Solutions	**Homework 1**
MATH 402 Autumn 2017	**Oct 05, 2017**

Problem 1 (1.3.1) *Symmetries :* $\{e, r, r^2, a, b, c\}$
Multiplication table:

Table 1: Mutiplication Table of Symmetries of an Equilateral Triangle

	e	r	r^2	a	b	c
e	e	r	r^2	a	b	c
r	r	r^2	e	c	a	b
r^2	r^2	e	r	b	c	a
a	a	b	c	e	r	r^2
b	b	c	a	r^2	e	r
c	c	a	b	r	r^2	e

Problem 2 (1.3.3) *(a) By the multiplication table:*

$$a = a$$

$$ra = d$$

$$r^2a = rd = b$$

$$r^3a = r(r^2a) = rb = c$$

Thus a complete list of the symmetries if

$$\{e, r, r^2, r^3, a, ra, r^2a, r^3a\}$$

(b) By the multiplication table:

$$r(ar) = rc = a \Rightarrow ar = r^{-1}a$$

Note $r^4 = e$, *we know* $r^{-1} = r^3$. *Therefore*

$$ar = r^{-1}a = r^3a.$$

(c) First notice $ar^k = r^{-k}a$ *for* $k = 1$ *by (b). Assume* $ar^k = r^{-k}a$ *holds for any knn. Then for* $k = n + 1$, *we know*

$$ar^{k+1} = (ar^k)r = r^{-k}ar = r^{-k}r^{-1}a = r^{-(k+1)}a,$$

where the second equality is by assumption for $k = n$ *and the third equality is by (b).*
(d) Since any element in the list of symmetries can be represented as r^ia^j *where* $0 \le i \le 3$ *and*

$0 \leq j \leq 1$ *(We use the convention* $a^0 = r^0 = e$ *here). Then for any two elements* $r^i a^j$ *and* $r^m a^n$ *where* $0 \leq i, m \leq 3$ *and* $0 \leq j, n \leq 1$. *If* $j = 0$*:*

$$(r^i a^j)(r^m a^n) = r^{i+m} a^n = r^s a^n$$

where $i + m \equiv s \ mod(4)$ *and* $0 \leq s \leq 3$. *If* $j = 1$*,*

$$(r^i a^j)(r^m a^n) = r^i (a r^m) a^n = r^i r^{-m} a a^n = r^{i-m} a^{n+1} = r^s a^t$$

where $i - m \equiv s \ mod(4)$, $0 \leq s \leq 3$ *and* $n + 1 \equiv t \ mod(1)$, $0 \leq t \leq 1$.

Problem 3 (1.4.2) *Let* R_1 *and* R_2 *be the matrix corresponding to rotation by* $2\pi/3$ *and* $4\pi/3$ *respectively. Therefore*

$$R_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}, R_1 \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

The second equality can be rewritten as

$$R_1(-\frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\sqrt{3}}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

Since R_1 *is linear transformation, we obtain*

$$R_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

Note all the transforms are done within the plane $z = 0$. R_1 *should make the third coordinate unchanged.*

$$R_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence we know

$$R_1 = R_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$R_2 = R_1^2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For A which represents the flip along the axis through $\begin{bmatrix}1\\0\\0\end{bmatrix}$, similarly

$$A\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}, A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\-1\\0\end{bmatrix}, A\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\-1\end{bmatrix}$$

and we have

$$A=A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}.$$

$$B=R_1^2A=R_2A=\begin{bmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\0&0&-1\end{bmatrix}$$

$$C=R_1A=\begin{bmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\0&0&-1\end{bmatrix}$$

You can check the rest of the relations holds in the multiplication table in 1.3.1.

Problem 4 (1.4.8) *Notice each symmetry of equilateral triangle can be viewed as a permution of the three vertices. Hence the total number of symmetries cannot exceed the total number of permutations of vertices, which equals to* $3! = 6$*. On the other hand, we already find six diffierent symmetries of equilateral triangle. Therefore the number of symmetries of an equilateral triangle is exactly 6.*