Final - Math 308G

14th December 2016

NAME (last - first): ________________________________

- Do not open this exam until you are told to begin. You will have 1 hour and 50 minutes for the exam.
- This final contains 7 questions for a total of 100 points in 13 pages.
- You are allowed to have one double sided, handwritten note sheet and a non-programmable calculator.
- Show all your work. With the exception of True/False questions, if there is no work supporting an answer (even if correct) you will not receive full credit for the problem.

Question	Points	Score
1	15	
2	15	
3	15	
4	10	
5	10	
6	20	
7	15	
Total:	100	

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ Date: ______________

Question 1. (15 points) Decide whether the following statements are true or false. For this question you don't need to show any work.

(a) [1 point] If the columns of a matrix A span the codomain of the associated linear transformation T_A, then T_A is 1-to-1.

○ True ○ False

(b) [1 point] If a matrix A is invertible then A^T is invertible.

○ True ○ False

(c) [1 point] If $\mathcal{B}$ is a basis for a subspace S then $\vec{0} \in \mathcal{B}$.

○ True ○ False

(d) [1 point] If A and B are equivalent $m \times n$ matrices then $A \cdot e_1 = B \cdot e_1$.

○ True ○ False

(e) [1 point] Let S be a subspace of $\mathbb{R}^n$ of dimension m and let $\mathcal{A}$ and $\mathcal{B}$ be two basis of S. Then the matrix of change of basis $M_{\mathcal{A},\mathcal{B}}$ is a square $m \times m$ matrix.

○ True ○ False

(f) [1 point] If B is the reduced echelon form of a square matrix A then $\det A = \det B$.

○ True ○ False

(g) [1 point] If v is an eigenvector of a matrix A, then $A \cdot v \in \operatorname{span}\{v\}$.

○ True ○ False

(h) [1 point] If v is a linear combination of $u_1, \ldots, u_m$ and $A = [u_1, \ldots, u_m]$, then $v \in \operatorname{range}(T_A)$.

○ True ○ False

(i) [1 point] If λ is an eigenvalue of A with multiplicity m, then $\dim E_\lambda(A) \geq m$.

○ True ○ False

(j) [1 point] If T_A is 1-to-1 then $\operatorname{nullity}(A) = 0$.

○ True ○ False

(k) [1 point] If $\det(A) = \det(B)$ then A and B are equivalent matrices.

○ True ○ False

(l) [1 point] If S is a subspace of $\mathbb{R}^n$ then the only vector contained both in S and $S^\perp$ is the zero vector.

○ True ○ False

(m) [1 point] If λ_i is an eigenvalue of a matrix A, then $p_\lambda(1 - \lambda_i) = 0$.

○ True ○ False

(n) [1 point] If λ is an eigenvalue of A, then $\operatorname{nullity}(A - \lambda I) = 0$.

○ True ○ False

(o) [1 point] If $S = \operatorname{span}\{v\}$ and $u \in S^\perp$ then $v \in \operatorname{span}\{u\}^\perp$.

○ True ○ False

Question 2. (15 points) Prove the following propositions.

(a) [5 points] Let A be a $n \times n$ matrix. Prove that if A is invertible then A^{-1} is invertible and

$$\det A^{-1} = \frac{1}{\det A}.$$

(b) [5 points] Prove that if λ is an eigenvalue of A then $1/\lambda$ is an eigenvalue of A^{-1}.[Hint: start by writing down what does it mean for λ to be an eigenvalue and manipulate the expression.]

(c) [5 points] Using the previous point prove that if A is diagonalizable and invertible then A^{-1} is also diagonalizable. [Hint rephrase the previous part using eigenvectors instead of eigenvalues]

Question 3. (15 points) Let A be the following matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & 0 & -2 \end{pmatrix}$$

(a) [2 points] Without doing any computation, what is $\det(A)$? Can you find an eigenvalue of A?

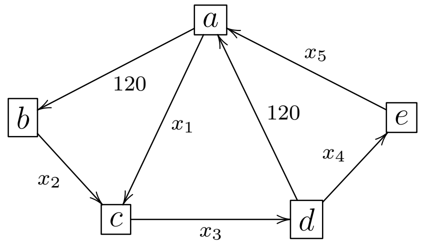
(b) [4 points] Find all the eigenvalues of A and their multiplicities.

(c) [5 points] Compute bases for all the eigenspaces of A.

(d) [3 points] Is A diagonalizable? If yes write the invertible matrix P and the diagonal matrix D such that $A = P \cdot D \cdot P^{-1}$ and exhibit a basis $\mathcal{B}$ of $\mathbb{R}^4$ made of eigenvectors of A.

(e) [1 point] Let $\mathcal{B}$ be the basis of $\mathbb{R}^4$ of the previous part. Is $\mathcal{B}$ an orthogonal basis of $\mathbb{R}^4$?

Question 4. (10 points) Consider the following graph, modelling water pipes and their junction in a home water network.



Anywhere that pipes meet, the total amount of water coming into that junction must be equal to the amount going out, otherwise we would quickly run out of water, or we would have a buildup of water. The junction (c) has just been added to the network and you are been assigned to compute the total amount of water (in gallons) that should run in each pipe to balance the existing junctions, i.e. in every junction the total amount of water entering the junction should equal the total amount of water exiting the junction.

(a) [3 points] Write down the linear system associated to the problem of finding x_1, x_2, x_3, x_4 and x_5, and its associated augmented matrix.

(b) [4 points] Reduce the matrix to echelon form and find the solution set of the system.

(c) [1 points] How many solutions does the linear system have? What is the dimension of the solution set?

(d) [2 points] If your system has free parameters, what value of the parameters make sense for the problem? What are the values of the parameters that make x_1 have the minimum possible value?

Question 5. (10 points) Let T_A and B be respectively the following linear transformation and matrix

$$T_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 - 2x_4 \\ -x_1 - 3x_2 + x_3 + 2x_4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

(a) [2 points] Write the matrix A such that $T_A(x) = A \cdot x$, and identify domain and codomain of both T_A and T_B, the linear transformation associated to B.

(b) [4 points] Decide which of the compositions $T_A \circ T_B$ and $T_B \circ T_A$ make sense, and for those that make sense, determine domain codomain and compute the associated matrices.

(c) [2 points] One can check (but you don't have to!) that

$$\text{null}(B \cdot A) = \text{span}\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \}$$

Given this information, is $T_B \circ T_A$ 1-to-1, onto and/or invertible? Is 0 and eigenvalue of $B \cdot A$? Why? (You don't need to do any computation to check this).

(d) [2 points] Which particular linear transformation of the plane is $T_A \circ T_B$ (dilation, rotation, etc.)? Is it 1-to-1, onto and/or invertible? Sketch the image under $T_A \circ T_B$ of the triangle with vertices $(0,0), (1,0)$ and $(0,1)$.

Question 6. (20 points) Given a matrix A we know that the following 5 vectors form a basis of $\mathbb{R}^5$ of eigenvectors of A (with the corresponding eigenvalues):

$$\begin{array}{cccccc}
 & \begin{pmatrix}1\\0\\0\\-1\\0\end{pmatrix} & \begin{pmatrix}0\\1\\1\\2\\2\end{pmatrix} & \begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\1\\1\\0\\1\end{pmatrix} \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\text{eigenvalues} & 1 & 2 & 2 & 0 & -1
\end{array}$$

(a) [2 points] Write down a basis for the null space of A. Is T_A 1-to-1, onto and/or invertible?

(b) [3 points] Without doing any computation, what is the rank of $A - 2I_5$? [Hint: what is the null space of that matrix?]

(c) [5 points] Is A diagonalizable? If yes find the invertible matrix P and the diagonal matrix D such that $A = P \cdot D \cdot P^{-1}$.

(d) [5 points] Find all the vectors $x \in \mathbb{R}^5$ such that $T_A(x) = -x$. [Hint: rewrite $T_A(x) = -x$ using the matrix A and the fact that $-x = -I_5 \cdot x$].

(e) [5 points] Let S be the subspace of $\mathbb{R}^5$ spanned by the first, third and fourth vector of the given basis of $\mathbb{R}^5$, i.e.

$$S = \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \}$$

Find an orthogonal basis for S and compute a basis of $S^{\perp}$.

Question 7. (15 points) Let S be the span of the following vectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad u_4 = \begin{pmatrix} 3 \\ 0 \\ -3 \\ 0 \end{pmatrix} \qquad u_5 = \begin{pmatrix} 5 \\ -1 \\ -6 \\ -2 \end{pmatrix}$$

(a) [4 points] Find a basis $\mathcal{B}$ of S and compute its dimension.

(b) [3 points] Let A be the matrix whose columns are $u_1, \ldots, u_5$. Is 0 an eigenvalue of A? If yes compute a basis of the corresponding eigenspace.

(c) [4 points] Let $\mathcal{A}$ be another basis of S (you don't need to check this) given by

$$\mathcal{A} = \{\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$$

Compute the coordinates of u_1, u_2 and u_3 with respect to the basis $\mathcal{A}$ and the matrix of change of basis $M_{\mathcal{A},\mathcal{B}}$.

(d) [3 points] Compute the matrix $M_{\mathcal{B},\mathcal{A}}$ and the coordinates of α_1, α_2 and α_3 with respect to the basis $\mathcal{B}$.

(e) [1 point] What is $(M^T_{\mathcal{B},\mathcal{A}})^{-1}$.