Final

for Math 308, Winter 2017

NAME (last - first): ____________________________________

- Do not open this exam until you are told to begin. You will have 110 minutes for the exam.
- This exam contains 7 questions for a total of 94 points in 15 pages.
- You are allowed to have one double sided, handwritten note sheet and a non-programmable calculator.
- Show all your work. With the exception of True/False questions, if there is no work supporting an answer (even if correct) you will not receive full credit for the problem.

Do not write on this table!

Question	Points	Score
1	10	
2	8	
3	18	
4	16	
5	16	
6	16	
7	10	
Total:	94	

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ________________________________ Date: ______________

Question 1. (10 points) Decide whether the following statements are true or false. For this you don't need to show any work.

(a) [1 point] If u and v are two non-zero orthogonal vectors in a subspace S then $\dim S \geq 2$.

$\surd$ **True** ◯ False

(b) [1 point] If A is a square matrix $\det A^T = \det A$.

$\surd$ **True** ◯ False

(c) [1 point] The solution set of a linear system is a subspace.

◯ True $\surd$ **False**

(d) [1 point] If A is a $m \times n$ matrix, $\mathrm{col}(A)$ is a subspace of $\mathbb{R}^m$.

$\surd$ **True** ◯ False

(e) [1 point] If $\mathcal{B}$ is a basis of a subspace S and $u \in \mathcal{B}$ then $4u \in \mathcal{B}$.

◯ True $\surd$ **False**

(f) [1 point] If a $n \times n$ matrix A is diagonalizable then A is invertible.

◯ True $\surd$ **False**

(g) [1 point] If λ is an eigenvalue of a $n \times n$ matrix A, $\mathrm{rank}(A - \lambda I_n) = n$.

◯ True $\surd$ **False**

(h) [1 point] In a subspace of dimension n there are at most n linearly independent vectors.

$\surd$ **True** ◯ False

(i) [1 point] If v is an eigenvector of A, then $A \cdot v$ and v are orthogonal.

◯ True $\surd$ **False**

(j) [1 point] If v and w are non-zero eigenvectors of A with the same eigenvalue λ then $v - w$ is an eigenvector of A.

$\surd$ **True** ◯ False

Question 2. (8 points) For any of the following question, give an explicit example.

(a) [2 point] An orthogonal basis of $\mathbb{R}^3$ containing the vector $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

Solution:

$$\left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} \right\}$$

(b) [1 point] A subspace S of $\mathbb{R}^5$ of dimension 2

Solution:

$$S = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(c) [2 point] Two non-zero linear transformation $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1 \circ T_2(x) = \underline{0}$ for every $x \in \mathbb{R}^2$.

Solution:

$$T_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \qquad T_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

(d) [1 point] A 3×3 diagonalizable matrix.

Solution:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(e) [1 point] A non-zero 3×3 matrix with $\det = 0$.

Solution:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(f) [1 point] A 2×2 matrix with e_2 as an eigenvector.

Solution:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Question 3. (18 points) Let A be the following matrix

$$A = \begin{pmatrix} 3 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

(a) [3 points] Compute the cofactors C_{12} and C_{22} of $(A - \lambda I_3)$.

Solution: By definition

$$C_{12} = (-1)^{1+2} \det M_{12} = -\det \begin{pmatrix} 2 & 2 \\ 2 & 3-\lambda \end{pmatrix} = -(6 - 2\lambda - 4) = 2\lambda - 2 = 2(\lambda - 1)$$

and

$$C_{22} = (-1)^{2+2} \det M_{22} = \det \begin{pmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix} = (3-\lambda)^2 - 4 = \lambda - 6\lambda + 5 = (\lambda-1)(\lambda-5)$$

(b) [4 points] Compute the characteristic polynomial $p_A(\lambda)$ of A, the eigenvalues of A and their multiplicities.

Solution: Using the previous computation we expand the determinant on the second column. The only remaining cofactor that we need is C_{32}

$$C_{32} = (-1)^{3+2} \det M_{32} = -\det \begin{pmatrix} (3-\lambda) & 2 \\ 2 & 2 \end{pmatrix} = -(2(3-\lambda)-4) = -(6-2\lambda-4) = 2(\lambda-1)$$

Using this we obtain

$$\begin{aligned} p_A(\lambda) = \det(A - \lambda) = \det \begin{pmatrix} -\lambda+3 & -2 & 2 \\ 2 & -\lambda-1 & 2 \\ 2 & -2 & -\lambda+3 \end{pmatrix} &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= -4(\lambda-1) + (-1-\lambda)(\lambda-1)(\lambda-5) - 4(\lambda-1) = -(\lambda-1)((1+\lambda)(\lambda-5)+8) \\ &= -(\lambda-1)(\lambda-1)(\lambda-3) = -(\lambda-1)^2(\lambda-3) \end{aligned}$$

Therefore the eigenvalues of A are 3 and 1 with multiplicities 1 and 2 respectively.

(c) [2 points] Given your previous computation, is A invertible? Why or why not?

Solution: Since 0 is not an eigenvalue of A by the Big Theorem A is invertible.

(d) [6 points] Determine a basis for every eigenspace of A.

Solution:

$$E_3(A) = \mathrm{null}(A - 3I_3) = \mathrm{null} \begin{pmatrix} 0 & -2 & 2 \\ 2 & -4 & 2 \\ 2 & -2 & 0 \end{pmatrix} = \mathrm{null} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore $E_3(A)$ is given by the solutions of

$$\begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases} \qquad \Longrightarrow \qquad E_3(A) = \text{span}\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \}$$

In the same way

$$E_1(A) = \text{null}(A - 1I_3) = \text{null} \begin{pmatrix} 2 & -2 & 2 \\ 2 & -2 & 2 \\ 2 & -2 & 2 \end{pmatrix} = \text{null} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which correspond to the solutions of

$$\begin{cases} x_1 = x_2 - x_3 \end{cases} \qquad \Longrightarrow \qquad E_1(A) = \text{span}\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \}$$

(e) [3 points] Is A diagonalizable? Why or why not? If it is diagonalizable, find P and D such that $A = P \cdot D \cdot P^{-1}$.

Solution: Yes A is diagonalizable since

$$\dim E_3(A) + \dim E_1(A) = 3$$

therefore there exists a basis of eigenvectors and A is diagonalizable. Given the previous computations we can take

$$P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Question 4. (16 points) Let A and T_B be the following matrix and linear transformation (where $a \in \mathbb{R}$):

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 3\,a+1 & 0 & -a & -2\,a \end{pmatrix} \qquad T_B \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 + 3x_2 + x_3 \\ x_1 - x_3 \end{pmatrix}$$

(a) [2 points] Compute the matrix B associated to the linear transformation T_B, i.e. such that $T_B(x) = B \cdot x$ and identify domain and codomain of both T_A, the linear transformation associated to A, and T_B.

Solution: We have

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

Moreover $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ and $T_B : \mathbb{R}^3 \to \mathbb{R}^4$.

(b) [4 points] For every composition that makes sense, compute the matrices associated to $T_A \circ T_B$ and $T_B \circ T_A$ and identify domain and codomain.

Solution: Since $\operatorname{codom} T_B = \operatorname{dom} T_A$ the composition $T_A \circ T_B : \mathbb{R}^3 \to \mathbb{R}^4$ make sense and the matrix associated to it is $A \cdot B$ that we compute as follows:

$$A \cdot B = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 3\,a+1 & 0 & -a & -2\,a \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & -2 & -1 \\ 1 & 1 & a \end{pmatrix}$$

On the other hand $\operatorname{codom} T_A = \operatorname{dom} T_B$ hence the composition $T_B \circ T_A : \mathbb{R}^4 \to \mathbb{R}^4$ is associated to the matrix $B \cdot A$

$$B \cdot A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 3\,a+1 & 0 & -a & -2\,a \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\,a & -1 & -a & -2\,a \\ 3\,a-1 & -1 & -a & -2\,a \\ -3\,a & 2 & a & 2\,a \end{pmatrix}$$

(c) [2 points] Let $C = B \cdot A$ with $a = 0$. Can T_C be onto? Explain why or why not.

Solution: Substituting $a = 0$ in the above expression we find that

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

Therefore $\operatorname{rank}(C) = \dim \operatorname{row}(C) = 2$ since there are three linearly dependent rows (the first, the second and the fourth row). This implies that $\dim \operatorname{range} T_C = \dim \operatorname{col} C = \dim \operatorname{row} C = 2$ is strictly smaller than $\dim \operatorname{codom} T_C = 4$, hence T_C is cannot be onto.

(d) [4 points] Let $D = A \cdot B$. Compute for which values of a, D has rank 3.

Solution: We begin by computing the reduced echelon form of D as follows

$$D = \begin{pmatrix} 1 & 3 & 2 \\ -1 & -2 & -1 \\ 1 & 1 & a \end{pmatrix} \xrightarrow[\text{III}-\text{I}]{\text{II}+\text{I}} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -2 & a-2 \end{pmatrix} \xrightarrow[\text{III}+2\text{II}]{\text{I}-3\text{II}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & a \end{pmatrix}$$

From this we can infer directly that when $a \neq 0$ the REF of D has three leading terms, one for each column; on the other hand if $a = 0$ there is a column without leading term, and a row of zeroes. Therefore $\operatorname{rank}(D) = \dim \operatorname{col}(D) = 3$ whenever $a \neq 0$.

(e) [4 points] For the values of a you found above, compute basis for $\operatorname{row}(D)$ and $\operatorname{col}(D)$.

Solution: Given the computation above

$$\mathcal{B}_{\operatorname{row}(C)} = \{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \} \qquad \mathcal{B}_{\operatorname{col}(C)} = \{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ a \end{pmatrix} \}$$

which are bases since we are assuming $a \neq 0$.

Question 5. (16 points) You are asked to compute all the quartics (i.e. functions of one variable given by a polynomial of degree 4) of the type

$$f(x) : ax^4 + bx^3 + cx^2 + dx + e$$

such that $f(x) = 1$ whenever $x = 0, -1, 1$.

(a) [2 points] Write down the linear system associated to the problem.

Solution: Inserting the values of x given by the problem we get

$$\begin{cases} e = 1 \\ a - b + c - d + e = 1 \\ a + b + c + d + e = 1 \end{cases}$$

(b) [2 points] Before solving the system can you tell if there the system is consistent? If yes, will the solution make sense for the problem? [Hint: can you guess a solution?]

Solution: The system is not homogeneous however one can see that taking $a = b = c = d = 0$ and $e = 1$ yields a solution therefore the system is consistent. However this particular solution will not make sense since it corresponds to $f(x) = 1$ which is not a polynomial of degree 4.

(c) [4 points] Solve the system using the Gauss-Jordan algorithm.

Solution:

$$\left(\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{\text{II}-\text{I}} \left(\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{1/2\text{II}}$$
$$\left(\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

From the computation we see that the solution set is given by

$$\left\{ \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} -s_1 \\ 0 \\ s_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -s_2 \\ 0 \\ s_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} : s_1, s_2 \in \mathbb{R} \right\}$$

(d) [3 points] The solution set is a subset inside some $\mathbb{R}^n$; for which n? What is its dimension? Is it a subspace? Why or why not?

Solution: The solution set is a subset of $\mathbb{R}^5$ of dimension 2, since it has two free parameters. However it is not a subspace since it does not contain the zero vector.

(e) [4 points] Assume that we want to find a quartic in which all the values $f(0), f(1)$ and $f(-1)$ are equal to 0 instead. Write down the corresponding linear system. Without computing the solutions, express the solution set as the null space of a matrix.

Solution: The corresponding linear system will read

$$\begin{cases} e = 0 \\ a - b + c - d + e = 0 \\ a + b + c + d + e = 0 \end{cases}$$

Considering

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

one sees that the solution set can be written as $\text{null}(A)$.

(f) [1 point] Will the solution set of the new system be a subspace inside some $\mathbb{R}^n$? Why or why not?

Solution: The solution set will be a subspace inside $\mathbb{R}^5$ since the solution set of an homogeneous linear system is always a subspace. Moreover it is equivalent to the null space of the matrix, and we know that the null space of any matrix is a subspace.

Question 6. (16 points) Let $S = \text{row}(A)$ where A is the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ -1 & 2 & 2 & -3 & 1 \\ 2 & 1 & 1 & 5 & 0 \end{pmatrix}$$

(a) [4 points] Compute a basis for S and its dimension.

Solution: To compute a basis we reduce A to reduced echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ -1 & 2 & 2 & -3 & 1 \\ 2 & 1 & 1 & 5 & 0 \end{pmatrix} \xrightarrow[\text{III}-2\text{I}]{\text{II}+\text{I}} \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 0 & 3 & 3 & -1 & 2 \\ 0 & -1 & -1 & 1 & -2 \end{pmatrix} \xrightarrow{-\text{III}\leftrightarrow\text{II}}$$
$$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -1 & 2 \end{pmatrix} \xrightarrow[\text{III}-3\text{II}]{\text{I}-\text{II}} \begin{pmatrix} 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

This shows that $S = \text{row}(A)$ has dimension three and a basis is given by

$$\mathcal{B}_S = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{pmatrix} \}$$

(b) [2 points] Is the basis you found in the previous part an orthogonal basis? Why?

Solution: No it is not since the dot product between the first two vectors is $-5 \neq 0$.

(c) [4 points] Compute a basis for $S^\perp$.

Solution: We know that $S^\perp = \text{null}(A)$ and we can use the previous computation since $S^\perp = \text{null}(A')$ where

$$A' = \begin{pmatrix} 1 & 0 & 0 & 3 & -1 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Using the reduced echelon form, the null space is given by the solution set of

$$\begin{cases} x_1 + 5x_5 = 0 \\ x_2 + x_3 = 0 \\ x_4 - 2x_5 = 0 \end{cases}$$

Therefore using the free variables x_5 and x_3 we have

$$S^{\perp} = \text{null}(A') = \text{span}\{\begin{pmatrix}-5\\0\\0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\\0\\0\end{pmatrix}\}$$

(d) [4 points] Use the previous computation to show that the following set is an orthogonal basis for S

$$\{\begin{pmatrix}1\\0\\0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-2\end{pmatrix}\}$$

Solution: The three vectors are clearly linearly independent. So we only need to show that they are in S. To do this we compute the dot product with the basis of $S^{\perp}$ that we found in the previous point. And we see that

$$\begin{pmatrix}1\\0\\0\\2\\1\end{pmatrix} \cdot \begin{pmatrix}-5\\0\\0\\2\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}-5\\0\\0\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\\-2\end{pmatrix} \cdot \begin{pmatrix}-5\\0\\0\\2\\1\end{pmatrix} = 0$$

and

$$\begin{pmatrix}1\\0\\0\\2\\1\end{pmatrix} \cdot \begin{pmatrix}0\\-1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}0\\-1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\\-2\end{pmatrix} \cdot \begin{pmatrix}0\\-1\\1\\0\\0\end{pmatrix} = 0$$

One can see immediately the three vectors are pairwise orthogonal.

(e) [2 points] Given the previous points write an orthogonal basis of $\mathbb{R}^5$ that contains the vectors of a basis of S.

Solution: Since the orthogonal vectors are linearly independent the following is a basis of $\mathbb{R}^5$

$$\{\begin{pmatrix}1\\0\\0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-2\end{pmatrix}, \begin{pmatrix}-5\\0\\0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\\0\\0\end{pmatrix}\}$$

The basis is orthogonal since both basis are orthogonal and they come from bases of S and $S^{\perp}$ respectively.

Extra Credit Question - 10 points

Question 7. (10 points) (a) [2 points] Find a 2×2 matrix A such that $A^5 = I_2$ and A is not the identity matrix. [Think about the relationship between T_A and T_{A^5}]

> **Solution:** It is enough to take as A the matrix associated to the rotation by $2\pi/5$. Since matrix multiplication corresponds to composition A^5 will be the matrix associated to the composition of 5 successive rotation of angle $2\pi/5$ which gives the rotation by an angle of 2π which is the identity in the plane. Therefore
>
> $$A = \begin{pmatrix} \cos\frac{2\pi}{2} & -\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{pmatrix}$$

(b) [8 points] Consider the following 3×3 matrix

$$B = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where A is the matrix you found in the previous exercise. Let $T_B : \mathbb{R}^3 \to \mathbb{R}^3$ be the corresponding linear transformation. For each of the following subspaces S, discuss and explain whether $T_B(S) = S$ (note that this only means that for every element $s \in S$, $T_B(s) \in S$, but not necessarily $T_B(s) = s$) or not.

1. S is the x-axis;

2. S is the z-axis;

3. S is the xy-plane;

4. S is the yz-plane;

Solution: The linear transformation T_B rotates by an angle $2\pi/5$ every vector in the xy-plane since for every v in the xy-plane, the z-coordinate is 0 and

$$B \cdot \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} A \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

This shows that if $S = xy$-plane then $T_B(S) = S$ but this does not hold for S equal the x-axis.

When S is the z-axis we have that

$$S = \operatorname{span} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad T_B \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

hence $T_B(S) = S$. For the yz-plane by the previous computation we know that the y axis get rotated by $2\pi/5$ and therefore when S is the yz plane $T_B(S)$ is not equal to S.