Symmetries of the Tetrahedron

How much symmetry has a tetrahedron? Consider a regular tetrahedron T and, for simplicity, think only of rotational symmetry. Figure 1.1 shows two axes. One, labelled L, passes through a vertex of the tetrahedron and through the centroid of the opposite face; the other, labelled M, is determined by the midpoints of a pair of opposite edges. There are four axes like L and two rotations about each of these, through $2\pi/3$ and $4\pi/3$, which send the tetrahedron to itself. The sense of the rotations is as shown: looking along the axis from the vertex in question the opposite face is rotated anticlockwise. Of course, rotating through $2\pi/3$ (or $4\pi/3$) in the opposite sense has the same effect on T as our rotation through $4\pi/3$ (respectively $2\pi/3$). As for axis M, all we can do is rotate through π, and there are three axes of this kind. So far we have $(4 \times 2) + 3 = 11$ symmetries. Throwing in the identity symmetry, which leaves T fixed and is equivalent to a full rotation through 2π about any of our axes, gives a total of twelve rotations.

We seem to have answered our original question. There are precisely twelve rotations, counting the identity, which move the tetrahedron onto itself. But this is not the end of the story. A flat hexagonal plate with equal sides also has twelve rotational symmetries (Fig. 1.2), as does a right regular pyramid on a twelve sided base (Fig. 1.3). For the plate we have five rotations (through $\pi/3$, $2\pi/3$, π, $4\pi/3$, and $5\pi/3$) about the axis perpendicular to it which passes through its centre of gravity. In addition there are three axes of symmetry determined by pairs of opposite corners, three determined by the midpoints of pairs of opposite sides, and we can rotate the plate through π about each of these. Not forgetting the identity, our total is again twelve. The pyramid has only one axis of rotational symmetry. It joins the apex of the pyramid to the centroid of its base, and there are twelve distinct rotations (through $k\pi/6$, $1 \leqslant k \leqslant 12$, in some chosen sense) about this axis. Despite the fact that we

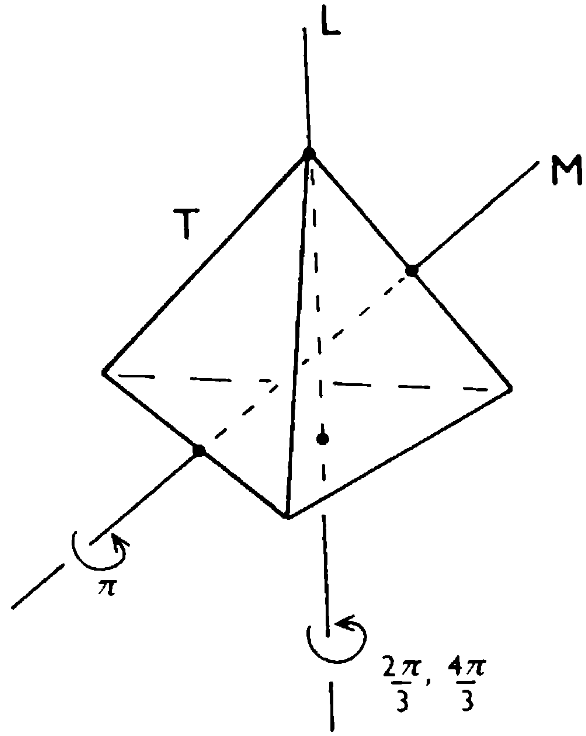

Figure 1.1

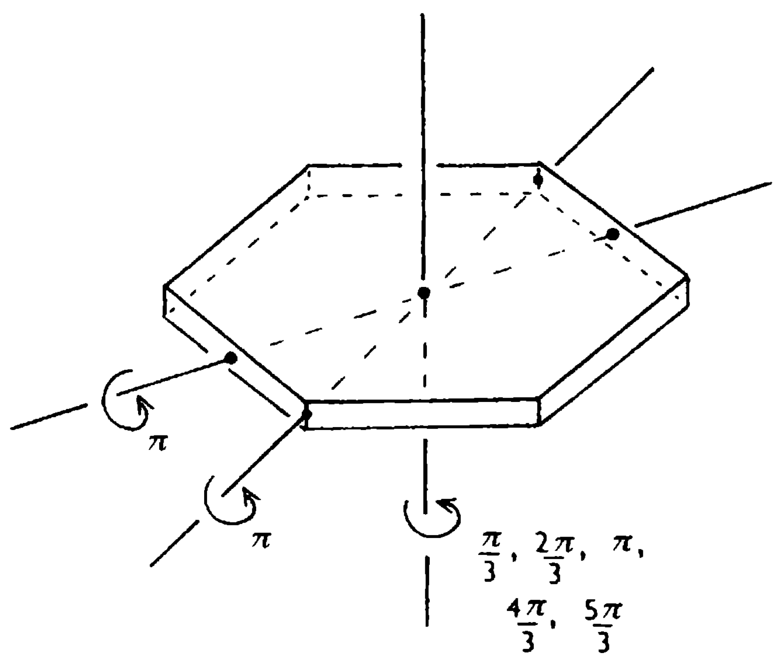

Figure 1.2

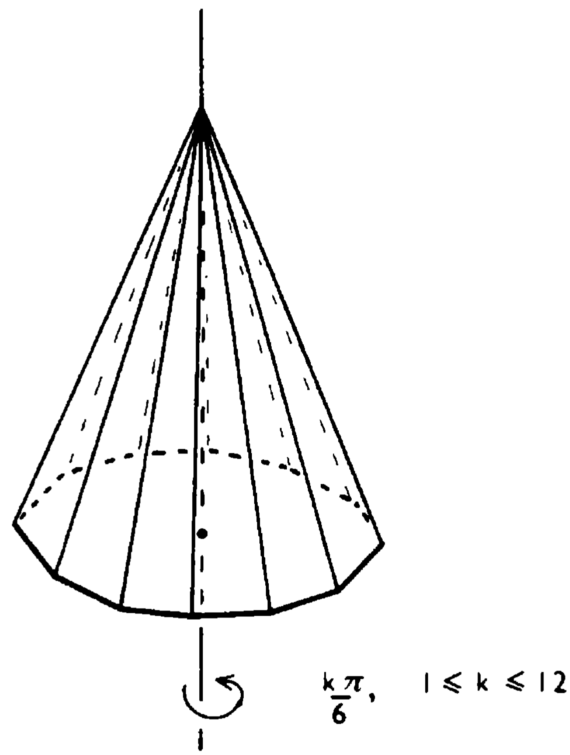

Figure 1.3

have counted twelve rotations in each case, the tetrahedron, the plate, and the pyramid quite clearly do not exhibit the same symmetry.

The most striking difference is that the pyramid possesses just one axis of symmetry. A rotation of $\pi/6$ about this axis has to be repeated (in other words, combined with itself) twelve times before the pyramid returns to its original position. Indeed, by suitable repetition of this basic rotation we can produce all the other eleven symmetries. However, no single rotation of the plate or the tetrahedron when repeated will give us all the other rotations.

If we look more carefully we can spot other differences, all of which have to do, in one way or another, with the way in which our symmetries combine. For example, the symmetries of the pyramid all *commute* with each other. That is to say, if we take any two and perform one rotation after the other, the effect on the pyramid is the same no matter which one we choose to do first. (These rotations all have the same axis, so if, for the sake of argument, we rotate through $\pi/3$ then through $5\pi/6$, we obtain rotation through $7\pi/6$, which is also the result of $5\pi/6$ first followed by $\pi/3$.) This is not the case for the tetrahedron or the plate. We recommend an experiment with the tetrahedron. Labelling the vertices of T as in Figure 1.4 enables us to see clearly the effect of a particular symmetry. Think of the rotations r ($2\pi/3$ about axis L in the sense indicated) and s (π about axis M). Performing first r then s takes vertex 2 back to its initial position and gives a rotation about axis N. But first s then r moves 2 to the place originally occupied by 4, and so cannot be the same rotation. Do

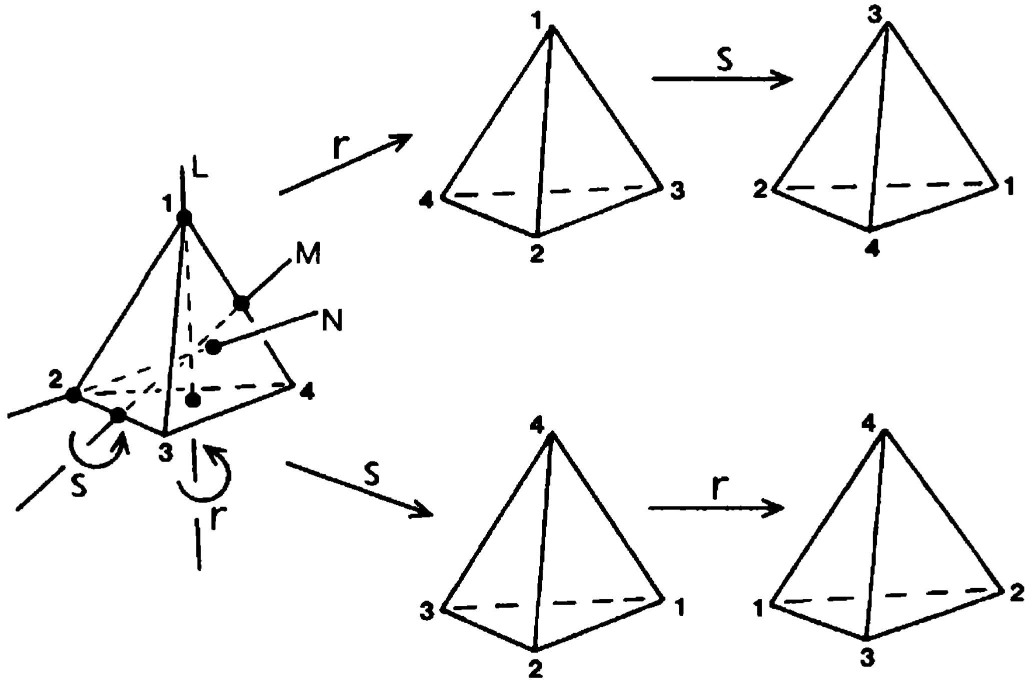

Figure 1.4

not fall into the trap of carrying the axis of s along with you as you do r first. Both r and s should be thought of as rigid motions of space, each of which has an axis that is *fixed* in space, and each of which rotates T onto itself.

Here is a third observation. There is only one rotation of the pyramid which, when combined once with itself, gives the identity; namely, the unique rotation through π. The plate has seven such symmetries and the tetrahedron three. These three rotations through π of the tetrahedron commute with one another, but only one of the seven belonging to the plate commutes with all the other six. Which one? Experiment until you find out.

To obtain a decent measure of symmetry, simply counting symmetries is not enough; we must also take into consideration how they combine with each other. It is the so-called symmetry group which captures this information and which we now attempt to describe.

The set of rotational symmetries of T has a certain amount of "algebraic structure". Given two rotations u and v we can *combine* them, by first doing v, then doing u, to produce a new rotation which also takes T to itself, and which we write uv. (Our choice of uv rather than vu is influenced by the convention for the composition of two functions, where fg usually means first apply g, then apply f.) The *identity* rotation, which we denote by e, behaves in a rather special way. Applying first e then another rotation u, or first u then e, always gives the same result as just applying u. In other words $ue = u$ and $eu = u$ for every symmetry u of T. Each rotation u has a so-called *inverse* u^{-1}, which is also a symmetry of T and which satisfies $u^{-1}u = e$ and $uu^{-1} = e$. To obtain u^{-1}, just rotate about the same axis and through the same angle as for u, but in the opposite sense. (For example, the inverse of the rotation r is rr, because applying r three times gives the identity.) Finally, if we take three of our rotations u,v, and w, it does not matter whether we first do w then the composite rotation uv, or whether we apply vw first and then u. In symbols this reduces to $(uv)w = u(vw)$ for any three (not necessarily distinct) symmetries of T.

The twelve symmetries of the tetrahedron together with this algebraic structure make up its rotational symmetry group.