

Computational Physics Assignment 1: N-Body Simulation

R. Abele^a and N. Matera^a

^a*Eberhard Karls University of Tuebingen*

Abstract

The following exercises showcase several integrators to solve the n-body problem. We were given initial conditions in the form of particle positions and velocities.

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1. Background

The following exercises showcase the implementation of several integrators to simulate the n-body problem. For simplicity, the total mass M as well as the gravitational constant G are set to 1. The coordinates and velocities of the particles are normalized so that the simulation is entered and does not drift over time.

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We were provided with initial conditions in the form of masses, positions, and velocities for $n = 2, 3, 100, 1000$ 3-dimensional particles.

2. The Program

The first part of our program, found under `01-nbody/code/data_normalization` directory, finds the center of mass and calculates the velocity of the center of mass. This information is then used to shift all of the positions and velocities of the particles so that the system does not drift over time.

The main part of the assignment, the actual simulation of planetary movement, can be found in `01-nbody/code/simulation`. This simulation works by first calculating the attractive forces between all of the objects and then determining by how much every particle should be moved in a given time step considering the particle's mass and current velocity. The velocities of the particles are also updated accordingly and the cycle repeats for the next time step.

Extrapolation of future particle positions and velocities using the known information can be performed with varying degrees of accuracy, varying not only the size of the time step, but also the integrator used.

2.1. Integrators

Integrators, as the name suggests integrate the changes in particle position over time and can be used to predict a future position. There are many kinds of integrators, but the ones implemented in this assignment are the *Euler*, *Euler-Cromer*, *velocity-Verlet*, *Heun*, *Hermite*, *iterated Hermite*, *Heun*, and *Runge-Kutta 4* integrators.

Implementing most of the integrators involves directly following the directions presented in the manual, but a small calculation involving the jerk and high-order

acceleration derivatives was required to determine the final form of the Hermite integrator. From the manual, we know that:

$$\frac{1}{2}a_n^{(2)} = -3\frac{a_n - a_{n+1}^p}{\Delta t^2} - \frac{2\dot{a}_n + \dot{a}_{n+1}^p}{\Delta t}, \quad (1a)$$

$$\frac{1}{6}a_n^{(3)} = 2\frac{a_n - a_{n+1}^p}{\Delta t^3} + \frac{\dot{a}_n + \dot{a}_{n+1}^p}{\Delta t^2}. \quad (1b)$$

Furthermore, we are given that the *corrected* positions and velocities calculated by the Hermite integrator can be written as follows:

$$v_{n+1}^c = v_{n+1}^p + \frac{1}{6}a_n^{(2)}\Delta t^3 + \frac{1}{24}a_n^{(3)}\Delta t^4 \quad (2a)$$

$$r_{n+1}^c = r_{n+1}^p + \frac{1}{24}a_n^{(2)}\Delta t^4 + \frac{1}{120}a_n^{(3)}\Delta t^5 \quad (2b)$$

Some rearranging and substitution results in the following for the corrected velocity and position:

$$v_{n+1}^c = v_{n+1}^p + \frac{1}{12}\left(-6a_n\Delta t + 6a_{n+1}^p\Delta t - 5\dot{a}_n\Delta t^2 - \dot{a}_{n+1}^p\Delta t^2\right) \quad (3a)$$

$$r_{n+1}^c = r_{n+1}^p + \frac{1}{60}\left(-9a_n\Delta t^2 + 9a_{n+1}^p\Delta t^2 - 7\dot{a}_n\Delta t^3 - 2\dot{a}_{n+1}^p\Delta t^3\right). \quad (3b)$$

A modified version of the Hermite integrator, known as the iterated Hermite integrator can be achieved by first calculating the corrected velocity and using it to calculate the predicted position, creating a set of equations with the following form:

$$v_{n+1}^c = v_n + \frac{1}{2}\left(a_{n+1}^p + a_n\right)\Delta t + \frac{1}{12}\left(\dot{a}_{n+1}^p - \dot{a}_n\right)\Delta t^2 \quad (4a)$$

$$r_{n+1}^c = r_n + \frac{1}{2}(v_{n+1}^c + v_n)\Delta t + \frac{1}{12}(a_{n+1}^p - a_n)\Delta t^2 \quad (4b)$$

3. Results

3.1. Two-Body Calculations

The assignment requires us to calculate and plot the following quantities for the two-body system as a function of time:

$$\log\left|\frac{E - E^{\text{start}}}{E^{\text{start}}}\right|, \log\left|\frac{\mathbf{e} - \mathbf{e}^{\text{start}}}{\mathbf{e}^{\text{start}}}\right|, \log\left|\frac{a_e - a_e^{\text{start}}}{a_e^{\text{start}}}\right|, \quad (5)$$

with E , \mathbf{e} , and a_e being the energy, Runge-Lenz vector, and semi-major axis respectively. The plotting of \mathbf{e} , however, required a slight modification of the following form:

$$\log\left|\frac{\mathbf{e} - \mathbf{e}^{\text{start}}}{\mathbf{e}^{\text{start}}}\right| \rightarrow \log|\mathbf{e} - \mathbf{e}^{\text{start}}|. \quad (6)$$

This modification was made to avoid division by zero when calculating the Runge-Lenz vector of a circular system (as is the case for the given two-body system).

After this small explanation for divergence from the given procedure, we proceed by presenting our findings.

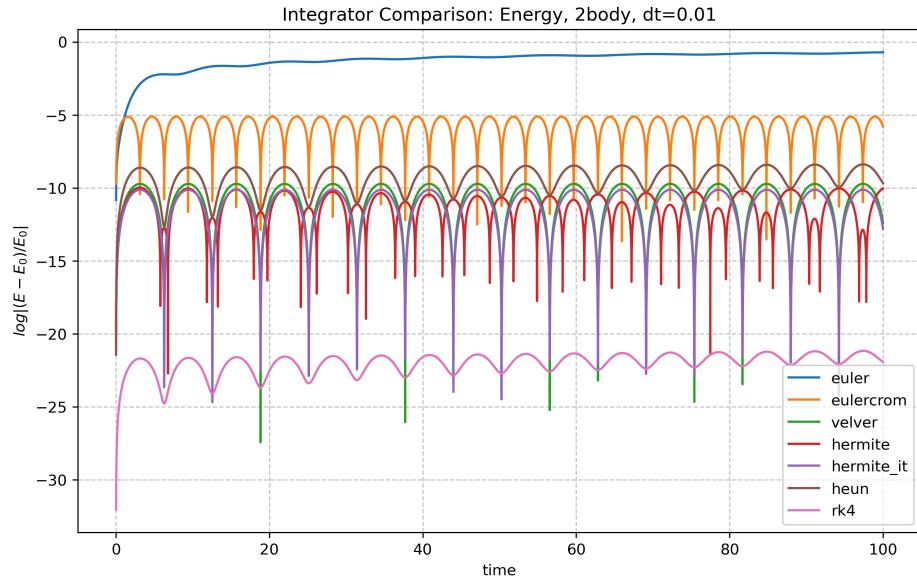


Figure 1: Plot of relative energy change over time in a two-body system with a time step of 0.01.

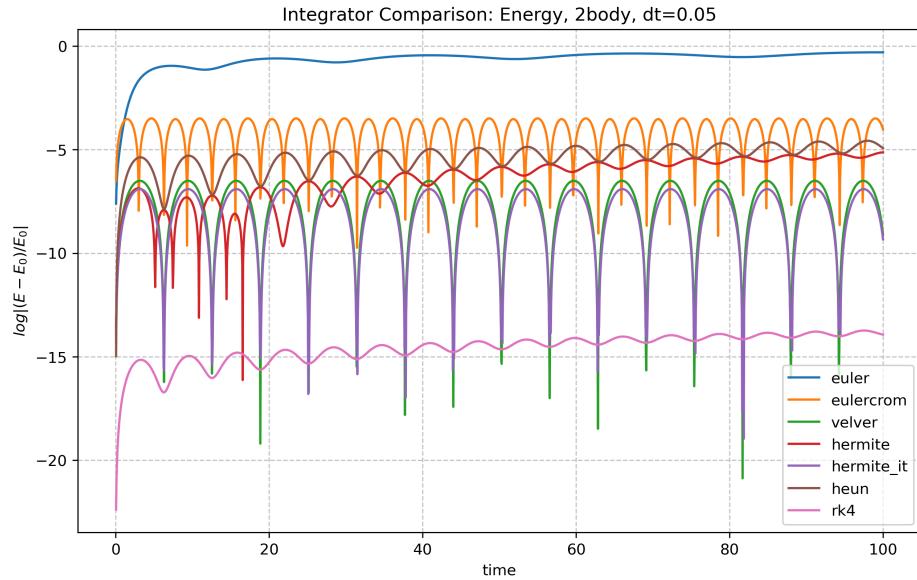


Figure 2: Plot of relative energy change over time in a two-body system with a time step of 0.05.

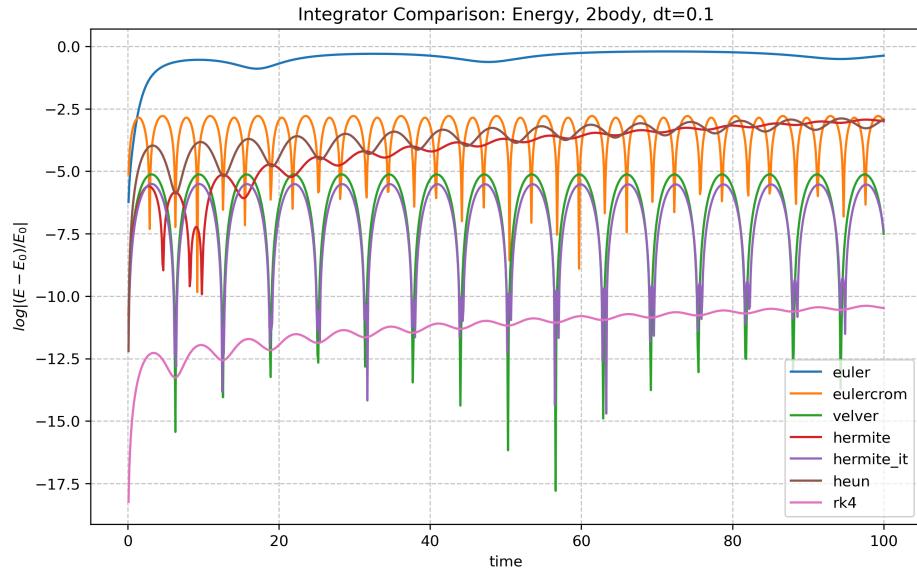


Figure 3: Plot of relative energy change over time in a two-body system with a time step of 0.1.

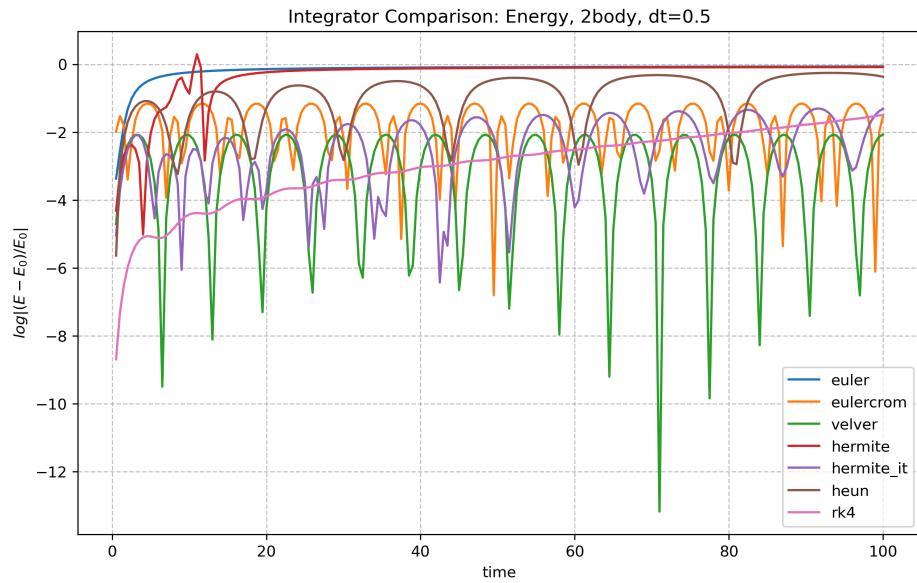


Figure 4: Plot of relative energy change over time in a two-body system with a time step of 0.5.

When comparing the relative change energy in the above plots, one can see that variations in relative energy change are smaller for more accurate integrators, particularly Runge-Kutta 4, and for smaller time steps.

Simply comparing figures 1 and 2, one can see that the scaling of the y-axis already differs by ten orders of magnitude. Looking at the results for the Euler-Cromer integrator as a specific example, one can see that the smaller time step of 0.01 results in a reduced peak variation of about 2 orders of magnitude.

In addition to these results, we also see that symplectic integrators, specifically Euler-Cromer and Velocity-Verlet, do not become less accurate over time. Even the generally most accurate integrator, Runge-Kutta 4, will show an increase in the energy change over time, eventually exceeding even the simple Euler-Cromer integrator. This takes a long time though and results are amplified for larger time steps, as in figures 3 and 4.

For a two-body system, it is also considered worthwhile to look at the eccentricity (or the Runge-Lenz vector) and the size of the semi-major axis of the system as a measure of simulation accuracy, denoted as \mathbf{e} and a_e here.

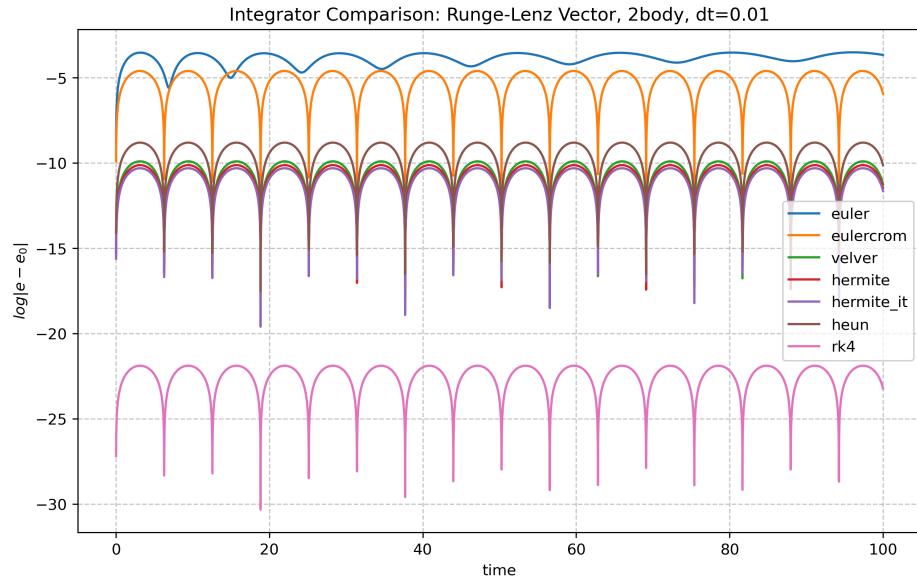


Figure 5: Plot of Runge-Lenz vector change over time in a two-body system with a time step of 0.01.

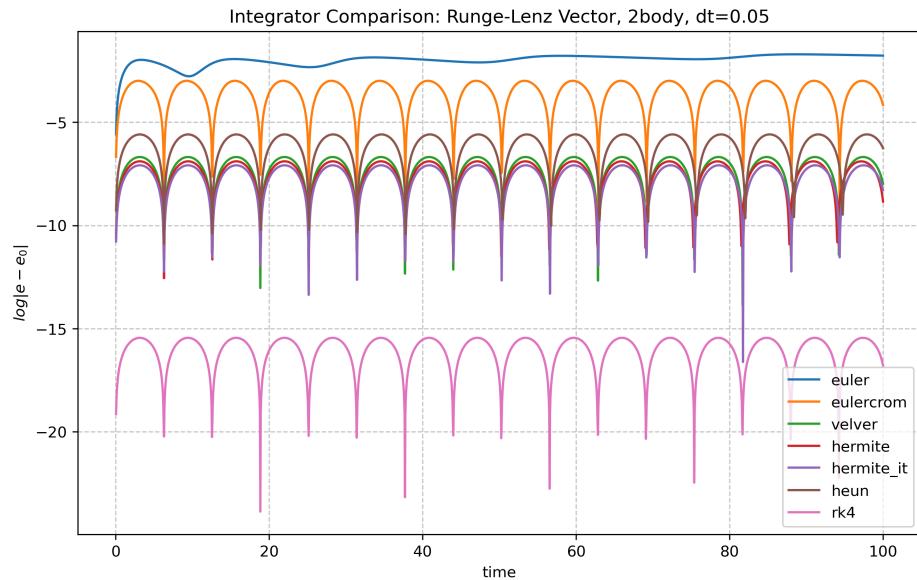


Figure 6: Plot of Runge-Lenz vector change over time in a two-body system with a time step of 0.05.

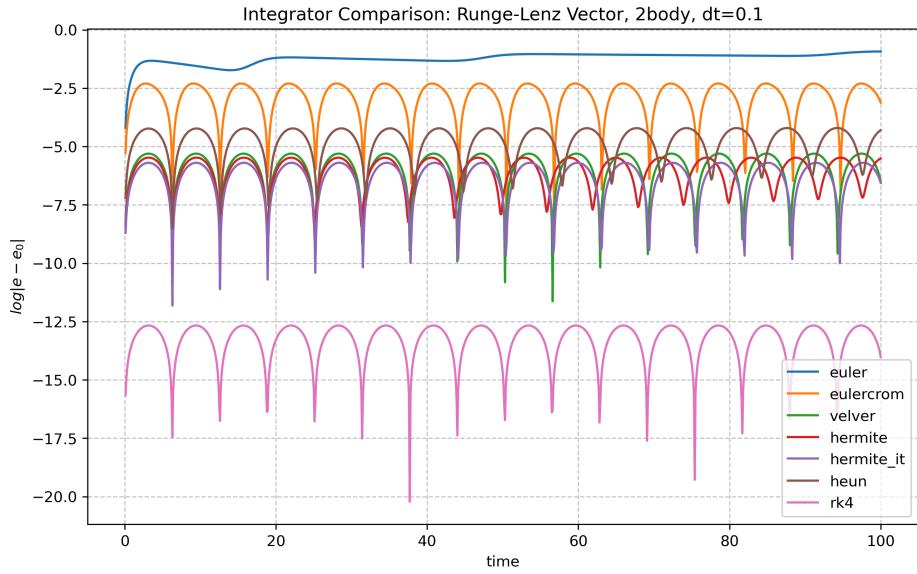


Figure 7: Plot of Runge-Lenz vector change over time in a two-body system with a time step of 0.1.

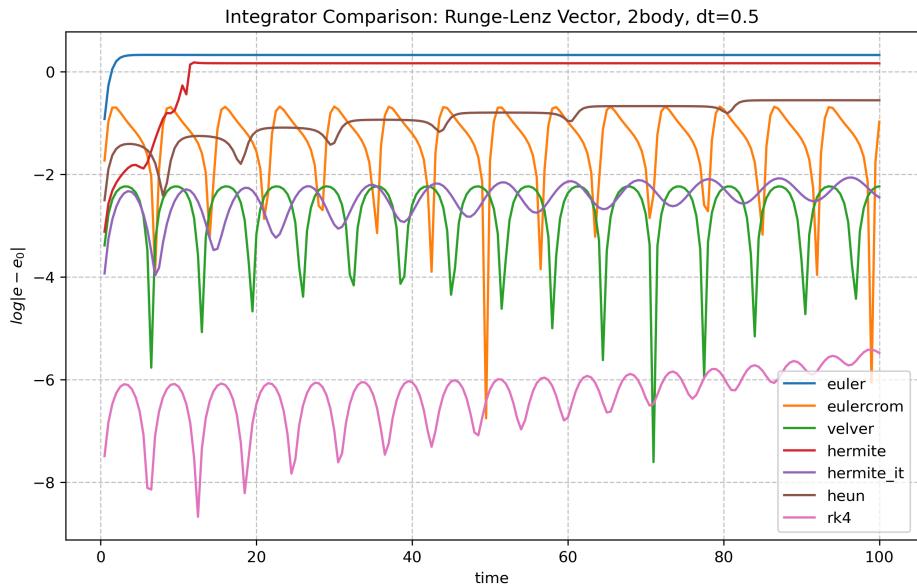


Figure 8: Plot of Runge-Lenz vector change over time in a two-body system with a time step of 0.5.

Looking at the above plots of the change in Runge-Lenz vector over time for different time steps, figures 5, 6, 7, and 8, we once again see an increase in accuracy with the reduction of time step size. Interestingly, the symplectic integrators also seem to keep variation over time constant as well, preventing the slow drift seen in non-symplectic integrators.

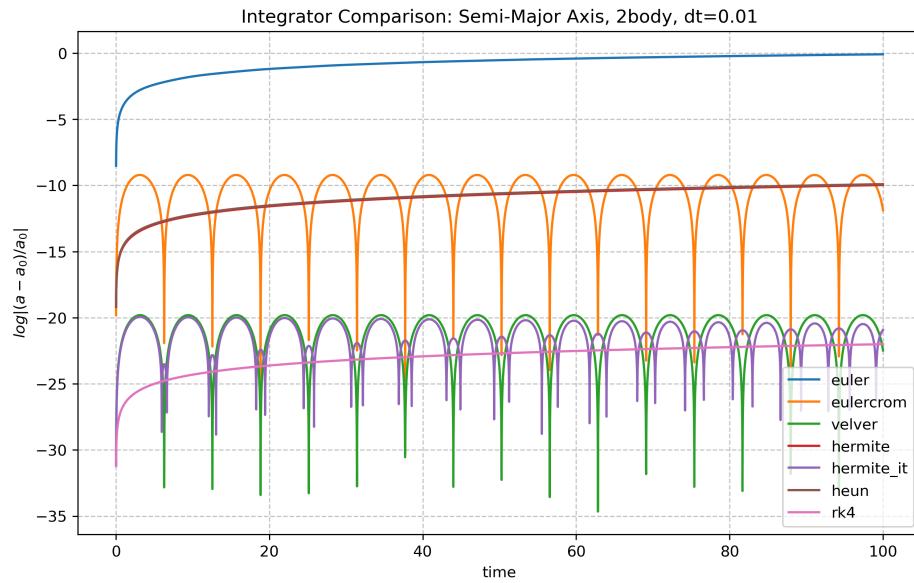


Figure 9: Plot of semi-major axis change over time in a two-body system with a time step of 0.01.

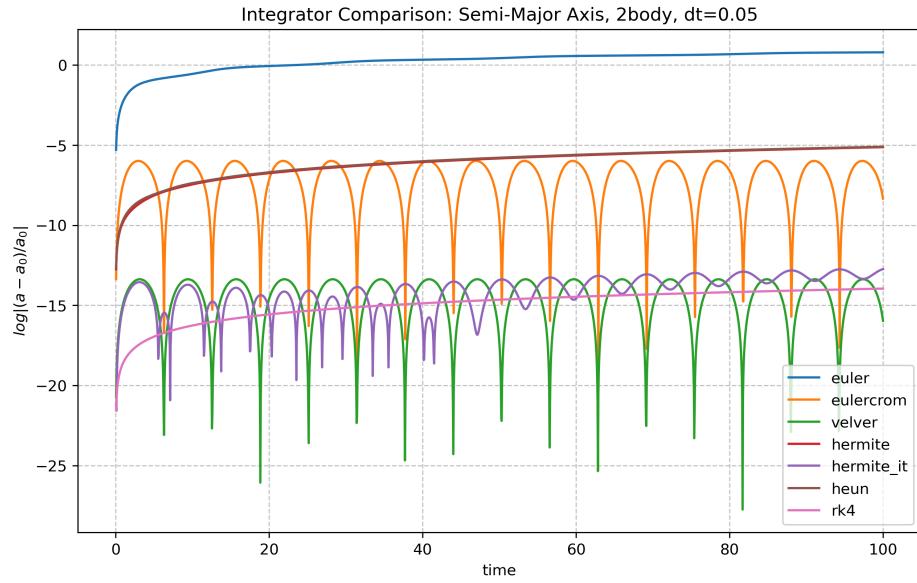


Figure 10: Plot of semi-major axis change over time in a two-body system with a time step of 0.05.

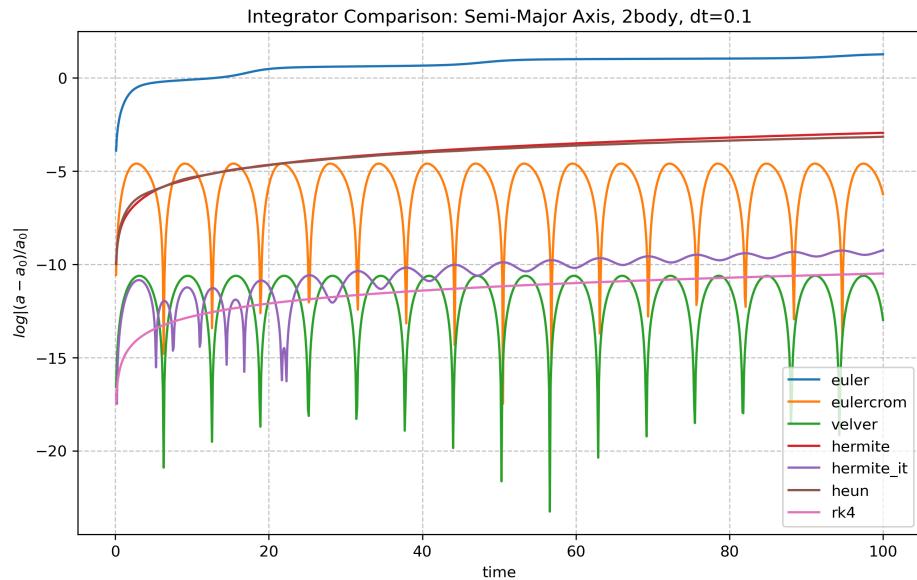


Figure 11: Plot of semi-major axis change over time in a two-body system with a time step of 0.1.

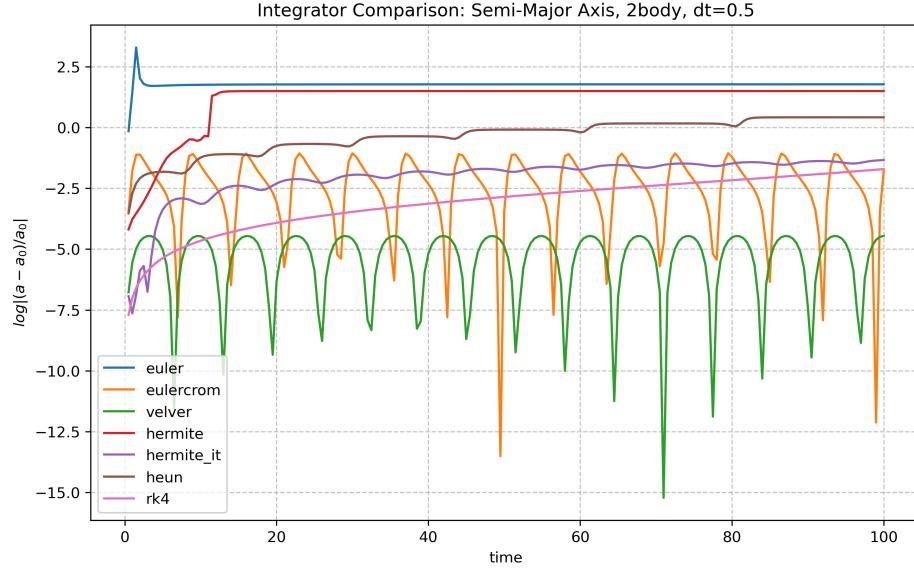


Figure 12: Plot of semi-major axis change over time in a two-body system with a time step of 0.5.

Looking at the results in figures 9, 10, 11, and 12, we see similar results as for the energy and Runge-Lenz vector over time.

Smaller time steps and more accurate integrators once again generally yield results with smaller changes in the semi-major axis over time, while the symplectic integrators maintain the same oscillations over time, avoiding drift.

3.2. N-Body Calculations

For the *N-Body* part of this assignment, we need only plot the logarithm of the relative change in energy as a function of time:

$$\log \left| \frac{E - E^{\text{start}}}{E^{\text{start}}} \right|. \quad (7)$$

We begin by looking at the data from a three-body system.

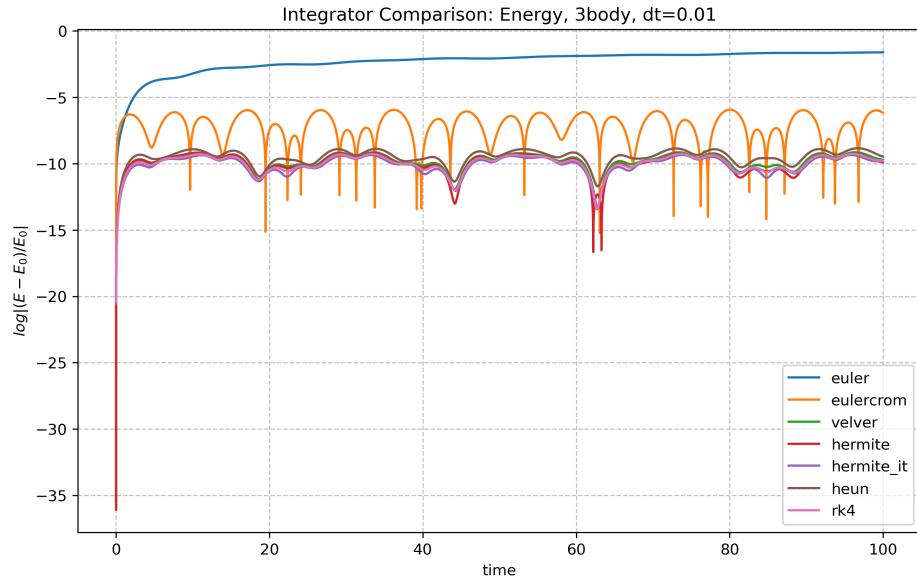


Figure 13: Plot of relative energy change over time in a three-body system with a time step of 0.01.

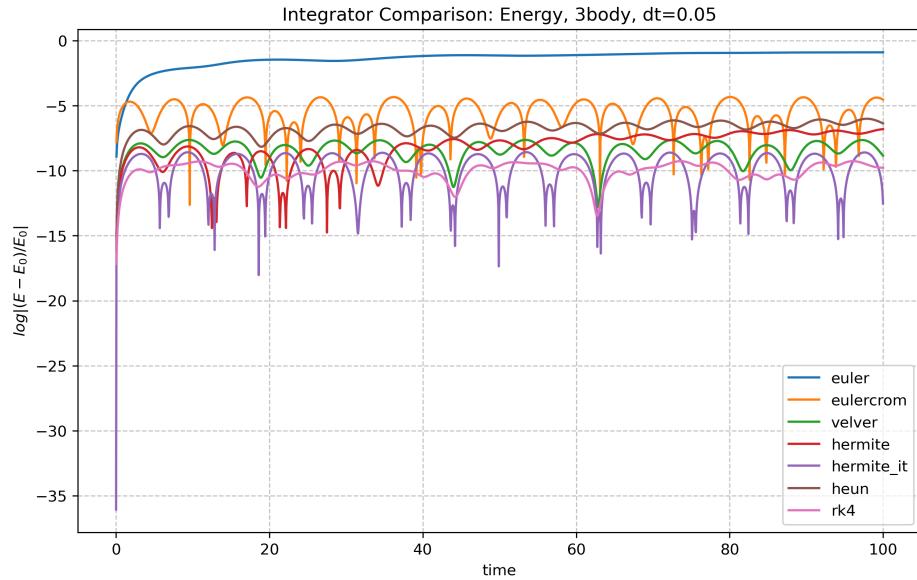


Figure 14: Plot of relative energy change over time in a three-body system with a time step of 0.05.

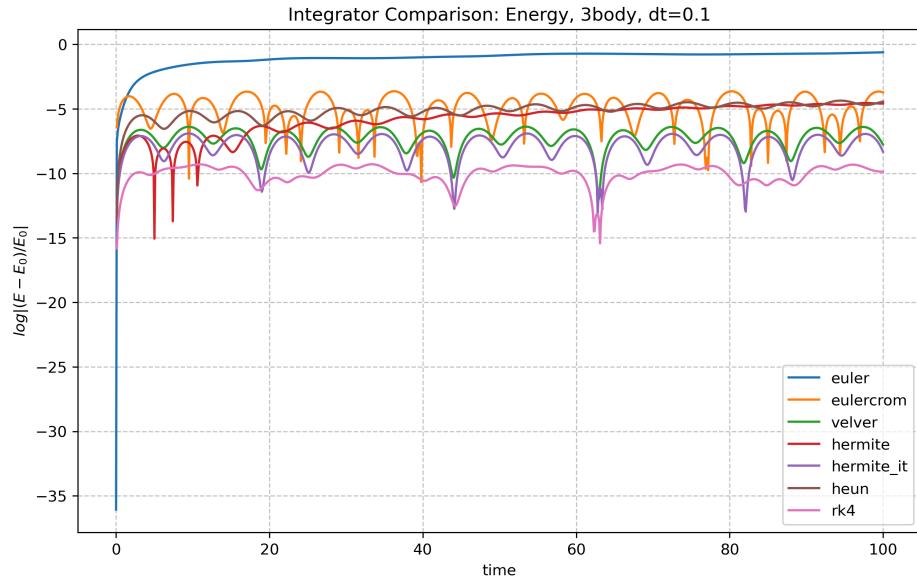


Figure 15: Plot of relative energy change over time in a three-body system with a time step of 0.1.

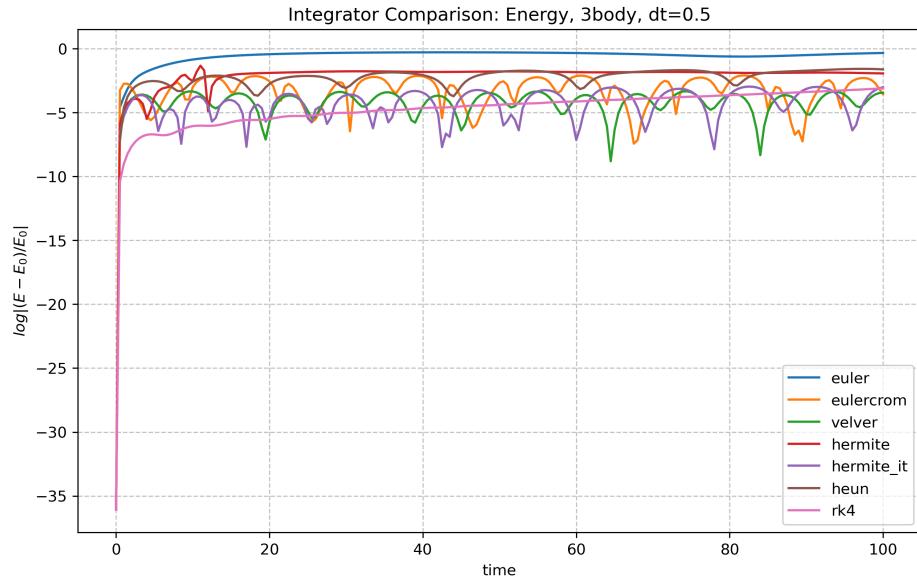


Figure 16: Plot of relative energy change over time in a three-body system with a time step of 0.5.

Looking at figure 13, we see that there is once again an oscillating pattern for relative change in energy for all except the simple Euler integrator. Looking at the plots with a greater time step, figures 14, 15, and 16, we see the performance differences of the integrators more clearly and the slow increase in error of all except the symplectic integrators.

We now continue with analysis of the one hundred-body system.

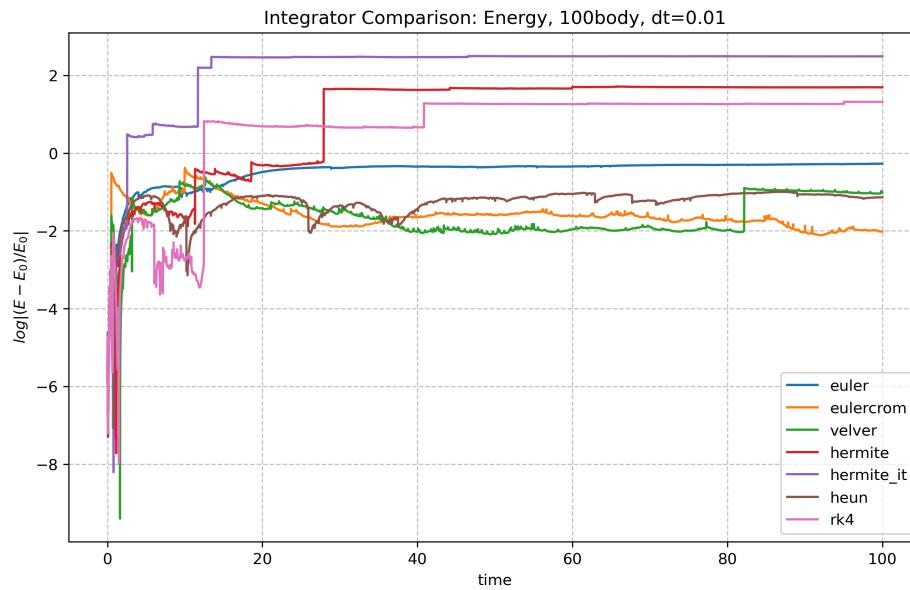


Figure 17: Plot of relative energy change over time in a one hundred-body system with a time step of 0.01.

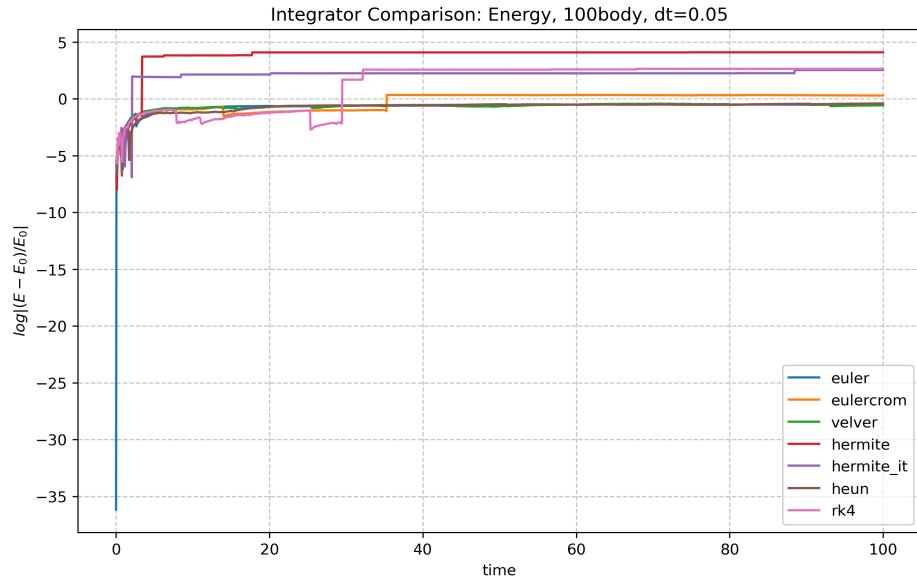


Figure 18: Plot of relative energy change over time in a one hundred-body system
with a time step of 0.05.

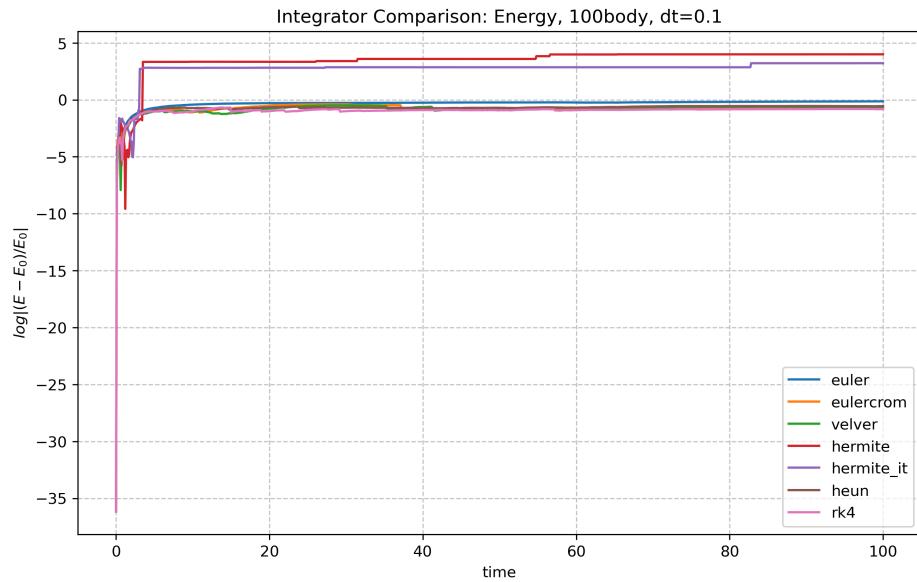


Figure 19: Plot of relative energy change over time in a one hundred-body system
with a time step of 0.1.

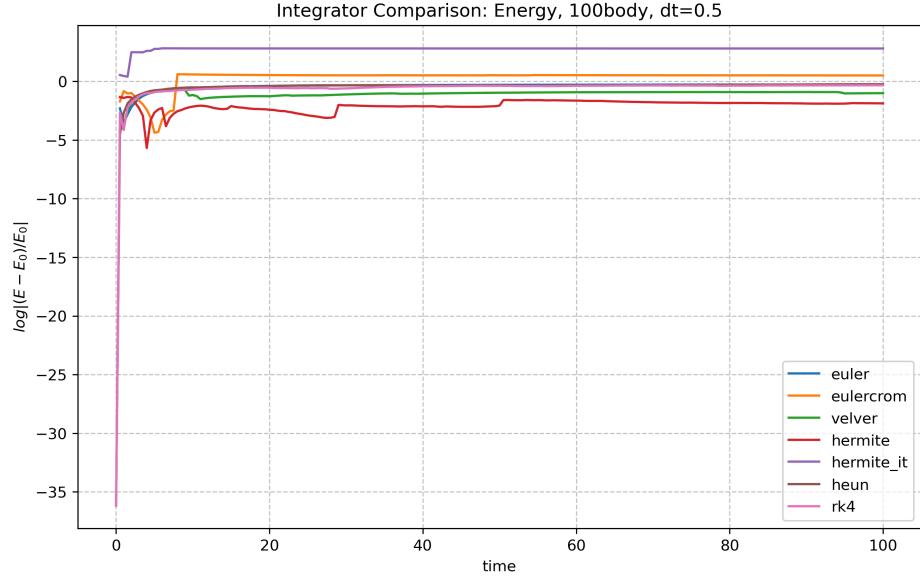


Figure 20: Plot of relative energy change over time in a one hundred-body system with a time step of 0.5.

As can be seen in the plots of the one hundred-body system above for varying time steps (see figures 17, 18, 19, 20), energy is not predictably conserved in the one hundred-body system. Accuracy in energy generally falls with time, as expected, but it seems to often increase in ‘explosive’ leaps, as seen most clearly in figure 17. These sudden increases in the relative energy hint at a limit with such simple n-body simulations - when masses are very close together, the force that they exert on each other increases in proportion to $\frac{1}{r}$. This can occasionally produce unrealistic results since the collision of planets is not accounted for and they can move extremely close to each other.

Another surprising result seen in the data for the one hundred-body simulation is that the precision of the various integrators does not seem to follow the expected pattern of Euler being the worst and RK4 being the best. We are not sure if this is an implementation error or a fact of the complexity presented in a one hundred-

body simulation. It should be noted that the symplectic operators Euler-Cromer and Velocity-Verlet once again seem to do a decent job preserving energy over time, though even this does not follow this nice cyclic patterns observed in the two and three-body cases.

Similar results follow for the data of the one thousand-body simulation.

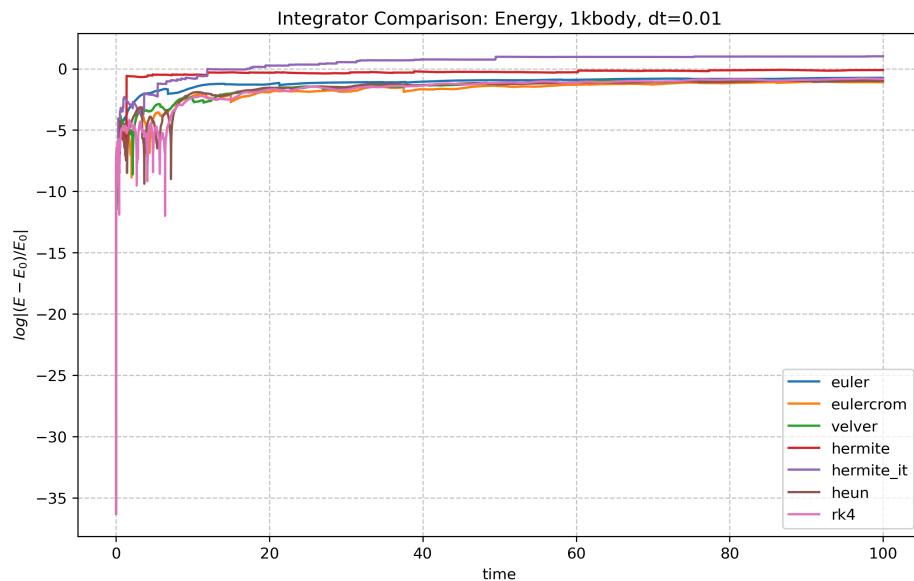


Figure 21: Plot of relative energy change over time in a one thousand-body system with a time step of 0.01.

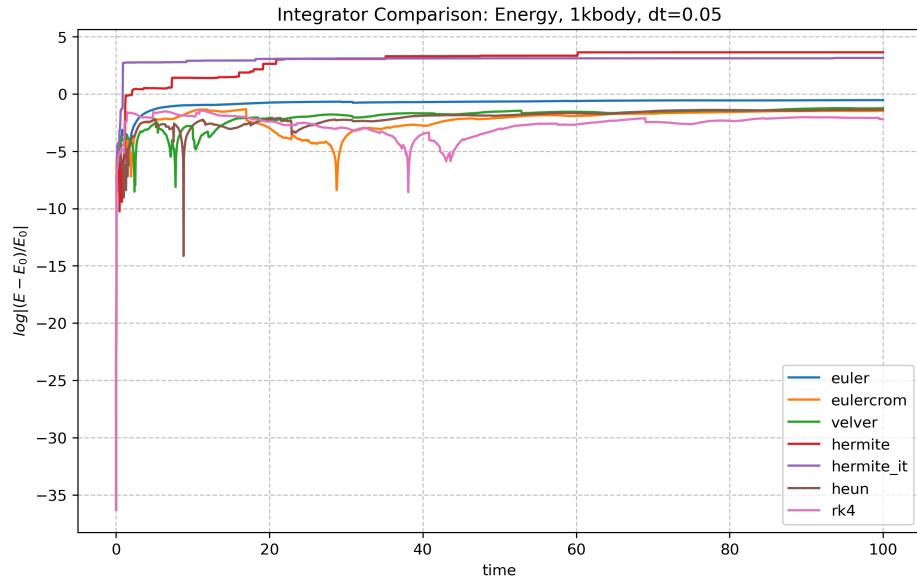


Figure 22: Plot of relative energy change over time in a one thousand-body system
with a time step of 0.05.

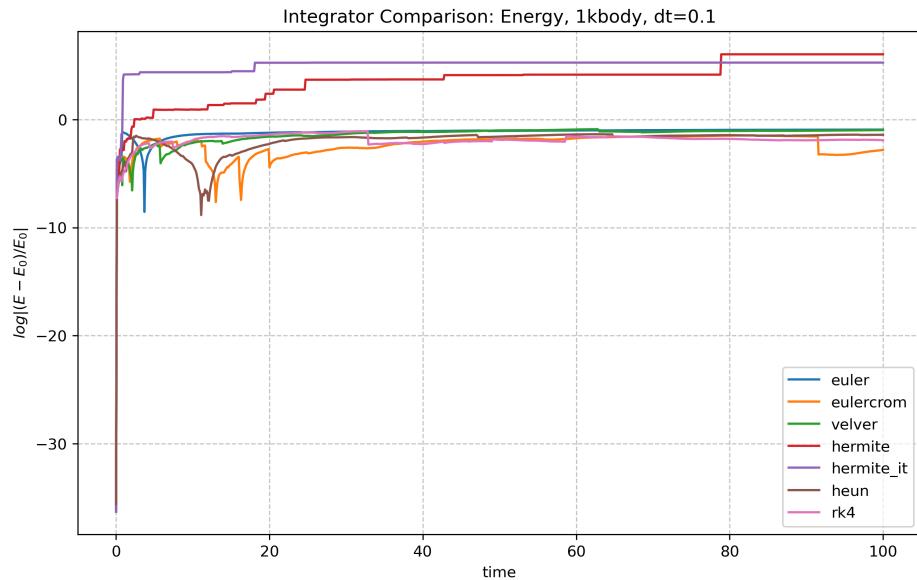


Figure 23: Plot of relative energy change over time in a one thousand-body system
with a time step of 0.1.

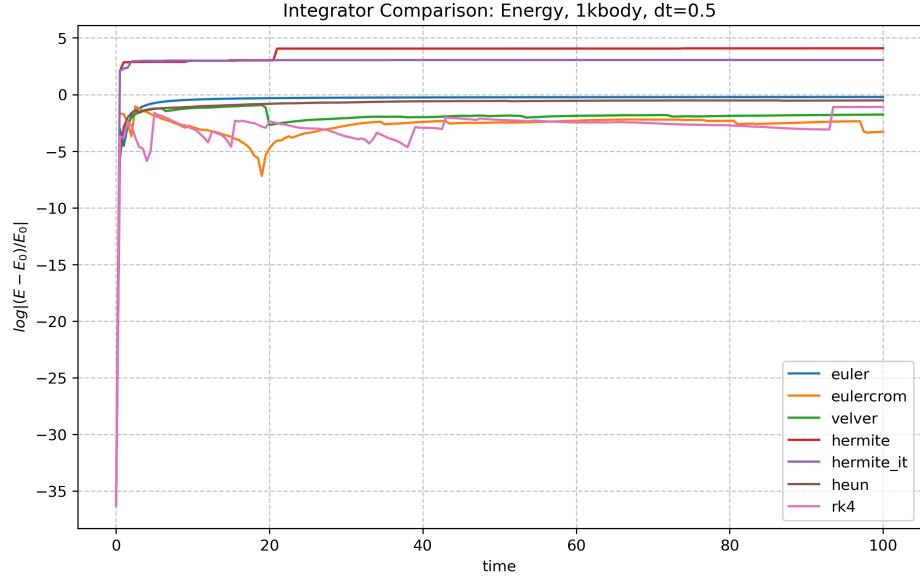


Figure 24: Plot of relative energy change over time in a one thousand-body system with a time step of 0.5.

Looking at the above figures 21, 22, 23, and 24 for the relative energy change of the one thousand-body system over time, we once again see sudden increases in energy difference and no clear pattern of which integrator is more accurate.

In the description of the assignment, we are also asked to discuss observed increase in computation required for running the simulations with an increasing number of particles N . Though we did not take exact measurements, the results seemed to follow the expected N^2 proportionality for calculation time, with the one thousand-body system taking about one hundred times longer than the one hundred-body system (corresponding directly to the ten-fold increase in particles).

4. Summary

All in all, the results of the simulation generally followed expected trends. Symplectic integrators generally minimized energy drift over time, though this

pattern did not seem to be very consistent for the largest systems. The simplest integrator, the Euler integrator usually performed the worst, while the most wide-spread integrator, the Runge-Kutta 4 integrator, performed the best.

Most of the results followed expected trends, though the one hundred and one thousand-body simulations seemed to show some irregularities, likely caused by particles getting too close to each other and exerting non-realistic forces.

In conclusion, this lab demonstrated the usefulness of various integrators and some of the shortcomings that they may come with.