Discrete Structures– Proofs: Direct and Contrapositive

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Plan for Today

- What is a proof?
- Direct proofs
- Proof by contrapositive

Warm Up

The ***existential quantifier*** is ∃ and is read "there is"

The ***universal quantifier*** is ∀ and is read "for all" or "every"

Work with your group to translate the following into English:

- $\forall x(E(x) \rightarrow (E(x+2))$
- $\forall x \exists y(\sin(x) = y)$
- $\forall y \exists x(\sin(x) = y$
- $\forall x, y(\mathrm{x}^3 = \mathrm{y}^3 \rightarrow \mathrm{x} = \mathrm{y})$

Defn: Proof

proof 1 of 3 **noun**

ˈprüf

Synonyms of *proof* ›

1 **a** : the cogency of evidence that compels acceptance by the mind of a truth or a fact

b : the process or an instance of establishing the validity of a statement especially by derivation from other statements in accordance with principles of reasoning

https://www.merriam-webster.com/dictionary/proof

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1 **a** : the cogency of evidence that compels acceptance by the mind of a truth or a fact

b : the process or an instance of establishing the validity of a statement especially by derivation from other statements in accordance with principles of reasoning

Proofs are simply the process of using what we know to be true to logically show that something new is also true.

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Proof Structure

Proofs are a form of communication. You need to write them with the understanding that they are primarily for other people. (Remember, the objective is to convincingly argue to someone that something is true.)

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Customs / Tips:

- Start with the word “Proof” and state what you are going to prove
- Write individual steps clearly, and keep them small
- End with QED, ■, //

Example Proof

Proof: We will show that for all integers, n, if n is even then n^2 is even.

Let n be any arbitrary integer.

Given the definition of even, we know $n = 2k$ for some integer k.

Therefore, $n^2 = (2k)^2$.

If we distribute the exponent, $n^2 = 4k^2 = 2(2\mathrm{k}^2)$.

Because $2k^2$ is an integer, and it is multiplied by 2, we know that n^2 is even. //

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End proof.

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Direct Proof

There are many types of proof. The most common is ***direct proof***.

Direct proof is usually used to prove implications.

To prove $P \rightarrow Q$, we will

1. Assume P is true
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Practice: Prove the following…

If two numbers a and b are even, then their sum, $a + b$, is even.

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Practice: Prove the following…

For all integers n, if n is even, then $8n$ is even.

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Practice: Prove the following…

The sum of two odd numbers is even.

Hint - Definition of odd: n is odd if $n = 2k + 1$ for some integer, k.

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Practice: Prove the following…

For all integers a, b, c, if $a|b$ and $b|c$ then $a|c$.

Note: $x|y$ is read “x divides y” and means that y is a multiple of x. In other words, x will divide into y without a remainder.

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Practice: Prove the following…

Let m and n be integers. If m and n are perfect squares, them mn is a perfect square.

Proof by Contrapositive

Recall that an implication ($P \rightarrow Q$) and its contrapositive ($\neg Q \rightarrow \neg P$) are logically equivalent.

This means we can prove the contrapositive of an implication to prove the implication itself.

To prove $P \rightarrow Q$, we will

1. Prove the contrapositive ($\neg Q \rightarrow \neg P$)
    1. Assume $\neg Q$ is true
    2. Deduce that $\neg P$ must then also be true

Proof by Contrapositive

Proof: We will show that for all integers n, if n^2 if even then n is even.

We will use proof by contrapositive. In other words, we will show if n is not even, then n^2 is not even.

By definition of odd, if n is odd we can write $n = 2k + 1$ for some integer k.

By substitution, $n^2 = (2k + 1)^2$

Using algebra we see: $(2k + 1)^2$

$$= (2k + 1)(2k + 1)$$

$$= 4k^2 + 2k + 2k + 1$$

$$= 2(2k^2 + k + k) + 1$$

$2k^2 + k + k$ is an integer, therefore we have shown n^2 is odd. //

Contrapositive

Start by stating what you will show.

For proofs other than direct, say what technique you will use

End proof.

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Practice: Prove the following…

For all integers a and b, if $a + b$ is odd, then a is odd or b is odd.

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Practice: Prove the following…

For every prime number p, if $p \neq 2$, then p is odd.

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Practice: Prove the following…

For all integers a and b, if $a^2 + b^2$ is off, then a or b is odd.

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Practice: Prove the following…

The game TENZI comes with 40 six-sided dice. Suppose you roll all 40 dice. Prove that there will be at least seven dice that land on the same number.

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Practice: Prove the following…

Suppose x, y are real numbers. If $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$.

Proof

Part of proof is choosing the technique.

Practice: Prove the following...

Suppose x, y are integers. If $5 \nmid xy$, then $5 \nmid x$ and $5 \nmid y$