Ejemplo 7 Folleto 4 Pág 13

$$f(z) = \frac{2}{(z-1)(z+1)}$$ para q' coverja $1 < |z+2| < 3$

$$f(z) = \frac{2}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

$A = 1$

$B = -1$

e^x

$\operatorname{Sen} x$

$\operatorname{Cos} x$

$\frac{1}{1+x}$

$\frac{1}{1-x}$

$$f(z) = \frac{1}{z-1} - \frac{1}{z+1}$$

$$f(z) = \frac{1}{z-1+2-2} - \frac{1}{z+1+2-2}$$

$$f(z) = \frac{1}{z-1+2-2} - \frac{1}{z+1+2-2}$$

$$1 < |z+2| < 3$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f(z) = \frac{1}{(z+2)-3} - \frac{1}{(z+2)-1}$$

$$f(z) = \frac{1}{-3\left(1 - \frac{(z+2)}{3}\right)} - \frac{1}{(z+2)\left[1 - \frac{1}{z+2}\right]}$$

$$f(z) = -\frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{z+2}{3}\right)^n - \frac{1}{z+2} \sum_{n=0}^{\infty} \left(\frac{1}{z+2}\right)^n$$

Ejemplo 7 $\quad f(z) = \dfrac{e^z - 1}{z^3}$ Parte de Taylor Sen Cos

Taylor $\quad e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$e^z - 1 = \cancel{1} + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \cancel{-1}$$

$$\frac{e^z - 1}{z^3} = \frac{z}{z^3} + \frac{z^2}{z^3 2!} + \frac{z^3}{z^3 3!} + \frac{z^4}{z^3 4!} + \cdots$$

$$\frac{e^z-1}{z^3} = \frac{z}{z^3} + \frac{z^2}{z^3 2!} + \frac{z^3}{z^3 3!} + \frac{z^4}{z^3 4!} + \frac{z^5}{z^3 5!} + \frac{z^6}{z^3 6!} + \cdots$$

Residuo $= \frac{1}{2}$

$$\frac{e^z-1}{z^3} = \underbrace{\frac{1}{z^2} + \frac{1}{2z}}_{\text{Principal}} + \frac{1}{6} + \underbrace{\frac{z}{4!} + \frac{z^2}{5!} + \frac{z^3}{6!} + \cdots}_{\text{Analítica}}$$

$\boxed{z=0}$

Polo

$$\lim_{z\to 0} \frac{e^z-1}{z^3} = \frac{0}{0} \Rightarrow \lim_{z\to 0} \frac{e^z}{3z^2} = \frac{1}{0} = \infty$$

Polo

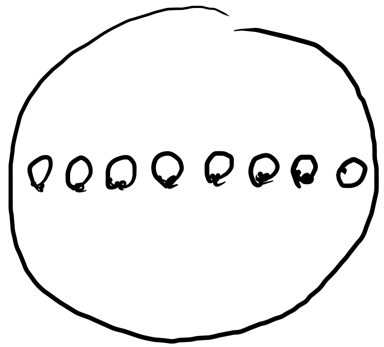
Hay un polo en $z=0$

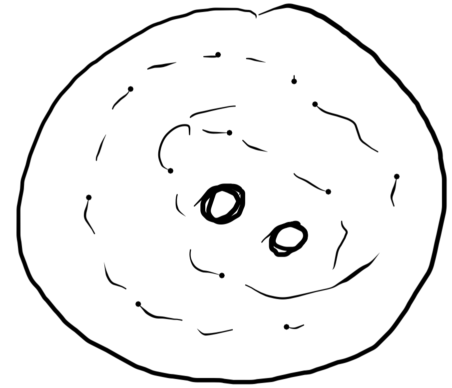
Folleto 5

Singularidades → aisladas *

Singularidades → no aisladas

\# que indefinen una función.





$$f(z) = \frac{z-2}{z-3}$$

$z-3=0$

$z=3$?

$$\lim_{z\to 3}\left(\frac{z-2}{z-3}\right) = \frac{1}{0} = \infty$$

$\boxed{z=3 \text{ ES un Polo}}$

Singularidad

$$\lim_{z\to \boxed{a}} f(z) = \begin{cases} \text{polo} & \infty \\ \text{evitable} & \# \\ \text{esencial} & \nexists \end{cases}$$

indefine

Cuando tengo la serie de Laurent:

- Evitable: NO HAY POTENCIAS NEGATIVAS (ANALÍTICA)

$$f(z) = a_0 + a_1(z-a)^1 + a_2(z-a)^2 + a_3(z-a)^3 + \dots$$

- Polos: Cuando usted puede contar las potencias negativas

$$f(z) = \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$$

- Esencial: infinitas potencias negativas

Residuos a_{-1}

- **Evitable** $\Rightarrow$ Solo potencias positivas $a_{-1} = 0$
- **Polo** $\Rightarrow$ Puedo contar las potencias negativas

Simple $\Rightarrow \lim\limits_{z \to a} (z-a)\boxed{f(z)} = \#$ Residuo es el #

Doble $\Rightarrow \lim\limits_{z \to a} (z-a)^2 f(z) = \#$

Triple $\Rightarrow \lim\limits_{z \to a} (z-a)^3 f(z) = \#$

$\vdots$

$$a_{-1} = \frac{1}{(m-1)!} \cdot \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$

orden del polo

- **Esencial** $\Rightarrow$ Solo desarrollando la serie Laurent

<u>Ejemplo</u>: Determine el tipo de singularidad y calcule el residuo

1) $f(z) = \frac{\operatorname{Sen} z}{z}$

$z = 0$ → evitable / Polo / Esencial.

$\lim_{z \to a} f(z) = \begin{cases} \exists \text{ Evit} \\ \infty \text{ POLO} \\ \nexists \text{ Esen.} \end{cases}$

$\lim_{z \to 0} \frac{\operatorname{Sen} z}{z} = \frac{0}{0} \Rightarrow \lim_{z \to 0} \frac{\operatorname{Cos} z}{1} = 1 \Rightarrow \exists$ <u>Evitable</u> $a_{-1} = 0$

$z = 0$ Singularidad evitable Residuo $= 0$.

Taylor

$$\operatorname{Sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\operatorname{Sen} z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{x^7}{7!}$$

$$\frac{\operatorname{Sen} z}{z} = \frac{z}{z} - \frac{z^3}{z\,3!} + \frac{z^5}{z\,5!} - \frac{z^7}{z\,7!}$$

$$\frac{\operatorname{Sen} z}{z} = \underbrace{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots}_{\text{Analítica}} \Rightarrow \text{Evitable} \quad \underline{a_{-1} = 0}$$

Ejemplo 2 : $f(z) = \dfrac{1}{z^4 - 16}$

$z^4 - 16 = 0$

$z = \sqrt[4]{16}$

Teorema MoivRE

$z^4 - 16$

$\downarrow \quad \downarrow$

$z^2 \quad 4$

$\Rightarrow (z^2+4)(z^2-4)$

$z^2 - 4$: z, 2 $\Rightarrow (z-2)(z+2)$

$z-2=0$, $z=2$

$z+2=0$, $z=-2$

z^2+4: $(z^2 - -4)$

$\downarrow \quad \downarrow$

$z \quad 2i$

$(z-2i)(z+2i)$

$\sqrt{-4}$ $\quad -1 = i^2$

$\sqrt{4i^2} = 2i$

$\rightarrow$ En $z = 2$

Singularidades $\begin{cases} z=2 \\ z=-2 \\ z=2i \\ z=-2i \end{cases}$

$\lim_{z\to 2} \dfrac{1}{(z+2)(z-2)(z+2i)(z-2i)} = \dfrac{1}{0} = \infty$

En $z=2$ Hay un polo

La Función $f(z) = \dfrac{1}{z^4 - 16}$ $\Rightarrow f(z) = \dfrac{1}{(z-2)(z+2)(z-2i)(z+2i)}$

- En $z=2 \Rightarrow$ Polo

$$\lim_{z \to 2} \cancel{(z-2)} \frac{1}{\cancel{(z-2)}\underbrace{(z+2)}\underbrace{(z-2i)}(z+2i)} = \frac{1}{4(2-2i)(2+2i)} = \frac{1}{4(4+4)} = \frac{1}{32}$$

$z=2$ Hay un polo simple

Residuo $a_{-1} = \dfrac{1}{32}$

- $z = 2i$ Polo. Simple

$$\lim_{z\to 2i} \cancel{(z-2i)} \frac{1}{\underbrace{(z-2)}\,\underbrace{(z+2)}\,\cancel{(z-2i)}\,\underbrace{(z+2i)}} = \frac{1}{\underbrace{(2i-2)(2i+2)}\cdot 4i}$$

$$= \frac{1}{4i(-4-4)} = \frac{1}{-32i} \cdot \frac{i}{i} = \frac{i}{32}$$

En $z=2i$ Polo Simple Residuo $\frac{i}{32}$

Ejemplo 3

$$f(z) = \frac{1}{\underbrace{z(1-e^z)}}$$

$$z(1-e^z) = 0$$

$$\underbrace{z=0}_{z=0} \qquad 1-e^z=0$$

$$z=0.$$

$$\lim_{z\to 0} \frac{1}{z(1-e^z)} = \frac{1}{0} = \infty.$$

En $z=0$ Hay un polo → orden ?

Polo

Evitable ?

Esencial

$$\lim_{z\to 0} (z-0)\frac{1}{z(1-e^z)}$$

$$\lim_{z\to 0} \cancel{z}\cdot\frac{1}{\cancel{z}(1-e^z)} = \frac{1}{0} = \infty$$

$$\lim_{z\to 0} (z-0)^2 \frac{1}{z(1-e^z)} \Rightarrow \lim_{z\to 0} \frac{z^2}{z(1-e^z)} \Rightarrow$$

$$\lim_{z\to 0} \frac{z}{(1-e^z)} = \frac{0}{0} \Rightarrow L'H$$

$$\lim_{z\to 0} \frac{1}{-e^z} = -1 \Rightarrow \text{Polo de Orden } 2.$$

Pág 3 $\quad a_{-1} = \frac{1}{(m-1)!} \cdot \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m \cdot f(z) \right]$

$$a_{-1} = \frac{1}{(m-1)!} \cdot \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m \cdot f(z) \right]$$

$$f(z) = \frac{1}{z(1-e^z)}$$

$z = 0$

Polo orden 2

$$a_{-1} = \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d}{dz} \left[(z-0)^2 \cdot \frac{1}{z(1-e^z)} \right]$$

$$a_{-1} = 1 \lim_{z \to 0} \frac{d}{dz} \left[z^2 \cdot \frac{1}{z(1-e^z)} \right]$$

$$a_{-1} = \frac{1}{1} \lim_{z \to 0} \frac{d}{dz}\left[z^{\cancel{2}} \cdot \frac{1}{\cancel{z}(1-e^z)} \right]$$

$$a_{-1} = \lim_{z \to 0} \frac{d}{dz}\left[\frac{z}{1-e^z} \right]$$

Derivada cocíente

$$a_{-1} = \lim_{z \to 0} \left[\frac{1 \cdot (1-e^z) - z(-e^z)}{(1-e^z)^2} \right]$$

$$a_{-1} = \lim_{z \to 0} \left[\frac{1-e^z + ze^z}{(1-e^z)^2} \right]$$

$$a_{-1} = \lim_{z\to 0} \left[\frac{1 - e^z + ze^z}{(1-e^z)^2} \right] = \frac{0}{0}$$

Producto.

L' Hopital.

$$a_{-1} = \lim_{z\to 0} \left[\frac{\cancel{-e^z} + \cancel{1 \cdot e^z} + z \cdot e^z}{2(1-e^z)' \cdot -e^z} \right] \Rightarrow \lim_{z\to 0} \frac{z \cdot \cancel{e^z}}{-2\cancel{e^z}(1-e^z)} = \frac{0}{0}$$

$$a_{-1} = \lim_{z\to 0} \frac{z}{-2+2e^z} = \frac{1}{2e^z} = \frac{1}{2}$$

$z=0$ Polo de orden 2 y el residuo es $\frac{1}{2}$ //