Name: ______________________________

Math 308
Autumn 2016
MIDTERM - 1
9/21/2016

Instructions: The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to **50** points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		6
2		6
3		14
4		12
5		12
TOTAL		50

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:**______________

Question 1. [6 points] Decide whether the following statements are true or false. If the statement is true, you don't need to prove it or even explain why it's true (i.e. just write "True"). If it's false, provide a counter-example.

(a) [1 point] If a matrix is in reduced row echelon form then the number of leading terms is smaller than or equal to the number of columns in the matrix.

(b) [1 point] If $\text{span}\{\vec{u}_1, \ldots, \vec{u}_m\} = \mathbb{R}^n$ and $m > n$ then the vectors $\vec{u_1}, \ldots, \vec{u}_m$ are linearly dependent.

(c) [1 point] Every homogeneous system of linear equations has *exactly* (no more, no less) one solution.

(d) [1 point] Every set of 4 distinct vectors in $\mathbb{R}^2$ spans $\mathbb{R}^2$.

(e) [1 point] If $\vec{u} \in \mathbb{R}^n$ is not a multiple of $\vec{v} \in \mathbb{R}^n$ then $\text{span}\{\vec{u}, \vec{v}\} \neq \text{span}\{\vec{u}\}$.

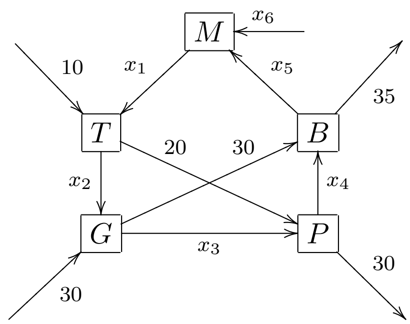
(f) [1 point] Given a system of three linear equations in three variables the solution set is at most two dimensional.

Question 2. [6 points] Given the vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m, \vec{b} \in \mathbb{R}^n$, prove that the two following statements are equivalent:

(a) $\vec{b} \in \operatorname{span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m\}$;

(b) The linear system corresponding to the matrix $[\vec{a}_1 \;\; \vec{a}_2 \;\; \ldots \;\; \vec{a}_m \mid \vec{b}]$ has at least one solution.

Question 3. [14 points] A truck rental company is considering opening 5 new rental stations in the cities of Milan, Turin, Genoa, Brescia and Parma. Sara is in charge of testing whether the proposed configuration of stations is compatible with estimated demand in each city, considering that sometimes trucks are rented in one city and driven to another. The proposed configuration can be modeled by the following diagram:



In the diagram, each arrow is labeled with the number of trucks entering or leaving the city in a given day (where some would be coming from outside the system). The arrows connecting two stations measure the number of trucks that need to be moved from one station to the other in order to have enough trucks to meet demand. In addition, each city's station has a capacity. At the end of a day a station cannot hold more trucks than its capacity because the station could not fit them in a secured garage, but it also cannot hold fewer trucks than its capacity because they want to keep up with estimated demand. In other words, the number of trucks in a given city at the end of a given day must remain constant.

Sara is tasked with finding the number of trucks that need to be moved between the stations to balance the system, assuming the following values for the capacities:

$$M = 5 \qquad T = 5 \qquad G = 5 \qquad P = 5 \qquad B = 5$$

(a) [5 points] Write a linear system that represents the problem Sara has to solve.
[*Hint*:] In Milan there are two arrows pointing inwards, labeled x_5 and x_6 and one arrow pointing outwards labeled x_1, therefore, since the capacity for Milan is 5, we get for the node labeled M the equation: $x_5 + x_6 - x_1 = 5$.

(b) [2 points] Write down the augmented matrix associated the linear system described above.

(c) [7 points] Sara inserts this augmented matrix into a linear algebra computer program and computes its reduced echelon form. The output of her computer looks like:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & \vert & 45 \\ 0 & 1 & 0 & 0 & -1 & 0 & \vert & 30 \\ 0 & 0 & 1 & 0 & -1 & 0 & \vert & 25 \\ 0 & 0 & 0 & 1 & -1 & 0 & \vert & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & \vert & 50 \end{pmatrix}$$

Using the above matrix write down the solution of the system in part (a) and determine whether the solution set is empty, a unique point, or infinite and determine its dimension. If there is a free parameter, put limits on that parameter so that the system is physically meaningful.

Question 4. [12 points] Show that $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ is contained in

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Specifically, give coordinates $x_1,\ x_2,\ x_3$ and x_4 such that

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

Question 5. [12 points] For this question consider the following set of vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \; \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; \vec{v}_5 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

(a) [6 points] Is $\mathbb{R}^4$ spanned by the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Why or why not? [Hint: Compare these vectors to the vectors from the previous problem. Don't do more work than you have to!]

(b) [6 points] Let A be the matrix $[\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3 \; \vec{v}_4 \; \vec{v}_5]$ and let $\vec{x}$ be the 1×5 vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$. How many solutions are there to the matrix equation $A\vec{x} = \vec{0}$? Explain your reasoning.