Discrete Structures–Proofs: Sets

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Plan for Today

- Sets
  - Notation
  - Relationships
  - Operations

Warm Up: Induction

To perform a proof by ***induction*** *on* $P(n)$:

1. Start with your ***base case***
   - Prove $P(n)$ is true for the smallest value of n possible
2. Perform the ***inductive step***
   - Prove that $P(k) \rightarrow P(k+1)$ for all k greater than or equal to the smallest possible value of n.
   - Note this is an if … then … proof, so we start by assuming $P(k)$ is true. This is called the ***inductive hypothesis*** for this type of proof.

Practice: Prove the following…

For all natural numbers, n, $6^n - 1$ is a multiple of 5.

Set Vocab

A ***set*** is an unordered collection of objects.

Ex. Snacks in your pantry

Ex. Natural numbers between 1 and 10 inclusive

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Notation:

- A set is denoted with {}
- We usually use a capital letter to represent a set

Ex. $A = \{chips, pretzels, cookies, peanuts\}$

Ex. $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Set Vocab

A ***set*** is an unordered collection of objects.

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Notation:

- We usually use a lowercase letter to represent an element of a set
- $\in$ is read “is an element of” and $\notin$ is read “is not an element of”

Ex. $cookies \in A$

Ex. $11 \notin B$

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Ex. $cookies \in A$

Ex. $11 \notin B$

- A set can be an element of another set.

Ex. $A = \{a, b, \{x, y, z\}\}$

A contains elements $a, b,$ and $\{x, y, z\}$

Set Vocab

Special Sets

There are some sets we use so often that we give them special names and symbols.

- $\emptyset$ is the ***empty set***, i.e. the set containing no elements ($\{\}$)
- $\mathbb{N}$ is the set of ***natural numbers***, $\{0, 1, 2, 3, \ldots\}$
- $\mathbb{Z}$ is the set of ***integers***, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Q}$ is the set of ***rational numbers***
- $\mathbb{R}$ is the set of ***real numbers***

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Practice: Write the following sets symbolically

- The set of classes you are taking this semester
- The set of all special sets listed above
- The set of even numbers

Set Builder Notation

Sometimes, listing all elements of a set is hard or imprecise

Ex. Let A be the set of all even natural numbers

$$A = \{0, 2, 4, 6, \dots\}$$

In these cases, we use ***set builder notation***

$$A = \{x \in \mathbb{N}: x\ is\ even\}$$

This is read, "A is the set of all natural numbers x such that x is even"

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Practice: Write the set builder notation for the following sets

- $\{x : x + 3 \in \mathbb{N}\}$
- $\{x \in \mathbb{N} : x + 3 \in \mathbb{N}\}$
- $\{x : x \in \mathbb{N} \vee -x \in \mathbb{N}\}$
- $\{x^2 : x \in \mathbb{N}\}$

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Practice: List the elements or describe the following sets

- The set of natural numbers between 10 and 30 (including 30)
- The set of primes
- The set of odds
- The set of positive odds

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Relationships Between Sets

Sets A and B are ***equal***, if they have exactly the same elements

Ex. $\{1, 2, 3\} = \{1,\ \ 1+1, 1+1+1\}$

Set A is a ***subset*** of set B if every element in A is also an element of B

Ex. $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$

The ***power set*** of A is the set of all subsets of A

Ex. $A = \{1, 2, 3\}$,

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

empty set is always a subset!

Relationships Between Sets

Sets A and B are ***equal***, if they have exactly the same elements

Practice:

What is the power set of $A = \{a, b\}$?
What is the power set of $A = \{a, b, c\}$?
What is the power set of $B = \{a, b, c, d\}$?

The ***power set*** of A is the set of all subsets of A

Ex. $A = \{1, 2, 3\}$,

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empty set is always a subset!

Cardinality

The size of a set (i.e. the number of elements in a set) is called it ***cardinality***.

Ex. $\mathrm{A} = \{a, b\}, |A| = 2$

$A = \emptyset, |A| = 0$

Practice: What are the cardinalities of the power sets you found?

power set of $A = \{a, b\}$?
power set of $A = \{a, b, c\}$?
power set of $B = \{a, b, c, d\}$?

Do you notice a pattern?

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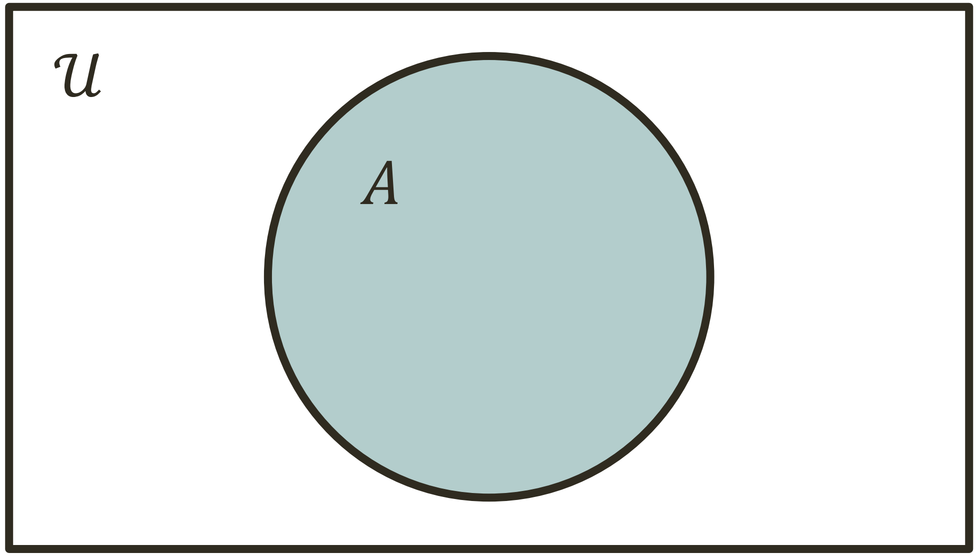
Let $|A| = n$, the cardinality of $\mathcal{P}(A) = 2^n$.

Set Operations

Often, when we deal with sets we have some notion of what "everything" is. We call this the ***universe***, $\mathcal{U}$.

We visualize sets within the universe:



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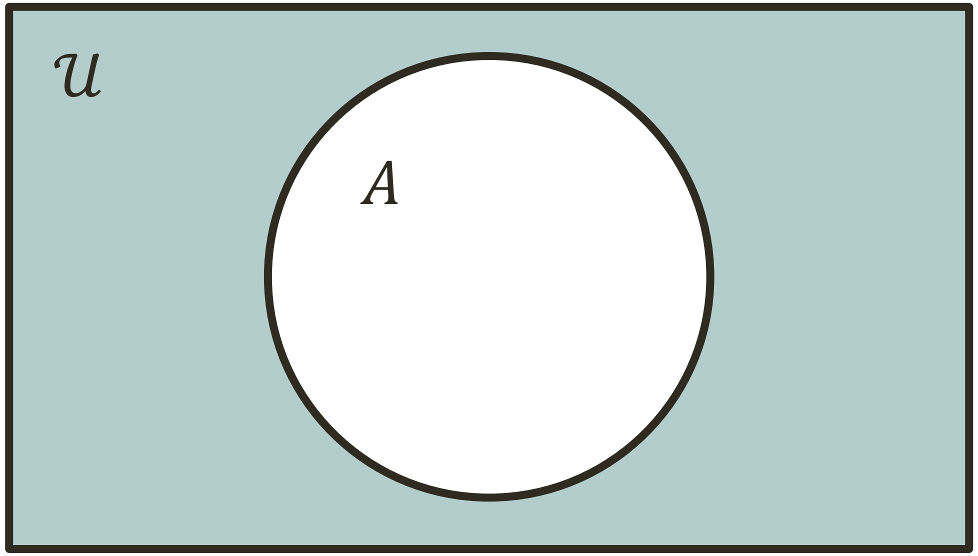
The ***complement*** of A (denoted $\overline{\boldsymbol{A}}$) is the set of all elements in the universe that are not in A



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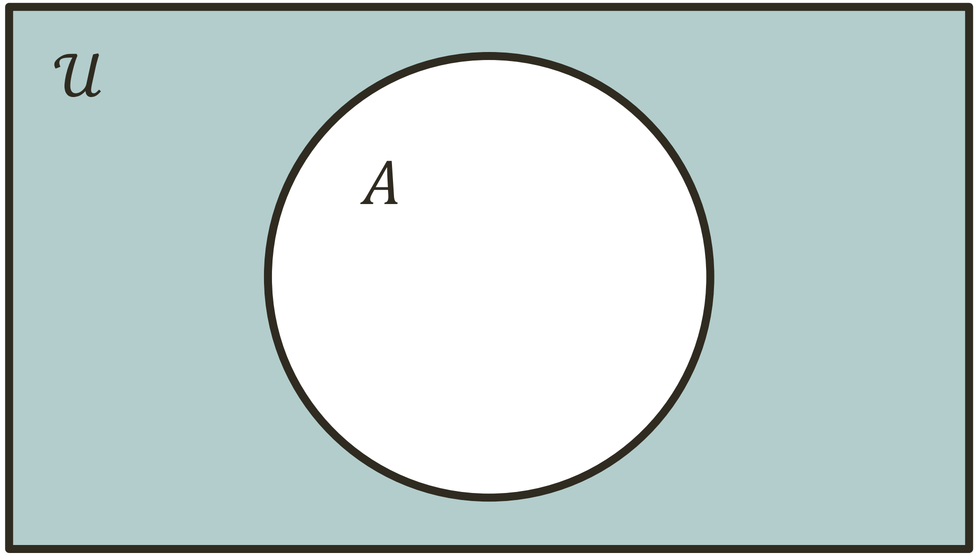
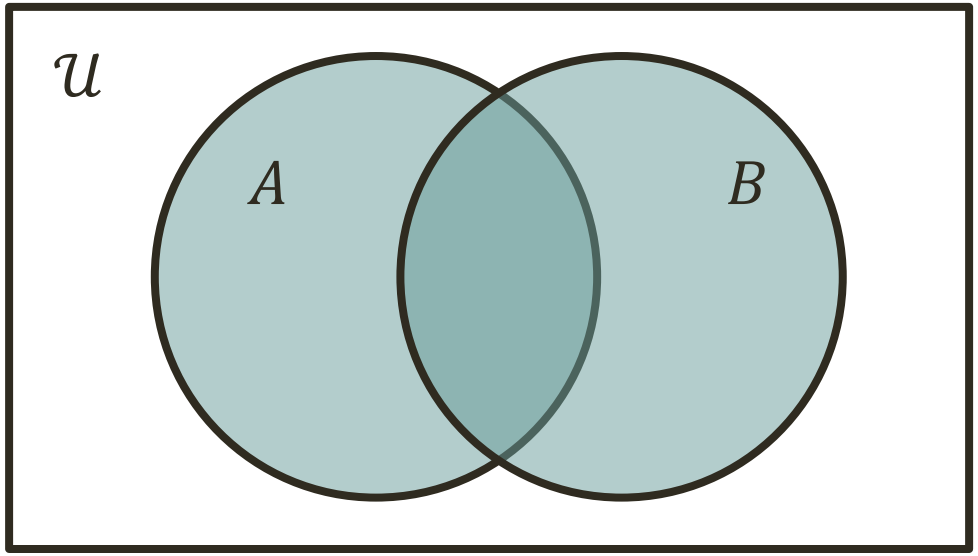


Practice: Let $A = \{\mathrm{x} : \mathrm{x} \in \mathbb{N}\ and\ x = 2k\ for\ some\ k \in \mathbb{N}\}$
What is $\overline{\boldsymbol{A}}$?

Set Operations

Let A, B be sets.

The ***union*** of A and B (denoted $\boldsymbol{A} \cup \boldsymbol{B}$) is the set of all elements in A, or B, or both



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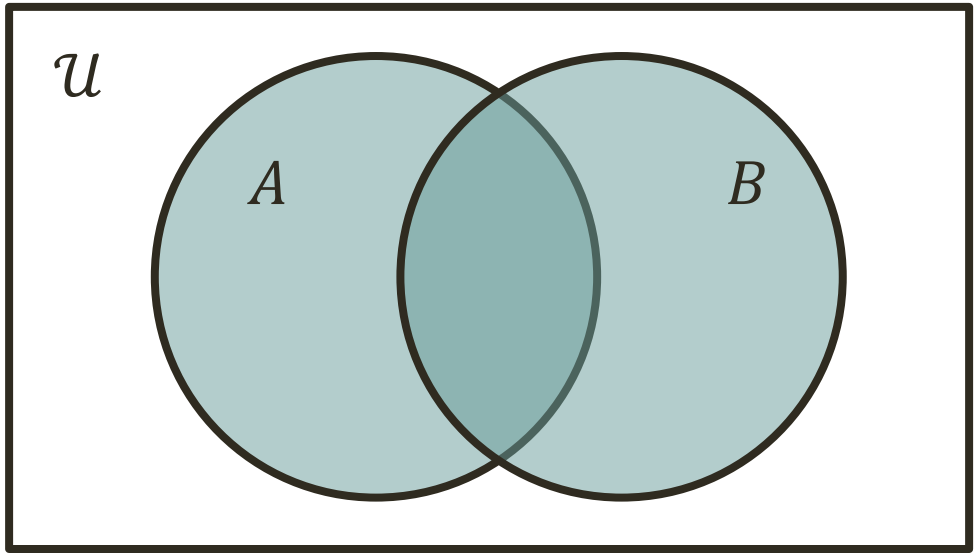

Practice: Let $A = \{\mathrm{x} : \mathrm{x} \in \mathbb{N}\ and\ x = 2k\ for\ some\ k \in \mathbb{N}\}$, and $B = \{\mathrm{x} : \mathrm{x} \in \mathbb{N}\ and\ x = 3k\ for\ some\ k \in \mathbb{N}\}$
What is $A \cup B$?

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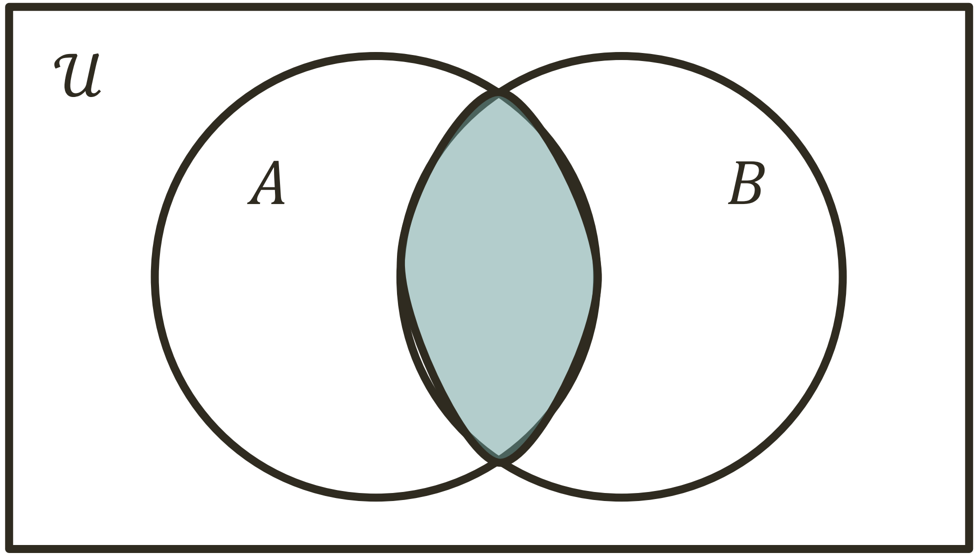
The ***intersection*** of A and B (denoted $\boldsymbol{A \cap B}$) is the set of all elements in A and B



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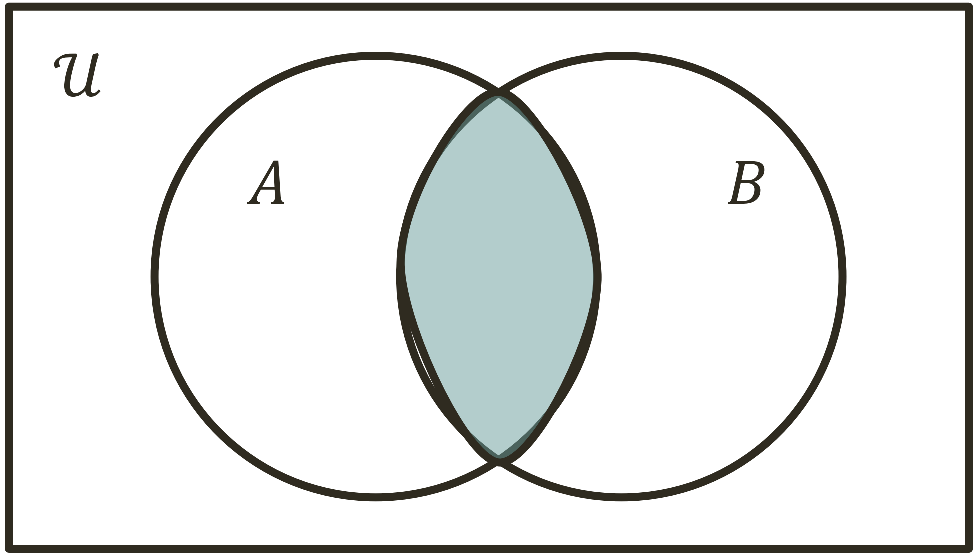

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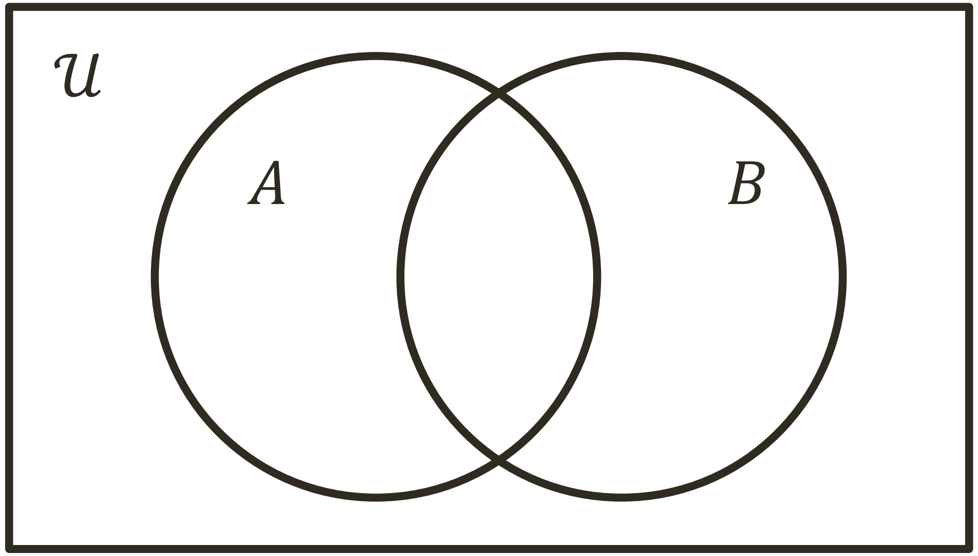
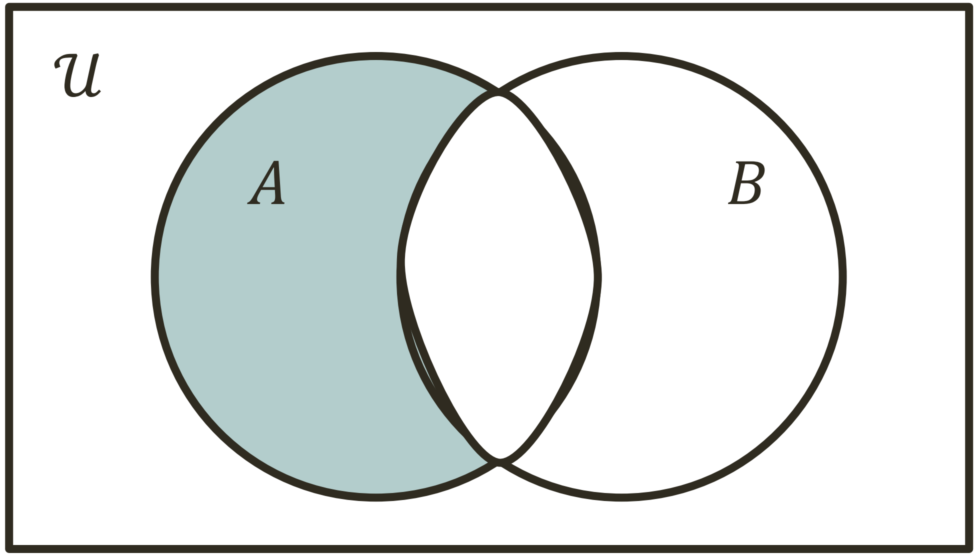


Practice: Let $A = \{x : x \in \mathbb{N}\ and\ x = 2k\ for\ some\ k \in \mathbb{N}\}$, and $B = \{x : x \in \mathbb{N}\ and\ x = 3k\ for\ some\ k \in \mathbb{N}\}$
What is $A \cap \bar{B}$? Visualize it with a Venn diagram.

Set Operations

Let A, B be sets.

The ***set difference*** of A and B (denoted $\boldsymbol{A \backslash B}$) is the set of all elements in A that are not in B



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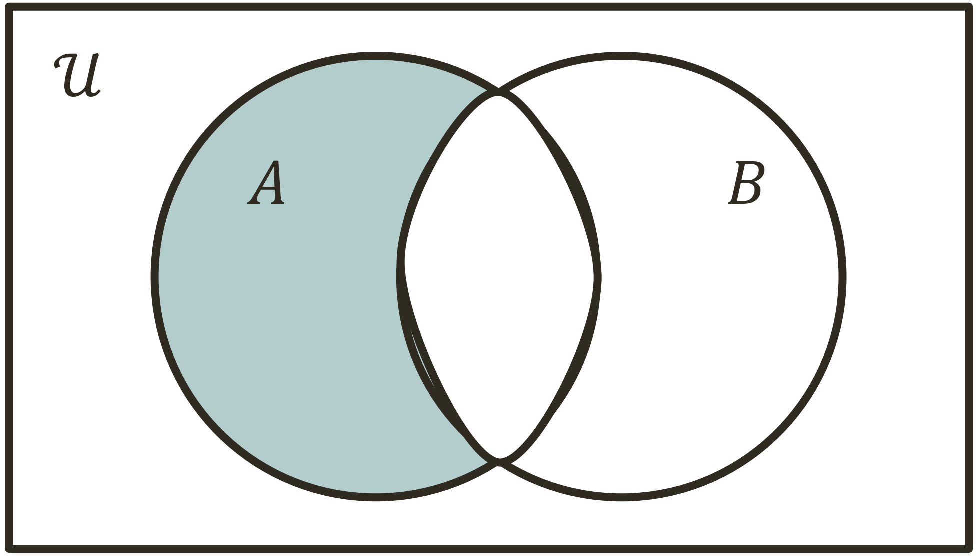

Practice: Is set difference of A and B equivalent to $A \cap \bar{B}$?

Set Operations

Let A, B be sets.

The ***cartesian product*** of A and B (denoted $\boldsymbol{A \times B}$) is the set of all ordered pairs where the first element is from A and the second is from B.

$$A \times B = \{(a, b): a \in A \text{ and } b \in B\}$$

Practice: What will the cardinality of $A \times B$ be?

Set Operations

Practice: Let $A = \{1, 2, 3, 4, 5, 6\}, B = \{2, 4, 6\}, C = \{1, 2, 3\}, D = \{7, 8, 9\}, \mathcal{U} = \{1, 2, \ldots, 10\}$. Find:

1. $A \cup B$
2. $A \cap B$
3. $B \cap C$
4. $A \cap D$
5. $\overline{B \cup C}$
6. $A \backslash \mathrm{B}$
7. $(D \cap \bar{C}) \cup \overline{A \cap B}$
8. $\emptyset \cup C$
9. $\emptyset \cap C$
10. $B \times C$

Visualize each with a Venn diagram.