Discrete Structures– Sequences Pt. 2

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Plan for Today

- Sequences
  - Sums
  - Solving recurrences

Warm Up: Arithmetic Sequences

If the terms of a sequence differ by a constant, we say the sequence is ***arithmetic***.

If the initial term (a_0) of the sequence is a and the ***common difference*** is d, then we have,

Recursive definition: $a_n = a_{n-1} + d$, $a_0 = a$

Closed formula: $a_n = a + dn$

Find the recursive definition and closed formula for this sequence:

8, 14, 20, 26, …

Warm Up: Geometric Sequences

If the terms of a sequence differ by a constant ratio, we say the sequence is ***geometric***.

If the initial term (a_0) of the sequence is a and the ***common ratio*** is r, then we have,

Recursive definition: $a_n = r * a_{n-1}$, $a_0 = a$

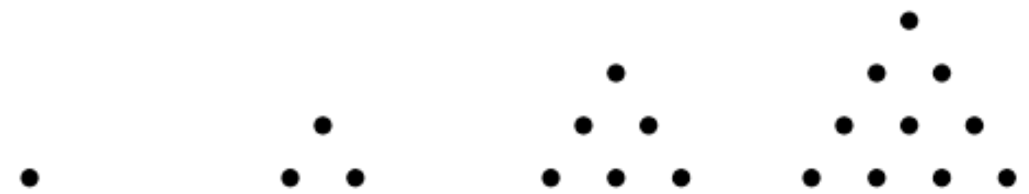
Closed formula: $a_n = a * r^n$

Find the recursive definition and closed formula for this sequence:

8, 32, 128, 512, …

Consider the ***triangular numbers***

Motivation



$T_1 = 1$ $T_2 = 3$ $T_3 = 6$ $T_4 = 10$

Is this sequence arithmetic, geometric, or neither?

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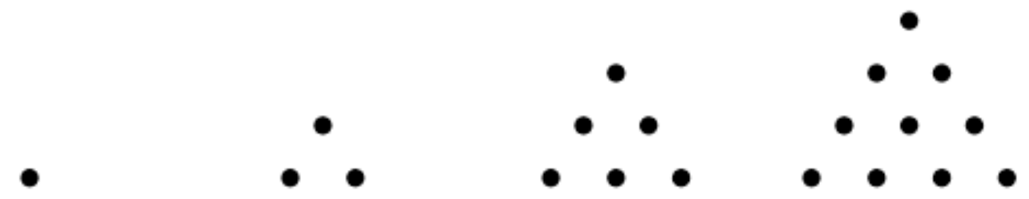


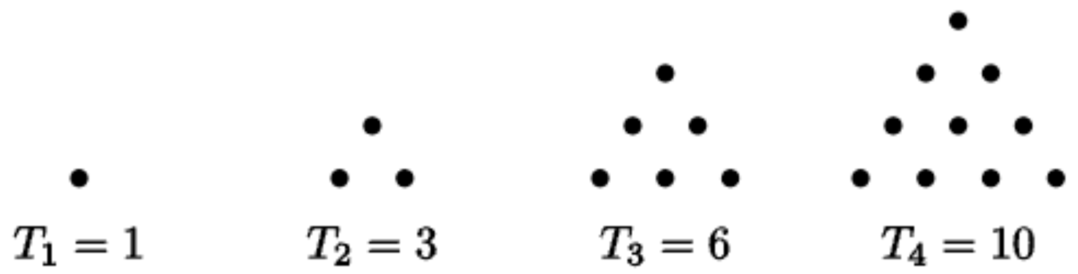
$T_1 = 1 \qquad T_2 = 3 \qquad T_3 = 6 \qquad T_4 = 10$

Is this sequence arithmetic, geometric, or neither?

What if you look at the sequence of differences between consecutive terms?

Motivation

Consider the ***triangular numbers***



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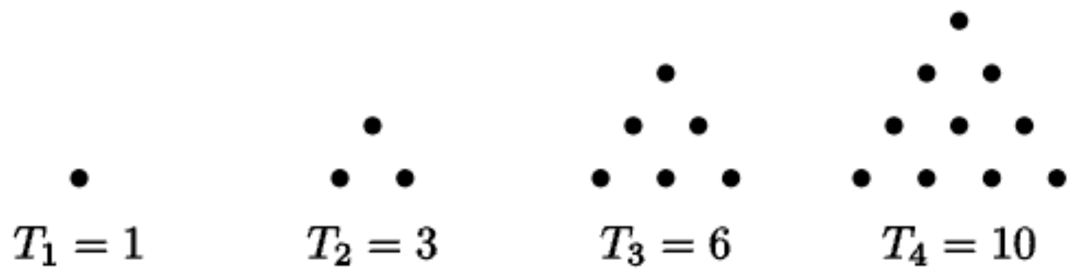

(T_n) is a ***sequence of partial sums***.

If we know how to add terms of arithmetic (or geometric) sequences, we can find closed formulas for sequences whose differences are terms of an arithmetic (or geometric) sequence.

Summing Arithmetic Sequences

Ex. Find the sum of the first 100 integers:
$1, 2, 3, \ldots, 98, 99, 100$

Notice that if we add pairs starting from the smallest and largest to the next smallest and largest to the next, we get the same number.

$$1 + 100 = 101$$
$$2 + 99 \quad = 101$$
$$3 + 98 \quad = 101$$

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$1 + 100 = 101$

$2 + 99 \quad = 101$

$3 + 98 \quad = 101$

If we continue, we end up adding 50 pairs (half the number of terms) all equal to our last term (100^{th}) + 1.

So, $T_n = \frac{n(n+1)}{2}$

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The partial sum, S_n, of any arithmetic sequence can be computed by ***reversing terms and adding, then multiplying by*** $\boldsymbol{\frac{n}{2}}$.

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$$d = 3, a_0 = 2, a_n = 2 + 3n$$

$$\begin{aligned} S &= 2 \quad + 5 \quad + 8 \quad + \cdots + 464 + 467 + 470 \\ S &= 470 + 467 + 464 + \cdots + 8 \quad + 5 \quad + 2 \\ 2S &= 472 + 472 + 472 + \cdots + 472 + 472 + 472 \end{aligned}$$

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We know $2 + 3n = 470, n = 156$, so there are 157 terms

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So, $2S = 157 * 472, S = \frac{157}{2} * 472 = 37052$

Summing Arithmetic Sequences

Find the sum of $5 + 9 + 13 + 17 + 21 + \cdots + 533$

Ex. $2 + 5 + 8 + 11 + 14 + \cdots + 470$

$$d = 3, a_0 = 2, a_n = 2 + 3n$$

$$\begin{aligned}
S &= 2 \quad + 5 \quad + 8 \quad + \cdots + 464 + 467 + 470 \\
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Summing Arithmetic Sequences

T
b

Find a closed formula for $6+10+14+\cdots+(4n-2)$

Ex. $2+5+8+11+14+\cdots+470$

$$d=3, a_0=2, a_n=2+3n$$

$$S = 2 \quad +5 \quad +8 \quad +\cdots+464+467+470$$
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Start here

Summing Arithmetic Sequences

Find a closed formula for $(a_n)_{n\geq 0}$: $2, 3, 7, 14, 24, 37, \ldots$

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$$d = 3, a_0 = 2, a_n = 2 + 3n$$

$$S = 2 \quad + 5 \quad + 8 \quad + \cdots + 464 + 467 + 470$$
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The partial sum, S_n, of any geometric sequence can be computed by ***multiplying by the common ratio, shifting, then subtracting***.

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$$S = 24573$$

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Find a closed formula for $S(n) = 2 + 10 + 50 + \cdots + 2 * 5^n$

Solving Recurrence Relations

Converting a recursive definition into a closed formula is called ***solving a recurrence relation***.

Sometimes, you can use your knowledge of known sequences to determine the closed formula for a recurrence relation.

Other times, finding the closed formula is more difficult.

Solving Recurrence Relations

Characteristic roots

Given a recurrence relation $a_n + \alpha a_{n-1} + \beta a_{n-2}$, the ***characteristic polynomial*** is

$$x^2 + \alpha x + \beta$$

giving the ***characteristic equation:***

$$x^2 + \alpha x + \beta = 0$$

If r_1 and r_2 are two distinct roots of the characteristic polynomial, then the solution to the recurrence relation is

$$a_n = a r_1^n + b r_2^n$$

where a, b are constants determined by the initial conditions.

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$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$
$$x^2 - 7x + 10 = 0$$
$$(x-2)(x-5) = 0$$

So, $x = 2, x = 5$ are the characteristic roots and

$$a_n = a2^n + b5^n$$

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$$2 = a2^0 + b5^0 = a + b$$

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$$a = \frac{7}{3}, b = -\frac{1}{3}$$

So, $a_n = \frac{7}{3}2^n - \frac{1}{3}5^n$

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Note: You might need to use the quadratic formula to find roots:

$$ax^2 + bx + c = 0, x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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