RESEARCH STATEMENT

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1. Overview

My research interests lie at the intersection of Number Theory and Algebraic Geometry, more precisely in the field of *Diophantine Geometry*. This term, which was originally coined by Serge Lang, describes the study of Diophantine equations, i.e. systems of equations with solutions in the integers and rational numbers, using ideas and techniques from Algebraic Geometry. One of the most intriguing and influential open problems in this field is to determine whether there exist infinitely many integral points on algebraic varieties defined over $\mathbb{Q}$, or equivalently infinitely many solutions to the system of equations defining the variety. Since the Siegel Theorem, and later Faltings proof of Mordell Conjecture, this task has been completed for the case of algebraic curves. In particular, it has been shown that whenever an affine curve has genus greater than zero it contains only finitely many integral points. For algebraic surfaces the problem turns out to be much more subtle and challenging and, although recently deep results have been achieved, a complete solution seems to be, at present, beyond hope. Nevertheless, a number of conjectures have been stated and serve as both focal points as well as the direction towards which Diophantine geometry research is moving. Among these conjectures, one of the most important, and the one around which my research is based, is the conjecture due independently to Paul Vojta [Voj] and Serge Lang [Lan] which (in the surface case) reads as follows:

Conjecture 1.1 (Lang-Vojta for number fields)**.** *Let X be a smooth affine surface defined over a number field k. Let $\tilde{X}$ be a smooth projective variety containing X as an open subset. Let $D = \tilde{X} \setminus X$ is the divisor at infinity and $K = K_{\tilde{X}}$ is a canonical divisor of $\tilde{X}$. Suppose that D is a normal crossing divisor. Then if $D + K$ is big, for every ring of S-integers $\mathcal{O}_S \subset k$, the set of S-integral points $X(\mathcal{O}_S)$ is not Zariski-dense.*

Motivated by this conjecture, in my Ph.D. thesis I specialized in studying the distribution of integral points on log-general type varieties defined over function fields of curves, replacing the number field k with a function field $\mathcal{K}(\mathcal{C})$. The focus of my thesis has been on the study of algebraic hyperbolicity for complements of normal crossing divisors in the projective plane, i.e. the existence of a bound for the degree of affine curves in affine surfaces whose compactification is $\mathbb{P}^2$, in terms of their Euler characteristic. Much of my work so far has been focused on affine surfaces of type $\mathbb{P}^2 \setminus D$ where D is a normal crossing divisor of degree 4 in $\mathbb{P}^2$. In my thesis, I have proven the following two results:

- The non-split version of Lang-Vojta conjecture for function fields for complements of a conic and two lines in $\mathbb{P}^2$ (see 3.1).

Date: October 20^{th}, 2013.

- Lang-Vojta conjecture for function fields for complements of a very general plane curve of degree four with normal crossing singularities (see 3.2).

The second result provides first examples of the function field version of Lang-Vojta Conjecture for complements of irreducible quartics in $\mathbb{P}^2$, and, together with previous results, proves the conjecture for $\mathbb{P}^2$ under the hypothesis of very genericity of the divisor at infinity.

2. Introduction

Given an affine algebraic surface X defined over a number field k, conjecture 1.1 predicts degeneracy of S-integral points, where S is a finite set of places containing the Archimedean ones. Using the strong analogy between number fields and function fields, one can restate the Lang-Vojta conjecture and try to attack this problem using geometric methods which are unavailable in the number field case. The conjecture we are interested in is the following:

Conjecture 2.1 (Lang-Vojta for function fields)**.** *Let X be a smooth affine surface defined over $\mathbb{C}$, and let $\tilde{X}$ be a smooth projective variety containing X as an open subset. Let $K = K_{\tilde{X}}$ be its canonical divisor and $D := \tilde{X} \setminus X$ the divisor at infinity. Suppose that D has normal crossing and $D + K$ is big. Then given a smooth curve $\tilde{\mathcal{C}}$ and a finite subset $S \subset \tilde{\mathcal{C}}$, there exists a bound for the degree for the images of non-constant morphisms $\tilde{\mathcal{C}} \setminus S \to X$ in terms of the Euler characteristic of $\tilde{\mathcal{C}} \setminus S$.*

The surface $X = \tilde{X} \setminus D$ is called a surface of *log-general type*, a term which, similarly to general type surfaces, denotes an affine surface such that the divisor $D + K$, formed by the canonical of its completion plus the divisor at infinity, is big. The Euler characteristic of the affine curve $\mathcal{C}$ is defined in the following way: the normalization of $\mathcal{C}$ is of the form $\tilde{\mathcal{C}} \setminus S$, for a (unique) smooth complete curve $\tilde{\mathcal{C}}$ and a finite subset S. We define the *Euler characteristic* of $\mathcal{C}$ to be (minus) the Euler characteristic of $\tilde{\mathcal{C}} \setminus S$, i.e.

$$\chi(\mathcal{C}) := \chi_S(\tilde{\mathcal{C}}) = 2g(\tilde{\mathcal{C}}) - 2 + \sharp S$$

Conjecture 2.1 asks for boundedness of the degree of affine curves on a surface of log-general type in terms of their Euler characteristic in the sense just defined. This property, in analogy with the complex analytic case, is also called *algebraic hyperbolicity*; the term refers to the fact that in a hyperbolic complex manifold which is algebraic, the degree of compact irreducible curves is bounded above by a constant times its Euler characteristic (see for example [Dem]).

One of the most studied cases is when $\tilde{X} = \mathbb{P}^2_{\mathbb{C}}$, where conjecture 2.1 predicts algebraic hyperbolicity for the complement of a plane curve of degree at least 4. In this setting the case of a four-component divisor of degree 4 has been known for a long time, and follows as an application of Stothers-Mason's abc Theorem (see [BM]). Using their new proof of Siegel's Theorem via Schmidt Subspace Theorem, Corvaja and Zannier in [CZ1] proved an extension to surface (even in the number field case) further generalized by Levin in [Lev]. Both these results essentially depend on the high number of components of the divisor at infinity. In [CZ2] Corvaja and Zannier proved the split case of Conjecture

2.1 in the case where the divisor D is a sum of a conic and two lines in general position, providing the first example in which the logarithmic irregularity is equal to the dimension (equivalently the number of components is one more than the dimension). This has been generalized by the same authors in the very recent paper [CZ3] for ramified covers of $\mathbb{G}_m^2$.

3. Thesis Work

3.1. **Fibered threefold and the non-split case.** The first result I have obtained in my thesis [Tur2] is the non-split version of the theorem of [CZ2], i.e. the case of Lang-Vojta's conjecture for the complement of a conic and two lines in $\mathbb{P}^2$, where now the equation of the divisor depends on a parameter. Geometrically, this corresponds to the data of an (affine) threefold X fibered over the curve $\mathcal{C}$, where each fiber is isomorphic to $\mathbb{P}^2 \setminus D$ and D is a divisor consisting of a conic and two lines. The situation is made explicit in the following diagram:

(1)
$$\begin{tikzcd} X \arrow[d, "\pi"] \\ \mathcal{C} \arrow[u, bend left, "\sigma"] \arrow[r, "\lambda"] & \mathbb{P}^1 \end{tikzcd}$$

Here σ is a section of the projection π and λ is a rational function of the cross-ratio of the four singular points of the divisor on the fiber over P. In fact, one can prove that isomorphism classes of degree four and three components divisors in $\mathbb{P}^2$ can be described by exactly one point in $\mathbb{P}^1$: this point is precisely the value of the function

$$\overline{\lambda} := \frac{\beta(P_1, P_2, P_3, P_4)^2 + 1}{\beta(P_1, P_2, P_3, P_4)}$$

where $P_1, \dots, P_4$ are the four singular points of the divisor lying in the conic, and β is the cross-ratio with respect to the conic. This implies that a complete description of the fibered threefold X can be given by a rational function of $\overline{\lambda}$, hence λ uniquely characterizes X (see [Tur1] for details) .

As in [CZ2] the problem can be reduced to solving an equation and bounding the height of its solutions using the Corvaja and Zannier gcd method. In this case, using the description sketched before of the threefold X, the equation that describes this setting reads as follows:

$$y^2 = u_1^2 + \lambda u_1 + u_2 + 1$$

Bounding the degree of sections $\sigma : \mathcal{C} = \tilde{\mathcal{C}} \setminus S \to X$ is then equivalent to bounding the heights of solutions to the previous equation. Thus the problem relies on describing solutions to the equation in so-called S-units u_1, u_2 (elements of $\kappa(\tilde{\mathcal{C}})$ with poles and zeros contained in S) and S-integer y (regular function on $\mathcal{C}$). Once these are fixed, one can follow the strategy used in [CZ2] paying particular attention to the presence of the non-constant rational function λ. In the end, an application of the function field version of the gcd theorem gives the following:

Theorem 3.1. *[[Tur1]] Let $\tilde{\mathcal{C}}, S, X$ as above. Let $\sigma : \mathcal{C} \to X$ be a nonconstant section for the fibration $\pi : X \longrightarrow \mathcal{C} \xrightarrow{\lambda} \mathbb{P}^1$ where each fiber is isomorphic to $\mathbb{P}^2 \setminus D$. Then the degree of the curve $\sigma(\tilde{\mathcal{C}})$, in a suitable projective embedding of the variety X, verifies*

$$deg(\sigma(\tilde{\mathcal{C}})) \leq 2^{13} \cdot \left(58 \cdot \chi_S(\tilde{\mathcal{C}}) + 28 H_{\tilde{\mathcal{C}}}(\lambda)\right) + 8 H_{\tilde{\mathcal{C}}}(\lambda) \tag{2}$$

3.2. **Very general quartics.** The second goal of my Ph.D. thesis has been the proof of Lang-Vojta conjecture for complements of $\mathbb{P}^2$. After [CZ2] (and even more after [CZ3]) the pre-existing methods, deeply affected by the number of components of the divisor at infinity, were no longer able to deal with either the case of two conics or the case of a generic quartic, i.e. cases in which the logarithmic irregularity is strictly smaller than the dimension of the variety. Moreover the complex analytic analogues, which are usually viewed as possible hints to the solutions, either require at least three components like in [NWY] or a high degree for the divisor D as in [Rou].

My strategy has been to consider the three component divisor as a deformation of the general quartic curve. This idea has been applied by Chen in [Che2], but the author obtained the result for divisors of degree at least five, and his methods do not apply in the degree four case. What was lacking was a systematic way to treat intersections for families in the logarithmic category. Recently, starting with seminal work by Li ([Li1] and [Li2]), and more recently with the development of the framework of Logarithmic Geometry, the behaviour of log-stable maps and their counting via Gromov-Witten theory has become one of the central themes of research, with major contributions (in the algebraic category) coming from the works, among others, of Gross and Siebert [GS], Chen [Che1] and Abramovich and Chen [AC].

Using these recent developments of Gromov-Witten invariants in the logarithmic category, one can make the following idea precise: consider a flat family whose special fiber is the complement of a conic and two lines in general position in $\mathbb{P}^2$, and the generic fiber is the complement of a quartic. If a curve, which intersects the quartic, can be moved to the special fiber, keeping track of the tangency multiplicities, the Corvaja and Zannier theorem can be applied on the special fiber, and hence a bound for the degree in terms of the Euler characteristic can be recovered. Following this idea, the main result I obtained in my thesis is the following:

Theorem 3.2 (Geometric Lang-Vojta in $\mathbb{P}^2$, [Tur2])**.** *Let D be a very general reduced curve in $\mathbb{P}^2$ of degree four with simple normal crossing singularities. Then $\mathbb{P}^2 \setminus D$ is algebraically hyperbolic.*

In order to prove theorem 3.2, my strategy has been the following: first I have extended the main theorem of [CZ2] to log-stable maps, allowing one to consider degenerate cases in which the curve has several connected components with some of them mapped onto a component of D. Notice that the presence of the divisor S here naturally gives a pointed structure to the curves involved. Moreover D, being a reduced simple normal crossing divisor, gives $\mathbb{P}^2$ a logarithmic structure $\mathcal{M}_D$, which is Deligne-Faltings, as defined in [AC]. Secondly, I have reformulated the theorem of [CZ2] in terms of the vanishing of certain logarithmic Gromov-Witten invariants on the stack $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$, of log-stable maps to a

Deligne-Faltings pair. This can be done by observing that if a fixed homology class in $\mathbb{P}^2$ violates the main theorem of [CZ2], there are no (stable) maps from abstract curves to this class. This can be made precise using logarithmic Gromov-Witten invariants and hence the theorem for the conic and two lines gives vanishing of certain invariants for maps to the log variety $(\mathbb{P}^2, \mathcal{M}_D)$. Finally, using the properness of the stack $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ coming from [Che1] and [AC] (or more generally from [GS]) one gets Theorem 3.2 (detailed proofs in [Tur2]).

4. Future Work

In my future work I plan to focus on two important strengthenings of Theorem 3.2. The first one deals with the problem of removing the very general hypothesis which is not present in [CZ2] or [BM]. My result gives genericity once the genus and the number of points is fixed; the necessity of considering very general quartics comes from the fact that we need to take countable unions over all genus and number of points. I plan to study in further detail the minimal condition required for the vanishing of logarithmic Gromov-Witten invariants: in this direction a better understanding of the virtual dimension of the stacks $\mathcal{K}_\Gamma$ with respect to their dimension is the first step that should be taken. Once the virtual dimension is known, a detailed comparison with the dimension of the stack could give indication towards the best strategy in order to move from very general quartics to general ones.

The second generalization could be obtained by working out the same strategy used to prove the recent theorem of [CZ3]: in this paper the two authors proved the Lang-Vojta conjecture for function fields for log-general type surfaces that possess a finite map to the two dimensional algebraic torus $\mathbb{G}_m^2$. In this case, one is led to consider families of divisors in varieties other than the projective plane. Using the same strategy as in [Tur2], one can likely prove Lang-Vojta conjecture for very general divisors of the expected degree.

In connection with this last generalization, there is also another problem that I would like to study, namely the extension to the non-split case of the main theorem of [CZ3]. The tools and the proofs given in [Tur1] can be used to deal with more general fibered threefolds, where each fiber possesses a finite map to $\mathbb{G}_m^2$. In this case one recovers an extra divisor from the ramification of these maps, which itself is moving together with the fibration over the curve. The strategy taken in my thesis can be modified to deal with this problem.

Another possible application of the main theorem of my thesis could be its implications in the complex analytic category. We noticed before how results about degeneracy of holomorphic curves in log-general type surfaces possess the same limitation as the results regarding function fields available before logarithmic Gromov-Witten invariants were defined. In these last years similar constructions have been investigated by Parker in [Par2] and [Par1] in the symplectic category and can possibly be applied, in the same way as the algebraic analogues have been in my thesis, to solve the Lang-Vojta conjecture for holomorphic maps to complements of $\mathbb{P}^2$.

I also would like to broaden the range of my research to study questions in other branches of algebraic geometry, number theory and related fields.

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