Folleto 4 Serie Laurent

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$f(z) = \ldots a_{-3}(z-a)^{-3} + a_{-2}(z-a)^{-2} + a_{-1}(z-a)^{-1} + a_0(z-a)^0 + a_1(z-a) + a_2(z-a)^2 + \ldots$$

$$f(z) = \underbrace{\ldots \frac{a_{-3}}{(z-a)^3} + \frac{a_{-2}}{(z-a)^2} + \frac{\boxed{a_{-1}}}{(z-a)^1}}_{\text{Rama principal}} + a_0 + \underbrace{a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3}_{\text{Rama analítica}} + \ldots$$

a_{-1} = Residuo

Series de Taylor

$0! = 1$

1) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$3! = 1 \cdot 2 \cdot 3$

$4! = 1 \cdot 2 \cdot 3 \cdot 4$

2) $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^4}{8!} - \cdots$

3) $\operatorname{Sen} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

4) $$\frac{1}{1+\boxed{x}} = \sum_{n=0}^{\infty} (-1)^n \boxed{x}^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

5) $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

6) $$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

7) $$\frac{1}{(1-x)^2}$$

8) $$(1+x)^{\alpha}$$

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NOTAS

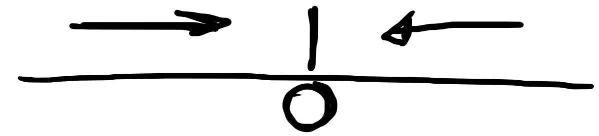
$|z| <$ # : Saca a FC el #

$|z| >$ # : Saca a FC la "z"

Cálculo II

Ejemplo 1

$$f(z) = \frac{1}{z-2}$$ alrededor de cero



1) e^{x}

2) $\cos x$

3) $\operatorname{Sen} x$

4) $\dfrac{1}{1+x}$

5) $\dfrac{1}{1-x}$

Taylor

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^{n}$$

a) $|z| < 0$

$$f(z) = \frac{1}{z-2} \Rightarrow f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} \Rightarrow$$

$$f(z) = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}}$$

$$f(z) = -\frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{n}$$

$$f(z) = -\frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

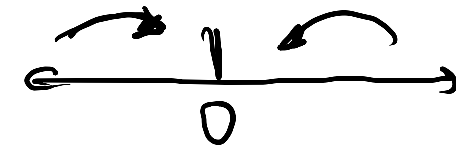
$$f(z) = -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \frac{z^4}{16} + \cdots\right]$$

$$f(z) = -\frac{1}{2} \underbrace{- \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \frac{z^4}{32} + \cdots}_{\text{Analítico}}$$

$a_{-1} = 0$

Residuo $= 0$

Desarrolle la serie de Laurent $f(z) = \frac{1}{z-2}$ alrededor 0



b)

$|z| > 0$

$$f(z) = \frac{1}{z-2} \Rightarrow f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)}$$

e^x

$\cos x$

$\operatorname{Sen} x$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x}$$

$$\Rightarrow f(z) = \frac{1}{z} \cdot \frac{1}{1-\frac{2}{z}}$$

$$\Rightarrow f(z) = \frac{1}{z} \cdot \sum_{n=\infty}^{\infty} \left(\frac{2}{z}\right)^n$$

$$\Rightarrow f(z) = \frac{1}{z} \cdot \sum_{n=\infty}^{\infty} \left(\frac{2}{z}\right)^n$$

$$\Rightarrow f(z) = \frac{1}{z}\left[1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \frac{16}{z^4} + \cdots\right]$$

$$\Rightarrow f(z) = \underbrace{\frac{1}{z^1} + \frac{2}{z^2} + \frac{4}{z^3} + \frac{8}{z^4} + \frac{16}{z^5} + \cdots}_{\text{Parte principal.}}$$

Residuo

$a_{-1} = 1$

Ejemplo 2 $\quad f(z) = \frac{1}{z-2}$ para que converja en $|z-3| > 1$

$$f(z-3) = \frac{1}{z-3+3-2} = \frac{1}{(z-3)+1}$$

$$\frac{1}{1-x}$$

$$f(z) = \frac{1}{(z-3)+1} = \frac{1}{(z-3)\left[1 + \frac{1}{z-3}\right]}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$f(z) = \frac{1}{(z-3)} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z-3}\right)^n$$

$$f(z)=\frac{1}{(z-3)}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z-3}\right)^n$$

$$f(z)=\frac{1}{z-3}\left[1-\frac{1}{z-3}+\frac{1}{(z-3)^2}-\frac{1}{(z-3)^3}+\frac{1}{(z-3)^4}-\cdots\right]$$

$$f(z)=\underbrace{\frac{1}{(z-3)^1}-\frac{1}{(z-3)^2}+\frac{1}{(z-3)^3}-\frac{1}{(z-3)^4}+\frac{1}{(z-3)^5}-\cdots}_{\text{Principal}}$$

Residuo = 1

Ejemplo 3

$$f(z) = \frac{1}{(z+1)(z+3)}$$

e^x

$\operatorname{Sen} x$

$\operatorname{Cos} x$

$$\frac{1}{1+x}$$

a) $1 < |z| < 3$

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{A}{(z+1)} + \frac{B}{(z+3)}$$

Fracciones Parciales

$$\frac{1}{1-x}$$

$A = \frac{1}{2}$ $B = -\frac{1}{2}$

$$f(z) = \frac{1}{2} \cdot \frac{1}{z+1} - \frac{1}{2} \cdot \frac{1}{z+3}$$

$$f(z) = \frac{1}{2}\cdot\underbrace{\frac{1}{z+1}}_{|z|>1} - \frac{1}{2}\cdot\underbrace{\frac{1}{z+3}}_{|z|<3}$$

a) $1 < |z| < 3$

$|z| > 1 \Rightarrow$ FC "z"

$|z| < 1 \Rightarrow$ FC #

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$f(z) = \frac{1}{2}\cdot\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\cdot\frac{1}{3\left(1+\frac{z}{3}\right)}$$

$$f(z) = \frac{1}{2}\cdot\frac{1}{z}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{6}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\left(\frac{z}{3}\right)^n$$

$$f(z) = \frac{1}{2}\cdot\frac{1}{z}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{6}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\left(\frac{z}{3}\right)^n$$

$$f(z) = \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} + \cdots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \frac{z^4}{81} + \cdots\right]$$

$$f(z) = \underbrace{\frac{1}{2z^{1}} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \frac{1}{2z^5}}_{\text{Principal}} - \frac{1}{6} + \underbrace{\frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots}_{\text{Analítico}}$$

Residuo $= \frac{1}{2}$

b) $f(z) = \frac{1}{2} \cdot \frac{1}{z+1} - \frac{1}{2} \cdot \frac{1}{z+3}$ $\qquad |z| > 3$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n$$

$$f(z) = \frac{1}{2} \cdot \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2} \cdot \frac{1}{z\left(1+\frac{3}{z}\right)}$$

$$f(z) = \frac{1}{2z} \cdot \underbrace{\sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{1}{z}\right)^n} - \frac{1}{2z} \underbrace{\sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{3}{z}\right)^n}$$

$$f(z) = \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} + \dots\right] - \frac{1}{2z}\left[1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \dots\right]$$

$$f(z)=\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}+\cdots\right]-\frac{1}{2z}\left[1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+\cdots\right]$$

$$f(z)=\cancel{\frac{1}{2z}}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots+\cancel{-\frac{1}{2z}}+\frac{3}{2z^2}-\frac{9}{2z^3}+\frac{27}{2z^4}-\cdots$$

$$f(z)=\underbrace{\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\cdots}_{\text{Principal}}$$

Residuo = 0.

c) $0 < |z+1| < 2$

$$f(z) = \frac{1}{2} \cdot \frac{1}{z+1} - \frac{1}{2} \cdot \frac{1}{z+3}$$

$$f(z) = \frac{1}{2} \cdot \frac{1}{z+1} - \frac{1}{2} \cdot \frac{1}{z+1-1+3}$$

$$f(z) = \frac{1}{2} \cdot \frac{1}{(z+1)} - \frac{1}{2} \cdot \frac{1}{(z+1)+2}$$

$|z+1| < 2$

$$f(z) = \left\{ \frac{1}{2} \cdot \frac{1}{z+1} - \frac{1}{2} \cdot \frac{1}{2} \left[\frac{1}{1 + \frac{z+1}{2}} \right] \right\}$$

$$f(z) = \left\{ \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{2}\left[\frac{1}{1+\frac{z+1}{2}}\right]\right\} \qquad \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1) x^n$$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{4}\left[1 - \frac{z+1}{2} + \frac{(z+1)^2}{4} - \frac{(z+1)^3}{8} + \cdots\right]$$

$$f(z) = \underbrace{\frac{1}{2(z+1)}}_{\text{Principal}} - \frac{1}{4} + \underbrace{\frac{z+1}{8} - \frac{(z+1)^2}{16} + \frac{(z+1)^3}{32} - \cdots}_{\text{Analítica}}$$

$$\boxed{\text{Residuo} = \frac{1}{2}}$$

Ejemplo 4 $\quad f(z) = \dfrac{2z^2+2z+2}{z^3+z}$ para $|z|>1$.

$i^2=-1$

$$f(z) = \frac{2z^2+2z+2}{z^3+z}$$

$$z^3+z$$

$$z(z^2+1)$$

$$z(z^2-(-1))$$

$$z(z^2-i^2)$$

$$\downarrow \qquad \downarrow$$

$$z \qquad i$$

$$z(z-i)(z+i)$$

e^x

$\cos x$

$\operatorname{sen} x$

$\dfrac{1}{1+x}$

$\dfrac{1}{1-x}$

$\ln(1+x)$

$\dfrac{1}{(1-x)^2}$

$(1+x)^\alpha$

$$f(z) = \frac{2z^2+2z+2}{z^3+z} \Rightarrow \frac{2z^2+2z+2}{z(z-i)(z+i)}$$

$$\frac{2z^2+2z+2}{\cancel{z(z-i)(z+i)}} = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}$$

$$= \frac{A(z-i)(z+i) + Bz(z+i) + Cz(z-i)}{\cancel{z(z-i)(z+i)}}$$

$$2z^2+2z+2 = Az^2 + \cancel{Azi} - \cancel{izA} + A + Bz^2 + Bzi + Cz^2 - Czi$$

$$2z^2+2z+2=Az^2+\cancel{Az\bar{i}}-\cancel{izA}+A+Bz^2+Bzi+Cz^2-Czi$$

$$2z^2=Az^2+Bz^2+Cz^2 \Rightarrow 2=A+B+C \Rightarrow \cancel{2}=\cancel{2}+B+C$$

$$0=B+C$$

$$\boxed{-C=B}$$

$$2z=Bzi-Czi \Rightarrow 2=Bi-Ci$$

$$2=A \qquad \Rightarrow A=2$$

Entonces $B=-C$

$$\boxed{B=-i}$$

$$2=-Ci-Ci$$

$$2=-2Ci$$

$$\frac{2}{-2i}=C$$

$$\frac{-1\cdot i}{i\cdot i}=C$$

$$\frac{-i}{-1}=C$$

$$\boxed{i=C}$$

$A = 2$

$B = -i$

$C = i$

$$f(z) = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f(z) = \frac{2}{z} - \frac{i}{z-i} + \frac{i}{z+i}$$

$|z| > 1$

$$\frac{i}{z-i} = \frac{i}{|z|\left(1-\frac{i}{z}\right)} = \frac{i}{|z|} \sum_{n=0}^{\infty} \left(\frac{i}{z}\right)^n = \sum_{n=0}^{\infty} \frac{i}{z}\left(\frac{i}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{i}{z}\right)^{n+1}$$

$$= \sum_{n=1}^{\infty} \left(\frac{i}{z}\right)^n.$$

$$\frac{i}{z+i} = \frac{i}{|z|\left(1+\frac{i}{z}\right)} = \frac{i}{|z|}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{i}{z}\right)^n .$$

$$\boxed{\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{i}{z}\left(\frac{i}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{i}{z}\right)^{n+1}$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{i}{z}\right)^n .$$

$$f(z) = \frac{2}{z} - \frac{i}{z-i} + \frac{i}{z+i}$$

$$f(z) = \frac{2}{z'} - \sum_{n=1}^{\infty} \left(\frac{i}{z}\right)^n + \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{i}{z}\right)^n$$

Ejemplo 5

$$f(z) = \frac{z^2+2z+i}{\underbrace{(z-i)}^3}$$

$z-i=0$

$\boxed{z=i}$

$\underbrace{z-i}=0$

$$f(z) = \frac{z^2+2z+i}{(z-i)^3} = \frac{(z+i-i)^2+2(z-i+i)+i}{(z-i)^3} \Rightarrow$$

$$f(z) = \frac{((z-i)+i)^2+2(z-i)+2i+i}{(z-i)^3}$$

$$f(z) = \frac{(z-i)^2+2i(z-i)-1+2(z-i)+3i}{(z-i)^3}$$

$$f(z)=\frac{(z-i)^2+2i(z-i)-1+2(z-i)+3i}{(z-i)^3}$$

$$f(z)=\frac{(z-i)^2}{(z-i)^3}+\frac{2i(z-i)}{(z-i)^3}-\frac{1}{(z-i)^3}+\frac{2(z-i)}{(z-i)^3}+\frac{3i}{(z-i)^3}$$

$$f(z)=\frac{1}{z-i}+\frac{2i}{(z-i)^2}-\frac{1}{(z-i)^3}+\frac{2}{(z-i)^2}+\frac{3i}{(z-i)^3}$$

$$f(z)=\underbrace{\frac{1}{(z-i)^1}+\frac{(2+2i)}{(z-i)^2}+\frac{(3i-1)}{(z-i)^3}}_{\text{Principal}}$$

Residuo

$a_{-1}=1$

Ejemplo 6

$$f(z) = \frac{1}{z(z-1)(z-2)} \qquad 1 < |z| < 2$$

$$\frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$A = \frac{1}{2}$ $\quad B = -1$

$C = \frac{1}{2}$

$$f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2} \cdot \frac{1}{z-2}$$

$$\frac{1}{1-x}$$

$$\frac{1}{1+x}$$

$$f(z) = \frac{1}{2z} - \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2} \cdot \frac{1}{-2\left[1-\frac{z}{2}\right]}$$

$$f(z) = \frac{1}{2z} - \frac{1}{\underbrace{z\left(1-\frac{1}{z}\right)}} + \frac{1}{2} \cdot \frac{1}{-2\left[1-\frac{z}{2}\right]}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)\, x^n \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f(z) = \frac{1}{2z} - \underbrace{\frac{1}{z} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n} - \frac{1}{4} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n .$$

$$f(z) = \frac{1}{2z} - \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^{n+1} - \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$f(z) = \frac{1}{2z} - \sum_{n=1}^{\infty} \left(\frac{1}{z}\right)^n - \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

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