These are notes for the class Math 340 at UW. The notes are a *WORK IN PROGRESS*

Notation. We will use the notation $A := B$ whenever we *define* A by means of B.
We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ the set of the natural, integer, rational, real and complex numbers. The notation $\mathbb{Z}_{>0}$ denotes the set of all positive integers.

1. Logic notation

We use the following notation in this course: $\forall$ means *for all*, and $\exists$ means *exists* as quantifiers (you should have seen this is math 300). We will also use the symbol $\Rightarrow$ for implication.

2. Sets and Relations

In an informal way we will use the term *set* to indicate a collection of objects. Such collection might be finite or infinite. In this sense $X = \{a, b, c\}$ will be a set consisting of three objects that we will call its *elements*, and we will use the notation $a \in X$ to indicate that a is an element of X. On the other hand the set of all integers, usually denoted by the letter $\mathbb{Z}$, is the infinite set

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

Its elements are all the integers and we might write the tautology $1 \in \mathbb{Z}$.

Given two sets X and Y we say that Y is a subset of X, denoted by $Y \subset X$ if every element of Y is also an element of X (in logic terms this can be written as $\forall y \in Y \Rightarrow y \in X$). For example if $X = \{a, b, c\}$ and $Y = \{b, c\}$ then $Y \subset X$ since every element of Y is an element of X. On the other hand the set $Z = \{a, f\}$ is not a subset of X since $f \in Z$, i.e. f is an element of Z but $f \notin X$, i.e. f is not an element of X (sometimes this is denote by $Z \not\subset X$).

If X and Y are two sets we say that $X = Y$ if $X \subset Y$ and $Y \subset X$ (this is also the strategy one should use when trying to prove that 2 sets are equal).

Standard operation between sets are: union, intersection, power set, difference that we quickly recall here: given two sets X and Y we have

- the union $X \cup Y = \{a : a \in X \text{ or } a \in Y\}$;
- the intersection $X \cap Y = \{a : a \in X \text{ and } a \in Y\}$;
- the difference $X \setminus Y = \{a : a \in X \text{ and } a \notin Y\}$;
- the power set $\mathcal{P}(X) := \{A : A \subset X\}$.

Given two sets X and Y, their *Cartesian product*, often called just *product*, is the set $X \times Y$ defined by

$$X \times Y = \{(a, b) : a \in X, b \in Y\}$$

i.e. is the set of all ordered pair (a, b) such that the first element is in X and the second element is in Y. Note that the order here is fundamental; in particular the sets $X \times Y$ and $Y \times X$ are in general different sets.

Given a set X a *partition* of X is a collection of subsets $\mathcal{Y} = \{Y_i\}_{i \in I} = \{Y_1, Y_2, \dots\} \subset \mathcal{P}(X)$, i.e. a subset of the power set of X, such that

(1) The union of all elements in the partition verifies $\cup \mathcal{Y} = \cup_{i \in I} Y_i := Y_1 \cup Y_2 \cup \dots = X$;
(2) For every two elements Y_i and Y_j in the partition, $Y_i \cap Y_j = \emptyset$ (i.e. they are disjoint).

We will focus on the notion of relation.

Definition 2.1. Given two sets X and Y a *relation* $\mathscr{R}$ is a subset of the set $X \times Y$, i.e. $\mathscr{R} \subset X \times Y$[1]. We say that two elements $a \in X$ and $b \in Y$ are $\mathscr{R}$-related if $(a, b) \in \mathscr{R}$. The notations for this are the following

$$(a, b) \in \mathscr{R} \;\Leftrightarrow\; a \mathscr{R} b \;\Leftrightarrow\; a \sim_{\mathscr{R}} b$$

Thus one can describe a relation either as a subset of the Cartesian product $X \times Y$ or by describing which elements are $\mathscr{R}$-related.

A relation $\mathscr{R} \subset X \times X$ is called an *equivalence relation* (on X) if it satisfies the following properties:

Reflexivity: : for every $a \in X$ one has $a \sim_{\mathscr{R}} a$;

Symmetry: : if $a \sim_{\mathscr{R}} b$ then $b \sim_{\mathscr{R}} a$;

Transitivity: : if $a \sim_{\mathscr{R}} b$ and $b \sim_{\mathscr{R}} c$ then $a \sim_{\mathscr{R}} c$.

One way to think of a relation on a set X is as a law that singles out couples of elements of X (this is equivalent to specify a subset of $X \times X$ after all). For example given the set of all integers $\mathbb{Z}$ one can define a relation as $x \sim y$ if $x = y$ for every $x, y \in \mathbb{Z}$. This indeed is a relation and corresponds to the subset of $\mathbb{Z} \times \mathbb{Z}$ given by $\{(a, a) : a \in Z\}$, i.e. the subset of $\mathbb{Z} \times \mathbb{Z}$ consisting of all the couples in which both elements are the same. This relation is a formal way of describing the same kind of abstract operation you do when you write $2 = 2$: in fact one can use the symbol $=$ instead of $\sim$ in this precise setting. Moreover one can also see that such relation is in fact an equivalence relation since $=$ verifies both Reflexivity, symmetry and transitivity.

Exercise 2.2.

(1) Show that for every two sets X, Y the following law describes a relation: $a \sim b$ for every $a \in X$ and $b \in Y$. Prove that if $X = Y$ such relation is an equivalence relation.

(2) Show that for every $n \in \mathbb{Z}_{>0}$ the following is an equivalence relation on $\mathbb{Z}$:

$$x \equiv_n y \text{ if and only if } x - y = n \cdot t \text{ for some } t \in \mathbb{Z}$$

(you should read $\equiv_n$ as "congruent modulo n").

(3) Given the set $X = \{a, b\}$ write down all possible equivalence relations on X.

Definition 2.3. Given an equivalent relation $\sim$ on a set X and an element $a \in X$ the *equivalence class* of a for $\sim$ is the subset

$$[a]_\sim := \{x \in X : x \sim a\}.$$

Given the relation $\sim$ the *quotient set* of X with respect to $\sim$ is the set

$$X_{/\sim} := \{[a]_\sim : a \in X\}$$

i.e. is the set whose elements are the equivalence classes of elements of X for $\sim$.

Exercise 2.4. Prove the following facts:

- If $\sim$ is an equivalence relation on X, then for every $a, b \in X$, $[a]_\sim = [b]_\sim$ if and only if $a \sim b$;
- The quotient set $X_{/\sim}$ is a subset of the power set of X and it is a partition of X.

3. Functions

The informal definition of a function $f : X \to Y$ is a law that associates to every element of the domain X a unique element of the codomain Y. One can formalize this by using relations.

Definition 3.1. Given two sets X and Y a *function* with domain X and codomain Y, notation $f : X \to Y$ is a relation on $X \times Y$ such that for every $a \in X$ there is exactly one pair containing a.

In other words a function $f : X \to Y$ is defined by a set of pairs in $X \times Y$; this is equivalent to the usual definition of function since having the pair (a, b) in the relation amounts to the fact that f sends a to b, which we usually write $f(a) = b$, or $f : a \mapsto b$. The requirement that there is exactly one pair for every element of the domain is equivalent to the fact that a function associates to every element of the domain a unique element of the codomain.

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[1] this is sometimes called the graph of the relation but we will not make this distinction here.

The set of pairs inside $X \times Y$ is called the *graph* of the function $f : X \to Y$. We can rephrase the definition by saying that to specify a function f is the same as describing its domain, its codomain and its graph.

Given $a \in X$, $A \subset X$, $B \subset Y$ and $b \in Y$ we can define

$$\begin{aligned}
&f(a) - \text{ the image of } a\\
&f(A) = \{f(a) : a \in A\} - \text{ the image of } A\\
&f^{-1}(b) = \{a \in X : f(a) = b\} - \text{ the pre-image of } b\\
&f^{-1}(B) = \{a \in X : f(a) \in B\} - \text{ the pre-image of } B
\end{aligned}$$

Given a function $f : X \to Y$ the function is called

(1) *injective* if $f(a) = f(b)$ implies $a = b$;

(2) *surjective* if for every $b \in Y$ there is $a \in X$ such that $f(a) = b$;

(3) *bijective* if it is injective and surjective.

Given two functions $f : X \to Y$ and $g : Y \to Z$ such that the domain of g is the same as the codomain of f (it would be sufficient that the domain of g contains the domain of f) the composition of f and g is the function $g \circ f : X \to Z$ defined as $g \circ f(x) = g(f(x))$.

Exercise 3.2. Prove the following facts:

- Two sets X and Y have the same cardinality (i.e. the same number of elements) if there exists a bijective function $f : X \to Y$;
- If X is a finite set and $f : X \to X$ is an injective function then f is also surjective.
- A set X is infinite is there exists an injective function $f : X \to X$ which is not surjective.
- If f and g are two injective functions then $g \circ f$ is injective.
- If f and g are two surjective functions then $g \circ f$ is surjective.
- If $g \circ f$ is injective then f is injective.
- If $g \circ f$ is surjective then g is surjective.