# Bayesian Learning of Finite Asymmetric Gaussian Mixtures by MH-within-Gibbs

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**Abstract.** Asymmetric Gaussian mixture (AGM) model has been proven to be more flexible than the classic Gaussian mixture model from many aspects. In contrast with previous efforts that have focused on maximum likelihood estimation, this paper introduces a fully Bayesian learning method using Metropolis-Hastings (MH) within Gibbs sampling method to learn AGM model. We show the merits of the proposed model using synthetic data and a challenging intrusion detection application.

**Keywords - Asymmetric Gaussian Mixture, Metropolis-Hastings, Gibbs sampling, MCMC, Intrusion detection**

## 1 Introduction

Along with the development of information-based industries, network security problems are becoming increasingly important today. In order to address this challenge, many data mining methodologies were proposed including both classification-based [1] and clustering-based [2] ones. However, classification-based solutions generally perform ineffectively for dynamic and variate attacking methods because changes of the intrusion patterns cannot be automatically adapted by supervised learning algorithms. Consequently, unsupervised approaches are more favorable for modern intrusion-detection.

As an efficient clustering approach, Gaussian mixtures model (GMM) [3] is widely deployed because of its outstanding suitability in several domains such as computer vision, pattern recognition and data mining. In this paper, we choose asymmetric Gaussian mixture (AGM) model [4] for modeling because it uses two variance parameters for left and right parts of each distribution in the mixture which allows to accurately model non-Gaussian datasets including asymmetric ones.

A challenging issue when deploying mixture models is the learning of the model's parameters. The estimation of the parameters of mixture distributions can be accomplished by using maximum-likelihood-based expectation maximization (EM) [5] algorithm. However, EM has some drawbacks such as overfitting and dependency on initialization [6] [7]. Therefore, an alternative is the fully Bayesian approach, based for instance on Markov Chain Monte Carlo (MCMC) methods, which has been found to be useful in many applications by considering parameters priors which can avoid overfitting problems. As a sampling-based learning approach, the main difficulty of MCMC method is that, under some circumstances, direct sampling is not always straightfoward. As widely deployed implementations of MCMC method, Metropolis-Hastings (Hastings, 1970) [8] and Gibbs sampling (Geman and Geman, 1984) [9] methods can be introduced to solve this problem through applying proposal priors and posteriors and sampling one parameter by giving the others. By combining the advantages of both sampling techniques together, the Metopolis-Hastings within Gibbs method [6] is selected as the learning algorithm for AGM model.

The rest of this paper is organized as follows. Section 2 illustrates the AGM model and its Bayesian learning process. Section 3 is devoted to experimental results using both synthetic data and a real application (network intrusion detection). Finally, Section 4 concludes the paper.

## 2 Bayesian Model

### 2.1 Asymmetric Gaussian Mixture Model

Assuming that the AGM model has $M$ components then the likelihood function (Elguebaly and Bouguila, 2013) [4] is defined as follows:

$$p(\mathcal{X}|\Theta) = \prod_{i=i}^{N} \sum_{j=1}^{M} p_j p(X_i|\xi_j) \tag{1}$$

where $\mathcal{X} = (X_1, ..., X_N)$ is the set of $N$ observations, $\Theta = \{p_1, ..., p_M, \xi_1, ..., \xi_M\}$ represents the parameters set. $p_j$ $(0 < p_j \leq 1$ and $\sum_{j=1}^{M} p_j = 1)$ is the weight for each component in the mixture model and $\xi_j$ is the AGD parameters of mixture component $j$. Giving $X = (x_1, ..., x_d)$, the probability density function (Elguebaly and Bouguila, 2013) [4] can be defined as follows:

$$p(X|\xi_j) \propto \prod_{k=1}^{d} \frac{1}{(\sigma_{l_{jk}} + \sigma_{r_{jk}})} \times \begin{cases} \exp\left[-\frac{(x_k-\mu_{jk})^2}{2(\sigma_{l_{jk}})^2}\right] & if\ x_k\ <\ \mu_{jk} \\ \exp\left[-\frac{(x_k-\mu_{jk})^2}{2(\sigma_{r_{jk}})^2}\right] & if\ x_k\ \geqslant\ \mu_{jk} \end{cases} \tag{2}$$

where $\xi_j = (\mu_j, \sigma_{lj}, \sigma_{rj})$ is the set of parameters of component $j$ and $\mu_j = (\mu_{j1}, ..., \mu_{jd})$ is the mean, $\sigma_{lj} = (\sigma_{lj1}, ..., \sigma_{ljd})$ and $\sigma_{rj} = (\sigma_{rj1}, ..., \sigma_{rjd})$ are the left and right standard deviations for AGD. To be more specific, $x_k \sim N(\mu_{jk}, \sigma_{ljk})$ $(x_k < \mu_{jk})$ and $x_k \sim N(\mu_{jk}, \sigma_{rjk})$ $(x_k \geqslant \mu_{jk})$ for each dimension.

In order to simplify the Bayesian learning process, we introduce a $M$-dimensional membership vector $Z$. For each observation $X_i, 1 < i < N, Z_i = (Z_{i1}, ..., Z_{iM})$ which indicates to which specific component $X_i$ belongs to (Bouguila, Ziou and Monga, 2006) [10], such that:

$$Z_{ij} = \begin{cases} 1 & \text{if } X_i \text{ belongs to component } j \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

in other words, $Z_{ij} = 1$ only if observation $X_i$ has the highest probability of belonging to component $j$ and accordingly, for other components, $Z_{ij} = 0$.

By combining the Eq. (1) and Eq. (3) together we derive the complete likelihood function:

$$p(\mathcal{X}, Z|\Theta) = \prod_{i=1}^{N}\prod_{j=1}^{M}(p_j p(X_i|\xi_j))^{Z_{ij}} \tag{4}$$

### 2.2 Learning Algorithm

Before describing MH-within-Gibbs learning steps, the priors and posteriors need to be specified. Frist, we denote the postorior probability of membership vector Z as $\pi(Z|\Theta, \mathcal{X})$ (Elguebaly and Bouguila, 2011) [11]:

$$Z^{(t)} \sim \pi(Z|\Theta^{(t-1)}, \mathcal{X}) \tag{5}$$

the number of observations belonging to a specific component $j$ can be calculated using $Z^{(t)}$ as follows:

$$n_j^{(t)} = \sum_{i=1}^{N} Z_{ij} \;\; (j = 1, ..., M) \tag{6}$$

thus $n^{(t)} = (n_i^{(t)}, ..., n_M^{(t)})$ represents the number of observations belonging to each mixture component.

Since the mixture weight $p_j$ satisfies the following conditions ($0 < p_j \leq 1$ and $\sum_{j=1}^{M} p_j = 1$), a natural choice of the prior is Dirichlet distribution as follows [12]:

$$\pi(p_j^{(t)}) \sim \mathcal{D}(\gamma_1, ..., \gamma_M) \tag{7}$$

where $\gamma_j$ is known hyperparameter. Consequently, the posterior of the mixture weight $p_j$ is:

$$p(p_j^{(t)}|Z^{(t)}) \sim \mathcal{D}(\gamma_1 + n_1^{(t)}, ..., \gamma_M + n_M^{(t)}) \tag{8}$$

Direct sampling of mixture parameters $\xi \sim p(\xi|Z, \mathcal{X})$ could be difficult so Metropolis-Hastings method should be deployed using proposal distributions for $\xi^{(t)} \sim q(\xi|\xi^{(t-1)})$. To be more specific, for parameters of AGM model which are $\mu$ , $\sigma_l$ and $\sigma_r$, we choose proposal distributions as follows:

$$\mu_j^{(t)} \sim \mathcal{N}_d(\mu_j^{(t-1)}, \Sigma) \tag{9}$$

$$\sigma_{lj}^{(t)} \sim \mathcal{N}_d(\sigma_{lj}^{(t-1)}, \Sigma) \tag{10}$$

$$\sigma_{rj}^{(t)} \sim \mathcal{N}_d(\sigma_{rj}^{(t-1)}, \Sigma) \tag{11}$$

the proposal distributions are $d$-dimensional Gaussian distributions with $\Sigma$ as $d$ x $d$ identity matrix which makes the sampling a random walk MCMC process.

As the most important part of Metropolis-Hastings method, at the end of each iteration, for new generated mixture parameter set $\Theta^{(t)}$, an acceptance ratio $r$ needs to be calculated in order to make a decision whether they should be accepted or discarded for the next iteration. The acceptance ratio $r$ is given by:

$$r = \frac{p(\mathcal{X}|\Theta^{(t)})\pi(\Theta^{(t)})q(\Theta^{(t-1)}|\Theta^{(t)})}{p(\mathcal{X}|\Theta^{(t-1)})\pi(\Theta^{(t-1)})q(\Theta^{(t)}|\Theta^{(t-1)})} \tag{12}$$

where $\pi(\Theta)$ is the proposed prior distribution which can be decomposed to $d$-dimensional Gaussian distributions such that $\mu \sim \mathcal{N}_d(\eta, \Sigma)$ and $\sigma_l, \sigma_r \sim \mathcal{N}_d(\tau, \Sigma)$ given known hyperparameters $\eta$ and $\tau$. Since mixture weight $p$ has been computed previously during the Gibbs sampling part, it should not be included in Eq. (12). Further information about the calculation of acceptance ratio $r$ is explained in Appendix A.

Once acceptance ratio $r$ is derived by Eq. (15), we compute acceptance probability $\alpha = min[1, r]$ [13]. Then $u \sim U_{[0,1]}$ is supposed to be generated randomly. If $\alpha < u$, the proposed move should be accepted and parameters should be updated by $p^{(t)}$ and $\xi^{(t)}$ for next iteration. Otherwise, we discard $p^{(t)}$, $\xi^{(t)}$ and set $p^{(t)} = p^{(t-1)}$, $\xi^{(t)} = \xi^{(t-1)}$.

We summarize the MH-within-Gibbs learning process for AGM model in the following steps:

**Input:** Data observations $\mathcal{X}$ and components number $M$
**Output:** AGM mixture parameter set $\Theta$

1. Initialization
2. Step $t$: For $t = 1, \ldots$
   **Gibbs sampling part**
   (a) Generate $Z^{(t)}$ from Eq. (5)
   (b) Compute $n_j^{(t)}$ from Eq. (6)
   (c) Generate $p_j^{(t)}$ from Eq. (8)
   **Metropolis-Hastings part**
   (d) Sample $\xi_j^{(t)}$ ($\mu_j^{(t)}, \sigma_{lj}^{(t)}, \sigma_{rj}^{(t)}$) from Eqs. (9) (10) (11)
   (e) Compute acceptance ratio $r$ from Eq. (15)
   (f) Generate $\alpha = min[1, r]$ and $u \sim U_{[0,1]}$
   (g) If $\alpha \geq u$ then $\xi^{(t)} = \xi^{(t-1)}$

## 3 Experimental Results

### 3.1 Design of Experiments

We apply the AGM model to both synthetic data and intrusion detection application. For synthetic data validation, testing observations will be generated from AGD with known component number $M$ and experimental results will be evaluated by calculating Euclidean distances between the estimated and actual mixture parameters. In intrusion detection section, we select NSL-KDD dataset [14] as testing database. K-means algorithm will be introduced as initialization and comparison of AGM model and the result analysis will be based on statistics derived from confusion matrices of both methods.

### 3.2 Synthetic Data

The main goals of this section are feasibility analysis and efficiency evaluation of the AGM learning algorithm.

Observation number is set to 300 spliting into two groups ($M = 2$). Hyperparameters are set accordingly that $\gamma_j = 1$ [15] for sampling mixture weight $p_j$ from Eq. (8). $\eta$ and $\tau$ are $d$-dimensional zero vectors in prior distributions of mixture parameter set $\xi$.

Different proposed component numbers ($M' = 1, \ldots, 5$) are applied during the AGM learning process and the statistics are summarized in Table 1. In order to distinguish the best value of the proposed component number $M'$, we introduce marginal likelihood [6] as a measurement where the best fit $M'$ should have the maximum marginal likelihood value. Obviously, the best result is derived when proposed component number $M'$ equals to original component number $M$ ($M' = M = 2$) shown as Fig. 1. The probability density diagrams are plotted for both original and estimated AGM components and the polylines show the trace of accepted moves for each component.

In terms of the best fit result , the accuracy is evaluated by calculating the Euclidean distance between original and estimated mixture parameter sets $\xi$ and $\hat{\xi}$ (Table 2). In summary, the estimation of mean is accurate because the Euclidean distance between $\mu_j$ and $\hat{\mu}_j$ is slight but the distance between standard deviation $\sigma_{lj}, \sigma_{rj}$ and $\hat{\sigma}_{lj}, \hat{\sigma}_{rj}$ is significant. Since membership vector $Z$ is involved for hard-clustering so this difference will not affect the clustering result too much.

**Table 1.** AGM Learning Statistics ($M = 2,\ M' = 1, \ldots, 5,\ iterations = 300$)

| **Component Number $M'$** | ***Moves Accepted*** | ***Acceptance Ratio*** | ***ML***[a] |
|---|---|---|---|
| 1 | 22 | 7.33% | -1596.143 |
| 2 | 11 | 3.67% | -1500.370 |
| 3 | 14 | 4.67% | -1684.518 |
| 4 | 63 | 21.00% | -1522.148 |
| 5 | 39 | 13.00% | -1517.533 |

[a] Marginal likelihood.

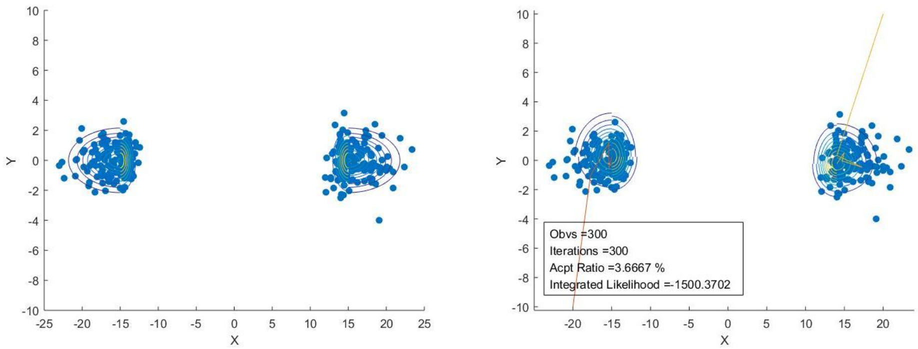


**Fig. 1.** Original synthetic observations and learning result ($M' = M = 2$)

**Table 2.** Accuracy Analysis ($M' = M = 2$)

| **Component Number** $j = 1$ | **Mean** $(\mu_j)$ | **Left Standard deviation** $(\sigma_{lj})$ | **Right Standard deviation** $(\sigma_{rj})$ |
|---|---|---|---|
| $\xi$ | [-15.00, 0.00] | [10.00, 1.00] | [1.00, 1.00] |
| $\hat{\xi}$ | [-14.99, 0.25] | [4.77, 1.13] | [2.31, 1.88] |
| Euclidean Distance | 0.246 | 5.236 | 1.581 |
| **Component Number** $j = 2$ | **Mean** $(\mu_j)$ | **Left Standard deviation** $(\sigma_{lj})$ | **Right Standard deviation** $(\sigma_{rj})$ |
| $\xi$ | [15.00, 0.00] | [1.00, 1.00] | [10.00, 1.00] |
| $\hat{\xi}$ | [14.02, -0.24] | [2.04, 1.04] | [5.70, 1.59] |
| Euclidean Distance | 1.010 | 1.036 | 4.338 |

### 3.3 Intrusion Detection

We select NSL-KDD [14], an improved KDDCUP'99 data set, as the intrusion detection testing target. Before applying the AGM model onto the dataset, the data pre-processing is needed because discrete enumerated values must be translated to numerical ones and be normalized properly in order to have an accurate result.

For better discriminating high-frequency discrete values, firstly we substitute enumerated values with their numbers of occurrences. Having all numerical data in hand, we apply feature scaling method to normalize numerical values between 0 to 1 thus we could use unified proposal distributions during random walk MCMC sampling step for every dimension (Table 3).

K-means clustering algorithm [16] is chosen for the initialization of mixture parameters for learning step and also for the comparison of accuracy. Network operations with total amount of 25192 (20% of NSL-KDD dataset) are clustered into two groups with 11743 intrusions and 13449 normal behaviors indicating component number $M' = 2$. The comparison of confusion matrices between K-means and AGM model (Table 4) reveals the fact that based on a less accurate initialization given by K-means (60.85%), AGM model makes a significant improvement with a much higher accuracy (80.47%). At the same time, it has high precision percentage (96.86%) and low false positive rate (4.26%) illustrating AGM model is capable of effectively detecting intrusions from background noises. Compared with K-means algorithm, AGM model has a higher false negative rate (28.58%) which means it tends to strictly identify normal behaviors as intrusions which could be mitigated by reducing dimensions of dataset using feature selection methodologies.

**Table 3.** Translation and Normalization of Internet Protocols (Enumerated Values)

| **Internet Protocols** | ***Number of Occurrences*** | ***Normalized Values*** |
|---|---|---|
| ICMP | 1655 | 0 |
| UDP | 3011 | 0.071867 |
| TCP | 20526 | 1 |

**Table 4.** Confusion Matrices and Statistics of K-means and AGM Model

**K-means**

| | ***NF*** [a] | ***F*** [b] |
|---|---|---|
| ***NF*** | 2445 | 9298 |
| ***F*** | 565 | 12884 |

**AGM Model**

| | ***NF*** | ***F*** |
|---|---|---|
| ***NF*** | 11374 | 369 |
| ***F*** | 4552 | 8897 |

| | ***K-means*** | ***AGM Model*** |
|---|---|---|
| ***Accuracy*** | 60.85% | 80.47% |
| ***Precision*** | 20.82% | 96.86% |
| ***False Positive Rate*** | 41.92% | 4.26% |
| ***False Negative Rate*** | 18.77% | 28.58% |

[a]Non fault-prone, [b]Fault-prone.

## 4 Conclusion and Future Work

This paper illustrated a new intrusion detection approach by applying asymmetric Gaussian mixtures with a fully Bayesian learning model. The Bayesian learning algorithm is based on MH-within-Gibbs which is a variant of sampling-based MCMC method. The K-means is designated for initialization and comparison of AGM model based on KDDCUP'99 intrusion detection dataset. According to the synthetic and intrusion detection experiments, the AGM model is proved as an effective approach for unsupervised clustering application such as intrusion detection. The experimental results demonstrate clearly that the AGM model outperforms K-means algorithm from many statistical features. In order to improve the accuracy and flexibility of the AGM model, we plan to extend the Bayesian learning process and introduce model selection and feature selection methodologies to adapt to different natures of applications.

## Appendix A

### 4.1 Derivation of Acceptance Ratio $r$ by Eq. (12)

The derivation of acceptance ratio $r$ is based on the assumption that mixture parameters are independent from each other which means that:

$$\begin{aligned}\pi(\Theta) &= \pi(p,\xi) = \pi(\xi)\\ &= \prod_{j=1}^{M} \pi(\mu_j)\pi(\sigma_{lj})\pi(\sigma_{rj})\\ &= \prod_{j=1}^{M} \mathcal{N}_d(\mu_j|\eta,\Sigma)\mathcal{N}_d(\sigma_{lj}|\tau,\Sigma)\mathcal{N}_d(\sigma_{rj}|\tau,\Sigma)\end{aligned} \tag{13}$$

in Eq. (14), since the mixture weigh $p$ is generated following Gibbs sampling method whose acceptance ratio is always 1, it should be excluded from Metropolis-Hastings estimation step. Accordingly, apply the same rule to the proposal distribution as well:

$$\begin{aligned}q(\Theta^{(t)}|\Theta^{(t-1)}) &= q(\xi^{(t)}|\xi^{(t-1)})\\ &= \prod_{j=1}^{M} \mathcal{N}_d(\mu_j^{(t)}|\mu_j^{(t-1)},\Sigma)\mathcal{N}_d(\sigma_{lj}^{(t)}|\sigma_{lj}^{(t-1)},\Sigma)\mathcal{N}_d(\sigma_{rj}^{(t)}|\sigma_{rj}^{(t-1)},\Sigma)\end{aligned} \tag{14}$$

by combining Eqs. (2) (4) (9) (10) (11) (14) and (15), equation (12) can be written as follows:

$$\begin{aligned}r &= \frac{p(\mathcal{X}|\Theta^{(t)})\pi(\Theta^{(t)})q(\Theta^{(t-1)}|\Theta^{(t)})}{p(\mathcal{X}|\Theta^{(t-1)})\pi(\Theta^{(t-1)})q(\Theta^{(t)}|\Theta^{(t-1)})}\\ &= \prod_{i=i}^{N}\prod_{j=1}^{M}\Big(\frac{p(X_i|\mu_j^{(t)},\sigma_{lj}^{(t)},\sigma_{rj}^{(t)})}{p(X_i|\mu_j^{(t-1)},\sigma_{lj}^{(t-1)},\sigma_{rj}^{(t-1)})}\\ &\times \frac{\mathcal{N}_d(\mu_j^{(t)}|\eta,\Sigma)\mathcal{N}_d(\sigma_{lj}^{(t)}|\tau,\Sigma)\mathcal{N}_d(\sigma_{rj}^{(t)}|\tau,\Sigma)}{\mathcal{N}_d(\mu_j^{(t-1)}|\eta,\Sigma)\mathcal{N}_d(\sigma_{lj}^{(t-1)}|\tau,\Sigma)\mathcal{N}_d(\sigma_{rj}^{(t-1)}|\tau,\Sigma)}\\ &\times \frac{\mathcal{N}_d(\mu_j^{(t-1)}|\mu_j^{(t)},\Sigma)\mathcal{N}_d(\sigma_{lj}^{(t-1)}|\sigma_{lj}^{(t)},\Sigma)\mathcal{N}_d(\sigma_{rj}^{(t-1)}|\sigma_{rj}^{(t)},\Sigma)}{\mathcal{N}_d(\mu_j^{(t)}|\mu_j^{(t-1)},\Sigma)\mathcal{N}_d(\sigma_{lj}^{(t)}|\sigma_{lj}^{(t-1)},\Sigma)\mathcal{N}_d(\sigma_{rj}^{(t)}|\sigma_{rj}^{(t-1)},\Sigma)}\end{aligned} \tag{15}$$

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