Counting

GIU Berlin

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Basic Counting Principles

The Product Rules

- **The Product Rule:** Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.
- **The Extended Product Rule:** Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_m$ in a sequence. If each of the tasks T_i for $i = 1, 2, \ldots, m$ can be done in n_i ways regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to carry out the procedure.

The Basics of Counting

Basic Counting Principles

Example 1 (Product Rule)

- The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
- How many different bit strings of length 5 are there?
- How many functions are there from a set with m elements to a set with n elements?
- How many one-to-one functions are there from a set with m elements to one with n elements?
- How many strings of eight English letters are there:
  - if letters can be repeated?
  - if no letter can be repeated?

The Basics of Counting

Basic Counting Principles

- The product rule is often phrased in terms of sets in this way: If $A_1, A_2, ..., A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set.
- To relate this to the product rule, note that the task of choosing an element in the Cartesian product $A_1 \times A_2 \times ... \times A_m$ is done by choosing an element in A_1, an element in A_2, . . . , and an element in A_m. By the product rule it follows that,

$$|A_1 \times A_2 \times ... \times A_m| = |A_1| \cdot |A_2| \cdot ... \cdot |A_m|.$$

The Basics of Counting

Basic Counting Principles

The Sum Rules

- **The Sum Rule:**If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.
- **The Extended Sum Rule:** Suppose that a task can be done in one of the n_1 ways, or in one of n_2 ways, or..., in one of n_m ways where where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i for all $1 \leq i < j \leq m$. Then the number of ways to do the task is $n_1 + n_2 + ... + n_m$.

The Basics of Counting

Basic Counting Principles

Example 2 (Sum Rule)

- In the discrete mathematics class there are 33 boys and 27 girls. If a representative of a class is to be chosen, how many possible choices are there?
- A student can choose a computer project from one of three lists. The three lists contain $23, 15$, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

The Basics of Counting

Basic Counting Principles

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Basic Counting Principles

- The sum rule can be phrased in terms of sets as: If $A_1, A_2, ..., A_m$ are disjoint finite sets, then the number of elements in the union of these sets is the sum of the numbers of elements in the sets.
- To relate this to our statement of the sum rule, note there are $|A_i|$ ways to choose an element from A_i for $i = 1, 2, ..., m$. Because the sets are disjoint, when we select an element from one of the sets A_i, we do not also select an element from a different set A_j.
- Consequently, by the sum rule, because we cannot select an element from two of these sets at the same time, the number of ways to choose an element from one of the sets, which is the number of elements in the union, is

$$|A_1 \cup A_2 \cup ... \cup A_m| = |A_1| + |A_2| + ... + |A_m|.$$

The Basics of Counting

More Complex Counting Problems

- Many counting problems cannot be solved using just the sum rule or just the product rule.
- However, many complicated counting problems can be solved using both of these rules in combination.

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Example 3

- The name of a variable in the GIU programming language is a string that can contain uppercase letters, lowercase letters, digits, or special characters $!, +, =$, and @. Further, the first character in the string must be a digit or a special character. If the name of a variable is determined by its first five characters, how many different variables can be named in GIU? (Note that the name of a variable may contain fewer than five characters.)

The Basics of Counting

The Inclusion-Exclusion Principle

The Basics of Counting

The Inclusion-Exclusion Principle

- Suppose that a task can be done in n_1 or in n_2 ways, but that some of the set of n_1 ways to do the task are the same as some of the n_2 other ways to do the task.
- In this situation, we cannot use the sum rule to count the number of ways to do the task. Adding the number of ways to do the tasks in these two ways leads to an overcount, because the ways to do the task in the ways that are common are counted twice.
- To correctly count the number of ways to do the two tasks, we add the number of ways to do it in one way and the number of ways to do it in the other way, and then subtract the number of ways to do the task in a way that is both among the set of n_1 ways and of n_2 ways.
- This technique is called the **principle of inclusion-exclusion**. Sometimes, it is also called the **subtraction principle** for counting.

The Basics of Counting

The Inclusion-Exclusion Principle

Example 4 (Inclusion-exclusion)

- In the discrete mathematics class there are 34 students who take Calculus, 40 students who take Programming. If the class has a total of 61 students how many of them take both subjects?
- How many bit strings of length eight either start with a 1 bit or end with the two bits 00?
- How many bit strings of length seven either begin with two 0-s or end with three 1-s?
- How many bit strings of length 10 either begin with three 0-s or end with two 0-s?
- How many bit strings of length 10 contain either five consecutive 0-s or five consecutive 1-s?

The Basics of Counting

Tree Diagrams

The Basics of Counting

Tree Diagrams

- A tree consists of a root, a number of branches leaving the root, and possible additional branches leaving the endpoints of other branches.
- To use trees in counting, we use a branch to represent each possible choice.
- We represent the possible outcomes by the leaves, which are the endpoints of branches not having other branches starting at them.

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- A tree consists of a root, a number of branches leaving the root, and possible additional branches leaving the endpoints of other branches.
- To use trees in counting, we use a branch to represent each possible choice.
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Example 5 (Tree diagram)

How many bit strings of length four do not have two consecutive 0-s?

The Basics of Counting

The Pigeonhole Principle

The Basics of Counting

The Pigeonhole Principle

Theorem 6 (The Pigeonhole Principle)

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

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- The pigeonhole principle is also called the Dirichlet drawer principle, after the nineteenth century German mathematician Dirichlet, who often used this principle in his work.
- The pigeonhole principle can be used to prove a useful corollary about functions.

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Corollary 7

A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one.

The Basics of Counting

The Pigeonhole Principle

Example 8

- Among any group of 367 people, there must be at least two with the same birthday.
- Out of 8 dates two fall on the same weekday.
- Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.
- Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends.

The Basics of Counting

The Generalized Pigeonhole Principle

Theorem 9 (The Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects, where $\lceil \cdot \rceil$ represents the ceiling function.

The Basics of Counting

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Example 10

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

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Example 10

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Solution 11

The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade.

The Basics of Counting

Some Elegant Applications of the Pigeonhole Principle

- In many interesting applications of the pigeonhole principle, the objects to be placed in boxes must be chosen in a clever way.

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Example 12 (Applications of the Pigeonhole Principle)

- During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.
- Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

The Basics of Counting

The Pigeonhole Principle

- Suppose that $a_1, a_2, ..., a_N$ is a sequence of real numbers. A **subsequence** of this sequence is a sequence of the form $a_{i_1}, a_{i_2}, ..., a_{i_m}$ with $1 \leq i_1 < i_2 < ... < i_m \leq N$
- A sequence is called **strictly increasing** if each term is larger than the one that precedes it, and it is called **strictly decreasing** if each term is smaller than the one that precedes it.

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- A sequence is called **strictly increasing** if each term is larger than the one that precedes it, and it is called **strictly decreasing** if each term is smaller than the one that precedes it.

Theorem 13

Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

The Basics of Counting

The Pigeonhole Principle

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The Pigeonhole Principle

- Other applications of the Pigeonhole Principle are seen in **Ramsey Theory** which is a part of combinatorics that deals with the distribution of subsets of elements of sets.

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Example 14

Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Then there are either three mutual friends or three mutual enemies in the group.

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Example 14

Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Then there are either three mutual friends or three mutual enemies in the group.

- The Ramsey number $R(m, n)$, where m and n are integers ≥ 2, denotes the minimum number of people at a party such that there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies.
- It is possible to prove some useful properties about Ramsey numbers, but for the most part it is difficult to find their exact values. Ramsey theory is an interesting area of current research in mathematics.

The Basics of Counting

Permutations and Combinations

The Basics of Counting

Permutations and Combinations

- A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set.
- An ordered arrangement of r elements of a set is called an **r-permutation**.
- The number of r-permutations of a set with n elements is denoted by $P(n, r)$. We can find $P(n, r)$ using the product rule.

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Example 15 (permutation)

Let $S = \{1, 2, 3, 4\}$ be a set then the ordered arrangement $(2, 3, 1, 4)$ is a permutation of S. The ordered arrangement $(2, 4)$ is a 2-permutation of S.

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Theorem 16

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are $P(n, r) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)$, r-permutations of a set with n distinct elements.

The Basics of Counting

Permutations and Combinations

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Permutations and Combinations

- **Combinations** are a way to count unordered selection of objects.
- An **r-combination** of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

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Example 17

In how many ways can you form a musical band of 5 people from a group of 10 people?

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Example 17

In how many ways can you form a musical band of 5 people from a group of 10 people?

Theorem 18

The number of r-combinations of a set with n elements, where n is a non-negative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

The Basics of Counting

Permutations and Combinations

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Permutations and Combinations

Corollary 2.1

If n and r are integers with $0 \leq r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Corollary 2.2

Let n and r be non-negative integers with $r \leq n$. Then

$$C(n, r) = C(n, n-r).$$

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The Basics of Counting

Permutations and Combinations

Example 19 (Permutations and Combinations)

- List all the permutations of $\{a, b, c\}$.
- How many permutations of $\{a, b, e, d, e, t, g\}$ end with a?
- A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?
- How many subsets with an odd number of elements does a set with 10 elements have?
- How many license plates consisting of three letters followed by three digits contain no letter or digit twice?
- There are six runners in the 100-yard dash. How many ways are there for three medals to be awarded if ties are possible?
- How many ways are there for a horse race with four horses to finish if ties are possible?

The Basics of Counting

Binomial Coefficients

The Basics of Counting

Binomial Coefficients

- The number of r-combinations from a set with n elements is often denoted by

$$C(n,r) = \binom{n}{r}. \tag{1}$$

- This number is also called a **binomial coefficient** because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a+b)^n$.
- Below we present a theorem that gives a power of a binomial expression as a sum of terms involving binomial coefficients.

The Basics of Counting

Binomial Coefficients

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- Below we present a theorem that gives a power of a binomial expression as a sum of terms involving binomial coefficients.

Theorem 20 (The Binomial Theorem)

Let x and y be variables, and let n be a non-negative integer. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

The Basics of Counting

Binomial Coefficients

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Binomial Coefficients

Example 21

- Find the expansion of $(x + y)^5$.
- What is the binomial coefficient of x^6y^3 in the expansion of $(x + y)^9$?

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Some Results:

Corollary 2.3

Let n be a non-negative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{2}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0. \tag{3}$$

The Basics of Counting

Binomial Coefficients

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Binomial Coefficients

Theorem 22 (Pascal's identity)

Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{4}$$

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- Pascal's Identity is the basis for a geometric arrangement of the binomial coefficients in a triangle.
- The$n$-th row in the triangle consists of the binomial coefficients $\binom{n}{k}, k = 0, 1, ..., n$.
- This triangle is known as **Pascal's triangle**.
- Pascal's Identity shows that when two adjacent binomial coefficients in this triangle are added, the binomial coefficient in the next row between these two coefficients is produced.

Theorem 23 (Vandermonde' s identity)

Let m, n, and r be non-negative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \cdot \binom{n}{k} \tag{5}$$

The Basics of Counting

Binomial Coefficients

Theorem 23 (Vandermonde' s identity)

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Theorem 24

Let n and r be non-negative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r} \tag{6}$$

Example 25

- How many terms are there in the expansion of $(x+y)^{100}$ after like terms are collected?
- What is the coefficient of x^9 in $(2-x)^{19}$?
- Give a formula for the coefficient of x^k in the expansion of $(x+1/x)^{100}$, where k is an integer.
- Let n be a positive integer. Show that $\binom{2n}{n+1}+\binom{2n}{n}=\binom{2n+2}{n+1}/2$.
- Let n and k be integers with $1 \leq k \leq n$. Show that

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}=\binom{2n+2}{n+1}/2-\binom{2n}{n}.$$

- Prove the Binomial Theorem using mathematical induction.

- Recall that a recursive definition of a sequence specifies one or more initial terms and a rule for determining subsequent terms from those that precede them.
- Such a rule is called a **recurrence relation**. Such relations can be used in studying and solving counting problems.

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- Such a rule is called a **recurrence relation**. Such relations can be used in studying and solving counting problems.

Definition 2.1 (Recurrence Relation)

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, ..., a_{n-1}$. for all integers n with $n \geq n_0$, where n_0 is a non-negative integer. A sequence is called a ***solution*** *of a recurrence relation if its terms satisfy the recurrence relation.*

Advanced Counting Techniques

Recurrence Relations

Example 26

Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every non-negative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \cdots$. Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

Example 26

Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every non-negative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \cdots$. Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

- The **initial conditions** for a sequence specify the terms that precede the first term where the recurrence relation takes effect.
- The recurrence relation and initial conditions uniquely determine a sequence.
- This is the case because a recurrence relation, together with initial conditions, provide a recursive definition of the sequence.
- Any term of the sequence can be found from the initial conditions using the recurrence relation a sufficient number of times.

Advanced Counting Techniques

Recurrence Relations

Example 27 (Compound Interest)

Suppose that a person deposits $\$10,000$ in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

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Example 28 (Rabbits and the Fibonacci Numbers)

Consider this problem, which was originally posed by Leonardo Pisano, also known as Fibonacci, in the thirteenth century in his book *Liber abaci*. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.

Advanced Counting Techniques

Solving Linear Recurrence Relations

Definition 2.2

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} \tag{7}$$

where $c_1, c_2, ..., c_k$ are real numbers, and $c_k \neq 0$.

Definition 2.2

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where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

- The recurrence relation is **linear** because the right-hand side is a sum of previous terms each multiplied by a function of n.
- The recurrence relation is **homogeneous** because no terms occur that are not multiples of the a_j-s.
- The coefficients of the terms of the sequence are all **constants**, rather than functions that depend on n.
- The **degree** is k because an is expressed in terms of the previous k terms of the sequence.

Advanced Counting Techniques

Solving Linear Recurrence Relations

Advanced Counting Techniques

Solving Linear Recurrence Relations

- A consequence of the second principle of mathematical induction is that a sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions
$$a_0 = C_0, a_1 = C_1, ..., a_{k-1} = C_{k-1}.$$

- A consequence of the second principle of mathematical induction is that a sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions

$$a_0 = C_0, a_1 = C_1, ..., a_{k-1} = C_{k-1}.$$

Example 29 (degree)

- The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one.
- The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two.
- The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five.

Advanced Counting Techniques

Solving Linear Recurrence Relations

- The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where r is a constant.
- Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}.$$

- Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution of $P(r) = 0$ where

$$P(r) = r^n - c_1 r^{n-1} - c_2 r^{n-2} - \ldots - c_k r^{n-k}.$$

$P(r)$ is called the **characteristic polynomial** and $P(r) = 0$ is the **characteristic equation**. The solutions of this equation are called the **characteristic roots** of the recurrence relation.

Advanced Counting Techniques

Solving Linear Recurrence Relations

- When $r = 0$ then we get the **trivial solution** of the recurrence relation.
- When $r \neq 0$ then we obtain the non-trivial solutions of the recurrence relation.
- The **non-trivial solutions** can be obtained from the solutions of the reduced polynomial of degree k.

$$\bar{P}(r) = r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k.$$

- Notice that any homogeneous linear recurrence relation of degree k has a corresponding reduced form of characteristic polynomial of degree k.

Advanced Counting Techniques

Solving Linear Recurrence Relations

Theorem 30 (distinct roots)

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

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Example 31

- Find an explicit formula for the Fibonacci numbers.
- What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

where $a_0 = 2$ and $a_1 = 7$?

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Theorem 32 (double root)

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

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Example 33

What is the solution of the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2}$$

where $a_0 = 2$ and $a_1 = 6$?

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Theorem 34 (distinct roots)

Let $c_1, c_2, ..., c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - ... - c_k = 0$$

has k distinct roots $r_1, r_2, ..., r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + ... + \alpha_k r_k^n$$

for $n = 0, 1, 2, ...,$ where $\alpha_1, \alpha_2, ..., \alpha_k$ are constants.

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Theorem 35 (repeated roots)

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for all $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if

$$a_n = \sum_{i=1}^{t} \sum_{j=0}^{m_t - 1} \alpha_{i,j}\, n^j\, r_i^n \quad \textit{for } n = 0, 1, 2, \cdots .$$

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Example 36 (distinct roots)

Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2, a_1 = 5$, and $a_2 = 15$.

Example 37 (repeated roots)

Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with the initial conditions $a_0 = 1, a_1 = -2$, and $a_2 = -1$.

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Solving Linear Recurrence Relations

Definition 2.3

A **linear non-homogeneous recurrence relation of degree k** *with constant coefficients is a recurrence relation of the form*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n) \tag{8}$$

where $c_1, c_2, \dots, c_k$ are real numbers, and $c_k \neq 0$ and $F(n)$ is a function not identically zero depending only on n. The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \tag{9}$$

is called the **associated homogeneous recurrence relation**.

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Solving Linear Recurrence Relations

- The key fact about linear non-homogeneous recurrence relations with constant coefficients is that every solution is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation.

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Solving Linear Recurrence Relations

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Theorem 38 (Linear non-homogeneous recurrence relation)

If $\{a_n^{(p)}\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$ where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}.$$

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Example 39 (non-homogeneous)

Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Solution 40

It is easy to see that $a_n^{(h)} = \alpha 3^n$, where α is a constant. For the particular solution notice that $F(n) = 2n$ is linear in n therefore we might guess that the particular solution has a linear form. Say $a_n^{(p)} = bn + c$. Plugging it in the recurrence relation one can derive that $b = -1, c = -3/2$. By the previous theorem we have

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha 3^n.$$

To find α use the initial condition $a_1 = 3$ to get that $\alpha = 11/6$.

Advanced Counting Techniques

Generating Functions

- Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series.
- Generating functions can be used to solve many types of counting problems, such as
  - the number of ways to select or distribute objects of different kinds, subject to a variety of constraints
  - the number of ways to make change for a dollar using coins of different denominations
  - the number of ways to partition a positive integer in terms of smaller positive integers.

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Generating Functions

Definition 2.4 (Generating Functions)

The **generating function** *for the sequence* $a_0, a_1 ..., a_k, ...$ *of real numbers is the infinite series*

$$G(x) = a_0 + a_1 x + a_2 x^2 + ... + a_k x^k + ... = \sum_{k=0}^{\infty} a_k x^k. \tag{10}$$

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$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k. \tag{10}$$

- We can define generating functions for finite sequences of real numbers by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0, a_{n+2} = 0$, and so on.
- The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

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Generating Functions

Theorem 41

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *and* $g(x) = \sum_{k=0}^{\infty} b_k x^k$. *Then*

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \tag{11}$$

and

$$f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k. \tag{12}$$

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Remark 2.1

In the case of series that do not converge, the statements in last theorem can be taken as definitions of addition and multiplication of generating functions.

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Generating Functions

Example 42 (Generating functions)

- $F(x) = \dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$
- $G(x) = \dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots$
- $H(x) = e^x = 1 + x + x^2/2! + x^3/3! + \ldots$
- $J(x) = (1+x)^n = C(n,0) + C(n,1)x + C(n,2)x^2 + \ldots + C(n,n)x^n$

Example 42 (Generating functions)

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- $J(x) = (1+x)^n = C(n,0) + C(n,1)x + C(n,2)x^2 + \ldots + C(n,n)x^n$

- Notice that if one knows a particular generating function then by applying differentiation or integration one is able to obtain other generating functions that encrypt information about other sequences of numbers. For instance $G(x) = F'(x)$ in the above example.

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Generating Functions

Definition 2.5

Let u be a real number and k a non-negative integer. Then the **extended binomial coefficient** $\binom{u}{k}$ *is defined by*

$$\binom{u}{k} = \begin{cases} u(u-1)(u-2)\cdot \ldots \cdot (u-k+1)/k! & : k > 0 \\ 1 & : k = 0. \end{cases}$$

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Theorem 43 (The extended binomial theorem)

Let x be a real number with $|x| < 1$ and let u be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k. \tag{13}$$

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Generating Functions

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise

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$\dfrac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\dfrac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\dfrac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\dfrac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=0}^{\infty} \dfrac{(-1)^{k+1}}{k} x^k = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

Note. The series for the last two generating functions can be found in most calculus books when power series are discussed.

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Generating Functions

Example 44 (counting problems)

- Find the number of solutions of the equation $u + v + w = 10$ where $3 \leq u, v, w \leq 5$.
- In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?
- Use generating functions to find the number of k-combinations of a set with n elements. Assume that the Binomial Theorem has already been established.
- Use generating functions to find the number of r-combinations from a set with n elements when repetition of elements is allowed.