Discrete Structures– Proofs: Contradiction and Counter Example

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Plan for Today

- Proof by contradiction
- Proof by counter example

Warm Up

Direct proof is usually used to prove implications.

To prove $P \rightarrow Q$, we will

1. Assume P is true
2. Deduce that Q must then also be true

Proof by contrapositive

To prove $P \rightarrow Q$, we will

1. Prove the contrapositive ($\neg Q \rightarrow \neg P$)
    1. Assume $\neg Q$ is true
    2. Deduce that $\neg P$ must then also be true

Work with a small group to prove if ab is an even number, then a, or b is even.

Proof by Contradiction

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If this implication is true, and Q is false, what does that tell us about $\neg P$?

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T	T	T
T	F	F
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If $\neg P$ is false, then P must be true.

This gives us another option for proof called ***proof by contradiction***.

Proof by Contradiction

To perform a ***proof by contradiction***:

1. Assume $\neg P$ is true
2. Show that this assumption leads to a contradiction
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   (i.e. if it impossible for P to be false, we know it must be true)

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and $a^2 = 2b^2$, which means a^2 is even and a is even.

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Thus, $\sqrt{2}$ is irrational. //

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Practice: Prove the following…

There are no integers x and y such that $x^2 = 4y + 2$.

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There are no integers x and y such that x is a prime greater than 5 and $x = 6y + 3$.

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Practice: Prove the following…

$\sqrt{3}$ is irrational.

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Practice: Prove the following...

For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even.

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This is true of all properties, principles, theorems, etc. you've proven before. They are fair game for future proofs. (Unless the property/principle/theorem/etc. is what you're being asked to prove.)

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1. Make sure you're trying to show something is false
2. Find an exception to the statement

Proof by Counter Example

Proof: We will show that the statement for all integers a and b, if a is odd or b is odd, then $a + b$ is odd.

Let $a = 1$, and $b = 3$.

Then $a + b = 1 + 3 = 4$.

4 is not odd; we have found an example where the statement is not true.

Thus, the statement is false.

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Practice: Prove the following...

For all integers a and b, if ab is a multiple of 6, then a is even and b is a multiple of 3.

Notes about proofs

Sometimes you will need to break a proof into ***cases***.

For example, if you were to prove this:

For any integer n, the number $(n^3 - n)$ is even.

You might look at a case where n is even and a care where n is odd.

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