Final

for Math 308 G, Spring 2017

NAME (last - first): ____

- Do not open this exam until you are told to begin. You will have 110 minutes for the exam.
- This exam contains 7 questions for a total of 110 points in 13 pages.
- You are allowed to have one double sided, handwritten note sheet and a non-programmable calculator.
- Show all your work. With the exception of True/False questions, if there is no work supporting an answer (even if correct) you will not receive full credit for the problem.

Do not write on this table!

Question	Points	Score
1	10	
2	12	
3	18	
4	20	
5	18	
6	22	
7	10	
Total:	110	

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ____ Date: ____________________

Question 1. (10 points) Decide whether the following statements are true or false. For this you don't need to show any work.

(a) [1 point] If $\mathcal{A}$ is a basis of $\mathbb{R}^n$ for every vector $x \in \mathbb{R}^n$, $[x]_{\mathcal{A}} = x$.

○ True √ **False**

(b) [1 point] If $\mathcal{A}$ and $\mathcal{B}$ are basis of a subspace S then $M_{\mathcal{A},\mathcal{B}}$ is a square matrix.

√ **True** ○ False

(c) [1 point] If $\mathcal{A}$ is a basis of a subspace S and $\mathcal{B}$ is a basis of $S^{\perp}$ then $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements.

○ True √ **False**

(d) [1 point] If λ is an eigenvalue of A then $A - \lambda I$ is invertible.

○ True √ **False**

(e) [1 point] If A is $n \times n$ there is a cofactor C_{ij} equal to zero.

○ True √ **False**

(f) [1 point] If A is upper triangular then $\det A$ is the product of the diagonal elements.

√ **True** ○ False

(g) [1 point] If u and v are two orthogonal vectors then $\mathrm{proj}_v(u) = 0$.

√ **True** ○ False

(h) [1 point] If λ is an eigenvalue of A then $E_\lambda(A)$ has dimension at least 1.

√ **True** ○ False

(i) [1 point] If $\hat{x}$ is a least square approximation of $Ax = b$ then $A\hat{x} = b$.

○ True √ **False**

(j) [1 point] If S is a subspace and $u \notin S$ then $\mathrm{proj}_S(u)$ is the vector of S closest to u.

√ **True** ○ False

Question 2. (12 points) For any of the following question, give an explicit example.

(a) [2 point] An orthogonal basis of $\mathbb{R}^3$ containing the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

Solution:

$$\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\-2\end{pmatrix}\}$$

(b) [2 point] A vector $v \in \operatorname{span}(e_1) \subset \mathbb{R}^2$ such that $\|v - e_2\| = \sqrt{5}$.

Solution:

$$v = \begin{pmatrix}2\\0\end{pmatrix}$$

(c) [2 point] Two bases $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{R}^3$ such that $M_{\mathcal{B},\mathcal{A}} = [e_1 + e_2, e_2 + e_3, e_3 + e_1]$.

Solution:

$$\mathcal{B} = \mathcal{E} \qquad \mathcal{A} = \{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\}$$

(d) [2 point] A vector $v \in \mathbb{R}^3$ and a basis $\mathcal{B}$ of $\mathbb{R}^3$ different from the standard one such that $[v]_{\mathcal{B}} = e_1$

Solution:

$$v = \begin{pmatrix}1\\2\\1\end{pmatrix} \qquad\qquad \mathcal{B} = \{v, e_2, e_3\}$$

(e) [2 point] A 3×3 which is diagonalizable but not diagonal.

Solution:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 2 & -1 \end{pmatrix}$$

(f) [2 point] A 2×2 matrix with e_2 as an eigenvector.

Solution:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Question 3. (18 points) Let A and T_B be the following matrix and linear transformation (where $a \in \mathbb{R}$):

$$A = \begin{pmatrix} 1 & 2 & a & 3 \\ 1 & 1 & 1 & a+1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad T_B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \\ x_2 \\ x_2 \end{pmatrix}$$

(a) [2 points] Compute the matrix B associated to the linear transformation T_B, i.e. such that $T_B(x) = B \cdot x$ and identify domain and codomain of both T_A, the linear transformation associated to A, and T_B.

Solution: We have

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Moreover $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ and $T_B : \mathbb{R}^2 \to \mathbb{R}^4$.

(b) [4 points] For every composition that makes sense, compute the matrices associated to $T_A \circ T_B$ and $T_B \circ T_A$ and identify domain and codomain.

Solution: Since $\operatorname{codom} T_B = \operatorname{dom} T_A$ the composition $T_A \circ T_B : \mathbb{R}^2 \to \mathbb{R}^3$ make sense and the matrix associated to it is $A \cdot B$ that we compute as follows:

$$A \cdot B = \begin{pmatrix} 1 & 2 & a & 3 \\ 1 & 1 & 1 & a+1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & a+3 \\ 2 & a+2 \\ 2 & 2 \end{pmatrix}$$

On the other hand $\operatorname{codom} T_A \neq \operatorname{dom} T_B$ hence the composition $T_B \circ T_A$ does not make sense.

(c) [4 points] Let $C = A \cdot B$. Compute for which values of a, C has rank 2.

Solution: We begin by computing the reduced echelon form of C as follows

$$C = \begin{pmatrix} 3 & a+3 \\ 2 & a+2 \\ 2 & 2 \end{pmatrix} \xrightarrow{\text{III}\leftrightarrow 1/2\text{I}} \begin{pmatrix} 1 & 1 \\ 3 & a+3 \\ 2 & a+2 \end{pmatrix} \xrightarrow[\text{III}-2\text{I}]{\text{II}-3\text{I}} \begin{pmatrix} 1 & 1 \\ 0 & a \\ 0 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & a \\ 0 & 0 \end{pmatrix}$$

From this we can infer directly that when $a \neq 0$ the REF of C has two leading terms, one for each column; on the other hand if $a = 0$ there is a column without leading term, and two row of zeroes. Therefore $\operatorname{rank}(C) = \dim \operatorname{col}(C) = 2$ whenever $a \neq 0$.

(d) [4 points] For the values of a you found above, compute basis for $\operatorname{row}(C)$ and $\operatorname{col}(C)^{\perp}$.

Solution: Given the computation above

$$\mathcal{B}_{\mathrm{row}(C)} = \{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\a\end{pmatrix}\}$$

which is a bases since we are assuming $a \neq 0$. On the other hand, $\mathrm{col}(C)^{\perp} = \mathrm{null}\, C^T$ hence

$$\mathrm{col}(C)^{\perp} = \mathrm{null}\begin{pmatrix} 3 & 2 & 2 \\ a+3 & a+2 & 2 \end{pmatrix} = \mathrm{null}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix} = \mathrm{span}\begin{pmatrix}-2\\2\\1\end{pmatrix}$$

(e) [4 points] Let v_1, v_2 be the columns of the REF of C with $a = 2$ and let $u_1 = e_1 + e_2$ and $u_2 = e_2$. Show that $\mathcal{B} = \{u_1, u_2\}$ is a basis of $\mathrm{span}\{v_1, v_2\}$ and compute the change of basis matrix $M_{\mathcal{B},\mathcal{A}}$ where $\mathcal{A} = \{v_1, v_2\}$.

Solution: It is easy to see that u_1 and u_2 are linearly independent and their dor product with e_3 is 0 (note that $\mathrm{span}\{u_1, u_2\}^{\perp} = \mathrm{span}\{e_3\}$); in particular they are a basis. Moreover we have

$$v_1 = u_1 - u_2 \qquad\qquad v_2 = u_1 + u_2$$

This shows that

$$M_{\mathcal{B},\mathcal{A}} = ([v_1]_{\mathcal{B}}, [v_2]_{\mathcal{B}}) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Question 4. (20 points) At each given year $\frac{2}{10}$ of the population of New York (N) moves to California while $\frac{3}{10}$ of Californians (C) move to New York (assuming the remaining part of the population does not move).

(a) [2 points] Denote by $N(n)$ and $C(n)$ the populations in New York and California respectively at year n. Write a linear system expressing the change of the population in New York and California from year n to year $n+1$. (Hint: express $N(n+1)$ using $N(n)$ and $C(n)$ and to the same with $C(n+1)$).

> **Solution:** Given the description above the linear system reads as follows
>
> $$\begin{cases} N(n+1) = \frac{8}{10}N(n) + \frac{3}{10}C(n) \\ C(n+1) = \frac{2}{10}N(n) + \frac{7}{10}C(n) \end{cases}$$

(b) [1 points] Write the linear system in vector form (i.e. express the vector $(N(n+1)C(n+1))$ as a matrix A multiplied by a vector).

> **Solution:**
>
> $$\begin{pmatrix} N(n+1) \\ C(n+1) \end{pmatrix} = \begin{pmatrix} \frac{8}{10} & \frac{3}{10} \\ \frac{2}{10} & \frac{7}{10} \end{pmatrix} \begin{pmatrix} N(n) \\ C(n) \end{pmatrix}$$

(c) [3 points] Denoted by A the matrix appearing above compute the eigenvalues of A and their multiplicities.

> **Solution:** The characteristic polynomial of A is given by
>
> $$p_A(\lambda) = (\lambda - 1) \cdot (\lambda - \frac{1}{2})$$
>
> Hence the eigenvalues are 1 and $\frac{1}{2}$ both with multiplicity 1.

(d) [1 points] Can you tell if A is diagonalizable without further computations? Explain.

> **Solution:** Since A has 2 eigenvalues of multiplicity 1 the corresponding eigenspaces will have both dimension 1 proving that A is diagonlizable.

(e) [4 points] Compute bases for all eigenspaces of A.

> **Solution:** For $\lambda = 1$
>
> $$E_1(A) = \text{null}(A - I) = \text{null}\begin{pmatrix} -\frac{2}{10} & \frac{3}{10} \\ \frac{2}{10} & -\frac{3}{10} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$
>
> Similarly for $\lambda = \frac{1}{2}$
>
> $$E_{\frac{1}{2}}(A) = \text{null}(A - \frac{1}{2}I) = \text{null}\begin{pmatrix} \frac{3}{10} & \frac{3}{10} \\ \frac{2}{10} & \frac{2}{10} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(f) [2 points] If A is diagonalizable find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$, and a basis $\mathcal{B}$ of $\mathbb{R}^2$ of eigenvectors of A.

Solution: By the computation above we can take

$$P = \begin{pmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{pmatrix} \qquad\qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

and the basis $\mathcal{B} = \{\begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$.

(g) [3 points] Assume that the population of New York and California in the year 2017 is respectively 20 and 40 milions. Express the vector $(N(2017), C(2017))$ as a linear combination of the vectors in the base $\mathcal{B}$.

Solution: We need to find x_1, x_2 such that

$$\begin{pmatrix} N(2017) \\ C(2017) \end{pmatrix} = \begin{pmatrix} 20 \\ 40 \end{pmatrix} = x_1 \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Using Gauss-Jordan we have

$$\left(\begin{array}{cc|c} \frac{3}{2} & -1 & 20 \\ 1 & 1 & 40 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 24 \\ 0 & 1 & 16 \end{array}\right)$$

This implies that

$$\begin{pmatrix} N(2017) \\ C(2017) \end{pmatrix} = 24 \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} + 16 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(h) [4 points] Using all the information you gatheres, compute the expected population in 20 years from now, i.e. compute $N(2037)$ and $C(2037)$. (Use the approximation $16/2^{20} \sim 0.00002$).

Solution: This is equivalent to compute

$$\begin{aligned} \begin{pmatrix} N(2037) \\ C(2037) \end{pmatrix} &= A^{20} \begin{pmatrix} N(2017) \\ C(2017) \end{pmatrix} = 24A^{20} \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} + 16A^{20} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= 24 \cdot 1^{20} \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} + 16 \cdot \frac{1}{2^{20}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 36 \\ 24 \end{pmatrix} + \begin{pmatrix} -0.00002 \\ 0.00002 \end{pmatrix} = \begin{pmatrix} 35.99998 \\ 24.00002 \end{pmatrix} \end{aligned}$$

Question 5. (18 points) Let A be the following matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

(a) [4 points] Compute the characteristic polynomial $p_A(\lambda)$ of A, the eigenvalues of A and their multiplicities.

Solution: Since A is diagonal, $A - \lambda I_3$ is also diagonal therefore

$$p_A(\lambda) = \det(A - \lambda I_3) = (1-\lambda)(2-\lambda)^2.$$

This shows that the eigenvalues of A are 1 and 2 of multiplicity 1 and 2 respectively.

(b) [2 points] Given your previous computation, is A invertible? Why or why not?

Solution: Since 0 is not an eigenvalue of A by the Big Theorem A is invertible.

(c) [6 points] Determine a basis for every eigenspace of A.

Solution:

$$E_1(A) = \operatorname{null}(A - 1I_3) = \operatorname{null}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \operatorname{null}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore $E_1(A)$ is given by the solutions of of $x_1 = x_2 = 0$ hence

$$E_1(A) = \operatorname{span}\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$$

In the same way

$$E_2(A) = \operatorname{null}(A - 2I_3) = \operatorname{null}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} = \operatorname{null}\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which correspond to the solutions of $x_1 + x_2 - x_3 = 0$ hence

$$E_2(A) = \operatorname{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\}$$

(d) [3 points] Is A diagonalizable? Why or why not? If it is diagonalizable, find P and D such that $A = P \cdot D \cdot P^{-1}$.

Solution: Yes A is diagonalizable since

$$\dim E_1(A) + \dim E_2(A) = 3$$

therefore there exists a basis of eigenvectors and A is diagonalizable. Given the previous computations we can take

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(e) [3 points] Let $B = A - I_3$. What is $\mathrm{rank}(B)$? (hint: try to compute the $\mathrm{nullity}(B)$ first and relate $\mathrm{null}(B)$ with eigenvectors of A).

Solution: If $v \in \mathrm{null}(B)$ then $B \cdot v = (A - I)v = 0$ which is equivalent to $A \cdot v = v$. Hence every vector in the null space is an eigenvector of A with eigenvalue 1. In particular $\mathrm{null}(B) = E_1(A)$ which implies $\mathrm{nullity}(B) = \dim E_1(A) = 1$. By the rank-nullity theorem this implies $\mathrm{rank}(B) = 3 - \mathrm{nullity}(B) = 3 - 1 = 2$.

Question 6. (22 points) You are asked to best fit parabola of the form

$$y = ax^2 + bx + c$$

passing trought the points $(1,1)$, $(0,1,)$, $(-1,1)$, $(0,0)$ and $(1,0)$.

(a) [1 points] Show that the linear system $Ax = b$ associated to finding a parabola passing through all the given points has no solution.

> **Solution:** Substituting for x and y we get the system
>
> $$\begin{cases} a+b+c=1 \\ c=1 \\ a-b+c=1 \\ c=0 \\ a+b+c=0 \end{cases}$$
>
> The system is clearly inconsistent since the second equation reads $c = 1$ while the fourth asks $c = 0$.

(b) [5 points] Let A as above and denote by u_1, u_2, u_3 the three columns of A. Compute the vectors

$$v_1 = u_1 \qquad \text{and} \qquad v_2 = u_2 - \mathrm{proj}_{v_1}(u_2) \qquad \text{and} \qquad v_3 = u_3 - \mathrm{proj}_{v_1}(u_3) - \mathrm{proj}_{v_2}(u_3)$$

(note the projections are onto v_1 and v_2!).

> **Solution:** We apply the formula of the projection as follows
>
> $$v_2 = u_2 - \mathrm{proj}_{v_1}(u_2) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \frac{u_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/3 \\ 0 \\ 1/3 \\ 0 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 0 \\ -4/3 \\ 0 \\ 2/3 \end{pmatrix}$$
>
> Similarly
>
> $$v_3 = u_3 - \mathrm{proj}_{v_1}(u_3) - \mathrm{proj}_{v_2}(u_3) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \underline{0} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

(c) [2 points] Show that $\{v_1, v_2, v_3\}$ is an orthogonal basis of $\mathrm{col}(A)$.

Solution: It is easy to see that $v_1 \cdot v_2 = v_2 \cdot v_3 = v_3 \cdot v_1 = 0$ so they form an orthogonal set. Moreover by definition v_1, v_2 and v_3 are linear combinations of u_1, u_2 and u_3 hence they all belong to $\mathrm{col}(A)$. Since they are orthogonal they are linearly independent and therefore they form a basis (since $\mathrm{rank}(A) \leq 3$).

(d) [5 points] Let A and b be as above. Compute $\hat{b} = \mathrm{proj}_{\mathrm{col}(A)}(b)$. (hint: you might want to use the basis you just computed).

Solution: Since $\{v_1, v_2, v_3\}$ is an orthogonal basis we have by definition

$$\mathrm{proj}_{\mathrm{col}(A)}(b) = \mathrm{proj}_{v_1}(b) + \mathrm{proj}_{v_2}(b) + \mathrm{proj}_{v_3}(b) = \frac{2}{3}v_1 + \frac{-1}{4}v_2 + \frac{1}{2}v_3 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 1/2 \\ 1/2 \end{pmatrix}$$

(e) [3 points] Write the normal equations for the problem of finding the least square approximation (do not solve them yet).

Solution: The Normal Equations are $A^T \cdot Ax = A^T \cdot b$ hence we compute

$$A^T \cdot A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 5 \end{pmatrix}$$

and

$$A^T \cdot b = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

Therefore the normal equations read

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 5 \end{pmatrix} \hat{x} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

(f) [5 points] Find the least square approximation $\hat{x}$ using the Gauss-Jordan algorithm (hint: you might consider swapping two lines).

Solution:

$$\begin{pmatrix} 3 & 1 & 3 & 2 \\ 1 & 3 & 1 & 0 \\ 3 & 1 & 5 & 3 \end{pmatrix} \xrightarrow{\text{I}\leftrightarrow\text{II}} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 3 & 1 & 3 & 2 \\ 3 & 1 & 5 & 3 \end{pmatrix} \xrightarrow[\text{III}-3\text{I}]{\text{II}-3\text{I}} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -8 & 0 & 2 \\ 0 & -8 & 2 & 3 \end{pmatrix} \xrightarrow[-1/8\text{III}]{-1/8\text{II}}$$

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & -1/4 \\ 0 & 1 & -1/4 & -3/8 \end{pmatrix} \xrightarrow[\text{III}-\text{II}]{\text{I}-3\text{II}} \begin{pmatrix} 1 & 0 & 1 & 3/4 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & -1/4 & -1/8 \end{pmatrix} \xrightarrow{-4\text{III}}$$

$$\begin{pmatrix} 1 & 0 & 1 & 3/4 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 1/2 \end{pmatrix} \xrightarrow{\text{I}-\text{III}} \begin{pmatrix} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 1/2 \end{pmatrix}$$

From the computation we see that the solution of the normal equations is given by

$$\hat{x} = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{2} \end{pmatrix}$$

(g) [1 points] Show that $A\hat{x} = \hat{b}$.

Solution:

$$A \cdot \hat{x} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 1/2 \\ 1/2 \end{pmatrix} = \hat{b}$$

Extra Credit Question - 10 points

Question 7. (10 points) Let A be a $m \times n$ matrix and B be a $n \times m$ matrix. Assume $n > m$ and $A \cdot B = C$ is an invertible matrix.

(a) [4 points] Show that nullity $B = 0$. (Hint: show that $\mathrm{null}(B) \subset \mathrm{null}(A \cdot B)$, i.e. if a vector is the null space of B then it has to be in the null space of $A \cdot B$.)

> **Solution:** If $v \in \mathrm{null}(B)$ then $Bv = \underline{0}$. This implies that $(A \cdot B)v = \underline{0}$ so $v \in \mathrm{null}(A \cdot B) = \mathrm{null}(C) = \{\underline{0}\}$, since C is invertible. Hence $v = \underline{0}$ which shoes that $\mathrm{nullity}(B) = 0$.

(b) [3 points] Show that rank $A = m$. (Hint: show that $\mathrm{rank}(A) = \mathrm{rank}(A \cdot B)$ by showing that the columns of $A \cdot B$ are linear combination of columns of A).

> **Solution:** If $B = [b_1, \ldots, b_m]$ then $A \cdot B = [A \cdot b_1, \ldots, A \cdot b_m]$ and $A \cdot b_i$ is a linear combination of columns of A. This shows that every column of $A \cdot B$ is a linear combination of columns of A hence $\mathrm{col}(A \cdot B) = \mathrm{col}(A)$ which implies $\mathrm{rank}(A) = \mathrm{rank}(A \cdot B) = \mathrm{rank}(C) = m$ since C is invertible.

(c) [2 points] Use this to show that if $T_A \circ T_B$ is an invertible linear transformation then T_B needs to be 1-to-1 and T_A needs to be onto.

> **Solution:** Since $T_A \circ T_B = T_{A \cdot B}$ is invertible this shows that $A \cdot B = C$ is an invertible matrix. We already showed that this implies that $\mathrm{nullity}(B) = 0$ which implies $\ker T_B = \{\underline{0}\}$ hence T_B is 1-to-1. Similarly we showed that $\mathrm{rank}(A) = m$ which implies $\dim \mathrm{range}(T_A) = m$; since $\mathrm{codom}\, T_A = \mathbb{R}^m$ this shows that T_A is onto.