Midterm 2

for Math 308 G, Spring 2017

NAME (last - first): ______________________________

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- This exam contains 5 questions for a total of 45 points in 7 pages.
- You are allowed to have one double sided, handwritten note sheet and a non-programmable calculator.
- Show all your work. With the exception of True/False questions, if there is no work supporting an answer (even if correct) you will not receive full credit for the problem.

Do not write on this table!

Question	Points	Score
1	6	
2	4	
3	14	
4	7	
5	14	
Total:	45	

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ Date: ____________________

Question 1. (6 points) Decide whether the following statements are true or false. For this you don't need to show any work.

(a) [1 point] If A, B are invertible $n \times n$ matrices then $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

$\surd$ **True** ◯ False

(b) [1 point] The function $T(x) = x + e_1$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ is a linear transformation.

◯ True $\surd$ **False**

(c) [1 point] If $\mathcal{B} = \{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$ then $u_1 + u_2 \in \mathcal{B}$.

◯ True $\surd$ **False**

(d) [1 point] If A is a 4×7 matrix row(A) is a subspace of $\mathbb{R}^7$.

$\surd$ **True** ◯ False

(e) [1 point] If $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is onto, then nullity$(A) \neq 0$.

◯ True $\surd$ **False**

(f) [1 point] If $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation of angle θ, then R_θ is not invertible.

◯ True $\surd$ **False**

Question 2. (4 points) For any of the following question, give an explicit example.

(a) [1 points] A matrix A with nullity$(A) = 1$ and rank$(A) = 2$.

Solution:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) [2 point] Two non-zero 2×2 matrices whose product is the zero matrix.

Solution:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

(c) [1 point] A linear transformation $\mathbb{R}^2 \to \mathbb{R}^3$ which is 1-to-1.

Solution:

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

Question 3. (14 points) Consider the following matrix and linear transfomration

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 + x_3 \\ x_1 + x_2 \\ x_2 + 2x_3 \\ x_2 - x_3 \end{pmatrix}$$

(a) [2 points] Indentify the matrix B such that $T = T_B$ and compute domain and codomain of both T_A and T_B

Solution:

$$B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

and $T_A : \mathbb{R}^2 \to \mathbb{R}^4$, $T_B : \mathbb{R}^3 \to \mathbb{R}^4$.

(b) [3 points] For every composition $T_A \circ T_B$ and $T_B \circ T_A$ that make sense compute domain, codomain and the associated matrix.

Solution: The only composition that make sense is $T_A \circ T_B$ since $\operatorname{codom}(T_B) = \operatorname{dom}(T_A)$. We have $T_A \circ T_B : \mathbb{R}^3 \to \mathbb{R}^2$. The associated matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}$$

(c) [3 points] Compute bases for $\operatorname{row}(C)$ and $\operatorname{col}(C)$ where $C = A \cdot B$.

Solution: The REF for C is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Therefore it follows that bases for $\operatorname{row}(C)$ and $\operatorname{col}(C)$ are given by

$$\mathcal{B}_{\operatorname{row}(C)} = \{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\} \qquad \mathcal{B}_{\operatorname{col}(C)} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}\}$$

(d) [2 points] Given the above computation what is $\operatorname{rank}(C)$? And $\operatorname{nullity}(C)$?

Solution: We have $\operatorname{rank}(C) = \dim \operatorname{col}(A) = 2$. Using the rank-nullity theorem we can compute

$$\operatorname{nullity}(C) = \#\text{ cols of } C - \operatorname{rank}(C) = 3 - 2 = 1.$$

(e) [2 points] Use all the information you gathered for C to motivate whether $T_A \circ T_B$ is 1-to-1 and/or onto and/or invertible.

Solution: Since $\text{nullity}(C) > 0$ then $T_A \circ T_B$ is not 1-to-1. Since $\text{rank}(C)$ is equal to $\dim \text{codom}(T_A \circ T_B)$ this implies that the range of $T_A \circ T_B$ is equal to the codomain and therefore the function is onto. Since it is not 1-to-1 it is not invertible.

Question 4. (7 points) Consider the following matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

(a) [2 points] Let λ be any real number. Compute $\det(A - \lambda I_2)$. Recall that I_2 is the identity matrix. (Hint: your expression should involve λ).

> **Solution:** We have
>
> $$\det(A - \lambda I_2) = \det \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 9$$

(b) [2 points] Determine for which values of λ the matrix $A - \lambda I_2$ is not invertible.

> **Solution:** $A - \lambda I_2$ is not invertible if its determinant is zero. Therefore it is not invertible if $(1-\lambda)^2 - 9 = 0$. This simplifies as
>
> $$(1-\lambda)^2 - 9 = (\lambda - 4)(\lambda + 2) = 0 \qquad \Rightarrow \lambda = 4, -2$$

(c) [3 points] For all the values of λ you computed in the previous part, find a basis for $\mathrm{null}(A - \lambda I_2)$.

> **Solution:** For $\lambda = 4$ we have
>
> $$\mathrm{null}(A - 4I_2) = \mathrm{null} \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} = \mathrm{null} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$
>
> This corresponds to the solution of the system $x_1 - x_2 = 0$ which is given by
>
> $$\{\begin{pmatrix} s \\ s \end{pmatrix} : s \in \mathbb{R}\} = \mathrm{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \mathcal{B}_{\mathrm{null}(A-4I_2)} = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$$
>
> In the same way for $\lambda = -2$ we have
>
> $$\mathrm{null}(A + 2I_2) = \mathrm{null} \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \mathrm{null} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$
>
> This corresponds to the solution of the system $x_1 + x_2 = 0$ which is given by
>
> $$\{\begin{pmatrix} s \\ -s \end{pmatrix} : s \in \mathbb{R}\} = \mathrm{span} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \mathcal{B}_{\mathrm{null}(A+2I_2)} = \{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$$

Question 5. (14 points) Consider the following 5 vectors in $\mathbb{R}^3$

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad u_4 = \begin{pmatrix} 4 \\ \sqrt{2} \\ 0 \end{pmatrix} \qquad u_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Let A be the matrix whose columns are $[u_1, u_2, \dots, u_5]$; then the reduce echelon form of A is given by the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & \sqrt{2}-4 & 0 \\ 0 & 0 & 1 & -\sqrt{2}+4 & 1 \end{pmatrix}$$

(a) [2 points] Using the above REF find a basis $\mathcal{B}$ for $S = \operatorname{span}(u_1, u_2, \dots, u_5)$ and show that $\dim S = 3$.

Solution: The REF has leading terms in the first three columns this implies that a basis of S is given by $\mathcal{B} = \{u_1, u_2, u_3\}$. This shows that $\dim S = 3$.

(b) [4 points] Let $v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be a vector of $\mathbb{R}^3$. Express v as a linear combination of the elements of the basis you found above. (Your answer should involve x_1, x_2 and x_3).

Solution: To write v as a linear combination of u_1, u_2 and u_3 we need to find c_1, c_2 and c_3 such that $v = c_1u_1 + c_2u_2 + c_3u_3$. This is equivalent to the linear system

$$\begin{aligned} c_1 &= x_1 \\ c_1 + c_2 &= x_2 \\ c_2 + c_3 &= x_3 \end{aligned}$$

Solving for c_1, c_2, c_3 gives the solution

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 + x_2 \\ x_1 - x_2 + x_3 \end{pmatrix}$$

which shows that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \cdot u_1 + (-x_1 + x_2) \cdot u_2 + (x_1 - x_2 + x_3) \cdot u_3$$

(c) [3 points] Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(u_1) = e_1 \qquad T(u_2) = e_2 \qquad T(u_3) = e_3$$

Compute $T(v)$ for the same v as in the previous part? (Hint: use the expression you just computed and the properties of linear transformations).

Solution: Since $v = x_1 \cdot u_1 + (-x_1 + x_2) \cdot u_2 + (x_1 - x_2 + x_3) \cdot u_3$ we have that

$$\begin{aligned} T(v) &= T(x_1 \cdot u_1 + (-x_1 + x_2) \cdot u_2 + (x_1 - x_2 + x_3) \cdot u_3) \\ &= x_1 \cdot T(u_1) + (-x_1 + x_2) \cdot T(u_2) + (x_1 - x_2 + x_3) \cdot T(u_3) \\ &= x_1 \cdot e_1 + (-x_1 + x_2) \cdot e_2 + (x_1 - x_2 + x_3) \cdot e_3 = \begin{pmatrix} x_1 \\ -x_1 + x_2 \\ x_1 - x_2 + x_3 \end{pmatrix} \end{aligned}$$

(d) [3 points] Using the previous part write down the matrix B such that $T = T_B$. Show that B is invertible by computing its determinant.

Solution: Given the formula for $T(v)$ above we can see that

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Since B is diagonal we have that $\det(B) = 1$ hence B is invertible.

(e) [2 points] Compute B^{-1}.

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{III}-\text{I}]{\text{II}+\text{I}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

This shows that

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$