Discrete Structures– Functions

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Plan for Today

- Functions
  - Describing
  - Properties
  - Image and Inverse Image

Warm Up: Sets

A ***set*** is an unordered collection of objects

The ***power set*** of A is the set of all subsets of A

Ex. $A = \{1, 2, 3\}$,

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Practice: What is the power set of

- $\{x \in \mathbb{N} : 10|x \ and \ x < 40 \}$
- $\{x : x \in \mathbb{Z} \ and \ |x| < 3\}$
- $\{x^2 : x \in \mathbb{N} \ and \ x \leq 2\}$

What is the cardinality of each power set?

Definitions

A ***function*** is a rule that assigns each input exactly one output.

We call the output the ***image*** of the input.

The set of all inputs for a function is called the ***domain***.

The set of allowable outputs is called the ***codomain***.

We write $f: X \rightarrow Y$ to represent a function with the name f, domain X, and codomain Y.

Definitions

A ***function*** is a rule that assigns each input exactly one output.

The assignment rule for a function is usually given by a formula describing how to compute the output for any input.

Ex.

$f: \mathbb{N} \to \mathbb{N}$ is defined by $f(x) = x^2 + 3$

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Practice: Describe the set of outputs for this function using set builder notation.

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The set of outputs for a function is called the ***range***.

Practice: What is the difference between codomain and range?

Definitions

A ***function*** is a rule that assigns each input exactly one output.

Ex.

$$f : \mathbb{Z} \to \mathbb{Z},\ f(n) = 3n$$

$$g : \{1, 2, 3\} \to \{a, b, c\},\ g(1) = c, g(2) = a, g(3) = a$$

$h : \{1, 2, 3, 4\} \to \mathbb{N}$,

x	1	2	3	4
$h(x)$	3	6	9	12

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Practice: ID the domain, codomain, and range for each function above.

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A ***function*** is a rule that assigns each input exactly one output.

Input can map to the same output as long as each input maps to *exactly one* output

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Ex.

$f\colon \mathbb{Z} \to \mathbb{Z},\ f(n) = 3n$

f and h are not the same because their domain and codomains differ

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Definitions

A ***function*** is a rule that assigns each input exactly one output.

Practice: Are the following functions? Why or why not?

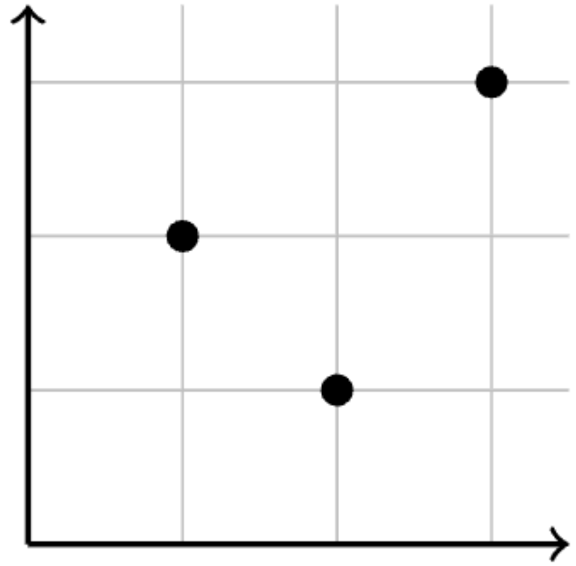
1. $f: \mathbb{N} \to \mathbb{N}, f(n) = \frac{n}{2}$
2. Consider the rule that matches each person to their phone number.

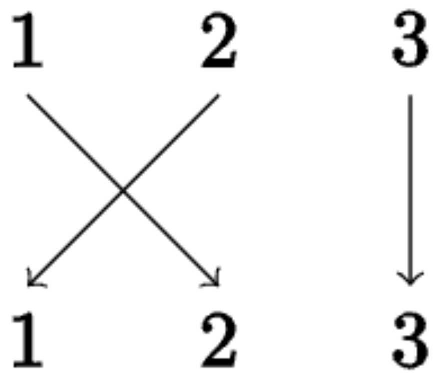
Describing Functions

We can describe functions many ways.

- algebraically, with a formula
- numerically, with a table
- graphically
- in words
- and more!

Ex. $f: \{1, 2, 3\} \to \{1, 2, 3\}$





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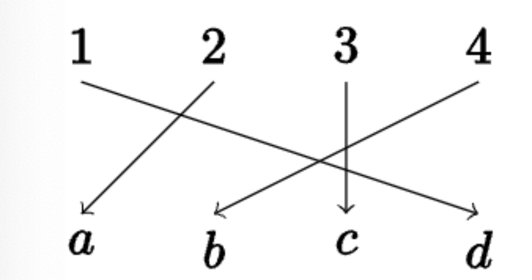
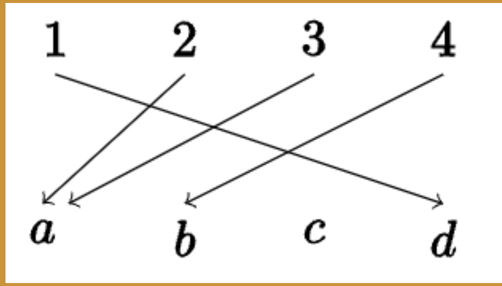
Practice: Are the following functions? Why or why not?

$$X = \{1, 2, 3, 4\}, Y = \{a, b, c, d\}$$

$f : X \to Y$



$g : X \to Y$



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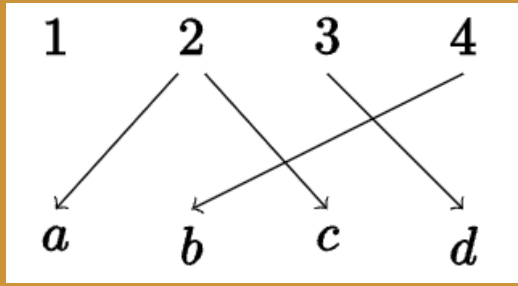

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Ex. $f\colon \{1,2,3\} \to \{1,2,3\}$

$$f(x) = \begin{cases} x+1 & if\ x=1 \\ x-1 & if\ x=2 \\ x & if\ x=3 \end{cases}$$

x	0	1	2	3	4
f(x)	3	3	2	4	1

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Ex. $f: \{1, 2, 3\} \to \{1, 2, 3\}$

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$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 4 & 1 \end{pmatrix}$$

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$$\Downarrow$$

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Ex. $f:\{1,2,3\} \to$

f(

Matrix notation is especially helpful for determining if something is a function

Ex. h from before:

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & a,c? & d & b \end{pmatrix}$$

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 4 & 1 \end{pmatrix}$$

Describing Functions

Often (especially in discrete!) we are interested in functions with domain $\mathbb{N}$.

These can be hard to write in table notation Ex.

$f: \mathbb{N} \rightarrow \mathbb{N}$

x	0	1	2	3	…
$f(x)$	0	1	4	9	…

Here, we can guess the output for 5, 6, etc. but cannot be positive.

In these cases, explicit rules are better. We call explicit rules like this ***closed formulas*** for the function.

Describing Functions

Often (especially in discrete!) we are interested in functions with domain $\mathbb{N}$.

We can also define this type of function (with domain $\mathbb{N}$) *recursively*.

For a function $f: \mathbb{N} \to \mathbb{N}$, a ***recursive definition*** consists of an *initial condition* together with a *recurrence relation*. The initial condition is explicitly given the value of $f(0)$. The recurrence relation is a formula for $f(n+1)$ in terms of $f(n)$ (or possibly n itself).

Ex. $f: \mathbb{N} \to \mathbb{N}, f(0) = 0, f(n+1) = f(n) + 2n + 1$

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Practice: What is $f(6)$?

Describing Functions

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Ex. $f: \mathbb{N} \to \mathbb{N}, f(0) = 0, f(n+1) = f(n) + 2n + 1$

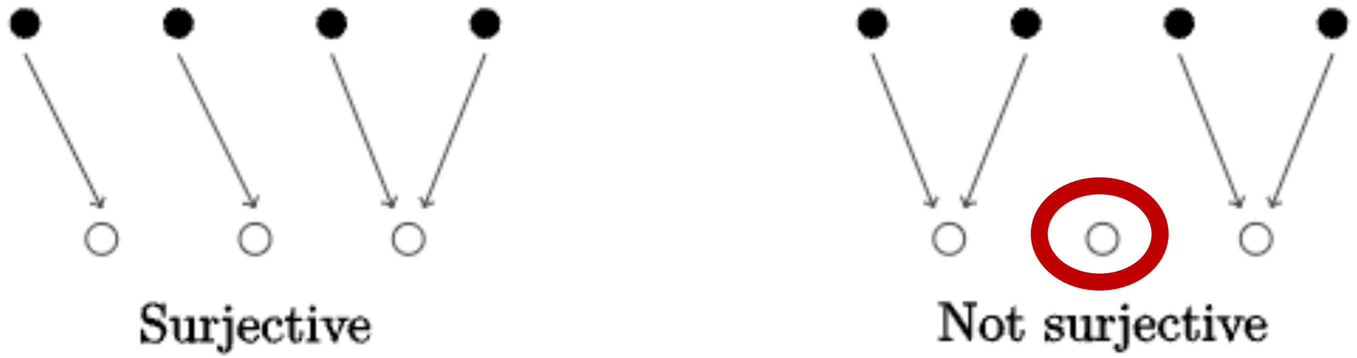
Practice: What is the recursive definition for:

$h: \mathbb{N} \to \mathbb{N}$ defined by $h(n) = n!$
(Remember, $0! = 1, n! = 1 * 2 * 3 * \cdots * (n-1)(n)$)

Function Properties

When a function maps the domain to everything in the codomain (i.e the codomain is the range) we say it is ***onto***.

An onto function is a ***surjective*** function.



Function Properties

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Practice: Are these functions surjective?

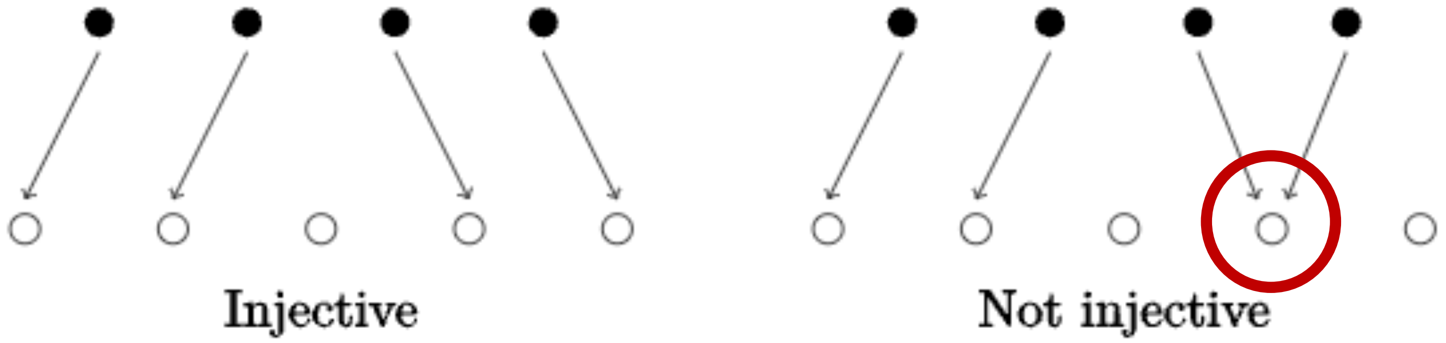
$$f : \mathbb{Z} \to \mathbb{Z}, f(n) = 3n$$

$$g : \{1, 2, 3\} \to \{a, b, c\}, g = \begin{pmatrix} 1 & 2 & 3 \\ c & a & a \end{pmatrix}$$

Function Properties

When a function is such that each element of the codomain is the image of at most one element of the domain, we say it is ***one-to-one***.

A one-to-one function is an ***injective*** function.



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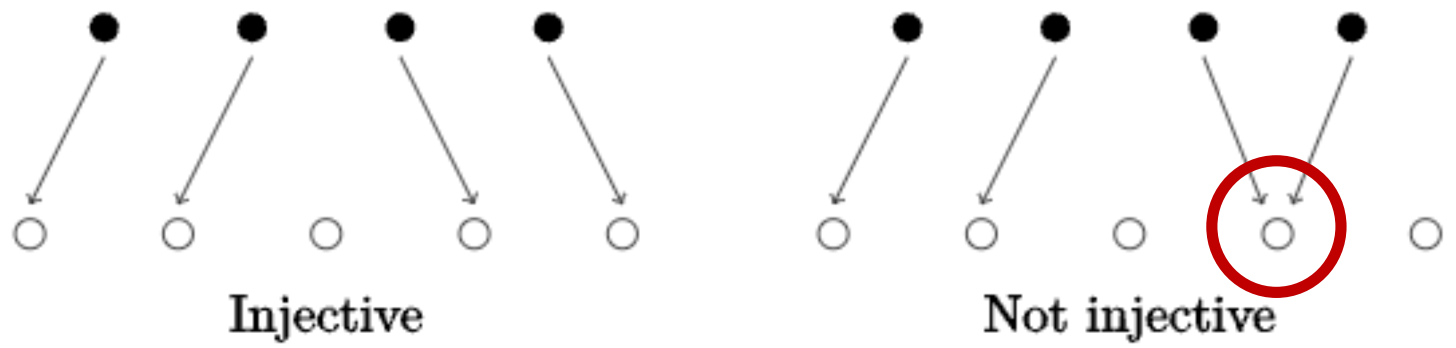

Practice: Are these functions injective?

$$f: \mathbb{Z} \rightarrow \mathbb{Z}, f(n) = 3n$$
$$g: \{1, 2, 3\} \rightarrow \{a, b, c\}, g = \begin{pmatrix} 1 & 2 & 3 \\ c & a & a \end{pmatrix}$$

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When a function maps the domain to everything in the codomain (i.e the codomain is the range) we say it is ***onto***.

When a function is such that each element of the codomain is the image of at most one element of the domain, we say it is ***one-to-one***.

An onto *and* one-to-one function is a ***bijective*** function.

Ex. $h: \{1, 2, 3\} \rightarrow \{1, 2, 3\}$,

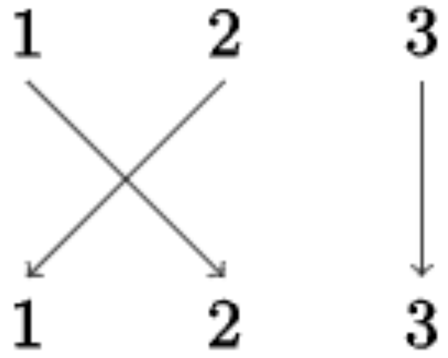

Image and Inverse Image

Consider the function $f: X \to Y$

We refer to elements in the domain f as x, and corresponding elements in the codomain as $f(x)$ (i.e. the image of x).

Let $A \subseteq X$, $f(A)$ denotes the ***image of $\boldsymbol{A}$ under $\boldsymbol{f}$***,

$f(A) = \{f(a) \in Y: a \in A\}$

Or in other words, $f(A)$ is the set of elements in Y that are the image of elements from A.

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Let $\mathrm{B} \subseteq Y$, $f^{-1}(B)$ denotes the ***inverse image of B under f***, $f^{-1}(B) = \{\mathrm{x} \in X: f(x) \in B\}$

Or in other words, $f^{-1}(B)$ is the set of elements in X whose images are elements in B.

Note: f^{-1} *is not a function*. Inverse functions only exist for bijections.

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Practice: Consider the function $f: \{1, 2, 3, 4, 5, 6\} \to \{a, b, c, d\}$, $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & a & b & b & b & c \end{pmatrix}$

Find: $f(\{1, 2, 3\}), f^{-1}(\{a, b\}), f^{-1}(\{d\})$

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Practice: Consider the function $\mathrm{g}: \mathbb{Z} \to \mathbb{Z}$, $\mathrm{g}(n) = n^2 + 1$

Find: $\mathrm{g}(1)$, $g(\{1\})$, $g^{-1}(\{1\})$, $g^{-1}(\{2\})$, $g^{-1}(\{3\})$

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Practice: Find a function f: $\{1, 2, 3, 4, 5\} \rightarrow \mathbb{N}$ such that $|f^{-1}(7)| = 5$