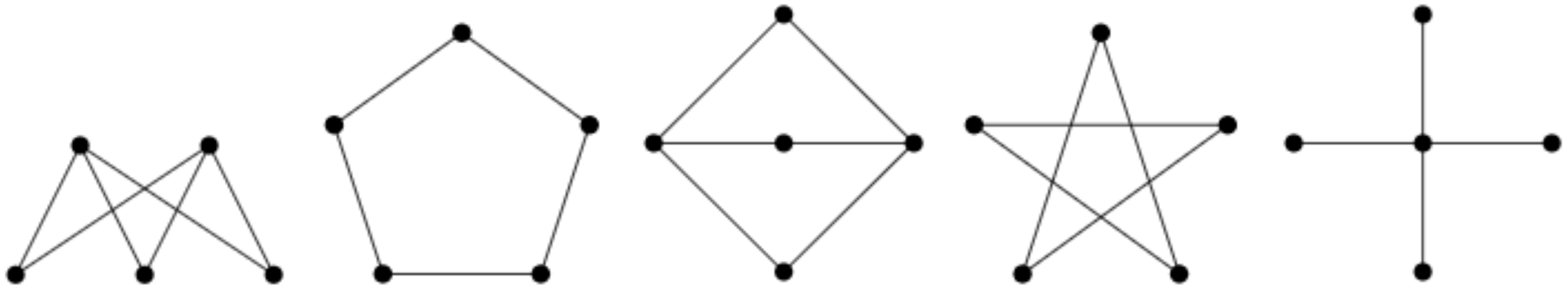
Discrete Structures– Graphs: Definitions

Dr. Ab Mosca (they/them)

Plan for Today

- Graph definitions

Are any of the graphs below the same? If you said yes, which ones and why?



Warm Up: Visualizing

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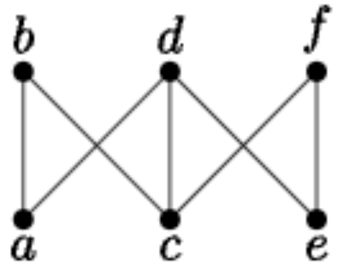
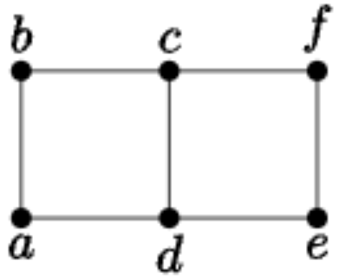
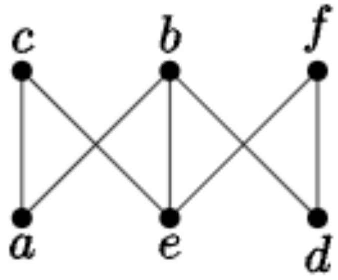
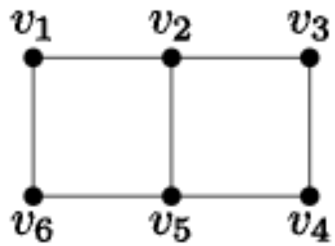
How about these with labels?

Are any of the graphs below the same? If you said yes, which ones and why?



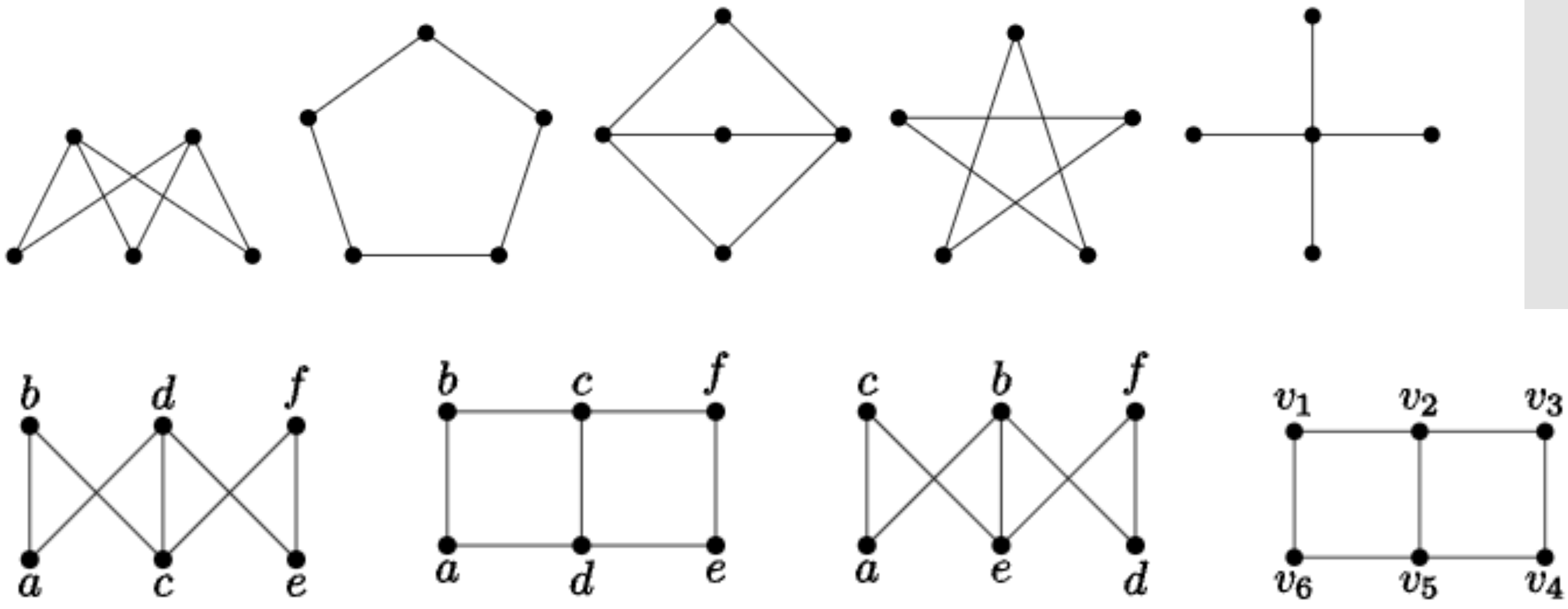






These examples are ***drawings*** of graphs.

Definition



Mathematically, a ***graph*** is an ordered pair, $G = (V, E)$, consisting of a nonempty set, V (called ***vertices***), and a set E (called ***edges***) of two-elements subsets of V.

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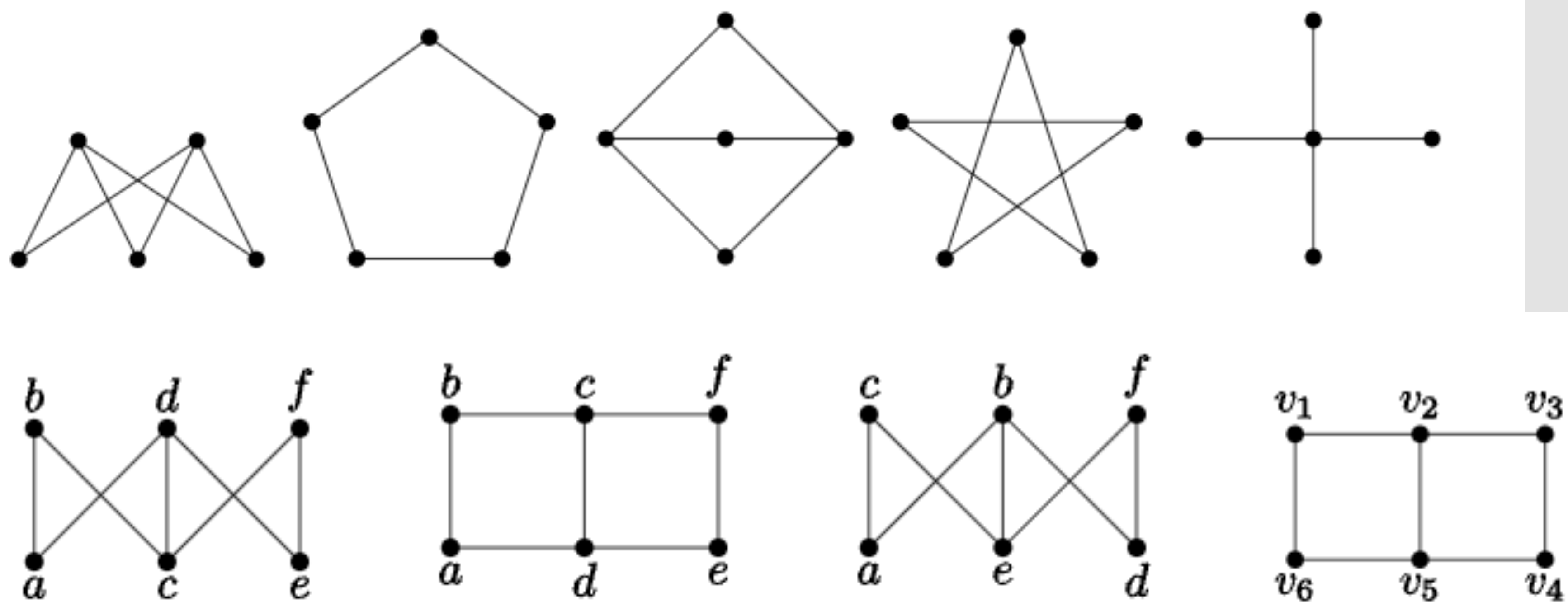

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Note: you may also hear vertices called ***nodes***.

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Ex. Graph 1:

$$V = \{a, b, c, d, e\},$$

$$E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{d, e\}\}$$

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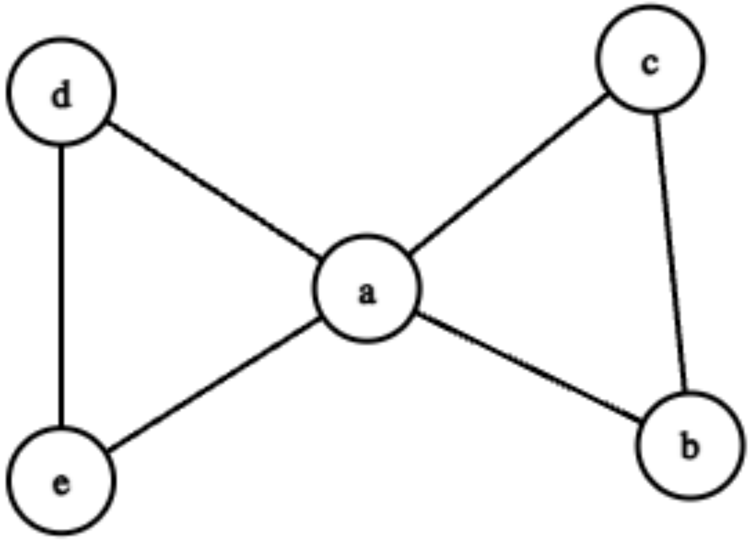
To visualize this graph we might draw it:



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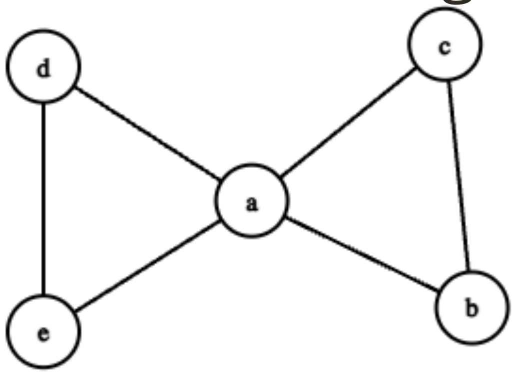

Draw this graph:

$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{\{v_1, v_3\}, \{v_1, v_5\}, \{v_2, v_4\},$
$\{v_2, v_5\}, \{v_3, v_5\}, \{v_4, v_5\}\}$

Relationships Between Graphs

Two ***graphs are equal*** if their sets of vertices and edges are equal.

Drawings of the same graph can differ.

Ex. $(\{a, b, c, d, e\},\ \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\})$

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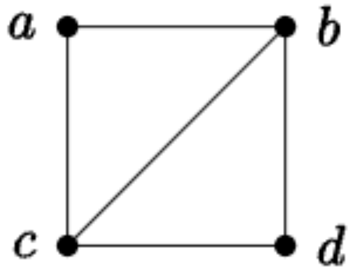

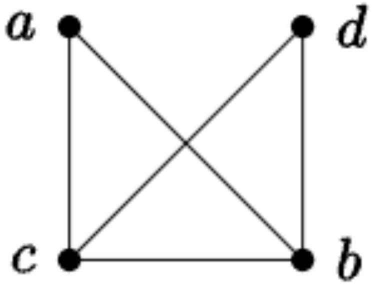

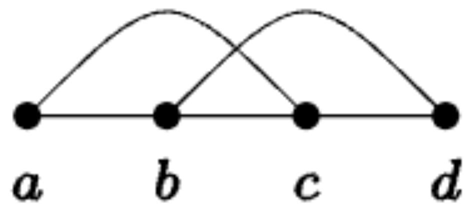

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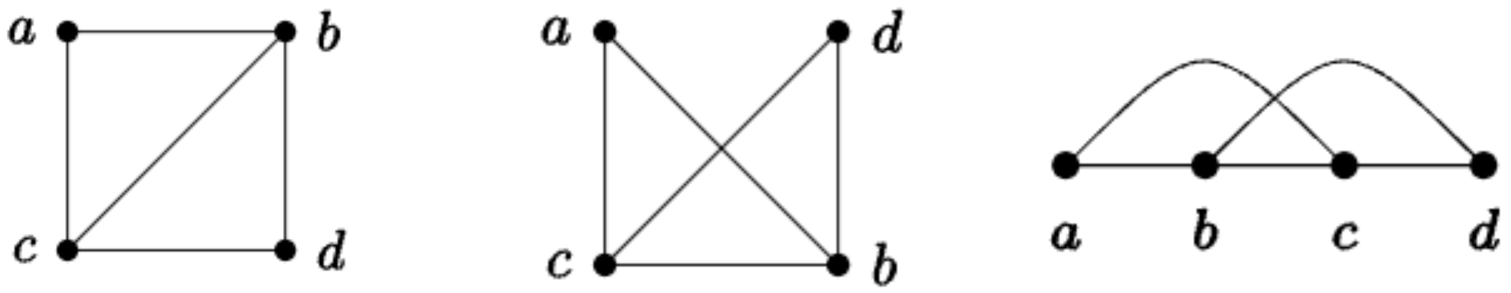

Are these graphs equal?

$G_1 = (\{a, b, c\}, \{\{a, b\}, \{b, c\}\})$

$G_2 = (\{a, b, c\}, \{\{a, c\}, \{c, b\}\})$

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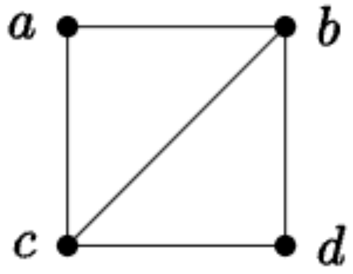

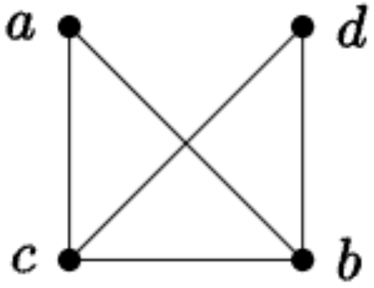

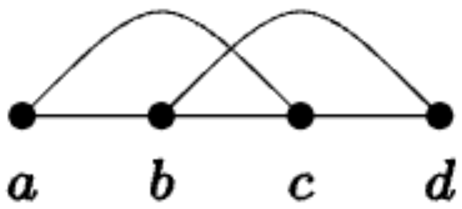

Are these graphs equal?

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$$G_2 = \left(\{u,v,w\},\{\{u,v\},\{u,w\},\{v,w\}\}\right)$$

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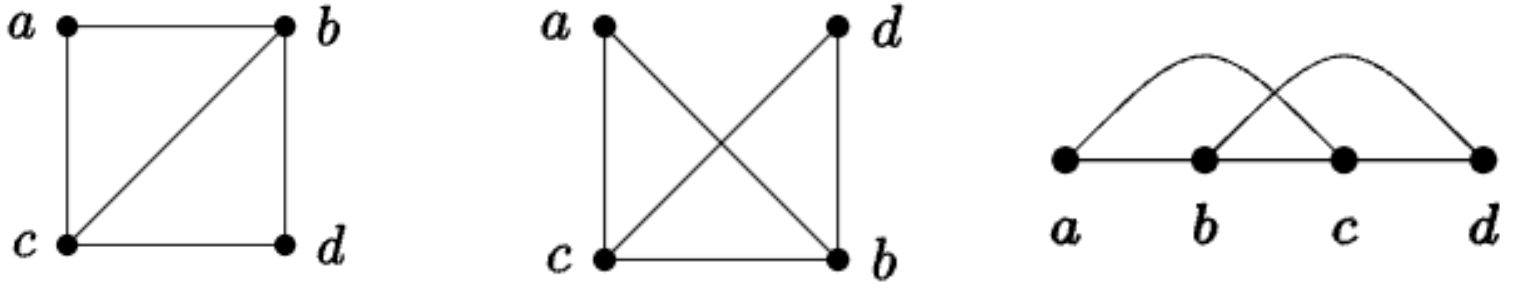

Would you call these graphs the same?

$G_1 = \left(\{a, b, c\}, \{\{a, b\}, \{a, c\}, \{b, c\}\}\right)$

$G_2 = \left(\{u, v, w\}, \{\{u, v\}, \{u, w\}, \{v, w\}\}\right)$

Relationships Between Graphs

An ***isomorphism*** between two graphs G_1 and G_2 is a bijection $f: V_1 \to V_2$ between the vertices of the graphs such that $\{a, b\}$ is an edge in G_1 if and only if $\{f(a), f(b)\}$ is an edge in G_2.

Two graphs are ***isomorphic*** if there is an isomorphism between them. In this case we write $G_1 \cong G_2$.

Are these graphs isomorphic?

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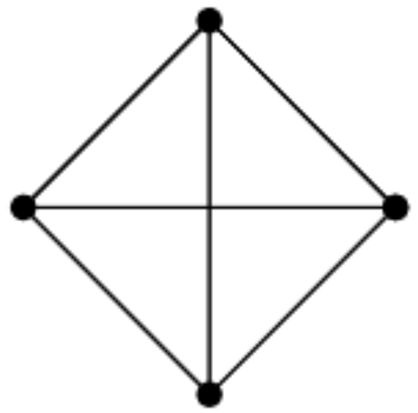
A collection of isomorphic graphs is called an ***isomorphism class***.

Ex. This graph with any labels for V



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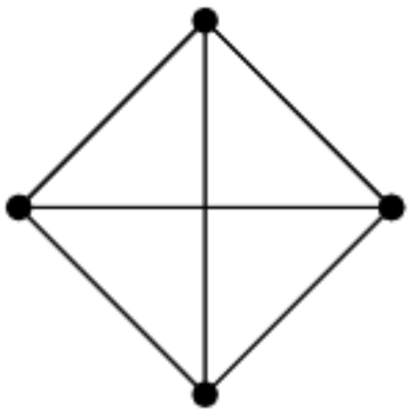

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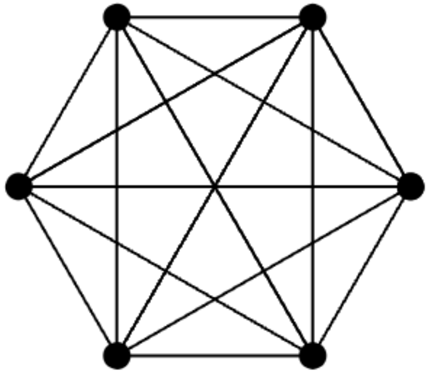
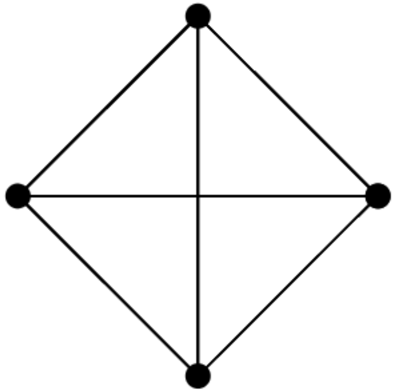
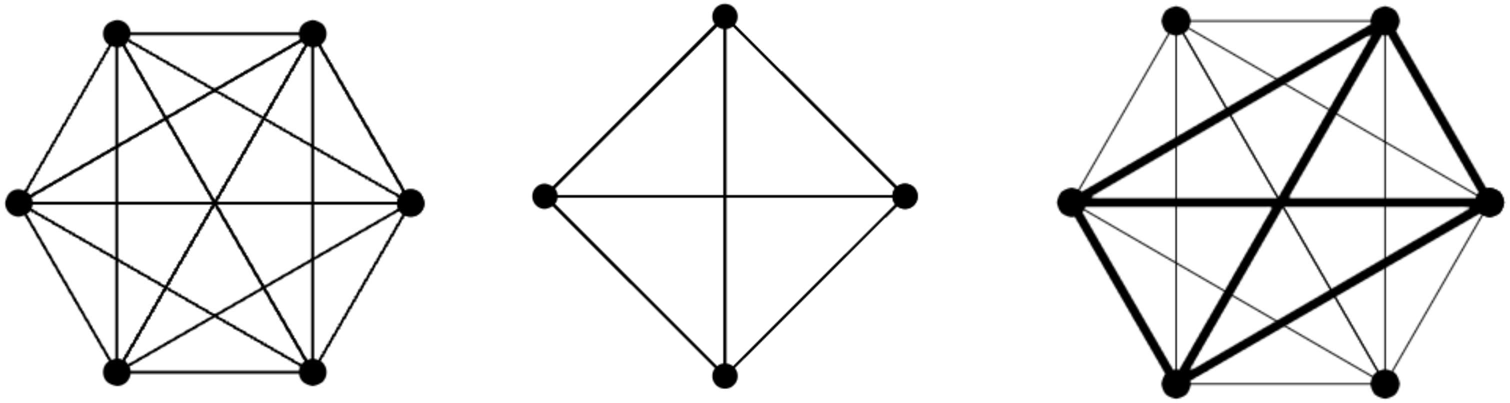




Do you think these graphs are related? If so, how?

Relationships Between Graphs



We say that $G' = (V', E')$ is a ***subgraph*** of $G = (V, E)$, and write $G' \subseteq G$, provided $V' \subseteq V$ and $E' \subseteq E$.

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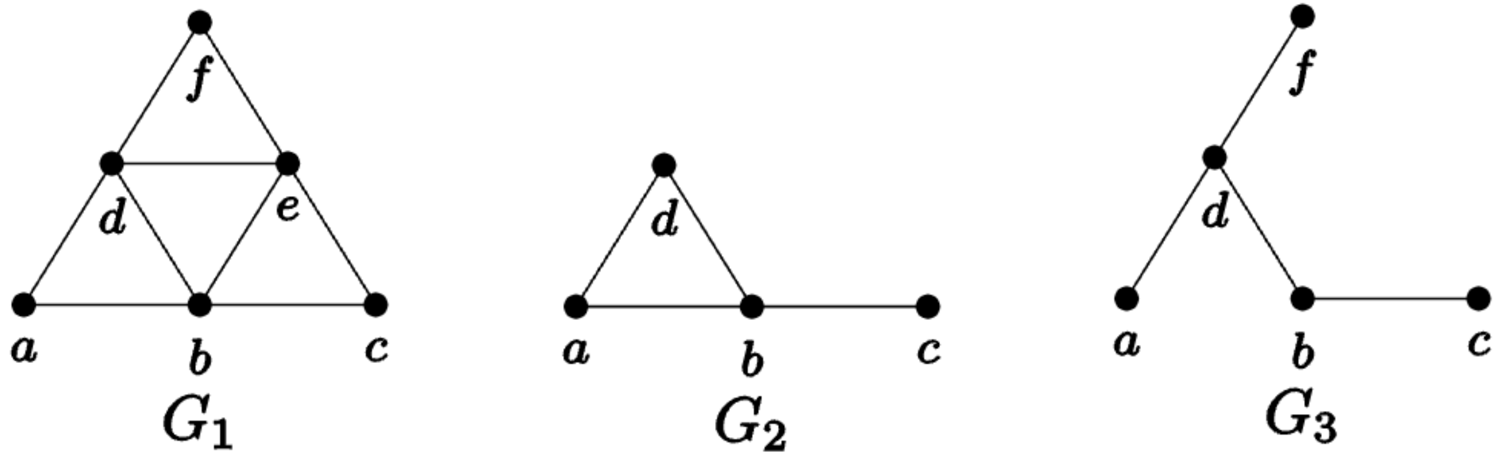

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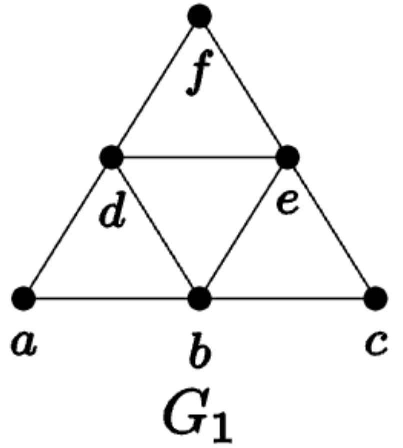

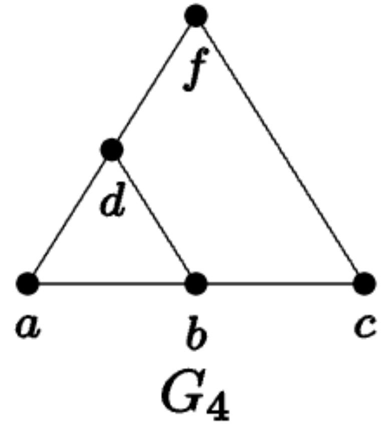

Is G_4 a subgraph of G_1?

Special Graphs

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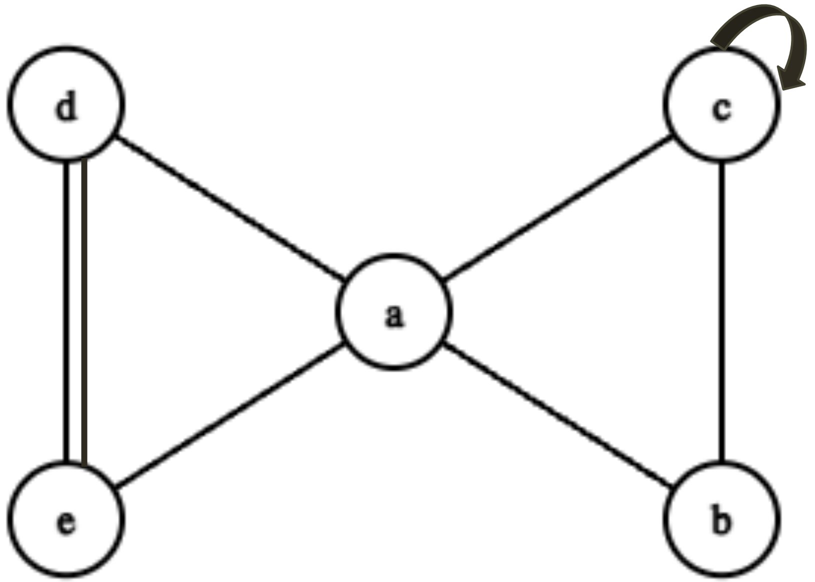
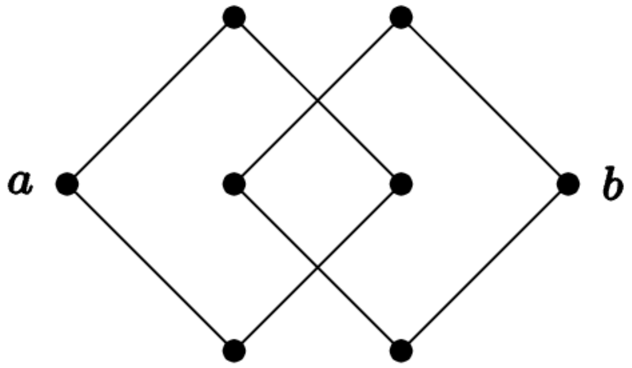
A ***multigraph***, is a graph that allows double (or more) edges, and for a vertex to be connected to itself.



Special Graphs

A ***connected*** graph is a graph where you can get from any vertex to any other vertex by following some path of edges.



Is this graph connected?

Special Graphs

A graph that includes all possible edges is called ***complete***. In other words, a graph is complete if every pair of vertices is connected by an edge.

K_n is the complete graph on n vertices.

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Each vertex in K_n is adjacent to $n - 1$ other vertices.

How many edges does K_n have?

Handshake Lemma

In any graph, the sum of the degrees of vertices in the graph is always twice the number of edges.

This can be written symbolically as: $\sum_{v \in V} d(v) = 2e$

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Given a ***degree sequence*** for a graph (a list of every degree of every vertex in the graph), we can use the handshake lemma to find the number of edges in the graph.

How many vertices and edges must the graph with the degree sequence (4, 4, 3, 3, 3, 2, 1) have?

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At a recent math seminar, 9 mathematicians greeted each other by shaking hands. Is it possible that each mathematician shook hands with exactly 7 people at the seminar?

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Proposition

We can generalize the previous example into a proposition:

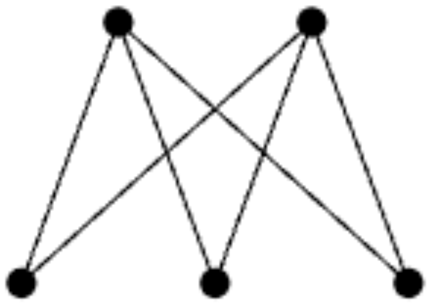
In any graph, the number of vertices with odd degree must be even.

Prove this proposition.

Bipartite

We say a graph is ***bipartite*** if the vertices can be divided into two sets, A and B, with no two vertices in A adjacent and no two vertices in B adjacent. The vertices in A can be adjacent to some or all of the vertices in B.

If each vertex in A is adjacent to all the vertices in B, then the graph is a ***complete bipartite graph***, and gets the special name: $K_{m,n}$ where $|A| = m, |B| = n$.



$K_{2,3}$

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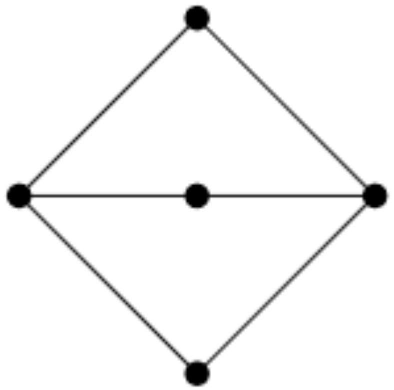

Is this graph bipartite?

Named Graphs

K_n: The complete graph on n vertices.

$K_{m,n}$: The complete bipartite graph with sets of m and n vertices.

C_n: The cycle on n vertices, just one big loop.

P_n: The path on $n+1$ vertices (so n edges), just one long path.