Discrete Structures– Proofs: Induction

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Plan for Today

- Proof by induction

Warm Up

To perform a ***proof by contradiction***:

1. Assume $\neg P$ is true
2. Show that this assumption leads to a contradiction
3. As a result, the only conclusion is that P is true

   (i.e. if it impossible for P to be false, we know it must be true)

Practice: Prove the following…

The sum of a rational number and an irrational number is irrational.

Motivation

You need to mail a package, but don't yet know how much postage you will need. You have a large supply of 8-cent stamps and 5-cent stamps. Which amounts of postage can you make exactly using these stamps? Which amounts are impossible to make?

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How did you try to solve this?

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Ex. If I tell you $P(43)$ is true, can you tell me the value of $P(44)$?

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Ex. If I tell you $P(43)$ is true, can you tell me the value of $P(44)$?

> Assume $P(43)$ includes at least three 5-cent stamps.
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> I can remove those 3 stamps and replace them with two 8-cent stamps. That'll increase the value by 1.

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- Assume $P(43)$ includes at least three 5-cent stamps.
- I can remove those 3 stamps and replace them with two 8-cent stamps. That'll increase the value by 1.
- If there aren't at least three 5-cent stamps, there we must have at least three 8-cent stamps.
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This is called a ***recursion*** (more on these later!)

Specifically, recursion says

- $P(k+1)$ is true if $P(k)$ is also true.

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This is called a ***recursion*** (more on these later!)

Specifically, recursion says

- $P(k+1)$ is true if $P(k)$ is also true.

We need to start the process with a true $P(k)$ (called the ***base case***) and we can build up from that initial condition.

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I can make 28 cents with four 5-cent stamps and one 8-cent stamp.

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But what if I told you that $P(n)$ is true for all $n \geq 28$?

I can make 28 cents with four 5-cent stamps and one 8-cent stamp.

Does the trick we used before (swapping three 5-cent stamps for two 8-cent stamps, or three 8-cent stamps for five 5-cent stamps) work for all numbers greater than 28?

Convince me.

Hint: Are you sure you have at least three 5-cent stamps *and* 8-cent stamps to make 28 cents?

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The proof we just did was an inductive proof.

We showed that $P(28)$ is true.

Then we showed that for any k greater than 28, if $P(k)$ is true then $P(k+1)$ is also true.

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Therefore, $P(n)$ is true for all $n \geq 28$.

[because we know $P(28)$ is true, and if that's true $P(28+1)$ is true and if $P(29)$ is true then $P(29+1)$ is true…]

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We showed that $P(28)$ is true.

This is called our base case

Then we showed that for any k greater than 28, if $P(k)$ is true then $P(k+1)$ is also true.

This is called the inductive step.

Therefore, $P(n)$ is true for all $n \geq 28$.

[because we know $P(28)$ is true, and if that's true $P(28+1)$ is true and if $P(29)$ is true then $P(29+1)$ is true…]

Induction

To perform a proof by ***induction*** *on* $P(n)$:

1. Start with your ***base case***
   - Prove $P(n)$ is true for the smallest value of n possible
2. Perform the ***inductive step***
   - Prove that $P(k) \to P(k+1)$ for all k greater than or equal to the smallest possible value of n.
   - Note this is an if … then … proof, so we start by assuming $P(k)$ is true. This is called the ***inductive hypothesis*** for this type of proof.

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Practice: Prove the following…

For each natural number, $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

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Practice: Prove the following…

For all natural numbers, n, $6^n - 1$ is a multiple of 5.

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Practice: Prove the following…

$n^2 > 2^n$ for all integers, $n \geq 5$

Logic Wrap Up

Take a look at in-class activity 1 (ic-01) on the course website. We’ll work on this project in class for the rest of today, and Thursday.