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FOLLETO 4

SERIES DE LAURENT

Si f(z) no es analítica en z = a, pero f(z) es nanlítica en un anillo $r < |z - a| < R$ entonces para todo z en el anillo $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$

Démosle valores a n:

$$f(z) = \underbrace{\frac{a_{-3}}{(z-a)^3} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)}}_{\textbf{Rama principal}} + \underbrace{a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots}_{\textbf{Parte analítica Serie Taylor}}$$

Como se calculan a_n

$$a_n = \frac{1}{2\pi i}\oint \frac{f(t)}{(t-a)^{n+1}} \qquad n = 0, \pm 1, \pm 2$$

C: cualquier circunferencia dentro del anillo

$$|z-a| = \rho \qquad r < \rho < R$$

$a_{-1} = \dfrac{1}{2\pi i}\oint f(t)\,dt$ recibe el nombre de residuo de f(z) en z = a

Recordemos las series de Taylor de algunas funciones centradas en x = 0:

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} ....$$

$$sen(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} ....$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n$$

$$(1+x)^{\alpha} = \sum_{n=0}^{\alpha} \frac{\alpha!}{n!\,(n-\alpha)!} \cdot x^n \quad \alpha \in N$$

Notas

$|z| < \#$: **Saca a factor común el número para hacer** $1 - \frac{z}{2}$

$|z| > \#$: **Saca a factor común "z"**

Ejemplo 1

Desarrolle la serie de Laurent $f(z) = \dfrac{1}{z-2}$ alrededor de cero

$$f(z) = \frac{1}{z-2} \quad \text{alrededor de cero}$$

$$f(z) = \frac{1}{z\left(1 - \frac{2}{z}\right)} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

Ejemplo 2

Desarrolle la serie de Laurent $f(z) = \dfrac{1}{z-2}$ para que converja en $|z-3| > 1$

$$f(z) = \frac{1}{z-2} \quad \text{alrededor de } 3 \rightarrow |z-3| > 1$$

$$f(z) = \frac{1}{z-3+3-2} = \frac{1}{z-3+1} = \frac{1}{(z-3)\left[1 + \frac{1}{z-3}\right]}$$

$$= \frac{1}{z-3} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{1}{z-3}\right)^n$$

$$= \frac{1}{z-3}\left[1 - \frac{1}{z-3} + \frac{1}{(z-3)^2} - \frac{1}{(z-3)^3} + \frac{1}{(z-4)^3} + \cdots\right]$$

$$= \underbrace{\frac{1}{z-3} - \frac{1}{(z-3)^2} + \frac{1}{(z-3)^3} + \frac{1}{(z-3)^4} + \cdots}_{\text{Principal}}$$

Ejemplo 3

Determine la expansión de la serie de Laurent de $f(z) = \dfrac{1}{(z+1)(z+3)}$ en cada caso:

a) $1 < |z| < 3$

b) $|z| > 3$

c) $0 < |z+1| < 2$

Respuestas

a) $1 < |z| < 3$

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} = \frac{A(z+3)+B(z+1)}{(z+1)(z+3)}$$

$$= \frac{Az + 3A + Bz + B}{(z+1)(z+3)}$$

$A + B = 0$ $\qquad$ $A = 1/2$

$3A + B = 1$ $\qquad$ $B = -1/2$

$$f(z) = \frac{1}{2} \cdot \frac{1}{z+1} - \frac{1}{2} \cdot \frac{1}{z+3}$$

$$\frac{1}{z+1} = \frac{1}{z\left(1 + \frac{1}{z}\right)} = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z}\right)^n$$

$$\frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} + \dots\right)$$

$$\frac{1}{z+3} = \frac{1}{3\left(1+\frac{z}{3}\right)} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$= \frac{1}{3}\left(1 - \frac{1}{3}z + \frac{1}{9}z^2 - \frac{1}{27}z^3 + \dots\right)$$

Entonces:

$$f(z) = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3}$$

$$f(z) = \frac{1}{2}\cdot\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{2}\cdot\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$f(z) = \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$f(z) = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right) - \frac{1}{6}\left(1 - \frac{1}{3}z + \frac{1}{9}z^2 - \frac{1}{27}z^3 + \dots\right)$$

$$f(z) = \underbrace{\frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \dots}_{\text{Parte principal}} \underbrace{- \frac{1}{6} + \frac{1}{18}z - \frac{1}{54}z^2 + \frac{1}{162}z^3 + \dots}_{\text{Parte analítica}}$$

b) $|z| > 3$

$$f(z) = \frac{1}{2} \cdot \frac{1}{z+1} - \frac{1}{2} \cdot \frac{1}{z+3}$$

$$f(z) = \frac{1}{2z} \cdot \frac{1}{\left(1 + \frac{1}{z}\right)} - \frac{1}{2z} \cdot \frac{1}{\left(1 + \frac{3}{z}\right)}$$

$$f(z) = \frac{1}{2z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z}\right)^n - \frac{1}{2z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{z}\right)^n$$

$$f(z) = \frac{1}{2z} \left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots \right) \frac{-1}{2z} \left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \dots \right)$$

$$f(z) = \cancel{\frac{1}{2z}} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \dots \cancel{-\frac{1}{2z}} + \frac{3}{2z^2} - \frac{9}{2z^3} + \frac{27}{2} z^4 + \dots$$

$$f(z) = \underbrace{\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} + \dots}_{\text{Parte principal}}$$

c) $0 < |z+1| < 2$

$$f(z) = \frac{1}{2} \cdot \frac{1}{z+1} - \frac{1}{2} \cdot \frac{1}{z+3}$$

$$f(z) = \frac{1}{2} \cdot \frac{1}{z+1} - \frac{1}{2} \cdot \frac{1}{z+1-1+3}$$

$$f(z) = \frac{1}{2} \cdot \frac{1}{z+1} - \frac{1}{2} \cdot \frac{1}{z+1+2}$$

$$\frac{1}{z+1+2} = \frac{1}{2\left(\frac{z+1}{2} + 1\right)} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z+1}{2}\right)^n$$

$$= \frac{1}{2} \left[1 - \frac{z+1}{2} + \frac{(z+1)^2}{4} - \frac{(z+1)^3}{8} + \frac{(z+1)^4}{16} + \cdots \right]$$

$$= \frac{1}{2} - \frac{z+1}{4} + \frac{(z+1)^2}{8} - \frac{(z+1)^3}{16} + \frac{(z+1)^4}{32} + \cdots$$

Entonces

$$f(z) = \frac{1}{2} \cdot \frac{1}{z+1} - \frac{1}{2} \cdot \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z+1}{2}\right)^n$$

$$f(z) = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\left[\frac{1}{2} - \frac{z+1}{4} + \frac{(z+1)^2}{8} - \frac{(z+1)^3}{16} + \frac{(z+1)^4}{32} + \dots\right]$$

$$f(z) = \underbrace{\frac{1}{2(z+1)}}_{\text{Principal}} \underbrace{- \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \frac{(z+1)^3}{32} + \dots}_{\text{Analítico}} \quad //$$

Ejemplo 4:

Desarrolle la serie de Laurent $f(z) = \dfrac{2z^2+2z+2}{z^3+z}$ para $|z|>1$

$|z-0|>1$

↳ Centrada en cero

$z^3 + z = 0$

$z(z^2+1) = 0 \qquad z=0 \qquad z=i \qquad z=-i$

$$\frac{2z^2+2z+2}{z^3+z} = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}$$

$$= \frac{A(z-i)(z+i) + Bz(z+i) + Cz(z-i)}{z(z-i)(z+i)}$$

$$= A(z^2+1) + Bz^2 + Bzi + Cz^2 - Czi$$

$$2z^2+2z+2 = Az^2 + A + Bz^2 + Bzi + Cz^2 - Czi$$

$$A + B + C = 2$$
$$Bi - Ci = 2$$
$$A = 2$$

$$B + C = 0 \Rightarrow B = -C.$$
$$Bi - Ci = 2 \Rightarrow -2Ci = 2$$

$$C = \frac{2}{-2i}$$
$$C = -\frac{1}{i} \cdot \frac{i}{i}$$
$$C = \frac{-i}{-1} \Rightarrow C = i$$
$$B = -C \Rightarrow B = -i$$

$$f(z) = \frac{2}{z} - \frac{i}{z-i} + \frac{i}{z+i}$$

Cuando $z = 0$

$$f(z) = \frac{2}{z}.$$

$$\frac{-i}{z-i} = \frac{-i}{z\left(1-\frac{i}{z}\right)} = -\frac{i}{z} \sum_{n=0}^{\infty} \left(\frac{i}{z}\right)^2$$
$$= \sum_{n=0}^{\infty} -\frac{i}{z} \cdot \left(\frac{i}{z}\right)^n \Rightarrow \sum_{n=0}^{\infty} -\frac{i^{n+1}}{z^{n+1}}$$
$$\sum_{n=1}^{\infty} \left(-\frac{i}{z}\right)^n$$

$$\frac{i}{z+i} = \frac{i}{z\left(1+\frac{i}{z}\right)} = \frac{i}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{z}\right)^n =$$

$$\sum_{n=0}^{\infty} \frac{i}{z} (-1)^n \left(\frac{i}{z}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{i^{n+1}}{z^{n+1}} =$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{i^n}{z^n}$$

$$f(z) = \frac{2}{z} - \frac{i}{z-i} + \frac{i}{z+i}$$

$$f(z) = \frac{2}{z} - \sum_{n=1}^{\infty} \left(\frac{-i}{z}\right)^n + \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{i^n}{z^n}$$

Ejemplo 5

Calcular la serie de Laurent en los valores donde la función no es analítica

$$f(z) = \frac{z^2+2z+i}{(z-i)^3}$$

$$z-i=0$$

$z=i$ No es analítica en $z=i$

$$\frac{z^2+2z+i}{(z-i)^3} = \frac{(z-i+i)^2+2(z-i+i)+i}{(z-i)^3}$$

$$= \frac{(z-i)^2 + 2i(z-i) - 1 + 2(z-i) + 2i + i}{(z-i)^3}$$

$$\frac{(z-i)^2}{(z-i)^3} + \frac{2i(z-i)}{(z-i)^3} - \frac{1}{(z-i)^3} + \frac{2(z-i)}{(z-i)^3} + \frac{3i}{(z-i)^3}$$

$$\frac{1}{(z-i)} + \frac{2i}{(z-i)^2} - \frac{1}{(z-i)^3} + \frac{2}{(z-i)^2} + \frac{3i}{(z-i)^3}$$

$$\underbrace{\frac{(3i-1)}{(z-i)^3} + \frac{(2i+2)}{(z-i)^2} + \frac{1}{(z-i)}}_{\text{Rama principal}}$$

$$a_{-3} = (3i-1) \qquad a_{-2} = 2i+2 \qquad a_{-1} = 0$$

Residuo es cero en $z=i$

Ejemplo 6

Calcular la serie de Laurent para que la función $f(z)=\dfrac{1}{z(z-1)(z-2)}$ para que converja en $1<|z|<2$

Desarrollo para $1 < |z| < 2$.

$$\frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$$= \frac{A(z-1)(z-2) + Bz(z-2) + Cz(z-1)}{z(z-1)(z-2)}$$

$$= A(z^2-2z-z+2) + Bz^2 - 2Bz + Cz^2 - Cz$$
$$= Az^2 - 3Az + 2A + Bz^2 - 2Bz + Cz^2 - Cz$$

$$A + B + C = 0 \qquad A = \frac{1}{2} \qquad C = \frac{1}{2}$$
$$-3A - 2B - C = 0$$
$$2A = 1 \qquad B = -1$$

$$f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

$$\frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$\frac{1}{z-2} = \frac{1}{2\left(\frac{z}{2}-1\right)} = \frac{-1}{2}\,\frac{1}{\left(1-\frac{z}{2}\right)} = \frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$f(z) = \frac{1}{z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

$$f(z) = \frac{1}{z} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2}\cdot\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$f(z) = \frac{1}{z} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

Ejemplo 7

Calcular la serie de Laurent para que la función $f(z) = \frac{2}{(z-1)(z+2)}$ para que converja en $1 < |z+2| < 3$

$$f(z) = \frac{2}{(z-1)(z+1)} \qquad 1 < |z+2| < 3$$

$\downarrow$

alrededor de -2

$$\frac{2}{(z-1)(z+1)} = \frac{A}{(z-1)} + \frac{B}{(z+1)} = \frac{A(z+1) + B(z-1)}{(z-1)(z+1)}$$

$$2 = Az + A + Bz - B \qquad \begin{matrix} A+B=0 \\ A-B=2 \end{matrix} \qquad \begin{matrix} A=1 \\ B=-1 \end{matrix}$$

$$f(z) = \frac{1}{(z-1)} - \frac{1}{z+1}$$

$$\frac{1}{(z-1+2-2)} = \frac{1}{(z+2-3)} = \frac{1}{3\left[\frac{z+2}{3} - 1\right]} =$$

$$\frac{-1}{3\left[1-\frac{z+2}{3}\right]} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z+2}{3}\right)^n$$

$$\frac{1}{(z+1+2-2)} = \frac{1}{(z+2-1)} = \frac{1}{(z+2)\left[1-\frac{1}{z+2}\right]} =$$

$$\frac{1}{z+2}\sum_{n=0}^{\infty}\left(\frac{1}{z+2}\right)^n$$

Ejemplo 8

Calcule la serie de Laurent de $f(z)=\frac{e^z-1}{z^3}$

$$f(z) = \frac{e^z-1}{z^3}$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$

$$e^z - 1 = \cancel{1} + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \cancel{-1}$$

$$= \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots}{z^3}$$

$$= \underbrace{\frac{1}{z^2} + \frac{1}{z \cdot 2!}}_{\text{rama principal}} + \frac{1}{3!} + \underbrace{\frac{z}{4!} + \frac{z^2}{5!}}_{\text{Rama analítica}} + \ldots$$

Práctica

Desarrolle las siguientes series de Laurent

1. $f(z) = \dfrac{e^z}{z^4}$ en z = 0

2. $f(z) = \dfrac{1 - Cos\, z}{z^4}$ en z = 0

3. $f(z) = \dfrac{1}{(z-1)^2 (z-3)}$ en z = 0

4. $f(z) = z \cos\left(\dfrac{1}{z}\right)$

5. $f(z) = z^2\, sen\left(\dfrac{1}{z}\right)$

Respuestas

1) $f(z) = \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{2!\, z^2} + \frac{1}{3!\, z^3} + \frac{1}{4!} + \frac{1}{5!} z + \ldots$

2) $f(z) = \frac{1}{2!\, z^2} + \frac{1}{4!} - \frac{1}{6!} z^2 + \frac{1}{8!} z^4 + \ldots$

3) $\sum_{n=0}^{\infty} \left(-\frac{5}{9}(n+1) + \frac{1}{3} - \frac{1}{9} \right) z^n$

4) $f(z) = z - \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{24}\left(\frac{1}{z^3}\right) - \frac{1}{720}\left(\frac{1}{z^5}\right) + \cdots$

5) $f(z) = z - \frac{1}{6}\left(\frac{1}{z}\right) + \frac{1}{120}\left(\frac{1}{z^3}\right) - \cdots$