

Problem Set 8 - Ordinary Differential Equations

Exercise 1. Runge Kutta Proofs

For this exercise we are asked to solve choose two of six possible exercises. The two exercises that I chose are as follows:

- (a) Derive the 3rd order Runge-Kutta formula

$$y_{n+1} = y_n + \frac{1}{9}(2k_1 + 3k_2 + 4k_3)$$

with

$$\begin{aligned} k_1 &= hf(x, y) \\ k_2 &= hf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_1\right) \\ k_3 &= hf\left(x + \frac{3}{4}h, y + \frac{3}{4}k_2\right) \end{aligned}$$

- (b) Prove that in the special case that $y' = f(x)$ the 4th order RK formula is equivalent to Simpson's rule.

Solution: 1

- (a) The general form of a 3rd order RK method is given by the form

$$y_{n+1} = y_n + w_1k_1 + w_2k_2 + w_3k_3$$

with w_n being weights and k_n being intermediate slopes. These k_n values are defined by the general form

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \alpha_2h, y_n + \beta_{21}k_1) \\ k_3 &= hf(x_n + \alpha_3h, y_n + \beta_{31}k_1 + \beta_{32}k_2) \end{aligned}$$

The values of w, k, α , and β are chosen to most closely approximate a Taylor of y_{n+1} . The general Taylor expansion takes the following form:

$$y_{n+1} = y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{6}y'''(x_n) + \mathcal{O}(h^4)$$

knowing that $y' = \frac{dy}{dx} = f$, we can rewrite the derivatives of y as

$$\begin{aligned} y''(x) &= \frac{d}{dx} f(x, y) = f_x + f_y y' = f_x + f_y f \\ y'''(x) &= \frac{d}{dx} (f_x + f_y f) \\ &= f_{xx} + 2f_{xy} f + f_{yy} f^2 + f_y (f_x + f_y f) \end{aligned}$$

with help from the chain rule in multivariable calculus.

We now continue by calculating the Taylor series of the RK formula so that we can compare coefficients. We know that the general form for a Taylor series in two variables is

$$f(x + h, y + k) = \sum_{i=0}^{\infty} \frac{1}{i!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(x, y).$$

Applying this formula we get

$$\begin{aligned} k_2 &= h[f(x_n, y_n) + \alpha_2 h f_x + \beta_{21} k_1 f_y + \frac{1}{2} (\alpha_2 h)^2 f_{xx} + \alpha_2 h \beta_{21} k_1 f_{xy} \\ &\quad + \frac{1}{2} (\beta_{21} k_1)^2 f_{yy} + \dots] \\ k_3 &= h[f(x_n, y_n) + \alpha_3 h f_x + (\beta_{31} k_1 + \beta_{32} k_2) f_y + \frac{1}{2} (\alpha_3 h)^2 f_{xx} \\ &\quad + \alpha_3 h (\beta_{31} k_1 + \beta_{32} k_2) f_{xy} + \frac{1}{2} (\beta_{31} k_1 + \beta_{32} k_2)^2 f_{yy} + \dots] \end{aligned}$$

Comparing the values of h, h^2, h^3 , and h^4 , we can develop the following system of equations:

$$\begin{aligned} w_1 + w_2 + w_3 &= 1 \\ w_2 \alpha_2 + w_3 \alpha_3 &= \frac{1}{2} \\ w_2 \alpha_2^2 + w_3 \alpha_3^2 &= \frac{1}{3} \\ w_3 \beta_{32} \alpha_2 &= \frac{1}{6} \end{aligned}$$

From the specific formulations of k_i in the assignment we see that

$$\alpha_2 = \frac{1}{2}, \alpha_3 = \frac{3}{4}, \beta_{21} = \frac{1}{2}, \beta_{31} = 0, \beta_{32} = \frac{3}{4}.$$

Inserting these values into the above system of equations leads to

$$\begin{aligned}w_1 + w_2 + w_3 &= 1 \\ \frac{1}{2}w_2 + \frac{3}{4}w_3 &= \frac{1}{2} \\ \left(\frac{1}{2}\right)^2 w_2 + \left(\frac{3}{4}\right)^2 w_3 &= \frac{1}{3} \\ \frac{3}{4} \cdot \frac{1}{2}w_3 &= \frac{1}{6}.\end{aligned}$$

Solving this system of equations yields

$$\begin{aligned}w_1 &= \frac{2}{9} \\ w_2 &= \frac{1}{3} \\ w_3 &= \frac{4}{9}.\end{aligned}$$

When these values are instulted into the RK formula, we finally obtain

$$\begin{aligned}y_{n+1} &= y_n + \frac{2}{9}k_1 + \frac{1}{3}k_2 + \frac{4}{9}k_3 \\ &= y_n + \frac{1}{9}(2k_1 + 3k_2 + 4k_3)\end{aligned}$$

as desired.

(b) Given that $y' = f(x)$, we can say that

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x) dx$$

We want to show that the above integral approximation using RK4 is equal to the approximation given by Simpson's Rule:

$$\int_{x_n}^{x_{n+1}} f(x) dx \approx \frac{h}{6} \left(f(x_n) + 4f\left(x_n + \frac{h}{2}\right) + f(x_{n+1}) \right)$$

We begin by writing the RK4 method for $y' = f(x)$:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

with

$$\begin{aligned}k_1 &= hf(x_n) \\k_2 &= hf\left(x_n + \frac{h}{2}\right) \\k_3 &= hf\left(x_n + \frac{h}{2}\right) \\k_4 &= hf(x_n + h)\end{aligned}$$

Substituting in the values of k and combining like-terms, we obtain

$$y_{n+1} = y_n + \frac{h}{6} \left(f(x_n) + 4f\left(x_n + \frac{h}{2}\right) + f(x_n + h) \right)$$

Comparing the above expression to Simpson's Rule, we immediately see that they are equivalent, thus completing the proof.

Exercise 2. 1st Order ODEs

Consider the Initial Value Problems (INPs) for the 1st order differential equations below. Solve them for the interval $x \in [0, 30]$ using the Heun's, RK4, and Adams methods.

(a) $y' = y \cos(x + y), y(0) = 1$

(b) $y' = \sin(xy) \cos x + y, y(0) = 1$

(c) $y' = xy - x^2, y(0) = 1$

(d) $y' = y(3 - y), y(0) = 1$

(e) $y' = y^2 - x^2, y(0) = \frac{1}{2}$

(f) $y' = \frac{1}{3x-2y+1}, y(0) = 0$

Solution: 2

The RK4, Heun, and Adams methods were implemented in C code and plotted with a Gnuplot script, both of which can be found in the `problem2` directory. The implemented routines are summarized below with a stepsize h .

The general form of RK4 is given by the following formula:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + \mathcal{O}(h^5)$$

with

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right) \\ k_3 &= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right) \\ k_4 &= hf(x_n + h, y_n + k_3) \end{aligned}$$

Heun's method can be implemented by first finding an intermediate \tilde{y}_{i+1} and then calculating the final approximation y_{i+1} as follows:

$$\begin{aligned} \tilde{y}_{i+1} &= y_i + hf(t_i, y_i) \\ y_{i+1} &= y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})] \end{aligned}$$

Finally, Adams' method is given by the scheme

$$y_{n+1} = y_n + \frac{h}{2}(y'_n + y'_{n+1})$$

The equations were numerically solved using $n = 1000$ steps and plotted below.

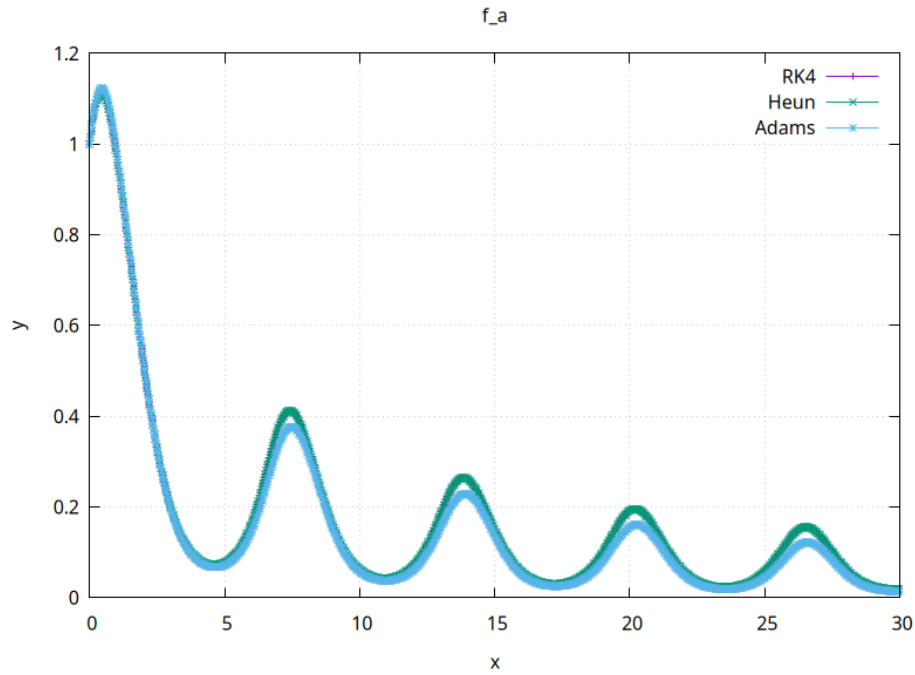


Figure 1: Function A

Exercise 3. - Nonlinear ODEs

For this problem we consider Bernoulli non-linear ODEs in the form

$$y'(x) = b(x)y^\alpha(x) + a(x)y(x)$$

with a constant $\alpha > 0$.

Write a program to numerically integrate a single first-order ODE using

- RK4
- Adams-Moulton, without a corrective step
- Adams-Moulton with a corrective step

(a) Test the program on the DE

$$y' = y + x^2 - 2x + \sin(x)$$

with $y_0 = 0.1$ on the interval $x \in [0, 2]$.

- (i) Plot the absolute analytical error, E at $x = 2$ as a function of the number of steps taken $N \in [10, 5000]$ in a log-plot and discuss the convergence order.

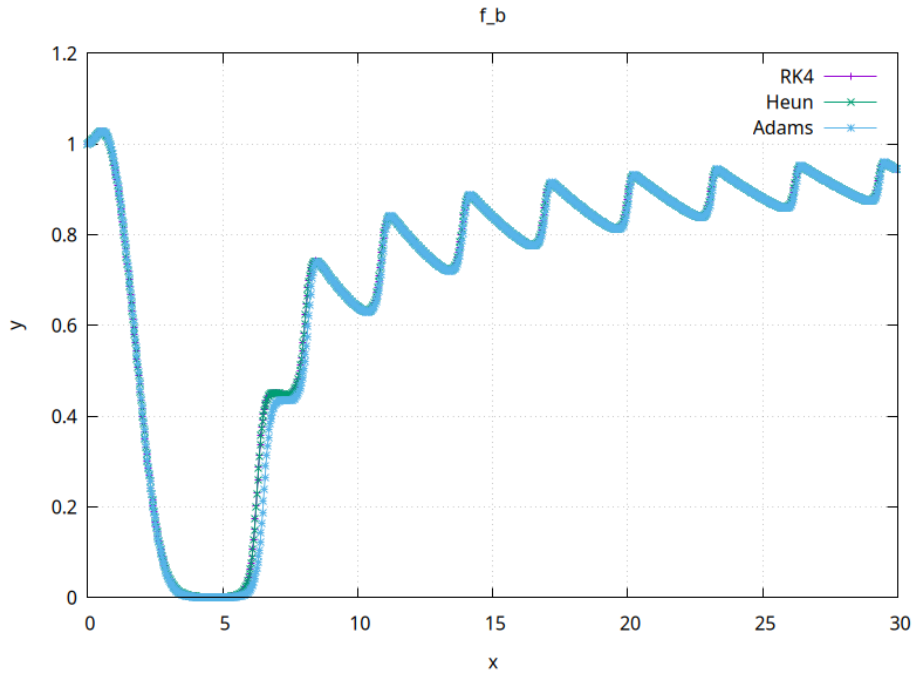


Figure 2: Function B

- (ii) Plot the error vs the number of function evaluations ($4N$ for RK4, N for Adams without corrections, $2N$ as a minimum for Adams with corrections). Which method is the most efficient?
- (b) The evolution of the velocity of a body flowing through a non-ideal fluid can be described by a Bernoulli DE

$$v'(t) + \mu(t)v(t) - \omega(t)^2 v^3(t) = 0$$

with $\omega(t)$ and $\mu(t)$ being two positive functions. Assume the initial condition $v_0 = 2$.

- (i) Plot the absolute analytical error at $t = 1.5$ with the constant parameters $\omega(t) = \omega_0 = 0.707$, $\mu(t) = \mu_0 = 2$.
- (ii) Keep $\mu_0 = 2$ constant and vary ω_0 in the vicinity of $\frac{1}{\sqrt{2}}$. How does the solution behave?
- (iii) Consider $\omega(t) = \omega_0 \tan(t)$ with $\omega_0 = \frac{1}{\sqrt{2}}$ and $\mu_0 = 2$ initially. Slowly increase μ_0 towards 1.5 - comment on what you see.

Solution: 3

To solve this problem the code from **problem2** was copied and expanded. The methodology of RK4 is explained in the previous problems on this sheet. The Adams-Moulton method for

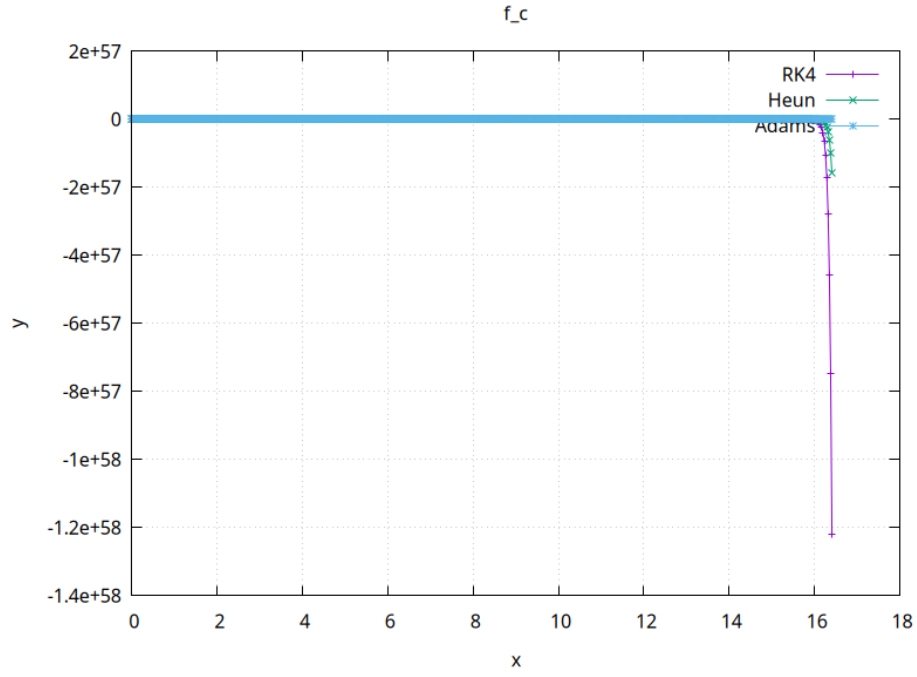


Figure 3: Function C

ODEs is given by

$$y_{k+1} = y_k + \frac{h}{24}(9f_{k+1} + 19f_k - 5f_{k-1} + f_{k-2}) + \mathcal{O}(h^5)$$

f_{k+1} of course depends on y_{k+1} , making this an implicit method. This can be remedied by using predictor-corrector techniques, such as evaluating y_{k+1} as

$$y_{k+1} = y_k + \frac{h}{24}(55f_k - 59f_{k-1} + 37f_{k-2} - 9f_{k-3}) + \mathcal{O}(h^5).$$

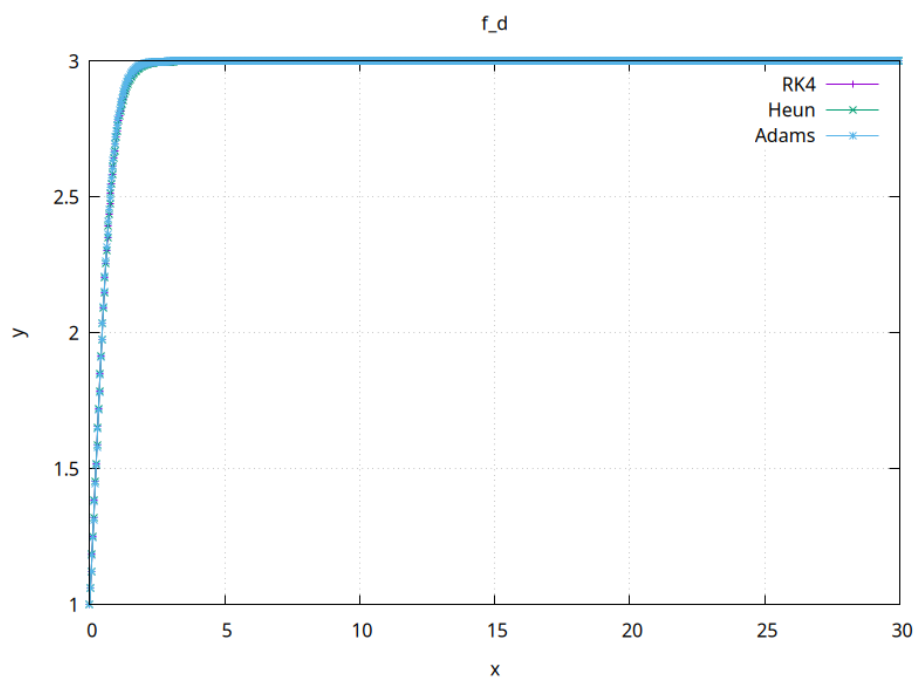


Figure 4: Function D

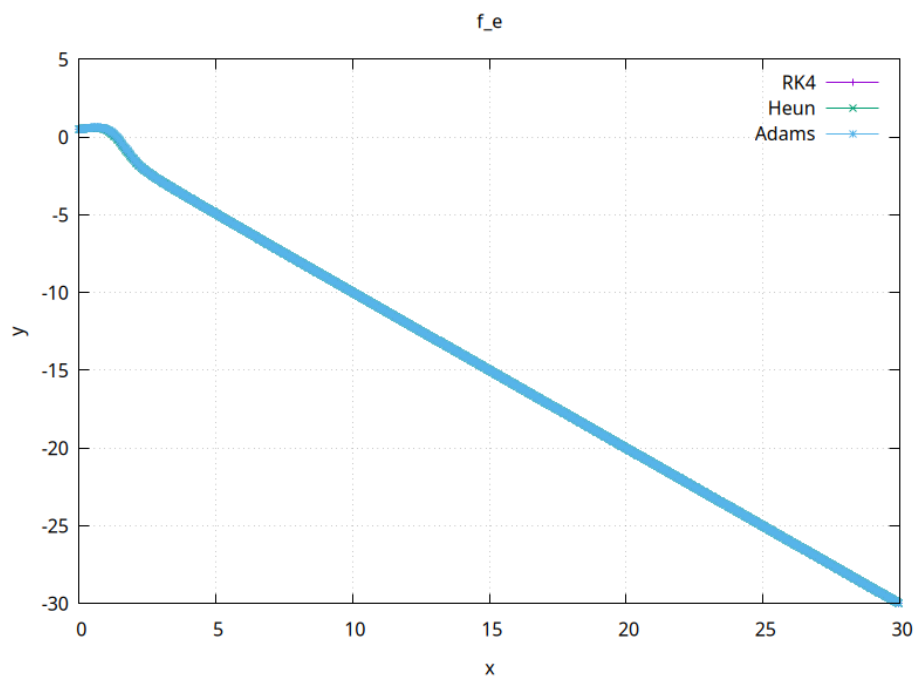


Figure 5: Function E

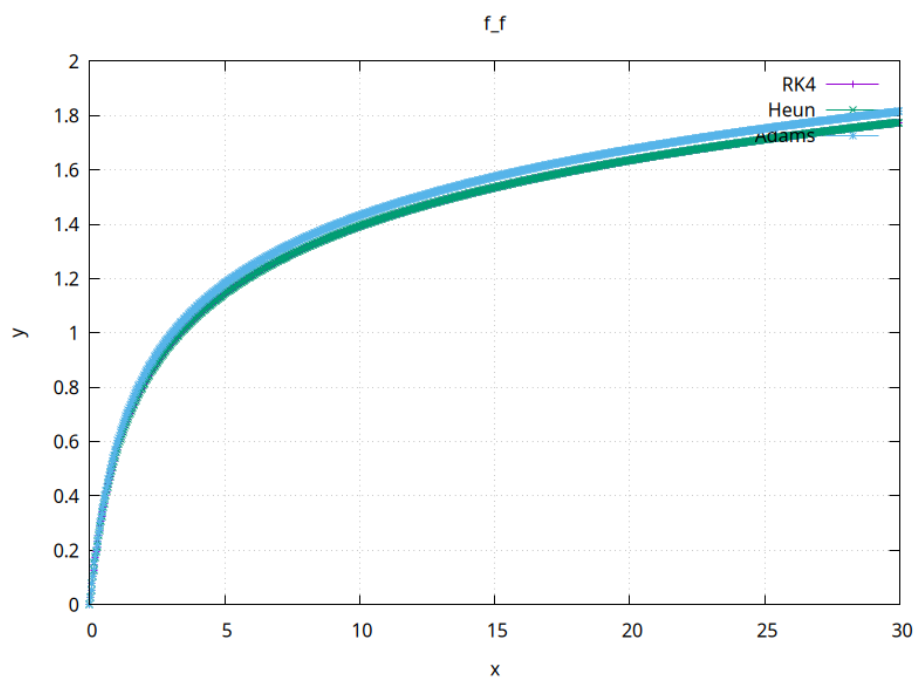


Figure 6: Function F