrix} a & b & c & x \\ 0 & 0 & 0 & 2 \\ d & e & f & y \\ g & h & i & z \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6, \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$$

$$\begin{vmatrix} b & c & a \\ e & f & d \\ h & i & g \end{vmatrix} \overset{C_2 \leftrightarrow C_3}{=} -\begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix} \overset{C_1 \leftrightarrow C_2}{=} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$$

$$\begin{vmatrix} b & h & e \\ c & i & f \\ a & g & d \end{vmatrix} \overset{A^T}{=} \begin{vmatrix} b & c & a \\ h & i & g \\ e & f & d \end{vmatrix} \overset{L_2 \leftrightarrow L_3}{=} -\begin{vmatrix} b & c & a \\ e & f & d \\ h & i & g \end{vmatrix} = -3,$$

$$\begin{vmatrix} 6b & 2c & 2a \\ 3e & f & d \\ 3h & i & g \end{vmatrix} = 2 \begin{vmatrix} 3b & c & a \\ 3e & f & d \\ 3h & i & g \end{vmatrix} = 6 \begin{vmatrix} b & c & a \\ e & f & d \\ h & i & g \end{vmatrix} = -18.$$

3. Considere as seguintes matrizes:

$$A_1 = \begin{bmatrix} 5 & -4 & 3 \\ 0 & -1 & 0 \\ 1 & -3 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1/2 & 4 & 1 & 3 \\ 0 & -1 & 0 & 3/2 \\ -1 & 0 & 2 & -2 \\ 3 & 0 & -1 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & -2 & 4 & 0 \\ -1 & -2 & 3 & 1 \\ 0 & 4 & -4 & -5 \end{bmatrix}.$$

(a) Calcule $|A_1|$, $|A_2|$ e $|A_3|$.
$|A_1| = -7$, $|A_2| = 54$, $|A_3| = 0$.

(b) Classifique os sistemas $A_1X = B$, $A_3X = 0$ para qualquer matriz B de tipo 4×1.
$|A_1| \neq 0 \Rightarrow C(A_1) = 3 = n.\ A_1X = B$ PD
$|A_3| = 0 \Rightarrow C(A_3) < 4 = n.\ A_3X = 0$ PI

(c) Calcule $Adj(A_1)$, $Adj(A_2)$, $Adj(A_3)$.

$$Adj(A_1) = \begin{bmatrix} \begin{vmatrix} -1 & 0 \\ -3 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 1 & -3 \end{vmatrix} \\ -\begin{vmatrix} -4 & 3 \\ -3 & 2 \end{vmatrix} & \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 5 & -4 \\ 1 & -3 \end{vmatrix} \\ \begin{vmatrix} -4 & 3 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 5 & 3 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 5 & -4 \\ 0 & -1 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -2 & -1 & 3 \\ 0 & 7 & 0 \\ 1 & 11 & -5 \end{bmatrix}.$$

$$Adj(A_2) = \begin{bmatrix} 4 & 16 & 8 & 20 \\ \frac{15}{2} & -24 & -\frac{21}{4} & -3 \\ 7 & 28 & \frac{55}{2} & 8 \\ 5 & 20 & -\frac{7}{2} & -2 \end{bmatrix}, \ Adj(A_3) = \begin{bmatrix} -2 & -2 & -2 & -2 \\ 20 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 \\ 8 & 8 & 8 & 8 \end{bmatrix}.$$

(d) Quais das matriz dadas são invertíveis? Em tais casos, calcule a inversa.
$det A_1 \neq 0$, então A_1 invertível.

$$A_1^{-1} = Adj(A_1)/|A_1| = -\frac{1}{7} \begin{bmatrix} -2 & -1 & 3 \\ 0 & 7 & 0 \\ 1 & 11 & -5 \end{bmatrix}.$$

$det A_2 \neq 0$, então A_2 invertível. $A_2^{-1} = \frac{1}{54} \begin{bmatrix} 4 & 16 & 8 & 20 \\ \frac{15}{2} & -24 & -\frac{21}{4} & -3 \\ 7 & 28 & \frac{55}{2} & 8 \\ 5 & 20 & -\frac{7}{2} & -2 \end{bmatrix}$

$det A_3 = 0$, então A_3 não invertível.

4. Usando a regra de Cramer, resolva os sistemas:

$$(i) \begin{cases} \quad\ \ 2x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 1 \\ x_1 - x_2 + 2x_3 = 2 \end{cases} \qquad (ii) \begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

(i) $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$, $\det A = -4$. $A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$, $\det A_1 = 0$

$A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, $\det A_2 = 0$. $A_3 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$, $\det A_2 = -4$.

$(x_1, x_2, x_3) = (0, 0, 1)$.
(ii) $(x_1, x_2, x_3) = (-\frac{2}{3}, 3, -\frac{5}{3})$.

5. Seja $A_t = \begin{bmatrix} 5 & t & 3 \\ t & -1 & 0 \\ 1 & -3 & -2 \end{bmatrix}$ uma matriz de entradas reais. Determine os valores de t para os quais A_t é invertível, recorrendo ao cálculo do determinante.

   A_t é invertível $\Longleftrightarrow \det(A_t) \neq 0$.

   $\det(A_t) = 3 \det \begin{bmatrix} t & -1 \\ 1 & -3 \end{bmatrix} - 2 \det \begin{bmatrix} 5 & t \\ t & -1 \end{bmatrix} = 2t^2 - 9t + 13$.

   $\Delta = b^2 - 4ac = -23 < 0$. $\det(A_t) \neq 0$ para todo t em reais.

6. Calcule uma função polinomial de grau 2 cujo gráfico contenha os pontos $(1, 1)$, $(1/2, -2)$ e $(-1, 1)$.

   Seja o polinomial de grau 2 é $y = ax^2 + bx + c$.

   $1 = a + b + c$

   $-2 = a(\frac{1}{2})^2 + \frac{1}{2}b + c$

   $1 = a - b + c$

   $(a, b, c) = (4, 0, -3)$, o polinomial é $y = 4x^2 - 3$.