Computational Physics Labwork: Numerical Hydrodynamics

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Hydrodynamical Equations

The Euler equations in hydrodynamics read in the conservative form

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 && (1)\\
\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) &= -\nabla p + \rho \mathbf{k} && (2)\\
\frac{\partial (\rho \epsilon)}{\partial t} + \nabla \cdot (\rho \epsilon \mathbf{u}) &= -p \nabla \cdot \mathbf{u} && (3)
\end{aligned}$$

$\mathbf{u}$: velocity, $\mathbf{k}$: specific external forces, ϵ specific internal energy
These equations describe the conservation of mass, momentum and energy.
The closure condition is given by the equation of state (EOS)

$$p = (\gamma - 1)\, \rho \epsilon. \tag{4}$$

Using the EOS, we can also reformulate the energy equation (3) in an equation for the pressure

$$\frac{\partial p}{\partial t} + \nabla \cdot (p \mathbf{u}) = -(\gamma - 1) p\, \nabla \cdot \mathbf{u}. \tag{5}$$

Write out the divergence terms on the left hand side and use the continuity equation for the momentum and energy equation to obtain

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\rho = -\rho\, \nabla \cdot \mathbf{u} \tag{6}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\, \nabla p + \mathbf{k} \tag{7}$$

$$\frac{\partial p}{\partial t} + (\mathbf{u} \cdot \nabla)p = -\gamma p \nabla \cdot \mathbf{u}. \tag{8}$$

Since all terms are functions of the location ($\mathbf{r}$) and time (t), e.g., the density $\rho(\mathbf{r}, t)$, we can use the total derivative on the left hand side.
For the continuity equation, we get

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\rho = -\rho\, \nabla \cdot \mathbf{u}. \tag{9}$$

The operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \tag{10}$$

is called Lagrangian derivative or material derivative and corresponds to the total time derivative d/dt.

Now with the help of the Lagrangian derivative, we can write

$$
\begin{aligned}
\frac{D\rho}{Dt} &= -\rho \nabla \cdot \mathbf{u} && (11) \\
\frac{D\mathbf{u}}{Dt} &= -\frac{1}{\rho} \nabla p + \mathbf{k} && (12) \\
\frac{Dp}{Dt} &= -\gamma p \nabla \cdot \mathbf{u} && (13)
\end{aligned}
$$

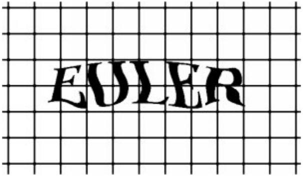
These equations describe the evolution of the quantities in a reference frame that moves with the flow.
This so-called Lagrangian representation is quite useful for radial star oscillations, which is a 1D problem with moving mass shells.

Numerical Hydrodynamics: Challenge

Consider the full evolution of the time dependent hydrodynamical equations.
The non-linear partial differential equations are solved numerically
from continuum to disretisation

How to solve the hydro eqs...

grid schemes (Euler)



fixed grid
- flow through grid cells

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p$$

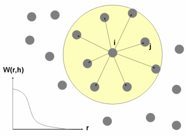
Methods:
- Finite Differences
  no conservation properties
- Finite Volume
  conservation properties
- Riemann-Solver
  wave properties
- Problem: Discontinuities

particle schemes (Lagrange)

moving grid or particles
- fluid moves the grid or the particles

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p$$

Methods: Smoothed Particle Hydrodynamics



'smeared-out particles'
useful for open boundaries, self-gravity

Consider the 1D Euler eqs

They describe the conservation of mass, momentum and energy

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{14}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} = -\frac{\partial p}{\partial x} \tag{15}$$

$$\frac{\partial(\rho \epsilon)}{\partial t} + \frac{\partial(\rho \epsilon u)}{\partial x} = -p\frac{\partial u}{\partial x}. \tag{16}$$

ρ: density
u: velocity
p: pressure
ϵ: internal specific energy (energy/mass)

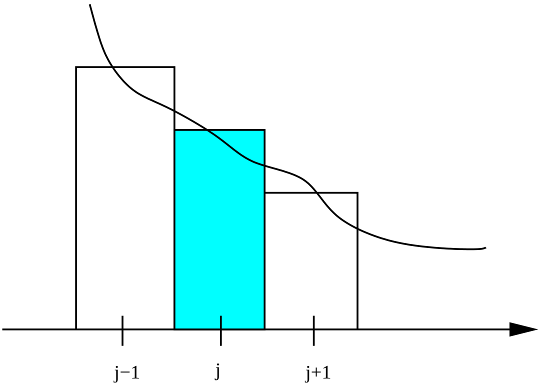
With EOS

$$p = (\gamma - 1)\rho\epsilon \tag{17}$$

γ: adiabatic exponent

PDE in both time and space

$\rightarrow$ Discretisation is required for both time and space!



Consider the function: $\psi(x, t)$
discretise in space
with the grid

$$\Delta x = \frac{x_{max} - x_{min}}{N}$$

ψ_j^n denotes the value in cell center of $\psi(x, t)$ at gridpoint x_j at time t^n

$$\psi_j^n = \psi(x_j, t^n) \approx \frac{1}{\Delta x} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \psi(x, n\Delta t) dx$$

ψ_j^n is piecewise constant. j spatial index, n timestep.

Consider the equation

$$\frac{\partial \psi}{\partial t} = \mathcal{L}(\psi(x,t)) \tag{18}$$

with the spatial differential operator $\mathcal{L}$.
Usual discretisation (1st order in time), at time $t = t^n = n\Delta t$

$$\frac{\partial \psi}{\partial t} \approx \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t} = \frac{\psi^{n+1} - \psi^n}{\Delta t} = L(\psi^n). \tag{19}$$

Now at the location of gridpoint x_j

$$\psi_j^{n+1} = \psi_j^n + \Delta t L(\psi_k^n). \tag{20}$$

$L(\psi_k^n)$: discretised differential operator $\mathcal{L}$ (here explicit) - k in $L(\psi_k)$: set of spatial indices
- e.g., for a 2nd order scheme $k \in \{j-2, j-1, j, j+1, j+2\}$
(information from left and right of the grid point is required, 5-point stencil)

$$\frac{\partial \mathbf{A}}{\partial t} = \mathcal{L}_1(\mathbf{A}) + \mathcal{L}_2(\mathbf{A}) \tag{21}$$

$\mathcal{L}_i(\mathbf{A}), i = 1, 2$ are single differential operators acting on values $\mathbf{A} = (\rho, u, \epsilon)$.
For ideal 1D hydro

$$\mathcal{L}_1 : \text{advection}$$

$$\mathcal{L}_2 : \text{pressure, external forces}$$

Divided into several substeps

$$\begin{aligned} \mathbf{A}^1 &= \mathbf{A}^n + \Delta t L_1(\mathbf{A}^n) \\ \mathbf{A}^{n+1} = \mathbf{A}^2 &= \mathbf{A}^1 + \Delta t L_2(\mathbf{A}^1) \end{aligned} \tag{22}$$

L_i is the differential operator to $\mathcal{L}_i$.

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\frac{\partial(\rho u)}{\partial x} \\
\frac{\partial(\rho u)}{\partial t} &= -\frac{\partial(\rho u u)}{\partial x} \\
\frac{\partial(\rho \epsilon)}{\partial t} &= -\frac{\partial(\rho \epsilon u)}{\partial x}
\end{aligned}$$

In conserved form

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial f(\mathbf{u})}{\partial x} = 0 \tag{23}$$

It is for $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{f} = (f_1, f_2, f_3)$:

$\mathbf{u} = (\rho, \rho u, \rho\epsilon)$ und $\mathbf{f} = (\rho\, u, \rho u u, \rho \epsilon u)$.

This first advection step yields: $\rho^n \to \rho^1 = \rho^{n+1}, \quad u^n \to u^1, \quad \epsilon^n \to \epsilon^1$

Conservation of momentum

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{24}$$

$$u_j^{n+1} = u_j - \Delta t \frac{1}{\bar{\rho}_j^{n+1}} \frac{(p_j - p_{j-1})}{\Delta x} \quad \text{for} \quad j = 2, \ldots, N \tag{25}$$

Conservation of energy

$$\frac{\partial \epsilon}{\partial t} = -\frac{p}{\rho}\frac{\partial u}{\partial x} \tag{26}$$

$$\epsilon_j^{n+1} = \epsilon_j - \Delta t \frac{p_j}{\rho_j^{n+1}} \frac{(u_{j+1} - u_j)}{\Delta x} \quad \text{for} \quad j = 1, \ldots, N \tag{27}$$

on the right hand side, we use the current values for u, ϵ and p, which are u^1, p^1, ϵ^1.

This step yields: $u^1 \to u^{n+1}$, $\epsilon^1 \to \epsilon^{n+1}$

The continuity equation reads

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \tag{28}$$

where $F^m = \rho u$ denotes the mass flux.

Using $\rho \to \psi$ and a constant speed $u \to a = \mathrm{const.}$, one obtains the linear advection eq

$$\frac{\partial \psi}{\partial t} + a\frac{\partial \psi}{\partial x} = 0. \tag{29}$$

For a constant velocity a, the solution is given by a wave moving to the right

$$\text{with} \quad \psi(x, t = 0) = f(x) \quad \text{it is.} \quad \psi(x, t) = f(x - at)$$

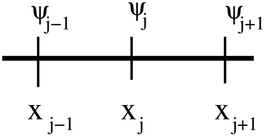
Here, $f(x)$ denotes the initial condition at time $t = 0$, which gets transported by advection with constant velocity a.

Numerical Hydrodynamics: Linear advection

FTCS: *Forward Time Centered Space* scheme

$$\frac{\partial \psi}{\partial t} + a \frac{\partial \psi}{\partial x} = 0 \tag{30}$$

with grid



we write

$$\left.\frac{\partial \psi}{\partial t}\right|_j^n = \frac{\psi_j^{n+1} - \psi_j^n}{\Delta t} \tag{31}$$

$$\left.\frac{\partial \psi}{\partial x}\right|_j^n = \frac{\psi_{j+1}^n - \psi_{j-1}^n}{2\,\Delta x}, \tag{32}$$

and it follows

$$\psi_j^{n+1} = \psi_j^n - \frac{a \Delta t}{2 \Delta x} \left(\psi_{j+1}^n - \psi_{j-1}^n \right). \tag{33}$$

Seems well justified but is unstable for all time step sizes Δt!

Upwind scheme I

$$\frac{\partial \psi}{\partial t} + \frac{\partial a \psi}{\partial x} = 0 \qquad (34)$$

oder

$$\frac{\partial \psi}{\partial t} + a \frac{\partial \psi}{\partial x} = 0 \qquad (35)$$

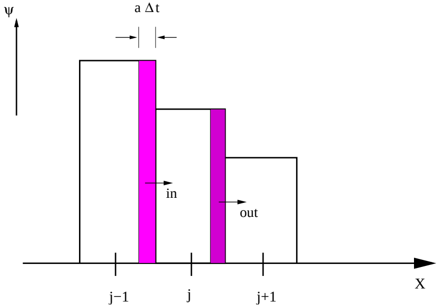
a: constant (velocity) > 0
$\psi(x, t)$, arbitrary transport value

Change of ψ in cell j

$$\psi_j^{n+1} \Delta x = \psi_j^n \Delta x + \Delta t (F_{in} - F_{out}). \qquad (36)$$

The flux F_{in} for constant ψ_j is given by

$$F_{in} = a\, \psi_{j-1}^n \qquad (37)$$
$$F_{out} = a\, \psi_j^n. \qquad (38)$$



Pink regions are transported in neighbouring cells.

Upwind scheme Information comes from upstream

Extension for non-constant states

$$F_{in} = a\,\psi_I\left(x_{j-1/2} - \frac{a\Delta t}{2}\right). \qquad (39)$$

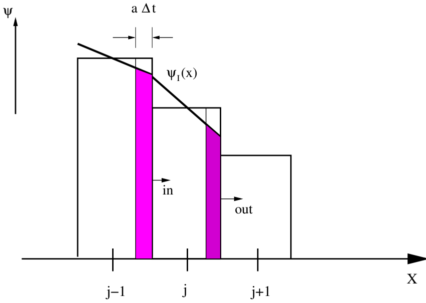
$\psi_I(x)$ polynomial of interpolation

Using linear interpolation (just a line)

$$\underbrace{\underbrace{F_{in} = a\left[\psi^n_{j-1}\right.}_{\text{1st order}} \left. + \frac{1}{2}(1-\sigma)\Delta\psi_{j-1}\right]}_{\text{2nd order}} \qquad (40)$$

with $\sigma \equiv a\Delta t/\Delta x$

$$\Delta\psi_j \approx \left.\frac{\partial\psi}{\partial x}\right|_{x_j} \Delta x$$



$$\psi_I(x) = \psi^n_j + \frac{x - x_j}{\Delta x}\Delta\psi_j \quad (41)$$

$\Delta\psi_j$ some approximation to the derivative, see next slide

2nd order upwind

$\psi_I(x)$ is calculated in the centers of the pink regions

Numerical Hydrodynamics: Approximations for the derivative

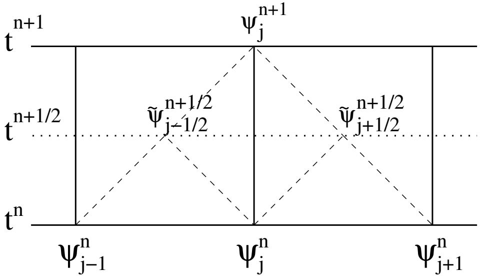
Different schemes:

a) $\Delta\psi_j = 0$ upwind, 1st order, piecewise constant
b) $\Delta\psi_j = \frac{1}{2}\left(\psi_{j+1} - \psi_{j-1}\right)$ Fromm, centred derivative
c) $\Delta\psi_j = \psi_j - \psi_{j-1}$ Beam-Warming, upwind derivative
d) $\Delta\psi_j = \psi_{j+1} - \psi_j$ Lax-Wendroff, downwind derivative

Often 2nd order upwind (van Leer scheme) Geometric mean (conserves monotonicity)

$$\Delta\psi_j = \begin{cases} 2\frac{(\psi_{j+1}-\psi_j)(\psi_j-\psi_{j-1})}{(\psi_{j+1}-\psi_{j-1})} & \text{if} \quad (\psi_{j+1}-\psi_j)(\psi_j-\psi_{j-1}) > 0 \\ 0 & \text{else} \end{cases} \tag{42}$$

The derivatives are calculated at the specific substep of the timestep, see below.



Schematic overview for Lax-Wendroff

makes use of spatial and time centred differences

2nd order in space and time

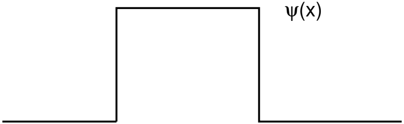
Doing two steps:

Predictor step (at time $t^{n+1/2}$)

$$\tilde{\psi}_{j+1/2}^{n+1/2} = \frac{1}{2}\left(\psi_j^n + \psi_{j+1}^n\right) - \frac{\sigma}{2}\left(\psi_{j+1}^n - \psi_j^n\right). \tag{43}$$

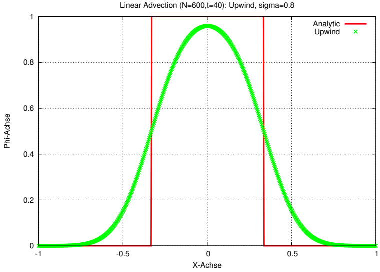
Followed by corrector step (to time t^{n+1})

$$\psi_j^{n+1} = \psi_j^n - \sigma\left(\tilde{\psi}_{j+1/2}^{n+1/2} - \tilde{\psi}_{j-1/2}^{n+1/2}\right). \tag{44}$$

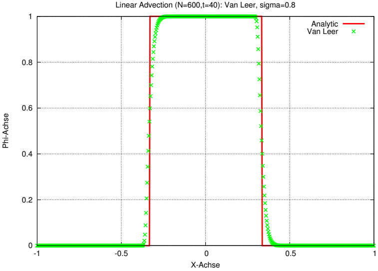


Rectangular function:
width 0.6 in interval $[-1, 1]$
velocity $a = 1$, until $t = 40$
periodic boundary conditions
$\sigma = a\Delta t/\Delta x = 0.8$
(Courant number)

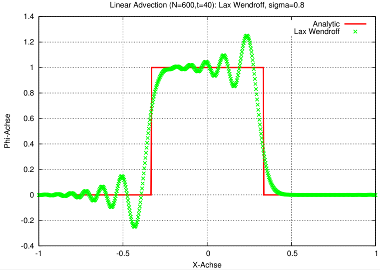
upwind - (diffusive)



Van Leer



Lax-Wendroff - (dispersive)



Assume a Fourier series for the solution (von Neumann 1940/50). Consider only one term of the series and check its growth

$$\psi_j^n = V^n \, e^{i\theta j}, \tag{45}$$

with the definition of θ using the grid size Δx and the total length L

$$\theta = \frac{2\pi\Delta x}{L}. \tag{46}$$

Now, consider the upwind scheme with $\sigma = a\Delta t/\Delta x$

$$\psi_j^{n+1} - \psi_j^n + \sigma(\psi_j^n - \psi_{j-1}^n) = 0. \tag{47}$$

Inserting (45) yields

$$V^{n+1} e^{i\theta j} = V^n e^{i\theta j} + \sigma V^n \left[e^{i\theta(j-1)} - e^{i\theta j} \right].$$

Divide by V^n and $e^{i\theta j}$ eventually gives

$$\frac{V^{n+1}}{V^n} = 1 + \sigma \left(e^{-i\theta} - 1 \right). \tag{48}$$

The square of the absolute value finally is

$$
\begin{aligned}
\lambda(\theta) \equiv \left| \frac{V^{n+1}}{V^n} \right|^2 &= \left[1 + \sigma\left(e^{-i\theta} - 1\right)\right]\left[1 + \sigma\left(e^{i\theta} - 1\right)\right] \\
&= 1 + \sigma\left(e^{-i\theta} + e^{i\theta} - 2\right) - \sigma^2\left(e^{-i\theta} + e^{i\theta} - 2\right) \\
&= 1 + \sigma(1-\sigma)(2\cos\theta - 2) \\
&= 1 - 4\sigma(1-\sigma)\sin^2\left(\frac{\theta}{2}\right). \qquad (49)
\end{aligned}
$$

The scheme is stable if the amplification factor $\lambda(\theta)$ is smaller than one. Hence, the upwind scheme is stable for $0 < \sigma < 1$, because then $\lambda < 1$, or

$$
\Delta t < f_{\mathsf{CFL}} \frac{\Delta x}{a} \qquad (50)
$$

with the Courant number $f_{\mathsf{CFL}} < 1$.

Theorem: *Courant-Friedrich-Levy* There is no explicit, consistent, stable finite difference scheme which is conditionless stable ($\forall \Delta t$).

Modified Eq I

Consider the upwind scheme with $\sigma = a\Delta t/\Delta x$

$$\psi_j^{n+1} - \psi_j^n + \sigma(\psi_j^n - \psi_{j-1}^n) = 0. \tag{51}$$

Replace differences with derivatives, (Taylor expansion up to 2nd order)

$$\frac{\partial \psi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 \psi}{\partial t^2}\Delta t^2 + \mathcal{O}(\Delta t^3) + \sigma\left(\frac{\partial \psi}{\partial x}\Delta x - \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2}\Delta x^2\right) + \mathcal{O}(\Delta t \Delta x^2) = 0. \tag{52}$$

Divide by Δt, replace σ

$$\frac{\partial \psi}{\partial t} + a\frac{\partial \psi}{\partial x} + \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial t^2}\Delta t - a\frac{\partial^2 \psi}{\partial x^2}\Delta x\right) + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2) = 0. \tag{53}$$

Use the wave equation $\psi_{tt} = a^2 \psi_{xx}$ to obtain the modified equation (index M)

$$\frac{\partial \psi_M}{\partial t} + a\frac{\partial \psi_M}{\partial x} = \frac{1}{2} a \Delta x \,(1-\sigma)\, \frac{\partial^2 \psi_M}{\partial x^2}. \tag{54}$$

This means, the FDE adds a diffuse term to the original PDE.

with the coefficient of diffusion

$$D = \frac{1}{2} a \Delta x \left(1 - \sigma\right). \tag{55}$$

Only for positive D, $(D > 0)$ this is a diffusive term, which means $\sigma < 1$ Hence, for the upwind scheme, we have get diffusion.
Lax-Wendroff yields

$$\frac{\partial \psi_M}{\partial t} + a \frac{\partial \psi_M}{\partial x} = \frac{\Delta t^2 a}{\sigma} \left(\sigma^2 - 1\right) \frac{\partial^3 \psi_M}{\partial x^3}. \tag{56}$$

This eq has the form

$$\psi_t + a \psi_x = \mu \psi_{xxx}, \tag{57}$$

where

$$\mu = \frac{\Delta t^2 a}{\sigma} \left(\sigma^2 - 1\right). \tag{58}$$

This causes dispersion: Waves are too slow $(\mu < 0)$, we get oscillations behind discontiuities.

According to the analysis on the last slides, the size of the timestep Δt has to be limited to obtain stable numerical evolution. In case of linear advection with speed a, it is

$$\Delta t < \frac{\Delta x}{a}. \tag{59}$$

In the more general case, the important transport speed of information is given by the sound speed (c_s), and we obtain the Courant-Friedrich-Lewy condition

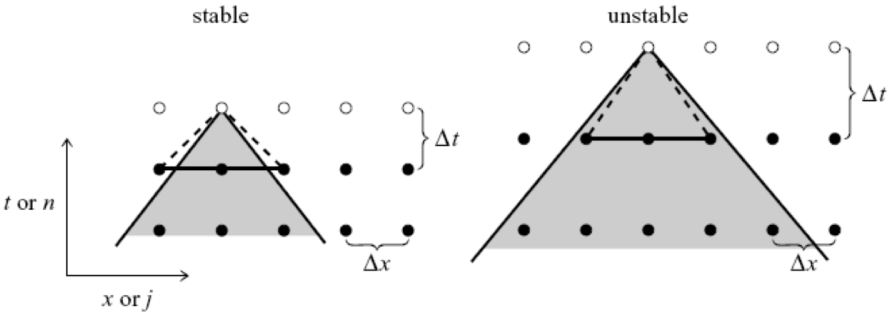
$$\Delta t < \frac{\Delta x}{c_s + |\mathbf{u}|}. \tag{60}$$

This means essentially, information is not allowed to change over the size of one grid cell during one timestep. To enforce this, one uses

$$\Delta t = f_C \frac{\Delta x}{c_s + |\mathbf{u}|}, \tag{61}$$

with the Courant number $f_C < 1$.

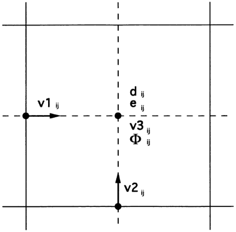
Size of time step - graphical



The numerical region of dependency (dashed area) has to be larger than the physical region of dependency (gray area): $\Delta x/\Delta t > a$.

The total information of the sound cone has to be considered.

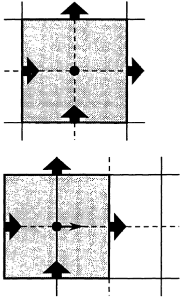
grid definitions (in 2D, staggered):
scalars in cell centre
(here: $\rho, \epsilon, p, v_3, \psi$)
vectors at interfaces
(here: v_1, v_2)



Fluxes through cell interfaces:

top: mass flux

bottom: x-momentum (grid moved!)



taken from ZEUS-2D: A radiation magnetohydrodynamics code for astrophysical flows in two space dimensions. I in The Astrophysical Journal Suppl., von Jim Stone und Mike Norman, 1992.

Use operator splitting und directional splitting: x and y directions are treated one after other. First x-scans, then y-scans.

Numerical schemes should reflect conservation properties.
-Write down the eqs in conserved form.
Numerical schemes should reflect wave properties.
- shock-capturing schemes, Riemann solver
Numerical schemes have to be able to treat discontinuities
- diffusion (⇒ stability), either explicitly (artificial viscosity) or implicitly (by construction of scheme)

Free codes on the web:

`ZEUS:` http://www.astro.princeton.edu/~jstone/zeus.html
classical upwind code, 2nd order, staggered grid, RMHD

`ATHENA:` https://trac.princeton.edu/Athena/
ZEUS successor: Riemann solver, centred grid, MHD

`NIRVANA:` http://nirvana-code.aip.de/
3D, AMR, Finite Volume code, MHD

`PLUTO:` http://plutocode.ph.unito.it/
3D, relativistic, Riemann solver/Finite Volume, MHD

`GADGET:` http://www.mpa-garching.mpg.de/galform/gadget/
SPH-Code, tree code, self-gravity

Consider the 1D equation (motion in x-direction):
Using Euler eqs with EOS $p = (\gamma - 1)\rho\epsilon$, we obtain

$$\left.\begin{array}{rcr} \frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} & = & 0 \\ \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} & = & -\frac{\partial p}{\partial x} \\ \frac{\partial(\rho\epsilon)}{\partial t} + \frac{\partial(\rho\epsilon u)}{\partial x} & = & -p\frac{\partial u}{\partial x} \end{array}\right\} \quad \Longrightarrow \quad \begin{array}{rcl} \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x} & = & 0 \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} & = & 0 \\ \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} & = & 0 \end{array},$$

or written in vectorial form

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}\,\frac{\partial \mathbf{W}}{\partial x} = 0, \tag{62}$$

with

$$\mathbf{W} = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} \qquad \text{und} \qquad \mathbf{A} = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{pmatrix} \tag{63}$$

The eqs are nonlinear and coupled!
De-coupling by diagonalisation of $\mathbf{A}$.

Eigenvalues (EV)

$$\det(\mathbf{A}) = \begin{vmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & 1/\rho \\ 0 & \gamma p & u-\lambda \end{vmatrix} = (u-\lambda) \begin{vmatrix} u-\lambda & 1/\rho \\ \gamma p & u-\lambda \end{vmatrix}$$

$$= (u-\lambda)\left[(u-\lambda)^2 - \gamma p/\rho\right] = 0. \quad (64)$$

We get

$$\begin{aligned} \lambda_0 &= u \\ \lambda_\pm &= u \pm c_s \end{aligned} \quad (65)$$

with sound speed

$$c_s^2 = \frac{\gamma p}{\rho}. \quad (66)$$

The EVs provide characteristic velocities, which are the speeds of information transport. It is combined from fluid- (u) and sound speed (c_s). Three real EVs which are the components of the diagonalised matrix

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \Lambda \quad (67)$$

$\mathbf{Q}$ is given by the eigenvectors to the EVs (λ_i, $i = 0, +, -$) and Λ a diagonal matrix.

$\mathbf{Q}$ can be calculated as

$$\mathbf{Q} = \begin{pmatrix} 1 & \frac{1}{2}\frac{\rho}{c_s} & -\frac{1}{2}\frac{\rho}{c_s} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2}\rho c_s & -\frac{1}{2}\rho c_s \end{pmatrix} \quad \text{and} \quad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{c_s^2} \\ 0 & 1 & \frac{1}{\rho c_s} \\ 0 & 1 & -\frac{1}{\rho c_s} \end{pmatrix}.$$

Now, consider the vector equation

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{W}}{\partial x} = 0 \tag{68}$$

and

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \Lambda.$$

Define:

$$d\mathbf{v} \equiv \mathbf{Q}^{-1} d\mathbf{W} \quad \text{and hence} \quad d\mathbf{W} = \mathbf{Q}\, d\mathbf{v} \tag{69}$$

Multiply eq. (68) with $\mathbf{Q}^{-1}$

$$\frac{\partial \mathbf{v}}{\partial t} + \Lambda \frac{\partial \mathbf{v}}{\partial x} = 0. \tag{70}$$

$v = (v_0, v_+, v_-)$ are charakteristic variables: $v_i = \mathrm{const.}$ on the curves

$$\frac{dx}{dt} = \lambda_i$$

Hydrodynamics: What is variable v_0?

From the definition

$$dv_0 = d\rho - \frac{1}{c_s^2}\, dp \tag{71}$$

$$\frac{\partial v_0}{\partial t} + \lambda_0 \frac{\partial v_0}{\partial x} = 0 \qquad \text{with} \qquad \lambda_0 = u \tag{72}$$

What is dv_0?
From thermodynamics (First law for specific quantities):

$$T ds = d\epsilon + p\, d\left(\frac{1}{\rho}\right) = d\epsilon - \frac{p}{\rho^2}\, d\left(\frac{1}{\rho}\right), \tag{73}$$

with $p = (\gamma - 1)\rho\epsilon$, $\epsilon = c_v T$, $\gamma = c_p/c_v$ one obtains

$$ds = -\frac{c_p}{\rho}\left[d\rho - \frac{dp}{c_s^2}\right] = -\frac{c_p}{\rho} dv_0 \tag{74}$$

$$\Longrightarrow \qquad \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} = 0, \tag{75}$$

which means s is $const.$ along streamlines, or

$$\frac{ds}{dt} = 0. \tag{76}$$

For the other characteristic variables, it is

$$\frac{\partial v_\pm}{\partial t} + (u \pm c_s)\,\frac{\partial v_\pm}{\partial x} = 0, \tag{77}$$

with

$$dv_\pm = du \pm \frac{1}{\rho c_s} dp, \tag{78}$$

it follows

$$v_\pm = u \pm \int \frac{dp}{\rho c_s}. \tag{79}$$

Assume constant entropy everywhere (i.e., $p = K\rho^\gamma$)

$$\Longrightarrow \quad v_\pm = u \pm \frac{2c_s}{\gamma - 1}. \tag{80}$$

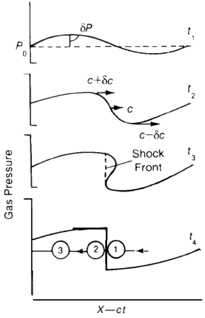
Riemann invariants: $v_\pm = \mathrm{const.}$ along curves

$$\frac{dx}{dt} = u \pm c_s.$$

Linearisation of the Euler eqs yields wave equation for a perturbation

$$\frac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \frac{\partial^2 \rho_1}{\partial x^2} \qquad (81)$$

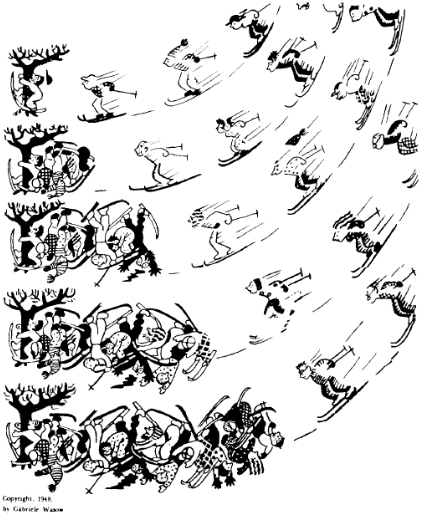
however!: c_s s not constant $\Rightarrow$ steepening



$\Rightarrow$ discontinuities

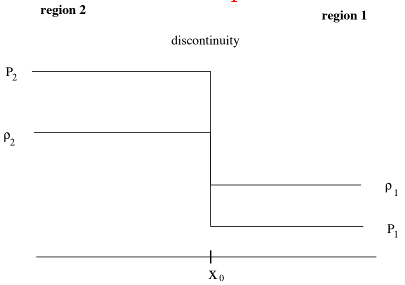
$\equiv$ Jump: sub-supersonic

Example for receeding shockwave

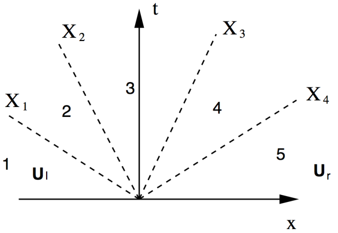


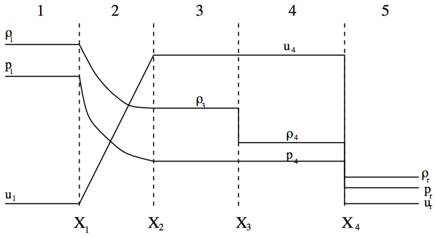
Discontinuity in the initial data in a tube at location x_0, (1D)

Riemann problem



pressure (p) and density (ρ) jump,
we obtain - a shock to the right (X_4)
(supersonic $u_{sh} > c_s$)
- a contact discontinuity
density jump (along X_3)
- a rarefaction wave
(between X_1 and X_2)

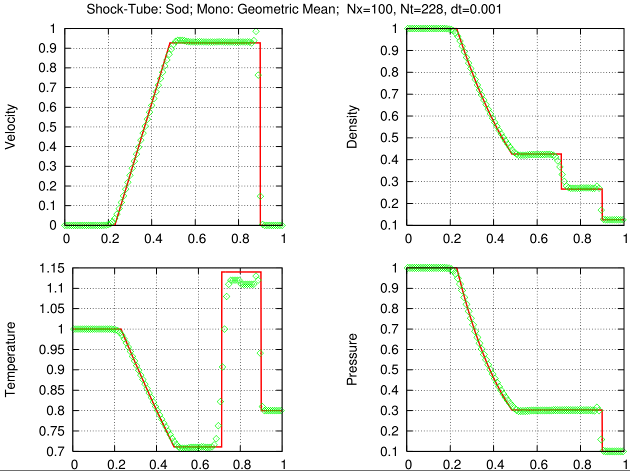




Standard testproblem for numerical hydrodynamis, $x \in [0,1]$ with $X_0 = 0.5$, $\gamma = 1.4$

$\rho_1 = 1.0, p_1 = 1.0, \epsilon_1 = 2.5, T_1 = 1$, and $\rho_2 = 0.1, p_2 = 0.125, \epsilon_2 = 2.0, T_2 = 0.8$

Solution using van Leer scheme at $t = 0.228$



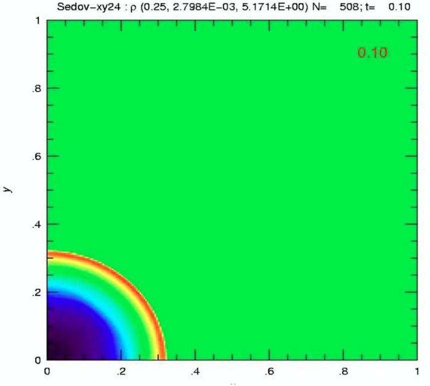
red: exact
green: numerical

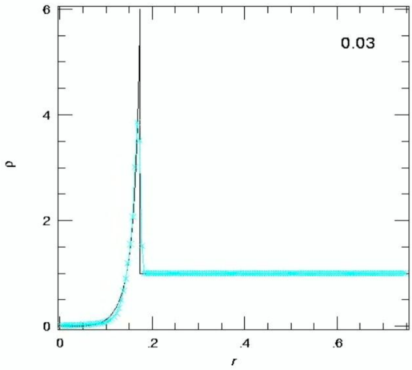
A model for exploding bombs (Sedov & Taylor, 1950er), analytical solution by Sedov

Standard test problem for ¿ 1D, for $x, y \in [0,1] \times [0,1]$

Energy deposity at origin, $E = 1$, in $\rho = 1$, $\gamma = 1.4$, 200×200 grid points

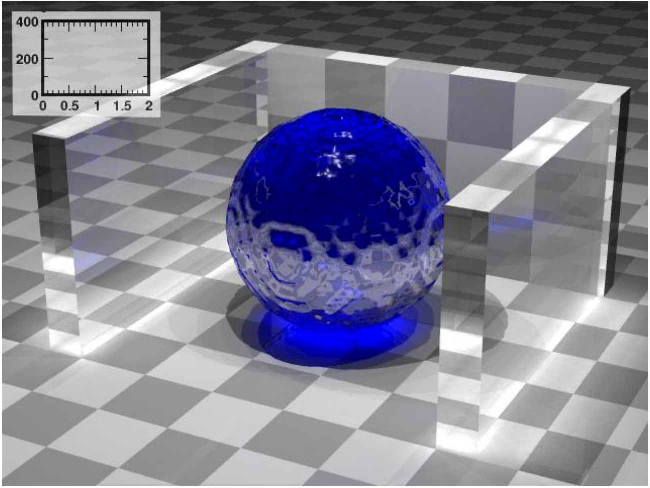
Solution using van Leer scheme, plotted quantity is density





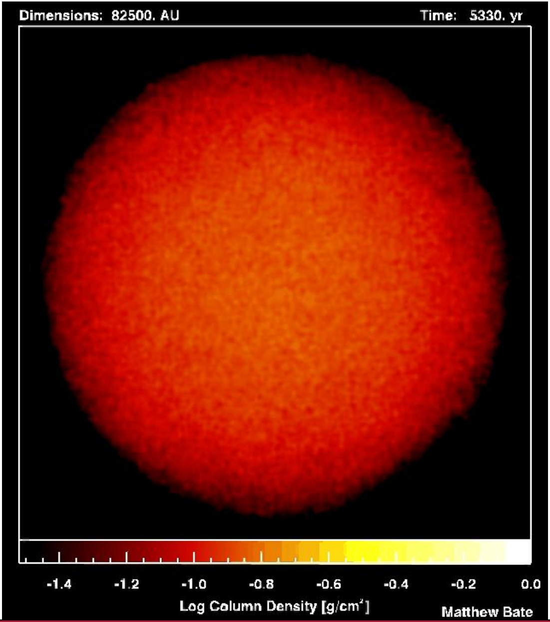
Liquid drop: SPH

Sphere of water (R=30cm), basin (1x1 m, 60cm height) including surface tension, time in seconds (TU-München, 2002)



(url)

Examples: Stellar formation: SPH



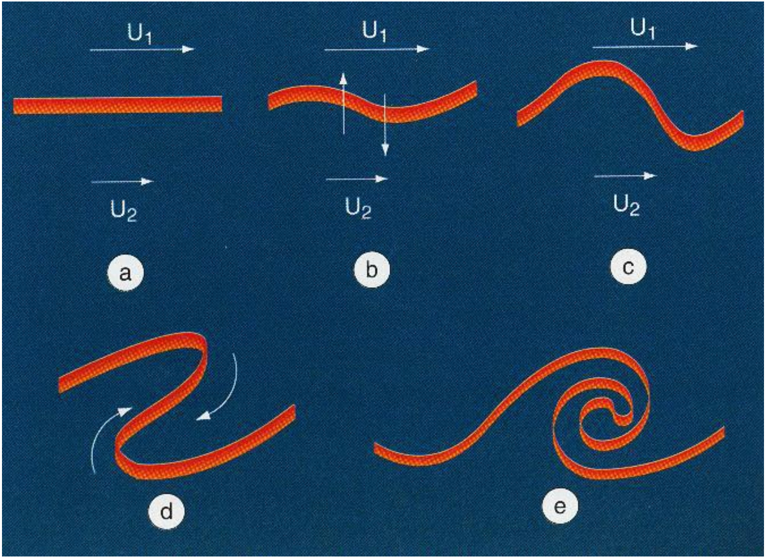
molecular cloud

mass:
50 $M_\odot$

diameter:
1.2 ly $=$ 76,000 au

temperature:
10 K (M. Bate, 2002)

Kelvin-Helmholtz instability



KHI is a shear instability: jump in tangential component leads to instability

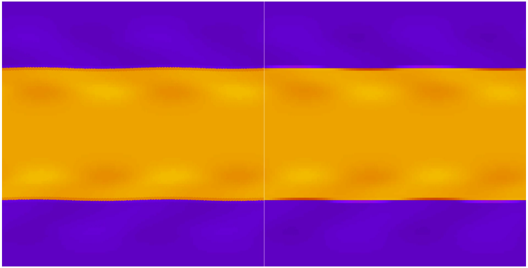
Examples: KHI in atmosphere

(Boulder (NCAR), USA)

KHI: Simulation

direct comparison: moving vs. fixed grid

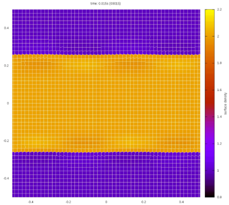
left: moving grid (Voronoi tesselation)
right: fixed squared grid (Euler)



(Kevin Schaal, Tübingen)

Youtube channel

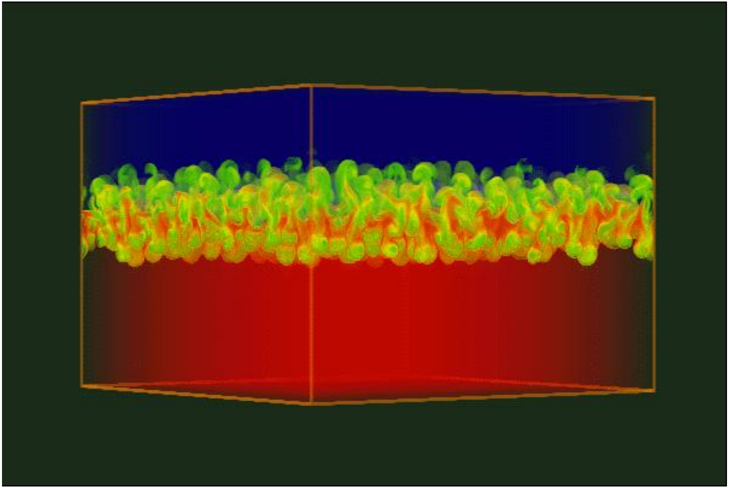
with moving grid



Rayleigh-Taylor instability

PPM Code, 128 Knoten, ASCI Blue-Pacific ID System at LLNL

512^3 grid points (LLNL, 1999)



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