RESEARCH INTERESTS - AMOS TURCHET

1. Introduction

My research interests lie at the intersection of Number Theory and Algebraic Geometry, more precisely in the field of *Diophantine Geometry*. This term, which was originally coined by Serge Lang, describes the study of Diophantine equations, i.e. systems of polynomial equations to be solved in integral or rational numbers, using ideas and techniques from Algebraic Geometry. One of the most intriguing and influential open problems in this field is to determine whether there exist infinitely many solutions to systems of Diophantine equations or equivalently infinitely many integral or rational points on algebraic varieties defined over number fields. Since the Siegel Theorem (1929), and later Faltings' proof of the Mordell Conjecture 1983 [Fal], this task has been completed for the case of algebraic curves. In particular, it has been shown that whenever an affine curve has genus greater than zero it contains only finitely many integral points. These deep results already show a distinctive feature of Diophantine Geometry: a geometric property of curves, namely the genus, *governs* the arithmetic, namely the finiteness of the set of integral points.

For algebraic surfaces the problem turns out to be much more subtle and challenging and, although recently deep results have been achieved, a complete solution seems to be at present beyond hope. Nevertheless, a number of conjectures have been stated and serve as both focal points as well as the direction towards which Diophantine geometry research is moving. Among these conjectures, one of the most important, and the one around which my research is based, is a conjecture due to Paul Vojta which, in the surface case, can be reformulated using ideas of Serge Lang as follows:

Conjecture 1.1. *[Arithmetic Lang-Vojta]* -[Voj],[Lan1]- *Let κ be a number field, S a finite set of places containing the archimedean ones. Let X be a quasi-projective surface defined over κ and let $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{\kappa,S}$ be a model of X over the S-integers. Then, if X is of logarithmic general type, $\mathcal{X}(\mathcal{O}_{\kappa,S})$ is not Zariski dense.*

Here a quasi projective variety is said to be of log-general type if there exists a desingularization $X_1 \to X$ and a compactification $\tilde{X}$ of X_1 such that the boundary divisor $D = \tilde{X} \setminus X_1$ has normal crossing singularities and $K_{\tilde{X}} + D$ is big. Since the spectrum of the ring of S-integers has dimension 1, the model $\mathcal{X}$ can be thought of as an arithmetic threefold where $\mathcal{X}(\mathcal{O}_{\kappa,S})$ corresponds bijectively to the set of sections $\operatorname{Spec} \mathcal{O}_{\kappa,S} \to \mathcal{X}$. This geometric viewpoint, together with well known analogies between number fields and function fields of curves, gives raise to an analogue statement in the so-called geometric setting.

Conjecture 1.2 (Geometric Lang-Vojta). *Let κ be a function field of a non-singular projective curve $\mathcal{C}$ over an algebraically closed field of characteristic 0 and let S be a finite set of points of $\mathcal{C}$. Let X be a quasi-projective surface defined over κ and let $\mathcal{X} \to \mathcal{C} \setminus S$ be a model of X over $\mathcal{C} \setminus S$. If X is of logarithmic general type then $\mathcal{X}$ is weakly algebraically hyperbolic, i.e. there exists a bound, in a suitable projective embedding, for the degree of non-constant sections $f : \mathcal{C} \setminus S \to \mathcal{X}$ of the form:*

$$\deg f(\mathcal{C}) \leq A \cdot \max\{1, 2g(\mathcal{C}) - 2 + \#S\}$$

for a suitable constant $A > 0$.

From Conjecture 1.2, considering the case $\mathcal{C} = \mathbb{P}^1_{\mathbb{C}}$ and $S = \{\infty\}$, one can see a direct link with questions on (algebraic) hyperbolicity in the sense of Brody and Demailly [Dem], and algebraic degeneracy in the sense of Green-Griffith, which conjecturally fit in a bigger picture of similarities (see [Lan2]).

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2. Results and future work

2.1. **Ph.D. Thesis.** Motivated by Conjecture 1.2 I specialized in studying the distribution of integral points on log-general type varieties defined over function fields of curves. The focus of my Ph.D. thesis [Tur2] has been on the study of algebraic hyperbolicity for complements of normal crossing divisors in the projective plane, i.e. the existence of a bound for the degree of affine curves in affine surfaces whose compactification is $\mathbb{P}^2$, in terms of their Euler characteristic. In my thesis, I have proven the following two results:

- the non-split version of Lang-Vojta conjecture for function fields for complements of a conic and two lines in $\mathbb{P}^2$;
- a reformulation of Lang-Vojta conjecture using the theory of log-stable maps, introducing a program that aims at completing the proof for $\mathbb{P}^2$ under very generic hypotheses.

The first result studies integral points of a fibered threefold and the problem can be reduced to solving a Diophantine equation and bounding the height of its solutions using the Corvaja and Zannier gcd method as in [CZ1]. In this case, using the geometric description of the threefold X, one is led to the study of the following Diophantine solution equation:

$$y^2 = u_1^2 + \lambda u_1 + u_2 + 1.$$

Conjecture 1.2 in this setting is then equivalent to a bound of the heights of the solutions to the previous equation. Thus the problem relies on describing solutions in so-called S-units u_1, u_2 (elements of the function field with poles and zeros contained in the set S) and S-integer y (regular function on the complement of S). In the end, an application of the function field version of the gcd theorem gives the following result.

Theorem 2.1. -[Tur1]- *Let $\mathcal{C}$ be a smooth projective curve defined over an algebraically closed field of characteristic 0 and let S be a finite set of points of $\mathcal{C}$. Let $\mathcal{X}$ be an affine threefold fibered on $\mathcal{C} \setminus S$ where each fiber is isomorphic to $\mathbb{P}^2 \setminus D$ where D is a divisor consisting of a conic and two lines. Let $\sigma : \mathcal{C} \setminus S \to \mathcal{X}$ be a non-constant section for the fibration. Then the degree of the curve $\sigma(\mathcal{C})$, in a suitable projective embedding of the variety $\mathcal{X}$, verifies*

$$deg(\sigma(\mathcal{C})) \leq 2^{13} \cdot \Big(58 \cdot (2g(\mathcal{C}) - 2 + \#S) + 28 H_{\mathcal{C}}(\lambda)\Big) + 8 H_{\mathcal{C}}(\lambda), \tag{2.1}$$

where $\lambda : \mathcal{C} \to \mathbb{P}^1$ is a function describing the geometry of $\mathcal{X}$.

For the second result, differently from the usual techniques applied in the study of Lang-Vojta Conjecture, I developed a method to attack the Conjecture using Logarithmic Geometry in the sense of Kato and Illusie ([Ill], [Kat]): rephrasing the problem using the theory of logarithmic stable maps as defined by Abramovich and Chen ([Che], [AC]), I was able to formulate a deformation argument that reduces the problem for complements of irreducible quartics to the known cases where D has at least three irreducible components.

The argument starts by extending a result for algebraic hyperbolicity of $(\mathbb{P}^2, D)$, where D is a divisor with sufficiently many irreducible components, to arbitrary log-stable maps in the sense of Abramovich and Chen [AC] (or equivalently Gross and Siebert [GS]). These are maps

$$f : (\mathcal{C}, \mathcal{M}_S) \to (\mathbb{P}^2, \mathcal{M}_D)$$

in the log category, where $\mathcal{M}_S$ is the divisorial log structure on $\mathcal{C}$ given by the set S (a sheaf of monoids similar to the sheaf of differentials with logarithmic poles) and $\mathcal{M}_D$ is the divisorial log structure on $\mathbb{P}^2$ given by the divisor D. The curve $\mathcal{C}$ here is a stable curve in the usual sense, in particular it can be reducible with nodes as worst singularities. Moreover, f might map some of the components of $\mathcal{C}$ onto components of D or even contract some of them. I was able to prove that even for such maps there exists a bound for the degree of the image in terms of the Euler characteristic of $\mathcal{C}$ using the log-stability condition. Once this extension is achieved, one observes that (weakly) algebraic hyperbolicity is equivalent to the emptiness of countably many moduli spaces of log-stable maps for specific discrete data. In particular the expression

$$\deg f(\mathcal{C}) \leq A \cdot \max\{1, 2g(\mathcal{C}) - 2 + \#S\}$$

can be seen as a restriction on the discrete data defining the moduli space of log-stable maps to $(\mathbb{P}^2, \mathcal{M}_D)$ since the degree of $f(\mathcal{C})$ corresponds to the degree of the homology class $\beta = f_*[\mathcal{C}] \in H^2(\mathbb{P}^2, \mathbb{Z})$, and $\#S$ is related to the n marking points of $\mathcal{C}$. In particular one can prove that if

$$\deg \beta > A \cdot \max\{1, 2g - 2 + n\}$$

then the moduli space $\mathcal{M}_\Gamma(\mathbb{P}^2, \mathcal{M}_D)$, with $\Gamma = (g, n, \beta, \vec{c})$, is empty for all choices of the multiplicities of intersection $\vec{c}$. The final step is a delicate deformation argument applied to the divisor D and uses the properness of the stacks $\mathcal{M}_\Gamma$ proved by Abramovich and Chen [AC] and Gross and Siebert [GS]; this step requires to carefully study possible deformation of covers and the relationship between the various log structure that enter the picture. While a partial answer was given in my thesis [Tur2], I am currently working on giving a complete treatment in the general setting.

2.2. **Uniformity.** In the seminal paper [Fal], Faltings proved that on a projective curve $\mathcal{C}$ defined over a number field K of genus $g = g(\mathcal{C}) \geq 2$ the set $\mathcal{C}(K)$ is finite. In higher dimensions there is a conjectural analogue formalized as follows.

Conjecture 2.2 (Bombieri-Lang (surfaces), Lang (dim $n > 0$), [Lan2])**.** *Let X be an algebraic variety defined over a number field K. If X is of general type, then the set $X(K)$ is not Zariski dense.*

The fundamental observation of Caporaso, Harris, and Mazur in [CHM] was that Conjecture 2.2 implies that the cardinality of the set $\mathcal{C}(K)$ in Faltings' Theorem is not only finite, but is also uniformly bounded by a constant $N = N(g, K)$ that does *not* depend on the curve $\mathcal{C}$. Similar results were obtained for surfaces of general type by Hassett in [Has].

All these results are based on a geometric property of families of varieties of general type, namely that for every family $X \to B$ of varieties of general type there exists a $n \gg 0$ such that the fibered power X^n_B dominates a variety of general type. This non-trivial fact, known as Fibered Power Theorem (or Correlation Theorem), has been proved to hold in arbitrary dimension by Abramovich in [Abr1].

The corresponding problem for integral points has been treated by Abramovich, who introduced the notion of stably integral points which are points that remain integral after semistable reduction (since the naive translation fails for integral points due to the arbitrariness of the choice of a model). Abramovich proved that, assuming Lang-Vojta Conjecture 1.1 in arbitrary dimension, stably integral points are uniform in elliptic curves [Abr2] and, together with Matzuki, in principally polarized abelian surfaces [AM].

In a current joint project with Kenneth Ascher, we extended part of these results to arbitrary log-general type surfaces using constructions coming from Log-MMP and the theory of stable pairs. In particular we proved a version for pairs of the fibered power theorem that reads as follows.

Theorem 2.3. -[AT2]- *Let $(X, D) \to B$ be a family of stable pairs with integral and log canonical general fiber over a smooth projective variety B such that the complement of D in the general fiber has canonical singularities. Then there exists an integer $n > 0$, a positive dimensional pair (W, Δ) of log general type, and a morphism $(X^n_B, D_n) \to (W, \Delta)$.*

Using such Theorem, we were able to extend uniformity results conditionally on Lang-Vojta Conjecture to arbitrary curves of log general type, and we are currently working on extensions in higher dimensions [AT1].

2.3. **Future work.** There are natural continuations of my work that are currently ongoing projects, e.g. [AT1] and the preparation of a second article based on the second part of my Thesis.

Moreover most of the results achieved are suitable for extensions and further investigation; in particular [Tur1] provides a method that can possibly be applied to more general situations, like a non-isotrivial version of [CZ2]. This would require a careful analysis of Diophantine equations in S-units and S-integers and suitable bounds for the heights of their solutions. In [AT2] and in the upcoming [AT1] there are open questions that are worth investigating: for example explicit bounds for the integer n appearing in Theorem 2.3 could lead to important consequences in the arithmetic implications regarding uniformity.

Finally, in a different direction, the themes of my research have potentially a lot of connections with areas that at a first glance might seem far from Diophantine Geometry. For example, via the link between Conjecture 1.1 and Nevanlinna Theory discovered by Vojta, or with the relation with Hyperbolicity questions, there is a deep and far-reaching connection to Complex Analysis. Moreover, the theory of stable pairs that is used and investigated in [AT2] has being recently proved to correspond to K-stability á la Donaldson, and is therefore linked to questions on the existence of Kähler metrics in complex manifolds. Logarithmic Geometry has also a very interesting link with Physics in the sense of String Theory and more specifically of Mirror Symmetry, where the work of Gross and Siebert (with their program) uses ideas and techniques of log-stable maps and their moduli spaces. Similarly, and possibly even more naturally, the fact that my research has used and has dealt with different aspects of Algebraic Geometry and Number Theory, makes the connection with some active research areas (Moduli Spaces of higher dimensional varieties and pairs, Logarithmic Geometry, Minimal Model Program, Semistable Reduction, Height Theory) very strong.

It seems, therefore, natural to broaden my research to study also questions coming from different branches of Number Theory, Algebraic Geometry and related fields.

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