HASSE PRINCIPLE AND BRAUER–MANIN OBSTRUCTION

Useful references: [Poo17], [Sko01], [CTS21], [GS17].

1. The Hasse principle

1.1. **Introduction.** For concreteness we'll work over $\mathbb{Q}$, but everything works the same over a general number field. Let $X/\mathbb{Q}$ be a variety (say finite type, separated and geometrically integral).

Question: When is $X(\mathbb{Q}) \neq \emptyset$?
A necessary condition is that $X(\mathbb{R}) \neq \emptyset$ and that $X(\mathbb{Q}_p) \neq \emptyset$ for all primes p. For example, the conic

$$C : \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{P}^2_{\mathbb{Q}}$$

has no $\mathbb{Q}$-points since $C(\mathbb{R}) = \emptyset$. We say a variety X satisfies the *Hasse principle* if the convernse holds, i.e. if $X(\mathbb{R}) \neq \emptyset$, and $X(\mathbb{Q}_p) \neq \emptyset$ for all p, implies that $X(\mathbb{Q}) \neq \emptyset$. Roughly, the game is to identify families of varieties that all satisfy the Hasse principle. Then for some X in the family one knows that checking everywhere local solubility (i.e. that $X(\mathbb{R}) \neq \emptyset$ and $X(\mathbb{Q}_p) \neq \emptyset$ for all p) is sufficient to establish the existence of rational points on X. An example of this situation is provided by the Hasse–Minkowski theorem.

Theorem 1.1 (Hasse–Minkowski theorem)**.** *Let $Q = \sum_{i<j} a_{ij} x_i x_j$ be a non-degenerate quadratic form in $(n+1)$-many variables $x_0, ..., x_n$, where the coefficients a_{ij} are rational numbers. Then the quadric*

$$\{Q = 0\} \subseteq \mathbb{P}^n_{\mathbb{Q}}$$

satisfies the Hasse principle. Put another way, let us say that Q represents 0 over a field K if there are $x_0, ..., x_n \in K$, not all 0, so that $Q(x_0, ..., x_n) = 0$. Then the theorem says that Q represents 0 over $\mathbb{Q}$ if and only if it does so over $\mathbb{R}$ and over $\mathbb{Q}_p$ for all primes p.

Example 1.2. The $n = 2$ case of the above result says (in particular) that conics

$$C_{a,b} : \{z^2 = ax^2 + by^2\} \subseteq \mathbb{P}^2_{\mathbb{Q}}$$

satisfy the Hasse principle (here $a, b \in \mathbb{Q}^\times$).

1.2. **Quadratic Hilbert symbols.** In the previous section we saw that determining when a conic is soluble over $\mathbb{Q}$ reduces to determining when conics a soluble over $\mathbb{Q}_p$ and over $\mathbb{R}$. The reals are easy and over the p-adics this can be done via Hensel's lemma. The answer is neatly packaged in terms of *Hilbert symbols.* Here we write $\mathbb{R} = \mathbb{Q}_\infty$ to ease notation.

Definition 1.3. For $p \leq \infty$ define the *Hilbert symbol*

$$(\ ,\ )_p : \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2} \times \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2} \longrightarrow \{\pm 1\}$$

by

$$(a, b)_p = \begin{cases} 1 & C_{a,b}(\mathbb{Q}_p) \neq \emptyset, \\ -1 & C_{a,b}(\mathbb{Q}_p) = \emptyset. \end{cases}$$

Here $C_{a,b}$ is the conic $\{z^2 = ax^2 + by^2\} \subseteq \mathbb{P}^2_{\mathbb{Q}_p}$.

Note that we have $(a,b)_\infty = -1$ if and only if $a < 0$ and $b < 0$. For any $p \leq \infty$, the Hilbert symbol $(\ ,\ )_p$ is symmetric, bilinear and non-degenerate (as a pairing on finite dimensional $\mathbb{F}_2$-vector spaces). Moreover, for any $a, b \in \mathbb{Q}^\times$ we have the product formula

$$\prod_{p \leq \infty} (a,b)_p = 1 \tag{1.4}$$

(this is a finite product). For odd primes p we have the *Tame symbol formula*

$$(a,b)_p = \left((-1)^{\mathrm{ord}_p(a)\mathrm{ord}_p(b)} \frac{a^{\mathrm{ord}_p(b)}}{b^{\mathrm{ord}_p(a)}}\right) \pmod p \in \mathbb{F}_p^\times / \mathbb{F}_p^{\times 2} \cong \{\pm 1\}.$$

The Hilbert symbol at $p = 2$ can e.g. be calculated from this and the product formula, though one can also calculate it directly via Hensel's lemma and some case work.

Remark 1.5. One has $(a,b)_p = 1$ if and only if b is a norm from $\mathbb{Q}_p(\sqrt{a})$. This property is naturally studied via local class field theory, which gives a convenient way to think about Hilbert symbols.

Example 1.6. Let p and q be distinct odd primes. Then one computes $(p,q)_\infty = 1$, $(p,q)_l = 1$ for $l \neq p, q$, $(p,q)_q = \left(\frac{p}{q}\right)$, $(p,q)_p = \left(\frac{q}{p}\right)$ and $(p,q)_2 = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}$. So (1.4) recovers quadratic reciprocity:

$$\left(\frac{p}{q}\right) \cdot \left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}.$$

1.3. **An example of a variety failing the Hasse principle.** Consider the variety

$$C/\mathbb{Q} : y^2 = 2x^4 - 34z^4.$$

This is most naturally viewed as a smooth genus 1 curve in weighted projective space $\mathbb{P}^2(1,2,1)$ over $\mathbb{Q}$. Here $\mathbb{P}^2(1,2,1)$ is the quotient of $\mathbb{A}^3 \setminus \{0\}$ by the equivalence relation

$$(x,y,z) \sim (\lambda x, \lambda^2 y, \lambda z)$$

for all $0 \neq \lambda$. Here the 'weighting' on y captures the natural 'invariance-under-weighted-scaling' satisfied by solutions to the non-homogeneous equation defining C.

Lemma 1.7. *The curve C is everywhere locally soluble.*

Proof. Clearly $C(\mathbb{R}) \neq \emptyset$. For $p \neq 2, 17$ the reduction of C modulo p is a smooth genus 1 curve; the Hasse bound shows that $C(\mathbb{F}_p) \neq \emptyset$. Any such point can be lifted to a $\mathbb{Q}_p$-point on C by Hensel's lemma, hence $C(\mathbb{Q}_p) \neq \emptyset$. For $p = 17$ we note that

$$2 \cdot 1^4 - 34 \cdot 1^4 = -32 \equiv 36 = 6^2 \pmod{17}.$$

By Hensel's lemma it follows that $\sqrt{-32} \in \mathbb{Q}_{17}$ and that $(1, \sqrt{-32}, 1) \in C(\mathbb{Q}_{17})$. For $p = 2$, set $f(t) = t^4 - 17$. Then $|f(3)|_2 = |64|_2 = 2^{-6}$. On the other hand, $|f'(3)|_2 = |4 \cdot 3^3|_2 = 2^{-2}$. Thus $|f(3)|_2 < |f'(3)|_2^2$ and by Hensel's lemma we see that there is $a \in \mathbb{Q}_2$ with $a^4 = 17$. Then $(a, 0, 1) \in C(\mathbb{Q}_2)$. $\square$

Proposition 1.8. *We have $C(\mathbb{Q}) = \emptyset$. In particular, the curve C violates the Hasse principle.*

Proof. Suppose we have a solution (x, y, z) over $\mathbb{Q}$. Without loss of generality we can suppose that x, y, z are coprime integers. Further, it's clear that $y \neq 0$ since 17 is not a 4^{th} power in $\mathbb{Q}$. Next, note that the coprimality assumption forces y to be coprime to 17. Let q be an odd prime dividing y. Reducing the defining equation modulo q we find $x^4 \equiv 17z^4 \pmod{q}$. In particular, 17 is a square modulo q. By quadratic reciprocity, q is a square modulo 17. Since also 2 and -1 are squares modulo 17 we conclude that y is a square modulo 17, say $y \equiv y_0^2 \pmod{17}$ for some $0 \neq y_0 \in \mathbb{F}_{17}$. We thus have

$$2x^4 \equiv y_0^4 \pmod{17}.$$

This is a contradiction since one readily checks that 2 is not a 4^{th} power modulo 17. □

Remark 1.9. In the argument above, the magic happens with the application of quadratic reciprocity. This is the one 'global' step in the proof.

Remark 1.10. The curve C is a 2-cover of the elliptic curve $E : y^2 = x^3 + 17x$. Since C is everywhere locally soluble, C represents an element of the 2-Selmer group $\mathrm{Sel}^2(E/\mathbb{Q})$. Proposition 1.8 shows that the image of C in $Ш(E/\mathbb{Q})[2]$ is non-trivial. See [Sil09, Chapter X.6].

2. The Brauer–Manin obstruction

2.1. **Introduction.** Let us rephrase the proof of Proposition 1.8 a bit. The reason for expressing things in this way will become clear later. Having examined C locally and established Lemma 1.7, we have

$$C(\mathbb{Q}) \subseteq \prod_{p \leq \infty} C(\mathbb{Q}_p) \neq \emptyset.$$

For each p (including ∞) consider the function

$$\alpha_p : C(\mathbb{Q}_p) \longrightarrow \{\pm 1\} \tag{2.1}$$

sending a point $P = (x, y, z)$ to the Hilbert symbol $(y, 17)_p$ (the scaling on $\mathbb{P}^2(1, 2, 1)$ changes y by squares which does not effect the Hilbert symbol). Note that this function as currently written is not defined when $y = 0$. However, there is a very natural definition of the function at such points. Indeed, if $(x, 0, z) \in C(\mathbb{Q}_p)$ then we have $2x^4 - 34z^4$ so that 17 is a square in $\mathbb{Q}_p$. Thus $(y, 17)_p = 1$ for all $y \neq 0$. So it makes sense to define $\alpha(P) = 1$ also when $y = 0$. Thus for any $P = (x, y, z) \in C(\mathbb{Q}_p)$ we set

$$\alpha_p(P) = \begin{cases} (y, 17)_p & y \neq 0, \\ 1 & y = 0. \end{cases}$$

Next, note that from the product formula (1.4), if $P \in C(\mathbb{Q})$ then we have

$$\prod_{p \leq \infty} \alpha_p(P) = 1.$$

Given this observation it is natural to consider the set

$$X = \Big\{ (P_p)_p \in \prod_{p \leq \infty} C(\mathbb{Q}_p) \,:\, \prod_{p \leq \infty} \alpha_p(P_p) = 1 \Big\}.$$

We then have

$$C(\mathbb{Q}) \subseteq X \subseteq \prod_{p\leq\infty} C(\mathbb{Q}_p). \tag{2.2}$$

We will show that $X = \emptyset$, showing that $C(\mathbb{Q}) = \emptyset$ also.

Remark 2.3. In the definition of X, it is not clear a priori that the product $\prod_{p\leq\infty} \alpha_p(P_p)$ is finite. To get round this one could instead define X to be the subset of $\prod_{p\leq\infty} C(\mathbb{Q}_p)$ for which the product is both finite and equal to 1, noting that (2.2) is still true. However, as we shall see, this is actually unnecessay: the product is always finite.

Lemma 2.4. *The function α_p is identically* 1 *on $C(\mathbb{Q}_p)$ for all $p \neq 17$. On $C(\mathbb{Q}_{17})$ the function α_{17} is identically -1. In particular we have $X = \emptyset$ and, consequently, $C(\mathbb{Q}) = \emptyset$.*

Proof. It's clear that α_2 and α_∞ are identically 1 since 17 is a square in both $\mathbb{R}^\times$ and $\mathbb{Q}_2^\times$. Let $p \neq 17$ be an odd prime. Suppose we have $(x, y, z) \in C(\mathbb{Q}_p)$ where, after scaling in $\mathbb{P}^2(1, 2, 1)$, x, y, z are in $\mathbb{Z}_p$ and not all divisible by p. If $\mathrm{ord}_p(y) > 0$ then $x^4 \equiv 17z^4 \pmod{p}$ so that 17 is a square modulo p. In this case 17 is a square in $\mathbb{Q}_p$ by Hensel's lemma and α_p is trivial. On the other hand, if $\mathrm{ord}_p(y) = 0$ then both entries of the Hilbert symbol $(y, 17)_p$ are units, so $(y, 17)_p = 1$.

We now turn to the case $p = 17$. Suppose $(x, y, z) \in C(\mathbb{Q}_{17})$ where again we assume without loss of generality that x, y, z are in $\mathbb{Z}_{17}$ and not all divisible by 17. We necessarily have $y \in \mathbb{Z}_{17}^\times$, so that

$$(y, 17)_{17} = \left(\frac{y}{17}\right).$$

If y is a square modulo 17 then the equation $y^2 = 2x^4 - 34z^4$ shows that 2 is a 4th modulo 17, which it is not. So we find $(y, 17)_{17} = -1$. □

Remark 2.5. The calcuation of the set X is a purely local (or 'everywhere local') problem. The product formula for Hilbert symbols provides the global component to the proof.

Very roughly speaking, the Brauer–Manin obstruction arises by formalising and generalising the proof given above to the case of arbitrary varieties.

2.2. **The Brauer group of a field.** Let K be a field. It what follows, rings will be associative with unit, but not necessarily commutative. For a ring R we denote by $Z(R)$ its *centre*,

$$Z(R) = \{r \in R \mid rx = xr \ \forall x \in R\}.$$

A *left ideal* of R is an additive subgroup $I \subseteq R$ such that $rI \subseteq I$ for all $r \in R$. We define *right ideals* analagously. A subset I of R is a *2-sided ideal* if it's both a left ideal and a right ideal. We say that a ring $R \neq 0$ is *simple* if it has no non-trivial 2-sided ideals (i.e. other than 0 and R). We write R^{opp} for the new ring whose underlying additive group is the same as that of R but whose multiplication is reversed, i.e. $r \cdot r' = r'r$ with the left-hand side taking place in R^{opp} and the right-hand side taking place in R. By a *K-algebra* we mean a ring R equipped with a (necessarily injective) homomorphism $K \to Z(R)$. We say that a K-algebra R is *central* if $Z(R) = K$.

Definition 2.6. A *central simple algebra* over K (CSA/K) is a finite dimensional (as a K-vector space) K-algebra A which is central, and simple as a ring.

Some examples of central simple algebras:

- K itself,
- the ring $M_n(K)$ of $n \times n$ matrices over K, for any $n \geq 1$,
- Hamilton's quaternions $\mathbb{H}$ over $K = \mathbb{R}$. That is, the 4-dimensional $\mathbb{R}$-algebra
$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$$
subject to the relations $i^2 = j^2 = -1$ and $ij = -ji$.
- The quaternion algebra $(a,b)_K$ over K (now any field), for $a, b \in K^\times$. That is, the 4-dimensional K-algebra
$$K \oplus Ki \oplus Kj \oplus Kij$$
subject to the relations $i^2 = a, j^2 = b$ and $ij = -ji$. Note that $(-1,-1)_\mathbb{R} = \mathbb{H}$.
- If A and B are central simple algebras over K then so are A^{opp} and $A \otimes_K B$.

Remark 2.7. The matrices
$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; IJ = \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix}$$
generate $M_2(K)$ as a K-vector space and satisfy $I^2 = 1, J^2 = 1$ and $IJ = -JI$. In particular, we see that $M_2(K) \cong (1,1)_K$.

More generally, one can show that the quaternion algebra $(a,b)_K$ is isomorphic to $M_2(K)$ if and only if conic
$$C_{a,b} : \{z^2 = ax^2 + by^2\} \subseteq \mathbb{P}^2_K$$
has a K-point.

Definition 2.8. Let A and A' be central simple algebras over K. We say that A and A' are *Brauer equivalent* if there are positive integers m, n such that $M_n(A) \cong M_m(A')$ as K-algebras. This is an equivalence relation on isomorphism classes of central simple algebras. We denote by $\mathrm{Br}(K)$ the set
$$\mathrm{Br}(K) = \{K\text{-alg iso classes of central simple algebras over } K\}/\sim.$$
For A a central simple algebra over K we denote by $[A]$ its class in $\mathrm{Br}(K)$.

One can show that $\mathrm{Br}(K)$ forms an abelian group with product $\otimes_K$, identity the class of K (hence also the class of $M_n(K)$ for any $n \geq 1$), and inverse $A \mapsto A^{\mathrm{opp}}$.

Remark 2.9. The class of a quaternion algebra $(a,b)_K$ in $\mathrm{Br}(K)$ is trivial if and only if $C_{a,b}$ has a K-point.

We have the following examples/facts of Brauer groups.

- For a finite field $\mathbb{F}_q$ we have $\mathrm{Br}(\mathbb{F}_q)$. For quaternion algebras this corresponds to the fact that every smooth conic over $\mathbb{F}_q$ has a $\mathbb{F}_q$-point.
- We have $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, generated by the class of $\mathbb{H}$. Denote by inv_∞ the unique injection $\mathrm{Br}(\mathbb{R}) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ (sending the class of $\mathbb{H}$ to $1/2$). For a quaternion algebra $A = (a,b)_\mathbb{R}$ we have $\mathrm{inv}_\infty(A) = 0$ if and only if $(a,b)_\infty = 1$.
- For p a prime, the *local invariant map* inv_p gives an isomorphism
$$\mathrm{inv}_p : \mathrm{Br}(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

For a quaternion algebra $A = (a, b)_{\mathbb{Q}_p}$ we have

$$\mathrm{inv}_p(A) = \begin{cases} 0 & (a,b)_p = 1, \\ 1/2 & (a,b)_p = -1. \end{cases}$$

- There is an exact sequence

$$0 \longrightarrow \mathrm{Br}(\mathbb{Q}) \longrightarrow \oplus_{p\leq\infty}\mathrm{Br}(\mathbb{Q}_p) \xrightarrow{\sum_p \mathrm{inv}_p} \mathbb{Q}/\mathbb{Z}.$$

  The first map sends the class of a central simple algebra A to the tuple $(A \otimes_{\mathbb{Q}} \mathbb{Q}_p)_p$. For quaternion algebras, exactness corresponds to two facts seen previously. First, injectivity of the first map shows that the conic $C_{a,b}$ $(a, b \in \mathbb{Q}^\times)$ has a $\mathbb{Q}$-point if and only if it has a $\mathbb{Q}_p$-point for all $p \leq \infty$. That is, that $C_{a,b}$ satisfies the Hasse principle. The fact that the sequence is a complex says that for any quaternion algebra $(a, b)_{\mathbb{Q}}$ we have

$$\sum_{p\leq\infty} \mathrm{inv}_p(a,b)_{\mathbb{Q}_p} = 0.$$

  This is precisely the product formula (1.4) for Hilbert symbols.

2.3. **The Brauer group of a variety.** Consider again the example of the curve $C : y^2 = 2x^4 - 34z^4$. The affine chart $\{z \neq 0\}$ has equation $y^2 = 2x^4 - 34$. Let y be the 'y-coordinate' function on this chart, viewed as an element of the function field $\mathbb{Q}(C)$. Then we can consider the quaternion algebra $A = (y, 17)_{\mathbb{Q}(C)}$. The function α_p of (2.1) is given by

$$P \longmapsto \mathrm{inv}_p(y(P), 17)_{\mathbb{Q}_p},$$

i.e. we form α_p by 'evaluating' A at points $P \in C(\mathbb{Q}_p)$. As we saw previously, there is a natural way to make sense of $(y(P), 17)_{\mathbb{Q}_p}$ even when y has a zero or pole at P. In general, given a field K, a variety X/K and a quaternion algebra $A' = (f, g)_{K(X)}$ over $K(X)$, we cannot always expect to be able to make sense of the quaternion algebra $(f(P), g(P))_K$ when P is a zero or pole of either f or g. Roughly speaking, the Brauer group of X is the subgroup of $\mathrm{Br}K(X)$ consisting of elements which 'make sense over X'. A formal definition is the following.

Definition 2.10. Let X be a smooth quasi-projective variety over a field K. An *Azumaya algebra* over X is a locally free sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ such that for each $x \in X$ the fibre $\mathcal{A} \otimes_{\mathcal{O}_X} k(x)$ is a central simple algebra over the residue field $k(x)$. Two Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ are *similar* if there are vector bundles $\mathscr{E}$ and $\mathscr{E}'$ on X, of positive rank at each $x \in X$, such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathbf{End}_{\mathcal{O}_X}(\mathscr{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \mathbf{End}_{\mathcal{O}_X}(\mathscr{E}').$$

The set of similarity classes of Azumaya algebras over X forms an abelian group with multiplication $\otimes_{\mathcal{O}_X}$, identity $\mathcal{O}_X$ and inverse $\mathcal{A} \mapsto \mathcal{A}^{\mathrm{opp}}$. We call this the *Brauer group* of X, denoted $\mathrm{Br}(X)$.

Remark 2.11. For a general scheme X, what is defined above is the *Azumaya–Brauer group* of X, denoted $\mathrm{Br}_{\mathrm{Az}}(X)$. The *Brauer group* of X is defined as $H^2_{\text{ét}}(X, \mathbb{G}_m)$. For smooth quasi-projective varities these are canonically isomorphic.

The following result makes contact with our previous informal definition.

Theorem 2.12. *Let X be a smooth, integral, quasi-projective variety over a field K of characteristic* 0*. For each prime divisor D on X, denote by ∂_D the residue map*

$$\partial_D : \mathrm{Br}K(X) \longrightarrow H^1(k(D), \mathbb{Q}/\mathbb{Z})$$

defined in [Poo17, Section 6.8]*.*

Then we have an exact sequence

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}K(X) \xrightarrow{\oplus_D \partial_D} \bigoplus_D H^1(k(D), \mathbb{Q}/\mathbb{Z})$$

where the sum is taken over all prime divisors D on X.

Remark 2.13. With the assumptions of the theorem, let $(f,g)_{K(X)}$ be a quaternion algebra over $K(X)$. Note that for a prime divisor D on X we can identify $H^1(k(D), \mathbb{Q}/\mathbb{Z})[2]$ with $k(D)^\times/k(D)^{\times 2}$ via Kummer theory. Then for any prime divisor D on X we have

$$\partial_D(f,g)_{K(X)} = (-1)^{\mathrm{ord}_D(f)\mathrm{ord}_D(g)} f^{\mathrm{ord}_D(g)} g^{-\mathrm{ord}_D(f)} \in k(D)^\times/k(D)^{\times 2}.$$

Example 2.14. Returning to the running example $C : y^2 = 2x^4 - 34z^4$, consider the quaternion algebra $(y, 17) \in \mathrm{Br}\mathbb{Q}(C)$. Then $\mathrm{div}(y) = D_1 - 2D_2$ where D_1 is the divisor $\{y = 0\}$ on the affine chart $y^2 = 2x^4 - 34$. The affine chart $z = 0$ has equation $z^2 = 2 - 34t^4$ where $t = 1/x$ and $z = yt^2$. Then D_2 is the divisor $\{t = 0\}$. Write $A = (y, 17)_{\mathbb{Q}(C)}$ and consider $\partial_D(A)$ for a prime divisor D. Clearly $\partial_D(A) = 0$ for $D \neq D_1, D_2$. Further, since y vanishes to order 2 along D_2 we have $\partial_{D_2}(A) = 0$. To compute $\partial_{D_1}(A)$ we compute

$$\mathbb{Q}(D_1) = \frac{\mathbb{Q}[x,y]}{(y^2 - 2x^4 + 34, y)} \cong \mathbb{Q}[x]/(x^4 - 17).$$

Then $\partial_{D_1}(A)$ is the class of

$$(-1)^{\mathrm{ord}_{D_1}(17)\mathrm{ord}_{D_1}(y)} y^0 17^{-1} = 1/17$$

in $\mathbb{Q}(D_1)^\times/\mathbb{Q}(D_1)^{\times 2}$, which is trivial. Indeed, 17 is even a 4^{th} power in $\mathbb{Q}(D_1)^\times$.

2.4. **The Brauer–Manin obstruction.** Let $\mathbb{A}_\mathbb{Q}$ be the adele ring of $\mathbb{Q}$, that is

$$\mathbb{A}_\mathbb{Q} = \{(x_p)_p \in \prod_{p \le \infty} \mathbb{Q}_p \,:\, x_p \in \mathbb{Z}_p \text{ for all but finitely many } p\}.$$

For a variety $X/\mathbb{Q}$ we have an inclusion

$$X(\mathbb{Q}) \subseteq X(\mathbb{A}_\mathbb{Q}).$$

Remark 2.15. If X is projective then we have

$$X(\mathbb{A}_\mathbb{Q}) = \prod_{p \le \infty} X(\mathbb{Q}_p).$$

More generally, for any X we can find a finite set S of primes and a finite type separated scheme $\mathcal{X}$ over $\mathbb{Z}[S^{-1}]$ having generic fibre X. Then we have

$$X(\mathbb{A}_\mathbb{Q}) = \{(x_p)_p \in \prod_{p \le \infty} X(\mathbb{A}_p) \,:\, x_p \in \mathcal{X}(\mathbb{Z}_p) \text{ for all but finitely many } p\}.$$

Given $\alpha \in \mathrm{Br}X$ and $(x_p)_p \in X(\mathbb{A}_\mathbb{Q})$ we can evaluate α at x_p to get $\alpha(x_p) \in \mathrm{Br}\mathbb{Q}_p$. One can show that $\alpha(x_p) = 0$ for all but finitely many p (using that $\mathrm{Br}\mathbb{Z}_p = 0$) so we have a well-defined map

$$e_\alpha : X(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

sending $(x_p)_p$ to $\sum_{p\leq\infty} \mathrm{inv}_p \alpha(x_p)$. Defining

$$X(\mathbb{A}_\mathbb{Q})^\alpha = \big\{x \in X(\mathbb{A}_\mathbb{Q}) : e_\alpha(x) = 0\big\}$$

we have

$$X(\mathbb{Q}) \subseteq X(\mathbb{A}_\mathbb{Q})^\alpha \subseteq X(\mathbb{A}_\mathbb{Q}),$$

as can be seen from the commutative diagram

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r] \arrow[d] & X(\mathbb{A}_\mathbb{Q}) \arrow[d] & \\ \mathrm{Br}\mathbb{Q} \arrow[r] & \bigoplus_p \mathrm{Br}\mathbb{Q}_p \arrow[r, "\sum_p \mathrm{inv}_p"] & \mathbb{Q}/\mathbb{Z}. \end{tikzcd}$$

Definition 2.16. We define the Brauer–Manin set $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \bigcap_{\alpha\in\mathrm{Br}X} X(\mathbb{A}_\mathbb{Q})^\alpha$. Clearly $X(\mathbb{Q}) \subseteq X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \subseteq X(\mathbb{A}_\mathbb{Q})$. We say there is a Brauer–Manin obstruction to the Hasse principle for X if $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \emptyset$ but $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$. For a class of varieties $\mathscr{C}$ we say that the Brauer–Manin obstruction to the Hasse principle is the only one if, for each $X \in \mathscr{C}$, the implication $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \neq \emptyset \Rightarrow X(\mathbb{Q}) \neq \emptyset$ holds.

Example 2.17. We have seen that the curve $C : y^2 = 2x^4 - 34z^4$ has a Brauer–Manin obstruction to the Hasse principle, arising from the quaternion algebra $\alpha = (y, 17) \in \mathrm{Br}C$.

Remark 2.18. For a variety $X/\mathbb{Q}_p$ and $\alpha \in \mathrm{Br}X$, the evaluation map $X(\mathbb{Q}_p) \to \mathrm{Br}\mathbb{Q}_p$ is locally constant for the p-adic topology on $X(\mathbb{Q}_p)$. In particular, for a variety $X/\mathbb{Q}$, the set $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}$ is closed in $X(\mathbb{A}_\mathbb{Q})$.

Definition 2.19. We say that there is a Brauer–Manin obstruction to weak approximation for X if $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}X} \subsetneq X(\mathbb{A}_\mathbb{Q})$. In this case we have

$$\overline{X(\mathbb{Q})} \subseteq X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}X} \subsetneq X(\mathbb{A}_\mathbb{Q}).$$

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