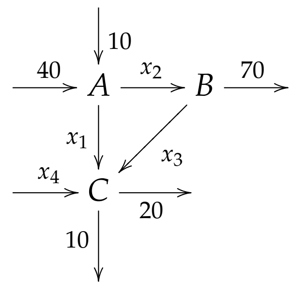
QUIZ 1 - MATH 308 G - OCTOBER 12, 2016

SOLUTION TYPE 2

Maria works for an IT company who is running a series of simulations for improving the traffic flow in Seattle. She is studying a simplify map of an intersection of three one-way roads that looks as follows:



The goal of Maria is to balance the graph, i.e. find the values of x_1, x_2, x_3 and x_4 such that at each node A, B and C the sum of the incoming traffic (arrows pointing *to* the node) equal the sum of the outgoing traffic (arrow pointing *away* from the node).

2 pt. Write down a linear system that represents the problem Maria is trying to solve;
Solution:

- Node A: $40 + 10 = x_1 + x_2$;
- Node B: $x_2 = 70 + x_3$;
- Node C: $x_4 + x_1 + x_3 = 10 + 20$

Therefore the corresponding linear system is

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 50 \\ x_2 - x_3 \qquad &= 70 \\ x_1 \qquad + x_3 + x_4 &= 30 \end{aligned}$$

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5 pt. Using the Gauss-Jordan algorithm solve the system.
Solution:

The augmented matrix representing the system is

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 70 \\ 1 & 0 & 1 & 1 & 30 \end{array}\right)$$

The Gauss-Jordan algorithm then gives:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 70 \\ 1 & 0 & 1 & 1 & 30 \end{array}\right) \overset{III-I}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 70 \\ 0 & -1 & 1 & 1 & -20 \end{array}\right) \overset{I-II}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -20 \\ 0 & 1 & -1 & 0 & 70 \\ 0 & -1 & 1 & 1 & -20 \end{array}\right)$$

$$\overset{III+II}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -20 \\ 0 & 1 & -1 & 0 & 70 \\ 0 & 0 & 0 & 1 & 50 \end{array}\right)$$

The last matrix is in Reduced Echelon Form and the corresponding linear system reads as follows:

$$\begin{aligned} x_1 \quad + x_3 \quad &= -20 \\ x_2 - x_3 \quad &= 70 \\ x_4 &= 50 \end{aligned}$$

We can therefore conclude that x_3 is a free variable, and after setting $x_3 = s_1$ we obtain that the solution of the system is

$$\begin{aligned} x_1 &= -20 - s_1 \\ x_2 &= 70 + s_1 \\ x_3 &= s_1 \\ x_4 &= 50 \end{aligned}$$

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1 pt. What is the dimension of the solution set? Why?
Solution:

The solution set is the set

$$\{(-20 - s_1, 70 + s_1, s_1, 50) : s_1 \in \mathbb{R}\}$$

Since there is one free parameter the solution set is a LINE in $\mathbb{R}^4$ and therefore it has dimension 1. ■

1 pt. Write down the solution set for the problem in vector form. *Solution:*

Taking the solution set as written in the previous answer we can rewrite it in vector form as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -20 - s_1 \\ 70 + s_1 \\ s_1 \\ 50 \end{pmatrix} = \begin{pmatrix} -20 \\ 70 \\ 0 \\ 50 \end{pmatrix} + s_1 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

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1pt. If there is a free parameter, for which values the solutions make sense for the problem.

Solution:

Since the map represents traffic flow, each of the variable must attain a positive value. This translates into the conditions

$$\begin{cases} -20 - s_1 \geq 0 \\ 70 + s_1 \geq 0 \\ s_1 \geq 0 \end{cases}$$

Since the first equations gives $s_1 \leq -20$ and the third is $s_1 \geq 0$ we see that there are no values of the free parameter for which the solution make sense. This will tell Maria that her model of the traffic flow cannot work, and she should consider inverting the direction of one road. ■