Práctica General

① Desarrolle la $\boxed{\text{serie de Laurent}}$ de la función $f(z) = \dfrac{4z-9}{(z+3)^2(z-1)}$ para que converja $2 < |z+5| < 6$.

Taylor:

e^x

$\cos x$

$\operatorname{Sen} x$

$\dfrac{1}{1+x}$

$\dfrac{1}{1-x}$

$\ln(1+x)$

$\dfrac{1}{(1-x)^2}$

$$f(z) = \frac{4z-9}{(z+3)^2(z-1)}$$

$$\frac{4z-9}{(z+3)^2(z-1)} = \frac{A}{(z+3)} + \frac{B}{(z+3)^2} + \frac{C}{z-1}$$

$$= \frac{A(z+3)(z-1) + B(z-1) + C(z+3)^2}{(z+3)^2(z-1)}$$

$$\frac{4z-9}{\cancel{(z+3)^2}(z-1)} = \frac{A(z+3)(z-1)+B(z-1)+C(z+3)^2}{\cancel{(z+3)^2}(z-1)}$$

(z^2+6z+9)

$$4z-9 = Az^2 - zA + 3zA - 3A + Bz - B + Cz^2 + 6zC + 9C$$

$$\begin{cases} A + C = 0 \\ 2A + B + 6C = 4 \\ -3A - B + 9C = -9 \end{cases}$$

$A = \frac{5}{16}$

$B = 21/4$

$C = -5/16$

x	y	z	d
1	0	1	0
2	1	6	4
-3	-1	9	-9

$$f(z) = \frac{4z - 9}{(z+3)^2 (z-1)}$$

$$f(z) = \frac{A}{(z+3)} + \frac{B}{(z+3)^2} + \frac{C}{(z-1)}$$

$$f(z) = \frac{5}{16} \cdot \frac{1}{z+3} + \frac{21}{4} \cdot \frac{1}{(z+3)^2} - \frac{5}{16} \cdot \frac{1}{z-1}$$

$$f(z) = \frac{5}{16} \cdot \frac{1}{z+5-5+3} + \frac{21}{4} \cdot \frac{1}{(z+5-5+3)^2} - \frac{5}{16} \cdot \frac{1}{(z+5-5-1)}$$

$$f(z) = \frac{5}{16} \cdot \frac{1}{(z+5)-2} + \frac{21}{4} \cdot \frac{1}{[(z+5)-2]^2} - \frac{5}{16} \cdot \frac{1}{(z+5-6)}$$

e^x

$\operatorname{Sen} x$

$\cos x$

$\frac{1}{1-x}$

$\frac{1}{1+x}$

$\frac{1}{(1+x)^2}$

$$F(z) = \frac{5}{16}\cdot\frac{1}{(z+5)-2} + \frac{21}{4}\cdot\frac{1}{[(z+5)-2]^2} - \frac{5}{16}\cdot\frac{1}{(z+5-8)} \qquad 2<|z+5|<6$$

$$\frac{1}{(z+5)-2} = \frac{1}{(z+5)\left[1-\frac{2}{(z+5)}\right]} = \sum_{n=0}^{\infty}\left(\frac{2}{z+5}\right)^n$$

$$\frac{1}{1+x}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$$

$$\frac{1}{[(z+5)-2]^2} = \frac{1}{\left[(z+5)\left(1-\frac{2}{z+5}\right)\right]^2}$$

$$= \frac{1}{(z+5)^2}\cdot\frac{1}{\left(1-\frac{2}{z+5}\right)^2} = \frac{1}{(z+5)^2}\cdot\sum_{n=0}^{\infty}(n+1)\left(\frac{2}{z+5}\right)^n$$

$$\frac{1}{(z+5)-6} = \frac{1}{6\left[\frac{z+5}{6} - 1\right]}$$

$$2 < \underbrace{|z+5|}< 6$$

$$= \frac{1}{-6\left[1 - \frac{z+5}{6}\right]} =$$

$$-\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z+5}{6}\right)^n.$$

$$\frac{1}{1+x}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f(z) = \frac{5}{16} \cdot \frac{1}{z+5} \sum_{n=0}^{\infty} \left(\frac{2}{z+5}\right)^n + \frac{21}{4} \cdot \frac{1}{(z+5)^2} \cdot \sum_{n=0}^{\infty} (n+1)\left(\frac{2}{z+5}\right)^n$$

$$- \frac{5}{16} \cdot \frac{-1}{6} \sum_{n=0}^{\infty} \left(\frac{z+5}{6}\right)^n$$

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2) Encontrar la parte principal y analítica de la serie de Laurent para $\dfrac{1}{z^2+9}$ en $|z-4|<5$

$$f(z) = \frac{1}{z^2+9}$$

z^2+9

$\underbrace{z^2}_{\downarrow\, z} - \underbrace{-9}_{\downarrow\, 3i}$ $(z-3i)(z+3i)$

e^x

$\cos x$

$\operatorname{sen} x$

$\dfrac{1}{1+x}$

$\dfrac{1}{1-x}$

$\dfrac{1}{(1-x)^2}$

$$f(z) = \frac{1}{(z-3i)(z+3i)} = \frac{A}{(z-3i)} + \frac{B}{(z+3i)}$$

$$f(z) = \frac{1}{(z-3i)(z+3i)} = \frac{A}{(z-3i)} + \frac{B}{(z+3i)}$$

$$\frac{1}{\cancel{(z-3i)(z+3i)}} = \frac{A(z+3i) + B(z-3i)}{\cancel{(z-3i)(z+3i)}}$$

$$1 = Az + 3Ai + Bz - 3Bi$$

$$A + B = 0 \Rightarrow A = -B$$

$$3Ai - 3Bi = 1$$

$$-3Bi - 3Bi = 1$$

$$-6Bi = 1$$

$$B = \frac{-1}{6i} \cdot \frac{i}{i} = \frac{+i}{+6}$$

$$B = \frac{i}{6} \Rightarrow A = -\frac{i}{6}$$

$$f(z) = -\frac{i}{6} \cdot \frac{1}{z-3i} + \frac{i}{6} \cdot \frac{1}{z+3i}$$

$$f(z) = -\frac{i}{6} \cdot \frac{1}{z-3i} + \frac{i}{6} \cdot \frac{1}{z+3i}$$

$$|z-4| < 5$$

$$|4-3i| = \sqrt{16+9}$$
$$= \sqrt{25}$$
$$= 5$$

$$f(z) = -\frac{i}{6} \cdot \frac{1}{(z-4+4-3i)} + \frac{i}{6} \cdot \frac{1}{(z-4+4+3i)}$$

$$f(z) = -\frac{i}{6} \cdot \frac{1}{(z-4)+(4-3i)} + \frac{i}{6} \cdot \frac{1}{(z-4)+(4+3i)}$$

$$f(z) = \frac{1}{(z-4)+(4-3i)} = \frac{1}{(4-3i)\left[\frac{z-4}{4-3i}+1\right]}$$

$$\frac{1}{(z-4)+(4-3i)} = \frac{1}{(4-3i)\left[\frac{z-4}{4-3i}+1\right]}$$

$$= \frac{1}{(4-3i)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-4}{4-3i}\right)^n$$

$$\frac{1}{1-x}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{(1-x)^2}$$

$$|z-4| < 5$$

$$4+3i$$

$$\sqrt{16+9}$$

$$5$$

$$\frac{1}{(z-4)+(4+3i)} = \frac{1}{(4+3i)\left[\frac{z-4}{4+3i}+1\right]}$$

$$= \frac{1}{4+3i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-4}{4+3i}\right)^n$$

$$f(z) = -\frac{i}{6} \cdot \frac{1}{(4-3i)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-4}{4-3i}\right)^n + \frac{i}{6} \cdot \frac{1}{4+3i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-4}{4+3i}\right)^n$$

$$f(z) = \frac{-i}{6(4-3i)} \left[1 - \frac{z-4}{4-3i} + \left(\frac{z-4}{4-3i}\right)^2 - \left(\frac{z-4}{4-3i}\right)^3 + \left(\frac{z-4}{4-3i}\right)^4 - \cdots \right]$$

$$+ \frac{i}{6(4+3i)} \left[1 - \frac{z-4}{4+3i} + \left(\frac{z-4}{4+3i}\right)^2 - \left(\frac{z-4}{4+3i}\right)^3 + \cdots \right]$$

$$f(z)=\frac{-i}{6(4-3i)}\left[1-\frac{z-4}{4-3i}+\left(\frac{z-4}{4-3i}\right)^2-\left(\frac{z-4}{4-3i}\right)^3+\left(\frac{z-4}{4-3i}\right)^4-\cdots\right]$$

$$+\frac{i}{6(4+3i)}\left[1-\frac{z-4}{4+3i}+\left(\frac{z-4}{4+3i}\right)^2-\left(\frac{z-4}{4+3i}\right)^3+\cdots\right]$$

$$f(z)=\frac{-i}{6(4-3i)}+\frac{i(z-4)}{6(4-3i)^2}-\frac{i(z-4)^2}{6(4-3i)^3}+\frac{i(z-4)^3}{6(4-3i)^4}-\cdots+$$

Parte analítica

$$\frac{i}{6(4+3i)}-\frac{i(z-4)}{6(4+3i)^2}+\frac{i(z-4)^2}{6(4+3i)^3}-\frac{i(z-4)^3}{6(4+3i)^4}$$

③ Encuentre la singularidad de la siguiente función e indique si es polo, evitable, o esensial. Encuentre el residuo.

$$f(z) = \frac{\operatorname{Sen} 5z}{2z^4}.$$

$$2z^4 = 0$$
$$z^4 = 0 \Rightarrow \boxed{z = 0}$$

$$\operatorname{Sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

e^x
Senx
Cosx

Parta de Taylor y arme la original

$$\operatorname{Sen} 5z = 5z - \frac{(5z)^3}{3!} + \frac{(5z)^5}{5!} - \frac{(5z)^7}{7!} + \ldots$$

$$\frac{\operatorname{Sen} 5z}{2z^4} = \frac{5z}{2z^4} - \frac{125z^3}{3!\,2z^4} + \frac{3125z^5}{5!\,2z^4} - \frac{78125z^7}{7!\,2z^4}$$

$$\frac{\operatorname{Sen} 5z}{2z^4} = \frac{5z}{2z^4} - \frac{125z^3}{3!\,2z^4} + \frac{3125z^5}{5!\,2z^4} - \frac{78125z^7}{7!\,2z^4}$$

$$\frac{\operatorname{Sen} 5z}{2z^4} = \frac{5}{2z^3} - \frac{125}{12z^1} + \frac{625z}{48} - 7{,}75z^3 + \dots$$

En $z=0$ hay un polo de orden 3 y residuo $a_{-1} = -125/12$ //

Laurent

- Evitable ⇒ Solo potencias positivas
- Polo ⇒ Potencias negativas contables. Orden es la mayor potencia (−)
- Esencial: Potencias negativas infinitas

Fracciones Parciales

$$\frac{1}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-b)}$$

$$\frac{1}{(x-a)^2(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$$

$$\frac{1}{(x^2-a)(x-b)^2(x-c)} = \frac{Ax+B}{x^2-a} + \frac{B}{(x-b)} + \frac{C}{(x-b)^2} + \frac{D}{(x-c)}$$

Serie Laurent

1) Me apoyo de TayloR, me ayudo con Fracciones Parciales

2) Todas las "z" deben estar en potencias de "z-a" que es donde converje

3) Agrupo en los "z-a"

4) Trabajo por separado
FC $\rightarrow$ letra $\Rightarrow |z-a|$ mayor a algo
$\searrow$ # $\Rightarrow |z-a|$ menor a algo

Luego me fijo a que serie de Taylor se parece

5) Monto y desarrolla sumatoria si es necesario