HOMEWORK 6 - MATH402B

DUE: WEDNESDAY NOVEMBER 29TH

(1) Goodman 2.7.9

(2) Goodman 2.7.10

(3) Goodman 2.7.12

(4) Let G be a group. Define
$$[G,G] := \langle \{[a,b] : a,b \in G\} \rangle$$
called the *commutator of* G: is the subgroup of G generated by the commutators. Recall that the commutator of two elements $a,b \in G$ is the element $[a,b] = a^{-1}b^{-1}ab$. Compute $[D_4, D_4]$.

(5) Let G be the following set
$$G = \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a,b,d \in \mathbb{R} \text{ and } ad \neq 0\}$$
Then G is a group under matrix multiplication. Verify that G is a non abelian group and that
$$N = \{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in R\}$$
is a normal subgroup of G. Prove that G/N is abelian. Is G/N cyclic?

(6) Discuss whether $\mathbb{Z} \times \mathbb{Z}$ is a cyclic group.

(7) Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$, i.e. $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Recall that $\mathbb{T}$ is a group under multiplication, and there is a surjective homomorphism $f : \mathbb{R} \to \mathbb{T}$ given by $f(t) = e^{2\pi i t}$ such that $\ker f = \mathbb{Z}$. Define $U_n = \{z \in C : z^n = 1\}$ the group of n-th roots of unity.

- Show that U_n is a finite cyclic subgroup of $\mathbb{T}$ of order n for every $n \geq 1$.
- By the correspondence theorem U_n corresponds to a subgroup of $\mathbb{R}$ containing $\mathbb{Z}$; describe the subgroup explicitly when $n = 3$;
- Let $U = \cup_{n \geq 1} U_n$. Prove that U is an infinite subgroup of $\mathbb{T}$ in which every element has finite order;
- What subgroup of $\mathbb{R}$ corresponds to U under the correspondence Theorem?

(8) Let G be a group and let $\Delta = \{(g,g) : g \in G\}$ be the diagonal subgroup of $G \times G$. Prove that Δ is normal in $G \times G$ if and only if G is abelian.