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Numerical Methods - Problem Set 6

Exercise 1.

In `problem 1` we are asked to determine which formula from `table 1` in the lecture notes on numerical integration will analytically be derived from combining the Simpson $\frac{1}{3}$ integration method with the Romberg integration method.

The five formulas in the lecture note table are as follows:

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f_0 + f_1) - \frac{1}{12}h^3 f^{(2)}(\zeta_1) \tag{1}$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(f_0 + 4f_1 + f_2) - \frac{1}{90}h^5 f^{(4)}(\zeta_1) \tag{2}$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3}{80}h^5 f^{(4)}(\zeta_1) \tag{3}$$

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) - \frac{8h^7}{945} f^{(6)}(\zeta_1) \tag{4}$$

$$\int_{x_0}^{x_5} f(x)\,dx = \frac{5h}{288}\left(19f_0 + 75f_1 + 50f_2 + 50f_3 + 75f_4 + 19f^5\right) - \frac{275h^7}{12096} f^{(6)}(\zeta_1) \tag{5}$$

Solution: 1

Using *Numerical Mathematics and Computing by Cheney et al.* 6th ed. and consulting Wikipedia for an equivalent relation shows that the Simpson's $\frac{1}{3}$ rule for n subintervals is given by the form

$$\int_{a=x_0}^{b=x_n} f(x)\,dx \approx \frac{h}{3}\sum_{i=1}^{n/2} f_{2i-2} + 4f_{2i-1} + f_{2i}$$

Similar to Aitken's acceleration and Richardson extrapolation, one can then use two approximations that were calculated with different intervals to calculate a more accurate approximation.

Using only two intervals yields the familiar formulation for the first order Simpson's $\frac{1}{3}$ rule:

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{3}\sum_{i=1}^{2/2} f_{2i-2} + 4f_{2i-1} + f_{2i}\\ &= \frac{h}{3}(f_0 + 4f_1 + f_2)\end{aligned}$$

or if we adjust the $h \to \frac{h}{2}$ and adjust the x values to reflect the altered interval:

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{3}(f_0 + 4f_2 + f_4)$$

Dividing the two intervals in half creates four subintervals, resulting in an approximation of the following form:

$$\begin{aligned}
\int_{x_0}^{x_4} f(x)\,dx &= \frac{h}{3}\sum_{i=1}^{4/2} f_{2i-2} + 4f_{2i-1} + f_{2i} = \frac{h}{3}\sum_{i=1}^{2} f_{2i-2} + 4f_{2i-1} + f_{2i} \\
&= \frac{h}{3}[(f_0 + 4f_1 + f_2) + (f_2 + 4f_3 + f_4)] \\
&= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4)
\end{aligned}$$

Approximations using the Romberg algorithm can then be carried out using the following formula

$$R(n,m) = R(n,m-1) + \frac{1}{4^m - 1}[R(n,m-1) - R(n-1,m-1)].$$

where $R(n,0)$ denotes the result with 2^n subintervals.

(Note: See *Numerical Mathematics and Computing by Cheney et al.* 6th ed. Pg. 205 for details)

We procede by setting $R(1,1)$ equal to Simpson's $\frac{1}{3}$ rule, and $R(2,1)$ equal to the Simpson's $\frac{1}{3}$ rule with halved intervals (note than $m = 1$ since $m = 0$ corresponds to the lower order integral approximation, the trapezoidal rule):

$$\begin{aligned}
R(2,2) &= R(2,1) + \frac{1}{4^2 - 1}[R(2,1) - R(1,1)] \\
&= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) \\
&\quad + \frac{1}{15}\left[\frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) - \frac{2h}{3}(f_0 + 4f_2 + f_4)\right] \\
&= \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4)
\end{aligned}$$

The obtained formula correspond to **formula 4** in the given table.

Exercise 2.

In `problem 2` we are calculate the integral

$$f(x) = \frac{2^x \sin(x)}{x}$$

using the following three numerical techniques

(a) Simpson's $\frac{1}{3}$ rule

(b) Simpson's $\frac{1}{3}$ rule with the Romberg algorithm

(c) Gauss-Legendre with four points (data can be taken from either table 2 in the lecture notes or from page 28)

Solution:

To solve this exercise a C program was created (see the folder `problem2`) with the name `integratorinator`. Executing this program with the appropriate command line arguments calculates the integral approximation using the appropriate scheme. Integrals were calculated on the interval $x_i = 0.1$ to $x_f = 1.1$.

Results were compared to an approximation given by Mathematica, yielding

$$I \approx 1.41891918440222$$

(a) Simpson's $\frac{1}{3}$ rule used the following integration approximation:

$$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2)$$

Running the program yielded

$$I \approx 1.418709$$

which differs from the answer given by Mathematica by about -0.0148424239%.

(b) Simpson's $\frac{1}{3}$ rule combined with the Rhomberg algorithm used the following integral approximation (see `problem 1`).

$$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4)$$

Running the program yielded

$$I \approx 1.41892$$

which differs from Mathematica's calculated value by about 0.0000248637%.

(c) Sample