# Midterm 1

for Math 308, Winter 2017

NAME (last - first): ________________________________________

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- This exam contains 5 questions for a total of 50 points in 9 pages.
- You are allowed to have one double sided, handwritten note sheet and a non-programmable calculator.
- Show all your work. With the exception of True/False questions, if there is no work supporting an answer (even if correct) you will not receive full credit for the problem.

Do not write on this table!

| Question | Points | Score |
|---|---|---|
| 1 | 6 | |
| 2 | 4 | |
| 3 | 10 | |
| 4 | 12 | |
| 5 | 18 | |
| Total: | 50 | |

## Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ________________________________________ Date: ____________________

**Question 1.** (6 points) Decide whether the following statements are true or false. For this you don't need to show any work.

(a) [1 point] If the augmented matrix of a linear system has more rows then columns, the system is inconsistent.

○ True   √ **False**

(b) [1 point] If $u \in \text{span}(v_1, v_2)$ then $u, v_1$ and $v_2$ are not linearly independent.

√ **True**   ○ False

(c) [1 point] If a linear system has six variables then the dimension of the solution set is always 6.

○ True   √ **False**

(d) [1 point] A matrix in reduced echelon form might have rows of zeros.

√ **True**   ○ False

(e) [1 point] Given any set of $m > n$ vectors $u_1, \ldots, u_m \in \mathbb{R}^n$, $\text{span}(u_1, \ldots, u_m) = \mathbb{R}^n$.

○ True   √ **False**

(f) [1 point] If a linear system is inconsistent, the reduced echelon form of the augmented matrix have at least two columns without leading term.

○ True   √ **False**

**Question 2.** (4 points) For any of the following question, give an explicit example.

(a) [1 point] Give an example of an inconsistent linear system with more variables than equations.

**Solution:**

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 = 2 \end{cases}$$

(b) [1 point] Give an example of four vectors in $\mathbb{R}^3$ whose span is NOT $\mathbb{R}^3$ and none of them is a multiple of another.

**Solution:**

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(c) [1 point] Give an example of a matrix in reduced echelon form with four columns, two of which do not contain leading terms.

**Solution:**

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(d) [1 points] Give an example of a set of three vectors that spans $\mathbb{R}^2$.

**Solution:**

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

**Question 3.** (10 points) This problem (with different constants) comes from a Chinese manuscript dating from over 2000 years ago, in which the solution was found by a method incredibly similar to Gauss-Jordan Elimination.

> *There are three grades of grain: top, medium and low. Three sheaves of top-grade, two sheaves of medium-grade and one sheaf of low-grade are 12 Dous(unit of dry measure in ancient China). Two sheaves of top-grade, three sheaves of medium-grade and one sheaf of low-grade are 12 Dous. One sheaf of top-grade, two sheaves of medium-grade and three sheaves of low-grade are 24 Dous. How many Dous does one sheaf of top-grade, medium-grade and low-grade grain yield respectively?*

(a) [2 points] Denoting by $x$ the Dous of one sheaf of top-grade grain, by $y$ the Dous of one sheaf of medium-grade grain and by $z$ the Dous in a sheaf of low-grade grain, write down the linear system associated to the problem.

**Solution:** The text gives three equations in the three variables, so the linear system looks like

$$\begin{cases} 3x + 2y + z = 12 \\ 2x + 3y + z = 12 \\ x + 2y + 3z = 24 \end{cases}$$

(b) [6 points] Solve the linear system using the Gauss-Jordan algorithm. (Hint: to get the first "1", swap two lines)

**Solution:** We start with the augmented matrix of the system and apply the algorithm:

$$\begin{pmatrix} 3 & 2 & 1 & 12 \\ 2 & 3 & 1 & 12 \\ 1 & 2 & 3 & 24 \end{pmatrix} \xrightarrow{\mathrm{I} \leftrightarrow \mathrm{III}} \begin{pmatrix} 1 & 2 & 3 & 24 \\ 2 & 3 & 1 & 12 \\ 3 & 2 & 1 & 12 \end{pmatrix} \xrightarrow[\mathrm{III}-3\mathrm{I}]{\mathrm{II}-2\mathrm{I}} \begin{pmatrix} 1 & 2 & 3 & 24 \\ 0 & -1 & -5 & -36 \\ 0 & -4 & -8 & -60 \end{pmatrix}$$

$$\xrightarrow[-\frac{1}{4}\mathrm{III}]{-\mathrm{II}} \begin{pmatrix} 1 & 2 & 3 & 24 \\ 0 & 1 & 5 & 36 \\ 0 & 1 & 2 & 15 \end{pmatrix} \xrightarrow[\mathrm{III}-\mathrm{II}]{\mathrm{I}-2\mathrm{II}} \begin{pmatrix} 1 & 0 & -7 & -48 \\ 0 & 1 & 5 & 36 \\ 0 & 0 & -3 & -21 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{3}\mathrm{III}} \begin{pmatrix} 1 & 0 & -7 & -48 \\ 0 & 1 & 5 & 36 \\ 0 & 0 & 1 & 7 \end{pmatrix} \xrightarrow[\mathrm{II}-5\mathrm{III}]{\mathrm{I}+7\mathrm{III}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 7 \end{pmatrix}$$

Therefore the unique solution is $x = 1$, $y = 1$ and $z = 7$.

(c) [2 points] Using the previous part, can you tell if

$$\begin{pmatrix} 12 \\ 12 \\ 24 \end{pmatrix} \in \text{span}\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \}$$

and why?

**Solution:** By the previous part we know that

$$\begin{pmatrix} 12 \\ 12 \\ 24 \end{pmatrix} = 1 \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + 7 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

which shows that the vector is a linear combination of the other three, and therefore lies inside their span (being the span the set of all possible linear combinations).

**Question 4.** (12 points) Consider the following 4 vectors in $\mathbb{R}^4$

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 1 \end{pmatrix} \quad u_4 = \begin{pmatrix} 7 \\ 11 \\ 9 \\ 12 \end{pmatrix}$$

(a) [2 points] Write down the linear system associated to the problem of determining whether the given vectors are linearly independent or not. [You don't have to solve the system yet!]. Can you tell something about the solution set without computing it?

**Solution:** The vectors are linearly independent if the only solution to

$$x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4 = \vec{0}$$

is the trivial solution. So the linear system associated to this problem is

$$\begin{aligned} x_1 + \phantom{2}x_2 + 2x_3 + \phantom{1}7x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 &= 0 \\ 2x_2 + 2x_3 + \phantom{1}9x_4 &= 0 \\ x_1 \phantom{+ 2x_2} + \phantom{2}x_3 + 12x_4 &= 0 \end{aligned}$$

Since the system is homogeneous the solution set will always contain at least one vector, namely the zero vector. If the vector are linearly dependent the solution set will be infinite.

(b) [8 points] Compute whether the vectors $u_1, u_2, u_3$ and $u_4$ are linearly independent. If they are not, give an explicit linear combination expressing one of them as a linear combination of the others.

**Solution:** To check whether the vectors are linearly independent we build up the matrix whose columns are the vectors and reduce it to reduced echelon form.

$$\begin{pmatrix} 1 & 1 & 2 & 7 \\ 1 & 2 & 3 & 11 \\ 0 & 2 & 2 & 9 \\ 1 & 0 & 1 & 12 \end{pmatrix} \xrightarrow[\mathrm{IV}-\mathrm{I}]{\mathrm{II}-\mathrm{I}} \begin{pmatrix} 1 & 1 & 2 & 7 \\ 0 & 1 & 1 & 4 \\ 0 & 2 & 2 & 9 \\ 0 & -1 & -1 & 5 \end{pmatrix} \xrightarrow[\substack{\mathrm{III}-2\mathrm{II} \\ \mathrm{IV}+\mathrm{II}}]{\mathrm{I}-\mathrm{II}} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$$\xrightarrow[\substack{\mathrm{II}-4\mathrm{III} \\ \mathrm{IV}-9\mathrm{III}}]{\mathrm{I}-3\mathrm{III}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The computation shows that there is a column without leading terms. This implies that the vectors are *not* linearly independent. To find the linear combination we write the solution of the homogeneous linear system associated to the matrix which reads as follows:

$$\begin{aligned} x_1 \phantom{+ x_2} + x_3 \phantom{+ x_4} &= 0 \\ x_2 + x_3 \phantom{+ x_4} &= 0 \\ x_4 &= 0 \end{aligned}$$

So setting the free variable $x_3 = s$ we get that the solution set is given by

$$\begin{aligned} x_1 &= -s \\ x_2 &= -s \\ x_3 &= \phantom{-}s \\ x_4 &= \phantom{-}0 \end{aligned}$$

This implies that

$$-s \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - s \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 2 \\ 3 \\ 2 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 7 \\ 11 \\ 9 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which can be rewritten, taking $s = 1$ as

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 2 \\ 1 \end{pmatrix}.$$

This gives a linear combination expressing $u_1$ as a linear combination of $u_2$ and $u_3$.

(c) [2 points] Does $\text{span}\{u_1, u_2, u_3, u_4\} = \mathbb{R}^4$? Why or why not?

**Solution:** No it does not. Since we are given 4 vectors in $\mathbb{R}^4$ the Big Theorem apply, which implies that their span is $\mathbb{R}^4$ exactly when they are linearly independent. Since we just proved that they are not linearly independent this shows that their span is not $\mathbb{R}^4$.

**Question 5.** (18 points) Consider the following four vectors in $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

(a) [8 points] Determine whether $\text{span}(v_1, v_2, v_3, v_4) = \mathbb{R}^3$.

**Solution:** To check whether the span of the vectors is $\mathbb{R}^3$ we consider the matrix whose columns are the vector and we reduce it to reduced echelon form.

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & -2 \end{pmatrix} \xrightarrow{\text{I}\leftrightarrow\text{III}} \begin{pmatrix} 1 & 0 & 1 & -2 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{II}-\text{I}} \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$
$$\xrightarrow{\frac{1}{2}\text{II}} \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{III}-\text{II}} \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there is a row of zeroes in the reduced echelon form of the matrix we can conclude that the span is not $\mathbb{R}^3$.

(b) [2 points] Are the vectors linearly independent? Why or why not?

**Solution:** Four vectors in $\mathbb{R}^3$ are never linearly independent. This can also be seen by the computation above, since the reduced echelon form of the matrix contains columns without leading terms.

(c) [6 points] If the vectors do not span $\mathbb{R}^3$ determine a plane or a line containing their span.

**Solution:** To obtain an equation for the span we re-do the same operation we did while computing the reduced echelon form with a generic vector (thought as the constant term of a linear system)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{\mathrm{I}\leftrightarrow\mathrm{III}} \begin{pmatrix} z \\ x \\ y \end{pmatrix} \xrightarrow{\mathrm{II}-\mathrm{I}} \begin{pmatrix} z \\ y-z \\ x \end{pmatrix} \xrightarrow{\frac{1}{2}\mathrm{II}} \begin{pmatrix} z \\ \frac{y-z}{2} \\ x \end{pmatrix} \xrightarrow{\mathrm{III}-\mathrm{II}} \begin{pmatrix} z \\ \frac{y-z}{2} \\ x-\frac{y-z}{2} \end{pmatrix}$$

For this vector to be in the span of $v_1, v_2, v_3$ and $v_4$ we saw that the last component needs to be zero (otherwise in the corresponding linear system we will get an equation of the form $0 = k$) so we get the equation

$$x - \frac{y-z}{2} = 0 \qquad \text{or equivalently} \qquad 2x - y + z = 0$$

We can check that all the four given vectors verify this equation, that identifies exactly their span. This shows that the span is a plane in $\mathbb{R}^3$.

(d) [2 points] Given the previous point, write down two vectors that do not lie in $\mathrm{span}\{v_1, v_2, v_3, v_4\}$.

**Solution:** [2in] It is enough to take a vector for which the previous equation is not satisfied, for example

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$