Interval Estimation and Hypothesis Testing

Presentation 4

Interval Estimators

- The Confidence Intervals give probabilistic guarantee for the obtained results
- Every time the actual value belongs or not to the c.i., but if we construct multiple c.i. they guarantee the inclusion of the actual parameter value with some probability

Interval Estimators

- Measure for the accuracy of the calculated estimator is the so called random interval
- We construct this interval knowing the distribution of the estimator

$$(\hat{\beta}_2 - \delta, \hat{\beta}_2 + \delta)$$

$$\Pr(\hat{\beta}_2 - \delta \le \beta_2 \le \hat{\beta}_2 + \delta) = 1 - \alpha,$$

$$\alpha, \delta > 0; \quad \alpha \in (0,1)$$

Interval Estimators

- If we assume Normal distribution for the residuals it can be shown that the estimators are also with Normal distribution:

$$Z = \frac{\hat{\beta}_2 - \beta_2}{\mathrm{se}\,(\hat{\beta}_2)}$$

$$= \frac{(\hat{\beta}_2 - \beta_2)\sqrt{\sum x_i^2}}{\sigma}$$

Interval Estimators

- Usually the actual value of σ^2 is unknown. If we use the LS Estimator instead, the distribution of the estimator for β is Student's t with n-2 degrees of freedom:

$$t = \frac{\hat{\beta}_2 - \beta_2}{\operatorname{se}(\hat{\beta}_2)} = \frac{(\hat{\beta}_2 - \beta_2)\sqrt{\sum x_i^2}}{\hat{\sigma}}$$

Interval Estimators

- Applying the c.i. definition we get the following expression:

$$\Pr\left(-t_{\alpha/2} \leq t \leq t_{\alpha/2}\right) = 1 - \alpha$$

$$\Pr\left[-t_{\alpha/2} \leq \frac{\hat{\beta}_2 - \beta_2}{\operatorname{se}(\hat{\beta}_2)} \leq t_{\alpha/2}\right] = 1 - \alpha$$

$$\Pr\left[\hat{\beta}_2 - t_{\alpha/2}\operatorname{se}(\hat{\beta}_2) \leq \beta_2 \leq \hat{\beta}_2 + t_{\alpha/2}\operatorname{se}(\hat{\beta}_2)\right] = 1 - \alpha$$

Interval Estimators

- Similarly:

$$\Pr[\hat{\beta}_1 - t_{\alpha/2}\,\mathrm{se}\,(\hat{\beta}_1) \le \beta_1 \le \hat{\beta}_1 + t_{\alpha/2}\,\mathrm{se}\,(\hat{\beta}_1)] = 1 - \alpha$$

Example 1: Construct 95% confidence interval for β_2 .

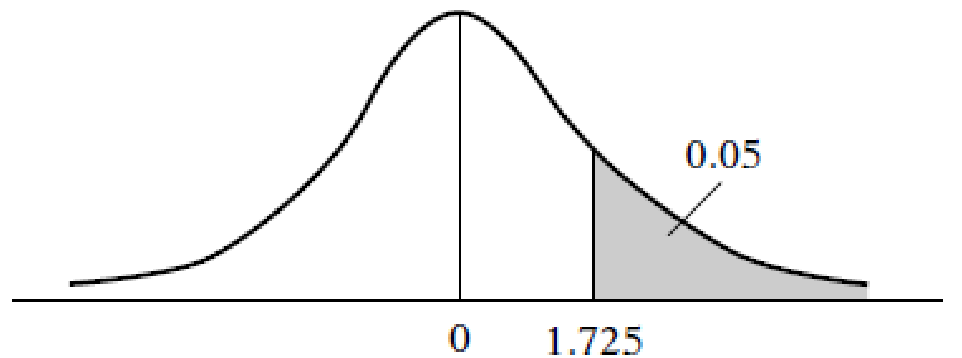
PERCENTAGE POINTS OF THE t DISTRIBUTION

Example

$\Pr(t > 2.086) = 0.025$

$\Pr(t > 1.725) = 0.05$ for df = 20

$\Pr(|t| > 1.725) = 0.10$



Pr / df	0.25 0.50	0.10 0.20	0.05 0.10	0.025 0.05	0.01 0.02	0.005 0.010	0.001 0.002
1	1.000	3.078	6.314	12.706	31.821	63.657	318.31
2	0.816	1.886	2.920	4.303	6.965	9.925	22.327
3	0.765	1.638	2.353	3.182	4.541	5.841	10.214
4	0.741	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	1.476	2.015	2.571	3.365	4.032	5.893
6	0.718	1.440	1.943	2.447	3.143	3.707	5.208
7	0.711	1.415	1.895	2.365	2.998	3.499	4.785
8	0.706	1.397	1.860	2.306	2.896	3.355	4.501
9	0.703	1.383	1.833	2.262	2.821	3.250	4.297
10	0.700	1.372	1.812	2.228	2.764	3.169	4.144
11	0.697	1.363	1.796	2.201	2.718	3.106	4.025
12	0.695	1.356	1.782	2.179	2.681	3.055	3.930
13	0.694	1.350	1.771	2.160	2.650	3.012	3.852
14	0.692	1.345	1.761	2.145	2.624	2.977	3.787

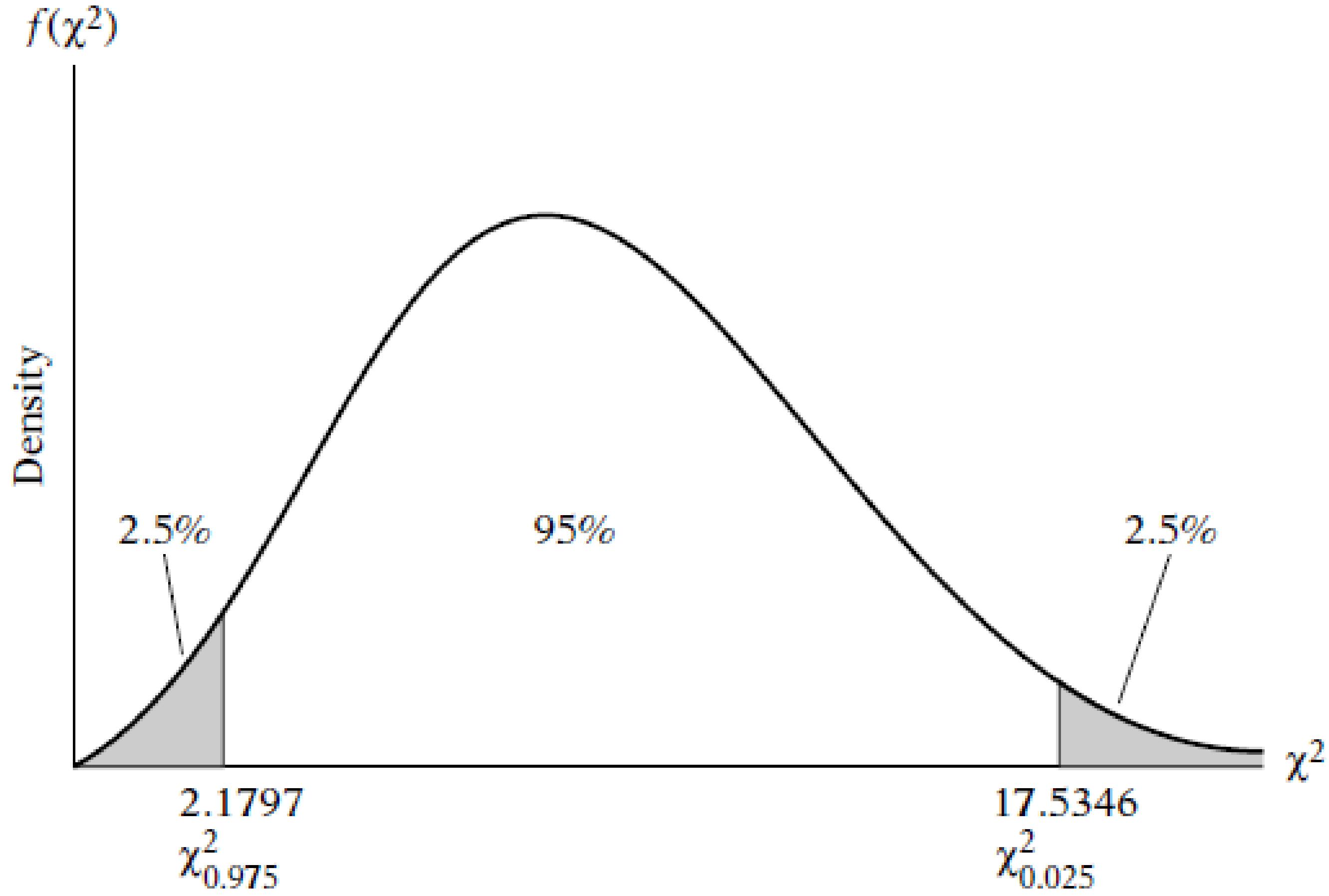
Interval Estimators for σ^2

$$\chi^2 = (n-2)\frac{\hat{\sigma}^2}{\sigma^2}$$

$$\Pr\left(\chi^2_{1-\alpha/2} \le \chi^2 \le \chi^2_{\alpha/2}\right) = 1-\alpha$$

$$\Pr\left[(n-2)\frac{\hat{\sigma}^2}{\chi^2_{\alpha/2}} \le \sigma^2 \le (n-2)\frac{\hat{\sigma}^2}{\chi^2_{1-\alpha/2}}\right] = 1-\alpha$$

Interval Estimators for σ^2



The 95% confidence interval for χ^2 (8 df).

Example 2

- Construct 95% confidence interval for σ and for σ^2

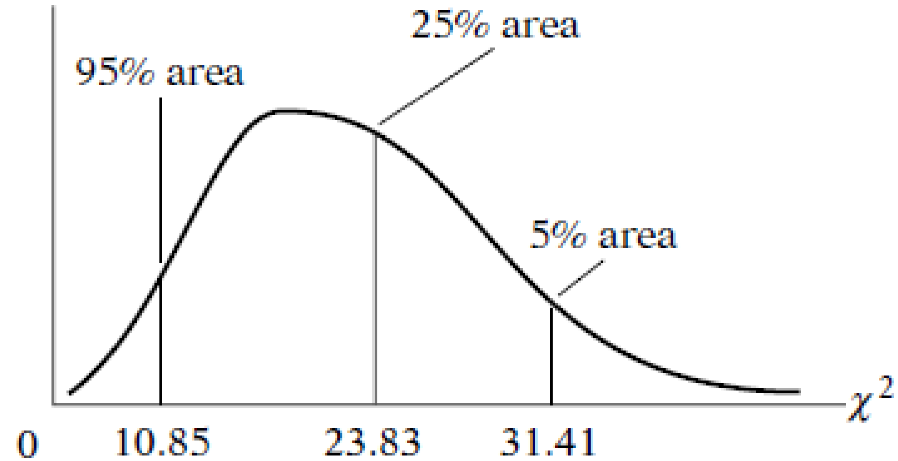
UPPER PERCENTAGE POINTS OF THE χ^2 DISTRIBUTION

Example

$\Pr(\chi^2 > 10.85) = 0.95$

$\Pr(\chi^2 > 23.83) = 0.25$ for df = 20

$\Pr(\chi^2 > 31.41) = 0.05$



Degrees of freedom \ Pr	.995	.990	.975	.950	.900
1	392704×10^{-10}	157088×10^{-9}	982069×10^{-9}	393214×10^{-8}	.0157908
2	.0100251	.0201007	.0506356	.102587	.210720
3	.0717212	.114832	.215795	.351846	.584375
4	.206990	.297110	.484419	.710721	1.063623
5	.411740	.554300	.831211	1.145476	1.61031
6	.675727	.872085	1.237347	1.63539	2.20413
7	.989265	1.239043	1.68987	2.16735	2.83311
8	1.344419	1.646482	2.17973	2.73264	3.48954
9	1.734926	2.087912	2.70039	3.32511	4.16816
10	2.15585	2.55821	3.24697	3.94030	4.86518
11	2.60321	3.05347	3.81575	4.57481	5.57779
12	3.07382	3.57056	4.40379	5.22603	6.30380
13	3.56503	4.10691	5.00874	5.89186	7.04150
14	4.07468	4.66043	5.62872	6.57063	7.78953

.750	.500	.250	.100	.050	.025	.010	.005
.1015308	.454937	1.32330	2.70554	3.84146	5.02389	6.63490	7.87944
.575364	1.38629	2.77259	4.60517	5.99147	7.37776	9.21034	10.5966
1.212534	2.36597	4.10835	6.25139	7.81473	9.34840	11.3449	12.8381
1.92255	3.35670	5.38527	7.77944	9.48773	11.1433	13.2767	14.8602
2.67460	4.35146	6.62568	9.23635	11.0705	12.8325	15.0863	16.7496
3.45460	5.34812	7.84080	10.6446	12.5916	14.4494	16.8119	18.5476
4.25485	6.34581	9.03715	12.0170	14.0671	16.0128	18.4753	20.2777
5.07064	7.34412	10.2188	13.3616	15.5073	17.5346	20.0902	21.9550
5.89883	8.34283	11.3887	14.6837	16.9190	19.0228	21.6660	23.5893
6.73720	9.34182	12.5489	15.9871	18.3070	20.4831	23.2093	25.1882
7.58412	10.3410	13.7007	17.2750	19.6751	21.9200	24.7250	26.7569
8.43842	11.3403	14.8454	18.5494	21.0261	23.3367	26.2170	28.2995
9.29906	12.3398	15.9839	19.8119	22.3621	24.7356	27.6883	29.8194
10.1653	13.3393	17.1170	21.0642	23.6848	26.1190	29.1413	31.3193

Hypotheses testing and interval estimators

- We can use interval estimators for hypotheses testing
- If we have the null hypothesis H_0: $\beta_2 = 0$
- The alternative hypothesis is that $\beta_2 <> 0$
- We can reject the H_0 (with level of significance a), if the constructed confidence interval does not contain 0

Hypotheses testing and interval estimators

- This type of hypotheses are called two-sided
- The confidence interval is called acceptance region and the tails are called rejection region

A test-of-significance approach

- The value under the null hypothesis is used to construct confidence interval
- We check whether the obtained estimator belongs to the constructed interval

A test-of-significance approach

$$H_0 : \beta_2 = \beta_2^*, \quad \alpha$$

$$H_1 : \beta_2 \neq \beta_2^*$$

$$\Pr\left[-t_{\alpha/2} \le \frac{\hat{\beta}_2 - \beta_2^*}{\operatorname{se}(\hat{\beta}_2)} \le t_{\alpha/2}\right] = 1 - \alpha$$

$$\Pr\left[\beta_2^* - t_{\alpha/2}\operatorname{se}(\hat{\beta}_2) \le \hat{\beta}_2 \le \beta_2^* + t_{\alpha/2}\operatorname{se}(\hat{\beta}_2)\right] = 1 - \alpha$$

A test-of-significance approach

- Example 3:
  Test the hypothesis that β_2=0 for M1 and 0.5 for M2 with level of significance 1%.

A test-of-significance approach

- For the One-Sided hypotheses the critical region consists from one of the tails. For example:

$$H_0: \beta \leq \beta_2^*, \quad \alpha$$
$$H_1: \hat{\beta} > \beta_2^*$$
$$t^* = \frac{\widehat{\beta}_2 - \beta_2^*}{se(\widehat{\beta}_2)}$$
$$Do\ not\ accept,\ if \quad \frac{\hat{\beta}_2 - \beta_2^*}{se(\hat{\beta}_2)} > t_\alpha$$

A test-of-significance approach

THE t TEST OF SIGNIFICANCE: DECISION RULES

Type of hypothesis	H_0: the null hypothesis	H_1: the alternative hypothesis	Decision rule: reject H_0 if
Two-tail	$\beta_2 = \beta_2^*$	$\beta_2 \neq \beta_2^*$	$\lvert t \rvert > t_{\alpha/2,\mathrm{df}}$
Right-tail	$\beta_2 \leq \beta_2^*$	$\beta_2 > \beta_2^*$	$t > t_{\alpha,\mathrm{df}}$
Left-tail	$\beta_2 \geq \beta_2^*$	$\beta_2 < \beta_2^*$	$t < -t_{\alpha,\mathrm{df}}$

Notes: β_2^* is the hypothesized numerical value of β_2.
$\lvert t \rvert$ means the absolute value of t.
t_α or $t_{\alpha/2}$ means the critical t value at the α or $\alpha/2$ level of significance.
df: degrees of freedom, $(n - 2)$ for the two-variable model, $(n - 3)$ for the three-variable model, and so on.
The same procedure holds to test hypotheses about β_1.

A test-of-significance approach

A SUMMARY OF THE χ^2 TEST

H_0: the null hypothesis	H_1: the alternative hypothesis	Critical region: reject H_0 if
$\sigma^2 = \sigma_0^2$	$\sigma^2 > \sigma_0^2$	$\dfrac{\text{df}(\hat{\sigma}^2)}{\sigma_0^2} > \chi^2_{\alpha,\text{df}}$
$\sigma^2 = \sigma_0^2$	$\sigma^2 < \sigma_0^2$	$\dfrac{\text{df}(\hat{\sigma}^2)}{\sigma_0^2} < \chi^2_{(1-\alpha),\text{df}}$
$\sigma^2 = \sigma_0^2$	$\sigma^2 \neq \sigma_0^2$	$\dfrac{\text{df}(\hat{\sigma}^2)}{\sigma_0^2} > \chi^2_{\alpha/2,\text{df}}$ or $< \chi^2_{(1-\alpha/2),\text{df}}$

Note: σ_0^2 is the value of σ^2 under the null hypothesis. The first subscript on χ^2 in the last column is the level of significance, and the second subscript is the degrees of freedom. These are critical chi-square values. Note that df is $(n-2)$ for the two-variable regression model, $(n-3)$ for the three-variable regression model, and so on.

A test-of-significance approach

- Example 4:
  Test the One-Sided hypotheses for M2:
- $\beta_2 >= 4$ and $\sigma <= 100$, $\alpha = 3\%$.

Type I and type II errors

- The *α* parameter gives the probability to reject true hypothesis (type I error)
- Type II error happens when we accept false hypothesis
- Decreasing type I error we increase type II error and vice-versa for fixed sample size
- Usually type I error is more important for us

P-value

- P-value is the exact level of significance
- Or p-value is the lowest significance level at which a null hypothesis can be rejected
- Calculate p-values for example 3 hypotheses