Practice Midterm 2

for Math 308-G, Autumn 2016

Total number of questions: 5

Total number of points: 50

Question 1. (10 points) Decide whether the following statements are true or false. For this you don't need to show any work (but for practicing you might want to try to do that).

(a) [1 point] If $A \cdot x = b$ is consistent, then $b \in \operatorname{range} T_A$ where T_A is the linear function $x \mapsto A \cdot x$.

$\surd$ **True** ◯ False

(b) [1 point] A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ does not change the length of the vectors, i.e. x and $T(x)$ have the same length for every $x \in \mathbb{R}^2$.

◯ True $\surd$ **False**

(c) [1 point] If $\mathcal{B}$ is a basis for a subspace S, then $\vec{0} \in \mathcal{B}$.

◯ True $\surd$ **False**

(d) [1 point] If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $\operatorname{range}(T) = \mathbb{R}^m$ then T is onto.

$\surd$ **True** ◯ False

(e) [1 point] If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is the linear function $x \to A \cdot x$ then A is a $n \times 1$ matrix.

◯ True $\surd$ **False**

(f) [1 point] If S is a subspace of dimension k then every set of m vectors of S with $m > k$ span S.

◯ True $\surd$ **False**

(g) [1 point] If A, B, C are $n \times n$ matrices and $A \cdot C = B \cdot C$, then $A = B$.

◯ True $\surd$ **False**

(h) [1 point] An upper triangular matrix is symmetric.

◯ True $\surd$ **False**

(i) [1 point] If $\mathcal{B}$ is a basis for a subspace S then $\dim S$ is bigger than 1.

◯ True $\surd$ **False**

(j) [1 point] The kernel of a linear transformation needs to contain the zero vector.

$\surd$ **True** ◯ False

Question 2. (10 points) Let A be a $n \times n$ matrix and let T_A be the corresponding linear transformation defined by $x \mapsto A \cdot x$. Prove that the following two statements are equivalent:

(a) T_A is 1-to-1;

(b) $\ker(T_A) = \{\vec{0}\}$.

Solution:

(a)⇒(b) Since T_A is a linear transformation we know that $\vec{0} \in \ker(T_A)$, so we have to show that any other vector in the kernel is equal to the zero vector. Suppose $\vec{v}$ is in $\ker(T_A)$. Then, since the kernel is the set $\{x : T_A(x) = \vec{0}\}$ we know that $T_A(\vec{v}) = \vec{0}$. But we also know that $T_A(\vec{0}) = \vec{0}$. So we found that there are two vectors $\vec{v}$ and $\vec{0}$ that have the same image. Since T_A is 1-to-1 this can happen only if $\vec{v} = \vec{0}$. So we proved that every vector $\vec{v}$ in the kernel of T_A is equal to the zero vector.

(b)⇒(a) We have to show that T_A is 1-to-1. This means that for every two vectors $\vec{v}$ and $\vec{w}$ if $T_A(\vec{v}) = T_A(\vec{w})$ then we must have $\vec{v} = \vec{w}$. Notice that the last assertion is equivalent to $\vec{v} - \vec{w} = \vec{0}$. Let's call $c = T_A(\vec{v}) = T_A(\vec{w})$. To prove that T_A is 1-to-1 we compute the image of $\vec{v} - \vec{w}$ as follows:

$$T_A(\vec{v} - \vec{w}) = T_A(\vec{v}) - T_A(\vec{w}) = c - c = \vec{0}$$

This implies that $\vec{v} - \vec{w}$ is in the kernel of T_A. But we are assuming that $\ker(T_A)$ consists of the zero vector alone. So $\vec{v} - \vec{w} = \vec{0}$ which is equivalent to $\vec{v} = \vec{w}$. We have shown that every two vectors that have the same image under T_A needs to be the same vector, thus proving that T_A is 1-to-1.

Question 3. (10 points) Let A be the following matrix:

$$A = \begin{pmatrix} 2 & 4 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and let T_A be the corresponding linear transformation, $T_A : x \mapsto A \cdot x$.

(a) [4 points] Determine a basis for $\text{null}(A)$.

Solution: $\text{null}(A)$ is the set of solutions of the homogeneous linear system $A \cdot x = 0$. To solve it we reduce A in echelon form obtaining

$$A = \begin{pmatrix} 2 & 4 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we see that the only solution of $A \cdot x = 0$ is the zero vector. Hence $\text{null}(A) = \{\vec{0}_{\mathbb{R}^4}\}$.

(b) [4 points] Determine a basis for $\text{range}(T_A)$.

Solution: $\text{range}(T_A)$ is spanned by the columns of A. By the previous computation we see that the column of A are linearly independent - since in reduced echelon form every column contains a leading term. Therefore

$$\text{range}(T_A) = \text{span}\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \}$$

(c) [1 point] Is T_A 1-to-1 and/or onto? Why?

Solution: By the first part we know that T_A is 1-to-1 since the kernel consists only of the zero vector. Form the second part we know that $\dim \text{range}(T_A) = 4$ since a basis has cardinality 4. This implies that T_A is not onto since the codomain is $\mathbb{R}^5$ and has dimension 5, therefore the range cannot equal the codomain.

(d) [1 point] What is $\text{rank}(A)$? And the nullity of A?

Solution: The rank(A) is the dimension of the row space or of the column space. Since the columns of A are linearly independent the column space has dimension 4 and therefore $\text{rank}(A) = 4$. The nullity is the dimension of $(A) = \ker(T_A)$ and since $\text{null}(A) = \{\vec{0}\}$ we have that the nullity is 0. This shows another time that

$$\text{rank}(A) + \text{nullity}(A) = \dim \text{dom}(T_A) = \dim \mathbb{R}^4 = 4$$

Question 4. (10 points) Let T be the linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + 2x_3 + x_4 \\ 2x_1 + 3x_2 + 4x_3 + 3x_4 \\ x_2 + x_3 + x_4 \\ x_1 + x_2 + x_3 \end{pmatrix}$$

(a) [2 points] Identify domain and codomain of T and compute the matrix A such that $T(x) = A \cdot x$.

Solution: From the formula given above we see that $T : \mathbb{R}^4 \to \mathbb{R}^4$ and

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

(b) [2 points] Suppose that the reduced echelon form of A is equal to the identity matrix I_4. Is T_A 1-to-1 and/or onto? What is the rank and the nullity of A?

Solution: If the reduced echelon form of A is the identity we know that the columns of A are linearly independent and they span $\mathbb{R}^4$. This implies that T_A is both 1-to-1 and onto. Moreover the dimension of the column space of A is 4, and therefore $\operatorname{rank}(A) = 4$. Since $\operatorname{rank}(A) + \operatorname{nullity}(A)$ is equal to the dimension of the codomain, which is 4 in our case, we know that the nullity of A is $4 - \operatorname{rank}(A) = 0$.

(c) [6 points] In the assumption of the previous part determine if T_A is invertible, and if it possible compute the matrix associated to the inverse transformation.

Solution: Since T_A is 1-to-1 and onto it is invertible and the matrix associated to the inverse is A^{-1}. To compute we perform the following operations.

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 \end{array}\right)$$

So

$$A^{-1} = \begin{pmatrix} -1 & 1 & -2 & 0 \\ -1 & 0 & 1 & 1 \\ 2 & -1 & 1 & 0 \\ -1 & 1 & -1 & -1 \end{pmatrix}$$

Question 5. (10 points) Let A and B be the following matrices

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 3 & 3 \\ 0 & 0 \end{pmatrix}$$

(a) [2 points] Identify the domain and the codomain of the linear transformations T_A and T_B associated to A and B.

Solution:

$$T_A : \mathbb{R}^5 \to \mathbb{R}^3 \qquad T_B : \mathbb{R}^2 \to \mathbb{R}^5$$

(b) [4 points] For all the compositions $T_A \circ T_B$ and $T_B \circ T_A$ explain whether they make sense and if they do, compute the matrix associated.

Solution: The only composition possible is $T_A \circ T_B$ since the codomain of T_B coincide with the domain of T_A. In this case the matrix associated is

$$A \cdot B = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 3 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 3 \\ 4 & 4 \end{pmatrix}$$

(c) [4 points] Compute whether the previous composition(s) are 1-to-1 and/or onto.

Solution: The columns of $A \cdot B$ are linearly independent, since they are neither zero nor multiples of each other. This implies that the columns of the matrix are a basis for the column space, and therefore the column space has dimension 2. Since the codomain is $\mathbb{R}^3$ this implies that the composition is not onto. The reduced echelon form of $A \cdot B$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

hence the homogeneous linear system $(A \cdot B)x = \vec{0}$ has only the trivial solution. This implies that the nullity of $A \cdot B$ is 0 and hence the composition is 1-to-1.