



Primitivas

1. Começemos por calcular $\int x^2 \sin x \, dx$.

Sejam

$$\begin{aligned} f'(x) &= \sin x & f(x) &= -\cos x \\ g(x) &= x^2 & g'(x) &= 2x \end{aligned}$$

Aplicando o método de primitivação por partes, obtemos

$$\int x^2 \sin x \, dx = -x^2 \cos x - \int -2x \cos x \, dx = -x^2 \cos x + \int 2x \cos x \, dx.$$

Vamos aplicar de novo o método de primitivação por partes para calcular $\int 2x \cos x \, dx$.

Sejam

$$\begin{aligned} f'(x) &= \cos x & f(x) &= \sin x \\ g(x) &= 2x & g'(x) &= 2 \end{aligned}$$

Aplicando o método de primitivação por partes, obtemos

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C, \quad C \in \mathbb{R}.$$

Vamos agora determinar o valor da constante C de modo a encontrar a primitiva que passa no ponto $(\frac{\pi}{2}, \pi)$. Tem-se que:

$$-\frac{\pi^2}{4} \cos \frac{\pi}{2} + \pi \sin \frac{\pi}{2} + 2 \cos \frac{\pi}{2} + C = \pi \Leftrightarrow C = 0.$$

Consequentemente, a primitiva que passa no ponto $(\frac{\pi}{2}, \pi)$ é

$$F(x) = -x^2 \cos x + 2x \sin x + 2 \cos x.$$

2. (a) $f(x) = \frac{2x^3}{3} - \frac{x^2}{2} - 8x + \frac{65}{6}.$

(b) $f(x) = \frac{5x}{4} - \frac{1}{8} \sin(2x).$

3. [Primitivas inmediatas]

$$\begin{aligned}
 (1) \quad \int (\sqrt{x} + 2)^2 dx &= \int (x + 4\sqrt{x} + 4) dx = \int x dx + \int 4x^{1/2} dx + \int 4 dx \\
 &= \frac{x^2}{2} + 4 \frac{x^{1/2+1}}{1/2+1} + 4x + C \\
 &= \frac{x^2}{2} + \frac{8}{3} x^{3/2} + 4x + C, \quad C \in \mathbb{R}
 \end{aligned}$$

$$(2) \quad \int (3x^2 - 2x^5) dx = x^3 - \frac{x^6}{3} + C, \quad C \in \mathbb{R}$$

$$(3) \quad \int (2x + 10)^{20} dx = \frac{1}{2} \int 2(2x + 10)^{20} dx = \frac{1}{2} \frac{(2x + 10)^{21}}{21} + C = \frac{(2x + 10)^{21}}{42} + C, \quad C \in \mathbb{R}$$

$$(4) \quad \int x^2 e^{x^3} dx = \frac{1}{3} \int 3x^2 e^{x^3} dx = \frac{1}{3} e^{x^3} + C, \quad C \in \mathbb{R}$$

$$(5) \quad \int x^4 (x^5 + 10)^9 dx = \frac{1}{5} \int 5x^4 (x^5 + 10)^9 dx = \frac{1}{5} \frac{(x^5 + 10)^{10}}{10} + C = \frac{(x^5 + 10)^{10}}{50} + C, \quad C \in \mathbb{R}$$

$$(6) \quad \int \frac{2x + 1}{x^2 + x + 3} dx = \ln(x^2 + x + 3) + C, \quad C \in \mathbb{R}$$

$$\begin{aligned}
 (7) \quad \int \sqrt{2x + 1} dx &= \int (2x + 1)^{1/2} dx = \frac{1}{2} \int 2(2x + 1)^{1/2} dx = \frac{1}{2} \frac{(2x + 1)^{3/2}}{3/2} + C \\
 &= \frac{1}{3} (2x + 1)^{3/2} + C, \quad C \in \mathbb{R}
 \end{aligned}$$

$$(8) \quad \int \frac{x}{3 - x^2} dx = -\frac{1}{2} \int \frac{-2x}{3 - x^2} dx = -\frac{1}{2} \ln |3 - x^2| + C, \quad C \in \mathbb{R}$$

$$(9) \quad \int \frac{1}{4 - 3x} dx = -\frac{1}{3} \int \frac{-3}{4 - 3x} dx = -\frac{1}{3} \ln |4 - 3x| + C, \quad C \in \mathbb{R}$$

$$(10) \quad \int \frac{1}{e^{3x}} dx = \int e^{-3x} dx = -\frac{1}{3} \int -3e^{-3x} dx = -\frac{1}{3} e^{-3x} + C, \quad C \in \mathbb{R}$$

$$\begin{aligned}
 (11) \quad \int \frac{-7}{\sqrt{1 - 5x}} dx &= \int -7(1 - 5x)^{-1/2} dx = \frac{-7}{-5} \int -5(1 - 5x)^{-1/2} dx \\
 &= \frac{7}{5} \frac{(1 - 5x)^{1/2}}{1/2} + C = \frac{14}{5} (1 - 5x)^{1/2} + C, \quad C \in \mathbb{R}
 \end{aligned}$$

$$\begin{aligned}
(12) \quad \int \frac{\sqrt{1+3\ln x}}{x} dx &= \int \frac{1}{x} (1+3\ln x)^{1/2} dx = \frac{1}{3} \int \frac{3}{x} (1+3\ln x)^{1/2} dx \\
&= \frac{1}{3} \frac{(1+3\ln x)^{3/2}}{3/2} + C = \frac{2}{9} (1+3\ln x)^{3/2} + C, \quad C \in \mathbb{R}
\end{aligned}$$

$$(13) \quad \int x \operatorname{sen}(x^2) dx = \frac{1}{2} \int 2x \operatorname{sen}(x^2) dx = -\frac{1}{2} \cos(x^2) + C, \quad C \in \mathbb{R}$$

$$(14) \quad \int \frac{1}{x(\ln^2 x + 1)} dx = \int \frac{\frac{1}{x}}{1 + \ln^2 x} dx = \operatorname{arctg}(\ln x) + C, \quad C \in \mathbb{R}$$

$$\begin{aligned}
(15) \quad \int \left(\frac{2}{x} - 3\right)^2 \frac{1}{x^2} dx &= -\frac{1}{2} \int -\frac{2}{x^2} \left(\frac{2}{x} - 3\right)^2 dx \\
&= -\frac{1}{2} \frac{\left(\frac{2}{x} - 3\right)^3}{3} + C = -\frac{1}{6} \left(\frac{2}{x} - 3\right)^3 + C, \quad C \in \mathbb{R}
\end{aligned}$$

$$(16) \quad \int \operatorname{sen}(\pi - 2x) dx = -\frac{1}{2} \int -2 \operatorname{sen}(\pi - 2x) dx = \frac{1}{2} \cos(\pi - 2x) + C, \quad C \in \mathbb{R}$$

$$(17) \quad \int \operatorname{th} x dx = \int \frac{\operatorname{sh} x}{\operatorname{ch} x} dx = \ln(\operatorname{ch} x) + C, \quad C \in \mathbb{R}$$

$$(18) \quad \int \operatorname{sen} x \cos x dx = \frac{\operatorname{sen}^2 x}{2} + C, \quad C \in \mathbb{R}$$

$$(19) \quad \int \operatorname{sen}(2x) \cos x dx = \int 2 \operatorname{sen} x \cos^2 x dx = -2 \int -\operatorname{sen} x \cos^2 x dx = \frac{-2 \cos^3 x}{3} + C, \quad C \in \mathbb{R}$$

$$\begin{aligned}
(20) \quad \int \operatorname{sen}^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right) dx &= \int \left(\frac{1 - \cos x}{2}\right) \left(\frac{1 + \cos x}{2}\right) dx = \frac{1}{4} \int (1 - \cos x)(1 + \cos x) dx \\
&= \frac{1}{4} \int (1 - \cos^2 x) dx = \frac{1}{4} \int \operatorname{sen}^2 x dx = \frac{1}{4} \int \frac{1 - \cos(2x)}{2} dx \\
&= \frac{1}{8} \int (1 - \cos(2x)) dx = \frac{1}{8} \int 1 dx - \frac{1}{16} \int 2 \cos(2x) dx \\
&= \frac{1}{8} x - \frac{1}{16} \operatorname{sen}(2x) + C, \quad C \in \mathbb{R}
\end{aligned}$$

$$\begin{aligned}
(21) \quad \int \operatorname{sen}^2 x dx &= \int \frac{1 - \cos(2x)}{2} dx = \frac{1}{2} \int 1 dx - \frac{1}{4} \int 2 \cos(2x) dx \\
&= \frac{1}{2} x - \frac{1}{4} \operatorname{sen}(2x) + C, \quad C \in \mathbb{R}
\end{aligned}$$

$$(22) \quad \int \cos^3 x \, dx = \int (1 - \sin^2 x) \cos x \, dx = \int \cos x \, dx - \int \cos x \sin^2 x \, dx \\ = \sin x - \frac{\sin^3 x}{3} + C, \quad C \in \mathbb{R}$$

$$(23) \quad \int \frac{x}{x^2 - 1} \, dx = \frac{1}{2} \int \frac{2x}{x^2 - 1} \, dx = \frac{1}{2} \ln |x^2 - 1| + C, \quad C \in \mathbb{R}$$

$$(24) \quad \int \frac{x}{\sqrt{x^2 - 1}} \, dx = \frac{1}{2} \int 2x (x^2 - 1)^{-1/2} \, dx = \frac{1}{2} \frac{(x^2 - 1)^{1/2}}{1/2} + C = \sqrt{x^2 - 1} + C, \quad C \in \mathbb{R}$$

$$(25) \quad \int \frac{1}{x} \sin(\ln x) \, dx = -\cos(\ln x) + C, \quad C \in \mathbb{R}$$

$$(26) \quad \int \frac{-3}{x (\ln x)^3} \, dx = -3 \int \frac{1}{x} (\ln x)^{-3} \, dx = -3 \frac{(\ln x)^{-2}}{-2} + C = \frac{3}{2 (\ln x)^2} + C, \quad C \in \mathbb{R}$$

$$(27) \quad \int \frac{e^x}{1 + e^{2x}} \, dx = \int \frac{e^x}{1 + (e^x)^2} \, dx = \operatorname{arctg}(e^x) + C, \quad C \in \mathbb{R}$$

$$(28) \quad \int \frac{e^x}{1 - 2e^x} \, dx = -\frac{1}{2} \int \frac{-2e^x}{1 - 2e^x} \, dx = -\frac{1}{2} \ln |1 - 2e^x| + C, \quad C \in \mathbb{R}$$

$$(29) \quad \int \frac{1}{\cos^2(7x)} \, dx = \frac{1}{7} \int \frac{7}{\cos^2(7x)} \, dx = \frac{1}{7} \operatorname{tg}(7x) + C, \quad C \in \mathbb{R}$$

$$(30) \quad \int (\sqrt{2x - 1} - \sqrt{1 + 3x}) \, dx = \int (2x - 1)^{1/2} \, dx - \int (1 + 3x)^{1/2} \, dx \\ = \frac{1}{2} \int 2(2x - 1)^{1/2} \, dx - \frac{1}{3} \int 3(1 + 3x)^{1/2} \, dx \\ = \frac{1}{2} \frac{(2x - 1)^{3/2}}{3/2} - \frac{1}{3} \frac{(1 + 3x)^{3/2}}{3/2} + C \\ = \frac{1}{3} (2x - 1)^{3/2} - \frac{2}{9} (1 + 3x)^{3/2} + C, \quad C \in \mathbb{R}$$

$$(31) \quad \int \frac{1}{x} (1 + (\ln x)^2) \, dx = \int \frac{1}{x} \, dx + \int \frac{1}{x} (\ln x)^2 \, dx \\ = \ln x + \frac{(\ln x)^3}{3} + C, \quad C \in \mathbb{R}$$

$$\begin{aligned}
(32) \quad \int \frac{2 + \sqrt{\operatorname{arctg}(2x)}}{1 + 4x^2} dx &= \int \frac{2}{1 + (2x)^2} dx + \int \frac{1}{1 + (2x)^2} (\operatorname{arctg}(2x))^{1/2} dx \\
&= \int \frac{2}{1 + (2x)^2} dx + \frac{1}{2} \int \frac{2}{1 + (2x)^2} (\operatorname{arctg}(2x))^{1/2} dx \\
&= \operatorname{arctg}(2x) + \frac{1}{2} \frac{(\operatorname{arctg}(2x))^{3/2}}{3/2} + C \\
&= \operatorname{arctg}(2x) + \frac{1}{3} (\operatorname{arctg}(2x))^{3/2} + C, \quad C \in \mathbb{R}
\end{aligned}$$

$$(33) \quad \int \frac{e^{\operatorname{arctg} x}}{1 + x^2} dx = e^{\operatorname{arctg} x} + C, \quad C \in \mathbb{R}$$

$$\begin{aligned}
(34) \quad \int \frac{\operatorname{sen} x}{\sqrt{1 + \cos x}} dx &= \int \operatorname{sen} x (1 + \cos x)^{-1/2} dx = - \int -\operatorname{sen} x (1 + \cos x)^{-1/2} dx \\
&= - \frac{(1 + \cos x)^{1/2}}{1/2} + C = -2 \sqrt{1 + \cos x} + C, \quad C \in \mathbb{R}
\end{aligned}$$

4. [Primitivação por partes]

(1) Calcule $\int \ln x \, dx$.

Sejam

$$\begin{aligned}
f'(x) &= 1 & f(x) &= x \\
g(x) &= \ln x & g'(x) &= \frac{1}{x}
\end{aligned}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - \int 1 \, dx = x \ln x - x + C, \quad C \in \mathbb{R}.$$

(2) Calcule $\int x \operatorname{sen}(2x) \, dx$.

Sejam

$$\begin{aligned}
f'(x) &= \operatorname{sen}(2x) & f(x) &= -\frac{1}{2} \cos(2x) \\
g(x) &= x & g'(x) &= 1
\end{aligned}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}
\int x \operatorname{sen}(2x) \, dx &= -\frac{1}{2} x \cos(2x) - \int -\frac{1}{2} \cos(2x) \cdot 1 \, dx = -\frac{1}{2} x \cos(2x) + \frac{1}{2} \int \cos(2x) \, dx \\
&= -\frac{1}{2} x \cos(2x) + \frac{1}{4} \int 2 \cos(2x) \, dx = -\frac{1}{2} x \cos(2x) + \frac{1}{4} \operatorname{sen}(2x) + C, \quad C \in \mathbb{R}.
\end{aligned}$$

(3) Calcule $\int \operatorname{arctg} x \, dx$.

Sejam

$$\begin{aligned} f'(x) &= 1 & f(x) &= x \\ g(x) &= \operatorname{arctg} x & g'(x) &= \frac{1}{1+x^2} \end{aligned}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned} \int \ln x \, dx &= x \operatorname{arctg} x - \int \frac{x}{1+x^2} \, dx = x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx \\ &= x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + C, \quad C \in \mathbb{R}. \end{aligned}$$

(4) Calcule $\int x \cos x \, dx$.

Sejam

$$\begin{aligned} f'(x) &= \cos x & f(x) &= \operatorname{sen} x \\ g(x) &= x & g'(x) &= 1 \end{aligned}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int x \cos x \, dx = x \operatorname{sen} x - \int \operatorname{sen} x \, dx = x \operatorname{sen} x + \cos x + C, \quad C \in \mathbb{R}.$$

(5) Calcule $\int \ln(1-x) \, dx$.

Sejam

$$\begin{aligned} f'(x) &= 1 & f(x) &= x \\ g(x) &= \ln(1-x) & g'(x) &= \frac{-1}{1-x} \end{aligned}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned} \int \ln(1-x) \, dx &= x \ln(1-x) - \int \frac{-x}{1-x} \, dx = x \ln(1-x) - \int \frac{1-x-1}{1-x} \, dx \\ &= x \ln(1-x) - \int 1 \, dx + \int \frac{1}{1-x} \, dx \\ &= x \ln(1-x) - \int 1 \, dx - \int \frac{-1}{1-x} \, dx \\ &= x \ln(1-x) - x - \ln(1-x) + C, \quad C \in \mathbb{R}. \end{aligned}$$

(6) Calcule $\int x \ln x \, dx$.

Sejam

$$\begin{aligned} f'(x) &= x & f(x) &= \frac{x^2}{2} \\ g(x) &= \ln x & g'(x) &= \frac{1}{x} \end{aligned}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int x \ln x \, dx &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx \\ &= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C, \quad C \in \mathbb{R}.\end{aligned}$$

(7) Calcule $\int x^2 \sin x \, dx$.

Sejam

$$f'(x) = \sin x \quad f(x) = -\cos x$$

$$g(x) = x^2 \quad g'(x) = 2x$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int x^2 \sin x \, dx = -x^2 \cos x - \int (-\cos x) 2x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx.$$

Apliquemos de novo o método de primitivação por partes para calcular $\int x \cos x \, dx$.

Sejam

$$f'(x) = \cos x \quad f(x) = \sin x$$

$$g(x) = x \quad g'(x) = 1$$

Aplicando de novo a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int x^2 \sin x \, dx &= -x^2 \cos x + 2 \int x \cos x \, dx \\ &= -x^2 \cos x + 2 \left(x \sin x - \int \sin x \, dx \right) \\ &= -x^2 \cos x + 2x \sin x - 2 \int \sin x \, dx \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C, \quad C \in \mathbb{R}.\end{aligned}$$

(8) Calcule $\int x \sin x \cos x \, dx$.

Sejam

$$f'(x) = \sin x \cos x \quad f(x) = \frac{\sin^2 x}{2}$$

$$g(x) = x \quad g'(x) = 1$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}
\int x \operatorname{sen} x \cos x \, dx &= x \frac{\operatorname{sen}^2 x}{2} - \int \frac{\operatorname{sen}^2 x}{2} \, dx \\
&= x \frac{\operatorname{sen}^2 x}{2} - \frac{1}{2} \int \frac{1 - \cos(2x)}{2} \, dx \\
&= x \frac{\operatorname{sen}^2 x}{2} - \int \frac{1}{4} \, dx + \frac{1}{4} \int \cos(2x) \, dx \\
&= x \frac{\operatorname{sen}^2 x}{2} - \int \frac{1}{4} \, dx + \frac{1}{8} \int 2 \cos(2x) \, dx \\
&= x \frac{\operatorname{sen}^2 x}{2} - \frac{x}{4} + \frac{1}{8} \operatorname{sen}(2x) + C, \quad C \in \mathbb{R}.
\end{aligned}$$

(9) Calcule $\int \ln^2 x \, dx$.

Sejam

$$f'(x) = 1 \qquad f(x) = x$$

$$g(x) = \ln^2 x \qquad g'(x) = 2 \frac{\ln x}{x}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \ln^2 x \, dx = x \ln^2 x - \int 2x \frac{\ln x}{x} \, dx = x \ln^2 x - 2 \int \ln x \, dx.$$

Pelo exercício 4.(1) temos que: $\int \ln x \, dx = x \ln x - x + C$, $C \in \mathbb{R}$. Então,

$$\int \ln^2 x \, dx = x \ln^2 x - 2x \ln x + 2x + C, \quad C \in \mathbb{R}.$$

(10) Calcule $\int e^x \cos x \, dx$. Este exercício é análogo ao Exemplo 5 do ficheiro Aula 14 Cálculo (T). A solução deste exercício é:

$$\int e^x \cos x \, dx = \frac{e^x \cos x + e^x \operatorname{sen} x}{2} + C, \quad C \in \mathbb{R}.$$

(11) Calcule $\int \arcsen x \, dx$. Resolvido na aula.

(12) Calcule $\int e^{\operatorname{sen} x} \operatorname{sen} x \cos x \, dx$.

Sejam

$$f'(x) = e^{\operatorname{sen} x} \cos x \qquad f(x) = e^{\operatorname{sen} x}$$

$$g(x) = \operatorname{sen} x \qquad g'(x) = \cos x$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}
\int e^{\operatorname{sen} x} \operatorname{sen} x \cos x \, dx &= e^{\operatorname{sen} x} \operatorname{sen} x - \int e^{\operatorname{sen} x} \cos x \, dx \\
&= e^{\operatorname{sen} x} \operatorname{sen} x - e^{\operatorname{sen} x} + C, \quad C \in \mathbb{R}.
\end{aligned}$$

(13) Calcule $\int \frac{\arcsen \sqrt{x}}{\sqrt{x}} dx$.

Sejam

$$f'(x) = \frac{1}{\sqrt{x}} = x^{-1/2} \quad f(x) = 2\sqrt{x}$$

$$g(x) = \arcsen \sqrt{x} \quad g'(x) = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned} \int \frac{\arcsen \sqrt{x}}{\sqrt{x}} dx &= 2\sqrt{x} \arcsen \sqrt{x} - \int \frac{1}{\sqrt{1-x}} dx \\ &= 2\sqrt{x} \arcsen \sqrt{x} - \int (1-x)^{-1/2} dx \\ &= 2\sqrt{x} \arcsen \sqrt{x} + 2\sqrt{1-x} + C, \quad C \in \mathbb{R}. \end{aligned}$$

(14) Calcule $\int x \operatorname{arctg} x dx$. Resolvido na aula.

(15) Calcule $\int x^2 \log x dx$.

Sejam

$$f'(x) = x^2 \quad f(x) = \frac{x^3}{3}$$

$$g(x) = \log x \quad g'(x) = \frac{1}{x}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned} \int x^2 \log x dx &= \frac{x^3}{3} \log x - \int \frac{x^2}{3} dx \\ &= \frac{x^3}{3} \log x - \frac{x^3}{9} + C, \quad C \in \mathbb{R}. \end{aligned}$$

(16) Calcule $\int \operatorname{sen}(\log x) dx$.

Sejam

$$f'(x) = 1 \quad f(x) = x$$

$$g(x) = \operatorname{sen}(\log x) \quad g'(x) = \frac{1}{x} \cos(\log x)$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \operatorname{sen}(\log x) dx = x \operatorname{sen}(\log x) - \int \cos(\log x) dx.$$

Apliquemos de novo o método de primitivação por partes para calcular $\int \cos(\log x) dx$.

Sejam

$$\begin{aligned}f'(x) &= 1 & f(x) &= x \\g(x) &= \cos(\log x) & g'(x) &= -\frac{1}{x}\sin(\log x)\end{aligned}$$

Aplicando de novo a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int \sin(\log x) dx &= x \sin(\log x) - [x \cos(\log x) - \int -\sin(\log x) dx] \\&= x \sin(\log x) - x \cos(\log x) - \int \sin(\log x) dx.\end{aligned}$$

Então,

$$\int \sin(\log x) dx = x \sin(\log x) - x \cos(\log x) - \int \sin(\log x) dx,$$

ou, de forma equivalente,

$$2 \int \sin(\log x) dx = x \sin(\log x) - x \cos(\log x).$$

Consequentemente,

$$\int \sin(\log x) dx = \frac{x \sin(\log x) - x \cos(\log x)}{2} + C, \quad C \in \mathbb{R}.$$

(17) Calcule $\int \operatorname{ch} x \sin(3x) dx$.

Sejam

$$\begin{aligned}f'(x) &= \operatorname{ch} x & f(x) &= \operatorname{sh} x \\g(x) &= \sin(3x) & g'(x) &= 3 \cos(3x)\end{aligned}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \operatorname{ch} x \sin(3x) dx = \operatorname{sh} x \sin(3x) - 3 \int \operatorname{sh} x \cos(3x) dx.$$

Apliquemos de novo o método de primitivação por partes para calcular $\int \operatorname{sh} x \cos(3x) dx$.

Sejam

$$\begin{aligned}f'(x) &= \operatorname{sh} x & f(x) &= \operatorname{ch} x \\g(x) &= \cos(3x) & g'(x) &= -3 \sin(3x)\end{aligned}$$

Aplicando de novo a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int \operatorname{ch} x \sin(3x) dx &= \operatorname{sh} x \sin(3x) - 3[\operatorname{ch} x \cos(3x) - \int -3 \operatorname{ch} x \sin(3x) dx] \\&= \operatorname{sh} x \sin(3x) - 3 \operatorname{ch} x \cos(3x) - 9 \int \operatorname{ch} x \sin(3x) dx.\end{aligned}$$

Então,

$$\int \operatorname{ch} x \operatorname{sen}(3x) dx = \operatorname{sh} x \operatorname{sen}(3x) - 3 \operatorname{ch} x \cos(3x) - 9 \int \operatorname{ch} x \operatorname{sen}(3x) dx,$$

ou, de forma equivalente,

$$10 \int \operatorname{ch} x \operatorname{sen}(3x) dx = \operatorname{sh} x \operatorname{sen}(3x) - 3 \operatorname{ch} x \cos(3x).$$

$$\text{Consequentemente, } \int \operatorname{ch} x \operatorname{sen}(3x) dx = \frac{\operatorname{sh} x \operatorname{sen}(3x) - 3 \operatorname{ch} x \cos(3x)}{10} + C, \quad C \in \mathbb{R}.$$

(18) Calcule $\int x^3 e^{x^2} dx$.

Sejam

$$f'(x) = x e^{x^2} \quad f(x) = \frac{1}{2} e^{x^2}$$

$$g(x) = x^2 \quad g'(x) = 2x$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned} \int x^3 e^{x^2} dx &= \frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} dx \\ &= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C, \quad C \in \mathbb{R}. \end{aligned}$$

5. [Primitivação por substituição]

(1) Resolvido na aula.

(2) Exercício análogo a um exercício resolvida na aula.

(3) Calcule $\int \sqrt{4+x^2} dx$, efetuando a substituição $x = 2 \operatorname{sh} t$, $t \geq 0$.

(i) Substituição:

Fazendo $x = 2 \operatorname{sh} t$, $t \geq 0$, tem-se que

$$\varphi(t) = 2 \operatorname{sh} t, \quad \varphi'(t) = 2 \operatorname{ch} t, \quad t = \operatorname{argsh}(x/2).$$

(ii) Cálculo da nova primitiva:

$$\begin{aligned} \int \sqrt{4+x^2} dx &= \int \sqrt{4+4 \operatorname{sh}^2 t} \cdot \underbrace{2 \operatorname{ch} t}_{\varphi'(t)} dt \\ &= \int 2\sqrt{1+\operatorname{sh}^2 t} \cdot 2 \operatorname{ch} t dt = \int 4\sqrt{\operatorname{ch}^2 t} \cdot \operatorname{ch} t dt = \int 4 \operatorname{ch}^2 t dt \\ &= 4 \int \frac{1+\operatorname{ch}(2t)}{2} dt = 4 \left(\frac{t}{2} + \frac{1}{4} \operatorname{sh}(2t) \right) + C, \quad C \in \mathbb{R} \\ &= 2t + \operatorname{sh}(2t) + C = 2t + 2 \operatorname{sh} t \operatorname{ch} t + C, \quad C \in \mathbb{R}. \end{aligned}$$

(iii) Regresso à variável inicial x :

Atendendo a que: $\operatorname{sh} t = \frac{x}{2}$, $t = \operatorname{argsh}(x/2)$ e a que

$$\operatorname{ch}^2 t - \operatorname{sh}^2 t = 1 \Rightarrow \operatorname{ch} t = \sqrt{1 + \operatorname{sh}^2 t} \quad (\operatorname{ch} t \geq 1)$$

obtemos que

$$\int \sqrt{4 + x^2} dx = 2 \operatorname{argsh}(x/2) + x \sqrt{1 + \frac{x^2}{4}} + C = 2 \operatorname{argsh}(x/2) + \frac{x}{2} \sqrt{4 + x^2} + C, \quad C \in \mathbb{R}.$$

(4) Resolvido na aula.

(5) Resolvido na aula.

(6) Resolvido na aula.

6. [Primitivação de funções racionais]

(1) Resolvido na aula.

(2) Resolvido na aula.

(3) Resolvido na aula.

(4) Resolvido na aula.

$$(5) \int \frac{x^2 - x + 2}{x(x^2 - 2)} dx = -2 \ln |x| + \ln |x - 1| + 2 \ln |x + 1| + C, \quad C \in \mathbb{R}.$$

$$(6) \int \frac{27}{x^4 - 3x^3} dx = \frac{9}{2x^2} + \frac{3}{x} - \ln |x| + \log |x - 3| + C, \quad C \in \mathbb{R}.$$

$$(7) \int \frac{x + 3}{(x - 2)(x^2 - 2x + 5)} dx = \ln |x - 2| - \frac{1}{2} \log |x^2 - 2x + 5| + C, \quad C \in \mathbb{R}.$$

$$(8) \int \frac{x + 1}{x(x^2 + 1)^2} dx = \ln |x| + \frac{1}{2(x^2 + 1)} - \frac{\ln(x^2 + 1)}{2} + \frac{\operatorname{arctg} x}{2} + \frac{x}{2(x^2 + 1)} + C, \quad C \in \mathbb{R}.$$

7. (1) Calcule $\int \frac{1}{(2 + \sqrt{x})^7 \sqrt{x}} dx$.

$$\begin{aligned} \int \frac{1}{(2 + \sqrt{x})^7 \sqrt{x}} dx &= \int \frac{1}{\sqrt{x}} (2 + \sqrt{x})^{-7} dx = 2 \int \frac{1}{2\sqrt{x}} (2 + \sqrt{x})^{-7} dx \\ &= 2 \frac{(2 + \sqrt{x})^{-6}}{-6} + C = -\frac{1}{3(2 + \sqrt{x})^6} + C, \quad C \in \mathbb{R}. \end{aligned}$$

(2) Calcule $\int \operatorname{tg}^2 x \, dx$.

$$\begin{aligned}\int \operatorname{tg}^2 x \, dx &= \int \frac{\operatorname{sen}^2 x}{\cos^2 x} \, dx = \int \frac{1 - \cos^2 x}{\cos^2 x} \, dx \\ &= \int \frac{1}{\cos^2 x} \, dx - \int 1 \, dx = \operatorname{tg} x - x + C, \quad C \in \mathbb{R}.\end{aligned}$$

(3) Calcule $\int \frac{x + (\operatorname{arcsen}(3x))^2}{\sqrt{1 - 9x^2}} \, dx$.

$$\begin{aligned}\int \frac{x + (\operatorname{arcsen}(3x))^2}{\sqrt{1 - 9x^2}} \, dx &= \int x(1 - 9x^2)^{-1/2} \, dx + \int \frac{1}{\sqrt{1 - 9x^2}} (\operatorname{arcsen}(3x))^2 \, dx \\ &= -\frac{1}{18} \int -18x(1 - 9x^2)^{-1/2} \, dx + \frac{1}{3} \int \frac{3}{\sqrt{1 - (3x)^2}} (\operatorname{arcsen}(3x))^2 \, dx \\ &= -\frac{1}{18} \frac{(1 - 9x^2)^{1/2}}{1/2} + \frac{(\operatorname{arcsen}(3x))^3}{9} + C \\ &= -\frac{1}{9} \sqrt{1 - 9x^2} + \frac{(\operatorname{arcsen}(3x))^3}{9} + C, \quad C \in \mathbb{R}.\end{aligned}$$

(4) Calcule $\int \frac{1}{1 + e^x} \, dx$.

$$\int \frac{1}{1 + e^x} \, dx = \int \frac{1 + e^x - e^x}{1 + e^x} \, dx = \int 1 \, dx - \int \frac{e^x}{1 + e^x} \, dx = x - \ln(1 + e^x) + C, \quad C \in \mathbb{R}.$$

(5) Calcule $\int \frac{1}{\cos^2 x \operatorname{sen}^2 x} \, dx$.

$$\begin{aligned}\int \frac{1}{\cos^2 x \operatorname{sen}^2 x} \, dx &= \int \frac{\cos^2 x + \operatorname{sen}^2 x}{\cos^2 x \operatorname{sen}^2 x} \, dx \\ &= \int \frac{1}{\operatorname{sen}^2 x} \, dx + \int \frac{1}{\cos^2 x} \, dx = -\cotg x + \operatorname{tg} x + C, \quad C \in \mathbb{R}.\end{aligned}$$

(6) Calcule $\int \frac{1}{x^2 \sqrt{4 - x^2}} \, dx$, efetuando a substituição $x = 2 \operatorname{sen} t$.

(i) Susbtituição:

Fazendo $x = 2 \operatorname{sen} t$, tem-se que

$$\varphi(t) = 2 \operatorname{sen} t, \quad \varphi'(t) = 2 \cos t, \quad t = \operatorname{arcsen}(x/2), \quad t \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[.$$

(ii) Cálculo da nova primitiva:

$$\begin{aligned}\int \frac{1}{x^2 \sqrt{4-x^2}} dx &= \int \frac{1}{4 \operatorname{sen}^2 t \sqrt{4-4 \operatorname{sen}^2 t}} \cdot \underbrace{2 \cos t}_{\varphi'(t)} dt \\&= \int \frac{1}{8 \operatorname{sen}^2 t \cos t} \cdot 2 \cos t dt \\&= \frac{1}{4} \int \frac{1}{\operatorname{sen}^2 t} dt = -\frac{1}{4} \cot g t + C, \quad C \in \mathbb{R}\end{aligned}$$

(iii) Regresso à variável inicial x :

Atendendo a que:

$$x = 2 \operatorname{sen} t \quad \text{e a que} \quad 1 + \cot g^2 t = \frac{1}{\operatorname{sen}^2 t}$$

obtem-se que

$$\int \frac{1}{x^2 \sqrt{4-x^2}} dx = -\frac{1}{4} \frac{\sqrt{4-x^2}}{x} + C, \quad C \in \mathbb{R}.$$