Trigonometry

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1 Ray

A ray is part of a line that consists of an endpoint and all points on the line to one side of the endpoint.

A ray is denoted symbolically by $\overrightarrow{PQ}$ where P is the endpoint and Q is another point on the ray to the right of P. A ray always has one endpoint which is always written first when naming the ray.

2 Angle

An angle is formed by rotating a ray about it's endpoint. The starting position is called the initial side of the angle, and the final position of the ray is called the terminal side. The common endpoint is called the vertex of the angle, and the vertex is often denoted by a capitlal letter such as A.

An angle is notated by $\angle A$ or by $\angle BAC$, where B is the initial side, A being the vertex, and C being the terminal side.

2.1 Standard Position

An angle is said to be at standard position if it's vertex is at the origin in the xy-plane, and the initial ray is directly on the x axis.

2.2 Measurement

The measure of an angle quantifies the direction and amount of rotation from the initial ray to the terminal ray. The measure is positive if the rotation is counterclockwise, and the measurement is negative if the rotation is clockwise. Often, degrees are used to measure the amount of rotation, one full rotation of a ray about its vertex is equivalent to $360°$

2.3 Angle Type Names

1. Right Angle:
   Right angles are angles at $90°$ measure.
2. Straight Angle:
   Straight angles are angles at $180°$ measure.
3. Acute Angle:
   Acute angles are angles at $0° < \theta < 90°$
4. Obtuse Angle:
   Obtuse angles are angles at $90° < \theta < 180°$

5. Complementary Angles:
   A pair of angles which sum to $90°$
6. Supplementary Angles:
   A pair of angles which sum to $180°$
7. Coterminal Angles:
   Two angles in standard position with the same initial side and same terminal side are called coterminal angles.

2.4 Units

1. Minutes / Arcminutes (min or $'$):
   A singular degree can be divided into 60 equal parts; each 1/60th is referred to as one arcminute / minute. $\left(\frac{1}{60}\right)^{\circ} = 1' \vee 1min$
2. Seconds / Arcseconds (sec or "):
   A singular arcminute can be divided into 60 equal parts; each 1/60th of such is referred to as one arcsecond. $\left(\frac{1}{60}\right)' = 1'' \vee 1sec$

3 Radian Measurement

3.1 Definition of a Radian

A central angle that incercepts an arc on the circle with length equal to the radius of the circle has a measure of one radian.

3.2 Angle Measurement

When two lines or rays cross a circle, the part of the circle between the intersection points is called the intercepted arc and is often denoted by s.

Any central angle can be measured in radians by dividing the length s of the intercepted arc by the radius r.

3.3 Review

1. π:
   is defined as the ratio of the circumference of a circle to its diameter d.

2. Circumference:
   Therefore the circumference is given by $C = \pi d$ or $C = 2\pi r$, where r is the radius of the circle, dividing this by the radius gives the number of radians in one revolution. The angular measure of one full rotation is 2π

3.4 Definition of Radian Angle Measure

The radian measure of a central angle θ subtended by an arc of length s on a circle of radius r is given by $\theta = \frac{s}{r}$. radian measure carries no units because it is measured as a ratio of two lengths with the same units (the units associated with s/r "cancel"). Thus, 2π radians is simply written as 2π. It is universally understood that the measure is in radians. Sometimes the notation "rad" is included for emphasis, but is not necessary.

1. 1 revolution: $2\pi = 360°$
2. $\frac{1}{2}$ revolution: $\pi = 180°$
3. $\frac{1}{4}$ revolution: $\frac{\pi}{2} = 90°$
4. $\frac{3}{4}$ revolution: $\frac{3\pi}{2} = 270°$

3.5 Converting Degrees to Radians

To convert to radians from degrees multiply the degree by $\frac{\pi}{180°}$

$$x° \cdot \frac{\pi}{180°} = x_{rad}$$

To convert from radians to degrees multiply the degree by $\frac{180°}{\pi}$

$$x_{rad} \cdot \frac{180°}{\pi} = x°$$

4 Trigonometric Functions

4.1 The Unit Circle

The unit circle consists of all points (x,y) that satisfy the equation $x^2 + y^2 = 1$

4.2 Functions

Let $P(x, y)$ be the point associated with a real number t measured along the circumference of the unit circle from the point (1,0).

1. sine:
   $\sin t = y$

2. cosine:
   $\cos t = x$

3. tangent:
   $\tan t = \frac{y}{x}(x \neq 0)$

4. cosecant:
   $\csc t = \frac{1}{y}(y \neq 0)$

5. secant:
   $\sec t = \frac{1}{x}(x \neq 0)$

6. cotangent:
   $\cot t = \frac{x}{y}(y \neq 0)$

If P is on the y-axis in a unit circle example, then x = 0 and the tangent and secant functions are undefined.

If P is on the x-axis in a unit circle example, then y = 0 and the cotangent and cosecant functions are undefined.

4.3 Functions of Real Numbers and Angles

If $\theta = t$ rad, then:

$$\sin t = \sin\theta$$
$$\cos t = \cos\theta$$
$$\tan t = \tan\theta$$
$$\csc t = \csc\theta$$
$$\cot t = \cot\theta$$
$$\sec t = \sec\theta$$

5 Using Trigonometric Functions to Determine Point Values on a Unit Circle

In a unit circle where the $radius = 1$ you can denote the equation of the circle using $x^2 + y^2 = r$, this means you can plug in $\sin t$'s or $\cos t$'s t value into the circle equation to produce the other value and thus the coordinate set for point locations, you will have to have a qualifier for the quadrant within which the real coordinate lies in order to determine the real $P(x, y)$ value.

6 Fundamental Trigonometric Identities

When computing the value of a trigonometric function the argument MUST be included.

1. $\sin t$ and $\csc t$ are reciprocals

$$\csc t = \frac{1}{\sin t} \vee \sin t = \frac{1}{\csc t}$$

2. $\cos t$ and $\sec t$ are reciprocals

$$\sec t = \frac{1}{\cos t} \vee \cos t = \frac{1}{\sec t}$$

3. $\tan t$ and $\cot t$ are reciprocals

$$\cot t = \frac{1}{\tan t} \vee \tan t = \frac{1}{\cot t}$$

4. $\tan t$ is the ratio of $\sin t$ and $\cos t$.

$$\tan t = \frac{\sin t}{\cos t}$$

5. $\cot t$ is the ratio of $\cos t$ and $\sin t$

$$\cot t = \frac{\cos t}{\sin t}$$

7 Pythagorean Identities

1.
$$\sin^2 t + \cos^2 t = 1$$

2.
$$\tan^2 t + 1 = \sec^2 t$$

3.
$$1 + \cot^2 t = \csc^2 t$$

8 Periodic Functions

A function f is periodic if $f(t + p) = f(t)$ for some constant p.
The smallest positive value p for which f is periodic is called the period of f.

8.1 Periodic Properties of Trigonometric Functions

The values of the six trigonometric functions of t are determined by the corresponding point $P(x, y)$ on the unit circle. The Circumference of the unit circle is 2π, adding (or subtracting) 2π to t results in the same terminal point (x, y). Consequently, the values of the trigonometric functions are the same for t and $t + 2n\pi$

1. Sine:
   $period = 2\pi$
   $property = \sin(t + 2\pi) = \sin t$

2. Cosine:
   $period = 2\pi$
   $property = \cos(t + 2\pi) = \cos t$

3. Cosecant:
   $period = 2\pi$
   $property = \csc(t + 2\pi) = \csc t$

4. Secant:
   $period = 2\pi$
   $property = \sec(t + 2\pi) = \sec t$

5. Tangent:
   $period = \pi$
   $property = \tan(t + \pi) = \tan t$

6. Cotangent:
   $period = \pi$
   $property = \cot(t + \pi) = \cot t$

If the period is p, then $f(t + p) = f(t)$. It is also true that $f(t + np) = f(t)$ for any integer n. That is, adding any integer multiple of the period to a domain element of a periodic function results in the same function value.

9 Even and Odd Properties of Trigonometric Functions

1. Sine:
   $\sin t = y$ and $\sin(-t) = -y$
   Odd Function: $\sin(-t) = -\sin t$

2. Cosine:
   $\cos t = x$ and $\cos(-t) = x$
   Even Function: $\cos(-t) = \cos t$

3. Cosecant:
   $\csc t = \frac{1}{y}$ and $\csc(-t) = \frac{1}{-y}$
   Odd Function: $\csc(-t) = -\csc t$

4. Secant:
   $\sec t = \frac{1}{x}$ and $\sec(-t) = \frac{1}{x}$
   Even Function: $\sec(-t) = \sec t$

5. Tangent:
   $\tan t = \frac{y}{x}$ and $\tan(-t) = \frac{-y}{x}$
   Odd Function: $\tan(-t) = -\tan t$

6. Cotangent:
   $\cot t = \frac{x}{y}$ and $\cot(-t) = \frac{x}{-y}$
   Odd Function: $\cot(-t) = -\cot t$

10 Trigonometric Functions of Any Angle

Let θ be an angle in standard position with point $P(x, y)$ on the terminal side, and let $r = \sqrt{x^2 + y^2} \neq 0$ represent the distance from $P(x, y)$ to $(0, 0)$. Then:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}(x \neq 0)$$
$$\csc\theta = \frac{r}{y}(y \neq 0)$$
$$\sec\theta = \frac{r}{x}(x \neq 0)$$
$$\cot\theta = \frac{x}{y}(y \neq 0)$$

11 Reference Angles

11.1 Definition

Let θ be an angle in standard position. The reference angle for θ is the acute angle θ' formed by the terminal side of θ and the horizontal axis.
The length of $\angle\theta'$s vertical leg is $|y|$.
The length of the horizontal leg is $|x|$.
The hypotenuse's length is $r = \sqrt{x^2 + y^2}$

11.2 Evaluate Trigonometric Functions using Reference Angles

$\angle\theta =$ standard position
$\angle\theta' =$ reference angle complementary to $\angle\theta$

$$\cos\theta = \frac{x}{r} \wedge \cos\theta' = \frac{adj}{hyp} = \frac{|x|}{r}$$

To Find the value of a trigonometric function of a given angle θ

1. Determine the function value of the reference angle θ'
2. Affix the appropriate sign based on the quadrant in which θ lies

12 Right Triangle Trigonometry 4.3

12.1 Anatomy of a Triangle

Consider a right triangle, the longest side of the triangle is called the hypotenuse, the remaining two legs are determined relative to the position of the considered angle. There is an acute angle θ, the leg opposite of this angle is considered the opposite leg, the adjacent leg is, similarly, considered the adjacent leg.

The six trigonometric functions take inputs relative to the acute angle θ to determine the ratios of the lengths of the sides of the triangle.

12.2 Definition of Trigonometric Functions of Acute Angles

$$sine = \sin\theta = \frac{opp}{hyp}$$

$$cosine = \cos\theta = \frac{adj}{hyp}$$

$$tangent = \tan\theta = \frac{opp}{adj}$$

$$cosecant = \csc\theta = \frac{hyp}{opp}$$

$$secant = \sec\theta = \frac{hyp}{adj}$$

$$cotangent = \cot\theta = \frac{adj}{opp}$$

The mnemonic device "SOH-CAH-TOA" may help you remember the ratios for $\sin\theta$, $\cos\theta$, $\tan\theta$, respectively.

12.3 Pythagorean Theorem

legs a, b;
hypotenuse c;

$$a^2 + b^2 = c^2$$

12.4 Cofunction Identities

Cofunctions of complementary angles are equal; the "co" prefixing any one of these functions stands for "complementary".

1. Sine and Cosine are Cofunctions:

$$\sin\theta = \cos(90^\circ - \theta)$$
$$\cos\theta = \sin(90^\circ - \theta)$$

2. Tangent and Cotangent are Cofunctions:

$$\tan\theta = \cot(90^\circ - \theta)$$
$$\cot\theta = \tan(90^\circ - \theta)$$

3. Secant and Cosecant are Cofunctions:

$$\sec\theta = \csc(90^\circ - \theta)$$
$$\csc\theta = \sec(90^\circ - \theta)$$

12.5 Angles of Elevation and Depression

An angle of elevation gives the measure upwards from a horizontal line of reference.
An angle of depression gives the measure downwards from a horizontal line of reference.

cos X sin X csc X sec X tan X cot X

13 Graphs of Trigonometric Functions

13.1 Graphing y = sine x and y = cos x

Recall that $\sin\theta = \frac{y}{r}$, thus if you measure $y = \sin x$ you will produce a wave about the x axis with a peak of 1 and a trough of -1 which resembles a semicircle alternating across the ±y-plane as it approaches $\pm\inf$ on x-plane.

The cos function performs similarly being that they are cofunctions, however they differ about the origin in that sin which represents the $\frac{y}{r}$ value for a unit circle example will start at 0 whereas cos, being that it starts at $\frac{x}{r}$ will start at 1

13.2 Trigonometric Function Transformations

Similarly to other functions, $\sin \wedge \cos$ can have function transformations performed upon their parent function to produce predictably altered graphs.
Phase Shift is another word for the horizontal translation of a trigonometric function.
Horizontal scaling causes period changes because it stretches or compresses the amount inputed into the parent function.
The function transformations seem to apply generically to trigonometric functions in a manner similar to basic algebraic functions.

13.3 Properties of General Sine and Cosine Functions

The Sine wave form starts halfway between it's crest and trough because the degree input 0 corresponds to point P(1,0) on the unit circle.

The Cosine wave form starts at the crest of it's wave because it represents the x-value on a unit circle of the degree inputed, i.e. $0 = 1$ because degree zero corresponds to the point P(1,0) on the unit circle.

Recall transformation format:

$$a \cdot f(b(x-h)) + k$$

Sine and Cosine functions measure period using the following format for convenience:

$$y = A \cdot f(Bx - C) + D$$

This is just a distributed form of the above, where the horizontal scaler has been distributed over the parent functions inner quantity.

Taking the above into account, you can easily determine the horizontal scaler and phase shift (horizontal translater) by reverse distributing the coefficient of the x-value, B.

In the following explanations I will be using the variable names from both of the above forms to refer to measurements.

1. Amplitude is determined by taking the absolute value of the vertical scaler variable $|a| \vee |A|$.

2. The period is determined by:

$$\frac{2\pi}{B}(B > 0)$$

   If $B < 0$ then rewrite the function using the odd and even properties before evaluation.

3. The phase shift is determined by h or:
$$\frac{C}{B}$$

4. The vertical shift is determined by D or k.

13.4 Producing a Function According to Given Specifications

Where crest is the highpoint and trough is the lowpoint.

$$x = crest$$
$$y = trough$$

1. Amplitude is found by halving the distance between the highest and lowest values.
2. Cosine Period: Twice the distance from the first maximum value to the minimum value.
3. Cosine Vertical Shift: The midpoint of the range from the highest to lowest value gives the vertical shift value:
$$-|A| + D \leq y \leq |A| + D$$
4. Phase Shift: Where the graph beings = y. find using normal $\frac{C}{B} = y$ algebraic deduction.

13.5 Graphing Cosecant Functions

In the cosecant function,
$$\frac{r}{y} \vee \frac{1}{\sin\theta}$$
a vertical asymptote appears with a period of π, because when the degree lies on the x-axis y is zero. Therefore the function,
$$f(x) = \csc x$$
Will have vertical asympotes at,
$$f(\pi), f(2\pi), f(3\pi), ...$$
and so on.

The relative minima and maxima of the sine function correspond to the relative minima and maxima of the base of the quadratic like in shape, functions which occur between the vertical asympotes.

13.6 Graphing Secant Functions

The Secant function, corresponding to the cosine function as it's reciprocal, is rather like the cosecant function however it's period of vertical asymptote production is different, occuring at every $\frac{\pi}{2}$ mark, i.e. along the y-axis. This is the main differentiator between the Secant function and the Cosecant function.

13.7 Graphing Tangent Functions

Recall that the $\tan\theta$ is an odd function. It's period is π. The function resembles a cube root function which repeats itself between vertical asympotes which lie along the x-axis and y-axis in terms of x and y unit circle measurements.

13.8 Universal Transformation Formulas

Transformation Format:

$$y = A \cdot f(Bx - C) + D$$

1. Amplitude:

$$Amp = |A|$$

2. Phase Shift:

$$PS = C$$

3. Vertical Translation:

$$VT = D$$

4. Period:

   $period_{parent}$ = the parent functions period

$$P = \frac{period_{parent}}{|B|}$$