Name: ____

Math 308
Autumn 2016
MIDTERM - 2
11/18/2016

Instructions: The exam is **9** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to **50** points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		10
2		10
3		10
4		10
5		10
TOTAL		50

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:**______________

Question 1. (10 points) Decide whether the following statements are true or false. For this you don't need to show any work (but you will need to justify the answers of the other questions).

(a) [1 point] If $A{\cdot}x = b$ is consistent, then $b \in \text{range}\,(T_A)$ where T_A is the linear transformation $x \mapsto A \cdot x$.

$\surd$ **True** ◯ False

(b) [1 point] The function $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T\left(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_1 x_2 \end{smallmatrix}\right]$ is a linear transformation.

◯ True $\surd$ **False**

(c) [1 point] If A is a $m \times n$ matrix, then $\text{nullity}(A) \leq n$

$\surd$ **True** ◯ False

(d) [1 point] If $\mathcal{B}$ is a basis for a subspace S, and $u, v \in \mathcal{B}$ then $u + v \in \mathcal{B}$.

◯ True $\surd$ **False**

(e) [1 point] If S is a subspace of $\mathbb{R}^n$, then any basis of S consists of at least n elements.

◯ True $\surd$ **False**

(f) [1 point] The set $\left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right] \in \mathbb{R}^4 : 2x_1 = x_3 + 1 \right\}$ is a subspace of $\mathbb{R}^3$.

◯ True $\surd$ **False**

(g) [1 point] For any matrix A, the transpose of A exists.

$\surd$ **True** ◯ False

(h) [1 point] If B is the reduced echelon form of A then T_A and T_B, their associated linear transformations, are equal.

◯ True $\surd$ **False**

(i) [1 point] If a linear transformation is 1-to-1 then it is also onto.

◯ True $\surd$ **False**

(j) [1 point] If a square matrix A is invertible then A^2 is also invertible.

$\surd$ **True** ◯ False

Question 2. (10 points) Let A be an $n \times n$ matrix. Answer the following questions without using the Big Theorem (you may, however, use other theorems from the text).

(a) [5 points] Prove that $rank(A) = n$ if and only if $nullity(A) = 0$.

(b) [5 points] Assuming either $rank(A) = n$ *or* $nullity(A) = 0$, prove that T_A, the linear transformation associated to A, is onto.

Solution: First we prove part (a).

($rank(A) = n \Rightarrow nullity(A) = 0$): Suppose that $rank(A) = n$. Using the Rank-Nullity Theorem we know that $n + nullity(A) = n$, so $nullity(A) = 0$.

($nullity(A) = 0 \Rightarrow rank(A) = n$): Suppose that $nullity(A) = n$. Using the Rank-Nullity Theorem we know that $rank(A) + 0 = n$, so $rank(A) = n$.

For part (b) we can assume either of the above statements, but it's much easier if we assume that $rank(A) = n$. Remember that $range(T_A)$ is equal to the span of the columns of A, and the dimension of the span of the columns of A is precisely $rank(A) = n$. Thus the range of T_A is n-dimensional and in the codomain $\mathbb{R}^n$, so much be *all* of the codomain.

Question 3. (10 points) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & -2 \end{bmatrix}$ and let T_B be the linear transformation given by $T_B\left(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x_2 \\ x_1 \\ x_1 \end{smallmatrix}\right]$. Denote the linear transformation associated to A by T_A.

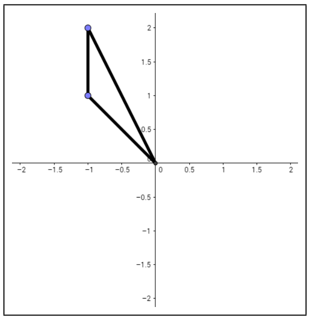
(a) [2 points] Find the matrix associated to T_B and the domains and codomains of T_A and T_B.

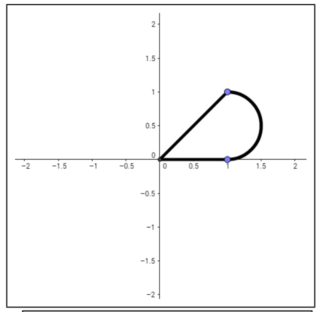
Solution: The matrix associated to T_B is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. Clearly T_B takes as input a vector of $\mathbb{R}^2$ and has as output a vector of $\mathbb{R}^3$, hence the domain of T_B is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$. Since A is a 2×3 matrix, the associated linear transformation T_A has domain $\mathbb{R}^3$ and codomain $\mathbb{R}^2$.

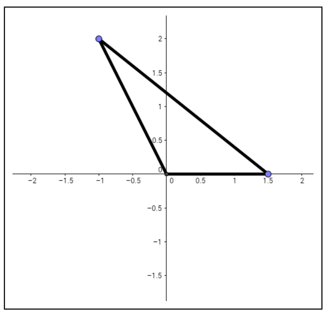
(b) [4 points] Write down matrices associated to the linear transformations $(T_A \circ T_B)$ and $(T_B \circ T_A)$. If no such matrix exists, say why.

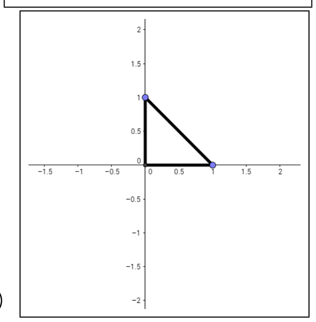
Solution: By multiplying B with A on each side we get that the matrix for $T_A \circ T_B$ is $\left[\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right]$ and the matrix for $T_B \circ T_A$ is $\left[\begin{smallmatrix} -1 & 0 & -2 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{smallmatrix}\right]$.

(c) [4 points] Apply the linear transformation $(T_A \circ T_B)^{-1}$ to the triangle in the plane with vertices at the points $(0,0)$, $(0,1)$ and $(1,0)$. Check the circle next to the picture that best represents the resulting shape. Explain your reasoning in the space below.

$\surd$ 

◯ 

◯ 

◯ 

Solution:
First we should determine what $(T_A \circ T_B)^{-1}$ is. By using the previous answer, we know that it's given by multiplying by the inverse of $\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$. The inverse of that matrix is $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$.
Recalling that a linear transformation always takes lines to lines, we know that the image of the source triangle must be one of the three triangles depicted (i.e. not the shape with the curved edge). Also, this tells us that we can just apply the linear transformation to each of the vertices to determine the answer, which we can do by just multiplying the above matrix by the vectors $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It's not hard to see that three resulting vectors are $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (notice that these are just the columns of the matrix representing the linear transformation). These points are the vertices of only the upper left triangle, so that must be the result.

Question 4. (10 points) Let A be the following matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 3 & 2 & 1 & 2 \\ 1 & 3 & 3 & 2 & 1 \\ 2 & 0 & 4 & 2 & -8 \end{pmatrix}$$

(a) [4 points] Compute a basis for row(A) and a basis for col(A).

Solution: To compute the basis for both the row space and the column space we reduce A to echelon form

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 3 & 2 & 1 & 2 \\ 1 & 3 & 3 & 2 & 1 \\ 2 & 0 & 4 & 2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & -4 & 2 & 2 & -10 \end{pmatrix} \sim$$
$$\begin{pmatrix} 1 & 0 & -1 & -2 & -1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From the reduced echelon form we can see that the non zero rows are a basis for row(A) and the columns of A corresponding to columns wiht leading terms are a basis for col(A), i.e.

$$\mathcal{B}_{\text{row}(A)} = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \} \qquad \mathcal{B}_{\text{col}(A)} = \{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \}$$

(b) [2 points] What are the rank and the nullity of A?

Solution: The rank of A is equal to the dimension of col(A) (or equivalently of row(A)) so it follows from the previous that rank(A) = 3.
By the rank-nullity Theorem we know that rank(A)+nullity(A) = number of columns of A. This implies that nullity(A) = $5 - 3 = 2$.

(c) [2 points] Compute a basis for the null space of A.

Solution: By the computation of the reduced echelon form we know that the null space is equivalent to the solution set of the system

$$\begin{cases} x_1 - x_4 - 2x_5 = 0 \\ x_2 + 2x_5 = 0 \\ x_3 + x_4 - x_5 = 0 \end{cases}$$

Since x_4 and x_5 are free variables, we set $x_4 = s_1$ and $x_5 = s_2$ and we get that the solution set is

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = s_1 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s_2 \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} : s_1, s_2 \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(d) [2 points] Let T_A be the associated linear transformation, i.e. $T_A(x) = A \cdot x$. Identify the domain and codomain of T_A and decide whether T_A is 1-to-1, onto and/or invertible.

Solution: Since A is 4×5 we have that $T_A : \mathbb{R}^5 \to \mathbb{R}^4$. The fact that $\text{nullity}(A) > 0$ implies that $\ker(T_A)$ contains more vectors than the zero vector, hence T_A is not 1-to-1. On the other hand, $\text{range}(T_A) = \text{col}(A)$ is a subspace of $\mathbb{R}^5$ of dimension 3, therefore $\text{range}(T_A) \neq \text{codom}(T_A) = \mathbb{R}^5$, so T_A is not onto. In particular T_A is not invertible.

Question 5. (10 points) Let S be $\text{span}\{u_1, u_2, u_3, u_4, u_5\}$ where

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix} \qquad u_5 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

(a) [3 points] Find a basis for S.

Solution: To find a basis we construct a matrix whose columns are the vectors $u_1, \ldots, u_5$ and we reduced it to echelon form.

$$\begin{pmatrix} 1 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 0 & 3 \\ 2 & 1 & 1 & 3 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & -2 & -4 & 1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & -2 & -4 & 1 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The vectors in the first three columns, i.e. u_1, u_2, u_3 are linearly independent, since the corresponding columns in the reduced echelon form contain a leading term, while the other two vectors are linearly dependent. Therefore a basis for S is given by $\{u_1, u_2, u_3\}$.

(b) [2 points] What is the dimension of S? Is the basis you found a basis for $\mathbb{R}^4$?

Solution: Since we showed that a basis for S is given by $\{u_1, u_2, u_3\}$ the dimension of S is 3. On the other hand, since $\dim \mathbb{R}^4 = 4$ we see that this is not a basis of $\mathbb{R}^4$.

(c) [1 points] Let A be the matrix whose columns are the vectors of the basis you found in the previous part. What can you say about the nullity of A without computing the null space?

Solution: Since the columns of A are linearly independent we know that $\text{rank}(A) = 3$. It follows from the Rank-Nullity Theorem that $\text{nullity}(A) = 3 - \text{rank}(A) = 3 - 3 = 0$.

(d) [4 points] Let $L = \text{span}\, e_1 \subset \mathbb{R}^3$ be the line corresponding to the x-axis. What is $T_A(L)$, the image of L under the linar transformation T_A, where as usual $T_A : x \to A \cdot x$?

Solution: Every element of L is of the form $r \cdot e_1$. Since T_A is a linear transformation we know that $T_A(r \cdot e_1) = r \cdot T_A(e_1)$. Moreover $T_A(e_1) = A \cdot e_1$ is the first column of the matrix A, i.e. u_1. This implies that

$$T_A(r \cdot e_1) = r \cdot T_A(e_1) = r \cdot u_1.$$

In particular every element in L is sent to a multiple of u_1. This shows that $T_A(L) = \text{span}(u_1)$.