Universidade do Minho
Departamento de Matemática

Licenciatura em Ciências da Computação
Análise
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Funções reais de n variáveis reais: limite e continuidade

1. Indique o domínio das seguintes funções:

   (a) $f(x,y) = \dfrac{1}{y-x}$

   (b) $f(x,y) = \log(x^2+y)$

   (c) $f(x,y) = \sqrt{9x^2+4y^2-36}$

   (d) $f(x,y) = \dfrac{1}{\sqrt{y^2-x^2}}$

2. Represente graficamente o domínio das funções definidas por:

   (a) $f(x,y) = \dfrac{x}{y-1}$

   (b) $f(x,y,z) = \log(y^2+z^2-1)$

   (c) $f(x,y) = \sqrt{xy}$

   (d) $f(x,y,z) = \sqrt{9-x^2-y^2-z^2}$

   (e) $f(x,y) = \dfrac{x}{\log(y^2-x)}$

   (f) $f(x,y) = \sqrt{x^2+y^2-2xy}$

   (g) $f(x,y) = \sqrt{-(x-y)^2}$

   (h) $f(x,y) = \sqrt{4-x^2} - \sqrt{y^2-4}$

   (i) $f(x,y) = \dfrac{x-y}{\sqrt{1-x^2-y^2}}$

   (j) $f(x,y) = \log(x^2) + \sqrt{y-x^2}$

   (k) $f(x,y) = \dfrac{e^x}{\sqrt{xy}}$

   (l) $f(x,y) = \dfrac{y}{\sqrt{x^2-y^2}}$

   (m) $f(x,y) = \sqrt{y\,\mathrm{sen}x}$

   (n) $f(x,y) = \dfrac{1}{\sqrt{x}}\log(2x-y)$

   (o) $f(x,y) = e^{\sqrt{x^2+y^2-4}}$

   (p) $f(x,y) = \cos\left(\dfrac{y}{x}\right)$

   (q) $f(x,y) = \log(y-x)$

   (r) $f(x,y) = \sqrt{y-x} + \sqrt{1-x}$.

3. Esboce o gráfico das seguintes funções:

   (a) $f : [0,1] \times [0,4] \longrightarrow \mathbb{R}$ tal que $f(x,y) = x$;

   (b) $f : [0,2\pi] \times [0,1] \longrightarrow \mathbb{R}$ tal que $f(x,y) = \mathrm{sen}x$;

   (c) $f : [-1,1] \times [-1,1] \longrightarrow \mathbb{R}$ tal que $f(x,y) = y^2$;

   (d) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ tal que $f(x,y) = \sqrt{x^2+y^2}$.

4. Determine e esboce algumas curvas de nível das seguintes funções:

   (a) $f(x,y) = x^2 - y^2$

   (b) $f(x,y) = 1 - \dfrac{x}{2} - \dfrac{y}{4}$

   (c) $f(x,y) = \sqrt{4-x^2-y^2}$

   (d) $f(x,y) = x - y + 5$

   (e) $f(x,y) = y - x^2$

   (f) $f(x,y) = -\sqrt{1-x^2-y^2}$

5. Descreva as superfícies de nível das seguintes funções:

   (a) $f(x,y,z) = x^2 + y^2 + z^2$;

   (b) $f(x,y,z) = x^2 + y^2$;

   (c) $f(x,y,z) = x + 2y + 3z$;

   (d) $f(x,y,z) = z - x^2 - y^2$.

6. Estude a existência dos seguintes limites:

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)\,, \quad \text{com} \quad f(x,y) = \begin{cases} 1 & \text{se } y = x^2 \\ 0 & \text{se } y \neq x^2 \end{cases}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y)\,, \quad \text{com} \quad g(x,y) = \begin{cases} 1 & \text{se } x^2 + y^2 \leq 1 \\ 7 & \text{se } x^2 + y^2 > 1 \end{cases}$

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} h(x,y)\,, \quad \text{com} \quad h(x,y) = \begin{cases} x^2 & \text{se } y = x \\ \operatorname{sen}(xy) & \text{se } y \neq x \end{cases}$

   (d) $\displaystyle\lim_{(x,y)\to(0,0)} k(x,y)\,, \quad \text{com} \quad k(x,y) = \begin{cases} 0 & \text{se } x \geq 0,\ y \geq 0 \\ x & \text{se } x \geq 0,\ y < 0 \\ y & \text{se } x < 0\,. \end{cases}$

7. Considere as funções:

   $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$ tal que $f(x,y) = \dfrac{2(x-1)y^2}{x^2+y^2}$;

   $g\colon \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$ tal que $g(x,y) = \dfrac{y^2 - 2y}{x^4 + y^2}$;

   $h\colon \{(x,y) \in \mathbb{R}^2 :\ y \neq -1\} \longrightarrow \mathbb{R}$ tal que $h(x,y) = \dfrac{x^2}{y+1}$.

   Calcule, caso existam, os seguintes limites:

   (a) $\displaystyle\lim_{(x,y)\to(1,1)} f(x,y)$ e $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$;

   (b) $\displaystyle\lim_{(x,y)\to(0,2)} g(x,y)$ e $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y)$;

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} h(x,y)$ e $\displaystyle\lim_{(x,y)\to(0,-1)} h(x,y)$.

8. Determine, caso existam, os seguintes limites:

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\operatorname{sen}\tfrac{1}{\sqrt{x^2+y^2}}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$

   (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

(e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

(f) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

(g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\operatorname{sen}(x-y)}{\cos(x-y)}$

(h) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^3}{3x^2+4y^6}$

(i) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2+(y-1)^2}$

(j) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-2x^2+3y}{x^2+y^2}$

(k) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{y^2z}{x^2+y^2+z^2}$

(l) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^3+yz^2}{x^4+y^2+z^4}$

(m) $\displaystyle\lim_{(x,y)\to(2,0)} \frac{(x-2)^2y^2}{(x-2)^2+y^2}$

(n) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

(o) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

(p) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^2+y^2)^2}\operatorname{sen}(x^2+y^2)$

9. Seja $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{se } (x,y)\neq(0,0)\,, \\ 0 & \text{se } (x,y)=(0,0)\,. \end{cases}$ Mostre que:

   (a) f é descontínua em $(0,0)$;

   (b) para cada $m\in\mathbb{R}$, a restrição $f|_\ell$ da função f à recta ℓ de equação $y=mx$ é contínua em $(0,0)$.

10. Estude a continuidade das seguintes funções:

   (a) $f(x,y) = e^{x^2+3y}$

   (b) $f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}} & \text{se} \quad (x,y)\neq(0,0) \\ 0 & \text{se} \quad (x,y)=(0,0) \end{cases}$

   (c) $f(x,y) = \begin{cases} xy^2\operatorname{sen}\left(\dfrac{1}{y}\right) & \text{se} \quad y\neq 0 \\ 0 & \text{se} \quad y=0 \end{cases}$

   (d) $f(x,y) = \begin{cases} \cos\left(\dfrac{\pi}{2}+\dfrac{x^2-y^2}{x^2+y^2}\right) & \text{se} \quad (x,y)\neq(0,0) \\ 0 & \text{se} \quad (x,y)=(0,0) \end{cases}$

   (e) $f(x,y) = \begin{cases} \dfrac{x+y}{\sqrt{x^2+y^2}} & \text{se} \quad (x,y)\neq(0,0) \\ 0 & \text{se} \quad (x,y)=(0,0) \end{cases}$

   (f) $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^4} & \text{se} \quad (x,y)\neq(0,0) \\ 0 & \text{se} \quad (x,y)=(0,0) \end{cases}$

(g) $f(x,y) = \begin{cases} \dfrac{x}{y} & \text{se} \quad y \neq 0 \\ 0 & \text{se} \quad y = 0 \end{cases}$

(h) $f(x,y) = \begin{cases} \dfrac{4xy(x^2-y^2)}{x^2+y^2} & \text{se} \quad (x,y) \neq (0,0) \\ 0 & \text{se} \quad (x,y) = (0,0) \end{cases}$

(i) $f(x,y) = \begin{cases} \dfrac{2x^3y^3}{(x^2+y^2)^2} & \text{se} \quad (x,y) \neq (0,0) \\ 0 & \text{se} \quad (x,y) = (0,0) \end{cases}$

(j) $f(x,y) = \begin{cases} \dfrac{x^3y}{x^6+y^2} & \text{se} \quad (x,y) \neq (0,0) \\ 0 & \text{se} \quad (x,y) = (0,0) \end{cases}$

(k) $f(x,y) = \begin{cases} x & \text{se} \quad x \in \mathbb{Q} \\ 0 & \text{se} \quad x \notin \mathbb{Q} \end{cases}$

(l) $f(x,y) = \begin{cases} 0 & \text{se} \quad 0 < y < x^2 \\ 1 & \text{se} \quad y \leq 0 \vee x^2 \leq y \end{cases}$

(m) $f(x,y) = \begin{cases} x & \text{se} \quad x \geq y \\ y & \text{se} \quad x < y \end{cases}$

(n) $f(x,y) = \begin{cases} x^2+y^2 & \text{se} \quad x^2+y^2 \leq 1 \\ 0 & \text{se} \quad x^2+y^2 > 1 \end{cases}$

(o) $f(x,y) = \begin{cases} \dfrac{x}{y-2} & \text{se} \quad y \neq 2 \\ 0 & \text{se} \quad y = 2 \end{cases}$

(p) $f(x,y) = \begin{cases} \dfrac{\operatorname{sen}\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\frac{1}{x^2+y^2}} & \text{se} \quad (x,y) \neq (0,0) \\ 0 & \text{se} \quad (x,y) = (0,0). \end{cases}$

11. Considere as funções

$$f\colon \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R} \quad \text{tal que} \quad f(x,y) = \frac{1-\cos(x^2+y^2)}{x^2+y^2};$$

$$g\colon \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R} \quad \text{tal que} \quad g(x,y) = \frac{xy}{x^2+y^2};$$

$$h\colon \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R} \quad \text{tal que} \quad h(x,y) = \frac{x^3y}{x^2+y^4}.$$

Verifique se as função f, g e h admitem prolongamentos contínuos a $\mathbb{R}^2$.