Discrete Structures– Propositional Logic Pt. 2

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Plan for Today

- Boolean Algebra
- Propositions
- Quantifiers

Warm Up

One way we model truth values of statements is with ***truth tables***.

conjunction

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

disjunction

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

implication

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

biconditional

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

negation

P	$\neg P$
T	F
F	T

We also use truth tables to show logical equivalence. Two molecular statements *P* and *Q* are ***logically equivalent*** provided *P* is true precisely when *Q* is true.

Work with 1-2 other people to show that $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$.

De Morgan’s Laws

Last time we showed

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You just showed

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De Morgan's Laws

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Notice that these laws show you how to "distribute" a negation into parentheses.

Implications are Disjunctions

Last time we also showed that the implication $P \to Q$ is logically equivalent to the disjunction $\neg P \vee Q$.

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That means at any time we see $P \rightarrow Q$ in a statement we can replace it with $\neg P \vee Q$ without changing the truth values of the statement. And vice versa.

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That means at any time we see $P \to Q$ in a statement we can replace it with $\neg P \vee Q$ without changing the truth values of the statement. And vice versa.

(The same is true of De Morgan's Laws, or any logically equivalent statements; we can always replace one with another without changing the truth values of a statement.)

Double Negation

A double negation cancels itself out.

In other words, $\neg\neg P$ is logically equivalent to P.

Boolean Algebra

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So far we know:

- De Morgan's Laws
  - $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$
  - $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$
- Implications are disjunctions
  - $P \rightarrow Q$ is logically equivalent to $\neg P \vee Q$
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Practice: Use Boolean Algebra to show $\neg(P \rightarrow Q)$ and $P \wedge \neg Q$ are logically equivalent.

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Practice: Use Boolean Algebra to show $\neg((\neg P \wedge Q) \vee \neg(R \vee \neg S))$ and $(Q \rightarrow P) \wedge (S \rightarrow R)$ are logically equivalent.

Variables and Statements

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Now we can represent the statement above as:

$$P(n) \rightarrow \neg P(n+7)$$

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Practice: Write the sentence “Primes greater than 2 are odd” symbolically.

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Notice that $P(n) \rightarrow \neg P(n+7)$ is *not a statement*. We have not specified anything about n; it can be any value.

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To make this predicate a statement (something that is always true or false), we need to quantify the variable, n.

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Ex.

$$\exists x(x < 0)$$

is read "There exists an x such that x is less than 0"

This statement says there exists a number less than 0.

Quantifiers

In mathematics we use two quantifiers: existential and universal.

The ***universal quantifier*** is ∀ and is read "for all" or "every"

Ex.

$$\forall x(x \geq 0)$$

is read "For all x, x is greater than or equal to 0"

This statement says every number is greater than or equal to 0.

Quantifiers

The ***existential quantifier*** is ∃ and is read "there exists"

The ***universal quantifier*** is ∀ and is read "for all" or "every"

Practice: Write the sentence "All primes greater than 2 are odd" symbolically.

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Consider this statement: $\forall x \exists y (y < x)$
Let the domain of x, y be the ***natural numbers*** (0, 1, 2, …)

What is the truth value of the statement?

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Consider this statement: $\exists x \forall y (y \geq x)$
Let the domain of x, y be the ***natural numbers*** (0, 1, 2, …)

What is this statement in English?
What is the truth value of the statement?

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Negation of Quantifiers

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$\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$

$\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$

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Practice: Rewrite the following statement without negation

$$\neg\exists x \forall y (x \leq y)$$

What is each version in English?

Back to Implications

In the statement $P \to Q$, we say

- P is the ***hypothesis*** (or antecedent)
- and Q is the ***conclusion*** (or consequent)

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In the statement $P \rightarrow Q$, we say

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Recall this statement is false only when P is T and Q is F

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We will use this later for proofs. We will assume our hypothesis is true, and show that this implies our conclusion is true.

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The ***converse*** of an implication $P \to Q$ is the implication $Q \to P$

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The ***converse*** of an implication $P \rightarrow Q$ is the implication $Q \rightarrow P$

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Practice: Come up with an example where $P \to Q$ is true and $Q \to P$ is false.

Back to Implications

The ***contrapositive*** of an implication $P \to Q$ is the statement $\neg Q \to \neg P$

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Practice: Show $P \to Q$ and $\neg Q \to \neg P$ are logically equivalent using truth tables or Boolean algebra.