Midterm 1

for Math 308 G, Spring

NAME (last - first): ____

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- This exam contains 4 questions for a total of 45 points in 7 pages.
- You are allowed to have one double sided, handwritten note sheet and a non-programmable calculator.
- Show all your work. With the exception of True/False questions, if there is no work supporting an answer (even if correct) you will not receive full credit for the problem.

Do not write on this table!

Question	Points	Score
1	8	
2	6	
3	12	
4	19	
Total:	45	

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ Date: ____________________

Question 1. (8 points) Decide whether the following statements are true or false or give an explicit example. For this you don't need to show any work.

(a) [1 point] If an homogenous linear system has a non-trivial solution, then it has only one solution.

○ True √ **False**

(b) [1 point] If u and v are linearly independent vectors, then $u - v$ is not the zero vector.

√ **True** ○ False

(c) [1 point] If a linear system has more variables than equations then the dimension of the solution set is (strictly) positive, i.e. > 0.

○ True √ **False**

(d) [1 point] If $u_1, \ldots, u_4$ are four vectors in $\mathbb{R}^3$, then $\text{span}(u_1, \ldots, u_4) = \mathbb{R}^3$.

○ True √ **False**

(e) [2 point] Give en example of a set of four distinct non-zero vectors spanning $\mathbb{R}^2$.

Solution: $\begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}2\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix} \quad \begin{pmatrix}0\\2\end{pmatrix}$

(f) [2 point] Give en example of two linearly independent vectors lying in the plane $x + y - z = 0$ in $\mathbb{R}^3$.

Solution: $\begin{pmatrix}1\\0\\1\end{pmatrix} \quad \begin{pmatrix}0\\1\\1\end{pmatrix}$

Question 2. (6 points) You are assigned to test an algorithm that computes the reduced echelon form of a given matrix. For each given input discuss whether the output is or is not the correct reduced echelon form. [DO NOT compute the REF]

(a) [2 points]

$$\text{INPUT} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 0 \\ 1 & 3 & 0 \end{pmatrix} \qquad \text{OUTPUT} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution: The input matrix contains a column of zeroes. This implies that the columns are not linearly independent and therefore the REF should contain at least one column without leading term. Since every column of the REF contain a leading term the output is not correct.

(b) [2 point]

$$\text{INPUT} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{OUTPUT} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

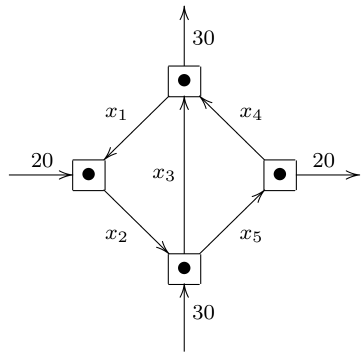
Solution: The columns of the input are linearly independent (the last two are not multiples of each other and the third one is the only one having a non-zero third component). This implies that all the columns of the REF should contain a leading term which is not the case. Therefore the output is not the correct REF.

(c) [2 point]

$$\text{INPUT} \begin{pmatrix} -1 & -1 & -1 & 4 & 6 \\ 0 & 1 & 1 & 3 & 5 \\ 0 & 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \qquad \text{OUTPUT} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution: The input is a triangular matrix which implies that the columns are both linearly independent and they span $\mathbb{R}^5$. This implies that the REF has all columns with leading term and no row of zeros hence it is the correct one.

Question 3. (12 points) Consider the following graph, modelling a traffic intersection.



(a) [3 points] Write down the linear system corresponding to the problem of balancing the graph, and the associated augmented matrix.

Solution:

$$\begin{cases} x_1 - x_2 = -20 \\ x_2 - x_3 - x_5 = -30 \\ x_4 - x_5 = -20 \\ x_1 - x_3 - x_4 = -30 \end{cases} \qquad \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -20 \\ 0 & 1 & -1 & 0 & -1 & -30 \\ 0 & 0 & 0 & 1 & -1 & -20 \\ 1 & 0 & -1 & -1 & 0 & -30 \end{pmatrix}$$

(b) [4 points] Reduce the matrix to REF using the Gauss-Jordan Algorithm. Identify the leading terms of the REF.

Solution:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -20 \\ 0 & 1 & -1 & 0 & -1 & -30 \\ 0 & 0 & 0 & 1 & -1 & -20 \\ 1 & 0 & -1 & -1 & 0 & -30 \end{pmatrix} \xrightarrow{\text{IV}-\text{I}} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -20 \\ 0 & 1 & -1 & 0 & -1 & -30 \\ 0 & 0 & 0 & 1 & -1 & -20 \\ 0 & 1 & -1 & -1 & 0 & -10 \end{pmatrix} \xrightarrow[\text{IV}-\text{II}]{\text{I}+\text{II}}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & -50 \\ 0 & 1 & -1 & 0 & -1 & -30 \\ 0 & 0 & 0 & 1 & -1 & -20 \\ 0 & 0 & 0 & -1 & 1 & 20 \end{pmatrix} \xrightarrow{\text{IV}+\text{III}} \begin{pmatrix} \mathbf{1} & 0 & -1 & 0 & -1 & -50 \\ 0 & \mathbf{1} & -1 & 0 & -1 & -30 \\ 0 & 0 & 0 & \mathbf{1} & -1 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) [1 points] Write down the solution set in vector form.

Solution: Setting the two free variables $x_3 = s_1$ and $x_5 = s_2$ the solution set is given by

$$\{\begin{pmatrix} s_1 + s_2 - 50 \\ s_1 + s_2 - 30 \\ s_1 \\ s_2 - 20 \\ s_2 \end{pmatrix} : s_1, s_2 \in \mathbb{R}\} = \{\begin{pmatrix} -50 \\ -30 \\ 0 \\ -20 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}\}$$

(d) [2 points] What is the dimension of the solution set. Describe it geometrically (i.e. is it a line, a plane or an hyperplane in which $\mathbb{R}^n$).

Solution: The dimension of the solution set is 2. Therefore it is a plane inside $\mathbb{R}^5$.

(e) [2 points] Which values of the free parameters (if any) make sense for the problem.

Solution: Since the problem is about traffic intersection, each x_i needs to be positive. This implies that

$$s_2 \geq 20 \qquad\qquad s_1 \geq 50 - s_2$$

Question 4. (19 points) Consider the following vectors in $\mathbb{R}^4$

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \quad u_4 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix} \quad u_5 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 4 \end{pmatrix}$$

(a) [4 points] Determine whether $\text{span}\{u_1, u_2, u_3, u_4, u_5\} = \mathbb{R}^4$.

Solution: We compute the REF of the matrix whose columns are the vectors in consideration

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & -1 & 1 \\ 1 & 0 & 2 & 2 & 4 \end{pmatrix} \xrightarrow[\text{IV}-\text{I}]{\text{III}-\text{I}} \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 & 4 \end{pmatrix} \xrightarrow[\substack{\text{III}-\text{II} \\ \text{IV}-\text{II}}]{\text{I}+\text{II}}$$
$$\begin{pmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow[-\text{III}]{\text{IV}+2\text{III}} \begin{pmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there is a row of zeroes in the REF the span is not equal to $\mathbb{R}^4$.

(b) [4 points] If the span is not equal to $\mathbb{R}^4$ compute the equation of a line, a plane or an hyperplane containing it.

Solution: To compute an equation for the span we applied the same transformation we did to obtain the REF to a generic vector in $\mathbb{R}^4$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \xrightarrow[\text{IV}-\text{I}]{\text{III}-\text{I}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 - x_1 \\ x_4 - x_1 \end{pmatrix} \xrightarrow[\substack{\text{III}-\text{II} \\ \text{IV}-\text{II}}]{\text{I}+\text{II}} \begin{pmatrix} x_1 + x_2 \\ x_2 \\ x_3 - x_1 - x_2 \\ x_4 - x_1 - x_2 \end{pmatrix} \xrightarrow[-\text{III}]{\text{IV}+2\text{III}} \begin{pmatrix} x_1 + x_2 \\ x_2 \\ x_1 - x_2 - x_3 \\ x_4 + 2x_3 - 3x_1 - 3x_2 \end{pmatrix}$$

This shows that a vector in the span will have to satisfy

$$x_4 + 2x_3 - 3x_1 - 3x_2 = 0$$

This is the equation of an hyperplane in $\mathbb{R}^4$.

(c) [2 points] Are the vectors u_1, u_2, u_3, u_4, u_5 linearly independent? Why?

Solution: They are not linearly independent since the number of vectors 5 is bigger than 4, where the vectors are in $\mathbb{R}^4$.

(d) [4 points] If they are not linearly independent, express at least one of them as a linear combination of the others. (Hint: you might want to use the computation you did above).

Solution: Given the REF computed above the associated homogeneous linear system is equivalent to

$$\begin{cases} x_1 \quad + 2x_3 \quad\quad 2x_5 = 0 \\ \quad x_2 + \ \ x_3 \quad + 2x_5 = 0 \\ \quad\quad\quad\quad x_4 + \ \ x_5 = 0 \end{cases}$$

Setting $x_3 = s_1$ and $x_5 = s_2$ the solutions are given by

$$\{ \begin{pmatrix} -2s_1 + -2s_2 \\ -s_1 - 2s_2 \\ s_1 \\ -s_2 \\ s_2 \end{pmatrix} : s_1, s_2 \in \mathbb{R} \}$$

Choosing $s_1 = 1$ and $s_2 = 0$ gives the solution $(-2, -1, 1, 0, 0)$ which implies

$$-2u_1 - u_2 + u_3 = \underline{0} \qquad \rightarrow \qquad u_3 = 2u_1 + u_2$$

(e) [5 points] If the vectors are not linearly independent can you find the maximal subset of u_1, u_2, u_3, u_4 and u_5 that is linearly independent (i.e. the largest set of linearly independent vectors among the five).

Solution: By the above computation $u_3 \in \text{span}(u_1, u_2)$. Similarly one can see that $u_5 \in \text{span}(u_1, u_2, u_4)$ (equivalently the columns with leading terms correspond to u_1, u_2 and u_4). This shows that $\{u_1, u_2, u_4\}$ is a set of linearly independent vectors (the REF of the matrix whose columns are the three vectors has been already computed and shows that each column has a leading term). On the other hand both u_3 and u_5 lie in the span of these three vectors hence this is the largest set of linearly independent vectors among the u_i's.