GIT AND DIOPHANTINE APPROXIMATION

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1. GIT I - Quotients of affine algebraic varieties

Goal of Mumford-Hilbert: Construct moduli spaces by constructing "quotients" of algebraic varieties by algebraic groups.

Notation: $k = \bar{k}$.

Example 1.1. G an algebraic group acting on X, algebraic variety. One can ask if the quotient is an algebraic variety. In general the answer is NO. As an example one can consider $G = \mathbb{C}^*$ acting on $\mathbb{C}^2 = X$ by $t * (x, y) = (tx, ty)$. Then $(0, 0)$ is a fixed point and lies in every orbit. In particular $X/G = \{0\} \cup \mathbb{P}^1(\mathbb{C})$ and $0 \in \overline{\{p\}}$ for all $p \in \mathbb{P}^1(\mathbb{C})$: so the quotient is not an algebraic variety since it is not separated.

1.1. The affine case.

G = k-reductive group acting on the affine algebraic variety $X = \operatorname{Spec} A$, with $A = k[x]$.

Theorem 1.2. *$A^G = \{f \in A : f(g.x) = f(x) \forall g \in G\}$ is a finitely generated k-algebra Moreover, denoted by $Y = \operatorname{Spec} A^G$ and by $\pi : X \to Y$ the morphism associated to $A^G \subset A$ the following hold:*

(1) *π is G-invariant and surjective.*

(2) *$\pi(x) = \pi(x')$ iff $\overline{G.x} \cap \overline{G.x'} \neq \emptyset$.*

(3) *$F \subset X$ is G-stable closed subset then $\pi(F) \subset Y$ is closed.*

(4) *$\pi^* : \mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$ is an isomorphism, where the map is given for every $V \subset Y$*

$$H^0(V, (\pi_* \mathcal{O}_X)^G) = H^0(\pi^{-1}(V), \mathcal{O}_X)^G.$$

(5) *For every G-invariant map $f : X \to Z$ there exists a unique $\tilde{f} : Y \to Z$ such that*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\pi"'] & Z \arrow[d, "\exists ! \tilde{f}"] \\ & Y \end{tikzcd}$$

(6) *$V \subset Y$ iff $\pi^{-1}(V) \subset X$ open.*

(7) *$\overline{G.x}$ contains exactly one closed point.*

Remark 1.3. $\mathrm{GL}_n \supset G$ is *reductive* if there is no normal connected unipotent subgroup different from the identity. If $\mathrm{char}\, k = 0$, this is equivalent to say that for every algebraic representation V there exists a unique G-equivariant projection $R : V \to V^G$. If $k = \mathbb{C}$ this is equivalent to $G(\mathbb{C})$ contains a Zariski dense compact subgroup. Over $\mathbb{C}$, if K is the dense compact subgroup, one gets

$$R(v) = \int_K g * v d\mu(v)$$

where μ is the Haar measure. The examples (that are going to appear in this course) are GL_n (where $K = U_n$) and SL_n (where $K = SU_n$).

Example 1.4. Let G be a reductive group acting algebraically on $\mathbb{C}^n$, i.e. there is an algebraic map $G \to \mathrm{GL}_n$, and let X be an algebraic variety in $\mathbb{C}^n$ which is G-stable. Then $\overline{G.x}$ contains a unique closed orbit and one defines $Y = X/\sim$ where

$$x \sim x' \text{ iff } \overline{G.x} \text{ and } \overline{G.x'} \text{contain the same closed orbit}$$

There is a quotient map $\pi : X \to Y$ and the topology is the quotient topology, i.e. $V \subset Y$ open if and only of $\pi^{-1}(V) \subset X$.

Theorem 1.5. *In these settings, Theorem 1.2 implies that Y is a complex affine algebraic variety with coordinate ring $\mathbb{C}[x]^G$.*

Example 1.6. Actions of $\mathbb{C}^*$ on $\mathbb{C}^2$:

(1) $t * (x, y) = (tx, ty)$. Then $\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}$ which means $Y = \{\mathrm{pt}\}$.

(2) $t * (x, y) = (tx, y)$. The invariants are $\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}[y]$ and so $Y = \mathbb{C}$ and $\pi : \mathbb{C} \to Y$ is the second projection.

(3) $t * (x, y) = (t^{-1}x, ty)$. The invariants are $\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}[xy]$ and so $Y = \mathbb{C}$ but $\pi : \mathbb{C} \to Y$ is given by $(x, y) \mapsto xy$.

Exercise 1.7. Find an action of $\mathbb{C}^*$ to $\mathbb{C}^3$ such that π is NOT open.

One can ask why restrict to reductive groups. The reason is that otherwise some of the results of the previous Theorems may fail. As an example consider SL_2 acting on $\mathbb{P}^1$. If one takes

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

this is the stabilizer of $(1 : 0)$. Then the quotient G/B is isomorphic to $\mathbb{P}^1$ which is not affine! The reason is that B is not reductive because it contains a unipotent normal subgroup namely

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

1.2. **Linear Algebra Revisited.** Take V to be the k vector space of dimesion n, $X = \mathrm{End}_k(V)$ and $G = \mathrm{GL}(V)$ acting on X by conjugation.

Lemma 1.8. *Let $\phi \in X$, then $\phi_{ss} \in \overline{G.\phi}$.*

Proof. In some basis we can write ϕ as an upper triangular matrix (a_{ij}) with $a_{ij} = 0$ if $i > j$. Consider $\lambda : \mathbb{G}_m \to \mathrm{GL}(V)$ defined as

$$\lambda(t) = \begin{pmatrix} t^n & 0 & 0 & 0 \\ 0 & t^{n-1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t \end{pmatrix}$$

then

$$\lambda(t)\phi\lambda(t)^{-1} = diag(a_{11}, \dots, a_{nn}) + (t^{j-1}a_{ij})$$

so it tends to $diag(a_{11}, \dots, a_{nn})$ for $t \to 0$ which belongs to $\overline{G.\phi}$. □

Corollary 1.9. *$G.\phi$ is closed if and only if ϕ is semi-simple (i.e. diagonal)*

So now we want to compute the invariants $\mathbb{C}[\mathrm{End}(V)]^{\mathrm{GL}(V)}$. Since $\phi \in \mathrm{End}(V)$ we have the characterist polynomials

$$P_\phi(T) = T^n - a_1(\phi)T^{n-1} + \cdots + (-1)^n a_n(\phi)$$

where a_i are $\mathrm{GL}(V)$-invariant polynomials.

Definition 1.10. Define

$$p : End(V) \to \mathbb{A}^n$$

sending $\phi \mapsto (a_1(\phi), \dots, a_n(\phi))$.

Theorem 1.11. *$(\mathbb{A}^n, p)$ satisfies proper (5) of Theorem 1.2, i.e. is the categorical quotient of X by G.*

Corollary 1.12. $\mathbb{C}[\mathrm{End}(V)]^{\mathrm{GL}(V)} = \mathbb{C}[a_1, \dots, a_n]$.

Proof. Define $\epsilon : \mathbb{A}^n \to \mathrm{End}(V)$ as

$$\epsilon(b_1, \dots, b_n) = \left(\begin{array}{ccc|c} & 0 & & b_1 \\ \hline 1 & & & b_2 \\ & \ddots & & \vdots \\ & & 1 & b_n \end{array}\right),$$

whose characteristic polynomial is

$$P_{\epsilon(b)}(T) = T^n - b_1T^{n-1} + \cdots + (-1)^n b_n,$$

Then one has $p \circ \epsilon = \mathrm{id}_{\mathbb{A}^n}$, i.e ϵ is a section of $p : \mathrm{End}(V) \to \mathbb{A}^n$.

In order to verify property (5) of Theorem 1.2 we start with

$$f : \mathrm{End}(V) \to Z$$

which is $\mathrm{GL}(V) = G$-invariant. Is enough to take $\tilde{f} := f \circ \epsilon$. We have to show that

$$f(\phi) = \tilde{f}(p(\phi)) \text{ for every } \phi \in X.$$

But $\tilde{f}(p(\phi)) = f \circ \epsilon \circ p(\phi)$. In general $\epsilon(p(\phi)) \neq \phi$, nevertheless they have the same characteristic polynomial. Therefore they have the same eigenvalues and so they have the same semisimple part up to conjugation. In partiucular this implies that

$$\overline{G.p} \cap \overline{G.\epsilon(p(\phi))} \neq \emptyset.$$

By G-invariance $f(\phi) = f(\epsilon(p(\phi)))$. $\square$