Math 308 Worksheet (Sections 5.1-5.2)

(1) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ -2 & 0 & 4 \end{bmatrix}^3 \begin{bmatrix} 8 & 0 & 3 \\ -1 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix}^{-1}.$$

(2) (Geometry Question) (Note: This problem is repeated in Chapter 3 with fewer parts.) Suppose we are given the unit square A in the plane with corners $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.

(a) Find a linear transformation T that sends A to the parallelogram B with corners $(0, 0)$, $(1, 2)$, $(2, 2)$ and $(1, 0)$.
(b) Where does T send the point $(1/2, 1/2)$, which was in A?
(c) Is the linear transformation T unique? Why or why not?
(d) What linear transformation T' would send A to itself?
(e) Calculate the area of B. Do you see a relationship between this area and the matrix of the linear transformation T? Similarly is there a relationship between the area of A and T'?
(f) Suppose we want to not only send A to B but also push B in the horizontal direction by one unit. What map can do this?
(g) Let L be the linear span of the side of B with corners $(0, 0)$ and $(1, 2)$. Write L in parametric form: $\mathbf{p} + \mathbf{q}\mathbf{t}$ where t varies in some range and $\mathbf{p}$, $\mathbf{q}$ are vectors. What is the range of t and what are $\mathbf{p}$ and $\mathbf{q}$?
(h) Find the point in A that maps under T to the point $(1/2, 1)$ on L. In your parametric representation of L, what is the representation of $(1/2, 1)$?
(i) How can you map A to a parallelogram C of area 4 while still keeping $(0, 0)$ and $(1, 0)$ as two of its corners?
(j) What is the general formula for the linear transformation that sends A to a parallelogram of area k while still keeping $(0, 0)$ and $(1, 0)$ as two of its corners?

(3) (The math world's worst formula for computing inverses)

Let $A = \begin{bmatrix} -2 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & -1 & 5 \end{bmatrix}$.

(a) Compute all nine cofactors of A, as well as $\det(A)$. Let B be the 3×3 matrix containing the cofactors, with each entry multiplied by the appropriate $\pm$ sign. So the ij-entry of B is $(-1)^{i+j} \det(M_{ij})$.

(b) Compute $A \cdot B^T$. You should get a diagonal matrix with the same number in every diagonal entry. In other words, a multiple of the identity matrix. What multiple is it (in terms of A)?

(c) Fill in the blank (with a scalar) to make this equation true:

$$A \cdot B^T = (\ ?\ ) \cdot I, \quad \text{therefore} \quad A^{-1} = \frac{1}{(\ ?\ )} \cdot B^T.$$

(d) A similar formula works for larger $n \times n$ matrices, involving computing all the cofactors of A. But this formula is *terrible* for computational purposes for finding A^{-1}. Why? Compare it to our other method.
(Note: Occasionally the formula is useful for theoretical purposes.)

(4) (Determinants and geometry)

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\pi/3$, i.e. $T(\vec{\mathbf{x}})$ = the rotation of $\vec{\mathbf{x}}$ by $\pi/3$ around $\vec{\mathbf{0}}$. Without computing any matrices, what would you expect $\det(T)$ to be? (Does T make areas larger or smaller?)
Guess, then check using the fact that the matrix for rotation by θ is

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

(b) Same question as (a), only this time let T be the transformation that reflects $\mathbb{R}^2$ over the line $y = x$. That is, $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} y \\ x \end{bmatrix}$.
Guess what $\det(T)$ should be, then check by finding the matrix for T and computing its determinant.

(c) Rotation matrices in $\mathbb{R}^3$ are more complicated than in $\mathbb{R}^2$ because you have to specify an axis of rotation, which could be any line through the origin. Nonetheless, what would you expect $\det(T)$ to be?
Look up the "basic 3D rotation matrices" on Wikipedia (`https://en.wikipedia.org/wiki/Rotation_matrix#In_three_dimensions`) and compute $\det(A)$ for each one.

(d) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be projection onto the xy-plane, so $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. What is $\det(T)$? Guess, then check using a matrix.

(5) (Determinants and interpolation)
Suppose we want to make a quadratic polynomial

$$y = f(x) = a_0 + a_1 x + a_2 x^2$$

that passes through three specified points $\mathbf{p}_1 = (p_1, q_1)$, $\mathbf{p}_2 = (p_2, q_2)$, $\mathbf{p}_3 = (p_3, q_3)$. Consider the equation

$$0 = \det \begin{bmatrix} 1 & x & x^2 & y \\ 1 & p_1 & p_1^2 & q_1 \\ 1 & p_2 & p_2^2 & q_2 \\ 1 & p_3 & p_3^2 & q_3 \end{bmatrix}.$$

The determinant above implicitly gives an equation $y = f(x)$ (it's easy to solve for y since no y^2, y^3, etc terms appear).

(a) Write out the matrix above, using $\mathbf{p}_1 = (0,0), \mathbf{p}_2 = (1,1), \mathbf{p}_3 = (3,5)$ for the constants p_i, q_i, but leaving x and y as variables.
Solve the equation $\det(A) = 0$ to get $y =$ a quadratic polynomial in x. Check directly that the parabola passes through $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$.

(b) Why does part (a) succeed? Examine the matrix A from part (a). If you plug in $(x, y) = \mathbf{p}_1 = (0, 0)$ to the first row of A, the first two rows will become the same. So, by the 'repeated rows' rule, the equation $\det(A) = 0$ must be true for those specific x, y values. What does this mean about the polynomial $y = f(x)$?

What about if you plug in $(x, y) = (1, 1)$ or $(3, 5)$? Why (in terms of determinants) must the equation $y = f(x)$ be satisfied by these points?

(c) Try to generalize: how could you use a determinant to make a cubic polynomial that passes through 4 given points? (It should require a 5×5 determinant.)