Funciones complejas

$f : \mathbb{C} \longrightarrow \mathbb{C}$

$x+yi \longrightarrow f(x+yi)$

coordenadas curvilíneas

$$f(x+yi) = u(x,y) + vi(x,y)$$

Ejemplo: Expresar $f(x) = x^2$ de la forma $u(x,y)+iv(x,y)$

$$(x+yi) \Rightarrow f(x+yi) = (x+yi)^2$$

$$= x^2 + 2xyi - y^2$$

$$= (x^2-y^2) + 2xyi$$

$u = x^2-y^2$

$v = 2xy$.

Encontrar $f(1,2)$

$$\boxed{f(z) = (x^2 - y^2) + 2xyi}$$

$u = x^2 - y^2 \quad v = 2xy$

$f(1,2) = (1-4) + 2 \cdot 1 \cdot 2i$

$f(1,2) = -3 + 4i$

Ejemplo: Expresar $f(x) = \ln z$ de la forma

$u(x,y) + i\,v(x,y)$

JAMÁS

$\cancel{\ln(x+yi) = \ln x + i \ln y}$

$z = x + yi$

$f(z) = \ln(z)$

polar.

$f(z) = \ln(x+yi)$

Euler

$e^{i\theta} = \cos\theta + i\,\mathrm{Sen}\,\theta$

$f(z) = \ln(r(\cos\theta + i\,\mathrm{Sen}\,\theta))$

$f(z) = \ln(r e^{i\theta})$

$f(z) = \ln r + \ln e^{i\theta}$

$f(z) = \ln r + i\theta \cancel{\ln e}$

$f(z) = \ln r + i\theta$

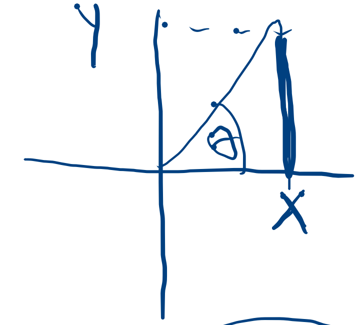
$u = \ln r \qquad v = \theta$

$$f(z) = \ln r + i\theta$$

$$u = \ln r \qquad v = \theta$$

$$z = x + yi$$

$$r = \sqrt{x^2 + y^2}$$



$$u = \ln\sqrt{x^2+y^2} \qquad v = \operatorname{Arctan}\left(\frac{y}{x}\right)$$

//

$$\tan\theta = \frac{y}{x}$$

Repasamos

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

$$x + yi \longrightarrow \boxed{f(x+yi)}\ w$$

$$w = u(x,y) + i\,v(x,y)$$

Funciones polinómica

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_2 x^2 + a_1 x + a_0$$

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots a_1 z + a_0$$

Racional

$$f(x) = \frac{P(x)}{Q(x)} \quad \Big| \quad f(z) = \frac{P(z)}{Q(z)}$$

Funcion exponencial : $f(x) = e^x$

$f(z) = e^z \Rightarrow e^{x+yi}$

w

Euler

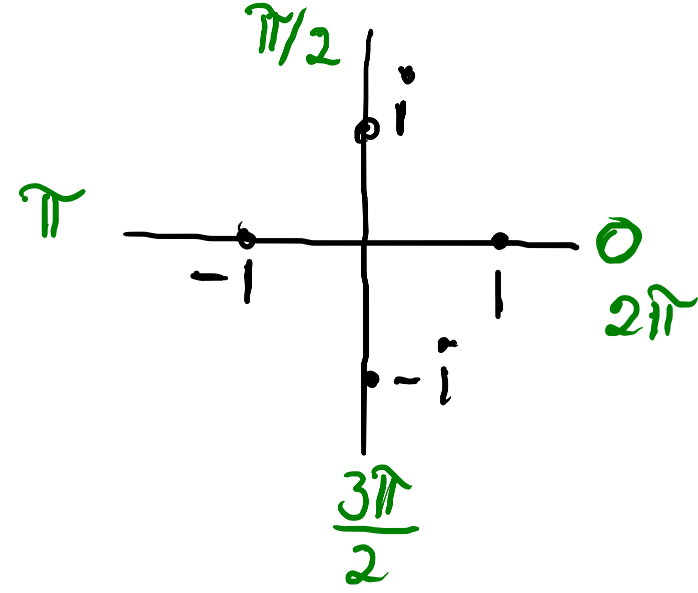
$e^{i\theta} = \cos\theta + i\,\text{Sen}\,\theta$

$w = e^x \cdot e^{yi}$

$w = e^x(\cos\theta + i\,\text{Sen}\,\theta)$

$w = e^x\cos\theta + ie^x\,\text{Sen}\,\theta$

Propiedades



1) $e^0 = 1$

2) $e^{i\pi} = -1$

3) $e^{i\cdot\frac{\pi}{2}} = i$

4) $e^{i\cdot\frac{3\pi}{2}} = -i$

5) $e^z \cdot e^w = e^{z+w}$

Funciones trigonométricas

$$\operatorname{Sen} z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \operatorname{Csc} z = \frac{2i}{e^{iz} - e^{-iz}}$$

$$\operatorname{Cos} z = \frac{e^{iz} + e^{-iz}}{2} \qquad \operatorname{Sec} z = \frac{2}{e^{iz} + e^{-iz}}$$

$$\operatorname{Tan} z = \frac{\operatorname{Sen} z}{\operatorname{Cos} z} \Rightarrow \frac{\dfrac{e^{iz} - e^{-iz}}{2i}}{\dfrac{e^{iz} + e^{-iz}}{2}} \Rightarrow \frac{\cancel{2}\left(e^{iz} - e^{-iz}\right)}{\cancel{2}i\left(e^{iz} + e^{-iz}\right)}$$

$$\tan z = \frac{\cancel{2}\,(e^{iz}-e^{-iz})}{\cancel{2}\,i\,(e^{iz}+e^{-iz})}$$

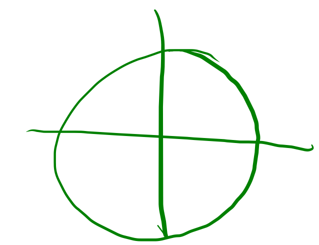
$$\tan z = \frac{e^{iz}-e^{-iz}}{i\,(e^{iz}+e^{-iz})} \Rightarrow \cot z = \frac{i\,(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}$$

6) <u>Función logarítmica</u>

polar

$$w = \ln(z) = \ln r + i\,(\theta \boxed{+2k\pi})$$

$$= \ln r + i\theta$$



OJO

1) $(2+3i)^{5}$ → Potencia Moivre

POLAR

2) $w = e^{(2+3i)}$ → Función exponencial

Potencia complejo

3) $[a]^{b}$ → complejo

a ↓ Complejo

$$[a]^{b} = \left\{ e^{b(\log a + 2k\pi i)} \right\}$$

$$[a]^b = \{ e^{b(\log a + 2k\pi i)} \}$$

$\ln \to \log$ **POLAR**

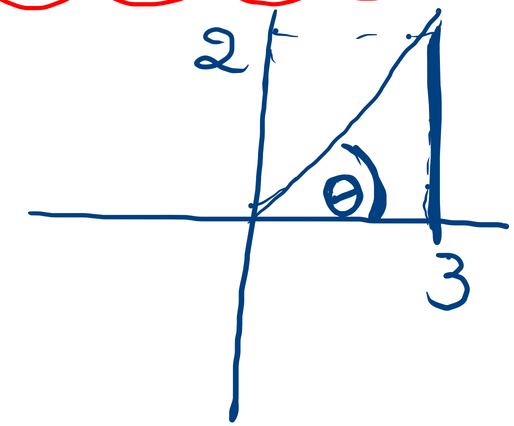
$\log = \ln r + i\theta$

Ejemplo extra

$$[3+2i]^{4i} = e^{4i(\log(3+2i))}$$

$$= e^{4i(\ln\sqrt{13} + i\cdot 34^\circ)}$$

$$= e^{4i\left(\ln\sqrt{13} + i\cdot \frac{17}{90}\pi\right)}$$



$r = \sqrt{(3)^2 + (2)^2}$

$r = \sqrt{13}$

$\tan\theta = \frac{2}{3} \Rightarrow \theta = 34^\circ$

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Ejemplo 4

a) $(1+i)^{i}$

$$[a]^{b} = e^{b(\log a + i\theta)} \Rightarrow \text{Variable complejas}$$

$$\log z = \ln r + i\theta$$

$z = 1+i$

$r = \sqrt{2}$ $\quad \theta = 45°$

$\theta = \dfrac{\pi}{4}$

$$(1+i)^{i} = e^{i(\log(1+i) + i\theta)}$$

$$= e^{i(\ln\sqrt{2} + i \cdot \pi/4)}$$

$$= e^{i\ln\sqrt{2} - \pi/4}$$

$$= e^{i \cdot \ln(2)^{1/2}} \cdot e^{-\pi/4}$$

$$(1+i)^i = e^{i \cdot \ln(2)^{1/2}} \cdot e^{-\pi/4}$$

$$= \underbrace{e^{i \frac{\ln(2)}{2}}} \cdot e^{-\pi/4}$$

$$= \left(\cos\left(\frac{\ln 2}{2}\right) + i \operatorname{Sen}\left(\frac{\ln 2}{2}\right)\right) \cdot e^{-\pi/4}$$

Euler

$$e^{i\theta} = \cos\theta + i \operatorname{Sen}\theta$$

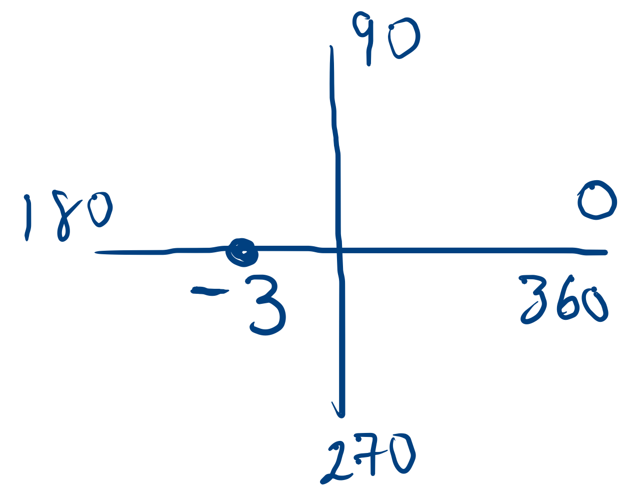
b) $\ln(1+i)$

$$\ln z = \ln r + i\theta$$

$$1+i \Rightarrow r = \sqrt{2} \qquad \theta = \frac{\pi}{4}$$

$$\ln(1+i) = \ln\sqrt{2} + i \cdot \frac{\pi}{4}$$

c) $\ln(-3)$



$$r = 3$$

$$\theta = \pi$$

$$\ln(-3) = \ln(3) + i \cdot \pi$$

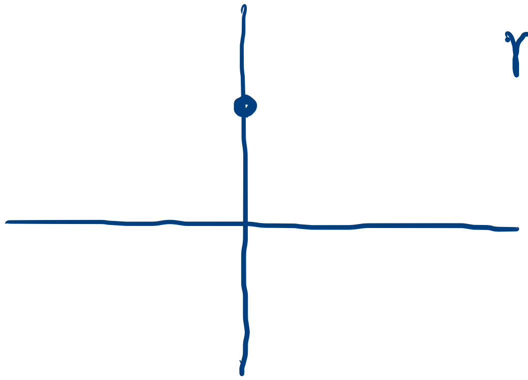
d) $\cosh^{-1}(i)$

$$\cosh^{-1}(z) = \ln\left(z + \sqrt{z^2-1}\right)$$

$$i^2 = -1 \qquad \sqrt{-2} = \sqrt{\cancel{i^2}\cdot 2} = i\cdot\sqrt{2}$$

$$\cosh^{-1}(i) = \ln\left(i + \sqrt{i^2-1}\right)$$
$$= \ln\left(i + \sqrt{-2}\right)$$
$$= \ln\left(\underline{\underline{i + i\sqrt{2}}}\right)$$
$$= \ln(1+\sqrt{2})\,i$$
$$= \ln(1+\sqrt{2}) + i\cdot\frac{\pi}{2}$$

$$\ln z = \ln r + i\cdot\theta$$



$r = 1+\sqrt{2}$

$r = 2,41$

$\theta = \frac{\pi}{2}$

e) $\operatorname{Tan}^{-1}(2i)$

$$\left\{ \operatorname{Tan}^{-1} z = \frac{1}{2i} \ln\left(\frac{1+iz}{1-iz}\right) \right.$$

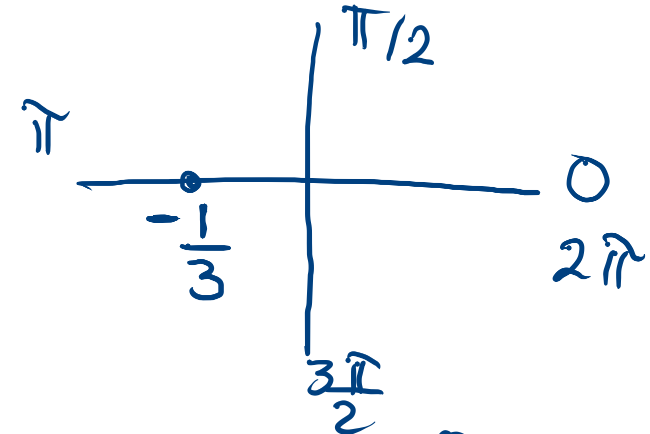
$$\operatorname{Tan}^{-1}(2i) = \frac{1}{2i} \ln\left(\frac{1+i\cdot 2i}{1-i\cdot 2i}\right)$$

$$\operatorname{Tan}^{-1}(2i) = \frac{1}{2i} \ln\left(\frac{-1}{3}\right)$$

$$\operatorname{Tan}^{-1}(2i) = \frac{1}{2i}\left(\ln\left(\frac{1}{3}\right) + i\cdot\pi\right)$$

$$= \frac{1}{2i}\left[\cancel{\ln(i)} - \ln(3)\right] + \frac{\pi}{2}$$

$$\ln z = \ln r + i\cdot\theta$$



$$r = \frac{1}{3} \qquad \theta = \pi$$

$$\tan^{-1}(2i) = \frac{1}{2i}\left[\cancel{\ln(i)} - \ln(3)\right] + \frac{\pi}{2}$$

$$\tan^{-1}(2i) = -\frac{1}{2i}\ln(3) + \frac{\pi}{2}$$

Ejemplo 2 Pág 6 Euler $e^{i\theta} = \cos\theta + i\,\mathrm{Sen}\,\theta$

$$e^{z} = 1+2i$$

$$e^{x+yi} = 1+2i$$

$$e^{x} \cdot e^{yi} = 1+2i$$

$$e^{x}\left(\cos y + i\,\mathrm{Sen}\,y\right) = 1+2i$$

$$e^{x}\cos y + i e^{x}\,\mathrm{Sen}\,y = 1+2i$$

I. $(e^{x}\cos y)^{2} = 1$ II $e^{x}\,\mathrm{Sen}\,y = 2$

$$e^{2x}\cos^{2} y = 1$$

$$e^{2x}\,\mathrm{Sen}^{2} y = 4$$

$$e^{2x}\cos^{2} y + e^{2x}\,\mathrm{Sen}^{2} y = 5$$

$$e^{2x}\left(\underline{\cos^{2} y + \mathrm{Sen}^{2} y}\right) = 5$$

$$e^{2x} = 5$$

$$e^{2x} = 5$$

$$\ln e^{2x} = \ln(5)$$

$$2x \cancel{\ln e} = \ln(5)$$

$$\boxed{x = \frac{\ln(5)}{2}}$$

$$\text{I. } e^x \cos y = 1 \qquad \text{II. } e^x \operatorname{Sen} y = 2$$

$$\frac{e^x \cos y = 1}{e^x \operatorname{Sen} y = 2} \qquad \frac{\cancel{e^x} \operatorname{Sen} y = 2}{\cancel{e^x} \cos y = 1}$$

$$\operatorname{Tan} y = 2$$

$$\boxed{y = \operatorname{Tan}^{-1}(2)}$$

$$z = x + yi$$

$$z = \frac{\ln(5)}{2} + \operatorname{Tan}^{-1}(2) \quad //$$

Ejemplo 2: $e^{z} = 1+2i$

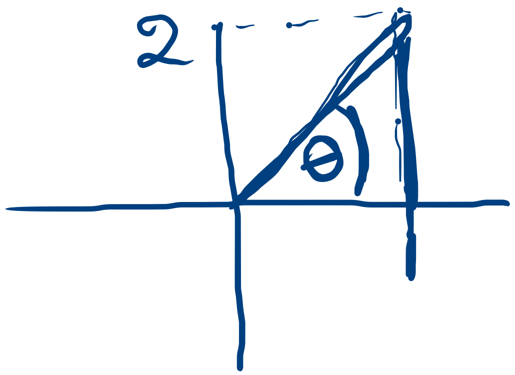
$$\ln e^{z} = \ln(1+2i)$$

$$z = \ln(\underline{1+2i})$$

POLAR

$$\boxed{\ln z = \ln r + i\theta}$$

$$z = \ln r + i\theta$$



$$z = \ln\sqrt{5} + i\cdot\operatorname{Tan}^{-1}(2)$$

$$r = \sqrt{(2)^2+(1)^2}$$

$$r = \sqrt{5}$$

$$z = \ln(5)^{1/2} + i\cdot\operatorname{Tan}^{-1}(2)$$

$$\operatorname{Tan}\theta = \frac{2}{1}$$

$$z = \frac{\ln(5)}{2} + i\cdot\operatorname{Tan}^{-1}(2)$$

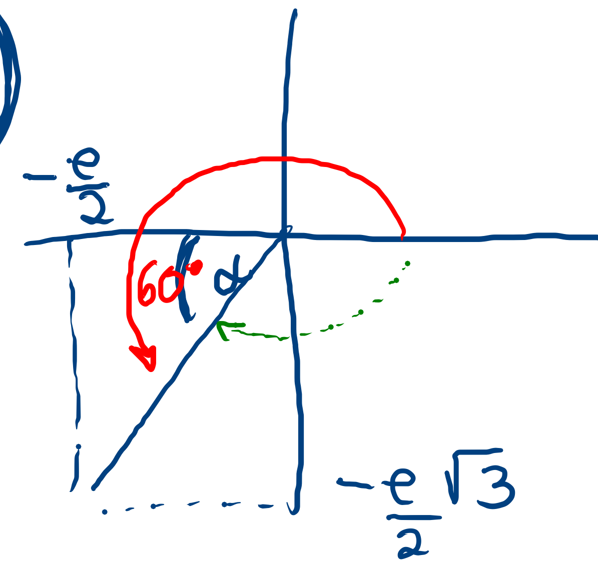
Ejemplo 3

$$\left[\frac{e}{2}\left(-1-\sqrt{3}\,i\right)\right]^{3\pi i}$$

Potencia compleja

$$[a]^{b} = e^{b(\log a)}$$

$$\left[-\frac{e}{2}-\frac{e\sqrt{3}}{2}\,i\right]^{3\pi i} = e^{3\pi i\left(\log\left(-\frac{e}{2}-\frac{e}{2}\sqrt{3}\,i\right)\right)}$$

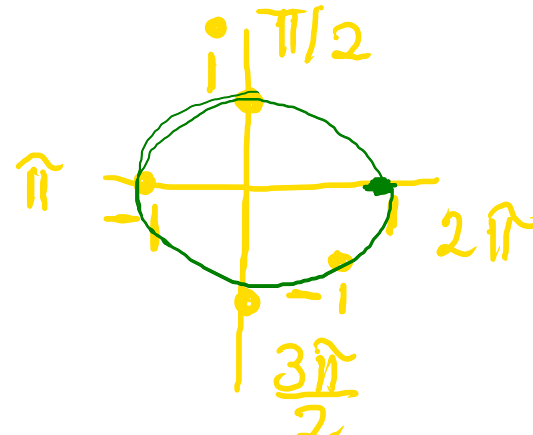


$$240^\circ = \frac{4}{3}\pi$$

$$= e^{3\pi i\left(\ln e + i\cdot\frac{4}{3}\pi\right)}$$

$$= e^{3\pi i - 4\pi^2} = e^{3\pi i}\cdot e^{-4\pi^2}$$

$$= -e^{-4\pi^2}$$



$r = e$

$r = 2{,}71\ldots$

Pág 11 Ejemplo 4

$\sqrt{\frac{-3}{4}}$ $\quad i^2 = -1$

a) Encuentre las raíces de $\cosh z = \frac{1}{2}$

$\cosh^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right)$

$\cosh z = \frac{1}{2}$

$z = \cosh^{-1}\left(\frac{1}{2}\right)$

$z = \ln\left(\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 - 1}\right)$

$z = \ln\left(\frac{1}{2} + \sqrt{\frac{-3}{4}}\right)$

$z = \ln\left(\underbrace{\frac{1}{2} + i\sqrt{\frac{3}{4}}}_{\text{POLAR}}\right)$

$r = 1$

$\theta = 60 \quad \frac{\pi}{3}$

$z = \cancel{\ln(1)} + i \cdot \frac{\pi}{3}$

$\boxed{z = i \cdot \frac{\pi}{3}}$

b) $e^{4z} = i$

$\ln e^{4z} = \ln i$

$4z \cancel{\ln e} = \ln i$

$4z = \ln i$

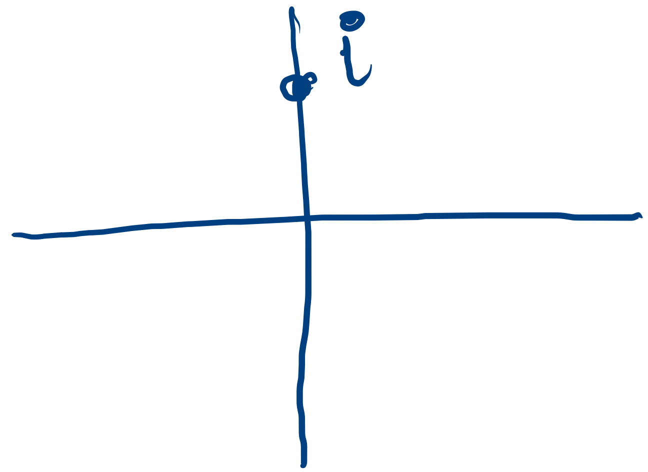
$z = \frac{1}{4} \ln(i)$

$z = \frac{1}{4}\left(\cancel{\ln 1} + i \cdot \frac{\pi}{2}\right) \Rightarrow \boxed{z = \frac{i \cdot \pi}{8}}$

$4z = \ln i$

$z = \frac{\ln i}{4} \Rightarrow \frac{1}{4} \ln i$

$\cancel{z = \ln\left(\frac{i}{4}\right)}$ JAMÁS



$r = 1 \qquad \theta = \frac{\pi}{2}$

Práctica 2

$u(x,y) + iv(x,y)$

$z = x + yi$

$\frac{x+3}{5} \quad \frac{x}{5} + \frac{3}{5}$

1) a) $\dfrac{1}{1-z}$

$$\frac{1}{1-(x+yi)} = \frac{1}{1-x-yi} = \frac{1}{(1-x)-yi} \cdot \frac{(1-x)+yi}{(1-x)+yi}$$

$$\frac{(1-x)+yi}{(1-x)^2+y^2} = \underbrace{\frac{1-x}{(1-x)^2+y^2}}_{u} + \underbrace{\frac{yi}{(1-x)^2+y^2}}_{v}$$

b) e^{3z}

$u(x,y) + i\,v(x,y)$

Euler

$e^{i\theta} = \cos\theta \cdot i \, \text{Sen}\,\theta$

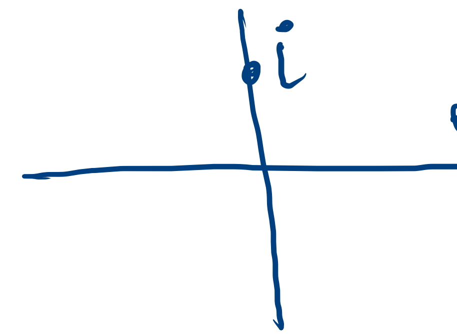
$e^{3(x+yi)} = e^{3x+3yi} = e^{3x} \cdot e^{3yi}$

$= e^{3x}(\cos 3y + i\,\text{Sen}\,3y)$

$= \underbrace{e^{3x}\cos 3y}_{u} + i\,\underbrace{e^{3x}\,\text{Sen}\,3y}_{v}$

⑤ Calcule $z = \frac{2}{i}\log\left(\frac{1+i}{1-i}\right)$

$$\frac{1+i}{1-i}\cdot\frac{(1+i)}{(1+i)} = \frac{(1+i)^2}{2} = \frac{1+2i-\cancel{1}}{2} = \frac{\cancel{2}i}{\cancel{2}} = i$$



$r = 1$

$\theta = \frac{\pi}{2}$

$$z = \frac{2}{i}\log(i)$$

$$z = \frac{2}{i}\left[\cancel{\ln(1)} + i\cdot\frac{\pi}{2}\right]$$

$$\boxed{z = \pi}$$

Folleto 3

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \Rightarrow \text{Reales}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta x} \Rightarrow \text{Complejos}$$

Funciones armónicas

NO VERLO

$$f(z) = u(x,y) + iv(x,y)$$

$$\underbrace{\text{I. } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{II. } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}_{\text{Analítica}}$$

Ejemplo 1

$f(z) = \bar{z}$ Halle $f'(z)$

$z = x + yi$

$\bar{z} = x - yi$

$\underline{u = x}$ $v = -y$

$\text{I. } \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ La función no es derivable

$1 = -1$ ✗

Ejemplo 2

$f(z) = |z|^2$ ¿Podrá derivarse?

$z = x + yi$

$|z| = \sqrt{x^2 + y^2}$

$u = \underline{x^2 + y^2}$ $v = 0$

$f(z) = x^2 + y^2$

Solo es derivable en $(0,0)$

I. $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$* \; 2x = 0$

$\boxed{x = 0}$

II. $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$2y = 0$

$\boxed{y = 0}$