

Trigonometry

Amos Thacker-Gwaltney

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1 Ray

A ray is part of a line that consists of an endpoint and all points on the line to one side of the endpoint.

A ray is denoted symbolically by \overrightarrow{PQ} where P is the endpoint and Q is another point on the ray to the right of P. A ray always has one endpoint which is always written first when naming the ray.

2 Angle

An angle is formed by rotating a ray about its endpoint. The starting position is called the initial side of the angle, and the final position of the ray is called the terminal side. The common endpoint is called the vertex of the angle, and the vertex is often denoted by a capital letter such as A.

An angle is notated by $\angle A$ or by $\angle BAC$, where B is the initial side, A being the vertex, and C being the terminal side.

2.1 Standard Position

An angle is said to be at standard position if its vertex is at the origin in the xy-plane, and the initial ray is directly on the x axis.

2.2 Measurement

The measure of an angle quantifies the direction and amount of rotation from the initial ray to the terminal ray. The measure is positive if the rotation is counterclockwise, and the measurement is negative if the rotation is clockwise. Often, degrees are used to measure the amount of rotation, one full rotation of a ray about its vertex is equivalent to 360°

2.3 Angle Type Names

1. Right Angle:
Right angles are angles at 90° measure.
2. Straight Angle:
Straight angles are angles at 180° measure.
3. Acute Angle:
Acute angles are angles at $0^\circ < \theta < 90^\circ$
4. Obtuse Angle:
Obtuse angles are angles at $90^\circ < \theta < 180^\circ$
5. Complementary Angles:
A pair of angles which sum to 90°
6. Supplementary Angles:
A pair of angles which sum to 180°
7. Coterminal Angles:
Two angles in standard position with the same initial side and same terminal side are called coterminal angles.

2.4 Units

1. Minutes / Arcminutes (min or '):
A singular degree can be divided into 60 equal parts; each 1/60th is referred to as one arcminute / minute. $\left(\frac{1}{60}\right)^\circ = 1' \vee 1min$
2. Seconds / Arcseconds (sec or ''):
A singular arcminute can be divided into 60 equal parts; each 1/60th of such is referred to as one arcsecond. $\left(\frac{1}{60}\right)' = 1'' \vee 1sec$

3 Radian Measurement

3.1 Definition of a Radian

A central angle that intercepts an arc on the circle with length equal to the radius of the circle has a measure of one radian.

3.2 Angle Measurement

When two lines or rays cross a circle, the part of the circle between the intersection points is called the intercepted arc and is often denoted by s .

Any central angle can be measured in radians by dividing the length s of the intercepted arc by the radius r .

3.3 Review

1. π :
is defined as the ratio of the circumference of a circle to its diameter d .
2. Circumference:
Therefore the circumference is given by $C = \pi d$ or $C = 2\pi r$, where r is the radius of the circle, dividing this by the radius gives the number of radians in one revolution. The angular measure of one full rotation is 2π

3.4 Definition of Radian Angle Measure

The radian measure of a central angle θ subtended by an arc of length s on a circle of radius r is given by $\theta = \frac{s}{r}$. radian measure carries no units because it is measured as a ratio of two lengths with the same units (the units associated with s/r "cancel"). Thus, 2π radians is simply written as 2π . It is universally understood that the measure is in radians. Sometimes the notation "rad" is included for emphasis, but is not necessary.

1. 1 revolution: $2\pi = 360^\circ$
2. $\frac{1}{2}$ revolution: $\pi = 180^\circ$
3. $\frac{1}{4}$ revolution: $\frac{\pi}{2} = 90^\circ$
4. $\frac{3}{4}$ revolution: $\frac{3\pi}{2} = 270^\circ$

3.5 Converting Degrees to Radians

To convert to radians from degrees multiply the degree by $\frac{\pi}{180^\circ}$

$$x^\circ \cdot \frac{\pi}{180^\circ} = x_{rad}$$

To convert from radians to degrees multiply the degree by $\frac{180^\circ}{\pi}$

$$x_{rad} \cdot \frac{180^\circ}{\pi} = x^\circ$$

4 Trigonometric Functions

4.1 The Unit Circle

The unit circle consists of all points (x,y) that satisfy the equation $x^2 + y^2 = 1$

4.2 Functions

Let $P(x, y)$ be the point associated with a real number t measured along the circumference of the unit circle from the point $(1, 0)$.

1. sine:
 $\sin t = y$
2. cosine:
 $\cos t = x$
3. tangent:
 $\tan t = \frac{y}{x} (x \neq 0)$
4. cosecant:
 $\csc t = \frac{1}{y} (y \neq 0)$
5. secant:
 $\sec t = \frac{1}{x} (x \neq 0)$
6. cotangent:
 $\cot t = \frac{x}{y} (y \neq 0)$

If P is on the y -axis in a unit circle example, then $x = 0$ and the tangent and secant functions are undefined.

If P is on the x -axis in a unit circle example, then $y = 0$ and the cotangent and cosecant functions are undefined.

4.3 Functions of Real Numbers and Angles

If $\theta = t$ rad, then:

$$\sin t = \sin \theta$$

$$\cos t = \cos \theta$$

$$\tan t = \tan \theta$$

$$\csc t = \csc \theta$$

$$\cot t = \cot \theta$$

$$\sec t = \sec \theta$$

5 Using Trigonometric Functions to Determine Point Values on a Unit Circle

In a unit circle where the *radius* = 1 you can denote the equation of the circle using $x^2 + y^2 = r$, this means you can plug in $\sin t$'s or $\cos t$'s t value into the circle equation to produce the other value and thus the coordinate set for point locations, you will have to have a qualifier for the quadrant within which the real coordinate lies in order to determine the real $P(x, y)$ value.

6 Fundamental Trigonometric Identities

When computing the value of a trigonometric function the argument MUST be included.

1. $\sin t$ and $\csc t$ are reciprocals

$$\csc t = \frac{1}{\sin t} \vee \sin t = \frac{1}{\csc t}$$

2. $\cos t$ and $\sec t$ are reciprocals

$$\sec t = \frac{1}{\cos t} \vee \cos t = \frac{1}{\sec t}$$

3. $\tan t$ and $\cot t$ are reciprocals

$$\cot t = \frac{1}{\tan t} \vee \tan t = \frac{1}{\cot t}$$

4. $\tan t$ is the ratio of $\sin t$ and $\cos t$.

$$\tan t = \frac{\sin t}{\cos t}$$

5. $\cot t$ is the ratio of $\cos t$ and $\sin t$

$$\cot t = \frac{\cos t}{\sin t}$$

7 Pythagorean Identities

- 1.

$$\sin^2 t + \cos^2 t = 1$$

- 2.

$$\tan^2 t + 1 = \sec^2 t$$

- 3.

$$1 + \cot^2 t = \csc^2 t$$

8 Periodic Functions

A function f is periodic if $f(t + p) = f(t)$ for some constant p .

The smallest positive value p for which f is periodic is called the period of f .

8.1 Periodic Properties of Trigonometric Functions

The values of the six trigonometric functions of t are determined by the corresponding point $P(x, y)$ on the unit circle. The Circumference of the unit circle is 2π , adding (or subtracting) 2π to t results in the same terminal point (x, y) . Consequently, the values of the trigonometric functions are the same for t and $t + 2n\pi$

1. Sine:
 $period = 2\pi$
 $property = \sin(t + 2\pi) = \sin t$
2. Cosine:
 $period = 2\pi$
 $property = \cos(t + 2\pi) = \cos t$
3. Cosecant:
 $period = 2\pi$
 $property = \csc(t + 2\pi) = \csc t$
4. Secant:
 $period = 2\pi$
 $property = \sec(t + 2\pi) = \sec t$
5. Tangent:
 $period = \pi$
 $property = \tan(t + \pi) = \tan t$
6. Cotangent:
 $period = \pi$
 $property = \cot(t + \pi) = \cot t$

If the period is p , then $f(t + p) = f(t)$. It is also true that $f(t + np) = f(t)$ for any integer n . That is, adding any integer multiple of the period to a domain element of a periodic function results in the same function value.

9 Even and Odd Properties of Trigonometric Functions

1. Sine:
 $\sin t = y$ and $\sin(-t) = -y$
Odd Function: $\sin(-t) = -\sin t$

2. Cosine:
 $\cos t = x$ and $\cos(-t) = x$
 Even Function: $\cos(-t) = \cos t$
3. Cosecant:
 $\csc t = \frac{1}{y}$ and $\csc(-t) = \frac{1}{-y}$
 Odd Function: $\csc(-t) = -\csc t$
4. Secant:
 $\sec t = \frac{1}{x}$ and $\sec(-t) = \frac{1}{x}$
 Even Function: $\sec(-t) = \sec t$
5. Tangent:
 $\tan t = \frac{y}{x}$ and $\tan(-t) = \frac{-y}{x}$
 Odd Function: $\tan(-t) = -\tan t$
6. Cotangent:
 $\cot t = \frac{x}{y}$ and $\cot(-t) = \frac{-x}{y}$
 Odd Function: $\cot(-t) = -\cot t$

10 Trigonometric Functions of Any Angle

Let θ be an angle in standard position with point $P(x, y)$ on the terminal side, and let $r = \sqrt{x^2 + y^2} \neq 0$ represent the distance from $P(x, y)$ to $(0, 0)$. Then:

$$\begin{aligned}\sin \theta &= \frac{y}{r} \\ \cos \theta &= \frac{x}{r} \\ \tan \theta &= \frac{y}{x} (x \neq 0) \\ \csc \theta &= \frac{r}{y} (y \neq 0) \\ \sec \theta &= \frac{r}{x} (x \neq 0) \\ \cot \theta &= \frac{x}{y} (y \neq 0)\end{aligned}$$

11 Reference Angles

11.1 Definition

Let θ be an angle in standard position. The reference angle for θ is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

The length of $\angle\theta'$'s vertical leg is $|y|$.

The length of the horizontal leg is $|x|$.

The hypotenuse's length is $r = \sqrt{x^2 + y^2}$

11.2 Evaluate Trigonometric Functions using Reference Angles

$\angle\theta$ = standard position

$\angle\theta'$ = reference angle complementary to $\angle\theta$

$$\cos\theta = \frac{x}{r} \wedge \cos\theta' = \frac{adj}{hyp} = \frac{|x|}{r}$$

To Find the value of a trigonometric function of a given angle θ

1. Determine the function value of the reference angle θ'
2. Affix the appropriate sign based on the quadrant in which θ lies

12 Right Triangle Trigonometry 4.3

12.1 Anatomy of a Triangle

Consider a right triangle, the longest side of the triangle is called the hypotenuse, the remaining two legs are determined relative to the position of the considered angle. There is an acute angle θ , the leg opposite of this angle is considered the opposite leg, the adjacent leg is, similarly, considered the adjacent leg.

The six trigonometric functions take inputs relative to the acute angle θ to determine the ratios of the lengths of the sides of the triangle.

12.2 Definition of Trigonometric Functions of Acute Angles

$$sine = \sin\theta = \frac{opp}{hyp}$$

$$cosine = \cos\theta = \frac{adj}{hyp}$$

$$tangent = \tan\theta = \frac{opp}{adj}$$

$$cosecant = \csc\theta = \frac{hyp}{opp}$$

$$secant = \sec\theta = \frac{hyp}{adj}$$

$$cotangent = \cot\theta = \frac{adj}{opp}$$

The mnemonic device "SOH-CAH-TOA" may help you remember the ratios for $\sin\theta$, $\cos\theta$, $\tan\theta$, respectively.

12.3 Pythagorean Theorem

legs a, b;
hypotenuse c;

$$a^2 + b^2 = c^2$$

12.4 Cofunction Identities

Cofunctions of complementary angles are equal; the "co" prefixing any one of these functions stands for "complementary".

1. Sine and Cosine are Cofunctions:

$$\sin \theta = \cos(90^\circ - \theta)$$

$$\cos \theta = \sin(90^\circ - \theta)$$

2. Tangent and Cotangent are Cofunctions:

$$\tan \theta = \cot(90^\circ - \theta)$$

$$\cot \theta = \tan(90^\circ - \theta)$$

3. Secant and Cosecant are Cofunctions:

$$\sec \theta = \csc(90^\circ - \theta)$$

$$\csc \theta = \sec(90^\circ - \theta)$$