Discrete Structures– Proofs: Induction

Dr. Ab Mosca (they/them)

Plan for Today

- Proof by induction

Warm Up

To perform a ***proof by contradiction*** on P:

1. Assume $\neg P$ is true
2. Show that this assumption leads to a contradiction
3. As a result, the only conclusion is that P is true

   (i.e. if it impossible for P to be false, we know it must be true)

Practice: Prove the following…

The sum of a rational number and an irrational number is irrational.

Motivation

You need to mail a package, but don't yet know how much postage you will need. You have a large supply of 8-cent stamps and 5-cent stamps. Which amounts of postage can you make exactly using these stamps? Which amounts are impossible to make?

Motivation

You need to mail a package, but don't yet know how much postage you will need. You have a large supply of 8-cent stamps and 5-cent stamps. Which amounts of postage can you make exactly using these stamps? Which amounts are impossible to make?

How did you try to solve this?

Motivation

Let’s let $P(n)$ be the statement: you can make n cents of postage using just 8-cent and 5-cent stamps.

Motivation

Let's let $P(n)$ be the statement: you can make n cents of postage using just 8-cent and 5-cent stamps.

If we want to find the value of $P(n)$ for some specific n, can we use smaller n's to get it?

> i.e. if we know the value of $P(n-1)$ can we get from that value to $P(n)$?

Motivation

Let's let $P(n)$ be the statement: you can make n cents of postage using just 8-cent and 5-cent stamps.

If we want to find the value of $P(n)$ for some specific n, can we use smaller n's to get it?

> i.e. if we know the value of $P(n-1)$ can we get from that value to $P(n)$?

Ex. If I tell you $P(43)$ is true, can you tell me the value of $P(44)$?

Motivation

Let's let $P(n)$ be the statement: you can make n cents of postage using just 8-cent and 5-cent stamps.

Ex. If I tell you $P(43)$ is true, can you tell me the value of $P(44)$?

> Assume $P(43)$ includes at least three 5-cent stamps.
>
> I can remove those 3 stamps and replace them with two 8-cent stamps. That'll increase the value by 1.

Motivation

Let's let $P(n)$ be the statement: you can make n cents of postage using just 8-cent and 5-cent stamps.

Ex. If I tell you $P(43)$ is true, can you tell me the value of $P(44)$?

> Assume $P(43)$ includes at least three 5-cent stamps.
>
> I can remove those 3 stamps and replace them with two 8-cent stamps. That'll increase the value by 1.
>
> If there aren't at least three 5-cent stamps, there we must have at least three 8-cent stamps.
>
> I can remove those 3 stamps and replace them with five 5-cent stamps. That'll increase the value by 1.

Motivation

Ex. If I tell you $P(43)$ is true, can you tell me the value of $P(44)$?

> Assume $P(43)$ includes at least three 5-cent stamps.
>
> I can remove those 3 stamps and replace them with two 8-cent stamps. That'll increase the value by 1.
>
> If there aren't at least three 5-cent stamps, there we must have at least three 8-cent stamps.
>
> I can remove those 3 stamps and replace them with five 5-cent stamps. That'll increase the value by 1.

Note: We don't say *how* to make 44-cents, just that we can on basis that we can make 43.

Motivation

Note: We don't say *how* to make 44-cents, just that we can on basis that we can make 43.

How do we know we can make 43? Because we know we can make 42, which we know because we can make 41, etc.

Motivation

Note: We don’t say *how* to make 44-cents, just that we can on basis that we can make 43.

How do we know we can make 43? Because we know we can make 42, which we know because we can make 41, etc.

This is called a ***recursion*** (more on these later!)

Specifically, recursion says

- $P(k+1)$ is true if $P(k)$ is also true.

Motivation

Note: We don't say *how* to make 44-cents, just that we can on basis that we can make 43.

How do we know we can make 43? Because we know we can make 42, which we know because we can make 41, etc.

This is called a ***recursion*** (more on these later!)

Specifically, recursion says

- $P(k+1)$ is true if $P(k)$ is also true.

We need to start the process with a true $P(k)$ (called the ***base case***) and we can build up from that initial condition.

Motivation

We know not all $P(n)$ are true.

But what if I told you that $P(n)$ is true for all $n \geq 28$?

Motivation

We know not all $P(n)$ are true.

But what if I told you that $P(n)$ is true for all $n \geq 28$?

I can make 28 cents with four 5-cent stamps and one 8-cent stamp.

Motivation

We know not all $P(n)$ are true.

But what if I told you that $P(n)$ is true for all $n \geq 28$?

I can make 28 cents with four 5-cent stamps and one 8-cent stamp.

Does the trick we used before (swapping three 5-cent stamps for two 8-cent stamps, or three 8-cent stamps for five 5-cent stamps) work for all numbers greater than 28?

Convince me.

Hint: Are you sure you have at least three 5-cent stamps *and* 8-cent stamps to make 28 cents?

Induction

The proof we just did was an inductive proof.

We showed that $P(28)$ is true.

Then we showed that for any k greater than 28, if $P(k)$ is true then $P(k+1)$ is also true.

Induction

The proof we just did was an inductive proof.

We showed that $P(28)$ is true.

Then we showed that for any k greater than 28, if $P(k)$ is true then $P(k+1)$ is also true.

Therefore, $P(n)$ is true for all $n \geq 28$.

[because we know $P(28)$ is true, and if that's true $P(28+1)$ is true and if $P(29)$ is true then $P(29+1)$ is true…]

Induction

The proof we just did was an inductive proof.

We showed that $P(28)$ is true.

This is called our base case

Then we showed that for any k greater than 28, if $P(k)$ is true then $P(k+1)$ is also true.

This is called the inductive step.

Therefore, $P(n)$ is true for all $n \geq 28$.

[because we know $P(28)$ is true, and if that's true $P(28+1)$ is true and if $P(29)$ is true then $P(29+1)$ is true…]

Induction

To perform a proof by ***induction** on* $P(n)$:

1. Start with your ***base case***
   - Prove $P(n)$ is true for the smallest value of n possible
2. Perform the ***inductive step***
   - Prove that $P(k) \rightarrow P(k+1)$ for all k greater than or equal to the smallest possible value of n.
   - Note this is an if … then … proof, so we start by assuming $P(k)$ is true. This is called the ***inductive hypothesis*** for this type of proof.

Induction

To perform a proof by ***induction*** *on* $P(n)$:

1. Start with your ***base case***
   - Prove $P(n)$ is true for the smallest value of n possible
2. Perform the ***inductive step***
   - Prove that $P(k) \rightarrow P(k+1)$ for all k greater than or equal to the smallest possible value of n.
   - Note this is an if … then … proof, so we start by assuming $P(k)$ is true. This is called the ***inductive hypothesis*** for this type of proof.

Practice: Prove the following…

For each natural number, $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Induction

To perform a proof by ***induction*** *on* $P(n)$:

1. Start with your ***base case***
   - Prove $P(n)$ is true for the smallest value of n possible
2. Perform the ***inductive step***
   - Prove that $P(k) \rightarrow P(k+1)$ for all k greater than or equal to the smallest possible value of n.
   - Note this is an if … then … proof, so we start by assuming $P(k)$ is true. This is called the ***inductive hypothesis*** for this type of proof.

Practice: Prove the following…

For all natural numbers, n, $6^n - 1$ is a multiple of 5.

Induction

To perform a proof by ***induction*** *on* $P(n)$:

1. Start with your ***base case***
   - Prove $P(n)$ is true for the smallest value of n possible
2. Perform the ***inductive step***
   - Prove that $P(k) \rightarrow P(k+1)$ for all k greater than or equal to the smallest possible value of n.
   - Note this is an if … then … proof, so we start by assuming $P(k)$ is true. This is called the ***inductive hypothesis*** for this type of proof.

Practice: Prove the following…

$n^2 > 2^n$ for all integers, $n \geq 5$

Logic Wrap Up

Take a look at in-class activity 1 (ic-01) on the course website. We’ll work on this project in class for the rest of today, and Thursday.