Geometric Lang-Vojta's conjecture in $\mathbb{P}^2$

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On Lang and Vojta's conjectures

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Basic Definition

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Fundamental problem

Describe geometric attributes of X that implies hyperbolicity.

Hyperbolicity for RS

In dimension 1, i.e. for Riemann surfaces, the following easy fact holds:

Theorem

If X is a Riemann surface of genus at least 2, every holomorphic map $\mathbb{C} \to X$ is constant. Therefore X is hyperbolic.

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Hyperbolicity for affine curves

There is an analogous results for affine curves, i.e. complements of a finite set of points in a RS.

Theorem

If X is a Riemann surface of genus $g(X)$, and S a finite set of points of X. Then

$$X \setminus S \text{ is hyperbolic} \iff 2g(X) - 2 + \#S > 0$$

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Q: What happens in higher dimensions?

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Conjecture (Kobayashi)

(a) *a very general hypersurface* $D \subset \mathbb{P}^{n+1}$ *with* $n \geq 2$ *of degree* $\deg D \geq 2n+1$ *is hyperbolic;*

(b) $\mathbb{P}^n \setminus D$ *with* $n \geq 2$ *is hyperbolic for a very general hypersurface* $D \subset \mathbb{P}^n$ *of degree* $\deg D \geq 2n+1$.

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Results in many important cases: works by Y.T. Siu, Demailly, El Goul, McQuillan, Diverio, Merker, Rousseau, Pacienza...

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Strategy: Lower the expectation.

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Theorem (Demailly)

Let X be a compact complex hyperbolic algebraic variety. Then there exists an $\epsilon > 0$ such that every compact irreducible curve $\mathcal{C} \subset X$ satisfies:

$$(\star) \qquad -\chi(\tilde{\mathcal{C}}) = 2g(\tilde{\mathcal{C}}) - 2 \geq \epsilon \deg \tilde{\mathcal{C}}$$

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Property $(\star)$ is defined in a pure algebraic way.

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Algebraic Hyperbolicity is directly related to Vojta conjectures.

Algebraic Hyperbolicity for affine varieties

Definition

Given a projective algebraic variety X and a normal crossing divisor D, $X \setminus D$ is said to be *algebraically hyperbolic* if there exists an $\epsilon > 0$ such that every compact irreducible curve $\mathcal{C} \subset X$ satisfies:

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We call $X \setminus D$ weakly algebraically hyperbolic if the previous bound reads

$$\max\{1, 2g(\tilde{\mathcal{C}}) - 2 + \#S\} \geq \epsilon \deg \tilde{\mathcal{C}}$$

The conjecture

Conjecture (Geometric Lang-Vojta)

Let $\tilde{X}$ be a smooth projective algebraic surface with canonical divisor $K_{\tilde{X}}$ and let D be a reduced normal crossing divisor on $\tilde{X}$. Let $X = \tilde{X} \setminus D$ be the complement of the support of D. If X is of log-general type, i.e. if $D + K_{\tilde{X}}$ is big, then X is weakly algebraically hyperbolic.

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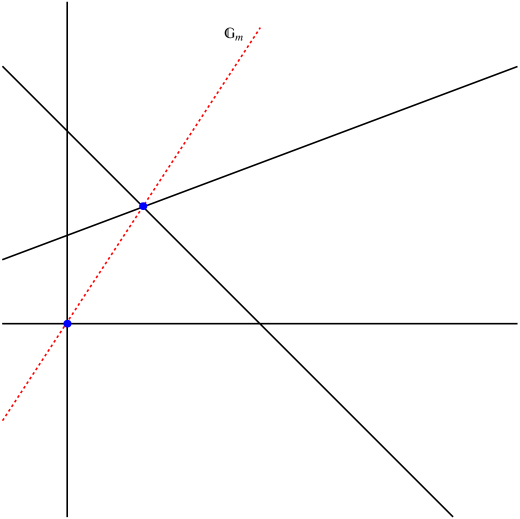
The "weakly" cannot be suppressed: there are examples of log general type surface which contains $\mathbb{G}_m$

Log general type surface, not AH



The case of $\mathbb{P}^2$

When $X = \mathbb{P}^2$ previous conjecture takes the form:

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Let D be a reduced plane curve with normal crossing and let $X = \mathbb{P}^2 \setminus D$. If $\deg D \geq 4$, then X is weakly algebraically hyperbolic.

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As in example before: in the case $\deg D = 4$ the complement X is not (algebraic) hyperbolic!

Known results

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All these results have structural obstruction for extensions to the remaining cases, namely D with less than three irreducible components and $\deg D = 4$.

Goal and ideas

Goal: prove Geometric LV for the complement in $\mathbb{P}^2$ of a (very) generic quartic (even irreducible!).

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A similar idea is used in Xi Chen's proof of LV Conjecture: he degenerate the boundary D to a union of hyperplanes and then applying the known results for $\mathbb{P}^2 \setminus \cup_{i=1,5} H_i$.

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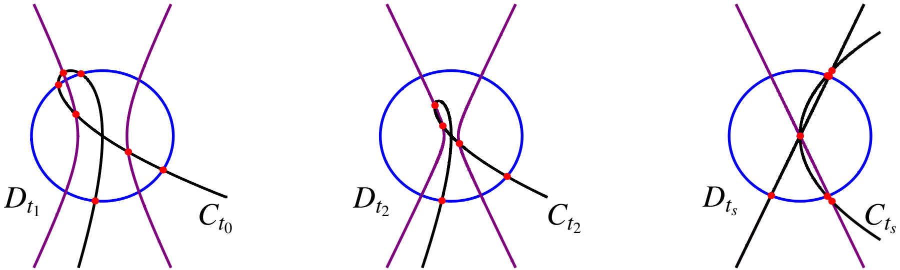
In our case there are problems that need to be addressed.

An example



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  Solution: use logarithmic Geometry (take care of multiplicities of intersection).

Main result

Theorem (T. - WAH for complements of quartics)

$\mathbb{P}^2 \setminus D$ is weakly algebraic hyperbolic for every simple normal crossing divisor D of degree 4 which flattly and log smoothly deforms to a conic and two lines.

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- Deformations need to be logarithmically smooth which is stronger than flat.
- Argument requires to work in logarithmic category.

Sketch of the argument

We start from Corvaja and Zannier's result (CZ in the sequel) for the complement of a conic and two lines that read as follows

Theorem (Corvaja and Zannier, 2008)

Let $X = \mathbb{P}^2 \setminus D$, where D is a quartic consisting of the union of a smooth conic and two lines in general position. Let $\tilde{\mathcal{C}}$ be a smooth complete algebraic curve and $S \subset \tilde{\mathcal{C}}$ a finite set of points. Then for every morphism $f : \tilde{\mathcal{C}} \to \mathbb{P}^2$ such that $f^{-1}(D) \subset S$ the following holds:

$$\deg f(\tilde{\mathcal{C}}) \leq 2^{15} \cdot 35 \cdot \max\{1, \chi_S(\tilde{\mathcal{C}})\}$$

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The main idea is to reformulate this results in terms of vanishing of certain moduli spaces of log stable maps $\mathcal{K}_\Gamma(\mathbb{P}^2, D)$.

Then use the properness of the stack $\mathcal{K}_\Gamma(\mathbb{P}^2, D)$ to extend the result for very general D deforming flattly to a conic and two lines.

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- Log Geometry gives a way to define point of intersection even for irreducible components coinciding with a component of the divisor.
- Each curve $\mathcal{C}$ has a "natural" log structure coming from the set S.

Fundamental example

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- The normal crossing condition can be rephrased in local terms if one considers the étale topology instead of the Zariski topology.

Logarithmic Scheme

Definition

A *logarithmic scheme* is a couple $(X, \mathcal{M})$ where X is a scheme and $\mathcal{M}$ is a sheaf of monoid on the étale site of X, called a log structure together with a morphisms of sheaves of monoids $\alpha : \mathcal{M} \to \mathcal{O}_X$ such that $\alpha^{-1}\mathcal{O}_X^* \to \mathcal{O}_X^*$ is an isomorphism.

A *morphism of log schemes* $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a couple $(f, f^\flat)$ where $f : \underline{X} \to \underline{Y}$ is a morphisms of schemes and $f^\flat : f^*\mathcal{M}_Y \to \mathcal{M}_X$ is a morphism of log structures on X.

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Here $f^*\mathcal{M}_X$ is the logarithmic structure associated to the map $f^{-1}(\mathcal{M}_Y) \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, called the inverse image.

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- Every pointed and at most nodal curve carries a “canonical” log structure

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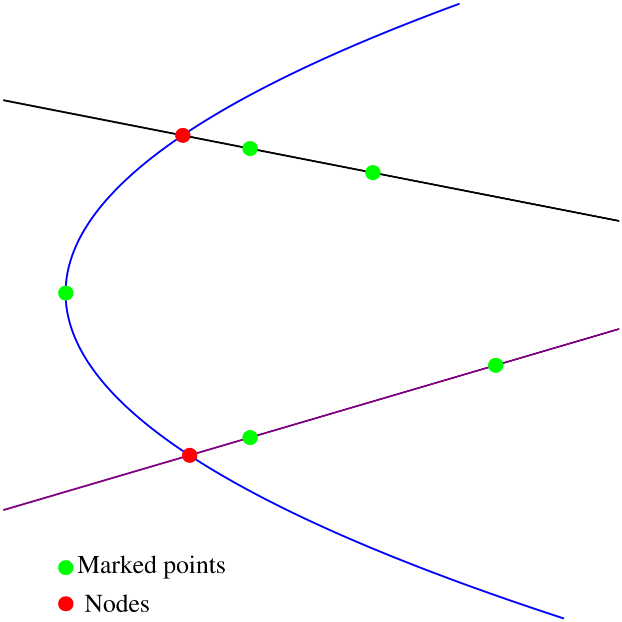
Moreover we would like to have a well defined notion of S in each irreducible component. Therefore nodes should count as two points (this can be made precise using log charts).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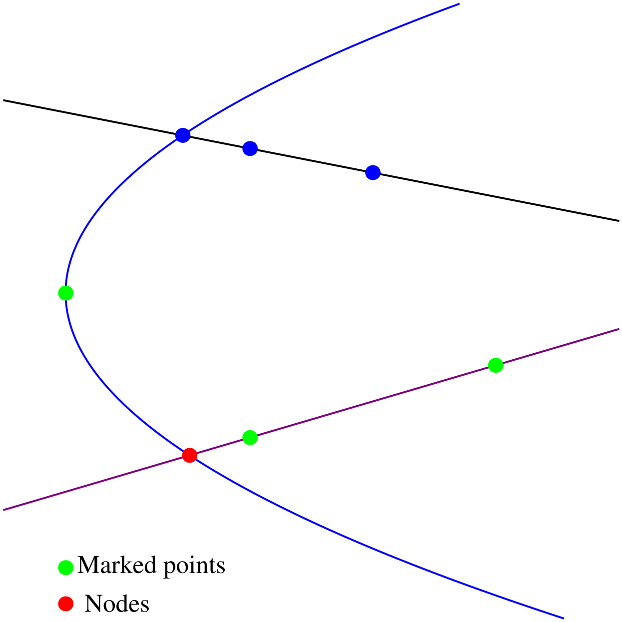
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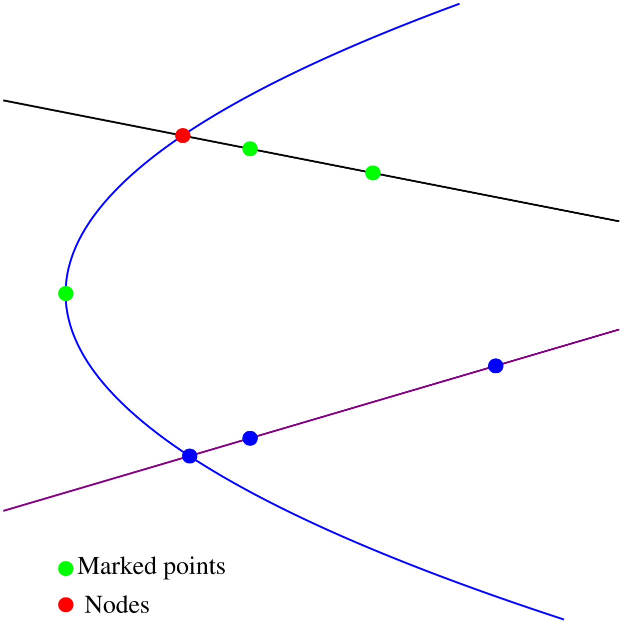
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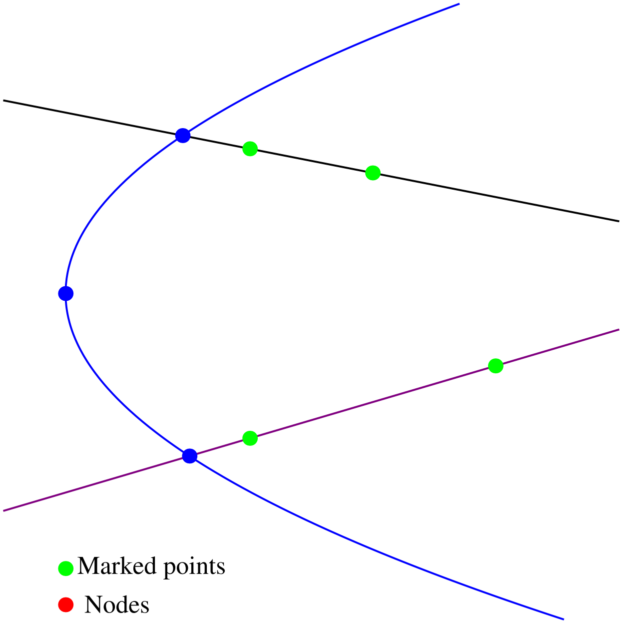
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2. For every irreducible component $\mathcal{C}_j$ of $\mathcal{C}$ such that $\mathcal{C}_j$ maps to the degree two irreducible component of D, $S_{\mathcal{C}_j}$ contains at least **four** points.

Extension to log stable maps

With previous definition the following extension of CZ holds

Proposition

Given $\tilde{\mathcal{C}}, S, D$ as above, let $\varphi : \tilde{\mathcal{C}} \to \mathbb{P}^2$ be a non-constant stable log-morphism such that $\varphi^{-1}(D) \subset S$. Then the degree of the image $\varphi(\tilde{\mathcal{C}})$ verifies:

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Here A is the same constant appearing in CZ Theorem.

Moduli interpretation

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Given a log-stable curve $\mathcal{C}$ as before, the degree of the image of every log-stable map to $(\mathbb{P}^2, \mathcal{M}_D)$ corresponds to the degree of the Chow class of the image curve.

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In particular for a map $f : (\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \to (\mathbb{P}^2, \mathcal{M}_D)$ if $f_*([\mathcal{C}]) = \beta$ for $\beta \in A^1(\mathbb{P}^2)$ then $\deg f(\mathcal{C}) = \deg \beta$.

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Hence once $\beta \in A^1(\mathbb{P}^2)$ $g = g(\beta)$ and $n = \#S$ are fixed a log-stable maps $f : (\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \to (\mathbb{P}^2, \mathcal{M}_D)$ from a genus g, n-marked curve exists only if $\deg \beta$ verifies the inequality of Proposition.

The moduli space of log stable maps

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Definition (Discrete data)

Let $\Gamma = (\beta, g, n, \vec{c})$ be a fourple consisting of the following data:

- $\beta \in H^2(X, \mathbb{Z})$ is a curve class;
- n, g are two non-negative integers (*marked points* and *genus*);
- $\vec{c}$ is a n-vector of non-negative integers (*multiplicities*) that verify:

$$\sum_{i=1}^{n} c_i = c_1(D) \cap \beta$$

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We denote by $\mathcal{K}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$ the moduli space of log-stable map to $(\mathbb{P}^2, \mathcal{M}_D)$ of type Γ.

CZ equivalent to $\mathcal{K}_\Gamma((\mathbb{P}^2, \mathcal{M}_D), \beta) = \emptyset$

Proposition

Given a curve $\mathcal{B} \subset X$ *such that* $S = \mathcal{B} \cap D$ *and* $\deg(\mathcal{B}) > A\chi_s(\mathcal{B})$, *let* β *denote the corresponding element of* $A^1(\mathbb{P}^2)$. *Then the moduli space* $\mathcal{K}_\Gamma((\mathbb{P}^2, \mathcal{M}_D), \beta)$ *is empty for* $g = g(\mathcal{B}), n = \#S$ *and every vector of multiplicities.*

Proposition

Suppose that for every plane curve $\mathcal{B}$ *with* $\deg \mathcal{B} > A\chi_{S_\mathcal{B}}\mathcal{B}$, *where* $S_\mathcal{B} = \mathcal{B} \cap D$, $\mathcal{K}_\Gamma((\mathbb{P}^2, \mathcal{M}_D), \beta)$ *are empty if* $g = g(\mathcal{B})$ *and* $n = \#S_\mathcal{B}$. *Then every log-stable map* f *from a genus* g *curve* $\mathcal{C}$ *to* X *with log-structure* $f^*\mathcal{M}_D$ *(and* $S = S_\mathcal{C} = f^{-1}(D)$*) verifies*

$$\deg f(\mathcal{C}) \leq A\chi_S(\mathcal{C})$$

Using properness of $\mathcal{K}_\Gamma((\mathbb{P}^2, \mathcal{M}_D), \beta)$

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With previous propositions this gives the conclusion.

Final comments

- Theorem uses in an essential way that $\mathcal{M}_D$ comes for a simple normal crossing divisor on D (i.e. $\mathbb{P}^2, \mathcal{M}_D$ is a Deligne-Faltings pair).

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- Bound obtained works only for minimal stable maps.
- Log smoothness hypothesis can possibly be removed (work in progress).
- Possibilities to extend to varieties other than $\mathbb{P}^2$.

Thank you for your attention