Practice Final

for Math 308-G, Autumn 2016

Total number of questions: 8

Total number of points: 175

Disclaimer: this is a collection of problems that could help you prepare for the Final. It is not intended to be representative of a Final. In particular there are more questions and more points that you will encounter in the "real" final.
The last two problems focus on an application of linear algebra to graphs, and the extension of the notion of change of basis to linear transformation. They can be a bit more hard then the other problems.

Question	Points	Score
1	20	
2	50	
3	10	
4	10	
5	20	
6	20	
7	15	
8	30	
Total:	175	

Question 1. (20 points) Decide whether the following statements are true or false. For this you don't need to show any work (but for practicing you might want to try to do that).

(a) [1 point] If the columns of a matrix A are linearly independent, then T_A the associated linear transformation, is onto.

○ True √ **False**

(b) [1 point] If λ_1 and λ_2 are two different eigenvalues of A and $v \in E_{\lambda_1}(A)$ $w \in E_{\lambda_2}(A)$ then v and w are linearly independent

√ **True** ○ False

(c) [1 point] If $\mathcal{B}$ is a basis for a subspace S and $u_1 \in \mathcal{B}$ then any multiple of u_1 is in $\mathcal{B}$.

○ True √ **False**

(d) [1 point] If $T : \mathbb{R}^n \to \mathbb{R}^m$ and A is the matrix such that $T = T_A$ then A has n rows.

○ True √ **False**

(e) [1 point] If A and B are equivalent $m \times n$ matrices then for any vector $v \in \mathbb{R}^n$, $A \cdot v = B \cdot v$

○ True √ **False**

(f) [1 point] If S is a subset of $\mathbb{R}^n$ of dimension m and $v \in S$, then for every basis $\mathcal{B}$ of S, $[v]_{\mathcal{B}}$ is a vector with m components.

√ **True** ○ False

(g) [1 point] If B is the reduced echelon form of a square matrix A then $\det A = \det B$.

○ True √ **False**

(h) [1 point] If v is an eigenvector of a matrix A, then $A \cdot v$ and v are linearly independent.

○ True √ **False**

(i) [1 point] If A is $n \times n$ matrix with n distinct eigenvalues, then there exists a basis of $\mathbb{R}^n$ made of eigenvectors of A.

√ **True** ○ False

(j) [1 point] If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has $\ker T = \{\vec{0}\}$ then for every $b \in \text{range}(T)$ there is exactly one $v \in \mathbb{R}^n$ such that $T(v) = b$.

√ **True** ○ False

(k) [1 point] If A is $n \times n$ matrix with n distinct eigenvalues, then there is *no* basis of $\mathbb{R}^n$ made of eigenvectors of A.

○ True √ **False**

(l) [1 point] If 0 is an eigenvalue of a matrix A then $\det A^2 = 0$.

√ **True** ○ False

(m) [1 point] If the column of A are linearly dependent, then there is always a non-trivial solution to $A \cdot x = \vec{0}$.

√ **True** ○ False

(n) [1 point] If $\mathcal{B}$ is a basis made of eigenvectors of A and $v \in \mathcal{B}$, then $[A \cdot v]_{\mathcal{B}}$ is a multiple of e_i for some i.

√ **True** ○ False

Question 2. (50 points) Prove the following propositions.

(a) [5 points] Let A be a $n \times n$ matrix. Prove that if A is invertible then A^{-1} is invertible and

$$\det A^{-1} = \frac{1}{\det A}.$$

Solution: A^{-1} is invertible since $A^{-1} \cdot A = I$. Using the properties of the determinant

$$1 = \det(I) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1})$$

from which it follows that $\det(A^{-1})$ is the inverse of $\det(A)$.

(b) [5 points] Using the previous part prove that if 0 is not an eigenvalue of A then the linear transformation associated to A^{-1} is 1-to-1.

Solution: Since 0 is not an eigenvalue of A it follows that A is invertible, which implies, by the previous part that A^{-1} is invertible. Since A is invertible the corresponding linear transformation $T_{A^{-1}}$ is invertible and therefore is in particular 1-to-1.

(c) [10 points] Prove that if $\lambda_1 \neq \lambda_2$ are two eigenvalues of A and $v_1 \in E_{\lambda_1}(A)$, $v_2 \in E_{\lambda_2}(A)$ both non zero, then v_1 and v_2 are linearly independent.

Solution: We want to show that v_1 and v_2 are linearly independent, i.e. for any linear combination $c_1 \cdot v_1 + c_2 \cdot v_2 = 0$ we should have $c_1 = c_2 = 0$. Assume we had a non trivial solution, say with $c_2 \neq 0$. Then $v_2 = -c_1/c_2 \cdot v_1$. From this it follows that

$$A \cdot v_2 = A \cdot -\frac{c_1}{c_2} v_1 = -\frac{c_1}{c_2} A \cdot v_1 = -\frac{c_1}{c_2} \lambda_1 v_1 = \lambda_1 \cdot \Big(-\frac{c_1}{c_2} v_1 \Big) = \lambda_1 v_2$$

On the other hand since $v_2 \in E_{\lambda_2}(A)$ we have that $A \cdot v_2 = \lambda_2 v_2$. Putting these two together we get

$$0 = A \cdot \vec{0} = A \cdot (v_2 - v_2) = A \cdot v_2 - A \cdot v_2 = \lambda_1 v_2 - \lambda_2 v_2 = (\lambda_1 - \lambda_2) v_2$$

and since $v_2 \neq 0$ it has to follow that $\lambda_1 - \lambda_2 = 0$ which implies $\lambda_1 = \lambda_2$ that contradicts the hypothesis.

(d) [10 points] Let A be a $m \times n$ matrix and let B be a $n \times k$ matrix. Assume that $A \cdot B = 0$. Prove that $\text{nullity}(A) \geq \text{rank}(B)$.

Solution: Since $A \cdot B = 0$ it follows that $T_A \circ T_B : \mathbb{R}^k \to \mathbb{R}^m$ is the linear transformation that sends everything to zero. In particular for every vector $v \in \mathbb{R}^k$ we have that $T_A(T_B(v)) = 0$. Call $w = T_B(v)$. Then clearly $w \in range(T_B) = \text{col}(B)$. On the other hand $T_A(w) = 0$ which means that $w \in \ker(T_A) = \text{null}(A)$. This shows that every vector in $\text{col}(B)$ is also a vector of $\text{null}(A)$. Let $u_1, \ldots, u_l$ be a basis for $\text{col}(B)$, which in particular says that $\text{rank}(B) = l$. Then $u_1, \ldots, u_l$ are linearly independent vectors in $\text{null}(A)$, which implies that $\text{nullity}(A) \geq l = \text{rank}(B)$.

(e) [20 points] Prove that if A is an $m \times n$ matrix, then $\text{nullity}(A) = \text{nullity}(A^T \cdot A)$. (HARD PROBLEM)

Solution: To prove this is enough to prove that $\text{null}(A) = \text{null}(A^T \cdot A)$, since then the dimensions have to agree. It is easy to see that if $x \in \text{null}(A)$ then $x \in \text{null}(A^T \cdot A)$, since $A \cdot x = \vec{0}$ implies that $A^T \cdot A \cdot x = \vec{0}$. This shows that $\text{null}(A^T \cdot A)$ contains $\text{null}(A)$. To finish the exercise we need to show that every element in $\text{null}(A^T \cdot A)$ belong also to $\text{null}(A)$ [This is a standard procedure in math proofs: if you want to show that two sets X and Y are the same you show first that $X \subset Y$ and then $Y \subset X$. The first assertion ensures that every element of X is contained in Y, while the second implies that every element of Y is contained in X. The two together show that X has to be equal to Y].
Assume that $x \in \text{null}(A^T \cdot A)$, i.e. $A^T \cdot A \cdot x = \vec{0}$, and call $w = A \cdot x$. Then $w \in \text{null}(A^T)$, i.e. $A^T \cdot w = \vec{0}$. If we expand

$$A^T \cdot w = \begin{pmatrix} (1^{st} \text{ row of } A^T) \cdot w \\ \vdots \\ (\text{ last row of } A^T) \cdot w \end{pmatrix} = \vec{0}$$

we see that this implies that if we dot any row of A^T with w we get 0. Now remember that by construction a row of A^T is a column of A. So this tells us that w dot any column of A is equal to zero. On the other hand $w = A \cdot x$ so $w \in col(A)$, i.e. we can write w as a linear combination of the columns of A

$$w = c_1 a_1 + \cdots + c_n a_n$$

We claim that this implies that w has length zero. Recall that the length of a vector is the square root of the dot product of the vector with itself. In order to show our claim is enough to prove that $w \cdot w$ is 0.

$$w \cdot w = (A \cdot x) \cdot w = (c_1 a_1 + \cdots + c_n a_n) \cdot w = c_1 a_1 \cdot w + \cdots + c_n a_n \cdot w = 0$$

since w dot any column of A is zero by what we proved above. This shows that w has length zero. But the only vector of $\mathbb{R}^n$ of length zero is the zero vector, so $w = \vec{0}$. This shows that $A \cdot x = w = \vec{0}$, i.e. $x \in \text{null}(A)$ finishing the proof.

Question 3. (10 points) Let A be the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 4 & 6 \\ 0 & 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 4 & 6 \end{pmatrix}$$

and let T_A be the corresponding linear transformation, $T_A : x \mapsto A \cdot x$.

(a) [5 points] Determine a basis for null(A).

Solution: null(A) is the set of solutions of the homogeneous linear system $A \cdot x = 0$. To solve it we reduce A in echelon form obtaining

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 4 & 6 \\ 0 & 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we see that the the linear system $A \cdot x = 0$ is equivalent to the system

$$\begin{cases} x_1 + x_4 + 2x_5 = 0 \\ x_2 + 2x_4 + x_5 = 0 \\ x_3 + x_4 = 0 \end{cases}$$

whose solution set is

$$\text{null}(A) = \text{span}\{ \begin{pmatrix} -1 \\ -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \}$$

(b) [2 points] Determine a basis for row(A) and range(T_A).

Solution: range(T_A) is spanned by the columns of A. By the previous computation we see that the columns of A are linearly dependent - since in reduced echelon form not every column contains a leading term. The only columns of the reduced echelon form containing a leading term are the first three. Therefore

$$\text{range}(T_A) = \text{span}\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \}$$

Using the reduced echelon form we can also find a basis of the row space, given by the non-zero rows, namely

$$\text{row}(A) = \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \}$$

(c) [1 point] Is T_A 1-to-1 and/or onto? Why?

Solution: By the first part we know that T_A is not 1-to-1 since the kernel has dimension 2 (recall that the kernel coincide with $\text{null}(A)$). From the second part we know that $\dim \text{range}(T_A) = 3$ since a basis has cardinality 3. This implies that T_A is not onto since the codomain is $\mathbb{R}^5$ and has dimension 5, therefore the range cannot equal the codomain.

(d) [1 point] What is $\text{rank}(A)$? And the nullity of A?

Solution: The $\text{rank}(A)$ is the dimension of the row space or of the column space. We showed in the second part that both $\text{row}(A)$ and $\text{col}(A)$ have dimension 3 therefore $\text{rank}(A) = 3$. The nullity is the dimension of $\text{null}(A) = \ker(T_A)$ and by the first part we have that the nullity is 2. This shows another time that

$$\text{rank}(A) + \text{nullity}(A) = \dim \text{dom}(T_A) = \dim \mathbb{R}^5 = 5$$

(e) [1 point] Without doing any computation, what is $\det(A)$?

Solution: Since T_A is not invertible, we have that A is not invertible either. This implies by the Big Theorem that $\det(A) = 0$.

Question 4. (10 points) Let E be the plane curve in $\mathbb{R}^2$ defined by the equation

$$E : \quad cy^2 = x^3 + ax + b$$

for some real numbers a, b and c.

(a) [2 points] What is the minimum number of points $P_i = (x_i, y_i)$ that are needed in order to have at most one E passing through all the P_i? (i.e. there is at most one value of a, b and c such that all the P_i are solution of the equation of E).

Solution: Given a point $P_i = (x_i, y_i)$, E passes through P_i if and only if (x_i, y_i) verifies the equation of E. This implies that every point gives a linear equation in (a, b, c). To have at most one solution we need to have at least three solution, so the minimum number of points is 3.

(b) [4 points] Find a linear system that corresponds to the problem of finding the equation of E passing through $(1, 1)$,$(2, 1)$ and $(0, 2)$.

Solution: Substituting each point into the equation of E one gets

$$\begin{cases} a + b - c = -1 \\ 2a + b - c = -8 \\ b - 4c = 0 \end{cases}$$

(c) [4 points] Solve the previous system and find the equation of E passing through the given points

Solution: We reduce the matrix of the system to echelon form as follows

$$\begin{pmatrix} 1 & 1 & -1 & | & -1 \\ 2 & 1 & -1 & | & -8 \\ 0 & 1 & -4 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & -7 \\ 0 & 1 & 0 & | & 8 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$

This implies that the solution of the system is given by $a = -7$, $b = 1$ and $c = 2$, hence the equation for E is

$$E : \quad 2y^2 = x^3 - 7x + 1$$

Curves like E are called *elliptic curves* and are a central object both in pure math and in application to cryptography.

Question 5. (20 points) Let T_A be the linear transformation

$$T_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 + x_3 \\ x_2 \\ x_1 + x_3 \\ x_1 + x_2 + x_3 \\ 2x_1 + x_3 \end{pmatrix}$$

and let B be the following matrices:

$$B = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & a & 1 & 0 & b \end{pmatrix}$$

(a) [2 points] Determine the matrix A associated to T_A and domain and codomain of the linear transformations T_A and T_B.

Solution: We have $T_A : \mathbb{R}^3 \to \mathbb{R}^5$ with

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

and $T_B : \mathbb{R}^5 \to \mathbb{R}^3$.

(b) [4 points] Compute whenever possible the matrix associated to $T_A \circ T_B$ and $T_B \circ T_A$.

Solution: Since A is 5×3 and B is 3×5 both compositions make sense and we have

$$T_A \circ T_B \mapsto A \cdot B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & a & 1 & 0 & b \end{pmatrix} = \begin{pmatrix} 3 & a & 2 & 2 & 2+b \\ 1 & 0 & 1 & 2 & 2 \\ 3 & 2+a & 2 & 0 & b \\ 4 & 2+a & 3 & 2 & 2+b \\ 4 & 4+a & 3 & 0 & b \end{pmatrix}$$

and

$$T_B \circ T_A \mapsto B \cdot A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & a & 1 & 0 & b \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 7 & 3 & 6 \\ 1+2b & 2+a & 3+b \end{pmatrix}$$

(c) [4 points] Determine for which values of a and b, if any, the linear transformation $T_B \circ T_A$ is invertible.

Solution: We compute the determinant of $B \cdot A$, which reads as $\det(B \cdot A) = 8a + 6b - 26$. Therefore the linear transformation is invertible if and only if $8a + 6b \neq 26$.

(d) [5 points] Determine for which values of a and b, if any, the linear transformation $T_A \circ T_B$ is 1-to-1 and/or onto, and compute bases and dimension of its range.

Solution: We begin by reducing to echelon form the matrix $A \cdot B$, which gives

$$\begin{pmatrix} 3 & a & 2 & 2 & 2+b \\ 1 & 0 & 1 & 2 & 2 \\ 3 & 2+a & 2 & 0 & b \\ 4 & 2+a & 3 & 2 & 2+b \\ 4 & 4+a & 3 & 0 & b \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & a-2 & a+b-2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 4-a & 4-a-b \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we see that for *any* value of a and b we have $\operatorname{rank}(A \cdot B) = 3$ and $\operatorname{nullity}(A \cdot B) = 2$. In particular, since the rank is strictly smaller than 5, $T_A \circ T_B$ is not onto, and since the nullity is bigger than 0, $T_A \circ T_B$ is not 1-to-1. From the reduced echelon form we can also deduce that

$$range(T_A \circ T_B) = \operatorname{span}\{ \begin{pmatrix} 3 \\ 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} a \\ 0 \\ 2+a \\ 2+a \\ 4+a \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \\ 3 \end{pmatrix} \}$$

(e) [5 points] Let $\mathcal{B}$ be the basis of $range(T_A \circ T_B)$ that you found in the previous point. Let $w = T_A \circ T_B(e_1 + e_2)$; compute the coordinate of the vector w with respect to $\mathcal{B}$. And compute the vector v such that

$$[v]_{\mathcal{B}} = (1, 2, 1)$$

Solution: Since w is the image of $(e_1 + e_2)$, this implies that w is the sum of the first two columns of $A \cdot B$, therefore the coordinates of w with respect to $\mathcal{B}$ is given by $[w]_{\mathcal{B}} = (1, 1, 0)^T$. Similarly we get

$$v = 1 \cdot \begin{pmatrix} 3 \\ 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} + 2 \cdot \begin{pmatrix} a \\ 0 \\ 2+a \\ 2+a \\ 4+a \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 5+2a \\ 2 \\ 9+2a \\ 11+2a \\ 15+a \end{pmatrix}$$

Question 6. (20 points) Let A be the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 \\ -1 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix}$$

(a) [10 points] Determine whether A is diagonalizable, and in that case find matrices P and D with P invertible and D diagonal, such that $A = PDP^{-1}$.

Solution: We need to determine whether there exists a basis of eigenvectors of A. For this we compute the characteristic polynomial of A as follows:

$$\det(A - \lambda I) = (\lambda - 1)^2(\lambda - 2)(\lambda + 1).$$

From this we see that A has three distinct eigenvalues $1, 2$ and -1 with multiplicities $2, 1$ and 1. Next we compute the eigenspaces.

$$E_1(A) = \text{null}(A - I) = \text{null}\begin{pmatrix} -1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix} = \text{null}\begin{pmatrix} -1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From which we see that

$$E_1(A) = \text{span}\{\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\}.$$

$$E_2(A) = \text{null}(A - 2I) = \text{null}\begin{pmatrix} -2 & 0 & 0 & 2 \\ -1 & -1 & 0 & 2 \\ 0 & 0 & -3 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \text{null}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can then deduce that

$$E_2(A) = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}\}.$$

Finally

$$E_{-1}(A) = \text{null}(A + I) = \text{null}\begin{pmatrix} 1 & 0 & 0 & 2 \\ -1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Which implies that

$$E_{-1}(A) = \text{span}\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\}.$$

From the computation we see that we can find 4 eigenvectors of A which are linearly dependent: it's enough to take the element of the bases we computed, since eigenvectors with different eigenvalues are linearly independent. Therefore the basis we are looking for is, for example,

$$\mathcal{B} = \{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \}$$

This tells us that A is diagonalizable, since there exists a basis of $\mathbb{R}^4$ whose elements are eigenvectors of A. Moreover

$$P = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(b) [2 points] Find a matrix Q such that $Q \cdot A \cdot Q^{-1}$ is a diagonal matrix.

Solution: It is enough to take $Q = P^{-1}$ and one can check that

$$Q \cdot A \cdot Q^{-1} = P^{-1} \cdot A \cdot P = P^{-1}(PDP^{-1})P = D$$

(c) [1 points] Let S be the subspace of $\mathbb{R}^3$ spanned by the first three columns of A. Find a basis $\mathcal{B}$ of S.

Solution: Since 0 is not an eigenvalue of A, A is invertible which implies all its columns are linearly independent. In particular the first three columns are linearly independent and therefore are a basis of S.

(d) [2 points] Verify that the following three vectors form a basis of S

$$u_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix} \quad u_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}$$

Solution: It's easy to see that all three vectors lie in S. In particular it is enough to show that they are linearly independent since S has dimension 3 by the previous point. To do this we show that if we remove the first component the corresponding vectors of $\mathbb{R}^3$ are linearly independent, since the matrix that has them as columns is invertible. This can be checked by computing

$$\det \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} \neq 0$$

(e) [4 points] Call $\mathcal{A}$ the basis formed by the vectors u_1, u_2 and u_3. Determine the change of basis matrix $M_{\mathcal{B},\mathcal{A}}$.

Solution: By the formula we have that

$$M_{\mathcal{B},\mathcal{A}} = ([u_1]_\mathcal{B}[u_2]_\mathcal{B}[u_3]_\mathcal{B}).$$

So we have to compute the coordinates of u_1, u_2 and u_3 with respect to $\mathcal{B}$. It is easy to see that

$$[u_1]_\mathcal{B} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad [u_2]_\mathcal{B} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad [u_3]_\mathcal{B} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}$$

Therefore

$$M_{\mathcal{B},\mathcal{A}} = \begin{pmatrix} 0 & 1 & -2 \\ 2 & 1 & -3 \\ 1 & 2 & 0 \end{pmatrix}$$

(f) [1 point] Use the previous point to compute the coordinate of the vector v with respect to the basis $\mathcal{B}$ given that $[v]_\mathcal{A}^T = (1, 1, 1)$. What is v?

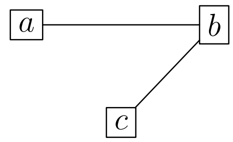
Solution:

$$[v]_\mathcal{B} = M_{\mathcal{B},\mathcal{A}}[v]_\mathcal{A} = \begin{pmatrix} 0 & 1 & -2 \\ 2 & 1 & -3 \\ 1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

Moreover

$$v = u_1 + u_2 + u_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Question 7. (15 points) A graph is a set of point, called vertices or nodes, connected by some lines, called edges (these are also called undirected graphs). Consider the following graph

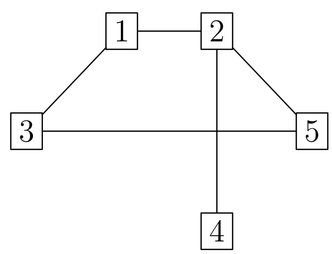


We have 3 vertices, namely a, b and c and two edges. To each graph we can associate a matrix, called the adjacency matrix. It has a row for each vertex, and the entry a_{ij} is either a 0 or a 1 according to whether there is an edge joining the nodes i and j or not. In the example above the adjacency matrix will be 3×3 since there are three nodes, and we have edges joining the nodes a and b, b and c, but not a and c so if we order the vertices in the natural order a, b and c the matrix looks like

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

For example the first row is the row corresponding to the node a, and you can read that there is only one edge that starts from a and connects a to the second node, namely b. This is why the element a_{12} is equal to 1. While there are no edges connecting a and c which explains why $a_{13} = 0$.

Consider the following graph G:



(a) [2 points] Write down the adjacency matrix A_G of G

Solution:

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

(b) [4 points] One of the properties of the adjacency matrix is the following: if you construct the product A_G^k the element in the position i, j counts the number of distinct paths of length k between the node i and the node j. Using this construction count the number of path of length 3 between the node 1 and the node 2, and between the node 3 and the node 4.

Solution: We "only" need to compute A_G^3 since we are looking for paths of length 3, therefore

$$A_G^3 = \begin{pmatrix} 0 & 5 & 4 & 0 & 0 \\ 5 & 0 & 0 & 3 & 5 \\ 4 & 0 & 0 & 2 & 4 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 5 & 4 & 0 & 0 \end{pmatrix}$$

The number of path of length 3 between the node 1 and the node 2 is given by $a_{1,2}$ of this matrix, i.e. the $1, 2$ entry, so it's 5, while the number of path of length 3 between the node 3 and the node 4 is given by $a_{3,4}$ entry, so it's 2.

(c) [4 points] Show that for any graph G the adjacency matrix is always symmetric.

Solution: This follows from the fact that if there is an edge joining the node i to the node j then the same edge clearly joins the node j with the node i. This implies that $a_{ij} = 1$ implies $a_{ji} = 1$. Since each entry of the matrix can only be 0 or 1 this proves that the matrix is symmetric.

(d) [5 points] Consider the linear transformation T_{A_G} associated to A_G. Determine whether T_{A_G} is 1-to-1, onto and invertible.

Solution: We reduce the matrix A_G to echelon form and we get

$$A_G \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we can see that A has rank 4 since there are four columns containing leading terms, and therefore it has nullity $5 - 4 = 1$. This implies that T_{A_G} is neither 1-to-1 (since the kernel has dimension 1) nor onto (since the range has dimension $4 \leq 5$). In particular is not invertible

Question 8. (30 points) Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ we can always find a matrix A such that $T = T_A$. We call A the matrix associated to the linear transformation T, with respect to the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ and we write $A = M_{\mathcal{E}_m,\mathcal{E}_n}(T)$ (where $\mathcal{E}_m$ is the standard basis of $\mathbb{R}^m$ and $\mathcal{E}_n$ is the standard basis of $\mathbb{R}^n$). The aim of this exercise is to relate diagonalization of the matrix A with the matrix of T_A with respect to a basis of eigenvectors of A.

(a) [3 points] Verify that

$$A = M_{\mathcal{E}_m,\mathcal{E}_n} = ([T(e_1)]_{\mathcal{E}_m}, [T(e_2)]_{\mathcal{E}_m}, \ldots, [T(e_n)]_{\mathcal{E}_m})$$

Solution: We proved in class that given T the columns of the matrix A such that $T = T_A$ are the image under T of the vectors e_i. The above formula then holds by recalling that coordinates with respect to the canonical basis are nothing but the same vector.

(b) [4 points] The construction can be generalized to arbitrary basis in the following way: if $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ is a basis of $\mathbb{R}^n$ and $\mathcal{A} = \{\alpha_1, \ldots, \alpha_m\}$ is a basis of $\mathbb{R}^m$ then the matrix associated to T with respect to the basis $\mathcal{B}$ and $\mathcal{A}$ is defined as

$$M_{\mathcal{A},\mathcal{B}}(T) = ([T(\beta_1)]_{\mathcal{A}}, [T(\beta_2)]_{\mathcal{A}}, \ldots, [T(\beta_n)]_{\mathcal{A}})$$

Show that $M_{\mathcal{A},\mathcal{B}}(T)$ has exactly m rows and n columns.

Solution: The number of columns of $M_{\mathcal{A},\mathcal{B}}(T)$ is equal to the number of vectors in $\mathcal{B}$. Since $\mathcal{B}$ is a basis of $\mathbb{R}^n$ there are exactly n such vectors and therefore the number of columns if n. On the other hand, the number of rows is given by the number of components of the vectors $[T(\beta_i)]_{\mathcal{A}}$. Since $\mathcal{A}$ is a basis of $\mathbb{R}^m$ each of the vectors has exactly m components, being the components the coefficients of a linear combination of elements of $\mathcal{A}$. This shows that the number of rows is exactly m.

(c) [3 points] Let T be the linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 + x_4 \\ x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_4 \\ x_4 - x_1 \end{pmatrix}$$

Identify domain and codomain of T and determine the matrix associated to the linear transformation with respect to the canonical bases of domain and codomain.

Solution: From the formula we see that $\mathrm{dom}(T) = \mathbb{R}^4$ and $\mathrm{codom}(T) = \mathbb{R}^5$. The matrix associated to T with respect to the canonical bases is

$$M_{\mathcal{E}_5,\mathcal{E}_4}(T) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

(d) [5 points] Let $\mathcal{B}$ be a basis of $\mathbb{R}^4$ formed by the following vectors:

$$\beta_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \beta_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Determine the matrix $M_{\mathcal{E}_n,\mathcal{B}}(T)$ where n is the dimension of the codomain of T.

Solution: From the definition

$$M_{\mathcal{E}_5,\mathcal{B}}(T) = ([T(\beta_1)]_{\mathcal{E}_5}, \dots, [T(\beta_4)]_{\mathcal{E}_5}) = (T(\beta_1), \dots, T(\beta_4)) = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -2 & 0 & 1 & 1 \end{pmatrix}$$

(e) [5 points] Let w be the vector $(2,2,2,2)^T \in \mathbb{R}^4$. Show that the following formula holds:

$$M_{\mathcal{E}_5,\mathcal{B}}(T) \cdot [w]_{\mathcal{B}} = M_{\mathcal{E}_5,\mathcal{E}_4}(T) \cdot M_{\mathcal{E}_4,\mathcal{B}}(\mathrm{id}_4) \cdot [w]_{\mathcal{E}_4}$$

Solution: We already computed two of the three matrices involved. The only one which is left is

$$M_{\mathcal{E}_4,\mathcal{B}}(\mathrm{id}_4) = ([\mathrm{id}_4(\beta_1)]_{\mathcal{E}_4}, \dots, [\mathrm{id}_4(\beta_4)]_{\mathcal{E}_4}) = (\beta_1, \dots, \beta_4)$$

Therefore the formula reads

$$\begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -2 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

(f) [2 points] Use the previous point to show that for every vector $v \in \mathbb{R}^4$

$$M_{\mathcal{E}_5,\mathcal{B}}(T) \cdot [v]_{\mathcal{B}} = M_{\mathcal{E}_5,\mathcal{E}_4}(T) \cdot M_{\mathcal{E}_4,\mathcal{B}}(\mathrm{id}_4) \cdot [v]_{\mathcal{B}}$$

Solution: By the above computation the product of the two matrices on the right equals the matrix on the left.

(g) [8 points] Let $\mathcal{A}$ be the basis of $\mathbb{R}^5$ given by $\{e_1, e_1 + e_2, e_2 + e_3, e_3 + e_4, e_4 + e_5\}$. Prove that

$$M_{\mathcal{A},\mathcal{B}}(T) = M_{\mathcal{A},\mathcal{E}_5}(\mathrm{id}_5) \cdot M_{\mathcal{E}_5,\mathcal{E}_4}(T) \cdot M_{\mathcal{E}_4,\mathcal{B}}(\mathrm{id}_4)$$

Solution: We begin by computing the matrix on the left. Using the formulas we get

$$M_{\mathcal{A},\mathcal{B}}(T) = ([T(\beta_1)]_{\mathcal{A}}, \ldots, [T(\beta_4)]_{\mathcal{A}} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 0 & -2 \\ -1 & -1 & 0 & 2 \\ 2 & 1 & -1 & -2 \\ -2 & 0 & 1 & 1 \end{pmatrix}$$

For the right hand side we already have computed $M_{\mathcal{E}_5,\mathcal{E}_4}(T)$ and $M_{\mathcal{E}_4,\mathcal{B}}(\mathrm{id}_4)$, so we are left we computing

$$M_{\mathcal{A},\mathcal{E}_5}(\mathrm{id}_5) = ([\mathrm{id}_5(e_1)]_{\mathcal{A}}, \ldots, [\mathrm{id}_5(e_5)]_{\mathcal{A}}) = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Computing the product yields the answer.

(h) [10 points] Now let A be the following matrix

$$A = \begin{pmatrix} 11 & 0 & -8 \\ 0 & -1 & 0 \\ 12 & 0 & -9 \end{pmatrix}$$

Prove that there exists a basis $\mathcal{B}$ of eigenvectors of A. Then show that $A = P \cdot D \cdot P^{-1}$ and

$$D = M_{\mathcal{B},\mathcal{B}}(T_A) \qquad P = M_{\mathcal{E}_3,\mathcal{B}}$$

Solution: We begin by computing the eigenvalues of A. The characteristic polynomial is (expanding the second row)

$$\det(A - \lambda I_3) = \det \begin{pmatrix} 11-\lambda & 0 & -8 \\ 0 & -1-\lambda & 0 \\ 12 & 0 & -9-\lambda \end{pmatrix} = -(\lambda+1)^2 + (\lambda - 3)$$

In particular the eigenvalues of A are -1 with multiplicity 2 and 3 with multiplicity 1. The corresponding eigenvalues are

$$E_{-1}(A) = \mathrm{null}(A + I_3) = \mathrm{null} \begin{pmatrix} 12 & 0 & -8 \\ 0 & 0 & 0 \\ 12 & 0 & -8 \end{pmatrix} = \mathrm{null} \begin{pmatrix} 3 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore the eigenspace is the set of solution of $3x_1 - 2x_3 = 0$ i.e.

$$E_{-1}(A) = \text{span}\{ \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}$$

Similarly for the eigenspace relative to 3 we get

$$E_3(A) = \text{null}(A + 3I_3) = \text{null} \begin{pmatrix} 8 & 0 & -8 \\ 0 & -4 & 0 \\ 12 & 0 & -12 \end{pmatrix} = \text{null} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the eigenspace is given by

$$E_3(A) = \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \}$$

This shows that the set

$$\{\beta_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \beta_2 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \beta_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}$$

Is a basis $\mathcal{B}$ of $\mathbb{R}^3$ formed by eigenvectors of A. So A is diagonalizable and we can write $A = PDP^{-1}$ with

$$P = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

On the other hand, using the formula above

$$\begin{aligned} M_{\mathcal{B},\mathcal{B}}(T_A) &= ([T_A(\beta_1)]_{\mathcal{B}}, [T_A(\beta_2)]_{\mathcal{B}}, [T_A(\beta_3)]_{\mathcal{B}}) = ([3\beta_1]_{\mathcal{B}}, [-\beta_2]_{\mathcal{B}}, [-\beta_3]_{\mathcal{B}}) \\ &= \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = D \end{aligned}$$

and also

$$M_{\mathcal{E},\mathcal{B}}(id) = ([id(\beta_1)]_{\mathcal{E}}, [id(\beta_2)]_{\mathcal{E}}, [id(\beta_3)]_{\mathcal{E}}) = (\beta_1, \beta_2, \beta_3) = P$$

This shows that

$$M_{\mathcal{E},\mathcal{E}}(T_A) = A = P \cdot D \cdot P^{-1} = M_{\mathcal{E},\mathcal{B}}(id) \cdot M_{\mathcal{B},\mathcal{B}}(T_A) \cdot M_{\mathcal{B},\mathcal{E}}(id)$$