Midterm 1

for Math 308, Winter 2017

NAME (last - first): ____

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- This exam contains 5 questions for a total of 50 points in 9 pages.
- You are allowed to have one double sided, handwritten note sheet and a non-programmable calculator.
- Show all your work. With the exception of True/False questions, if there is no work supporting an answer (even if correct) you will not receive full credit for the problem.

Do not write on this table!

Question	Points	Score
1	6	
2	4	
3	12	
4	12	
5	16	
Total:	50	

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ____ Date: ____________________

Question 1. (6 points) Decide whether the following statements are true or false. For this you don't need to show any work.

(a) [1 point] If S is a subspace of $\mathbb{R}^n$ then $\dim S \leq n$.

$\surd$ **True** ○ False

(b) [1 point] If $v \in \text{null}(A)$ is a non-zero vector, then T_A is not 1-to-1.

$\surd$ **True** ○ False

(c) [1 point] If $T = T_A$ is 1-to-1 then the columns of A span the codomain of T_A.

○ True $\surd$ **False**

(d) [1 point] If A is a $m \times n$ matrix, $\text{row}(A)$ is a subspace of $\mathbb{R}^n$.

$\surd$ **True** ○ False

(e) [1 point] If $\mathcal{B}$ is a basis of a subspace S and $u, v \in \mathcal{B}$ then $u + v \in \mathcal{B}$.

○ True $\surd$ **False**

(f) [1 point] The set $\{(a, b, c, d) \in \mathbb{R}^4 : a + b = cd\}$ is a subspace of $\mathbb{R}^4$.

○ True $\surd$ **False**

Question 2. (4 points) For any of the following question, give an explicit example.

(a) [1 point] A subspace S in $\mathbb{R}^3$ of dimension 2.

Solution:

$$S = \text{span}\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}$$

(b) [1 points] A 2×3 matrix with rank $= 1$.

Solution:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

(c) [1 point] A 3×3 matrix with nullity $= 0$.

Solution:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) [1 point] A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ which is 1-to-1 but not onto.

Solution:

$$T = T_A \qquad \text{where} \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Question 3. (12 points) Consider the following matrix and linear transformation

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 0 \\ 0 & 1 \end{pmatrix} \qquad T_B \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 \\ x_2 + x_4 \end{pmatrix}$$

(a) [3 points] Identify the matrix B associated to the linear transformation T_B and the domain and codomain of the linear transformations T_A and T_B.

> **Solution:** We have that
>
> $$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$
>
> Moreover $T_A : \mathbb{R}^2 \to \mathbb{R}^4$ and $T_B : \mathbb{R}^4 \to \mathbb{R}^2$.

(b) [4 points] Compute whenever possible the matrices associated to the compositions $T_A \circ T_B$ and $T_B \circ T_A$ and identify domain and codomain of the linear transformations.

> **Solution:** Both of the compositions make sense since $\operatorname{dom} T_A = \operatorname{codom} T_B$ and $\operatorname{dom} T_B = \operatorname{codom} T_A$. The matrices associated are
>
> $$T_A \circ T_B \to A \cdot B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 3 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$
>
> $$T_B \circ T_A \to B \cdot A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$$
>
> In particular $T_A \circ T_B : \mathbb{R}^4 \to \mathbb{R}^4$ and $T_B \circ T_A : \mathbb{R}^2 \to \mathbb{R}^2$.

(c) [2 points] Is the transformation $T_A \circ T_B$ invertible? Why or why not? (hint: you don't need to compute the inverse)

> **Solution:** We can see that $A \cdot B$ contains two equal columns, in particular the columns are not linearly independent. This implies that $T_A \circ T_B$ is not 1-to-1 and hence it cannot be invertible.

(d) [3 points] What is the image of the triangle with vertices $(0, 0)$ $(1, 1)$ and $(0, 1)$ under the transformation T_C where $C = (B \cdot A)^T$?

> **Solution:** We know that linear transformations send segments to segments, hence the image of the triangle will be another triangle whose vertices are the images of the

original vertices under the transformation. We need first to compute the matrix C.

$$C = (B \cdot A)^T = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}^T = \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix}$$

Now we can compute the image of the vertices as follows: first notice that the zero vector is always sent to the zero vector. For the other vertices we have

$$T_C \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad T_C \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

So the image of the triangle is the triangle with vertices $(0,0)$ $(4,3)$ and $(0,2)$.

Question 4. (12 points) Consider the following subspace of $\mathbb{R}^3$

$$S = \text{span}\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix} \}$$

(a) [6 points] Compute a basis $\mathcal{B}$ for S.

Solution: One can see immediately that the second vector is equal to the third and the fifth vector is the sum of the first and the fourth. So we expect the basis to be formed by the first, second and fourth vector. This can be seen by reducing to echelon form the following matrix

$$\begin{pmatrix} 1 & -2 & -2 & 3 & 4 \\ 1 & -1 & -1 & 2 & 3 \\ 1 & -1 & -1 & 3 & 4 \end{pmatrix} \xrightarrow[\text{III}-\text{I}]{\text{II}-\text{I}} \begin{pmatrix} 1 & -2 & -2 & 3 & 4 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow[\text{III}-\text{II}]{\text{I}+2\text{II}}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[\text{II}+\text{III}]{\text{I}-\text{III}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

From the computation we can see that the columns of the REF containing leading terms are the first, the second and the fourth. Therefore a basis for S is given by the corresponding columns of the original matrix, i.e.

$$\mathcal{B}_S = \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} \}$$

(b) [2 points] What is the dimension of S? Is $\mathcal{B}$ a basis for $\mathbb{R}^3$?

Solution: $\dim(S) = 3$ since $\mathcal{B}$ contains three vectors. This implies that $S = \mathbb{R}^3$ and $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

(c) [3 points] Let A be the matrix

$$A = \begin{pmatrix} 1 & -2 & -2 & 3 & 4 \\ 1 & -1 & -1 & 2 & 3 \\ 1 & -1 & -1 & 3 & 4 \end{pmatrix}$$

and T_A the corresponding linear transformation. What is the dimension of $\ker T_A$? (hint: what is the relationship between A and S?)

Solution: Since $\ker T_A = \text{null}(A)$ the dimension of the kernel is equal to $\text{nullity}(A)$. On the other hand we already computed the dimension of $\text{col}(A) = S$ which implies that $\text{rank}(A) = \dim \text{col}(A) = \dim S = 3$. Therefore we can conclude, by the rank-nullity theorem that

$$\text{nullity}(A) = 5 - \text{rank}(A) = 5 - 3 = 2$$

This shows that $\dim \ker T_A = 2$.

(d) [1 point] Given your answer at the previous point, is T_A 1-to-1? Why?

Solution: No, T_A is not 1-to-1 since $\ker T_A \neq \{\underline{0}\}$ because $\dim \ker T_A > 0$.

Question 5. (16 points) Consider the following matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 4 & -1 \\ 1 & 1 & -2 & 0 & 1 \\ 2 & 0 & 1 & 3 & 5 \\ 1 & 1 & -1 & 1 & 1 \end{pmatrix}$$

(a) [8 points] Compute bases for row(A) and col(A).

Solution: To compute the bases we reduce the matrix to echelon form as follows

$$\begin{pmatrix} 1 & 2 & 1 & 4 & -1 \\ 1 & 1 & -2 & 0 & 1 \\ 2 & 0 & 1 & 3 & 5 \\ 1 & 1 & -1 & 1 & 1 \end{pmatrix} \xrightarrow[\substack{\mathrm{III}-2\mathrm{I} \\ \mathrm{IV}-\mathrm{I}}]{\mathrm{II}-\mathrm{I}} \begin{pmatrix} 1 & 2 & 1 & 4 & -1 \\ 0 & -1 & -3 & -4 & 2 \\ 0 & -4 & -1 & -5 & 7 \\ 0 & -1 & -2 & -3 & 2 \end{pmatrix} \xrightarrow[\substack{-\mathrm{II} \\ -\mathrm{III}}]{-\mathrm{IV}} \begin{pmatrix} 1 & 2 & 1 & 4 & -1 \\ 0 & 1 & 3 & 4 & -2 \\ 0 & 4 & 1 & 5 & -7 \\ 0 & 1 & 2 & 3 & -2 \end{pmatrix}$$

$$\xrightarrow[\substack{\mathrm{III}-4\mathrm{II} \\ \mathrm{IV}-\mathrm{II}}]{\mathrm{I}-2\mathrm{II}} \begin{pmatrix} 1 & 0 & -5 & -4 & 3 \\ 0 & 1 & 3 & 4 & -2 \\ 0 & 0 & -11 & -11 & 1 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix} \xrightarrow[\substack{-\mathrm{IV} \\ \mathrm{III}\leftrightarrow\mathrm{IV}}]{-\mathrm{III}} \begin{pmatrix} 1 & 0 & -5 & -4 & 3 \\ 0 & 1 & 3 & 4 & -2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 11 & 11 & -1 \end{pmatrix} \xrightarrow[\substack{\mathrm{II}-3\mathrm{III} \\ \mathrm{IV}-11\mathrm{III}}]{\mathrm{I}+5\mathrm{III}}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow[\substack{\mathrm{II}-2\mathrm{IV} \\ -\mathrm{IV}}]{\mathrm{I}+3\mathrm{IV}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then a basis for row A is given by the non-zero rows of the REF, while a basis for col(A) is given by the rows of A corresponding to the columns in the REF having a leading term. This implies that two basis are

$$\mathcal{B}_{\mathrm{row}(A)} = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \} \qquad \mathcal{B}_{\mathrm{col}(A)} = \{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 5 \\ 1 \end{pmatrix} \}$$

(b) [2 points] What is rank(A)? What is nullity(A)?

Solution: Since rank(A) is the dimension of col(A) (or equivalently of row(A)) we know by the previous computation that rank(A) = 4. By the rank nullity theorem this implies that

$$\mathrm{nullity}(A) = \#\text{ cols of } A - \mathrm{rank}(A) = 5 - 4 = 1$$

(c) [3 points] Denoted by T_A the linear transformation associated to A (i.e. $T_A(x) = A \cdot x$), is T_A 1-to-1? Is it onto? Is it invertible?

Solution: T_A is 1-to-1 is equivalent to $\ker T_A = \{\underline{0}\}$. Moreover $\ker T_A = \mathrm{null}(A)$. From the previous point we know that $\dim \mathrm{null}(A) = \mathrm{nullity}(A) = 1 > 0$ so $\mathrm{null}(A) \neq \{\underline{0}\}$ which shows that $\ker T_A \neq \{\underline{0}\}$ which implies that T_A is not 1-to-1. T_A is onto is equivalent to $\mathrm{range}\, T_A = \mathrm{codom}\, T_A = \mathbb{R}^4$. Moreover $\mathrm{range}\, T_A = \mathrm{col}(A)$. From the previous point we know that $\dim \mathrm{col}(A) = \mathrm{rank}(A) = 4$ hence $\mathrm{col}(A) = \mathbb{R}^4$ (since the only subspace of dimension 4 inside $\mathbb{R}^4$ is $\mathbb{R}^4$ itself). This shows that $\mathrm{range}\, T_A = \mathbb{R}^4$ which implies that T_A is onto. Since T_A is not 1-to-1 it is not invertible.

(d) [1 points] How many elements will a basis of $\mathrm{null}(A)$ contain? (you don't need to do any computations for this)

Solution: Since $\mathrm{nullity}(A) = 1$ we know that $\dim \mathrm{null}(A) = 1$ hence every basis of $\mathrm{null}(A)$ will contain precisely one element.

(e) [2 points] Compute a basis for $\mathrm{null}(A)$ (hint: use the computation you already perform in part 1).

Solution: $\mathrm{null}(A)$ is the solution set of the homogeneous linear system associated to A. Since we already compute the REF of A we know that the system is equivalent to

$$\begin{cases} x_1 + x_4 = 0 \\ x_2 + x_4 = 0 \\ x_3 + x_4 = 0 \\ x_5 = 0 \end{cases}$$

Setting $x_4 = s$ the (unique) free variable we see that

$$\mathrm{null}(A) = \{ \begin{pmatrix} -s \\ -s \\ -s \\ s \\ 0 \end{pmatrix} : s \in \mathbb{R}\} = \mathrm{span}\{ \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \}$$

This is also a basis since the vector is not zero.