Final

for Math 308, Winter 2017

NAME (last - first): ___________________________________

- Do not open this exam until you are told to begin. You will have 110 minutes for the exam.
- This exam contains 7 questions for a total of 90 points in 13 pages.
- You are allowed to have one double sided, handwritten note sheet and a non-programmable calculator.
- Show all your work. With the exception of True/False questions, if there is no work supporting an answer (even if correct) you will not receive full credit for the problem.

Do not write on this table!

Question	Points	Score
1	10	
2	8	
3	18	
4	12	
5	12	
6	20	
7	10	
Total:	90	

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ___________________________________ Date: ________________

Question 1. (10 points) Decide whether the following statements are true or false. For this you don't need to show any work.

(a) [1 point] If v is an eigenvector of A, then $A \cdot v = \underline{0}$.

○ True √ **False**

(b) [1 point] If v and w are non-zero eigenvectors of A with the same eigenvalue λ then $v - w$ is an eigenvector of A.

√ **True** ○ False

(c) [1 point] If λ is an eigenvalue of a $n \times n$ matrix A, $\operatorname{rank}(A - \lambda I_n) = n$.

○ True √ **False**

(d) [1 point] In a subspace of dimension n there are at most n linearly independent vectors.

√ **True** ○ False

(e) [1 point] The solution set of a linear system is a subspace.

○ True √ **False**

(f) [1 point] If A is a $m \times n$ matrix, $\operatorname{col}(A)$ is a subspace of $\mathbb{R}^m$.

√ **True** ○ False

(g) [1 point] If u and v are two non-zero orthogonal vectors, $\dim \operatorname{span}\{u, v\} = 2$.

√ **True** ○ False

(h) [1 point] If A is a square matrix $\det A^T = \det A$.

√ **True** ○ False

(i) [1 point] If $\mathcal{B}$ is a basis of a subspace S and $u \in \mathcal{B}$ then $2u \in \mathcal{B}$.

○ True √ **False**

(j) [1 point] If a $n \times n$ matrix A is diagonalizable then A is invertible.

○ True √ **False**

Question 2. (8 points) For any of the following question, give an explicit example.

(a) [2 point] Two non-zero linear transformation $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1 \circ T_2$ is the linear transformation sending everything to the zero vector.

Solution:

$$T_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \qquad T_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

(b) [1 point] A 4×4 diagonalizable matrix.

Solution:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) [1 point] A non-zero 3×3 matrix with $\det = 0$.

Solution:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(d) [1 point] A 2×2 matrix with e_1 as an eigenvector.

Solution:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(e) [2 point] An orthogonal basis of $\mathbb{R}^3$ containing the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution:

$$\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \}$$

(f) [1 point] A subspace S of $\mathbb{R}^4$ of dimension 2

Solution:

$$S = \operatorname{span}\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \}$$

Question 3. (18 points) Let A be the following matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 \\ 0 & 2 & 0 & -1 \end{pmatrix}$$

(a) [4 points] Compute the characteristic polynomial $p_A(\lambda)$ of A, the eigenvalues of A and their multiplicities.

Solution:

$$p_A(\lambda) = \det(A - \lambda) = \det \begin{pmatrix} -\lambda+2 & 0 & 0 & 0 \\ 0 & -\lambda+1 & 0 & 0 \\ 0 & 2 & -\lambda+1 & -2 \\ 0 & 2 & 0 & -\lambda-1 \end{pmatrix}$$

$$= (2-\lambda)\det \begin{pmatrix} -\lambda+1 & 0 & 0 \\ 2 & -\lambda+1 & -2 \\ 2 & 0 & -\lambda-1 \end{pmatrix}$$

$$= (2-\lambda)(1-\lambda)\det \begin{pmatrix} -\lambda+1 & -2 \\ 0 & -\lambda-1 \end{pmatrix} = (2-\lambda)(1-\lambda)^2(1+\lambda)$$

Therefore the eigenvalues of A are $2, 1$ and -1 with multiplicities $1, 2, 1$ respectively.

(b) [2 points] Given your previous computation, is A invertible? Why or why not?

Solution: Since 0 is not an eigenvalue of A by the Big Theorem A is invertible.

(c) [9 points] Determine a basis for every eigenspace of A.

Solution:

$$E_2(A) = \text{null}(A - 2I_5) = \text{null}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 \\ 0 & 2 & 0 & -3 \end{pmatrix} = \text{null}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore $E_2(A)$ is given by the solutions of

$$\begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{cases} \qquad \Longrightarrow \qquad E_2(A) = \text{span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\}$$

In the same way

$$E_1(A) = \text{null}(A - 1I_5) = \text{null}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 2 & 0 & -2 \end{pmatrix} = \text{null}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which correspond to the solutions of

$$\begin{cases} x_1 = 0 \\ x_2 - x_4 = 0 \end{cases} \qquad \Longrightarrow \qquad E_1(A) = \text{span}\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \}$$

Finally

$$E_{-1}(A) = \text{null}(A + I_5) = \text{null} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 0 & 0 \end{pmatrix} = \text{null} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and therefore it is given by

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 - x_4 = 0 \end{cases} \qquad \Longrightarrow \qquad E_{-1}(A) = \text{span}\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \}$$

(d) [3 points] Is A diagonalizable? Why or why not? If it is diagonalizable, find P and D such that $A = P \cdot D \cdot P^{-1}$.

Solution: Yes A is diagonalizable since

$$\dim E_2(A) + \dim E_1(A) + \dim E_{-1}(A) = 4$$

therefore there exists a basis of eigenvectors and A is diagonalizable. Given the previous computations we can take

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Question 4. (12 points) Let $S = \text{row}(A)$ where A is the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 4 & 1 \\ 2 & 4 & 5 & 2 \\ 0 & 0 & 1 & 0 \\ 3 & 6 & 4 & 3 \end{pmatrix}$$

(a) [4 points] Compute a basis for S and its dimension (hint: compute the *reduced* echelon form).

Solution: To compute a basis we reduce A to reduced echelon form:

$$\begin{pmatrix} 1 & 2 & 4 & 1 \\ 2 & 4 & 5 & 2 \\ 0 & 0 & 1 & 0 \\ 3 & 6 & 4 & 3 \end{pmatrix} \xrightarrow[\text{IV}-3\text{I}]{\text{II}-2\text{I}} \begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -8 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This shows that $S = \text{row}(A)$ has dimension two and a basis is given by

$$\mathcal{B}_S = \{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \}$$

(b) [2 points] Is the basis you found in the previous part an orthogonal basis? Why?

Solution: Yes it is since

$$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0.$$

(c) [4 points] Compute a basis for $S^\perp$.

Solution: We know that $S^\perp = \text{null}(A')$ where

$$A' = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The matrix A' is already in reduced echelon form, so the null space is given by the solution set of

$$\begin{cases} x_1 + 2x_2 + x_4 = 0 \\ x_3 = 0 \end{cases}$$

Therefore

$$S^\perp = \text{null}(A') = \text{span}\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \}$$

(d) [2 points] Given the previous points exhibit a basis of $\mathbb{R}^4$ that extends the basis of S found previously, i.e. such that it contains the vectors in the basis of S. Is this basis orthogonal?

Solution: Since the orthogonal vectors are linearly independent the following is a basis of $\mathbb{R}^4$

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The basis is NOT orthogonal since

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 2 \neq 0$$

Question 5. (12 points) You are asked to compute all the plane conics in $\mathbb{R}^2$ with equation

$$\mathcal{C} : ay^2 + bx^2 + cx + dy + e = 0$$

passing trough the points $(0,0)$, $(0,-1)$, $(1,0)$ and $(1,1)$.

(a) [3 points] Write down the linear system associated to the problem.

Solution: We only need to insert the values of x and y given by the points and we get

$$\begin{cases} e = 0 \\ a - d + e = 0 \\ b + c + e = 0 \\ a + b + c + d + e = 0 \end{cases}$$

(b) [2 points] Before solving the system can you tell if there the system is consistent? If yes, will the solution make sense for the problem?

Solution: The system is homogeneous, hence consistent since there is always the trivial solution. However such solution will not make sense for the problem since it will give $0 = 0$ which is not a plane conic.

(c) [4 points] Solve the system using the Gauss-Jordan algorithm.

Solution:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{III}-\text{I}} \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{III}-\text{II}}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From the computation we see that the solution set is given by

$$\{ \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ -s \\ s \\ 0 \\ 0 \end{pmatrix} : s \in \mathbb{R} \}$$

(d) [3 points] Is the solution set a subspace inside some $\mathbb{R}^n$? If yes for which n? What is its dimension?

Solution: We can rewrite the solution set as

$$\operatorname{span}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

This shows that the solution set is a subspace of $\mathbb{R}^5$ of dimension 1.

Question 6. (20 points) Let A and T_B be the following matrix and linear transformation:

$$A = \begin{pmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & 1 & 3 \end{pmatrix} \qquad T_B \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 + 2x_2 + x_3 \\ x_1 - x_3 \\ -x_1 + x_3 \end{pmatrix}$$

(a) [4 points] Compute the matrix B associated to the linear transformation T_B, i.e. such that $T_B(x) = B \cdot x$ and identify domain and codomain of both T_A, the linear transformation associated to A, and T_B.

> **Solution:** We have
> $$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$
> Moreover $T_A : \mathbb{R}^5 \to \mathbb{R}^4$ and $T_B : \mathbb{R}^3 \to \mathbb{R}^5$.

(b) [6 points] For every composition that make sense, compute the matrices associated to $T_A \circ T_B$ and $T_B \circ T_A$ and identify domain and codomain.

> **Solution:** Since $\operatorname{codom} T_B = \operatorname{dom} T_A$ the composition $T_A \circ T_B : \mathbb{R}^3 \to \mathbb{R}^4$ make sense and the matrix associated to it is $A \cdot B$ that we compute as follows:
> $$A \cdot B = \begin{pmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 4 \\ 4 & 7 & 3 \\ 3 & 2 & -1 \\ -1 & 1 & 2 \end{pmatrix}$$
> On the other hand $\operatorname{codom} T_A \neq \operatorname{dom} T_B$ hence the composition $T_B \circ T_A$ does not make sense.

(c) [2 points] Let $C = A \cdot B$. Can T_C be onto? Explain why or why not.

> **Solution:** Since C is a 4×3 matrix $\operatorname{rank}(C) \leq 3$ which implies that $\dim \operatorname{range}(T_C) \leq 3$ while $\operatorname{codom} T_C = \mathbb{R}^4$. This shows that $\operatorname{range}(T_C)$ can never be equal to $\operatorname{codom} T_C$ so T_C cannot be onto.

(d) [6 points] Compute bases for $\operatorname{row}(C)$ and $\operatorname{col}(C)$.

Solution: We begin by computing the reduced echelon form of C as follows

$$C = \begin{pmatrix} 2 & 6 & 4 \\ 4 & 7 & 3 \\ 3 & 2 & -1 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow{1/2\mathrm{I}} \begin{pmatrix} 1 & 3 & 2 \\ 4 & 7 & 3 \\ 3 & 2 & -1 \\ -1 & 1 & 2 \end{pmatrix}$$

$$\underset{\substack{\mathrm{III}-3\mathrm{I} \\ \mathrm{IV}+\mathrm{I}}}{\xrightarrow{\mathrm{II}-4\mathrm{I}}} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -5 & -5 \\ 0 & -7 & -7 \\ 0 & 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From this we can infer directly that

$$\mathcal{B}_{\mathrm{row}(C)} = \{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \} \qquad \mathcal{B}_{\mathrm{col}(C)} = \{ \begin{pmatrix} 2 \\ 4 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \\ 2 \\ 1 \end{pmatrix} \}$$

(e) [2 points] What is $\mathrm{rank}(C)$? What is $\mathrm{nullity}(C)$?

Solution: Given the above computation $\mathrm{rank}(C) = 2$. This implies that $\mathrm{nullity}(C) = 3 - \mathrm{rank}(C) = 1$.

Extra Credit Question - 10 points

Question 7. (10 points) Suppose a matrix A has the following eigenvectors

$$u_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

with corresponding eigenvalues

$$\lambda_1 = 1 \qquad \lambda_2 = -1 \qquad \lambda_3 = \frac{1}{2}$$

(a) [1 points] Is $\mathcal{B} = \{u_1, u_2, u_3\}$ a basis of $\mathbb{R}^3$? Why?

> **Solution:** Yes $\mathcal{B}$ is a basis, since for example
>
> $$\det \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \neq 0$$

(b) [2 points] Show that $\operatorname{rank}(A - I_3) = \operatorname{rank}(A + I_3)$.

> **Solution:** Since $\mathcal{B}$ is a basis, we see that $\dim E_1(A) = \dim E_{-1}(A) = 1$. This implies that $\operatorname{nullity}(A - I_3) = \operatorname{nullity}(A + I_3) = 1$ hence
>
> $$\operatorname{rank}(A - I_3) = \operatorname{rank}(A + I_3) = 3 - \operatorname{nullity}(A - I_3) = 3 - 1 = 2$$

(c) [2 points] Let $v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Compute $A^{2017} \cdot v$. (Hint: express v in terms of the vectors u_i).

> **Solution:** Since $v = u_1 + u_2$ we have that
>
> $$A^{2017}v = A^{2017}u_1 + A^{2017}u_2 = (1)^{2017}u_1 + (-1)^{2017}u_2 = u_1 - u_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

(d) [5 points] Let $v_k = A^k \cdot v$ for any natural number k. Does the limit $\lim_{k\to\infty} v_k$ exists? If so what is it?

> **Solution:** Using the computation above
>
> $$v_k = A^k \cdot v = (1)^k u_1 + (-1)^k u_2 = u_1 + (-1)^k u_2$$

Therefore

$$\lim_{k\to\infty} v_k = u_1 + (\lim_{k\to\infty}(-1)^k)u_2$$

However the limit $\lim_{k\to\infty}(-1)^k$ does not exists (since it bounces between 1 and -1 indefinitely), therefore the limit $\lim_{k\to\infty} v_k$ does not exists.