Discrete Structures– Sequences

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Plan for Today

- Sequences
  - Describing
  - Arithmetic
  - Geometric

Warm Up: Counting and Proofs

A ***permutation*** is a (possible) rearrangement of objects. We write this $P(n,k)$, an call it a ***k-permutation of n elements***

$$P(n,k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)\ldots(n-(k-1))$$

A ***combination*** is the number of ways to choose k objects from n. We write this $C(n,k)$ or $\binom{n}{k}$, and read both ***n choose k***.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Consider the identity: $k\binom{n}{k} = n\binom{n-1}{k-1}$. Prove that this identity is true.

Motivation

You have a collection of 1X1 squares and 1X2 squares. You want to arrange these to. make a 1X15 strip.

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2. How are the 1X3 and 1X4 strips related to the 1X5 strips?
3. How many 1X15 strips can you make?
4. What if I asked you to find the number of 1X1000 strips? Would the method you used to calculate the number of 1X15 strips be helpful?

Definition

A ***sequence*** is an ordered list of numbers (think array!)

When we use variables to represent a sequence, we number them because order matters:

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Guess the next term in these sequences:

1. 3, -3, 3, -3, 3, …
2. 1, 5, 2, 10, 3, 15, …
3. 1, 2, 4, 8, 16, …
4. 1, 4, 9, 16, 25, 36, …

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Ex. $a_n = n^2$

$$a_n = \frac{n(n+1)}{2}$$

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^{-n}}{\sqrt{5}}$$

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A ***recursive*** (or ***inductive***) ***definition*** for a sequence $(a_n)_{n\in\mathbb{N}}$ consists of a ***recurrence relation*** (an equation relating a term of the sequence to previous terms) and an ***initial condition*** (a list of a few terms of the sequence).

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Ex. $a_n = 2a_{n-1}, a_0 = 1$

$a_n = 2a_{n-1}, a_0 = 27$

$a_n = a_{n-1} + a_{n-2}, a_0 = 0, a_1 = 1$

Find the 0^{th}, 1^{st}, and 5^{th} terms for each sequence

Common Sequences

- 1, 4, 9, 16, 25, …
  - ***square numbers***
  - For $(s_n)_{n\geq 1}, s_n = n^2$
- 1, 3, 6, 10, 15, 21, …
  - ***triangular numbers***
  - For $(T_n)_{n\geq 1}, T_n = \frac{n(n+1)}{2}$
- 1, 2, 4, 8, 16, 32, …
  - ***powers of two***
  - For $(a_n)_{n\geq 0}, a_n = 2^n$
- 1, 1, 2, 3, 5, 8, 13, …
  - ***Fibonacci numbers***
  - $F_n = F_{n-1} + F_{n-2}, \mathrm{F}_1 = \mathrm{F}_2 = 1$

Describing Sequences

- Finding the closed formula for a sequence is not always straightforward. There are many approaches.
- One option: Try to relate the sequence to a common sequence

Use $T_n = \frac{n(n+1)}{2}$ and $a_n = 2^n$ to find closed formulas for the following sequences. Assume each first term corresponds to $n = 0$.

- (b_n): $1, 2, 4, 7, 11, 16, 22, \ldots$
- (c_n): $3, 5, 9, 17, 33$
- (d_n): $0, 2, 6, 12, 20, 30, 42, \ldots$
- (f_n): $0, 1, 3, 7, 15, 31$

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Sam keeps track of how many push-ups she does each day of her “do lots of push-ups challenge.” Let $(a_n)_{n\geq 1}$ be the sequence that describes the number of push-ups done on the n^{th} day of the challenge. The sequence starts

$$3, 5, 6, 10, 9, 0, 12$$

Describe the sequence $(b_n)_{n\geq 1}$ that describes the total number of pushups done by Sam after the n^{th} day.

Describing Sequences

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- Some sequences naturally arise as the sum of terms of another sequence

Given any sequence $(a_n)_{n\geq 1}$ we can always form a new sequence $(b_n)_{n\geq 1}$ as

$$b_n = a_0 + a_1 + \cdots + a_n$$

$$b_n = \sum_{k=1}^{n} a_k$$

Since the terms of (b_n) are sums of the initial part of the sequence (a_n) we call (b_n) the ***sequence of partial sums*** of (a_n).

Describing Sequences

Rewrite these sums using $\sum$ notation

- $1 + 2 + 3 + 4 + \cdots + 100$
- $1 + 2 + 4 + 8 + \cdots + 2^{50}$
- $6 + 10 + 14 + \cdots + (4n - 2)$

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The multiplication version of this is:

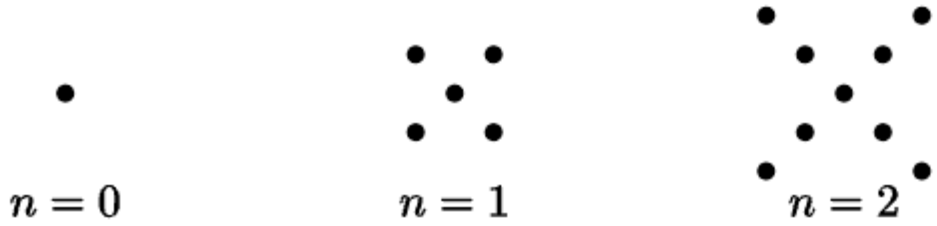
$$\prod_{k=1}^{n} a_k$$

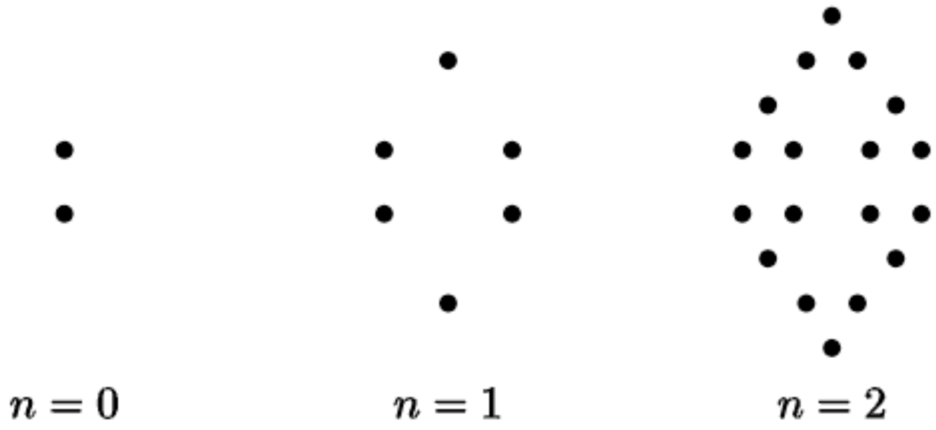
$$b_n = \sum_{k=1}^{n} a_k$$

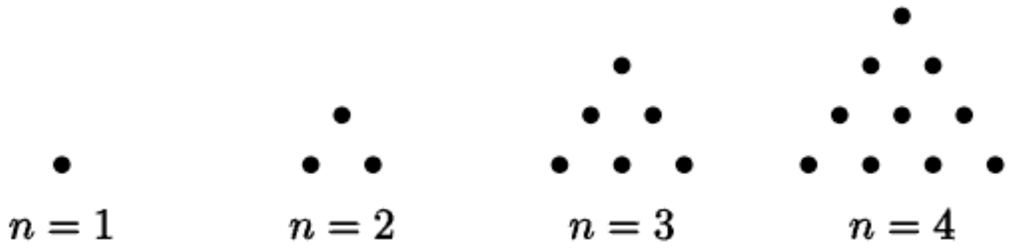
Since the terms of (b_n) are sums of the initial part of the sequence (a_n) we call (b_n) the ***sequence of partial sums*** of (a_n).

Closed Formulas

For the patterns of dots below, draw the next pattern in the sequence. Give a recursive definition and closed formula for the number of dots in the n^{th} pattern.







Arithmetic Sequences

If the terms of a sequence differ by a constant, we say the sequence is ***arithmetic***.

If the initial term (a_0) of the sequence is a and the ***common difference*** is d, then we have,

Recursive definition: $a_n = a_{n-1} + d$, $a_0 = a$

Closed formula: $a_n = a + dn$

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Find the recursive definitions and closed formulas for the arithmetic sequences below. Assume the first term listed is a_0.

- 2, 5, 8, 11, 14, ….
- 50, 43, 36, 29, …

Geometric Sequences

If the terms of a sequence differ by a constant ratio, we say the sequence is ***geometric***.

If the initial term (a_0) of the sequence is a and the ***common ratio*** is r, then we have,

Recursive definition: $a_n = r * a_{n-1}$, $a_0 = a$

Closed formula: $a_n = a * r^n$

Geometric Sequences

If the terms of a sequence differ by a constant ratio, we say the sequence is ***geometric***.

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Recursive definition: $a_n = r * a_{n-1}$, $a_0 = a$

Closed formula: $a_n = a * r^n$

Find the recursive definitions and closed formulas for the geometric sequences below. Assume the first term listed is a_0.

- 3, 6, 12, 24, 48, …
- 27, 9, 3, 1, 1/3, …