Practice Final

for Math 308-G, Autumn 2016

Total number of questions: 8

Total number of points: 175

Disclaimer: this is a collection of problems that could help you prepare for the Final. It is not intended to be representative of a Final. In particular there are more questions and more points that you will encounter in the “real” final.
The last two problems focus on an application of linear algebra to graphs, and the extension of the notion of change of basis to linear transformation. They can be a bit more hard then the other problems.

Question	Points	Score
1	20	
2	50	
3	10	
4	10	
5	20	
6	20	
7	15	
8	30	
Total:	175	

Question 1. (20 points) Decide whether the following statements are true or false. For this you don't need to show any work (but for practicing you might want to try to do that).

(a) [1 point] If the columns of a matrix A are linearly independent, then T_A the associated linear transformation, is onto.

○ True ○ False

(b) [1 point] If λ_1 and λ_2 are two different eigenvalues of A and $v \in E_{\lambda_1}(A)$ $w \in E_{\lambda_2}(A)$ then v and w are linearly independent

○ True ○ False

(c) [1 point] If $\mathcal{B}$ is a basis for a subspace S and $u_1 \in \mathcal{B}$ then any multiple of u_1 is in $\mathcal{B}$.

○ True ○ False

(d) [1 point] If $T : \mathbb{R}^n \to \mathbb{R}^m$ and A is the matrix such that $T = T_A$ then A has n rows.

○ True ○ False

(e) [1 point] If A and B are equivalent $m \times n$ matrices then for any vector $v \in \mathbb{R}^n$, $A \cdot v = B \cdot v$

○ True ○ False

(f) [1 point] If S is a subset of $\mathbb{R}^n$ of dimension m and $v \in S$, then for every basis $\mathcal{B}$ of S, $[v]_{\mathcal{B}}$ is a vector with m components.

○ True ○ False

(g) [1 point] If B is the reduced echelon form of a square matrix A then $\det A = \det B$.

○ True ○ False

(h) [1 point] If v is an eigenvector of a matrix A, then $A \cdot v$ and v are linearly independent.

○ True ○ False

(i) [1 point] If A is $n \times n$ matrix with n distinct eigenvalues, then there exists a basis of $\mathbb{R}^n$ made of eigenvectors of A.

○ True ○ False

(j) [1 point] If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has $\ker T = \{\vec{0}\}$ then for every $b \in \text{range}(T)$ there is exactly one $v \in \mathbb{R}^n$ such that $T(v) = b$.

○ True ○ False

(k) [1 point] If A is $n \times n$ matrix with n distinct eigenvalues, then there is *no* basis of $\mathbb{R}^n$ made of eigenvectors of A.

○ True ○ False

(l) [1 point] If 0 is an eigenvalue of a matrix A then $\det A^2 = 0$.

○ True ○ False

(m) [1 point] If the column of A are linearly dependent, then there is always a non-trivial solution to $A \cdot x = \vec{0}$.

○ True ○ False

(n) [1 point] If $\mathcal{B}$ is a basis made of eigenvectors of A and $v \in \mathcal{B}$, then $[A \cdot v]_{\mathcal{B}}$ is a multiple of e_i for some i.

○ True ○ False

Question 2. (50 points) Prove the following propositions.

(a) [5 points] Let A be a $n \times n$ matrix. Prove that if A is invertible then A^{-1} is invertible and

$$\det A^{-1} = \frac{1}{\det A}.$$

(b) [5 points] Using the previous part prove that if 0 is not an eigenvalue of A then the linear transformation associated to A^{-1} is 1-to-1.

(c) [10 points] Prove that if $\lambda_1 \neq \lambda_2$ are two eigenvalues of A and $v_1 \in E_{\lambda_1}(A)$, $v_2 \in E_{\lambda_2}(A)$ both non zero, then v_1 and v_2 are linearly independent.

(d) [10 points] Let A be a $m \times n$ matrix and let B be a $n \times k$ matrix. Assume that $A \cdot B = 0$. Prove that $\text{nullity}(A) \geq \text{rank}(B)$.

(e) [20 points] Prove that if A is an $m \times n$ matrix, then $\text{nullity}(A) = \text{nullity}(A^T \cdot A)$. (HARD PROBLEM)

Question 3. (10 points) Let A be the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 4 & 6 \\ 0 & 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 4 & 6 \end{pmatrix}$$

and let T_A be the corresponding linear transformation, $T_A : x \mapsto A \cdot x$.

(a) [5 points] Determine a basis for null(A).

(b) [2 points] Determine a basis for row(A) and range(T_A).

(c) [1 point] Is T_A 1-to-1 and/or onto? Why?

(d) [1 point] What is rank(A)? And the nullity of A?

(e) [1 point] Without doing any computation, what is det(A)?

Question 4. (10 points) Let E be the plane curve in $\mathbb{R}^2$ defined by the equation

$$E: \quad cy^2 = x^3 + ax + b$$

for some real numbers a, b and c.

(a) [2 points] What is the minimum number of points $P_i = (x_i, y_i)$ that are needed in order to have at most one E passing through all the P_i? (i.e. there is at most one value of a, b and c such that all the P_i are solution of the equation of E).

(b) [4 points] Find a linear system that corresponds to the problem of finding the equation of E passing through $(1, 1)$,$(2, 1)$ and $(0, 2)$.

(c) [4 points] Solve the previous system and find the equation of E passing through the given points

Question 5. (20 points) Let T_A be the linear transformation

$$T_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 + x_3 \\ x_2 \\ x_1 + x_3 \\ x_1 + x_2 + x_3 \\ 2x_1 + x_3 \end{pmatrix}$$

and let B be the following matrices:

$$B = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & a & 1 & 0 & b \end{pmatrix}$$

(a) [2 points] Determine the matrix A associated to T_A and domain and codomain of the linear transformations T_A and T_B.

(b) [4 points] Compute whenever possible the matrix associated to $T_A \circ T_B$ and $T_B \circ T_A$.

(c) [4 points] Determine for which values of a and b, if any, the linear transformation $T_B \circ T_A$ is invertible.

(d) [5 points] Determine for which values of a and b, if any, the linear transformation $T_A \circ T_B$ is 1-to-1 and/or onto, and compute bases and dimension of its range.

(e) [5 points] Let $\mathcal{B}$ be the basis of $range(T_A \circ T_B)$ that you found in the previous point. Let $w = T_A \circ T_B(e_1 + e_2)$; compute the coordinate of the vector w with respect to $\mathcal{B}$. And compute the vector v such that

$$[v]_{\mathcal{B}} = (1, 2, 1)$$

Question 6. (20 points) Let A be the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 \\ -1 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix}$$

(a) [10 points] Determine whether A is diagonalizable, and in that case find matrices P and D with P invertible and D diagonal, such that $A = PDP^{-1}$.

(b) [2 points] Find a matrix Q such that $Q \cdot A \cdot Q^{-1}$ is a diagonal matrix.

(c) [1 points] Let S be the subspace of $\mathbb{R}^3$ spanned by the first three columns of A. Find a basis $\mathcal{B}$ of S.

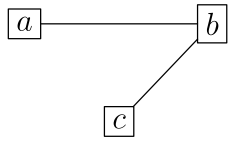
(d) [2 points] Verify that the following three vectors form a basis of S

$$u_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix} \quad u_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}$$

(e) [4 points] Call $\mathcal{A}$ the basis formed by the vectors u_1, u_2 and u_3. Determine the change of basis matrix $M_{\mathcal{B},\mathcal{A}}$.

(f) [1 point] Use the previous point to compute the coordinate of the vector v with respect to the basis $\mathcal{B}$ given that $[v]_{\mathcal{A}}^T = (1, 1, 1)$. What is v?

Question 7. (15 points) A graph is a set of point, called vertices or nodes, connected by some lines, called edges (these are also called undirected graphs). Consider the following graph

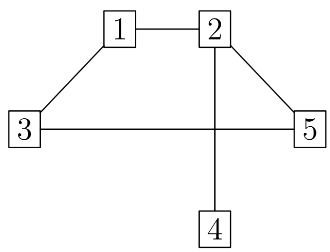


We have 3 vertices, namely a, b and c and two edges. To each graph we can associate a matrix, called the adjacency matrix. It has a row for each vertex, and the entry a_{ij} is either a 0 or a 1 according to whether there is an edge joining the nodes i and j or not. In the example above the adjacency matrix will be 3×3 since there are three nodes, and we have edges joining the nodes a and b, b and c, but not a and c so if we order the vertices in the natural order a, b and c the matrix looks like

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

For example the first row is the row corresponding to the node a, and you can read that there is only one edge that starts from a and connects a to the second node, namely b. This is why the element a_{12} is equal to 1. While there are no edges connecting a and c which explains why $a_{13} = 0$.

Consider the following graph G:



(a) [2 points] Write down the adjacency matrix A_G of G

(b) [4 points] One of the properties of the adjacency matrix is the following: if you construct the product A_G^k the element in the position i, j counts the number of distinct paths of length k between the node i and the node j. Using this construction count the number of path of length 3 between the node 1 and the node 2, and between the node 3 and the node 4.

(c) [4 points] Show that for any graph G the adjacency matrix is always symmetric.

(d) [5 points] Consider the linear transformation T_{A_G} associated to A_G. Determine whether T_{A_G} is 1-to-1, onto and invertible.

Question 8. (30 points) Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ we can always find a matrix A such that $T = T_A$. We call A the matrix associated to the linear transformation T, with respect to the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ and we write $A = M_{\mathcal{E}_m,\mathcal{E}_n}(T)$ (where $\mathcal{E}_m$ is the standard basis of $\mathbb{R}^m$ and $\mathcal{E}_n$ is the standard basis of $\mathbb{R}^n$). The aim of this exercise is to relate diagonalization of the matrix A with the matrix of T_A with respect to a basis of eigenvectors of A.

(a) [3 points] Verify that

$$A = M_{\mathcal{E}_m,\mathcal{E}_n} = ([T(e_1)]_{\mathcal{E}_m}, [T(e_2)]_{\mathcal{E}_m}, \ldots, [T(e_n)]_{\mathcal{E}_m})$$

(b) [4 points] The construction can be generalized to arbitrary basis in the following way: if $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ is a basis of $\mathbb{R}^n$ and $\mathcal{A} = \{\alpha_1, \ldots, \alpha_m\}$ is a basis of $\mathbb{R}^m$ then the matrix associated to T with respect to the basis $\mathcal{B}$ and $\mathcal{A}$ is defined as

$$M_{\mathcal{A},\mathcal{B}}(T) = ([T(\beta_1)]_{\mathcal{A}}, [T(\beta_2)]_{\mathcal{A}}, \ldots, [T(\beta_n)]_{\mathcal{A}})$$

Show that $M_{\mathcal{A},\mathcal{B}}(T)$ has exactly m rows and n columns.

(c) [3 points] Let T be the linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 + x_4 \\ x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_4 \\ x_4 - x_1 \end{pmatrix}$$

Identify domain and codomain of T and determine the matrix associated to the linear transformation with respect to the canonical bases of domain and codomain.

(d) [5 points] Let $\mathcal{B}$ be a basis of $\mathbb{R}^4$ formed by the following vectors:

$$\beta_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \beta_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Determine the matrix $M_{\mathcal{E}_n,\mathcal{B}}(T)$ where n is the dimension of the codomain of T.

(e) [5 points] Let w be the vector $(2, 2, 2, 2)^T \in \mathbb{R}^4$. Show that the following formula holds:

$$M_{\mathcal{E}_5,\mathcal{B}}(T) \cdot [w]_{\mathcal{B}} = M_{\mathcal{E}_5,\mathcal{E}_4}(T) \cdot M_{\mathcal{E}_4,\mathcal{B}}(\mathrm{id}_4) \cdot [w]_{\mathcal{E}_4}$$

(f) [2 points] Use the previous point to show that for every vector $v \in \mathbb{R}^4$

$$M_{\mathcal{E}_5,\mathcal{B}}(T) \cdot [v]_{\mathcal{B}} = M_{\mathcal{E}_5,\mathcal{E}_4}(T) \cdot M_{\mathcal{E}_4,\mathcal{B}}(\mathrm{id}_4) \cdot [v]_{\mathcal{B}}$$

(g) [8 points] Let $\mathcal{A}$ be the basis of $\mathbb{R}^5$ given by $\{e_1, e_1 + e_2, e_2 + e_3, e_3 + e_4, e_4 + e_5\}$. Prove that

$$M_{\mathcal{A},\mathcal{B}}(T) = M_{\mathcal{A},\mathcal{E}_5}(\mathrm{id}_5) \cdot M_{\mathcal{E}_5,\mathcal{E}_4}(T) \cdot M_{\mathcal{E}_4,\mathcal{B}}(\mathrm{id}_4)$$

(h) [10 points] Now let A be the following matrix

$$A = \begin{pmatrix} 11 & 0 & -8 \\ 0 & -1 & 0 \\ 12 & 0 & -9 \end{pmatrix}$$

Prove that there exists a basis $\mathcal{B}$ of eigenvectors of A. Then show that $A = P \cdot D \cdot P^{-1}$ and

$$D = M_{\mathcal{B},\mathcal{B}}(T_A) \qquad P = M_{\mathcal{E}_3,\mathcal{B}}$$