HOMEWORK 7 - MATH402B

DUE: WEDNESDAY NOVEMBER 22ND

(1) Let $X = \{a, b\}$ be a set. Write down all possible equivalence relations on X.

(2) Describe and classify all groups that contain no proper subgroups, i.e. the only subgroups are the group itself and the subgroup containing only the identity. (Hint: consider for an element of the group the subgroup generated by it).

(3) Let $SL(n, \mathbb{R})$ be the set of $n \times n$ matrices with real coefficients and determinant 1. We already proved in class that $SL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$. Compute the quotient $GL(n, \mathbb{R})/SL(n, \mathbb{R})$. (Hint, use the proof that $SL(n, \mathbb{R})$ is normal and the first homomorphism theorem).

(4) Use the First Homomorphism Theorem to give a different proof that every cyclic group is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_n$ for some n as follows: let $G = \langle a \rangle$ be a cyclic group and let $\varphi : \mathbb{Z} \to G$ be the function defined as $\varphi(k) = a^k$ for every $k \in \mathbb{Z}$.

- Show that φ is an homomorphism and compute its kernel (your answer should depends on the order of a).
- Use the first homomorphism theorem to deduce that $G \cong \mathbb{Z}$ or $G \cong \mathbb{Z}_n$.

(5) Show that A_4 contains subgroups of order 1,2,3,4,12 but NO subgroup of order 6. This shows that the "converse" of Lagrange Theorem can fail (i.e. if $d \mid |G|$ there might not be a subgroup of G of order d).

(6) Give an example of a cyclic subgroup of S_4 which is not normal. This shows that abelian subgroups of a group G might fail to be normal.

(7) Let $\mathcal{Q} = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ be the subset of $GL(2, \mathbb{C})$ where

$$1 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Show that

- $\mathbf{i}^4 = \mathbf{j}^4 = \mathbf{k}^4 = 1$, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$, $\mathbf{i}^3 = -\mathbf{i}$, $\mathbf{j}^3 = -\mathbf{j}$, $\mathbf{k}^3 = -\mathbf{k}$.
- $\mathcal{Q}$ is a non-abelian subgroup of $GL(2, \mathbb{C})$.
- The only subgroups of $\mathcal{Q}$ are $\langle \mathbf{i} \rangle$,$\langle \mathbf{j} \rangle$,$\langle \mathbf{k} \rangle$ and $C(\mathcal{Q})$ (i.e. the center of $\mathcal{Q}$). Moreover all subgroups are normal.
- Show that $\mathcal{Q} = \langle \mathbf{i} \rangle \cup \langle \mathbf{j} \rangle \cup \langle \mathbf{k} \rangle$.

The group $\mathcal{Q}$ is called the *quaternion group* and its a non-abelian group of order 8, where every subgroup is normal.