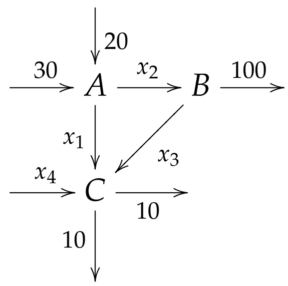
QUIZ 1 - MATH 308 G - OCTOBER 12, 2016

SOLUTION TYPE 1

Maria works for an IT company who is running a series of simulations for improving the traffic flow in Seattle. She is studying a simplify map of an intersection of three one-way roads that looks as follows:



The goal of Maria is to balance the graph, i.e. find the values of x_1, x_2, x_3 and x_4 such that at each node A, B and C the sum of the incoming traffic (arrows pointing *to* the node) equal the sum of the outgoing traffic (arrow pointing *away* from the node).

2 pt. Write down a linear system that represents the problem Maria is trying to solve;
Solution:

- Node A: $20 + 30 = x_1 + x_2$;
- Node B: $x_2 = 100 + x_3$;
- Node C: $x_4 + x_1 + x_3 = 10 + 10$

Therefore the corresponding linear system is

$$\begin{array}{ccccc} x_1 & +x_2 & & & = 50 \\ & x_2 & -x_3 & & =100 \\ x_1 & & +x_3 & +x_4 & = 20 \end{array}$$

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5 pt. Using the Gauss-Jordan algorithm solve the system.
Solution:

The augmented matrix representing the system is

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 100 \\ 1 & 0 & 1 & 1 & 20 \end{array}\right)$$

The Gauss-Jordan algorithm then gives:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 100 \\ 1 & 0 & 1 & 1 & 20 \end{array}\right) \overset{III-I}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & -1 & 1 & 1 & -30 \end{array}\right) \overset{I-II}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -50 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & -1 & 1 & 1 & -30 \end{array}\right)$$

$$\overset{III+II}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -50 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 0 & 1 & 70 \end{array}\right)$$

The last matrix is in Reduced Echelon Form and the corresponding linear system reads as follows:

$$\begin{aligned} x_1 \quad + x_3 \quad &= -50 \\ x_2 - x_3 \quad &= 100 \\ x_4 &= 70 \end{aligned}$$

We can therefore conclude that x_3 is a free variable, and after setting $x_3 = s_1$ we obtain that the solution of the system is

$$\begin{aligned} x_1 &= -50 - s_1 \\ x_2 &= 100 + s_1 \\ x_3 &= s_1 \\ x_4 &= 70 \end{aligned}$$

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1 pt. What is the dimension of the solution set? Why?
Solution:

The solution set is the set

$$\{(-50 - s_1, 100 + s_1, s_1, 70) : s_1 \in \mathbb{R}\}$$

Since there is one free parameter the solution set is a LINE in $\mathbb{R}^4$ and therefore it has dimension 1. ■

1 pt. Write down the solution set for the problem in vector form. *Solution:*

Taking the solution set as written in the previous answer we can rewrite it in vector form as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -50 - s_1 \\ 100 + s_1 \\ s_1 \\ 70 \end{pmatrix} = \begin{pmatrix} -50 \\ 100 \\ 0 \\ 70 \end{pmatrix} + s_1 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

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1pt. If there is a free parameter, for which values the solutions make sense for the problem.

Solution:

Since the map represents traffic flow, each of the variable must attain a positive value. This translates into the conditions

$$\begin{cases} -50 - s_1 \geq 0 \\ 100 + s_1 \geq 0 \\ s_1 \geq 0 \end{cases}$$

Since the first equations gives $s_1 \leq -50$ and the third is $s_1 \geq 0$ we see that there are no values of the free parameter s_1 for which the solution make sense. This will tell Maria that her model of the traffic flow cannot work, and she should consider inverting the direction of one road. ∎