

Discrete Structures— Sequences Pt. 2

Dr. Ab Mosca (they/them)

Plan for Today

- Sequences
 - Sums
 - Solving recurrences

Warm Up: Arithmetic Sequences

If the terms of a sequence differ by a constant, we say the sequence is ***arithmetic***.

If the initial term (a_0) of the sequence is a and the ***common difference*** is d , then we have,

Recursive definition: $a_n = a_{n-1} + d, a_0 = a$

Closed formula: $a_n = a + dn$

Find the recursive definition and closed formula for this sequence:

8, 14, 20, 26, ...

Warm Up: Geometric Sequences

If the terms of a sequence differ by a constant ratio, we say the sequence is **geometric**.

If the initial term (a_0) of the sequence is a and the **common ratio** is r , then we have,

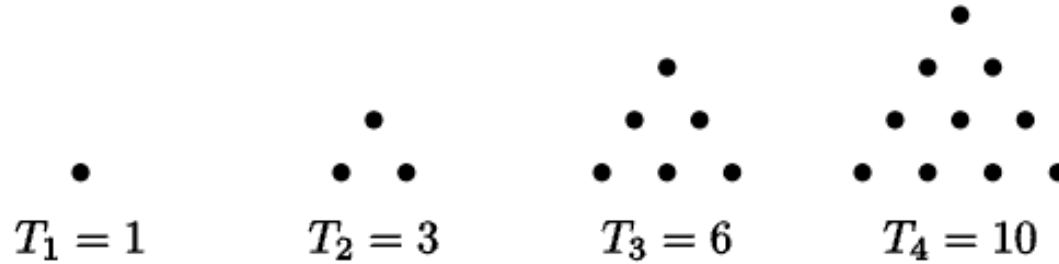
Recursive definition: $a_n = r * a_{n-1}, a_0 = a$

Closed formula: $a_n = a * r^n$

Find the recursive definition and closed formula for this sequence:

8, 32, 128, 512, ...

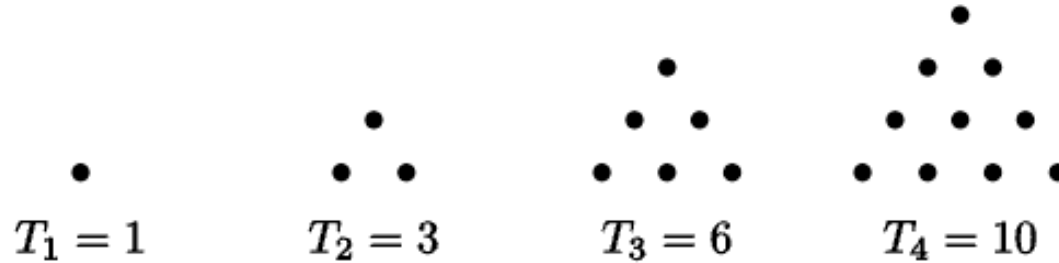
Consider the *triangular numbers*



Motivation

Is this sequence arithmetic, geometric, or neither?

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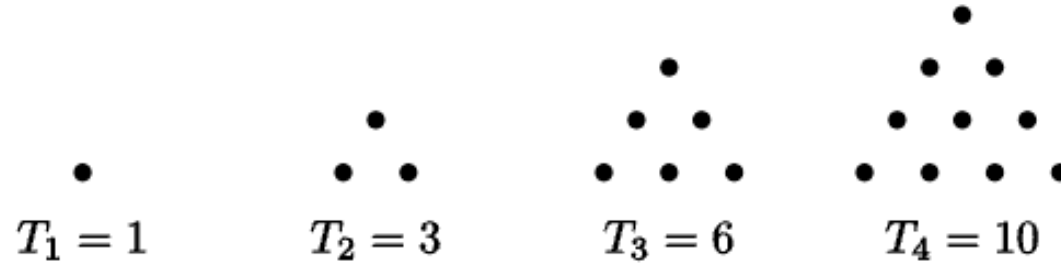


Motivation

Is this sequence arithmetic, geometric, or neither?

What if you look at the sequence of differences between consecutive terms?

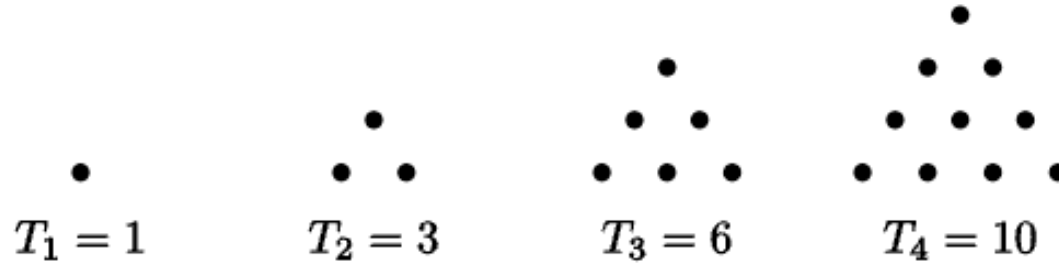
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Motivation

(T_n) is a ***sequence of partial sums***.

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(T_n) is a ***sequence of partial sums***.

If we know how to add terms of arithmetic (or geometric) sequences, we can find closed formulas for sequences whose differences are terms of an arithmetic (or geometric) sequence.

Summing Arithmetic Sequences

Ex. Find the sum of the first 100 integers:
1, 2, 3, ..., 98, 99, 100

Notice that if we add pairs starting from the smallest and largest to the next smallest and largest to the next, we get the same number.

$$1 + 100 = 101$$

$$2 + 99 = 101$$

$$3 + 98 = 101$$

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1, 2, 3, ..., 98, 99, 100

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$$1 + 100 = 101$$

$$2 + 99 = 101$$

$$3 + 98 = 101$$

If we continue, we end up adding 50 pairs (half the number of terms) all equal to our last term (100^{th}) + 1.

$$\text{So, } T_n = \frac{n(n+1)}{2}$$

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Ex. $2 + 5 + 8 + 11 + 14 + \cdots + 470$

$$d = 3, a_0 = 2, a_n = 2 + 3n$$

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Ex. $2 + 5 + 8 + 11 + 14 + \cdots + 470$

$$d = 3, a_0 = 2, a_n = 2 + 3n$$

$$S = 2 + 5 + 8 + \cdots + 464 + 467 + 470$$

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$$\text{So, } 2S = 157 * 472, S = \frac{157}{2} * 472 = 37052$$

Summing Arithmetic Sequences

Find the sum of $5 + 9 + 13 + 17 + 21 + \dots + 533$

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Summing Arithmetic Sequences

Find a closed formula for $6 + 10 + 14 + \cdots + (4n - 2)$

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Summing Arithmetic Sequences

Find a closed formula for $(a_n)_{n \geq 0}$: 2, 3, 7, 14, 24, 37, ...

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$$r = 2, a_0 = 3, a_n = 3 + 2^n$$

$$S = 3 + 6 + 12 + 24 + \cdots + 12288$$

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$$-S = 3 + 0 + 0 + 0 + \cdots + 0 \quad - 24576$$

$$S = 24573$$

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Find a closed formula for $S(n) = 2 + 10 + 50 + \cdots + 2 * 5^n$

Solving Recurrence Relations

Converting a recursive definition into a closed formula is called ***solving a recurrence relation***.

Sometimes, you can use your knowledge of known sequences to determine the closed formula for a recurrence relation.

Other times, finding the closed formula is more difficult.

Solving Recurrence Relations

Characteristic roots

Given a recurrence relation $a_n + \alpha a_{n-1} + \beta a_{n-2}$, the ***characteristic polynomial*** is

$$x^2 + \alpha x + \beta$$

giving the ***characteristic equation***:

$$x^2 + \alpha x + \beta = 0$$

If r_1 and r_2 are two distinct roots of the characteristic polynomial, then the solution to the recurrence relation is

$$a_n = ar_1^n + br_2^n$$

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$$(x - 2)(x - 5) = 0$$

So, $x = 2, x = 5$ are the characteristic roots and

$$a_n = a2^n + b5^n$$

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Solve for a, b with the initial conditions

$$2 = a2^0 + b5^0 = a + b$$

$$3 = a2^1 + b5^1 = 2a + 5b$$

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$$a = \frac{7}{3}, b = -\frac{1}{3}$$

$$\text{So, } a_n = \frac{7}{3}2^n - \frac{1}{3}5^n$$

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Solve the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$,
 $a_0 = 2, a_1 = 3$

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Note: You might need to use the quadratic formula to find roots:

$$ax^2 + bx + c = 0, x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 4$