Quiz 3

March 8, 2017

NAME (last,first): ______________________________

Question 1. (10 points) Let A be the following matrix:

$$A = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & -2 & 1 & 1 \end{pmatrix}$$

(a) [4 points] Compute the eigenvalues of A and their multiplicity

Solution: The characteristic polynomial of A is given by

$$\det(A - \lambda I_4) = \det \begin{pmatrix} -\lambda & -1 & 1 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & -1 & -\lambda + 1 & 0 \\ -1 & -2 & 1 & -\lambda + 1 \end{pmatrix} = -\lambda \cdot \det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda + 1 & 0 \\ -1 & 1 & -\lambda + 1 \end{pmatrix}$$

$$= -\lambda(1 - \lambda) \det \begin{pmatrix} -\lambda & 0 \\ -1 & -\lambda + 1 \end{pmatrix} = -\lambda(1 - \lambda)(-\lambda)(1 - \lambda) = \lambda^2 \cdot (1 - \lambda)^2$$

where we expanded the first determinants on the second row. This shows that the eigenvalues are $\lambda = 0$ and $\lambda = 1$ both with multiplicities 2.

(b) [4 points] Compute a basis for the eigenspace relative to the largest eigenvalue.

Solution: In this case the smallest eigenvalue is 1 hence we have to compute $E_1(A) = \text{null}(A - I_4)$. We notice that

$$A - I_4 = \begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & -2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

hence the null space is given by the solutions of the system

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 = 0 \end{cases}$$

whose solutions are given by $x_1 = s_1$, $x_2 = 0$, $x_3 = s_1$ and $x_4 = s_2$. Therefore we have that

$$E_1(A) = \text{null}(A - I_4) = \text{span}\{(1, 0, 1, 0), (0, 0, 0, 1)\}$$

which gives a basis for $E_1(A)$.

(c) [1 point] Assuming that $\dim E_0(A) = 2$ what is the nullity of A? Why?

Solution: Since $E_0(A) = \text{null}(A)$ we have that

$$\text{nullity}(A) = \dim \text{null}(A) = \dim E_0(A) = 2.$$

(d) [1 point] With the same assumption, i.e. $\dim E_0(A) = 2$, does there exists a basis of $\mathbb{R}^4$ made of eigenvectors of A?

Solution: Since the sum of the dimension of the eigenspaces is 4, a basis of $\mathbb{R}^4$ of eigenvectors of A exists.