Math 308 Discussion Problems #2 (Sections 2.1-2.3)

(1) When Jake works from $\vec{h}$ome, he typically spends 40 minutes of each hour on research, and 10 on teaching, and drinks half a cup of coffee. (The remaining time is spent on the internet.) For each hour he works in the math $\vec{d}$epartment, he spends around 20 minutes on research and 30 on teaching, and doesn't drink any coffee. Lastly, if he works at a $\vec{c}$offeeshop for an hour, he spends 25 minutes each on research and teaching, and drinks a cup of coffee.

(**Note:** be careful about units of minutes versus hours.)

(a) Last week, Jake spent 10 hours working from home, 15 hours working in his office in Padelford Hall, and 2 hours working at Cafe Allegro. Compute what was accomplished, and express the result as a vector equation.

(b) This week, Jake has 15 hours of research to work on and 10 hours of work related to teaching. He also wants 11 cups of coffee, because... of... very important reasons. How much time should he spend working from home, from his office, and from the coffeeshop?

(c) Describe the situation in part (b) as a vector equation and a matrix equation $A\vec{t} = \vec{w}$. What do the vectors $\vec{t}$ and $\vec{w}$ mean in this context? For which other vectors $\vec{w}$ does the equation $A\vec{t} = \vec{w}$ have a solution?

(d) Jake tries working in the math department $\vec{l}$ounge for an hour, and gets 30 minutes of research and 20 minutes of teaching work done, while having time to drink $\frac{1}{3}$ of a cup of coffee. Not bad. But Jake's colleague Vasu claims that there's no need to work in the lounge – the other options already give enough flexibility. Is he right? Explain mathematically.

(2) (after 2.1) Find a 3×4 matrix A, in *reduced* echelon form, with free variable x_3, such that the general solution of the equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 6 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

where s is any real number.

(3) (after 2.2) Find all values z_1 and z_2 such that $(2, -1, 3)$, $(1, 2, 2)$, and $(-4, z_1, z_2)$ do not span $\mathbb{R}^3$.

(4) (after 2.3) (a) Let $\mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{a_2} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, and $\mathbf{a_3} = \begin{bmatrix} t \\ -3 \\ -7 \end{bmatrix}$. Find all values of t for which there will be a unique solution to $\mathbf{a_1}x_1 + \mathbf{a_2}x_2 + \mathbf{a_3}x_3 = \mathbf{b}$ for every vector $\mathbf{b}$ in $\mathbb{R}^3$. Explain your answer.

(b) Are the vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ from part (a) linearly independent? Explain your answer.

(c) Let $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$ be as in (a). Let $\mathbf{a_4} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$. Without doing any further calculations, find all values of t for which there will be a unique solutions to $\mathbf{a_1}y_1 + \mathbf{a_2}y_2 + \mathbf{a_3}y_3 + \mathbf{a_4}y_4 = \mathbf{c}$ for every vector $\mathbf{c}$ in $\mathbb{R}^3$. Explain your answer.

(5) (after 2.3) (Geometry Question) Consider the infinite system of linear equations in two variables given by $ax + by = 0$ where (a, b) moves along the unit circle in the plane.

(a) How many solutions does this system have?
(b) How many linearly independent equations in the above system give you the same set of solutions? Write down two separate such linear systems, in vector form.
(c) What happens to the infinite linear system if you add to it the equation $0x + 0y = 0$?
(d) What happens to the infinite linear system if one of the equations slightly perturbs to $ax + by = c$ where c is a small positive number?
Explain all your answers in words.

(6) For each of the situations described below, **give an example** (if it's possible) or **explain why it's not possible**.

(a) A set of vectors that does not span $\mathbb{R}^3$. After adding one more vector, the set does span $\mathbb{R}^3$.
(b) A set of vectors that are linearly dependent. After adding one more vector, the set becomes linearly independent.
(c) A set of vectors in $\mathbb{R}^3$ with the following properties (four possibilities):

spans $\mathbb{R}^3$, linearly independent	spans $\mathbb{R}^3$, linearly dependent
doesn't span $\mathbb{R}^3$, linearly independent	doesn't span $\mathbb{R}^3$, linearly dependent

For each case that is *possible*, how many vectors could be in the set? (State any constraints, as in "there must be at least..." or "at most...")
(e) A system of equations with a unique solution. After adding another equation to the system, the new system has infinitely-many solutions.
(f) * A system of equations without any solutions. After deleting an equation, the system has infinitely-many solutions.