

## Math 308 Discussion Problems (Sections 6.1-6.2)

- (0) Google the word ‘eigenfaces’ and look at the Wikipedia page, which has some pictures. (They come from artificial intelligence research on computer vision).

Here, a vector  $\vec{v}$  represents an image. Basically  $\vec{v}$  is the list of RGB color values of each pixel in the image, so  $\vec{v} \in \mathbb{R}^N$  for some very large  $N$ . An ‘eigenface’ is an eigenvector for a matrix related to ‘image vectors’.

(This is not a math problem – just a neat application of linear algebra that’s outside the scope of Math 308.)

- (1) (Practice showing that something is a subspace). Suppose  $\lambda$  is an eigenvalue for the matrix  $A$ . Let  $S$  be the  $\lambda$ -eigenspace:

$$S = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\},$$

the set of all vectors  $\vec{v}$  satisfying the equation  $A\vec{v} = \lambda\vec{v}$ . In class, we said (roughly) that  $S$  is a subspace because  $S = \text{null}(A - \lambda I)$ . For this problem, instead show that  $S$  is a subspace by checking the three conditions on  $S$ .

- (2) Let  $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$ .

(a) Compare the eigenvalues and eigenvectors of  $A$ ,  $A^2$  and  $A^{-1}$ . Are they similar or different?

(b) Same questions as (a) but for  $A$ ,  $B$  and  $AB$ .

(c) Find an invertible matrix  $P$  and a diagonal matrix  $D$  so that  $A = PDP^{-1}$ . Then, compute  $A^{1000}$ . (Hint: you can use the method from the bonus problem of Midterm #2.)

(c) Suppose  $\vec{v}$  is an eigenvector of an arbitrary matrix  $M$ , with eigenvalue  $\lambda$ . Show (using matrix algebra) that  $\vec{v}$  is also an eigenvector of  $M + I$ , but with a different eigenvalue. What eigenvalue is it?

- (3) (Reflections and projections)

(a) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the transformation from Worksheet 4, problem 1:

$$T(\vec{x}) = \frac{1}{3} \begin{bmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \vec{x}.$$

Determine the eigenvalues of  $T$ , and find a basis for each eigenspace.

**Note:** You should find that the eigenvalues of  $T$  are 1 and  $-1$ .

(b) Remember that  $T$  is supposed to be ‘reflection across a plane  $S$ ’. Explain what the eigenvalues and eigenvectors from (a) mean geometrically. What is their relationship to  $S$ ? Why does it make sense for the eigenvalues to be 1,  $-1$ ?

(c) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the “averaging transformation”:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3}(x_1 + x_2 + x_3) \\ \frac{1}{3}(x_1 + x_2 + x_3) \\ \frac{1}{3}(x_1 + x_2 + x_3) \end{bmatrix}.$$

Find all eigenvalues and eigenspaces for  $T$ . Explain your answer (what does it mean in terms of ‘averaging’?)

(4) (Rotations)

(a) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be rotation by  $\pi/3$ . Compute the characteristic polynomial of  $T$ , and find any eigenvalues and eigenvectors. (You can look up the matrix for  $T$  from previous worksheets or your notes from class).

(b) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a rotation in  $\mathbb{R}^3$  by  $\pi/3$  around some chosen axis  $L$ , a line through the origin in  $\mathbb{R}^3$ . **Without computing any matrices**, explain why  $\lambda = 1$  is always an eigenvalue of  $T$ . What is the corresponding eigenspace?

(5) Find a  $3 \times 3$  matrix  $A$  with eigenvectors  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  with  $\lambda = 1$ ,  $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$  with

$\lambda = 2$  and  $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  with  $\lambda = 10$ . You may take for granted that  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is a basis for  $\mathbb{R}^3$ .

(**Hint:**  $A$  must be diagonalizable,  $A = PDP^{-1}$ . Figure out  $P$  and  $D$ , then compute  $A$  directly.)