Logarithms and Exponentials

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1 Definition of a Logarithm

if x and b are positive real numbers such that $b \not\equiv 1$ then $y = log_b x$ is called the logarithmic function base b, where:

$$(y = log_b x) = (b^y = x)$$

* Given $y = log_b x$, the value y is the exponent to which b must be raised to obtain x.

* The value of y is called the logarithm, b is called the base, and x is called the argument.

* The equations $y = log_b x$ and $b^y = x$ both define the same relationship between x and y. The expression $y = log_b x$ is called the logarithmic form and $x = b^y$ is called the exponential form.

2 The Use Case

Some exponential equations are impossible to solve by inspection.

in $5^x = 5$ the solution is $x = 1$
however, consider $5^x = 20$

We cannot find the value of x but we know it is between one and two. That said all exponential functions are one-to-one functions, so we can isolate x by performing the inverse operation of 5^x.

The inverse operation of an exponential function, base b, is the logarithmic function base b, which is defined above.

3 Converting Between Log. and Exp. Form

Questions:

$$log_2 16 = 4 \tag{1}$$

$$log_{10}(\frac{1}{100}) = -2 \tag{2}$$

$$log_7 1 = 0 \tag{3}$$

Answers:

$$(2^4 = 16) = (log_2 16 = 4) \tag{4}$$

$$(10^{-2} = \frac{1}{100}) = (log_{10}(\frac{1}{100}) = -2) \tag{5}$$

$$(7^0 = 1) = (log_7 1 = 0) \tag{6}$$

4 Common Logarithms

In many contexts the base of a logarithm will be excluded, this doesn't mean there is no base, it means that the base is ten. Base ten logarithms are called common logaritms. The notation for a log of base ten is the leftmost logarithm denoted below, the right log is for illustrating equivalency.

$$(y = logx) = (y = log_{10}x)$$

5 Natural Logarithm (ln)

The logarithmic function base e is called the natural logarithmic function. The natural logarithmic function is denoted below. You can think of the ln notation as log natural.

$$(y = \ln x) = (log_e x = y)$$

6 Basic Properties of Logarithms

1 * $log_b 1 = 0$ because $b^0 = 1$

2 * $log_b b = 1$ because $b^1 = b$

3 * $log_b b^x = x$ because $b^x = b^x$

4 * $b^{log_b x} = x$ because $log_b x = log_b x$

Properties 3 and 4 follow from the fact that a logarithmic function is the inverse of an exponential function of the same base.

Given $f(x) = b^x$ and $f^{-1}(x) = log_b x$

* $(f \circ f^{-1})(x) = b^{log_b x} = x$ (Property 4)
* $(f^{-1} \circ f)(x) = log_b(b^x) = x$ (Property 3)

7 Product Propery of Logarithms

By definition, $y = log_b x$ is equivalent to $b^y = x$. Becuase a logarithm is an exponent, the properties of exponents can be applied to logarithms. The first is called the product property of logarithms.

Let b, x, and y be positive real numbers where $b \neq 1$. Then

$$log_b(xy) = log_b x + log_b y$$

The logarithm of a product equals the sum of the logarithms of the factors.

Proof:

* Let $M = log_b x$, which implies $b^M = x$

* Let $N = log_b y$, which implies $b^N = y$

* Then $xy = b^M b^N = b^{M+N}$

Writing the expression $xy = b^{M+N}$ in logarithmic form, we have,

$$log_b(xy) = M + N$$

$$log_b(xy) = log_b x + log_b y$$

Example:

$$log_3(3 \cdot 9) = log_3 3 + log_3 9$$

$$log_3 27 = 1 + 2$$

$$3 = 3$$

8 Quotient Property of Logarithms

$\iff$ means if and only if

This property is derived from the quotient rule of exponents which tells us that

$$\frac{b^M}{b^N} = b^{M-N} \iff b \neq 0$$

. This property can be applied to logarithms due to the aforementioned relaltionship between a logarithm as an exponent value of its inverse exponential function. Let b, x, and y be positive real numbers where $b \neq 1$. Then

$$log_b \frac{x}{y} = log_b x - log_b y$$

The logarithm of a quotient equals the difference of the numerator and the logarithm of the denomiator.

Proof:

$$log(\frac{1,000,000}{100}) = log1,000,000 - log100$$
$$log10,000 = 6 - 2$$
$$4 = 4$$

9 Power Property of Logarithms

The power property of exponents tells us that

$$(b^M)^N = b^{MN}$$

The same principle can be applied to logarithms.

Let b and x be positive real numbers where $b \neq 1$. Let p be any real number.

$$log_b x^p = plog_b x$$

Proof:

$$log_2 4^2 = 2log_2 4$$
$$log_2 16 = 2 \cdot 2$$
$$4 = 4$$

10 Graphing

Since a logarithmic function $y = log_b x$ is the inverse of the corresponding function $y = b^x$, their graphs must be symmetric with respect to the line y = x.

The range of $y = b^x$ is the set of positive real numbers. As expected, the domain of its inverse function $y = log_b x$ is the set of positive real numbers.

When you have a logarithm of the form

a. $f(x) = log_2(2x + 4)$
b. $g(x) = log_2(5 - x)$
c. $h(x) = log_(x^2 - 9)$

(a) solution:

$$f(x) = log_2(2x + 4)$$

$$2x + 4 > 0$$

$$2x > -4$$

$$x > -2$$

The domain is $(-2, \infty)$
The vertical asymptote is $x = -2$

(b) solution:

$$g(x) = log_2(5 - x)$$

$$5 - x > 0$$

$$-x > -5$$

$$x < 5$$

The domain is $(-\infty, 5)$
The vertical asymptote is $x = -2$

(c) solution:

$$h(x) = log_(x^2 - 9)$$

$$x^2 - 9 > 0$$

$$(x - 3)(x + 3) = 0$$

The domain is $(-\infty, -3) \cup (3, \infty)$
The vertical asymptotes are $x = -3$ and $x = 3$

11 Change of Base Formula

Let a and b be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then for any positive real number x,

$$log_b x = \frac{log_a x}{log_a b}$$

The change-of-base formula converts a logarithm of one base to a ratio of logarithms of a different base. For the purpose of using a calculator, we often apply the change-of-base formula with base 10 or base e.

To derive the change-of-base formula, assume that a and b are positive real numbers with $a \neq 1$ and $b \neq 1$. Begin by letting $y = log_b x$. If $y = log_b x$, then

$$b^y = x$$

$$log_a b^y = log_a x$$

$$y \cdot log_a b = log_a x$$

$$y = \frac{log_a x}{log_a b}$$

$$log_b x = \frac{log_a x}{log_a b}$$

12 Examples

$$3^{2y-2} = \frac{1}{3}^{y-7}$$

express the fractional base as a negative exponent.

$$3^{2y-2} = (3^{-1})^{y-7}$$

$$3^{2y-2} = 3^{-y+7}$$

$$2y - 2 = -y + 7$$

$$2y = -y + 9$$

$$3y = 9$$

$$y = 3$$

$$10^{4+8y} + 3,500 = 138,000$$

$$10^{4+8y} = 134,500$$

$$log_{10}10^{4+8y} = log_{10}134,500$$

to simplify the lvalue:

$$log_{10}10^{4+8y} = (10^y = 10^{4+8y}) = 4 + 8y$$

back to the problem:

$$4 + 8y = log134,500$$

$$8y = log134,500 - 4$$

$$y = \frac{log134,500 - 4}{8}$$

13 Equivalence Property of Exponential Expressions

if b, x, and y are real numbers with $b > 0$ and $n \neq 1$ then $b^x = b^y$ implies that $x = y$.

if two exponential equations with the same base are equal, then their exponents must be equal.

14 Solving Exponential Equations with Logarithms

1: Isolate the exponential expression on one side of the equation.
2: Take a logarithm of the same base on both sides of the equation.
3: Use the power property of logarithms to "bring down" the exponent.
4: Solve the resulting equation.