Elementary Statistics – Probability

Dr. Ab Mosca (they/them)

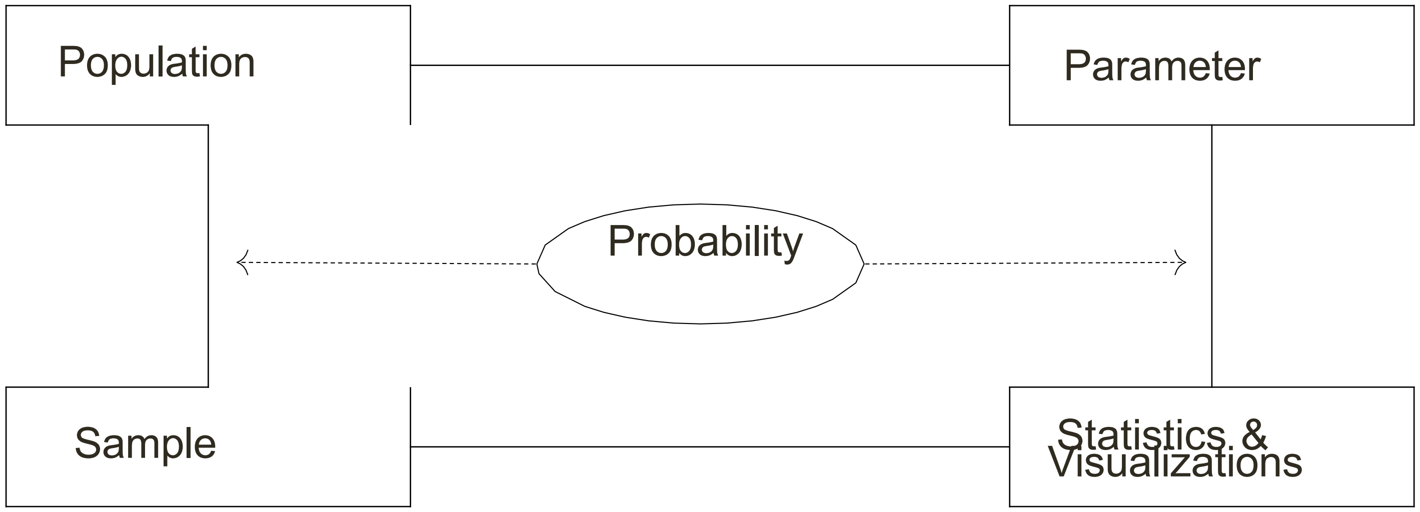
Slides based off slides courtesy of Kaitlyn Cook (https://www.smith.edu/people/kaitlyn-cook)

Plan for Today

- Probability
  - Rules
  - Independence

Recall: Course Overview

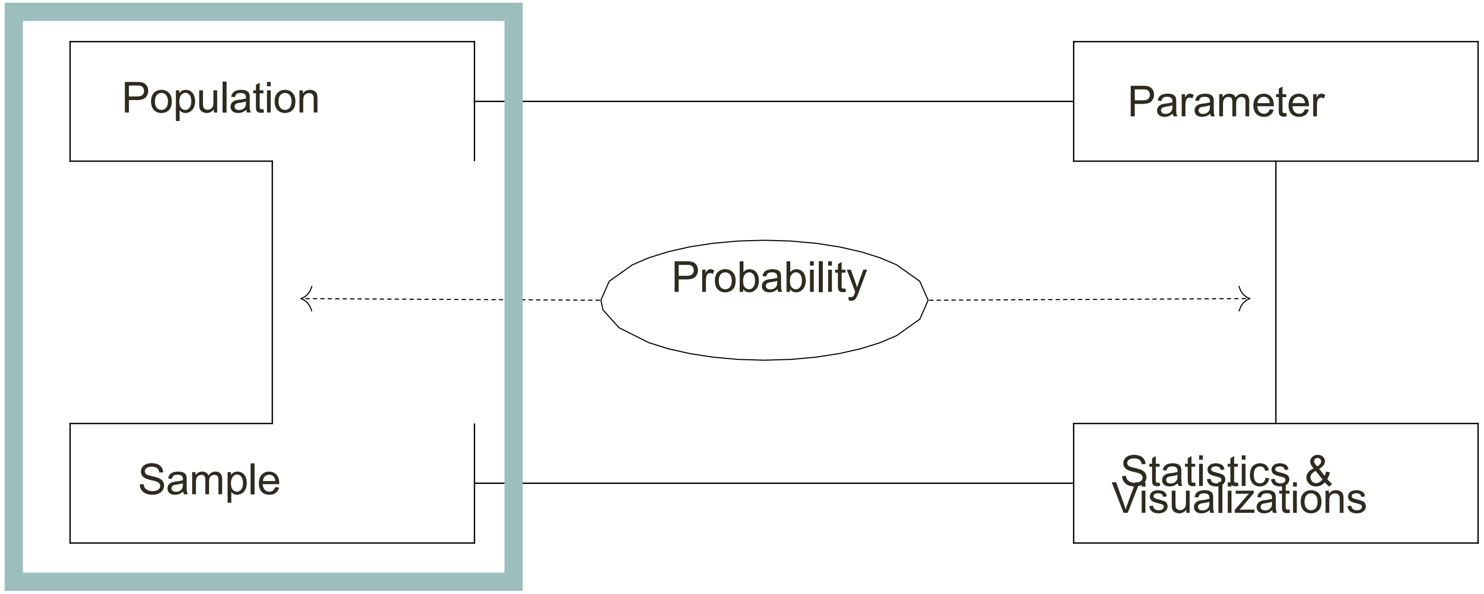
Given a statistical question. . .



- Population: the target group about which we wish to make claims or predictions
- Parameter: numerical summary of the population
- Sample: the data that we have at hand
- Statistic: numerical summary of the sample

Recall: Course Overview

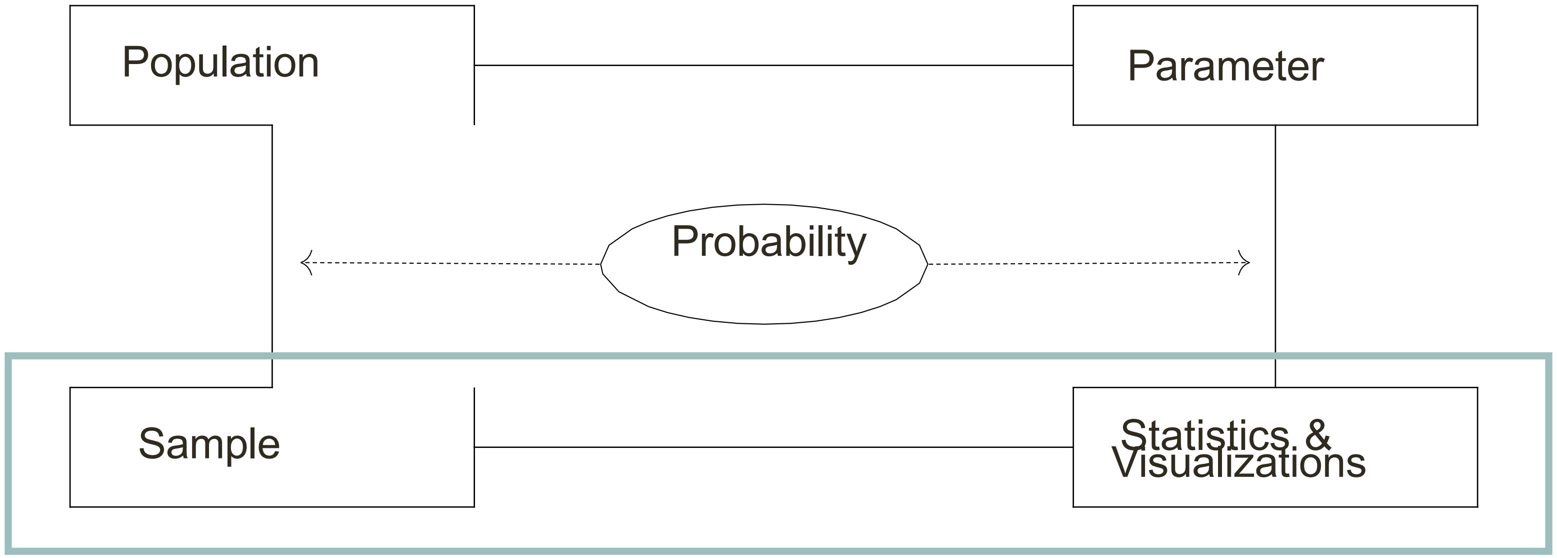
Given a statistical question…



Where do our data come from?

Recall: Course Overview

Given a statistical question…



How do we summarize and make sense of all this data (in a way that informs our research question)?

Course Overview

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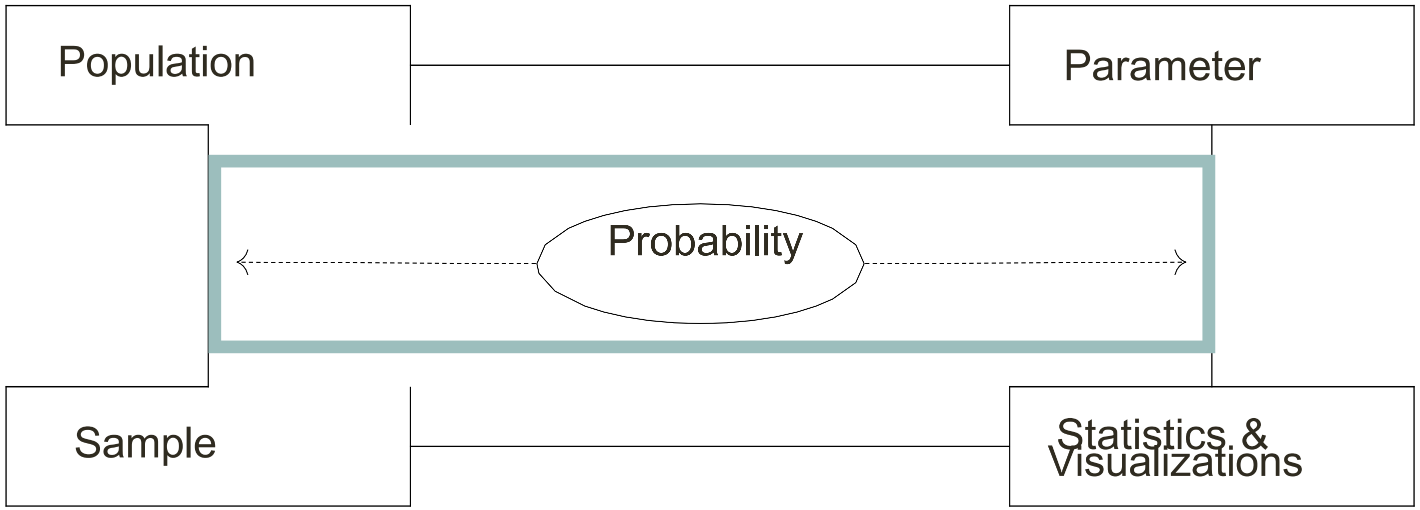

How can we use ideas from mathematics to relate our sample (and sample statistic) back to the population (and parameter of interest)?

What is Probability?

Suppose I have a fair coin and give it a toss. What is the chance that it lands heads up?

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Simulated Coin Tosses

One Coin Toss

```
## # A tibble: 1 x 2
##   tosses      n
##   <chr>   <int>
## 1 Tails       1
```

10 Coin Tosses

```
## # A tibble: 2 x 2
##   tosses      n
##   <chr>   <int>
## 1 Heads       3
## 2 Tails       7
```

100 Coin Tosses

```
## # A tibble: 2 x 2
##   tosses      n
##   <chr>   <int>
## 1 Heads      52
## 2 Tails      48
```

Notice how frequency of the event relates to outcomes.

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Suppose I have a fair coin and give it a toss. What is the chance that it lands heads up?

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```

Notice how frequency of the event relates to outcomes.
The ***probability of an event*** is the long-run relative frequency with which that events occurs if we were to repeat the random process (ex. flipping a coin) an infinite number of times.

What is Probability?

What about 10 sided-dice?

Frequencies when everyone tosses once:

Twice:

Three times:

Four times:

Probability Vocab

A ***random experiment*** is some activity, process, or experiment whose outcome is uncertain.

a) Call and ask a doctor whether they approve of a treatment
b) Forecast snow tomorrow
c) Conduct two consecutive coin flips

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Sample space is the collection (set) of _all possible outcomes_ of this experiment. (Denoted with S)

a) $S = \{Into\ It, Not\ Into\ It, Who\ is\ this?\}$
b) $S = \{Snows\ tomorrow, Does\ not\ snow\ tomorrow\}$
c) $S = \{(H,H),(H,T),(T,H),(T,T)\}$

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An ***event space*** is a collection of possible outcomes

a) Doctor's response: $\{Who\ is\ this?\}$
b) It snows tomorrow: $\{Snoes\ tomorrow\}$
c) First coin lands heads up: $\{(H,H),(H,T)\}$

Probability Vocab

A ***random experiment*** is some activity, process, or experiment whose outcome is uncertain.

Sample space is the collection (set) of *<u>all possible outcomes</u>* of this experiment. (Denoted with S)

An ***event space*** is a collection of possible outcomes

Practice: Suppose you roll the 10 sided dice you were given earlier. What is the sample space of that die roll? Give two examples of event spaces.

Suppose I have a shuffled deck of cards and deal 4 of them. What is the sample space of this experiment? Give two examples of event spaces.

Foundational Rules of Probability

Given a random experiment with sample space, S, a ***probability distribution*** lists:

- All possible outcomes of that experiment ($s \in S$) and
- The probabilities of each outcome ($P(\{s\})$)

To be valid, these probabilities must follow three rules:

1. ***Non-negative***: All probabilities must be positive
2. ***Sum to one***: Adding $P(\{s\})$ for all s must equal 1
3. ***Additive***: For any collection of events, A, $P(A)$ must equal the sum of the probabilities of each event in A

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Practice: Is the assignment of probabilities to each individual event below valid? Experiment: Choose a US-based movie at random and record both (i) whether or not it is G-rated and (ii) whether or not it had a box office gross of more than $100 million.

- $P(G\ rated\ and\ high\ box\ office\ earner) = 0.37$
- $P(G\ rated\ and\ low\ box\ office\ earner) = 0.63$
- $P(not\ G\ rated\ and\ high\ bbox\ office\ earner) = 0.22$
- $P(not\ G\ rated\ and\ low\ box\ office\ earner) = 0.78$

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Practice: Is the assignment of probabilities to each individual event below valid?
Experiment: deal a card from a well-shuffled deck of cards, where a standard deck contains 52 cards: 13 spades (♠), 13 clubs (|♣), 13 diamonds (♦), and 13 hearts (❤️)

- $P(spade) = 13/52$
- $P(club) = 13/52$
- $P(diamond) = 13/52$
- $P(heart) = 13/52$

Assigning Probabilities to Events

Problem

We don’t often have the time, energy, ability, etc. to repeat an experiment an infinite number of times. How do we assign probabilities to events in a realistic way?

With equally likely outcomes…

If the event A contains k outcomes and the sample space S contains n outcomes, then $P(A) = k/n$

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Practice: We flip a fair coin twice. Let A be the event that the coin lands on heads on the first toss. What is $P(A)$?

Hint: What is S? What is A?

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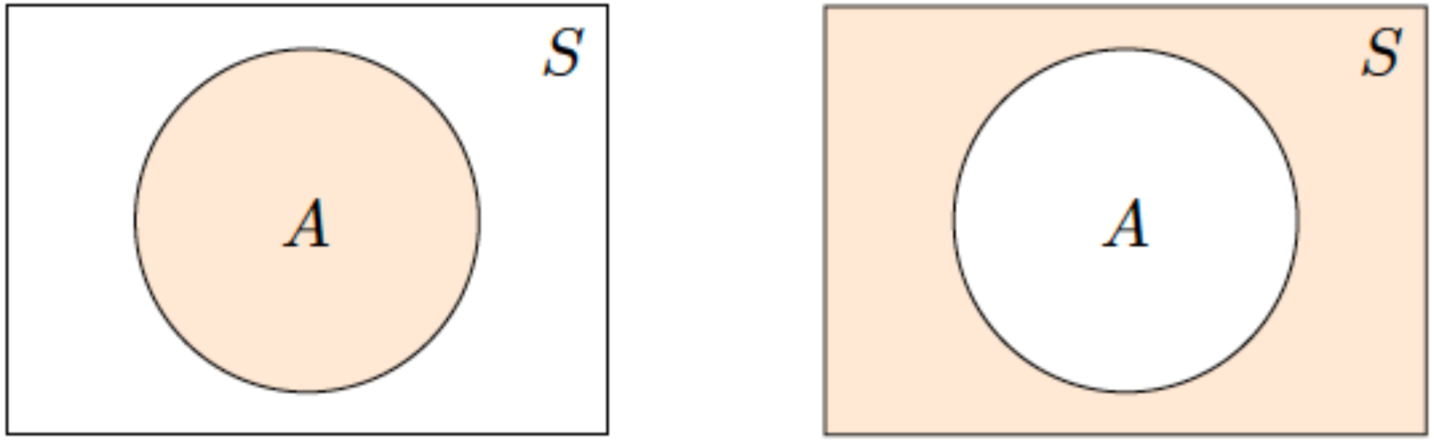
If the event A contains k outcomes and the sample space S contains n outcomes, then $P(A) = k/n$

With more complex events…
We use set operations to re-write the complex events in terms of simpler evens we know

Assigning Probabilities to Events

Let A be an event.

The probability of "***not A***" (A^c) is equivalent to 1 minus the probability of A



$$P(A^c) = 1 - P(A)$$

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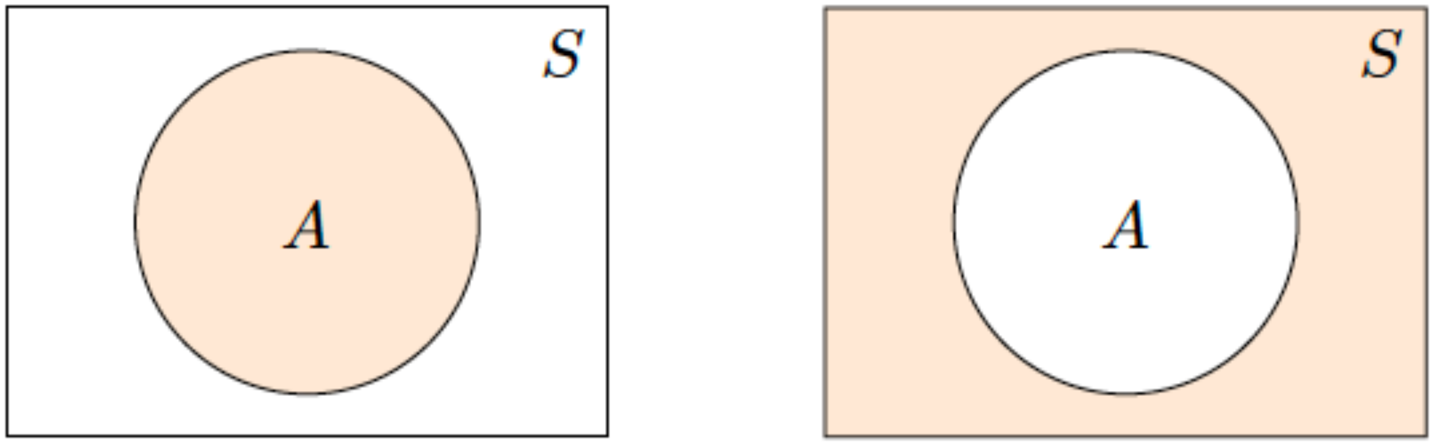

$$P(A^c) = 1 - P(A)$$

Experiment: Randomly pull a card from a deck. Let A be the event that the card is a king. What is A^c? What is $P(A^c)$?

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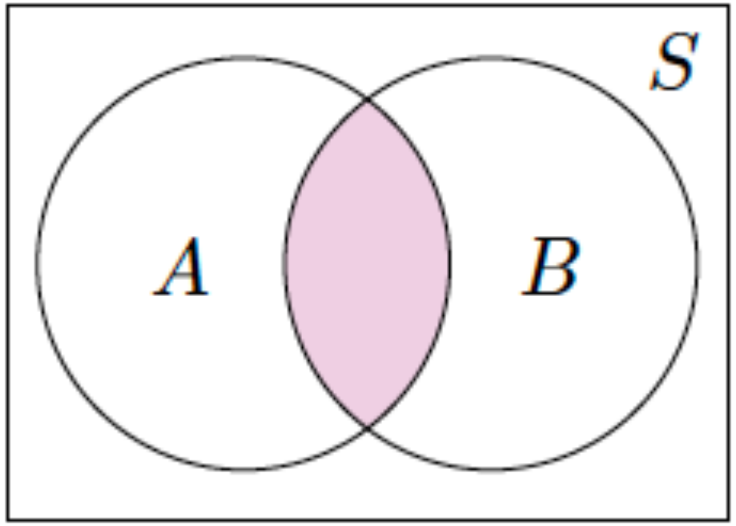
Let A and B be events.

The probability of A and B is the probability of the ***intersection*** (shared events) of the two



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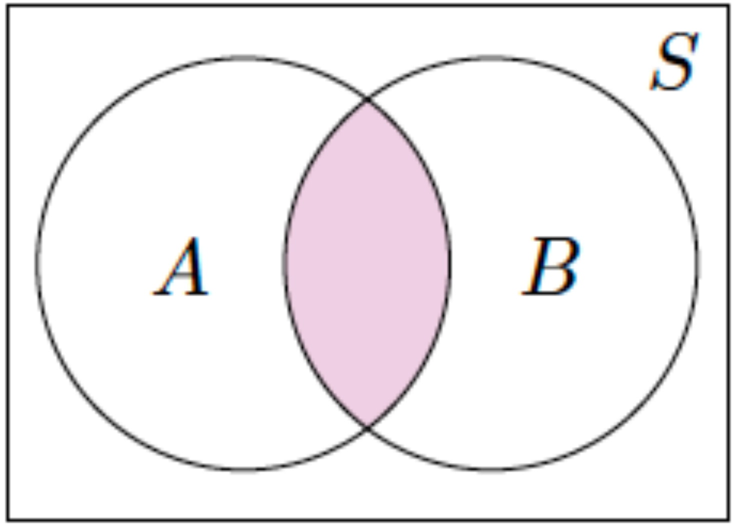

$$P(A \cap B)$$

Experiment: Randomly pull a card from a deck. Let A be the event that the card is a king, and B be the event that the card is a diamond. What is $A \cap B$? What is $P(A \cap B)$?

Assigning Probabilities to Events

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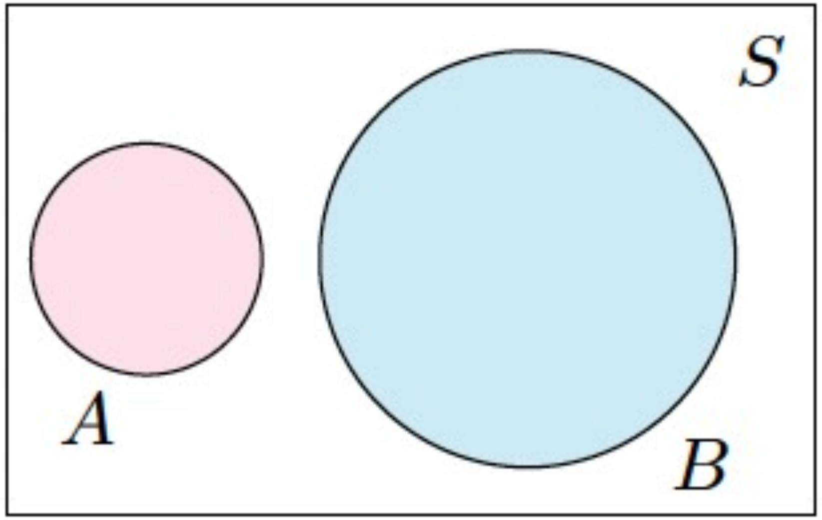
The probability of A and B is the probability of the ***intersection*** (shared events) of the two



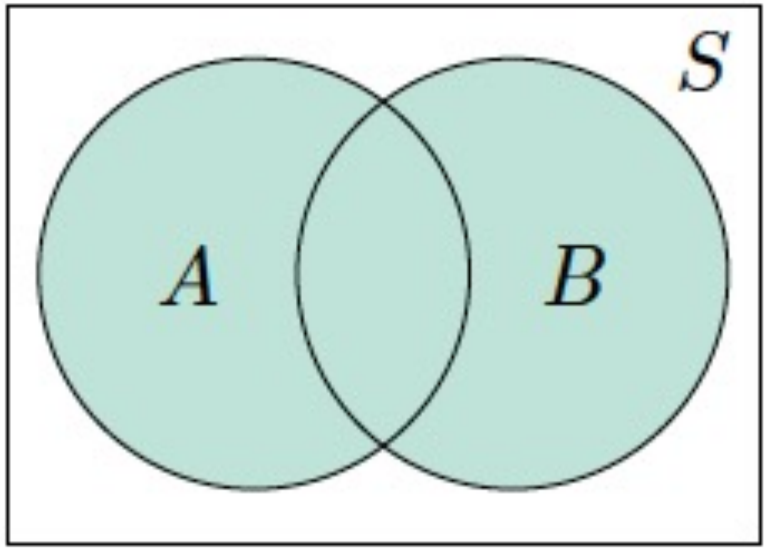
$P(A \cap B)$

If there is no intersection (overlap) between A and B we say these events are ***mutually exclusive***.

Assigning Probabilities to Events

Let A and B be events.

The probability of A or B is the probability of the ***union*** of the two. The union consists of all evens in either A or B or both.



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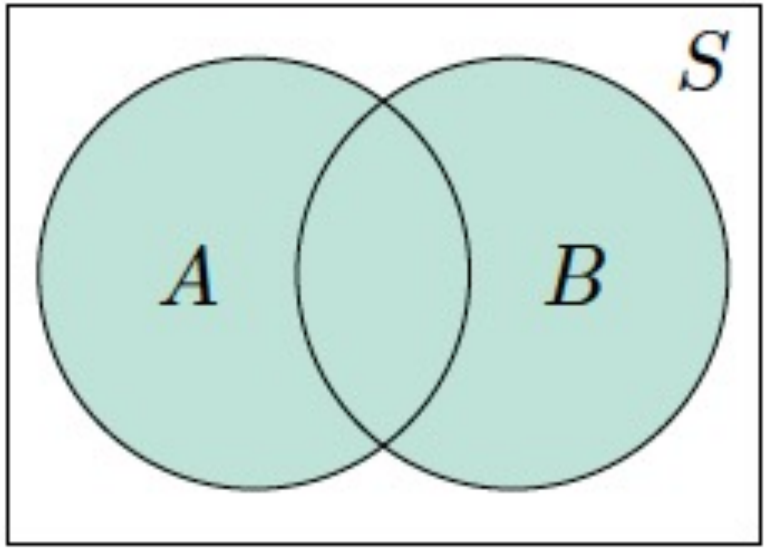


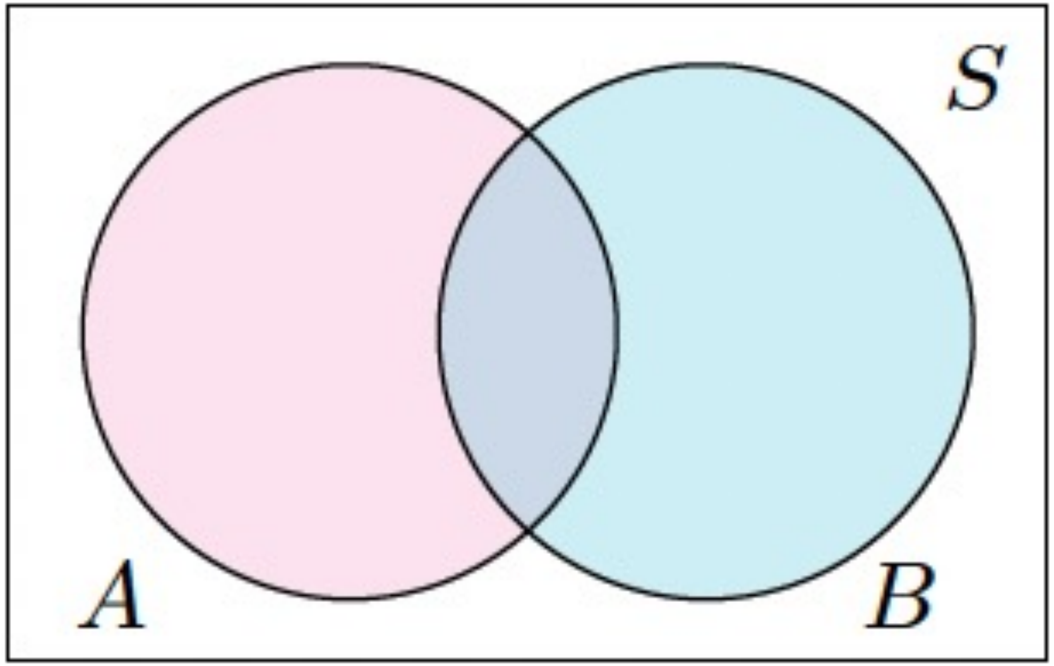
$$P(A \cup B)$$

Experiment: Randomly pull a card from a deck. Let A be the event that the card is a king, and B be the event that the card is a diamond. What is $A \cup B$? What is $P(A \cup B)$?

Assigning Probabilities to Events

Let A and B be events.

General Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



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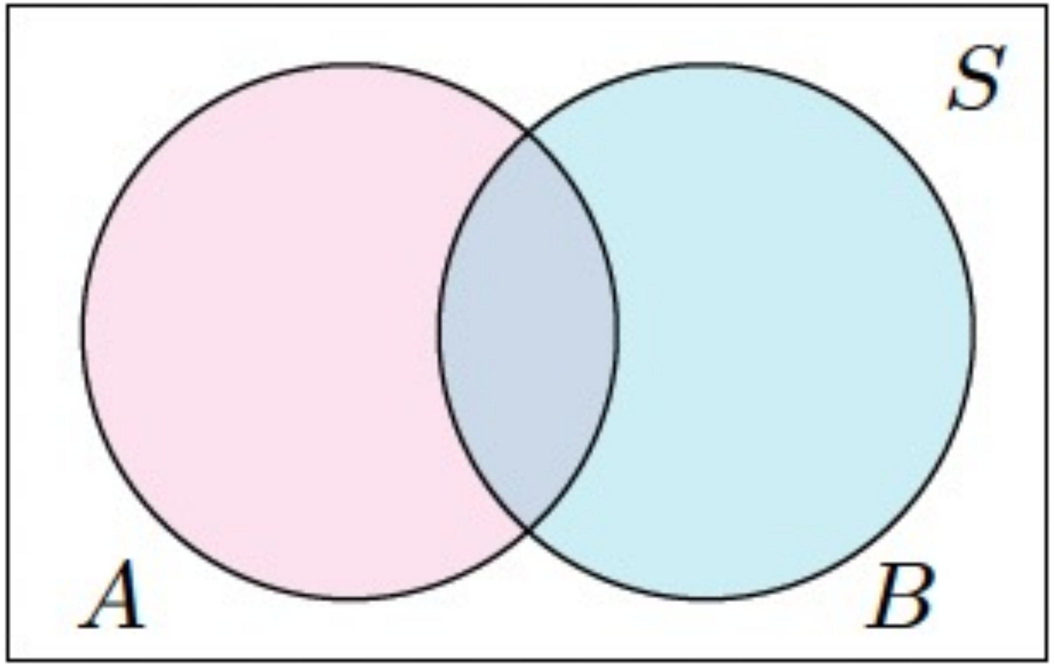

Practice: If A and B are mutually exclusive, what does $P(A \cup B)$ simplify to?

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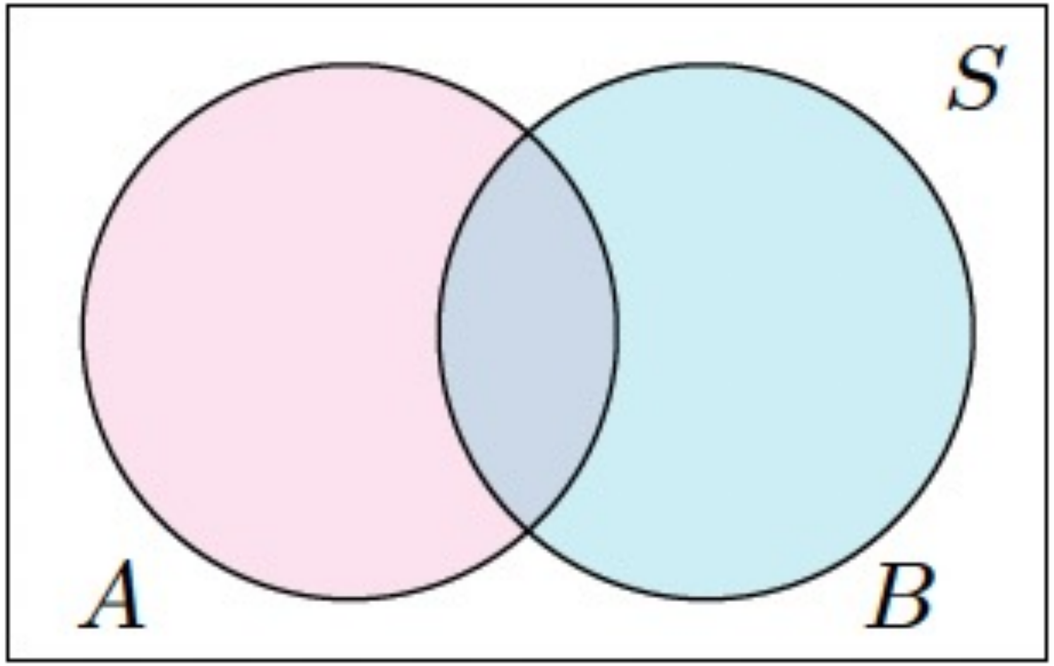

Practice: Suppose that 100 individuals were surveyed about their TV watching habits. 35 of those surveyed reported watching the TV show Survivor, 15 reported watching Big Brother, and 10 reported watching both. What percentage of the 100 individuals watched either Survivor or Big Brother?

Independance

Two events are ***independent*** if knowing the outcome of one provides no useful information about the outcome of the other

- Ex. In consecutive coin flips, knowing the first coin flip landed on heads *does not* provide any information on the determining what the second coin flip landed on.
  - Consecutive coin flips are independent
- Ex. In drawing cards from a deck without replacement, knowing that the first car draw was an ace *does* provide useful information for determining what the next card might be.
  - Consecutive card draws are not independent!

Assigning Probabilities to Events

Multiplication Rule for Independent Events:

If A and B are independent events:

$$P(A \cap B) = P(A) \times P(B)$$

More generally….

If there are k independent events, $A_1, A_2, A_3, \ldots, A_k$, the probability that all k occur is $P(A_1) \times P(A_2) \times P(A_3) \times \cdots \times P(A_k)$

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Multiplication Rule for Independent Events:

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Practice: Suppose you toss your 10 sided die twice in a row. What is the probability of getting a 2 both times?

Marginal, Joint, and Conditional Probabilities

Recall

- A ***contingency table*** summarizes the distribution of two categorical variables by displaying the number of observations falling into each unique combination of levels

	Box Office Gross		
MPAA Rating	Low	High	Total
Not Rated	21	0	21
G	41	25	66
PG	328	143	471
PG-13	856	252	1108
R	1207	124	1331
NC-17	13	0	13
Total	2466	544	3010

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Marginal Distribution

544 of the movies in the dataset were high box office earners:

$$\hat{p} = \frac{544}{3010} \approx 18\%$$

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Joint Distribution

25 of the movies in the dataset are rated G and were high box office earners:

$$\hat{p} = \frac{25}{3010} \approx 0.8\%$$

Marginal, Joint, and Conditional Probabilities

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Conditional Distribution

Among the movies in the dataset rated G, 25 were high box office earners:

$$\hat{p} = \frac{25}{66} \approx 37.9\%$$

Marginal, Joint, and Conditional Probabilities

Marginal probability captures information about a single event/process at a time

Joint probability considers how these two (or more) processes behave simultaneously

Conditional probability encodes how the probability of one event changes given that we know that the second event has occurred

Marginal, Joint, and Conditional Probabilities

Suppose we want to understand the relationship between two events: M (the event that a movie is a high box office earner) and G (the event that a movie is G rated)

Marginal probability captures information about a single event/process at a time

$$\boldsymbol{P(M) = 18\%}$$

Joint probability considers how these two (or more) processes behave simultaneously

$$\boldsymbol{P(M \cap G) = P(M\ and\ G) = 0.8\%}$$

Conditional probability encodes how the probability of one event changes given that we know that the second event has occurred

$$\boldsymbol{P(M|G) = P(M\ given\ G) = \frac{P(M \cap G)}{P(G)} = 38\%}$$

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Practice: A 1989 study of first-line therapies for cocaine dependency randomly assigned 72 chronic users into three groups: desipramine (antidepressant), lithium (standard treatment at the time), and placebo. Results of the study are summarized below:

	relapse	no relapse	total
desipramine	10	14	24
lithium	18	6	24
placebo	20	4	24
total	48	24	72

1. What is the probability that a patient did not relapse over the course of the study?
2. What is the probability that a patient received desipramine and relapsed?
3. Given that a patient received desipramine, what is the probability that they relapsed?