SOLUTIONS

Math 308
Autumn 2016
MIDTERM - 1
10/21/2016

Instructions: The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to **50** points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		6
2		6
3		14
4		12
5		12
TOTAL		50

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:**______________

Question 1. [6 points] Decide whether the following statements are true or false. If true, briefly explain why. If false, provide a counter-example.

(a) [1 points] If a matrix is in reduced row echelon form then the number of leading terms is smaller than or equal to the number of columns in the matrix.
Solution:

TRUE. If a matrix is in recued row echelon form each leading term is the only nonzero term in its column, in particular, there is at most one leading term for each column. □

(b) [1 points] If $\text{span}\{\vec{u}_1, \ldots, \vec{u}_m\} = \mathbb{R}^n$ and $m > n$ then the vectors $\vec{u_1}, \ldots, \vec{u}_m$ are linearly dependent.
Solution:

TRUE. It follows from the characterization of linearly dependent sets of vectors in $\mathbb{R}^n$ (Theorem 2.13 in Holt's book).

□

(c) [1 points] Every homogeneous system of linear equations has *exactly* (no more, no less) one solution.
Solution:

FALSE. Example:

$$\begin{aligned} x_1 + \ \ x_2 &= 0 \\ 2x_1 + 2x_2 &= 0 \end{aligned}$$

has infinitely many solutions. □

(d) [1 points] Every set of 4 distinct vectors in $\mathbb{R}^2$ spans $\mathbb{R}^2$.
Solution:

FALSE. Example:

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Then $\text{span}\{v_1, v_2, v_3, v_4\} = \text{span}\{v_1\} \neq \mathbb{R}^2$. □

(e) [1 points] If $\vec{u} \in \mathbb{R}^n$ is not a multiple of $\vec{v} \in \mathbb{R}^n$ then $\operatorname{span}\{\vec{u}, \vec{v}\} \neq \operatorname{span}\{\vec{u}\}$.
Solution:

TRUE. If $\operatorname{span}\{\vec{u}, \vec{v}\} = \operatorname{span}\{\vec{u}\}$ it would follow that $\vec{v} \in \operatorname{span}\{\vec{u}\}$ which would imply that there is a $c \in \mathbb{R}$ such that $\vec{v} = c\vec{u}$ contraddicting the assumption that $\vec{v}$ is not a multiple of $\vec{u}$. $\square$

(f) [1 points] Given a system of three linear equations in three variables the solution set is at most two dimensional.
Solution:

TRUE. The system can have at most two free variables, hence the dimension of the solution set is at most two. $\square$

Question 2. [6 points] Given the vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m, \vec{b} \in \mathbb{R}^n$, prove that the two following statements are equivalent:

(a) $\vec{b} \in \operatorname{span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m\}$;

(b) The linear system corresponding to the matrix $[\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_m \mid \vec{b}]$ has at least one solution.

Solution:

(a) $\Rightarrow$ **(b)**: if $\vec{b}$ is in the span of $\vec{a}_1, \ldots, \vec{a}_m$, by definition of *span*, there exists $c_1, \ldots, c_m \in \mathbb{R}$ such that

$$\vec{b} = c_1 \cdot \vec{a}_1 + c_2 \cdot \vec{a}_2 + \cdots + c_m \cdot \vec{a}_m$$

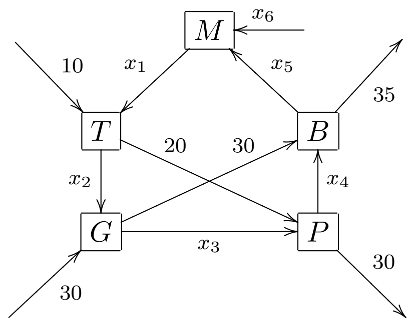
In particular $(c_1, c_2, \ldots, c_m)$ is a solution to the linear system associated to the matrix $[\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_m \mid \vec{b}]$.

(b) $\Rightarrow$ **(a)**: if the system corresponding to $[\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_m \mid \vec{b}]$ has at least a solution then there exists $c_1, c_2, \ldots, c_m \in \mathbb{R}$ such that:

$$c_1 \cdot \vec{a}_1 + c_2 \cdot \vec{a}_2 + \cdots + c_m \cdot \vec{a}_m = \vec{b}$$

which implies that $\vec{b}$ is a linear combination of the vectors $\vec{a}_1, \ldots, \vec{a}_m$, and therefore $\vec{b}$ is in the span of $\vec{a}_1, \ldots, \vec{a}_m$. $\square$

Question 3. [14 points] A truck rental company is considering opening 5 new rental stations in the cities of Milan, Turin, Genoa, Brescia and Parma. Sara is in charge of testing whether the proposed configuration of stations is compatible with estimated demand in each city, considering that sometimes trucks are rented in one city and driven to another. The proposed configuration can be modeled by the following diagram:



In the diagram, each arrow is labeled with the number of trucks entering or leaving the city in a given day (where some would be coming from outside the system). The arrows connecting two stations measure the number of trucks that need to be moved from one station to the other in order to have enough trucks to meet demand. In addition, each city's station has a capacity. At the end of a day a station cannot hold more trucks than its capacity because the station could not fit them in a secured garage, but it also cannot hold fewer trucks than its capacity because they want to keep up with estimated demand. In other words, the number of trucks in a given city at the end of a given day must remain constant.

Sara is tasked with finding the number of trucks that need to be moved between the stations to balance the system, assuming the following values for the capacities:

$$M = 5 \qquad T = 5 \qquad G = 5 \qquad P = 5 \qquad B = 5$$

(a) [5 points] Write a linear system that represents the problem Sara has to solve. [*Hint*:] In Milan there are two arrows pointing inwards, labeled x_5 and x_6 and one arrow pointing outwards labeled x_1, therefore for Milan we get the equation

$$x_5 + x_6 - x_1 = 5.$$

Solution:

Using the same technique explained in the hint we get the following equations:

$$\begin{aligned} (M) &\quad x_5 + x_6 - x_1 = 5 \\ (T) &\quad x_1 + 10 - (x_2 + 20) = 5 \\ (G) &\quad x_2 + 30 - (x_3 + 30) = 5 \\ (P) &\quad x_3 + 20 - (x_4 + 30) = 5 \\ (B) &\quad x_4 + 30 - (x_5 + 35) = 5 \end{aligned}$$

With a bit of rearranging we get the system in the usual form:

$$\begin{array}{rrrrrrrl} (M) & -x_1 & & & & +x_5 & +x_6 & = 5 \\ (T) & x_1 & -x_2 & & & & & = 15 \\ (G) & & x_2 & -x_3 & & & & = 5 \\ (P) & & & x_3 & -x_4 & & & = 15 \\ (B) & & & & x_4 & -x_5 & & = 10 \end{array}$$

□

(b) [2 points] Write down the augmented matrix associated the linear system described above. *Solution:*

$$\left(\begin{array}{cccccc|c} -1 & 0 & 0 & 0 & 1 & 1 & 5 \\ 1 & -1 & 0 & 0 & 0 & 0 & 15 \\ 0 & 1 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & -1 & 0 & 0 & 15 \\ 0 & 0 & 0 & 1 & -1 & 0 & 10 \end{array}\right)$$

□

(c) [7 points] Sara inserts this augmented matrix into a linear algebra computer program and computes its reduced echelon form. The output of her computer looks like:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & 45 \\ 0 & 1 & 0 & 0 & -1 & 0 & 30 \\ 0 & 0 & 1 & 0 & -1 & 0 & 25 \\ 0 & 0 & 0 & 1 & -1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 50 \end{array}\right)$$

Using the above matrix write down the solution of the system in part (a) and determine whether the solution set is empty, a unique point, or infinite and determine its dimension. If there is a free parameter, put limits on that parameter so that the system is physically meaningful.
Solution:

From the matrix we can tell that x_5 is a free variable, since the fifth column of the matrix does not contain any leading term. Therefore, setting $x_5 = s$ we get the following solution set:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 45+s \\ 30+s \\ 25+s \\ 10+s \\ s \\ 50 \end{pmatrix} = \begin{pmatrix} 45 \\ 30 \\ 25 \\ 10 \\ 0 \\ 50 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

for any choice of $s \in \mathbb{R}$. Therefore we see that the solution set is infinite at it has dimension 1 since it represents a line in $\mathbb{R}^6$ (or equivalently since there is only one free parameter). Since every variable represents the number of trucks we have that each must be positive. Hence the solution is physically meaningful only if $s \geq 0$. □

Question 4. [12 points] Show that $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ is contained in

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Specifically, give coordinates $x_1,\ x_2,\ x_3$ and x_4 such that

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

Solution:

To solve the system written above we write the associated augmented matrix associated to it and reduce it in reduced echelon form, as follos:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & | & -1 \\ 0 & 1 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & 1 & | & 2 \\ 0 & 0 & 0 & 1 & | & 1 \end{pmatrix} \underset{\sim}{I-II} \begin{pmatrix} 1 & 0 & 0 & 0 & | & -1 \\ 0 & 1 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & 1 & | & 2 \\ 0 & 0 & 0 & 1 & | & 1 \end{pmatrix} \underset{\sim}{II-III} \begin{pmatrix} 1 & 0 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & 0 & | & -2 \\ 0 & 0 & 1 & 1 & | & 2 \\ 0 & 0 & 0 & 1 & | & 1 \end{pmatrix} \underset{\sim}{III-IV} \begin{pmatrix} 1 & 0 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & 0 & | & -2 \\ 0 & 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & 1 & | & 1 \end{pmatrix}$$

From the computation we immediately see that the solution is $x_1 = -1$, $x_2 = -2$, $x_3 = 1$ and $x_4 = 1$, which proves that the initial vector is in the span of the other four vectors since

$$-\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

$\square$

Question 5. [12 points] For this question consider the following set of vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \vec{v}_5 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

(a) [6 points] Is $\mathbb{R}^4$ spanned by the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Why or why not? [Hint: Compare these vectors to the vectors from the previous problem. Don't do more work than you have to!] *Solution:*

By the computation of Question 4 we know that the matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]$ has the following reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore for every vector $\vec{b}$ the system associated to $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4 \mid \vec{b}]$ has always a solution. This implies that the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ is the all $\mathbb{R}^4$. □

(b) [6 points] Let A be the matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4\ \vec{v}_5]$ and let $\vec{x}$ be the 1×5 vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$. How many solutions are there to the matrix equation $A\vec{x} = \vec{0}$? Explain your reasoning. *Solution:*

Since the system is homogeneous there is always at least one solution, namely the trivial one. However, we showed in the previous question that

$$\vec{v}_5 = -\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 + \vec{v}_4$$

Using this we can see that

$$-\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 + \vec{v}_4 - \vec{v}_5 = \vec{0}.$$

So we found another solution to the homogenous linear system $A\vec{x} = \vec{0}$ namely

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

Therefore, since there are at least two solution, we can conclude that there are infinitely many. □