HOMEWORK 4 - MATH402B

DUE: WEDNESDAY OCTOBER 25TH

(1) Goodman 1.10.3

(2) Goodman 1.10.4

(3) Let $\mathcal{R}$ be the set of rational points on the unit circle (recall that the unit circle is the set of points (x, y) in $\mathbb{R}^2$ such that $x^2 + y^2 = 1$); these are the points having both coordinates rational numbers. Follow these steps to show $\mathcal{R}$ is a group under the given operation. Given two rational points (α, β) and (γ, λ) on the unit circle define the operation

$$(\alpha, \beta) \star (\gamma, \lambda) = (\alpha\gamma - \beta\lambda, \alpha\lambda + \beta\gamma)$$

   (a) Show that this is a well defined binary operation on $\mathcal{R}$. Show that $(1, 0)$ acts like an identity under $\star$. Given any (α, β) show that there exists (γ, λ) so that $(\alpha, \beta) \star (\gamma, \lambda) = (1, 0)$, i.e. every rational point has an inverse with respect to $\star$.

   (b) Compute $(\frac{3}{5}, \frac{4}{5}) \star (\frac{5}{13}, \frac{12}{13})$.

   (c) Describe the computation in the previous part in terms of the geometry of the unit circle.

(4) Goodman 2.1.5

(5) Goodman 2.1.12

(6) Goodman 2.1.15

(7) Consider the following two sets of matrices $A = \{\begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} : r \in \mathbb{R}\}$ and $N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\}$. Prove that they are subgroups of the group $SL_2(\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1\}$ with the usual matrix multiplication. Write two isomorphisms between the groups and another known group coming from sets of numbers (e.g. $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$).

(8) Goodman 2.2.4

(9) Let $(G, \cdot)$ be a group and consider the group $(G \times G, \cdot)$ as defined in class. Prove that $\Delta = \{(g, g) : g \in G\}$ is a subgroup of $G \times G$.