Universidad Técnica Nacional

UTN

Profesora: Jackeline Cascante Paniagua

FOLLETO 2

FUNCIONES Y TRASFORMACIONES CON VARIABLE COMPLEJA

Una función en variable compleja es homóloga a una función de variable real

$f : C \to C$

$z \to w \quad \Rightarrow f(z) = w$

Transformaciones: Si w = u + iv es una función unívoca de z = x + yi entonces se tiene que: u + iv = f (x + i y)

u= (x, y)   v= (x, y)   u, v: se conocen como coordenadas curvilíneas

Ejemplo: Expresar la función $f(x) = x^2$ de la forma u(x, y) + iv(x,y). Después calcule $f((1, 2))$

Sea $z = x + yi$

$f(z) = (x + yi)^2$

$w = f(z)$

$w = (x + yi)^2$

$w = x^2 + 2xyi - y^2$

$w = (x^2 - y^2) + 2xyi$

$u = x^2 - y^2 \qquad v = 2xy$

$u(1,2) = (1)^2 - (2)^2$

$u(1,2) = -3$

$v(1,2) = 2(1)(2)$

$v(1,2) = 4$

$f(1,2) = u(1,2) + iv(1,2)$

$\boxed{f(1,2) = -3 + 4i}$

Ejemplo: Expresar la función $f(x) = \ln z$ de la forma u(x, y) + iv(x,y).

$z = x + yi$

$f(z) = \ln(x + yi)$

$w = \ln(x + yi)$

$x + yi = r(\cos\theta + i\,\mathrm{Sen}\,\theta)$

$x + yi = re^{i\theta}$

$w = \ln(re^{i\theta})$

$w = \ln(r) + \ln(e^{i\theta})$

$w = \ln(\sqrt{x^2 + y^2}) + i\theta \ln e$

$w = \ln(\sqrt{x^2 + y^2}) + i\,\mathrm{Arctan}\left(\frac{y}{x}\right)$

$u = \ln(\sqrt{x^2 + y^2})$

$v = \mathrm{Arctan}\left(\frac{y}{x}\right)$

Funciones elementales

1. Polinómica: Son definidas por:

$w = a_0 z^n + a_1 z^{n-1} + \ldots\ldots + a_n = P(z)$

La trasformación $w = az + b$ es una transformación lineal

2. Función racional $w = \dfrac{P(z)}{Q(z)}$

3. Función exponencial

$w = e^z \Rightarrow w = e^{x+yi} \Rightarrow w = e^x \cdot e^{iy} \Rightarrow w = e^x(\mathrm{Cos}\,y + i\,\mathrm{Sen}\,y)$

$w = e^x\,\mathrm{Cos}\,y + i e^x\,\mathrm{Sen}\,y$

Propiedades

a) $e^0 = 1$

b) $e^{i\pi} = -1$

c) $e^{i\frac{\pi}{2}} = i$

d) $e^{i\frac{3\pi}{2}} = -i$

e) $e^z \cdot e^w = e^{z+w}$

f) $\dfrac{e^z}{e^w} = e^{z-w}$

g) $e^{-z} = \dfrac{1}{e^z}$

h) e^z tiene periodo $2k\pi i$

4. Funciones trigonométricas

$$sen\ z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\csc z = \frac{2i}{e^{iz} - e^{-iz}}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sec z = \frac{2}{e^{iz} + e^{-iz}}$$

¿Quién sería tan z y cot z?

$$\text{Tan } z = \frac{\text{Sen } z}{\text{Cos } z} \rightarrow \text{Tan } z = \frac{\dfrac{e^{iz} - e^{-iz}}{2i}}{\dfrac{e^{iz} + e^{-iz}}{2}} \rightarrow$$

$$\text{Tan } z = \frac{2(e^{iz} - e^{-iz})}{2i(e^{iz} + e^{-iz})} \rightarrow \text{Tan } z = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$\text{Cot } z = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

Identidades

a) $Sen^2 z + Cos^2 z = 1$

b) $1 + \tan^2 z = \sec^2 z$

c) $1 + \cot^2 z = \csc^2 z$

d) $Sen(-z) = -\,sen\,z$

e) $Cos(-z) = \cos z$

f) $Tan(-z) = -\tan z$

g) $Sen(z_1 \pm z_2) = Sen z_1 \cdot Cos z_2 \pm Cos z_1 \cdot Sen z_2$

h) $Cos(z_1 \pm z_2) = Cos z_1 \cdot Cos z_2 \mp Sen z_1 \cdot Sen z_2$

i) $Tan\,(z_1 \pm z_2) = \dfrac{Tan\,z_1 \pm Tan\,z_2}{1 \mp Tan\,z_1\,Tan\,z_2}$

j) $Sen(x + yi) = Sen x \cdot Cosh y \pm i\,Cos x \cdot Senh\,y$

k) $Cos(x + yi) = Cos x \cdot Cosh y \pm i\,Sen x \cdot Senh\,y$

5. Funciones hiperbólicas

$$senh\ z = \frac{e^z - e^{-z}}{2}$$

$$\operatorname{csc}h\ z = \frac{2}{e^z - e^{-z}}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$\operatorname{sec}h\ z = \frac{2}{e^z + e^{-z}}$$

$$Tanh\ z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$Coth\ z = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

Identidades

$$Cosh^2 z - Senh^2 z = 1$$

$$1 - \tanh^2 z = \operatorname{sec}h^2 z$$

$$\coth^2 z - 1 = \operatorname{csc}h^2 z$$

6. Funciones logarítmicas

$$w = \ln z = \ln r + i(\theta + 2k\pi) \quad k = 0, \pm 1. \pm 2,.....$$

7. Funciones trigonométricas inversas

$$sen^{-1} z = \frac{1}{i} \ln\left(i\, z + \sqrt{1 - z^2} \right)$$

$$\csc^{-1} z = \frac{1}{i} \ln\left(\frac{i + \sqrt{z^2 - 1}}{z} \right)$$

$$\cos^{-1} z = \frac{1}{i} \ln\left(z + \sqrt{z^2 - 1} \right)$$

$$\sec^{-1} z = \frac{1}{i} \ln\left(\frac{1 + \sqrt{1 - z^2}}{z} \right)$$

$$\tan^{-1} z = \frac{1}{2i} \ln\left(\frac{1 + iz}{1 - iz} \right)$$

$$\cot^{-1} z = \frac{1}{2i} \ln\left(\frac{z + i}{z - i} \right)$$

8. Funciones hiperbólicas inversas

$$senh^{-1} z = \ln\left(z + \sqrt{z^2 + 1} \right)$$

$$\csc h^{-1} z = \ln\left(\frac{1 + \sqrt{z^2 + 1}}{z} \right)$$

$$\cosh^{-1} z = \ln\left(z + \sqrt{z^2 - 1} \right)$$

$$\sec h^{-1} z = \ln\left(\frac{1 + \sqrt{1 - z^2}}{z} \right)$$

$$\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1 + z}{1 - z} \right)$$

$$\coth^{-1} z = \frac{1}{2} \ln\left(\frac{z + 1}{z - 1} \right)$$

9. Función potencia compleja

Es de la forma $w = [a]^b = \left\{ e^{b(\log a \,+\, 2k\pi i)};\ k \in Z \right\}$

Ejemplos 1 Cuaderno

Sea $w = f(z) = z^2$. Halle los valores de w en cada caso y grafíquelo

a) $z = -2 + i$ b) $z = 1 - 3i$ $R/ -8 -6i$

a) $z = -2 + i$

$w = z^2$
$w = (-2+i)^2$
$w = -1 - 4i + 4$
$w = 3 - 4i$

Ejemplo 2: Encontrar todas las "z" tales que $e^z = 1 + 2i$

Sea $z = x + yi$

$e^z = e^{x+yi}$

Como $e^z = 1 + 2i$ $e^{i\theta} = \cos\theta + i\,\text{Sen}\,\theta.$

$e^{x+yi} = 1 + 2i$

$e^x \cdot e^{yi} = 1 + 2i$

$e^x(\cos y + i\,\text{Sen}\,y) = 1 + 2i$

$e^x \cos y + ie^x\,\text{Sen}\,y = 1 + 2i$ Por igualdad de complejos

I. $e^x \cos y = 1$ II. $e^x\,\text{Sen}\,y = 2$

Elevando al cuadrado cada ecuación y sumando tenemos

$e^{2x}\cos^2 y = 1$
$e^{2x}\,\text{Sen}^2 y = 4$

$e^{2x}\cos^2 y + e^{2x}\,\text{Sen}^2 y = 5$

$e^{2x}(\cos^2 y + \text{Sen}^2 y) = 5$

$e^{2x} = 5$

$2x = \ln(5)$

$x = \frac{\ln(5)}{2}$

Ahora dividimos II ÷ I

$$\frac{e^x \operatorname{Sen} y}{e^x \operatorname{Cos} y} = \frac{2}{1}$$

$$\operatorname{Tan} y = 2 \quad \Rightarrow \quad y = \operatorname{Tan}^{-1}(2)$$
$$y \approx 63{,}43$$

$$z = x + yi$$

$$\boxed{z = \frac{1}{2}\ln(5) + i\operatorname{Tan}^{-1}(2)}$$

OTRA FORMA:

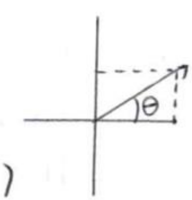
$$e^z = 1 + 2i$$

$$z \ln(e) = \ln(1+2i)$$
$$z = \ln(1+2i)$$

$1 + 2i$

$r = \sqrt{5}$

$\theta = \operatorname{Tan}^{-1}(2)$



$$z = \ln\sqrt{5} + i\operatorname{Tan}^{-1}(2)$$

$$\boxed{z = \frac{1}{2}\ln(5) + i\operatorname{Tan}^{-1}(2)}$$

Ejemplo 3

Hallar el valor principal de: $\left[\frac{e}{2}\left(-1-\sqrt{3}\,i\right)\right]^{3\pi i}$

$$\left(\frac{-e}{2} - \frac{e}{2}\sqrt{3}\,i\right)^{3\pi i}$$

$$e^{3\pi i\left(\log\left(-\frac{e}{2} - \frac{e}{2}\sqrt{3}i\right) + i2K\pi\right)}$$

Como es la principal $\Rightarrow$.

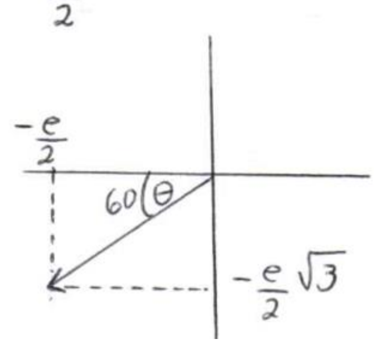
$$e^{3\pi i\left(\log\left(-\frac{e}{2} - \frac{e}{2}\sqrt{3}i\right)\right)}$$

Al ser potencia "z" debe estar en polares

$$z = -\frac{e}{2} - \frac{e}{2}\sqrt{3}i$$

$$\operatorname{Tan}\theta = \frac{\frac{e}{2}\sqrt{3}}{\frac{e}{2}} \Rightarrow \theta = 60$$

$$r = \sqrt{\left(-\frac{e}{2}\right)^2 + \left(-\frac{e}{2}\sqrt{3}\right)^2}$$



$$r = \sqrt{\frac{e^2}{4} + \frac{3e^2}{4}} \Rightarrow r = e$$

Giro positivo

$$180 + 60 = 240 = \frac{4}{3}\pi$$

Giro negativo

$$180 - 60 = 120 = \frac{2}{3}\pi$$

$$e^{3\pi i\left(\log\left(-\frac{e}{2} - \frac{e}{2}\sqrt{3}i\right)\right)}$$

Nota: $\ln z = \ln r + i(\theta + 2K\pi)$

Con $\theta = \frac{4}{3}\pi$

$e^{3\pi i\left(\ln e + i \cdot \frac{4}{3}\pi\right)}$

$e^{3\pi i - 4\pi^2}$

$e^{3\pi i} \cdot e^{-4\pi^2}$

$\boxed{-e^{-4\pi^2}}$

Con $\theta = -\frac{2}{3}\pi$

$e^{3\pi i\left(\ln e + i \cdot -\frac{2}{3}\pi\right)}$

$e^{3\pi i + 2\pi^2}$

$e^{3\pi i} \cdot e^{2\pi^2}$

$\boxed{-e^{2\pi^2}}$

$e^{\pi i} = -1$

Ejemplo 4

Calcule los siguientes valores

a) $(1+i)^i$

b) $\ln(1+i)$

c) $\ln(-3)$

d) $\cosh^{-1}(i)$

e) $Tan^{-1}(2i)$

a) $(1+i)^i$

$z = 1+i$

$r = \sqrt{2}$ $\theta = 45^\circ = \frac{\pi}{4}$

$e^{i\left(\log(1+i)\right)}$

$e^{i\left(\ln\sqrt{2} + i \cdot \pi/4\right)}$

$e^{\frac{i}{2}\ln(2) - \frac{\pi}{4}}$

$e^{\frac{i}{2}\ln(2)} \cdot e^{-\frac{\pi}{4}}$

$$\boxed{(1+i)^i = \left[\cos\left(\tfrac{1}{2}\ln(2)\right) + i\,\mathrm{Sen}\left(\tfrac{1}{2}\ln(2)\right)\right] e^{-\frac{\pi}{4}}}$$

b) $\ln(1+i)$.

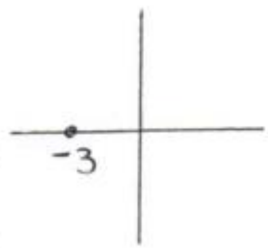
$1+i$

$r\ \sqrt{2} \qquad \Theta = \frac{\pi}{4}$

$$\ln(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4}\right)$$

$$\boxed{\ln(1+i) = \frac{1}{2}\ln(2) + i \cdot \frac{\pi}{4}}$$

c) $\ln(-3)$



$r = 3$

$\Theta = \pi$

$$\boxed{\ln(-3) = \ln(3) + i \cdot \pi}$$

d) $\cosh^{-1}(i)$

$$\cosh^{-1}(i) = \ln\left(i + \sqrt{i^2 - 1}\right)$$

$$\begin{aligned} \cosh^{-1}(i) &= \ln\left(i + \sqrt{-2}\right) \\ &= \ln\left(i + i\sqrt{2}\right) \\ &= \ln(1+\sqrt{2}) + i \cdot \frac{\pi}{2} \end{aligned}$$

$i + i\sqrt{2}$

$(1+\sqrt{2})\,i$

$r = 1+\sqrt{2}$

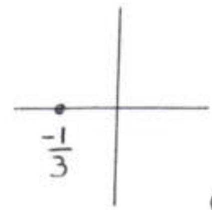
$\Theta = \frac{\pi}{2}$

$$\boxed{\cosh^{-1}(i) = \ln(1+\sqrt{2}) + i \cdot \frac{\pi}{2}}$$

e) $\operatorname{Tan}^{-1}(2i)$

$$\operatorname{Tan}^{-1}(2i) = \frac{1}{2i}\ln\left(\frac{1+i\cdot(2i)}{1-i\cdot(2i)}\right) = \frac{1}{2i}\ln\left(\frac{1-2}{1+2}\right)$$

$$\operatorname{Tan}^{-1}(2i) = \frac{1}{2i}\ln\left(-\frac{1}{3}\right)$$



$r = \frac{1}{3}$

$\theta = \pi$

$$\operatorname{Tan}^{-1}(2i) = \frac{1}{2i}\left[\ln\left(\frac{1}{3}\right) + i\cdot\pi\right]$$

$$\operatorname{Tan}^{-1}(2i) = \frac{1}{2i}\left[\ln(1) - \ln(3) + i\cdot\pi\right]$$

$$\boxed{\operatorname{Tan}^{-1}(2i) = \frac{-1}{2i}\ln(3) + \frac{\pi}{2}}$$ //

Ejemplo 4

Resuelva los siguientes ejercicios

a) Encuentre las raíces de la ecuación $\cosh z = \frac{1}{2}$

b) Resuelva la ecuación $e^{4z} = i$

c) Sea $w = f(z) = z\,(2-z)$. Halle el valor de w para z = - 1 + i

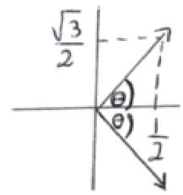
a) $\cosh z = \frac{1}{2}$

$$z = \cosh^{-1}\left(\frac{1}{2}\right)$$

$$z = \ln\left(\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 - 1}\right) \qquad \frac{1}{2} + i\sqrt{\frac{3}{4}}$$

$$z = \ln\left(\frac{1}{2} + \sqrt{-\frac{3}{4}}\right) \qquad r = 1$$

$$\theta = 60 = \frac{\pi}{3}$$



$$z = \ln\left(\frac{1}{2} + i\sqrt{\frac{3}{4}}\right)$$

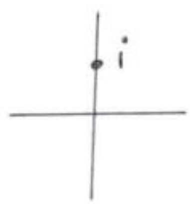
$$z = \ln(1) + i\cdot\frac{\pi}{3} \Rightarrow \boxed{z = \pm i\frac{\pi}{3}}$$

b) $e^{4z} = i$

$$\ln(e)^{4z} = \ln i \qquad i$$

$$4z = \ln i \qquad r = 1$$

$$z = \frac{1}{4}\ln i \qquad \theta = \frac{\pi}{2}$$



$$z = \frac{1}{4}\left(\ln 1 + i\cdot\frac{\pi}{2}\right)$$

$$\boxed{z = \frac{\pi}{8}i}$$

c) $w = z(2-z)$

$w = (-1+i)(2-(-1+i))$
$w = (-1+i)(2+1-i)$
$w = (-1+i)(3-i)$
$w = -3+i+3i+1$

$w = -2+4i$ //

Practica 2

1. Expresar cada función de la forma u(x, y) + i v(x, y)

a) $\frac{1}{1-z}$ $R/\ u = \frac{1-x}{(1-x)^2 + y^2}$ $v = \frac{y}{(1-x)^2 + y^2}$

b) e^{3z} $R/\ u = e^{3x}\,Cos\,3y$ $v = e^{3x}\,Sen3y$

c) $sen(2z)$ $R/\ u = Sen2x\ Cosh2y$ $v = Cos\,2x\ Senh2y$

2. Sea $w = f(z) = \frac{2z+1}{3z-2},\ z \neq \frac{2}{3}$. Halle $f\left(\frac{1}{z}\right)$ $R/\frac{2+z}{3-2z}$

3. Sea $w = f(z) = z(2-z)$ Hallar los valores de w para a) z= 2 – 2i $R/4+4i$

4. Halle los valores principales de:

a) ln (-4) $R/2\ln 2 + \pi\, i$

b) ln (3i) $R/\ \ln 3 + \frac{\pi}{2} i$

c) $\ln\left(\sqrt{3} - i\right)$ $R/\ \ln 2 + \frac{11\pi}{6} i$

d) $4\,senh\left(\frac{\pi}{3} i\right)$ $R/2\sqrt{3}\, i$

5. Calcule el número $z=\frac{2}{i}\ \log\left(\frac{1+i}{1-i}\right)$ $R/\ \pi\ +\ 4k\pi,\ k\in Z$

6. Resuelva las ecuaciones, encontrando la solución principal

a) $e^{z}=-2$ $R/\,z=\ln 2\ +\ \pi\, i$

b) senh z = i $R/\,z=\frac{\pi}{2}i$

c) $\log z=\frac{\pi}{2}i$ $R/\,z=\ i$

7. Pruebe que

a) $\log(-e\,i\,)=1+\frac{3\pi}{2}\,i$

b) $\log(1-i\,)=\frac{1}{2}\ln\ 2\ -\ \frac{\pi}{4}\,i$

8. Hallar el valor principal de

a) $i^{\,i}$ $R/\,e^{-\frac{\pi}{2}}$

b) $(-i\,)^{i}$ $R/\,e^{-\frac{3\pi}{2}}$ $e^{\frac{\pi}{2}}$

c) $(1-i\,)^{\,1\,+\,i}$ $R/\,e^{-\frac{7\pi}{4}}\ \cdot\ e^{\frac{\ln 2}{2}}\cdot Cos\left(\frac{7\pi}{4}+\frac{1}{2}\ln 2\right)$

9. Verifique que $\left(-1+\sqrt{3}\,i\right)^{\frac{3}{2}}=\pm 2\sqrt{2}$