Discrete Structures– Sequences Pt. 1

Dr. Ab Mosca (they/them)

Plan for Today

- Sequences
  - Describing
  - Arithmetic
  - Geometric

Warm Up: Counting and Proofs

A ***permutation*** is a (possible) rearrangement of objects. We write this $P(n,k)$, and call it a ***k-permutation of n elements***

$$P(n,k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)\ldots(n-(k-1))$$

A ***combination*** is the number of ways to choose k objects from n. We write this $C(n,k)$ or $\binom{n}{k}$, and read both ***n choose k***.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Consider the identity: $k\binom{n}{k} = n\binom{n-1}{k-1}$. Prove that this identity is true.

Motivation

You have a collection of 1X1 squares and 1X2 squares. You want to arrange these to make a 1X15 strip.

1. How many length 1X1 strips can you make? How many 1X2 strips? How many 1X3 strips? How many 1X4? Etc

Motivation

You have a collection of 1X1 squares and 1X2 squares. You want to arrange these to. make a 1X15 strip.

1. How many length 1X1 strips can you make? How many 1X2 strips? How many 1X3 strips? How many 1X4? Etc
2. How are the 1X3 and 1X4 strips related to the 1X5 strips?

Motivation

You have a collection of 1X1 squares and 1X2 squares. You want to arrange these to. make a 1X15 strip.

1. How many length 1X1 strips can you make? How many 1X2 strips? How many 1X3 strips? How many 1X4? Etc.
2. How are the 1X3 and 1X4 strips related to the 1X5 strips?
3. How many 1X15 strips can you make?

Motivation

You have a collection of 1X1 squares and 1X2 squares. You want to arrange these to. make a 1X15 strip.

1. How many length 1X1 strips can you make? How many 1X2 strips? How many 1X3 strips? How many 1X4? Etc.
2. How are the 1X3 and 1X4 strips related to the 1X5 strips?
3. How many 1X15 strips can you make?
4. What if I asked you to find the number of 1X1000 strips? Would the method you used to calculate the number of 1X15 strips be helpful?

Definition

A ***sequence*** is an ordered list of numbers (think array!)

When we use variables to represent a sequence, we use indices because order matters:

$$a_0, a_1, a_2, a_3, \ldots$$

Definition

A ***sequence*** is an ordered list of numbers (think array!)

When we use variables to represent a sequence, we use indices because order matters:

$$a_0, a_1, a_2, a_3, \ldots$$

To refer to an entire sequence we write:

$$(a_n)_{n\in\mathbb{N}} \text{ or } (a_n)_{n\geq 0} \text{ or } (a_n)$$

Definition

A ***sequence*** is an ordered list of numbers (think array!)

When we use variables to represent a sequence, we use indices because order matters:

$$a_0, a_1, a_2, a_3, \ldots$$

To refer to an entire sequence we write:

$$(a_n)_{n \in \mathbb{N}} \text{ or } (a_n)_{n \geq 0} \text{ or } (a_n)$$

Guess the next term in these sequences:

1. 3, -3, 3, -3, 3, …
2. 1, 5, 2, 10, 3, 15, …
3. 1, 2, 4, 8, 16, …
4. 1, 4, 9, 16, 25, 36, …

Definition

Guessing sequences from a few terms is imperfect, instead we need exact definitions.

A ***closed formula*** for a sequence $(a_n)_{n\in\mathbb{N}}$ is a formula for a_n using a fixed finite number of operations on n.

Definition

Guessing sequences from a few terms is imperfect, instead we need exact definitions.

A ***closed formula*** for a sequence $(a_n)_{n\in\mathbb{N}}$ is a formula for a_n using a fixed finite number of operations on n.

Ex. $a_n = n^2$

$$a_n = \frac{n(n+1)}{2}$$

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^{-n}}{\sqrt{5}}$$

Find the 0th, 1st, and 5th terms for each sequence

Definition

Guessing sequences from a few terms is imperfect, instead we need exact definitions.

A ***recursive*** (or ***inductive***) ***definition*** for a sequence $(a_n)_{n\in\mathbb{N}}$ consists of a ***recurrence relation*** (an equation relating a term of the sequence to previous terms) and an ***initial condition*** (a list of a few terms of the sequence).

Definition

Guessing sequences from a few terms is imperfect, instead we need exact definitions.

A ***recursive*** (or ***inductive***) ***definition*** for a sequence $(a_n)_{n\in\mathbb{N}}$ consists of a ***recurrence relation*** (an equation relating a term of the sequence to previous terms) and an ***initial condition*** (a list of a few terms of the sequence).

Ex. $a_n = 2a_{n-1}, a_0 = 1$

$a_n = 2a_{n-1}, a_0 = 27$

$a_n = a_{n-1} + a_{n-2}, a_0 = 0, a_1 = 1$

Find the 0^{th}, 1^{st}, and 5^{th} terms for each sequence

Common Sequences

- 1, 4, 9, 16, 25, …
  - ***square numbers***
  - For $(s_n)_{n\geq 1}, s_n = n^2$
- 1, 3, 6, 10, 15, 21, …
  - ***triangular numbers***
  - For $(T_n)_{n\geq 1}, T_n = \frac{n(n+1)}{2}$
- 1, 2, 4, 8, 16, 32, …
  - ***powers of two***
  - For $(a_n)_{n\geq 0}, a_n = 2^n$
- 1, 1, 2, 3, 5, 8, 13, …
  - ***Fibonacci numbers***
  - $F_n = F_{n-1} + F_{n-2}, \mathrm{F}_1 = \mathrm{F}_2 = 1$

Describing Sequences

- Finding the closed formula for a sequence is not always straightforward. There are many approaches.
- One option: Try to relate the sequence to a common sequence

Use $T_n = \frac{n(n+1)}{2}$ and $a_n = 2^n$ to find closed formulas for the following sequences. Assume each first term corresponds to $n = 0$.

- (b_n): $1, 2, 4, 7, 11, 16, 22, \ldots$
- (c_n): $3, 5, 9, 17, 33$
- (d_n): $0, 2, 6, 12, 20, 30, 42, \ldots$
- (f_n): $0, 1, 3, 7, 15, 31$

Describing Sequences

- Finding the closed formula for a sequence is not always straightforward. There are many approaches.
- Some sequences naturally arise as the sum of terms of another sequence

Start here Thursday

Describing Sequences

- Finding the closed formula for a sequence is not always straightforward. There are many approaches.
- Some sequences naturally arise as the sum of terms of another sequence

Sam keeps track of how many push-ups she does each day of her "do lots of push-ups challenge." Let $(a_n)_{n\geq 1}$ be the sequence that describes the number of push-ups done on the nth day of the challenge. The sequence starts

$$3, 5, 6, 10, 9, 0, 12$$

Describe the sequence $(b_n)_{n\geq 1}$ that describes the total number of pushups done by Sam after the nth day.

Describing Sequences

- Finding the closed formula for a sequence is not always straightforward. There are many approaches.
- Some sequences naturally arise as the sum of terms of another sequence

Given any sequence $(a_n)_{n\geq 1}$ we can always form a new sequence $(b_n)_{n\geq 1}$ as

$$b_n = a_0 + a_1 + \cdots + a_n$$

$$b_n = \sum_{k=1}^{n} a_k$$

Since the terms of (b_n) are sums of the initial part of the sequence (a_n) we call (b_n) the ***sequence of partial sums*** of (a_n).

Describing Sequences

Rewrite these sums using $\sum$ notation

- $1 + 2 + 3 + 4 + \cdots + 100$
- $1 + 2 + 4 + 8 + \cdots + 2^{50}$
- $6 + 10 + 14 + \cdots + (4n - 2)$

Given any sequence $(a_n)_{n\geq 1}$ we can always form a new sequence $(b_n)_{n\geq 1}$ as

$$b_n = a_0 + a_1 + \cdots + a_n$$

$$b_n = \sum_{k=1}^{n} a_k$$

Since the terms of (b_n) are sums of the initial part of the sequence (a_n) we call (b_n) the ***sequence of partial sums*** of (a_n).

Describing Sequences

- Finding the closed formula for a sequence is not always straightforward. There are many approaches.
- Some sequences naturally arise as the sum of terms of another sequence

Given any sequence $(a_n)_{n\geq 1}$ we can always form a new sequence $(b_n)_{n\geq 1}$ as

The multiplication version of this is:

$$\prod_{k=1}^{n} a_k$$

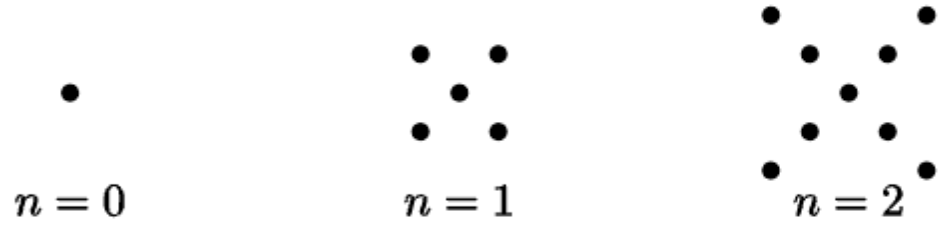
$$b_n = a_0 + a_1 + \cdots + a_n$$

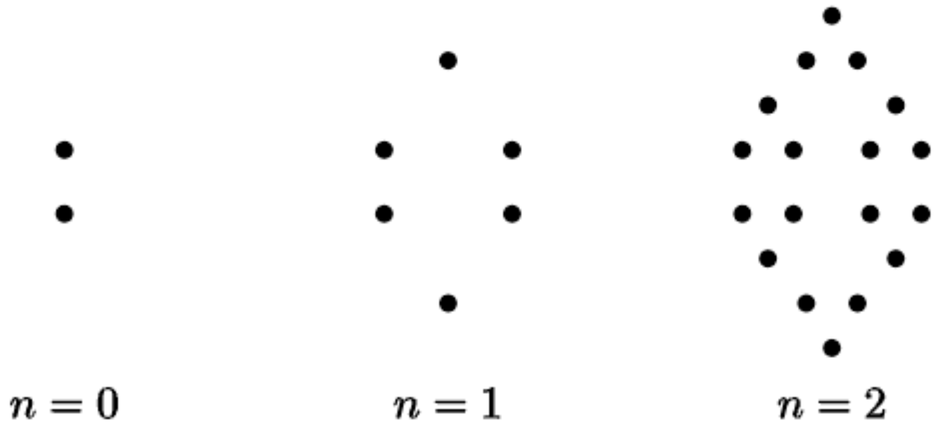
$$b_n = \sum_{k=1}^{n} a_k$$

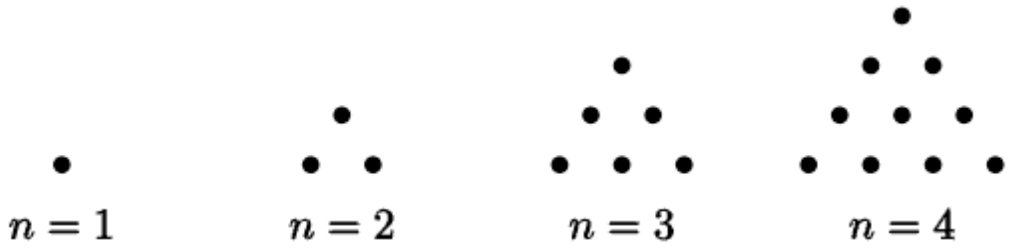
Since the terms of (b_n) are sums of the initial part of the sequence (a_n) we call (b_n) the ***sequence of partial sums*** of (a_n).

Closed Formulas

For the patterns of dots below, draw the next pattern in the sequence. Give a recursive definition and closed formula for the number of dots in the n^{th} pattern.







Arithmetic Sequences

If the terms of a sequence differ by a constant, we say the sequence is ***arithmetic***.

If the initial term (a_0) of the sequence is a and the ***common difference*** is d, then we have,

Recursive definition: $a_n = a_{n-1} + d$, $a_0 = a$

Closed formula: $a_n = a + dn$

Arithmetic Sequences

If the terms of a sequence differ by a constant, we say the sequence is ***arithmetic***.

If the initial term (a_0) of the sequence is a and the ***common difference*** is d, then we have,

Recursive definition: $a_n = a_{n-1} + d,\ a_0 = a$

Closed formula: $a_n = a + dn$

Find the recursive definitions and closed formulas for the arithmetic sequences below. Assume the first term listed is a_0.

- 2, 5, 8, 11, 14, ….
- 50, 43, 36, 29, …

Geometric Sequences

If the terms of a sequence differ by a constant ratio, we say the sequence is ***geometric***.

If the initial term (a_0) of the sequence is a and the ***common ratio*** is r, then we have,

Recursive definition: $a_n = r * a_{n-1}$, $a_0 = a$

Closed formula: $a_n = a * r^n$

Geometric Sequences

If the terms of a sequence differ by a constant ratio, we say the sequence is ***geometric***.

If the initial term (a_0) of the sequence is a and the ***common ratio*** is r, then we have,

Recursive definition: $a_n = r * a_{n-1}$, $a_0 = a$

Closed formula: $a_n = a * r^n$

Find the recursive definitions and closed formulas for the geometric sequences below. Assume the first term listed is a_0.

- 3, 6, 12, 24, 48, …
- 27, 9, 3, 1, 1/3, …