Final - Math 308G

14th December 2016

NAME (last - first): ____

- Do not open this exam until you are told to begin. You will have 1 hour and 50 minutes for the exam.
- This final contains 7 questions for a total of 100 points in 13 pages.
- You are allowed to have one double sided, handwritten note sheet and a non-programmable calculator.
- Show all your work. With the exception of True/False questions, if there is no work supporting an answer (even if correct) you will not receive full credit for the problem.

Question	Points	Score
1	15	
2	15	
3	15	
4	10	
5	10	
6	20	
7	15	
Total:	100	

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ____ Date: __________________

Question 1. (15 points) Decide whether the following statements are true or false. For this question you don't need to show any work.

(a) [1 point] If the columns of a matrix A span the codomain of the associated linear transformation T_A, then T_A is 1-to-1.

○ True √ **False**

(b) [1 point] If a matrix A is invertible then A^T is invertible.

√ **True** ○ False

(c) [1 point] If $\mathcal{B}$ is a basis for a subspace S then $\vec{0} \in \mathcal{B}$.

○ True √ **False**

(d) [1 point] If A and B are equivalent $m \times n$ matrices then $A \cdot e_1 = B \cdot e_1$.

○ True √ **False**

(e) [1 point] Let S be a subspace of $\mathbb{R}^n$ of dimension m and let $\mathcal{A}$ and $\mathcal{B}$ be two basis of S. Then the matrix of change of basis $M_{\mathcal{A},\mathcal{B}}$ is a square $m \times m$ matrix.

√ **True** ○ False

(f) [1 point] If B is the reduced echelon form of a square matrix A then $\det A = \det B$.

○ True √ **False**

(g) [1 point] If v is an eigenvector of a matrix A, then $A \cdot v \in \operatorname{span}\{v\}$.

√ **True** ○ False

(h) [1 point] If v is a linear combination of $u_1, \ldots, u_m$ and $A = [u_1, \ldots, u_m]$, then $v \in \operatorname{range}(T_A)$.

√ **True** ○ False

(i) [1 point] If λ is an eigenvalue of A with multiplicity m, then $\dim E_\lambda(A) \geq m$.

○ True √ **False**

(j) [1 point] If T_A is 1-to-1 then $\operatorname{nullity}(A) = 0$.

√ **True** ○ False

(k) [1 point] If $\det(A) = \det(B)$ then A and B are equivalent matrices.

○ True √ **False**

(l) [1 point] If S is a subspace of $\mathbb{R}^n$ then the only vector contained both in S and $S^\perp$ is the zero vector.

√ **True** ○ False

(m) [1 point] If λ_i is an eigenvalue of a matrix A, then $p_\lambda(1 - \lambda_i) = 0$.

○ True √ **False**

(n) [1 point] If λ is an eigenvalue of A, then $\operatorname{nullity}(A - \lambda I) = 0$.

○ True √ **False**

(o) [1 point] If $S = \operatorname{span}\{v\}$ and $u \in S^\perp$ then $v \in \operatorname{span}\{u\}^\perp$.

√ **True** ○ False

Question 2. (15 points) Prove the following propositions.

(a) [5 points] Let A be a $n \times n$ matrix. Prove that if A is invertible then A^{-1} is invertible and
$$\det A^{-1} = \frac{1}{\det A}.$$

Solution: A^{-1} is invertible since $A^{-1} \cdot A = I$. Using the properties of the determinant
$$1 = \det(I) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1})$$
from which it follows that $\det(A^{-1})$ is the inverse of $\det(A)$.

(b) [5 points] Prove that if λ is an eigenvalue of A then $1/\lambda$ is an eigenvalue of A^{-1}.[Hint: start by writing down what does it mean for λ to be an eigenvalue and manipulate the expression.]

Solution: Since λ is an eigenvalue of A we know that there exists a vector v such that $Av = \lambda v$. Using this expression and multiplying by A^{-1} on the left of both sides we get
$$A^{-1} \cdot Av = A^{-1} \cdot \lambda v \quad \longrightarrow \quad v = \lambda A^{-1} v$$
Dividing by λ on both sides gives
$$\frac{1}{\lambda} v = A^{-1} v$$
which tells us that $1/\lambda$ is en eigenvalue of A^{-1}.

(c) [5 points] Using the previous point prove that if A is diagonalizable and invertible then A^{-1} is also diagonalizable. [Hint rephrase the previous part using eigenvectors instead of eigenvalues]

Solution: By the previous part we know that if v is an eigenvector of A with eigenvalue λ then v is also an eigenvector of A^{-1} with eigenvalue $1/\lambda$. In particular if there exists a basis of $\mathbb{R}^n$ of eigenvectors of A then the same basis is also a basis of eigenvectors of A^{-1}, which prove that in particular A^{-1} is diagonalizable.

Question 3. (15 points) Let A be the following matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & 0 & -2 \end{pmatrix}$$

(a) [2 points] Without doing any computation, what is $\det(A)$? Can you find an eigenvalue of A?

Solution: Since A has a row of zeroes, it is not invertible. In particular $\det(A) = 0$ and 0 is an eigenvalue of A.

(b) [4 points] Find all the eigenvalues of A and their multiplicities.

Solution: We note that A is triangular, which implies that also $A - \lambda I_4$ is triangular, therefore the determinant is given by the products of the elements in the diagonal, namely

$$p_A(\lambda) = \det(A - \lambda I_4) = -\lambda(1-\lambda)^2(-2-\lambda)$$

From this one can read that the eigenvalues of A are $0, 1$ and -2 with multiplicities $1, 2$ and 1 respectively.

(c) [5 points] Compute bases for all the eigenspaces of A.

Solution:

$$E_0(A) = \mathrm{null}(A) = \mathrm{null}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & 0 & -2 \end{pmatrix} = \mathrm{null}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{span}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$E_1(A) = \mathrm{null}(A - I_4) = \mathrm{null}\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & -3 \end{pmatrix} = \mathrm{null}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \mathrm{span}\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\}$$

$$E_{-2}(A) = \mathrm{null}(A + 2I_4) = \mathrm{null}\begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ -3 & 3 & 0 & 0 \end{pmatrix} = \mathrm{null}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{span}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(d) [3 points] Is A diagonalizable? If yes write the invertible matrix P and the diagonal matrix D such that $A = P \cdot D \cdot P^{-1}$ and exhibit a basis $\mathcal{B}$ of $\mathbb{R}^4$ made of eigenvectors of A.

Solution: By the previous part we can find a basis of $\mathbb{R}^4$ made of eigenvectors of A, namely

$$\mathcal{B} = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \}.$$

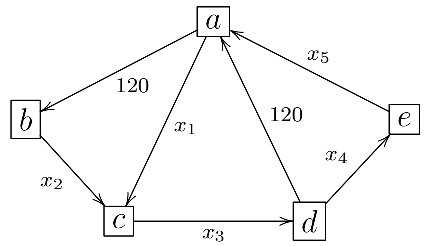
It follows that A is diagonalizable and

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

(e) [1 point] Let $\mathcal{B}$ be the basis of $\mathbb{R}^4$ of the previous part. Is $\mathcal{B}$ an orthogonal basis of $\mathbb{R}^4$?

Solution: No it is not, since the dot product between the first and the second vector is equal to 1. Recall that in an orthogonal basis every dot product between different vectors should be 0.

Question 4. (10 points) Consider the following graph, modelling water pipes and their junction in a home water network.



Anywhere that pipes meet, the total amount of water coming into that junction must be equal to the amount going out, otherwise we would quickly run out of water, or we would have a buildup of water. The junction (c) has just been added to the network and you are been assigned to compute the total amount of water (in gallons) that should run in each pipe to balance the existing junctions, i.e. in every junction the total amount of water entering the junction should equal the total amount of water exiting the junction.

(a) [3 points] Write down the linear system associated to the problem of finding x_1, x_2, x_3, x_4 and x_5, and its associated augmented matrix.

Solution: The linear system associated to the problem is

$$\begin{cases} (a) & x_5 + 120 = x_1 + 120 \\ (b) & 120 = x_2 \\ (c) & x_1 + x_2 = x_3 \\ (d) & x_3 = x_4 + 120 \\ (e) & x_4 = x_5 \end{cases}$$

After rearranging each equation one gets the following matrix:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 120 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

(b) [4 points] Reduce the matrix to echelon form and find the solution set of the system.

Solution: The reduced echelon form of the matrix reads as

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 120 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 120 \\ 0 & 0 & 1 & 0 & -1 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This implies that the solution set of the system is given by

$$\{x_1 = x_5, x_2 = 120, x_3 = 120 + x_5, x_4 = x_5\} = \{s \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 120 \\ 120 \\ 0 \\ 0 \end{pmatrix} : s \in \mathbb{R}\}$$

(c) [1 points] How many solutions does the linear system have? What is the dimension of the solution set?

Solution: The system has infinitely many solutions and the dimension of the solution set is 1, since there is exactly one free parameter.

(d) [2 points] If your system has free parameters, what value of the parameters make sense for the problem? What are the values of the parameters that make x_1 have the minimum possible value?

Solution: The unique free parameter needs to be non-negative, since every variable represents the quantity of water in gallons, and hence make sense only when it is non-negative. Since $x_1 = s$ the minimum possible value for x_1 is 0 and it is reached when $s = 0$.

Question 5. (10 points) Let T_A and B be respectively the following linear transformation and matrix

$$T_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 - 2x_4 \\ -x_1 - 3x_2 + x_3 + 2x_4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

(a) [2 points] Write the matrix A such that $T_A(x) = A \cdot x$, and identify domain and codomain of both T_A and T_B, the linear transformation associated to B.

Solution: The matrix A is the following matrix

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ -1 & -3 & 1 & 2 \end{pmatrix}$$

and we have, $\mathrm{dom}(T_A) = \mathbb{R}^4$, $\mathrm{codom}(T_A) = \mathbb{R}^2$, $\mathrm{dom}(T_B) = \mathbb{R}^2$ and $\mathrm{codom}(T_B) = \mathbb{R}^4$.

(b) [4 points] Decide which of the compositions $T_A \circ T_B$ and $T_B \circ T_A$ make sense, and for those that make sense, determine domain codomain and compute the associated matrices.

Solution: Since $\mathrm{dom}(T_A) = \mathrm{codom}(T_B)$ and $\mathrm{dom}(T_B) = \mathrm{codom}(T_A)$ both composition make sense and we have

$$T_A \circ T_B : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \mapsto \qquad A \cdot B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$T_B \circ T_A : \mathbb{R}^4 \to \mathbb{R}^4 \qquad \mapsto \qquad B \cdot A = \begin{pmatrix} 3 & 7 & -1 & -6 \\ -1 & -3 & 1 & 2 \\ 1 & 2 & 0 & -2 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

(c) [2 points] One can check (but you don't have to!) that

$$\mathrm{null}(B \cdot A) = \mathrm{span}\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \}$$

Given this information, is $T_B \circ T_A$ 1-to-1, onto and/or invertible? Is 0 and eigenvalue of $B \cdot A$? Why? (You don't need to do any computation to check this).

Solution: Given the expression for the null space of $B \cdot A$ we know that $\mathrm{nullity}(B \cdot A) = 2$ and therefore $\mathrm{rank}(B \cdot A) = 2$. In particular $T_B \circ T_A$ is not 1-to-1 since $\mathrm{nullity}(B \cdot A) > 0$ and it is not onto since $\mathrm{rank}(B \cdot A) < 4$; it follows that $T_B \circ T_A$ is not invertible. This implies that $B \cdot A$ is not invertible so 0 is an eigenvalue.

(d) [2 points] Which particular linear transformation of the plane is $T_A \circ T_B$ (dilation, rotation, etc.)? Is it 1-to-1, onto and/or invertible? Sketch the image under $T_A \circ T_B$ of the triangle with vertices $(0,0), (1,0)$ and $(0,1)$.

Solution: The linear transformation $T_A \circ T_B$ is a rotation of angle $\theta = \pi/2$, since

$$R_{\pi/2} = \begin{pmatrix} \cos \pi/2 & -\sin \pi/2 \\ \sin \pi/2 & \cos \pi/2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

It follows that it is an invertible linear transformation, being 1-to-1 and onto (the corresponding matrix has determinant 1) and the image of the given triangle is the same triangle rotated by $\pi/2$, i.e. the triangle with vertices $(0,0), (0,1)$ and $(-1,0)$.

Question 6. (20 points) Given a matrix A we know that the following 5 vectors form a basis of $\mathbb{R}^5$ of eigenvectors of A (with the corresponding eigenvalues):

$$\begin{array}{cccccc} & \begin{pmatrix}1\\0\\0\\-1\\0\end{pmatrix} & \begin{pmatrix}0\\1\\1\\2\\2\end{pmatrix} & \begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\1\\1\\0\\1\end{pmatrix} \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \text{eigenvalues} & 1 & 2 & 2 & 0 & -1 \end{array}$$

(a) [2 points] Write down a basis for the null space of A. Is T_A 1-to-1, onto and/or invertible?

Solution: The null space of A is the eigenspace relative to the eigenvalue 0. Since there is only one eigenvector in the basis with eigenvalue 0 we know that its span will be the entire null(A) and therefore it forms a basis of null(A). This implies that nullity(A) $= 1 > 0$ so T_A is nor 1-to-1 neither onto (by the Big Theorem). In particular is not invertible (which you can also see form the fact that 0 is an eigenvalue).

(b) [3 points] Without doing any computation, what is the rank of $A - 2I_5$? [Hint: what is the null space of that matrix?]

Solution: We know that the null space of $A - 2I_5$ is the eigenspace of the eigenvalue 2. Since there are two linearly independent eigenvectors with eigenvalue 2 we know that nullity$(A - 2I_5) = 2$ which implies that rank$(A - 2I_5) = 3$.

(c) [5 points] Is A diagonalizable? If yes find the invertible matrix P and the diagonal matrix D such that $A = P \cdot D \cdot P^{-1}$.

Solution: Since there is a basis of $\mathbb{R}^5$ of eigenvectors of A, A is diagonalizable. The two matrices are given by

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

(d) [5 points] Find all the vectors $x \in \mathbb{R}^5$ such that $T_A(x) = -x$. [Hint: rewrite $T_A(x) = -x$ using the matrix A and the fact that $-x = -I_5 \cdot x$].

Solution: $T_A(x) = -x$ is equivalent to $A \cdot x = -I \cdot x$ that can be rewritten as $(A+I)x = 0$. Therefore every vector with $T_A(x) = -x$ is in the null space of $A+I$, which corresponds to the eigenspace of the eigenvalue -1. By the information we know about A we can deduce that

$$\{x \in \mathbb{R}^5 : T_A(x) = -x\} = E_{-1}(A) = \text{span}\{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}\}$$

(e) [5 points] Let S be the subspace of $\mathbb{R}^5$ spanned by the first, third and fourth vector of the given basis of $\mathbb{R}^5$, i.e.

$$S = \text{span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$$

Find an orthogonal basis for S and compute a basis of $S^\perp$.

Solution: The three given vectors are already linearly independent and orthogonal to each other, so they form an orthogonal basis. To compute a basis for the orthogonal complement $S^\perp$ we construct the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and we compute a basis for the null space, which is given by $x_1 = x_4 = x_5 = 0$ which implies

$$S^\perp = \text{span}\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\}$$

Question 7. (15 points) Let S be the span of the following vectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad u_4 = \begin{pmatrix} 3 \\ 0 \\ -3 \\ 0 \end{pmatrix} \qquad u_5 = \begin{pmatrix} 5 \\ -1 \\ -6 \\ -2 \end{pmatrix}$$

(a) [4 points] Find a basis $\mathcal{B}$ of S and compute its dimension.

Solution: We reduce to echelon form the matrix whose columns are $u_1, \ldots, u_5$ and we get

$$\begin{pmatrix} 1 & 0 & 1 & 3 & 5 \\ 0 & 1 & 1 & 0 & -1 \\ -1 & 1 & 0 & -3 & -6 \\ 1 & 1 & 1 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 & -7 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we see that the columns containing a leading term are the first three therefore $S = \operatorname{span}\{u_1, u_2, u_3\}$ which is a basis and $\dim S = 3$.

(b) [3 points] Let A be the matrix whose columns are $u_1, \ldots, u_5$. Is 0 an eigenvalue of A? If yes compute a basis of the corresponding eigenspace.

Solution: Since the columns of A are linearly dependent A is not invertible and therefore 0 is an eigenvalue. The corresponding eigenspace $E_0(A)$ is $\operatorname{null}(A)$. We already computed the reduced echelon form of A and we can see that the equations defining $\operatorname{null}(A)$ are

$$\begin{cases} x_1 - x_5 = 0 \\ x_2 - 3x_4 - 7x_5 = 0 \\ x_3 + 3x_4 + 6x_5 = 0 \end{cases}$$

Using x_4 and x_5 as free variables we can write

$$E_0(A) = \operatorname{null}(A) = \operatorname{span}\{ \begin{pmatrix} 0 \\ 3 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ -6 \\ 0 \\ 1 \end{pmatrix} \}$$

(c) [4 points] Let $\mathcal{A}$ be another basis of S (you don't need to check this) given by

$$\mathcal{A} = \{\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \}$$

Compute the coordinates of u_1, u_2 and u_3 with respect to the basis $\mathcal{A}$ and the matrix of change of basis $M_{\mathcal{A},\mathcal{B}}$.

Solution: It is easy to see that $u_1 = \alpha_1 + \alpha_3$, $u_2 = \alpha_2 + \alpha_3$ and $u_3 = \alpha_1 + \alpha_2 - \alpha_3$ therefore

$$[u_1]_{\mathcal{A}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad [u_2]_{\mathcal{A}} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad [u_3]_{\mathcal{A}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

By the formula

$$M_{\mathcal{A},\mathcal{B}} = ([u_1]_{\mathcal{A}}, [u_2]_{\mathcal{A}}, [u_3]_{\mathcal{A}}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(d) [3 points] Compute the matrix $M_{\mathcal{B},\mathcal{A}}$ and the coordinates of α_1, α_2 and α_3 with respect to the basis $\mathcal{B}$.

Solution: The matrix we want is the inverse of the matrix we computed in the previous part, therefore we compute the inverse as follows:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

We conclude that

$$M_{\mathcal{B},\mathcal{A}} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Using this matrix we can compute the coordinates as follows:

$$[\alpha_1]_{\mathcal{B}} = M_{\mathcal{B},\mathcal{A}}[\alpha_1]_{\mathcal{A}} = M_{\mathcal{B},\mathcal{A}} e_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$[\alpha_2]_{\mathcal{B}} = M_{\mathcal{B},\mathcal{A}}[\alpha_2]_{\mathcal{A}} = M_{\mathcal{B},\mathcal{A}} e_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$[\alpha_3]_{\mathcal{B}} = M_{\mathcal{B},\mathcal{A}}[\alpha_3]_{\mathcal{A}} = M_{\mathcal{B},\mathcal{A}} e_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

(e) [1 point] What is $(M_{\mathcal{B},\mathcal{A}}^T)^{-1}$.

Solution: It is enough to observe that $M_{\mathcal{B},\mathcal{A}}$ is symmetric, and therefore $(M_{\mathcal{B},\mathcal{A}}^T)^{-1} = M_{\mathcal{B},\mathcal{A}}^{-1} = M_{\mathcal{A},\mathcal{B}}$.