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# Perpetual American Options and Implied Volatility

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## Abstract

This thesis analyzes perpetual American options and in particular aims to identify what kind of behavior the implied volatility of perpetual American put options display. Using two different approaches, we derive the value function for the American put option. Moreover, by assuming the volatility function of the underlying model to be decreasing we find that this induces an implied volatility skew. Furthermore, this claim is illustrated with numerical calculations.

# Contents

# 1 Introduction

The financial market is complex, with several different types of financial derivatives available for purchase. Among these derivatives are options, for which several different types exist. This thesis focuses on American options, in particular the American put option. An American put option gives the buyer the right, but not the obligation, to sell the underlying stock $X_t$ for a given strike price $K$ at any time point up to and including expiry. The payoff of this option is therefore $(K - X_t)^+$. The aim is to find the maximal value of the discounted payoff for an American option. Furthermore, when the American option has an infinite time horizon, it is called a perpetual American option. That the option has an infinite time horizon, will give a time-homogeneity to the problem, thereby simplifying the computations in comparison with its time-dependent counterpart.

The key parameter when pricing options is the volatility and the theoretical option price depends monotonically on the volatility. In theoretical situations, the volatility $\sigma$ is given and often also a constant. However in reality, the volatility is not often known, but what is known is the market prices of options for different strike prices. Thus, by calibrating one volatility for each market price, one obtains the *implied volatility*. The notion of implied volatility provides an efficient way of quoting prices of different options and also gives information about the market expectations of future volatility. Therefore, considering the benefit of the implied volatility, the aim of this thesis is to investigate the behavior of the implied volatility for perpetual American options.

The implied volatility is usually not constant, instead often resembling a ”smile” or a ”skew”. The focus in this thesis is the implied volatility skew, which is another way of saying that implied volatility is a decreasing function. An implied volatility skew can be induced by different factors. In this paper the main interest is a decreasing volatility function in the underlying model, however, both stochastic volatility models or jump processes can also be used to induce a skew in the implied volatility.

To investigate the properties of implied volatility in American options, we begin with computing the value function $V(x)$ of the option price with respect to the stock price $x$ using a free boundary method and optimal stopping times. Continuing from that, the value function $\hat{V}(K)$ of the option price with respect to the strike price $K$ can be obtained instead. In our main result, we show that a monotone local volatility function gives rise to a monotone implied volatility (as a function of $K$). In particular, local volatility models with decreasing volatility models may be used in markets exhibiting volatility skews. To prove our main result, we first derive technical properties of the decreasing fundamental solution $\varphi$ of the associated pricing ordinary differential equation. In the last section of this paper we use numerical software to find the implied volatility of the American put option when given an explicit deterministic volatility function. This gives a good illustration for the authenticity to the claim of the main theorem.

# 2 The free boundary problem of American options

## 2.1 Background

The geometric Brownian motion $X_t$ is characterized by the stochastic differential equation

$$dX_t = rX_t + \sigma X_t dB_t, \quad X_0 = x, \tag{2.1}$$

where $B_t$ is the standard Brownian motion with $B_0 = 0$, $r > 0$ is the risk-free rate and $\sigma > 0$ is the volatility.

The arbitrage free price of an American put option, without dividends, is given by

$$V(x) = \sup_{\tau} \mathbb{E}_x[e^{-r\tau}(K - X_\tau)^+], \tag{2.2}$$

where the supremum is taken over all stopping times of the geometric Brownian motion $X_t$. The function $V(x)$ is the value function of the American put option. Our first aim is to find the optimal stopping time $\tau^*$ and also the arbitrage free price.

Under the risk neutral measure, there exists a strong unique solution to equation (2.1), which is

$$X_t = x \exp\left((r - \frac{\sigma^2}{2})t + \sigma B_t\right),$$

with $x > 0$ and for $t \geq 0$. This is a Markov diffusion with infinitesimal generator

$$\mathscr{L}_X = rx\frac{\partial}{\partial x} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2}. \tag{2.3}$$

## 2.2 Perpetual American options

The aim of this section is to solve the optimal stopping problem given by equation (2.2). It is required to find the value function $V(x)$ and the optimal stopping time $\tau_b$. In this section information from Section 25.1 in [5] and Section 26.1 in [1] is used.

At all times, there are two possible alternatives for a perpetual American option, waiting or exercising. Since the payoff is $(K - X_t)^+$, it is easy to see that when $X_t \geq K$, waiting is always optimal and when $X_t < K$, it could be optimal to exercise the option. However in this case, there is also a possibility that waiting is optimal. Therefore there should exist some point $b \in (0, K)$ such that the stopping time $\tau_b = \inf\{t \geq 0 : X_t \leq b\}$ is optimal in equation (2.2). This $b$ is the boundary of the continuation region, and is not known beforehand, hence it is a free boundary problem.

The free boundary problem for the American put option is:

$$\mathscr{L}_X V = rV \qquad \text{for } x > b \tag{2.4}$$

$$V(x) = (K-x)^+ \qquad \text{for } x = b \tag{2.5}$$

$$V'(x) = -1 \qquad \text{for } x = b \tag{2.6}$$

$$V(x) > (K-x)^+ \qquad \text{for } x > b \tag{2.7}$$

$$V(x) = (K-x)^+ \qquad \text{for } 0 < x < b \tag{2.8}$$

We want to have continuity, and therefore the second condition is needed. Moreover, the intention is for the function to have a smooth fit, so the condition that $V' = -1$ at the point $b$ is added, which is the third condition.

Then, together with the definition of the infinitesimal generator for the geometric Brownian motion, as in equation (2.3), the first statement, equation (2.4), in this problem can be written as

$$\frac{\sigma^2}{2}x^2V''(x) + rxV'(x) - rV(x) = 0. \tag{2.9}$$

This is an ordinary differential equation of Euler type, and it has solutions on the form $V(x) = x^p$. Now (2.9) becomes a quadratic equation which can be easily solved to find the roots. One obtains that the general solution for this ordinary differential equation is

$$V(x) = C_1 x + C_2 x^{-\beta},$$

where $C_1$ and $C_2$ are some constants and $\beta = 2r/\sigma^2$. However, since $V(x) \leq K$, we must have $C_1 = 0$. Hence $V$ is on the form

$$V(x) = C_2 x^{-\beta}.$$

From the condition of the free-boundary problem regarding continuity, equation (2.5) stated that $V(b) = K - b$. Hence

$$K - b = C_2 b^{-\beta}.$$

This implies that

$$C_2 = \frac{K-b}{b^{-\beta}}. \tag{2.10}$$

To continue, the smooth fit condition is used, equation (2.6), such that

$$-C_2\beta b^{-\beta-1} = -1.$$

Inserting (2.10) and simplifying, one obtains

$$b = \frac{K\beta}{1+\beta}. \tag{2.11}$$

Therefore, with $C_2$ as in (2.10) and $C_1 = 0$, the value function is determined to be

$$V(x) = \begin{cases} (K-b)(\frac{x}{b})^{-\beta} & \text{if } x \in (b,\infty), \\ K-x & \text{if } x \in (0,b]. \end{cases} \tag{2.12}$$

Here the boundary value $b$ is as in equation (2.11).

It is noted that the value function $V$ is twice continuously differentiable on $(0,\infty)$ except at the boundary point $b$, i.e. $V \in C^2((0,b)\cup(b,\infty))$. At the point $b$, $V \in C^1$. It is also worth to note that $V$ is convex on $(0,\infty)$.

Now that the solution to the free boundary problem has been found, the next step is to verify that this is indeed the value of the optimal stopping problem (2.2).

**Theorem 2.1.** *The arbitrage-free price $V$ as stated in equation* (2.2) *is explicitly given by the system of equations* (2.12). *Moreover, with $b$ as in equation* (2.11), *the stopping time $\tau_b$ is optimal in the problem* (2.2).

*Proof.* The aim is to prove that the function $V$ as stated in this theorem is the function given in equation (2.2). Denote the value function from equation (2.2) by $V_*$ so the goal is to prove $V_*(x) = V(x)$ for all $x > 0$.

First, apply Itô's formula to $e^{-rt}V(X_t)$ and integrate to obtain

$$\begin{aligned} e^{-rt}V(X_t) = V(x) + \int_0^t e^{-rs}(\mathscr{L}_X V - rV)(X_s)\mathbb{I}(X_s \neq b)ds \\ + \int_0^t e^{-rs}\sigma X_s V'(X_s)dB_s. \end{aligned} \tag{2.13}$$

Denote $G(x) = (K-x)^+ = (K-x)\mathbb{I}\{x<K\}$, then

$$\begin{aligned} (\mathscr{L}_X G - rG)(x) &= -rx\mathbb{I}\{x<K\} - r(K-x)\mathbb{I}\{x<K\} \\ &= -rK\mathbb{I}\{x<K\} \end{aligned}$$

Since $K > 0$, it is obvious that $(\mathscr{L}_X G - rG)(x) < 0$. Now this result together with (2.4), the first condition in the free boundary problem, yields

$$(\mathscr{L}_X V - rV)(x) \leq 0, \tag{2.14}$$

which holds at every point on $(0, \infty)$ except at the boundary point $b$. Moreover, under the probability measure $\mathbb{P}_x$, it holds that $\mathbb{P}_x(X_s = b) = 0$ for all $s$ and all $x$. Therefore one can use the inequality (2.14) inserted in (2.13) to draw the conclusion that

$$e^{-rt} V(X_t) \leq V(x) + \int_0^t e^{-rs} \sigma X_s V'(X_s) dB_s. \tag{2.15}$$

Moreover, by (2.7) and (2.8), it is evident that

$$e^{-rt}(K - X_t)^+ \leq e^{-rt} V(X_t). \tag{2.16}$$

Denote by $M_t = \int_0^t e^{-rs} \sigma X_s V'(X_s) dB_s$. We see that $M = (M_t)_{t \geq 0}$ is a continuous local martingale. Then combining (2.15) and (2.16) we have:

$$e^{-rt}(K - X_t)^+ \leq e^{-rt} V(X_t) \leq V(x) + M_t. \tag{2.17}$$

Let $(\tau_n)_{n \geq 1}$ be a sequence of bounded stopping times of the martingale $M$. By the inequality (2.17), and for every stopping time $\tau$ of the process $X$, it follows that

$$e^{-r(\tau \wedge \tau_n)}(K - X_{\tau \wedge \tau_n})^+ \leq V(x) + M_{\tau \wedge \tau_n},$$

for all $n \geq 1$. Taking expectation, the martingale $M_{\tau \wedge \tau_n}$ vanishes, i.e.

$$\mathbb{E}_x[M_{\tau \wedge \tau_n}] = 0,$$

for all $n$. Hence

$$\mathbb{E}_x[e^{-r(\tau \wedge \tau_n)}(K - X_{\tau \wedge \tau_n})^+] \leq V(x).$$

Let $n \to \infty$ and by Fatou's lemma it follows that

$$\mathbb{E}_x[e^{-r\tau}(K - X_\tau)^+] \leq V(x),$$

Now, taking the supremum yields

$$\sup_\tau \mathbb{E}_x[e^{-r\tau}(K - X_\tau)^+] \leq V(x).$$

However, the left hand side in the inequality is exactly equation (2.2) and so in conclusion

$$V_*(x) \leq V(x)$$

for all $x > 0$.

To prove the desired equality, one needs to continue and prove the reverse inequality from what was just proved, i.e. $V_*(x) \geq V(x)$. Let $\tau_b$ be defined as above. Then by the same logic as in equation (2.13), one can write

$$e^{-r(\tau_b \wedge \tau_n)} V(X_{\tau_b \wedge \tau_n}) = V(x) + \int_0^{\tau_b \wedge \tau_n} e^{-rs} (\mathscr{L}_X V - rV)(X_s) \mathbb{I}(X_s \neq b) ds + \int_0^{\tau_b \wedge \tau_n} e^{-rs} \sigma X_s V'(X_s) dB_s. \quad (2.18)$$

By equation (2.4), which can be used since $X_s > b$, the second term in equation (2.18) vanishes and so one obtains

$$e^{-r(\tau_b \wedge \tau_n)} V(X_{\tau_b \wedge \tau_n}) = V(x) + \int_0^{\tau_b \wedge \tau_n} e^{-rs} \sigma X_s V'(X_s) dB_s. \quad (2.19)$$

Continue by taking expectations on both sides and end up with

$$\mathbb{E}_x \left[ e^{-r(\tau_b \wedge \tau_n)} V(X_{\tau_b \wedge \tau_n}) \right] = V(x),$$

for all $n \geq 1$, since the second term in equation (2.19) vanishes under expectation.

By equation (2.5), $V(X_{\tau_b}) = (K - X_{\tau_b})^+$, which implies that $e^{-r\tau_b} V(X_{\tau_b}) = e^{-r\tau_b}(K - X_{\tau_b})^+$. When $\tau_b = \infty$, this equation is 0.

Let $n \to \infty$. Then by the dominated convergence theorem it follows that

$$\mathbb{E}_x \left[ e^{-r\tau_b}(K - X_{\tau_b})^+ \right] = V(x).$$

Hence it is proved that $\tau_b$ is optimal in the perpetual American put problem (2.2). Consequently $V_*(x) = V(x)$ for all $x > 0$. $\square$

## 2.3 The American call option

The material in this subsection is taken from [1]. It is natural to question why it is only worthwhile to analyze the case with the perpetual American put option and not also the American call option. The American call option gives the buyer the option, but not the obligation, to buy the option at any time point for the strike price $K$.

Without dividends, it is never optimal to exercise the American call option early. The proof of this is as follows. For the American call option, the value function is

$$V(x) = \sup_{\tau} \mathbb{E}_x[e^{-r\tau}(X_\tau - K)^+].$$

That means that the American call option is in-the-money whenever $X_\tau > K$ and out-of-the-money whenever $X_\tau \leq K$.

Let $Z$ be the the reward process of this problem, that is,

$$Z_t = e^{-rt}(X_t - K)^+ = (e^{-rt}X_t - e^{-rt}K)^+.$$

We know that the process $e^{-rt}X_t$ is a martingale with respect to the risk neutral measure. However $e^{-rt}K$ is deterministic and also decreasing, so we can deduce that it is in fact a supermartingale. Therefore, $e^{-rt}X_t - e^{-rt}K$ is a submartingale. Moreover, $x \mapsto x^+$ is both convex and increasing. Putting all this together, we can conclude that the reward process is a submartingale. This follows because a convex, increasing function of a submartingale is itself a submartingale.

Hence, it is not optimal to exercise early and so the price of an American call option coincides with the European option.

# 3 Option price with respect to the strike price $K$

After finding the value function with respect to the stock price, a natural follow up is to find the value function with respect to the strike price $K$. Outside of theory, what is most often known is the market prices of the option with respect to different strike prices. Therefore it is natural to derive a formula for this function as well. A natural guess is that the value function $V(K)$ looks very similar to the function $V(x)$ derived in the previous section. This is the topic for this next section.

However, even though we will use the notation $\hat{V}(K)$ for the value function with respect to the strike price, and in the last section we used the notation $V(x)$ for the value function with respect to the stock price, the truth is that the value function for the option price is a function of both the strike price and the initial stock price. Hence to be extra correct, the true notation should be $V(x,K)$, because the value of the option price does of course not change depending on which method you use for deriving it.

While the approach in Section 2.2 is the most common one, we find it more convenient to follow another approach. This approach is better suited for studies of implied volatility as it produces the option value for different strikes rather than for a fixed strike but for several possible initial stock prices.

## 3.1 The value function $\hat{V}(K)$

In this section the material closely follows [4]. With the underlying stock denoted by $X_t$, which is characterized by the stochastic differential equation

$$dX_t = rX_t + \sigma(X_t)X_t dB_t, \quad X_0 = x_0,$$

the price of a perpetual American put with strike price $K$ is

$$\hat{V}(K) = \sup_{\tau} \mathbb{E}_{x_0}[e^{-r\tau}(K - X_\tau)]. \tag{3.1}$$

The supremum is taken over stopping times $\tau$, and the risk free rate $r$ is positive. The payoff function of the American option has properties which are useful in regards to the properties of the function $\hat{V}(K)$. The payoff function is bounded, monotone and convex from which the following proposition follows immediately.

**Proposition 3.1.** *The function $\hat{V} : (0,\infty) \to [0,\infty)$ satisfies the following conditions:*

*(i)* $(K - x_0)^+ \leq \hat{V}(K) \leq K$ *for all* $K$*;*

*(ii) The function* $\hat{V}$ *is non-decreasing and convex.*

**Example 3.2.** Assume the stock price follows a geometric Brownian motion, that is

$$dX_t = rX_t + \sigma X_t dB_t, \quad X_0 = x_0,$$

with a positive risk free rate $r$, constant volatility $\sigma > 0$ and no dividends. Then as seen in Section 2.2, the arbitrage free price of the perpetual American option with respect to the initial stock price $x_0$ is given by equation (2.12). This function depends on the strike price $K$ as well. Therefore one can obtain an expression for the value function with respect to the strike price by inverting the "if statements" to state some condition concerning the strike price $K$ instead. Since $b$ was found to be $b = K\beta/(1+\beta)$, the relation is

$$x_0 > \frac{K\beta}{1+\beta} \Leftrightarrow \frac{x_0(1+\beta)}{\beta} > K.$$

Denote $\hat{K} := x_0(1+\beta)/\beta$. Then for the perpetual American option modeled using the geometric Brownian motion, the value function with respect to the strike price is

$$\hat{V}(K) = \begin{cases} \frac{K}{1+\beta}\left(\frac{x_0(1+\beta)}{\beta K}\right)^{-\beta}, & \text{if } K < \hat{K}, \\ K - x_0, & \text{if } K \geq \hat{K}, \end{cases} \tag{3.2}$$

where $\beta = 2r/\sigma^2$ as before. Note that this value function depends more clearly on the strike price $K$ instead of the initial stock price $x_0$ as in (2.12). As stated before, the value function is in reality a function of both the initial stock price and the strike price, and both equation (2.12) and (3.2) will produce the same option price for the same initial stock price $x_0$ and strike price $K$.

Let us return to the general setting again. Continuing from the starting problem, equation (3.1), the following ordinary differential equation can be obtained

$$\frac{\sigma^2(x)}{2}x^2 u_{xx} + rxu_x - ru = 0, \tag{3.3}$$

for $x > 0$. This has two linearly independent solutions, $\psi$ and $\varphi$. The two solutions are linearly independent, and furthermore it is assumed that $\psi$ is positive and increasing and $\varphi$ is positive and decreasing, and they are unique up to multiplication with a positive constant. The solutions are then given by

$$\psi(x) = Cx,$$

and

$$\varphi(x) = Dx \int_x^\infty \frac{1}{y^2} \exp\left(-\int_{x_0}^y \frac{2r}{z\sigma(z)^2} dz\right) dy, \tag{3.4}$$

for some constants $C, D > 0$. Without loss of generality, let

$$D = \left(x_0 \int_{x_0}^\infty \frac{1}{y^2} \exp\left(-\int_{x_0}^y \frac{2r}{z\sigma(z)^2} dz\right) dy\right)^{-1}.$$

Then $\varphi(x_0) = 1$, which simplifies the solution and our computations in the future.

That $\psi$ is increasing is immediate. To continue, it is necessary to prove the assumption that $\varphi$ is decreasing.

**Lemma 3.3.** *The function $\varphi$ is strictly decreasing and strictly convex.*

*Proof.* This can be proved by analyzing the first derivative and second derivative of the function. Differentiation of $\varphi$ yields

$$\begin{aligned}
\varphi'(x) &= D \int_x^\infty \frac{1}{y^2} \exp\left(-\int_{x_0}^y \frac{2r}{z\sigma(z)^2} dz\right) dy + Dx \left(-\frac{1}{x^2} \exp\left(-\int_{x_0}^x \frac{2r}{z\sigma(z)^2} dz\right)\right) \\
&= D\left(\int_x^\infty \frac{1}{y^2} \exp\left(-\int_{x_0}^y \frac{2r}{z\sigma(z)^2} dz\right) dy - \frac{1}{x} \exp\left(-\int_{x_0}^x \frac{2r}{z\sigma(z)^2} dz\right)\right) \\
&= D\left(\int_x^\infty \frac{1}{y^2} \exp\left(-\int_{x_0}^y \frac{2r}{z\sigma(z)^2} dz\right) dy - \int_x^\infty \frac{1}{y^2} \exp\left(-\int_{x_0}^x \frac{2r}{z\sigma(z)^2} dz\right) dy\right) \\
&= D \int_x^\infty \frac{1}{y^2} \left( \exp\left(-\int_{x_0}^y \frac{2r}{z\sigma(z)^2} dz\right) - \exp\left(-\int_{x_0}^x \frac{2r}{z\sigma(z)^2} dz\right)\right) dy.
\end{aligned}$$

From this it is observed that $\varphi'(x) < 0$ so the function $\varphi$ is strictly decreasing. Differentiating again, one obtains

$$\varphi''(x) = \frac{2Dr}{x^2\sigma(x)^2} \exp\left(-\int_{x_0}^x \frac{2r}{z\sigma(z)^2} dz\right).$$

Hence $\varphi''(x) > 0$ and so $\varphi$ is strictly convex. □

Let $H_z$ be a hitting time given by $H_z = \inf\{t \geq 0 : X_t = z\}$. Moreover, since both $\varphi$ and $\psi$ are solutions to equation (3.3), we know that $e^{-rt}\varphi(X_t)$ and $e^{-rt}\psi(X_t)$ are local martingales. Hence $\mathbb{E}_{x_0}[e^{-rH_z}\varphi(X_{H_z})] = \varphi(x_0)$ for $z \leq x_0$ and $\mathbb{E}_{x_0}[e^{-rH_z}\psi(X_{H_z})] = \psi(x_0)$ for $z \geq x_0$.

This gives

$$\mathbb{E}_{x_0}[e^{-rH_z}] = \begin{cases} \frac{\varphi(x_0)}{\varphi(z)}, & \text{if } z < x_0, \\ \frac{\psi(x_0)}{\psi(z)}, & \text{if } z > x_0. \end{cases} \tag{3.5}$$

Define

$$\begin{aligned} \tilde{V}(K) &:= \sup_{z:z\leq x_0\wedge K} \mathbb{E}_{x_0}[e^{-rH_z}(K - X_{H_z})^+] \\ &= \sup_{z:z\leq x_0\wedge K} (K-z)\mathbb{E}_{x_0}[e^{-rH_z}] \\ &= \sup_{z:z\leq x_0\wedge K} \frac{(K-z)\varphi(x_0)}{\varphi(z)} \\ &= \sup_{z:z\leq x_0\wedge K} \frac{(K-z)}{\varphi(z)}. \end{aligned} \tag{3.6}$$

The third equality follows from the assumptions on the supremum, which clarifies that the first case in equation (3.5) should be used. Moreover, from the assumption that $\varphi(x_0) = 1$, the last equality follows.

It is clear that $\hat{V}(K) \geq \tilde{V}(K)$, since the supremum over all possible stopping times is clearly larger than or equal to the supremum over just the first hitting times. Moreover this inequality is actually an equality but this will be shown later.

As proved, the function $\varphi$ is strictly convex, which implies that there exists a unique $z = z(K) \leq x_0$, for each fixed $K$, such that the supremum of equation (3.6) is attained. With this unique $z$, equation (3.6) can therefore be simplified to

$$\tilde{V}(K) = \frac{K - z(K)}{\varphi(z(K))}. \tag{3.7}$$

The geometrical interpretation of the value $z = z(K)$ is that it is the unique value which maximizes the negative slope of the line segment through the points $(K,0)$ and $(z,\varphi(z))$, where $z$ is always less than or equal to $x_0$. This is visualized in Figure 1. The option price $\hat{V}(K)$ also has a geometrical interpretation. For a fixed $K \leq \hat{K}$, $\hat{V}(K)$ is the negative reciprocal of the slope of the unique tangent line of $\varphi$ which goes through $(K,0)$.

The intention is to find the point $\hat{K}$, also pictured in Figure 1. This is the smallest value of $K$ with $z(K) = x_0$, i.e. with the price function for the starting value.

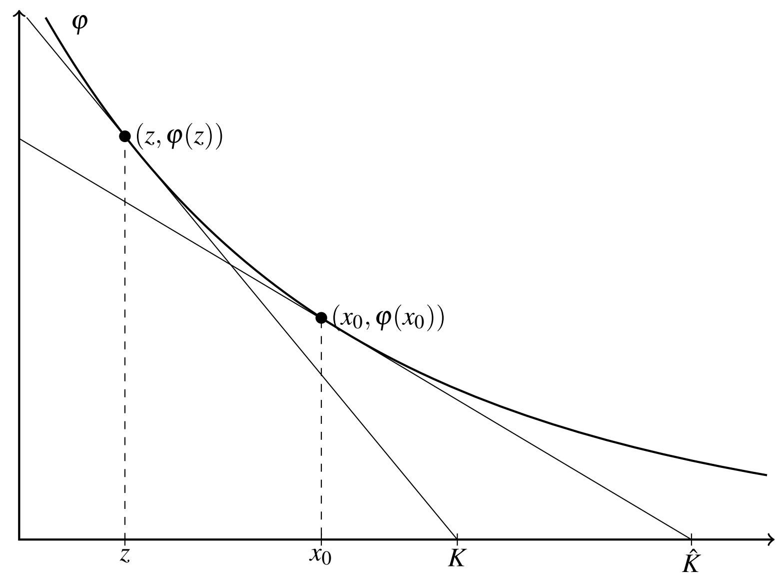


Figure 1: The geometrical visualization of how to find the option price $\hat{V}(K)$.

Define

$$\hat{K} := x_0 - \frac{1}{\varphi'(x_0)}. \tag{3.8}$$

As proved in Lemma 3.3, $\varphi$ is strictly convex. Hence if $K \geq \hat{K}$, then

$$\begin{aligned}\tilde{V} &= \sup_{z:z\leq x_0} \frac{K-z}{\varphi(z)} \\ &= \frac{K-x_0}{\varphi(x_0)} \\ &= K - x_0.\end{aligned}$$

Here the last equality follows from the fact that $\varphi(x_0) = 1$. For the other inequality, if $K \leq \hat{K}$, then $z \leq x_0$ always, so

$$\begin{aligned}\tilde{V}(K) &= \sup_{z:z\leq x_0} \frac{K-z}{\varphi(z)} \\ &= \sup_{z} \frac{K-z}{\varphi(z)}.\end{aligned} \tag{3.9}$$

From this equation it is seen that if one has strict inequality $K < \hat{K}$, then it follows that $\tilde{V} > (K - x_0)^+$.

**Lemma 3.4.** *The functions $\hat{V}$ and $\tilde{V}$ coincide, meaning,*

$$\hat{V}(K) = \sup_{z:z\leq x_0} \frac{K-z}{\varphi(z)}. \tag{3.10}$$

*Proof.* As stated previously, $\hat{V} \geq \tilde{V}$ always. To prove the other inequality, assume that $K \leq \hat{K}$. That is the same case as in equation (3.9) so then $\varphi(z) \geq (K-z)^+/\tilde{V}(K)$. Note that $e^{-rt}\varphi(X_t)$ is a local martingale, and also non-negative. That means that it is a supermartingale. Therefore, for any stopping time $\tau$ the following hold

$$\begin{aligned} 1 \geq \mathbb{E}_{x_0}[e^{-r\tau}\varphi(X_\tau)] \geq \mathbb{E}_{x_0}&[e^{-r\tau}(K - X_\tau)^+/\tilde{V}(K)] \\ &\Updownarrow \\ \hat{V} \geq \sup_{\tau}\, &\mathbb{E}_{x_0}[e^{-r\tau}(K - X_\tau)^+]. \end{aligned} \tag{3.11}$$

However, note that the right side in the equation (3.11) is exactly the definition of $\hat{V}$, hence $\tilde{V} \geq \hat{V}$.

Lastly, let $K > \hat{K}$ and then it follows that $\hat{V}(\hat{K}) = \hat{K} - x_0$. By Proposition 3.1 we can then draw the conclusion that $\hat{V}(K) = K - x_0$, which is exactly $\tilde{V}(K)$. Hence $\hat{V} = \tilde{V}$. □

**Example 3.5.** Return back to the case with the diffusion $X$ as a geometric Brownian motion. Then the decreasing function $\varphi(x)$ is

$$\varphi(x) = \left(\frac{x_0}{x}\right)^\beta,$$

with $\beta = 2r/\sigma^2$. Inserting this into the equation for $\hat{V}(K)$, equation (3.10), we can obtain the option price.

$$\begin{aligned} \hat{V}(K) &= \sup_{z:z\leq x_0} \frac{K-z}{\varphi(z)} \\ &= \sup_{z:z\leq x_0} \frac{K-z}{(x_0/z)^\beta} \\ &= x_0^{-\beta} \sup_{z:z\leq x_0} (K-z)z^\beta. \end{aligned}$$

Differentiation with respect to $z$ yields

$$\hat{V}' = x_0^{-\beta}\beta K z^{\beta-1} - x_0^{-\beta}(\beta+1)z^{\beta}.$$

Set the derivative equal to zero and rearrange to find the unique $z$ where the supremum is attained.

$$\begin{gathered} 0 = \beta K - (\beta+1)z \\ \Updownarrow \\ z = \frac{\beta K}{\beta+1} \end{gathered}$$

Define

$$z^* := \frac{\beta K}{\beta+1}.$$

Then if $z^* < x_0$, the supremum is attained for $z = z^*$ and if $z^* \geq x_0$, the supremum is attained for $z = x_0$. This coincides exactly with what we have already established in Example 3.2, and the price found in equation (3.2).

Note that it is possible that the process $X$ hits zero in finite time, under the assumptions we have made. In this case zero must be absorbing. We see that $X$ hits zero in finite time if and only if $\varphi(0) < \infty$. In this instance, set $\underline{K} = -\varphi(0)/\varphi'(0)$. Thus when $\varphi'(0) < \infty$, then $\underline{K} > 0$, and for all $K < \underline{K}$, let $z(K) = 0$ and $\hat{V}(K) = K/\varphi(0)$. Moreover, since $\varphi$ is strictly convex, then $\lim_{K \downarrow \underline{K}} z(K) = 0$.

In Proposition 3.1, two properties about the price function $\hat{V}$ were established. Let us continue this, with the following proposition.

**Proposition 3.6.** *The function $\hat{V} : [0,\infty) \to [0,\infty)$ satisfies the following conditions:*

*(i)* $\hat{V} > (K - x_0)^+$ *for all* $K \in (0, \hat{K})$ *and*
$\hat{V} = K - x_0$ *for all* $K \geq \hat{K}$*;*

*(ii)* $\hat{V}$ *is continuously differentiable on* $(0,\infty)$ *and twice continuously differentiable on* $(0,\infty)\setminus\{\underline{K}, \hat{K}\}$*;*

*(iii)* $\hat{V}$ *is strictly increasing on* $(0,\infty)$ *with a strictly positive second derivative on* $(\underline{K}, \hat{K})$.

*Proof.* Statement (i) holds for the function $\tilde{V}$, and as proved in Lemma 3.4, the functions $\hat{V}$ and $\tilde{V}$ coincide. We have $\underline{K} < K < \hat{K}$, and by (3.9) and since $\hat{V}$ and $\tilde{V}$ coincide as stated, we have

$$\hat{V}(K) = \sup_z \frac{K-z}{\varphi(z)} = \frac{K-z(K)}{\varphi(z(K))},$$

with $z(K) \in (0, x_0)$. We defined $z = z(K)$ to be the unique value, with $z \leq x_0$ such that $(K - z(K))/\varphi(z(K))$ is maximized. As stated before, the price $\hat{V}(K)$ is the negative reciprocal of the slope of the the tangent line which goes between $(z, \varphi(z))$ and $(0, K)$. That is

$$\begin{aligned} \hat{V}(K) = \frac{K-z(K)}{\varphi(z(K))} &= -\frac{1}{\varphi'(z(K))} \\ &\Updownarrow \\ \varphi(z(K)) &= -(K-z(K))\varphi'(z(K)). \end{aligned} \tag{3.12}$$

So from equation (3.12) and with the use of the implicit function theorem, one can conclude that $z(K)$ is continuously differentiable for $\underline{K} < K < \hat{K}$.

Now, since it is known that $z'(K)$ is differentiable, one can use this knowledge in the differentiation of the quotient $(K - z(K))/\varphi(z(K))$. Differentiate (3.7) with respect to $K$ to obtain

$$\begin{aligned} \hat{V}'(K) &= \frac{(1-z'(K))\varphi(z(K)) - (K-z(K))z'(K)\varphi'(z(K))}{(\varphi(z(K)))^2} \\ &= \frac{(1-z'(K))\varphi(z(K)) + z'(K)\varphi(z(K))}{(\varphi(z(K)))^2} \\ &= \frac{1}{\varphi(z(K)))}. \end{aligned} \tag{3.13}$$

The second equality here followed from (3.12). Remember that the current setting is $\underline{K} < K < \hat{K}$. Then $\hat{V}'(\hat{K}-) = 1/\varphi(x_0) = 1$, hence $\hat{V}$ is continuously differentiable at $\hat{K}$. Moreover, $\hat{V}'(\underline{K}+) = 1/\varphi(0+)$, hence $\hat{V}$ is continuously differentiable at the point $\underline{K}$ as well.

We know that $\varphi$ is continuously differentiable and $z(K)$ is continuously differentiable except at $\hat{K}$. Hence $\hat{V}$ is twice continuously differentiable on $(0,\infty)\backslash\{\underline{K}, \hat{K}\}$. The second statement is now proven.

In the setting $\underline{K} < K < \hat{K}$, differentiate (3.13).

$$\begin{aligned}\hat{V}''(K) &= \frac{-z'(K)\varphi'(z(K))}{(\varphi(z(K)))^2} \\ &= \frac{\varphi'(z(K)))^2}{(K - z(K))(\varphi(z(K)))^2\varphi''(z(K))}\end{aligned}$$

Hence $\hat{V}'' > 0$, so $\hat{V}$ has a strictly positive second derivative on $(\underline{K}, \hat{K})$.

$\square$

# 4 Implied volatility

Since the volatility of a process over a future interval is not often known, and cannot be directly observed, there are two different approaches to acquire the volatility. One possibility is to observe the past data of the stock price process and in that way obtain a so-called *historical volatility*. However, what is of interest, and what is most often used in financial settings, is what is called *implied volatility*. Since historical volatilities are constructed with past data, they are not necessarily useful in predicting the future behavior of stocks and options. Implied volatility, however, predicts how the stock price process will behave and is a good measurement on the behavior of the stock in the future.

Thus the next step is to find the implied volatility of the perpetual American option. The aim is to use the knowledge gained in the previous section about the value function with respect to the strike price, to obtain the (implied) volatility $\hat{\sigma}$. If it is possible, the goal is to find an explicit formula for the implied volatility. However in the case of perpetual American put options, this may be beyond the scope of this paper, and in that case, the aim is to find properties of the implied volatility instead.

## 4.1 The implied volatility skew

As written in [1], in standard computation with European options (and with geometric Brownian motions), the volatility is constant. Therefore by obtaining market prices for options with different strike prices, and then solving the Black-Scholes formula for $\sigma$, one can obtain the implied volatility $\hat{\sigma}$. This can be used as a test for the model, in the case of the European option with the geometric Brownian motion as the underlying asset, the implied volatility one obtains should be constant with respect to the strike price. However, as mentioned in [4], it has been observed that by using real market data, the implied volatility is non-constant and is similar to a smile. This is called the smile effect. It shall also be mentioned that in some cases, the implied volatility has been said to be skewed.

In this thesis we are interested in implied volatility skews, i.e. when the implied volatility as a function of the strike price is decreasing. This is exactly what is visualized in Figure 2.

In theory one can discuss several different cases which induces implied volatility skews. For example, these could be stochastic volatility models, jump processes and decreasing volatility models. In [2], Cont and Tankov describe the

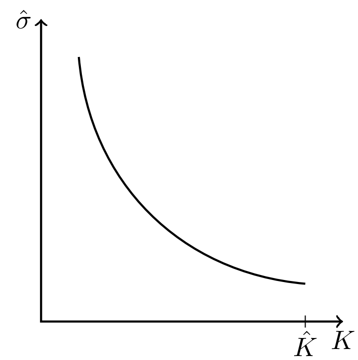


Figure 2: The implied volatility skew.

different behavior of the implied volatility concerning stochastic volatility models and jump processes.

First we use [2] to define stochastic volatility models. A stochastic volatility model is one such that the volatility function has randomness included in it, it is not a deterministic function as before. That is, the process $X$ is as usual characterized by the stochastic differential equation

$$dX_t = rX_t dt + \sigma_t dB_t,$$

but $(\sigma_t)_{t \geq 0}$ is a positive stochastic process now. In many cases, one chooses to use a mean-reverting stochastic process. A mean-reverting process is a process which oscillates around the mean.

The random process $\sigma_t$ is driven by a Brownian motion, just like the process $X_t$. It is the correlation between these Brownian motions which induces the implied volatility skews. It has been argued that a negative correlation between these Brownian motions will lead to an implied volatility skew. Moreover, Brownian motions have symmetric increments. Thus the correlation coefficient $\rho$ is responsible for the asymmetry in risk-neutral distribution, so therefore it is also responsible for the skew in the implied volatility. Note that stochastic volatility models can also induce an implied volatility smile, however we are not interested in that aspect in this thesis.

In the real world, a negative correlation coefficient is in most cases correlated to the *leverage effect*. This effect is what we call when a downward advancement is observed in the prices which causes upward moves in volatility. Thus it is natural to say that this leverage effect is a probable cause for the implied volatility skew. An example of a stochastic volatility model where the correlation coefficient may be non-zero is the Heston model.

Cont and Tankov in [2] also define another model which gives rise to implied volatility skews are jump diffusion models. One such risk-neutral model is the exponential Lévy model. Let $X$ be a Lévy process. A Lévy process is a process with stationary, independent increments. Then, an exponential-Lévy model is a model where the asset price $S_t$ can be represented by

$$S_t = S_0 \exp(rt + X_t).$$

Cont and Tankov write that implied volatility surfaces in exponential-Lévy models have some particular properties. In particular, they state that a jump distribution which is negatively skewed induces an implied volatility skew. This is further supported by the quote "the presence of a skew is attributed to the fear of large negative jumps by market participants", which Cont and Tankov state at [2, p. 10]. Note that with the use of jump process models, just as with stochastic models, both implied volatility smiles and skews can be present.

In addition to these stochastic models and jump process, another model which induces an implied volatility skew is a model which has a decreasing volatility function in the underlying asset. This is the topic for the next part of this thesis.

## 4.2 Properties of implied volatility

Assume an explicit volatility function $\sigma(x)$ is given. Then $\sigma(x)$ can be used in the solution $\varphi(x)$ to find the optimal exercise level and the corresponding option price, for a given strike price $K$.

Thus, given two strike prices, with $K_1 < K_2 < \hat{K}$, we can find the corresponding exercise levels $z_1$ and $z_2$. As stated in Section 3.1, the optimal exercise level $z_1$ maximizes the option value $V(K_1)$. This is equivalent to saying that $z_1$ maximizes the negative slope of the line segment between $(z_1, \varphi(z_1))$ and $(K_1, 0)$. Exactly the same argument can be made for the strike price $K_2$ and its optimal exercise level $z_2$. It follows that these line segments just described has the slopes $-1/V(K_1)$, and $-1/V(K_2)$, respectively.

It is natural to hypothesize that if we have the option price for a given strike price, another decreasing, convex function $\hat{\varphi}$ can be fitted, such that it induces the same option price for the same strike price but has the underlying model of a geometric Brownian motion instead. The geometric Brownian motion has a constant volatility hence one can find the implied volatility for this strike price in this way. The same argument can be made for the strike price $K_2$. Then we have obtained the implied volatilities for both $K_1$ and $K_2$. In Figure 3, the idea for this can be visualized.

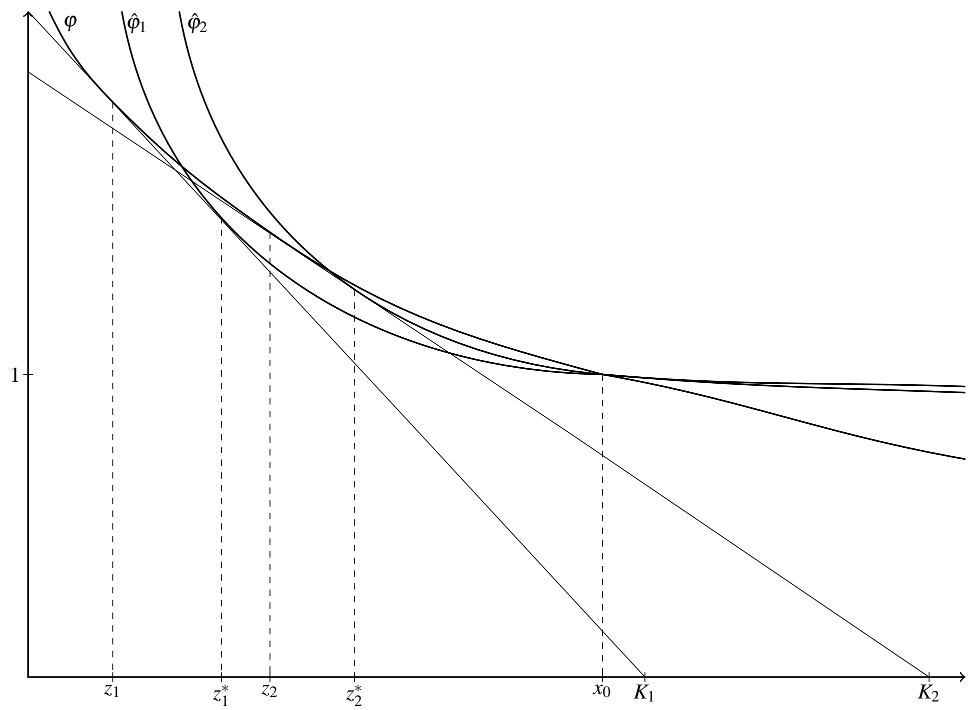


Figure 3: Visualization of the main idea for comparing implied volatilities, here with a decreasing volatility function $\sigma(x)$.

Let $\varphi$ be a function which induces some option prices, such as in equation (3.4). Let $\beta_1 = 2r/\hat{\sigma}_1^2$ and $\beta_2 = 2r/\hat{\sigma}_2^2$. Moreover let $\hat{\varphi}_1(x) = (x/x_0)^{-\beta_1}$ and $\hat{\varphi}_2(x) = (x/x_0)^{-\beta_2}$, be functions with constant volatility, which also satisfies equation (3.4). Saying that $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are of the form of equation (3.4), is the same as saying that they are solutions to the ordinary differential equation (3.3) with the diffusion coefficients being $\hat{\sigma}_1$ and $\hat{\sigma}_2$, respectively. First we can state a simple Lemma, which can be used on both the decreasing and increasing volatility cases.

**Lemma 4.1.** *If $\hat{\varphi}_1(x) \leq \hat{\varphi}_2(x)$ on $[0, x_0]$ then the volatility of $\hat{\varphi}_1(x)$ is higher or equal to the the volatility of $\hat{\varphi}_2(x)$.*

*Proof.* Denote the volatility of $\hat{\varphi}_1$ by $\hat{\sigma}_1$ and the volatility of $\hat{\varphi}_2$ by $\hat{\sigma}_2$. Note that it follows that $x/x_0 \in [0, 1]$, for all $x$ in the relevant domain $[0, x_0]$.

First of all, note that for $\beta_1 \neq \beta_2$, the equation

$$\left(\frac{x}{x_0}\right)^{-\beta_1} = \left(\frac{x}{x_0}\right)^{-\beta_2},$$

does not have a solution on $(0, x_0)$. Therefore the assumption that $\hat{\varphi}_1(x) \leq \hat{\varphi}_2(x)$ on $[0, x_0]$ can be simplified to saying $\hat{\varphi}_1(x) < \hat{\varphi}_2(x)$ on $(0, x_0)$ if $\beta_1 \neq \beta_2$, or else $\beta_1 = \beta_2$ and then $\hat{\varphi}_1 = \hat{\varphi}_2$.

Thus assume that $\hat{\varphi}_1(x) < \hat{\varphi}_2(x)$ on $(0, x_0)$. Then

$$\left(\frac{x}{x_0}\right)^{-\beta_1} < \left(\frac{x}{x_0}\right)^{-\beta_2}.$$

From this, it follows that

$$\left(\frac{x_0}{x}\right)^{\beta_1} < \left(\frac{x_0}{x}\right)^{\beta_2}.$$

Since all $x < x_0$, $(x_0/x) > 1$, hence it follows that $\beta_1 < \beta_2$. Then by the definition of $\beta_1$ and $\beta_2$, we must have $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$ and then immediately $\hat{\sigma}_1 > \hat{\sigma}_2$, which is the desired result. $\square$

### 4.2.1 A monotonicity result with respect to volatility

We want to say something about the ordering of two functions which are attached at the same boundary values but solve different ordinary differential equations.

**Lemma 4.2.** *On an interval $[a, b]$, let $u^1$ be the solution to*

$$\begin{aligned} \frac{{\sigma_1}^2(x)}{2} x^2 u^1_{xx} + rxu^1_x - ru^1 &= 0, \\ u^1(a) &= A, \\ u^1(b) &= B, \end{aligned}$$

*with $A, B \geq 0$ and let $u^2$ be the solution to*

$$\begin{aligned} \frac{{\sigma_2}^2(x)}{2} x^2 u^2_{xx} + rxu^2_x - ru^2 &= 0, \\ u^2(a) &= A, \\ u^2(b) &= B. \end{aligned}$$

*If $\sigma_2(x)$ dominates $\sigma_1(x)$ on the interval $[a,b]$, i.e. if $|\sigma_1(x)| \leq |\sigma_2(x)|$ for all $x$ in $[a,b]$, where $a$ and $b$ satisfy $aB < bA$, then*

$$u^1(x) \leq u^2(x) \quad \forall x \in [a,b].$$

*Proof.* See Proposition 4.3 in [3] and its proof. □

First let us note that it is fine to have $b = \infty$. This Lemma states that if two functions, $u^1$ and $u^2$, solve ordinary differential equations with the same boundary values, on an interval $[a,b]$, then the dominating diffusion coefficient in the underlying processes of the solutions will induce an inequality between the solutions. The added condition of $aB < bA$ is to assure convexity of the functions $u_1$ and $u_2$.

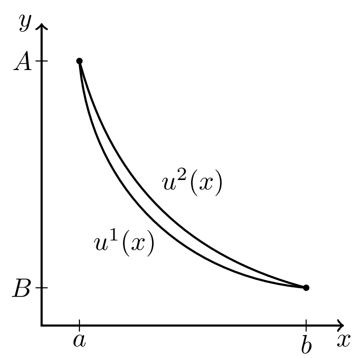


Figure 4: Two functions solving ordinary differential equations with different volatilities between the same boundary values.

## 4.3 Decreasing volatility

As briefly stated in Section 4, the hypothesis is that a decreasing volatility in the underlying asset model will induce an implied volatility skew. To be exact, we want the volatility to be non-increasing. Thus we arrive at the following theorem.

**Theorem 4.3.** *A decreasing volatility implies a decreasing implied volatility. That is, for strike prices $K_1 < K_2$, the following hold*

$$\sigma(\cdot) \text{ decreasing} \Rightarrow \hat{\sigma}(K_1) \geq \hat{\sigma}(K_2).$$

*Proof.* To be determined. □

Let $X$ and $Y$ be two price processes, satisfying the stochastic differential equations

$$\begin{aligned} dX_t &= rX_t dt + \sigma(X_t) X_t dB_t, & X_0 &= x_0, \\ dY_t &= rY_t dt + \gamma Y_t dB_t, & Y_0 &= x_0. \end{aligned}$$

Here $\sigma(x) > 0$ is a decreasing deterministic function and $\gamma > 0$ is a constant. Let $\varphi$ be a solution to the ordinary differential equation 3.3 with respect to the process $X$. Thus the ordinary differential equation with boundary conditions is the following:

$$\begin{aligned} \frac{\sigma^2(x)}{2} x^2 \varphi_{xx} + rx\varphi_x - r\varphi &= 0, \qquad (4.1) \\ \varphi(\infty) &= 0. \end{aligned}$$

Moreover let $f$ be the solution to the ordinary differential equation 3.3 with respect to the process $Y$, meaning

$$\begin{aligned} \frac{\gamma^2}{2} x^2 f_{xx} + rxf_x - rf &= 0, \qquad (4.2) \\ f(\infty) &= 0. \end{aligned}$$

Since $\gamma$ is a constant, the process $Y$ is a geometric Brownian motion and so we know that the function $f$ is on the form $f = (x/x_0)^{-2r/\gamma^2}$.

**Lemma 4.4.** *Let $\gamma > 0$ and $a > 0$ be given constants such that the function $g$ satisfies*

$$\frac{\gamma^2}{2} x^2 g_{xx} + rxg_x - rg = 0, \quad x > a, \qquad (4.3)$$

$$g(a) = \varphi(a), \qquad (4.4)$$

$$g(\infty) = 0. \qquad (4.5)$$

*If $g(b) = \varphi(b)$ for some $b > a$, then $g \leq \varphi$ on $[a, b]$.*

*Proof.* Denote by $A$ the intersection point of the functions $g$ and $\varphi$, $g(a) = \varphi(a) = A$. If there exist multiple intersections between $g$ and $\varphi$, then choose $a$ and $b$ so that either $g \leq \varphi$ or $g \geq \varphi$ on $[a, b]$. We will show that only the first case (i.e. $g \leq \varphi$)

can occur. The proof can be simplified into three different cases, depending on the relation between $\gamma$ and $\sigma(x)$.

Assume $g(b) = \varphi(b)$. Then if $\gamma \leq \sigma(b)$, certainly $|\gamma| \leq |\sigma(x)|$ for all $x$ in $[a,b]$, thus $g \leq \varphi$ on $[a,b]$, by Lemma 4.2.

If $\sigma(b) < \gamma < \sigma(a)$. Then for some $c \in [a,b]$, we have $\gamma = \sigma(c)$ for $x \in [c,d]$. Let $C$ denote the value at $\varphi(c)$, such that

$$\frac{\sigma^2(x)}{2}x^2\varphi_{xx} + rx\varphi_x - r\varphi = 0,$$
$$\varphi(c) = C,$$
$$\varphi(\infty) = 0.$$

Let $\varphi^\gamma$ be a solution that satisfies

$$\frac{\gamma^2}{2}x^2\varphi^\gamma_{xx} + rx\varphi^\gamma_x - r\varphi^\gamma = 0,$$
$$\varphi^\gamma(c) = C,$$
$$\varphi^\gamma(\infty) = 0.$$

Since $\sigma(x)$ is a non-increasing function, on the interval $[c,\infty)$, we have $|\gamma| \geq |\sigma(x)|$ for all $x$. Thus by Lemma 4.2, $\varphi^\gamma \geq \varphi$ on $[c,\infty)$.

We want $g$ to satisfy $g(b) = \varphi(b)$. Let $g = D\varphi^\gamma$. Then since $b \in [c,\infty)$

$$g(b) = D\varphi^\gamma(b) \geq D\varphi(b).$$

Thus $g(b) \geq D\varphi(b)$, meaning $D \leq 1$. If $D = 1$, then $g = \varphi^\gamma$. However, then

$$g(c) = \varphi^\gamma(c) = \varphi(c),$$

which is a contradiction because we assumed that $a$ is the first intersection point to the left of $b$ and $a < c$. Hence $D < 1$, and then

$$g(c) = D\varphi^\gamma(c) = D\varphi(c) < \varphi(c).$$

Thus $g$ lies below $\varphi$ at the point $c$. The point $a$ was chosen such that the interval $[a,b]$ is the shortest. Thus if $g(c) < \varphi(c)$ then $g$ must lie below $\varphi$ on the whole interval $(a,b)$.

Assume again $g(b) = \varphi(b)$. The last case is if $\gamma \geq \sigma(a)$. As stated before, we know that $\varphi$ satisfies the ordinary differential equation (4.2) and $g$ satisfies the ordinary differential equation (4.3) with boundary values (4.4) and (4.5). Then $|\gamma| \geq |\sigma(x)|$ on $[a,\infty)$, hence Lemma 4.2 tells us that $g \geq \varphi$ on $[a,\infty)$. Can $g$ be attached at $\varphi(b)$ too, without intersecting $\varphi$?

If $g$ touches $\varphi$ at $b$, then $g(b) = \varphi(b)$ and $g_x(b) = \varphi_x(b)$, so by (4.1) and (4.3),

$$\frac{\sigma^2(b)}{2} b^2 \varphi_{xx}(b) = \frac{\gamma^2}{2} b^2 g_{xx}(b).$$

But we know that $\gamma \geq \sigma(a) \geq \sigma(b)$ so

$$\frac{\gamma^2}{2} b^2 g_{xx}(b) \geq \frac{\sigma^2(b)}{2} b^2 g_{xx}(b).$$

Hence

$$\frac{\sigma^2(b)}{2} b^2 \varphi_{xx}(b) \geq \frac{\sigma^2(b)}{2} b^2 g_{xx}(b).$$

In the proof of Lemma 3.3, the second derivative of $\varphi$ is proven to always be positive, hence $\varphi_{xx}(b) \geq g_{xx}(b)$. Thus such a point $b$ cannot exist, unless $\varphi = g$. □

*Proof of Theorem 4.3.* Let $\varphi$ be a function with decreasing volatility which induces some option prices, such as in equation (3.4). Moreover let $\hat{\varphi}_1(x) = (x/x_0)^{-\beta_1}$ and $\hat{\varphi}_2(x) = (x/x_0)^{-\beta_2}$, be functions with constant volatility, which also satisfies equation (3.3) for $\sigma = \sigma_i$, $i = 1,2$. Again, $\beta_1 = 2r/\sigma_1^2$ and $\beta_2 = 2r/\sigma_2^2$. The goal is to prove that if $\hat{\varphi}_1(x)$ is the function fitted to the option price induced by $K_1$ and $\hat{\varphi}_2(x)$ is the function fitted to the option price induced by $K_2$, we have $\hat{\varphi}_1(x) \leq \hat{\varphi}_2(x)$ for $x \leq x_0$. Then Lemma 4.1 can be used to prove that $\hat{\sigma}(K_1) \geq \hat{\sigma}(K_2)$.

For a given $K_1$ there exists an optimal exercise level $z_1$ such that the slope between the points $(z_1, \varphi(z_1))$ and $(K_1, 0)$ is maximized. That is, the tangent line at $z_1$. Denote this tangent line by $y_1$. Similarly, for a given $K_2$ there exists an optimal exercise level $z_2$ such that the slope between the points $(z_2, \varphi(z_2))$ and $(K_2, 0)$ is maximized. This is the tangent line for $z_2$, denote this by $y_2$. Since it is assumed that $K_1 < K_2$, then $z_1 < z_2$.

Since $\varphi$ is a strictly convex function, the tangent lines $y_1$ and $y_2$ touch $\varphi$ in exactly one point, at $z_1$ and $z_2$, respectively. Moreover, also by the convexity of $\varphi$, the tangent lines must lie below $\varphi$, i.e. $y_1(x) < \varphi(x)$ for all $x \in [0, x_0]$ except at the point $z_1$, where there is equality between $y_1$ and $\varphi$. Also, $y_2(x) \leq \varphi(x)$ for all $x \in [0, x_0]$, with equality at $z_2$.

By definition $\hat{\varphi}_1(x)$ and $\hat{\varphi}_2(x)$ are also strictly convex, decreasing functions because they satisfy equation (3.4). Therefore they touch their respective tangent lines $y_1$ and $y_2$ at exactly one point and lie above them at all other times. Let $z_1^*$ denote the tangent point of $\hat{\varphi}_1$ that touches $y_1$ and let $z_2^*$ denote the point of $\hat{\varphi}_2$ where the function is equal to $y_2$. For $\beta_1 \neq \beta_2$, the equation

$$\left(\frac{x}{x_0}\right)^{-\beta_1} = \left(\frac{x}{x_0}\right)^{-\beta_2},$$

does not have a solution on $(0, x_0)$. Therefore the functions $\hat{\varphi}_1$ and $\hat{\varphi}_2$ cannot intersect anywhere on $(0, x_0)$ or they coincide.

We know that $z_1 < z_2$. Therefore the intersection points between the tangent lines, where $y_1(x) = y_2(x)$, must lie between $z_1$ and $z_2$. So for any $x \leq z_2$, we must have $y_1(x) < y_2(x)$.

Since we assumed that $\varphi$ has a decreasing volatility, we can use Lemma 4.4 to say that we know that $\hat{\varphi}_1$ lies below $\varphi$ on the interval $[x_1, x_0]$, if $x_1$ is an intersection point between $\hat{\varphi}_1$ and $\varphi$. Since the tangent line $y_1$ always is below $\varphi$ and $\hat{\varphi}_1$ because of convexity, we must have $z_1 \leq z_1^*$.

In exactly the same way, we can use Lemma 4.4 on $\varphi$ and $\hat{\varphi}_2$ with the intersection points $x_2$ and $x_0$, where $x_2$ is an intersection point between $\hat{\varphi}_2$ and $\varphi$. Thus we know that $\hat{\varphi}_2 \leq \varphi$ on $[x_2, x_0]$. The tangent line $y_2$ is always below $\varphi$ and $\hat{\varphi}_2$ because of convexity and so the conclusion is that $z_2 \leq z_2^*$.

So as stated, $z_2 \leq z_2^*$ and we know that $y_2 \leq \hat{\varphi}_2$ always, so of course $y_2 \leq \hat{\varphi}_2$ on $[x_2, x_0]$ in particular. However, $z_2 \leq x_2$, and so we must have $y_1 \leq y_2$ here, since $z_2$ is larger than the intersection point of $y_2$ and $y_1$. Hence $y_1 \leq \hat{\varphi}_2$ on $[x_2, x_0]$.

By another application of Lemma 4.4, we have $\varphi \leq \hat{\varphi}_2$ on $(0, x_2]$. Therefore $y_1(x) \leq \hat{\varphi}_2(x)$ for all $x \leq x_0$. Since $y_1$ is the tangent line for $\hat{\varphi}_1$, we must have $\hat{\varphi}_1 \leq \hat{\varphi}_2$ on $[0, x_0]$. Thus we have reached the desired inequality $\hat{\varphi}_1 \leq \hat{\varphi}_2$ on $[0, x_0]$ and now with this information, Lemma 4.1 tells us that the implied volatility for $K_1$ must be higher or equal to the implied volatility for $K_2$. □

## 4.4 Increasing volatility

In theory, it should be possible to draw some conclusion about the behavior of the implied volatility when we have an increasing volatility function. In practice, increasing volatility functions are not commonly used, as increasing deterministic functions often contradict the growth conditions stochastic processes should satisfy. This section closely follows the outline of Section 4.3 but with an increasing volatility instead of a decreasing one.

**Theorem 4.5.** *An increasing volatility implies an increasing implied volatility. That is, for strike prices $K_1 < K_2$, the following hold*

$$\sigma(\cdot) \text{ increasing} \Rightarrow \hat{\sigma}(K_1) \leq \hat{\sigma}(K_2).$$

*Proof.* To be determined. □

Let $X$ and $Y$ be two price processes, satisfying the stochastic differential equations

$$\begin{aligned} dX_t &= rX_t dt + \sigma(X_t) X_t dB_t & X_0 &= x_0, \\ dY_t &= rY_t dt + \gamma Y_t dB_t & Y_0 &= x_0. \end{aligned}$$

Here $\sigma(x) > 0$ is a increasing function and $\gamma > 0$ is a constant. Let $\varphi$ be a solution to the ordinary differential equation 3.3 with respect to the process $X$. The ordinary differential equation with boundary conditions is the following:

$$\frac{\sigma^2(x)}{2} x^2 \varphi_{xx} + rx\varphi_x - r\varphi = 0, \tag{4.6}$$
$$\varphi(\infty) = 0.$$

Moreover let $f$ be the solution to the ordinary differential equation 3.3 with respect to the process $Y$, meaning

$$\frac{\gamma^2}{2} x^2 f_{xx} + rxf_x - rf = 0,$$
$$f(\infty) = 0.$$

Since $\gamma$ was a constant, the process $Y$ is a geometric Brownian motion and so we know that the function $f$ is on the form $f = (x/x_0)^{-2r/\gamma^2}$.

**Lemma 4.6.** *Let $\gamma > 0$ and $a$ be given constants such that the function $g$ satisfies*

$$\frac{\gamma^2}{2} x^2 g_{xx} + rxg_x - rg = 0, \quad x > a, \tag{4.7}$$
$$g(a) = \varphi(a), \tag{4.8}$$
$$g(\infty) = 0. \tag{4.9}$$

*If $g(b) = \varphi(b)$ for some $b > a$, then $g \geq \varphi$ on $[a, b]$.*

*Proof.* The proof of this Lemma follows the proof of Lemma 4.4 very closely. Denote by $A$ the intersection point of the functions $g$ and $\varphi$, $g(a) = \varphi(a) = A$. If there exist multiple intersections between $g$ and $\varphi$, then choose $a$ and $b$ so that either $g \leq \varphi$ or $g \geq \varphi$ on $[a,b]$. We will show that only the first case (i.e. $g \geq \varphi$) can occur. The proof can be simplified into three different cases, depending on the relation between $\gamma$ and $\sigma(x)$.

Assume $g(b) = \varphi(b)$. Then if $\gamma \geq \sigma(b)$, certainly $|\gamma| \geq |\sigma(x)|$ for all $x$ in $[a,b]$, thus $g \geq \varphi$ on $[a,b]$, by Lemma 4.2.

If $\sigma(a) < \gamma < \sigma(b)$. Then for some $c \in [a,b]$, we have $\gamma = \sigma(c)$. Let $C$ denote the value at $\varphi(c)$, such that

$$\begin{aligned} \frac{\sigma^2(x)}{2}x^2\varphi_{xx} + rx\varphi_x - r\varphi = 0, \\ \varphi(c) = C, \\ \varphi(\infty) = 0. \end{aligned}$$

Let $\varphi^\gamma$ be a solution that satisfies

$$\begin{aligned} \frac{\gamma^2}{2}x^2\varphi^\gamma_{xx} + rx\varphi^\gamma_x - r\varphi^\gamma = 0, \\ \varphi^\gamma(c) = C, \\ \varphi^\gamma(\infty) = 0. \end{aligned}$$

Since $\sigma(x)$ is a increasing function, on the interval $[c,\infty)$, we have $|\gamma| \leq |\sigma(x)|$ for all $x$. Thus by Lemma 4.2, $\varphi^\gamma \leq \varphi$ on $[c,\infty)$.

We want $g$ to satisfy $g(b) = \varphi(b)$. Let $g = D\varphi^\gamma$. Then since $b \in [c,\infty)$

$$g(b) = D\varphi^\gamma(b) \leq D\varphi(b).$$

Thus $g(b) \leq D\varphi(b)$, meaning $D \geq 1$. If $D = 1$, then $g = \varphi^\gamma$. However, then

$$g(c) = \varphi^\gamma(c) = \varphi(c),$$

which is a contradiction because we assumed that $a$ is the first intersection point to the left of $b$ and $a < c$. Hence $D > 1$, and then

$$g(c) = D\varphi^\gamma(c) = D\varphi(c) > \varphi(c).$$

Thus $g$ lies above $\varphi$ at the point $c$. The point $a$ was chosen such that the interval $[a,b]$ is the shortest. Thus if $g(c) > \varphi(c)$ then $g$ must lie above $\varphi$ on the whole interval $(a,b)$.

Assume again $g(b) = \varphi(b)$. The last case is if $\gamma \leq \sigma(a)$. As stated before, we know that $\varphi$ satisfies the ordinary differential equation (4.2) and $g$ satisfies the ordinary differential equation (4.7) with boundary values (4.8) and (4.9). Since $\sigma$ is increasing, then $|\gamma| \leq |\sigma(x)|$ on $[a,\infty)$, hence Lemma 4.2 tells us that $g \leq \varphi$ on $[a,\infty)$. Is it possible that $g$ is attached at $\varphi(b)$ too, without intersecting $\varphi$?

If $g$ touches $\varphi$ at $b$, then $g(b) = \varphi(b)$ and $g_x(b) = \varphi_x(b)$, so by (4.6) and (4.7),

$$\frac{\sigma^2(b)}{2} b^2 \varphi_{xx}(b) = \frac{\gamma^2}{2} b^2 g_{xx}(b).$$

But we know that $\gamma \leq \sigma(a) \leq \sigma(b)$ so

$$\frac{\gamma^2}{2} b^2 g_{xx}(b) \leq \frac{\sigma^2(b)}{2} b^2 g_{xx}(b).$$

Hence

$$\frac{\sigma^2(b)}{2} b^2 \varphi_{xx}(b) \leq \frac{\sigma^2(b)}{2} b^2 g_{xx}(b).$$

In the proof of Lemma 3.3, the second derivative of $\varphi$ is proven to always be positive, hence $\varphi_{xx}(b) \leq g_{xx}(b)$. Thus such a point $b$ cannot exist, unless $\varphi = g$. □

*Proof of Theorem 4.5.* Same procedure as in decreasing volatility, except now we prove that $\hat{\varphi}_1$ and $\hat{\varphi}_2$ intersects their tangent lines to the left of the optimal exercise levels $z_1$ and $z_2$, respectively.

Let $\varphi$ be a function which induces some option prices, such as in equation (3.4). Now let the underlying volatility function of $\varphi$ be increasing instead. Just as in the decreasing case, let $\hat{\varphi}_1(x) = (x/x_0)^{-\beta_1}$ and $\hat{\varphi}_2(x) = (x/x_0)^{-\beta_2}$, be functions with constant volatility, which also satisfies equation (3.4) for $\sigma = \sigma_i$, $i = 1,2$. Again define $\beta_1 = 2r/\sigma_1^2$ and $\beta_2 = 2r/\sigma_2^2$. Let $x_1$ and $x_2$ be intersection points of $\varphi$ with $\hat{\varphi}_1(x)$ and $\hat{\varphi}_2(x)$, respectively.

The goal is to prove that if $\hat{\varphi}_1(x)$ is the function fitted to the option price induced by $K_1$ and $\hat{\varphi}_2(x)$ is the function fitted to the option price induced by $K_2$, then $\hat{\varphi}_2(x) \leq \hat{\varphi}_1(x)$ on for $x \leq x_0$. Then Lemma 4.1 can be used to prove that $\hat{\sigma}(K_1) \leq \hat{\sigma}(K_2)$.

Just as in the decreasing case, we have that for a given strike $K_1$ there exists an optimal exercise level $z_1$ such that the slope between the points $(z_1, \varphi(z_1))$ and $(K_1, 0)$ is maximized and this is the tangent line at $z_1$. Denote this tangent line by $y_1$. Similarly, for a given $K_2$ there exists an optimal exercise level $z_2$ such that the slope between the points $(z_2, \varphi(z_2))$ and $(K_2, 0)$ is maximized. This is the tangent line for $z_2$, denote this by $y_2$. Since it is assumed that $K_1 < K_2$, then $z_1 < z_2$.

Since $\varphi$ is a strictly convex function, the tangent lines $y_1$ and $y_2$ touch $\varphi$ in exactly one point, at $z_1$ and $z_2$, respectively. Moreover, also by the convexity of $\varphi$, the tangent lines must lie below $\varphi$, i.e. $y_1(x) < \varphi(x)$ for all $x \in [0, x_0]$ except at the point $z_1$, where there is equality between $y_1$ and $\varphi$. Similarly, $y_2(x) \leq \varphi(x)$ for all $x \in [0, x_0]$, with equality at $z_2$.

By definition $\hat{\varphi}_1(x)$ and $\hat{\varphi}_2(x)$ are also strictly convex, decreasing functions because they satisfy equation (3.4). Therefore they touch their respective tangent lines $y_1$ and $y_2$ at exactly one point and lie above them at all other times. Let $z_1^*$ denote the tangent point of $\hat{\varphi}_1$ that touches $y_1$ and let $z_2^*$ denote the point of $\hat{\varphi}_2$ that touches $y_2$. For $\beta_1 \neq \beta_2$, the equation

$$\left(\frac{x}{x_0}\right)^{-\beta_1} = \left(\frac{x}{x_0}\right)^{-\beta_2},$$

does not have a solution on $(0, x_0)$. Therefore the functions $\hat{\varphi}_1$ and $\hat{\varphi}_2$ cannot intersect anywhere on $(0, x_0)$ or they coincide.

We know that $z_1 < z_2$. Therefore the intersection points between the tangent lines, where $y_1(x) = y_2(x)$, must lie between $z_1$ and $z_2$. So for any $x \leq z_1$, we must have $y_2(x) < y_1(x)$.

In this Section, $\varphi$ has an increasing volatility, hence we can use Lemma 4.6 to say that $\hat{\varphi}_1$ lies above $\varphi$ on the interval $[x_1, x_0]$, if $x_1$ is an intersection point between $\hat{\varphi}_1$ and $\varphi$. Since the tangent line $y_1$ always is below $\varphi$ because of convexity, we must have $x_1 \leq z_1$, and then of course $z_1^* \leq x_1$. Hence $z_1^* \leq z_1$. To rule out the other case, assume $z_1 \leq x_1$ instead. Then because $y_1 \leq \hat{\varphi}_1$ always, there must exist another intersection between $\hat{\varphi}_1$ and $\varphi$ on $[z_1, x_0]$, which lies in between $z_1$ and $x_1$. Otherwise $\hat{\varphi}_1$ will intersect $y_1$. However, that contradicts Lemma 4.6, because then $\hat{\varphi}_1 \leq \varphi$ on the interval between the new intersection and $x_1$. Thus $x_1 \leq z_1$ certainly holds. Moreover, by another application of Lemma 4.6, we have $\hat{\varphi}_1 \leq \varphi$ on $(0, x_1]$.

In exactly the same way, we can use Lemma 4.6 on $\varphi$ and $\hat{\varphi}_2$ with the intersection points $x_2$ and $x_0$. Thus we know that $\varphi \leq \hat{\varphi}_2$ on $[x_2, x_0]$. The tangent line $y_2$ is always below $\varphi$ and $\hat{\varphi}_2$ because of convexity, i.e. $\hat{\varphi}_2$ can only touch $y_2$ to the left of $z_2$, hence $z_2 * \leq z_2$. With the same logic as in the case of $\hat{\varphi}_1$, we know that the case of $z_2 \leq z_2^*$ can be ruled out, because that creates another intersection point with $\varphi$, which is a contradiction to Lemma 4.6. In this case as well, we can apply Lemma 4.6, to say that $\hat{\varphi}_2 \leq \varphi$ on $(0, x_2]$.

So as stated, $z_1^* \leq z_1$ and we know that $y_1 \leq \hat{\varphi}_1$ always. Thus of course $y_1 \leq \hat{\varphi}_1$ on $[0, x_1]$ in particular. But $x_1 \leq z_1$, and so we must have $y_2 \leq y_1$ here, since we are to the left of the intersection point of $y_2$ and $y_1$. Hence $y_2 \leq \hat{\varphi}_1$ on $[0, x_1]$. Now since $y_2$ is the tangent line for $\hat{\varphi}_2$ we thus have $\hat{\varphi}_2 \leq \hat{\varphi}_1$ on $[0, x_1]$ and then since the functions cannot intersect on $(0, x_0)$, this inequality holds on the entire interval. Thus we have the desired inequality $\hat{\varphi}_2 \leq \hat{\varphi}_1$ on $[0, x_0]$ and now we can apply Lemma 4.1 to this. Lemma 4.1 tells us that the implied volatility for $K_2$ must be higher or equal to the implied volatility for $K_1$. □

# 5 Numerical approximations

## 5.1 The implied volatility for a given strike price $K$

Assume that we are given a price process $X_t$ which moves according to the CEV model, i.e. it has decreasing volatility $\sigma(x) = 1/\sqrt{x}$, such that the process follows the stochastic differential equation

$$dX_t = rX_t dt + \sqrt{X_t} dB_t,$$

where $B_t$ is a standard Brownian motion. Then using equation (3.4), we can obtain the explicit formula for $\varphi$ by inserting the explicit expression for $\sigma(x)$. Thus

$$\varphi(x) = Dx \exp(2rx_0) \int_x^\infty \frac{1}{y^2} \exp(-2ry) dy, \tag{5.1}$$

with the constant $D$ once again satisfying $\varphi(x_0) = 1$ such that

$$D = \left( \exp(2rx_0) \int_{x_0}^\infty \frac{1}{y^2} \exp(-2ry) dy \right)^{-1}.$$

Moreover, in Lemma 3.3 an expression for the derivative of the $\varphi$ function was derived. Therefore we can obtain

$$\varphi'(x) = D \exp(2rx_0) \int_x^\infty \frac{1}{y^2} \Big( \exp(-2ry) - \exp(-2rx) \Big) dy. \tag{5.2}$$

Now, with this explicit formula for $\varphi'$, we can use equation (3.8) to find $\hat{K}$ for this scenario, with the chosen values for $x_0$ and $r$. That is, we have

$$\hat{K} = x_0 - \frac{1}{\varphi'(x_0)}.$$

With this knowledge of $\hat{K}$, it is possible to choose a $K \leq \hat{K}$ and calculate the optimal exercise level $z(K)$ for this strike price. The optimal exercise level can then be used to compute the option price. The optimal exercise level $z(K)$ can be found by

$$(K - z(K))\varphi'(z(K)) + \varphi(z(K)) = 0,$$

and inserting this acquired $z(K)$ into

$$V(K) = \frac{K - z(K)}{\varphi(z(K))}, \tag{5.3}$$

we obtain the maximized option price.

Hence for a specific strike price $K$ it is possible to find the optimal exercise level and the option price for this perpetual American option. Denote the option price found in equation (5.3) by $V_\varphi(K)$, since it is obtained from the behavior of $\varphi$. Denote the optimal exercise price by $z(K)$ by $z$.

The next step is to find the implied volatility which corresponds to the option price $V_\varphi(K)$. As seen in Section 3.1, there exists a formula for finding the price of an American option when the underlying process is a geometric Brownian motion. This is seen in equation (3.2). Furthermore, denote by $z^*$ the optimal exercise level for the geometric Brownian motion fitted to the option price $V_\varphi(K)$ for the same strike $K$.

Hence for $K < \hat{K}$, set the option price $V_\varphi(K)$, which was obtained from equation (5.3), equal to the expression for the first case in equation (3.2). That is, for some $\beta$, the following equation holds:

$$V_\varphi(K) = \frac{K}{\beta+1}\left(\frac{x_0(\beta+1)}{\beta K}\right)^{-\beta}. \tag{5.4}$$

Using a numerical solver to solve equation (5.4) for $\beta$, we can now obtain a $\beta$ which is fitted exactly to $V_\varphi(K)$. Then the implied volatility $\hat{\sigma}$ can easily be found, such as

$$\beta = \frac{2r}{\hat{\sigma}^2} \Leftrightarrow \hat{\sigma} = \sqrt{\frac{2r}{\beta}}.$$

It is assumed that the implied volatility is positive, so only the positive root is considered.

However, when plotting the right hand side versus the left hand side of equation (5.4), it can be observed that there could exist multiple intersection points of these graphs. Hence some restriction to the domain of $\beta$ must be made. When computing the option price, the assumption was that the supremum $z$ satisfies $z \le x_0 \wedge K$. If $K < x_0$, then there is only one intersection because the right hand side of (5.4) is decreasing as a function of $\beta$, where $\beta > 0$. Note that for an option with geometric Brownian motion as the underlying process, the supremum $z$ is attained at $z = z^*$, if $z^* < x_0$, and $z = x_0$, if $z^* \ge x_0$. As seen before $z^* = \beta K/(\beta+1)$, and so

$$\frac{\beta K}{\beta + 1} \leq x_0$$
$$\Updownarrow$$
$$\beta \leq \frac{1}{(K/x_0) - 1}. \tag{5.5}$$

Using (5.5) as the maximum for the domain of $\beta$, there should now only exist one intersection point of the graphs and $\beta$ can be found using a numerical solver on equation (5.4) with the criteria for $\beta$ stated in equation (5.5).

Moreover, the solution to the ordinary differential equation (3.3) with constant diffusion coefficient, i.e. with a geometric Brownian motion as the underlying process, is on the form

$$\hat{\varphi}(x) = \left(\frac{x}{x_0}\right)^{-\frac{2r}{\hat{\sigma}^2}}. \tag{5.6}$$

Thus, after obtaining $\beta$ and subsequently the implied volatility which was the goal, we can use (5.6) to plot the corresponding $\hat{\varphi}$ function. Furthermore, for the fitted function $\hat{\varphi}$ the option price supremum is attained at $z^*(K) = \beta K/(\beta + 1)$. Figure 5 shows a visualization of how the $\varphi$ and $\hat{\varphi}$ solutions behave, as well as their respective optimal exercise levels $z$ and $z^*$ plotted at the tangent points of the functions with the tangent line which has the slope $-1/V_\varphi(K)$.

## 5.2 The implied volatility as a function of the strike price $K$

The procedure stated in Section 5.1 can be repeated for a vector of strike prices to obtain a plot of the implied volatility against the strike price. However, we are using a numerical program and have to discretize the $z$-axis, so it is not a continuous function. Nonetheless, the plots of the implied volatility against the strike prices will demonstrate a good idea of the behavior of the implied volatility with different $x_0$ and risk-free rates $r$.

As stated, this is only a numerical approximation of the implied volatility function, as the domain of $z$ is discretized. Therefore it is impractical to start with a vector of strike prices from some $z$ to $\hat{K}$, since there is a possibility that there is no exact solution for that specific strike price in the discretized vector of optimal exercise levels. For a given strike price, there is a risk that the optimal exercise is not found in the discretization of the $z$.

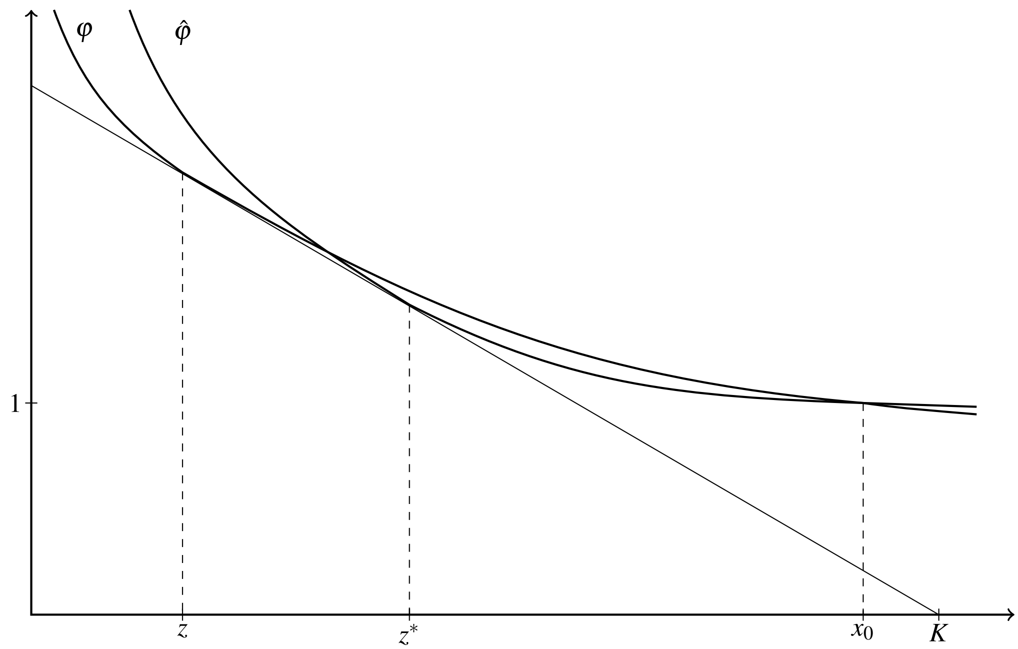


Figure 5: The functions $\varphi$ and $\hat{\varphi}$ plotted, with the optimal exercise levels $z$ and $z^*$, respectively.

Thus to make it easier to compute a vector of implied volatilities without the risk of not obtaining an solution, we instead want to start with a discretized vector of optimal exercise levels and use this to obtain the corresponding strike price vector. To obtain this vector, the explicit expressions stated in equations (5.1) and (5.2) for $\varphi$ and $\varphi'$ can be used in the following equation:

$$(K - z)\varphi'(z) + \varphi(z) = 0.$$

Thus, for each $z \in [0, x_0]$, it is possible to find the corresponding strike price $K$ and thus a discretized strike price vector can be found. Then for this vector of strike prices, an implied volatility can always be found.

Note that since $z \in [0, x_0]$, the corresponding strike prices will not exceed $\hat{K}$. Furthermore, as we start with $z = 0$, in addition to there existing an upper bound, $\hat{K}$, of the strike prices, there will also exist a lower bound, denoted by $\underline{K}$, which corresponds to $z(K) = 0$.

With this information, programming software can be implemented to compute the implied volatility for any given strike price. Thus we can generate plots of the implied volatility for different initial stock prices $x_0$ and different risk free rates $r$. For comparison, added in the plots is also the true volatility function $\sigma(x) = x^{-1/2}$ on the domain of $[\underline{K}, \hat{K}]$.

We see in Figures 6-9 that the implied volatility decreases, as the strike price increases. This aligns with the claim that an implied volatility skew should be present when the underlying model has a decreasing volatility function.

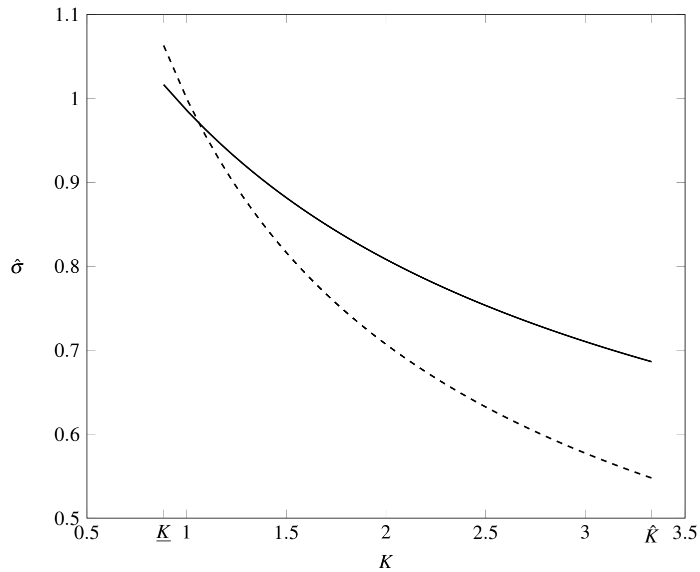


Figure 6: The solid line is the implied volatility as a function of $K$ with $x_0 = 1$ and $r = 0.1$ and the dashed line is the volatility function $\sigma(x) = x^{-1/2}$.

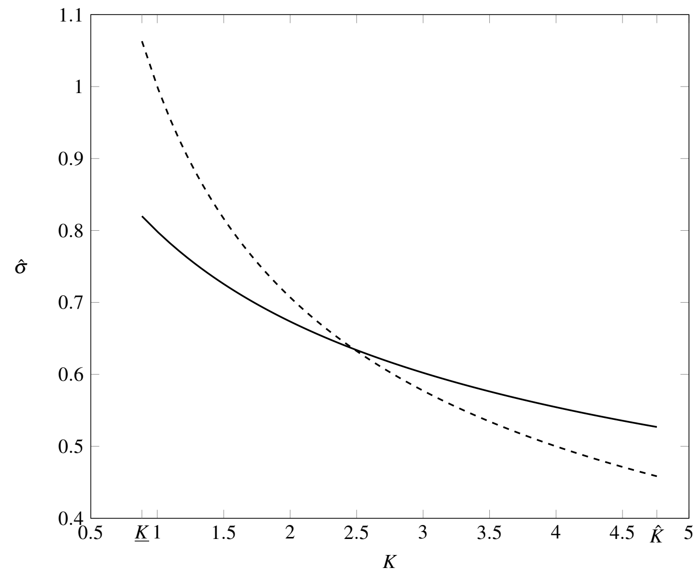


Figure 7: The solid line is the implied volatility as a function of $K$ with $x_0 = 2$ and $r = 0.1$ and the dashed line is the volatility function $\sigma(x) = x^{-1/2}$.

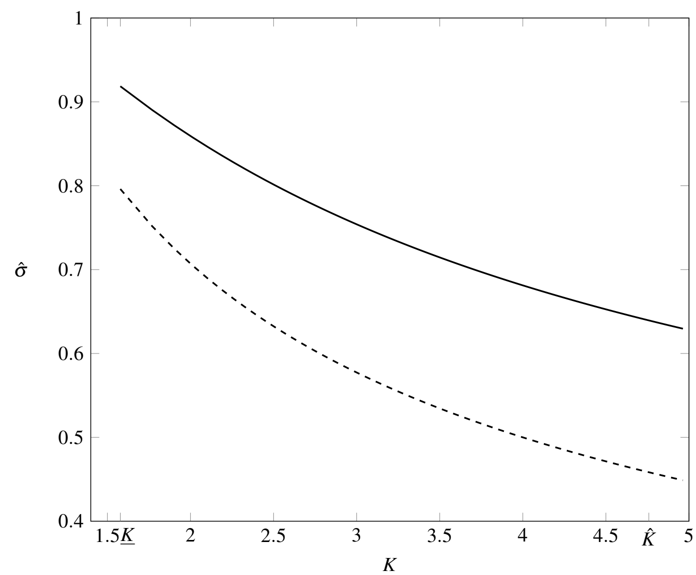


Figure 8: The solid line is the implied volatility as a function of $K$ with $x_0 = 1$ and $r = 0.05$ and the dashed line is the volatility function $\sigma(x) = x^{-1/2}$.

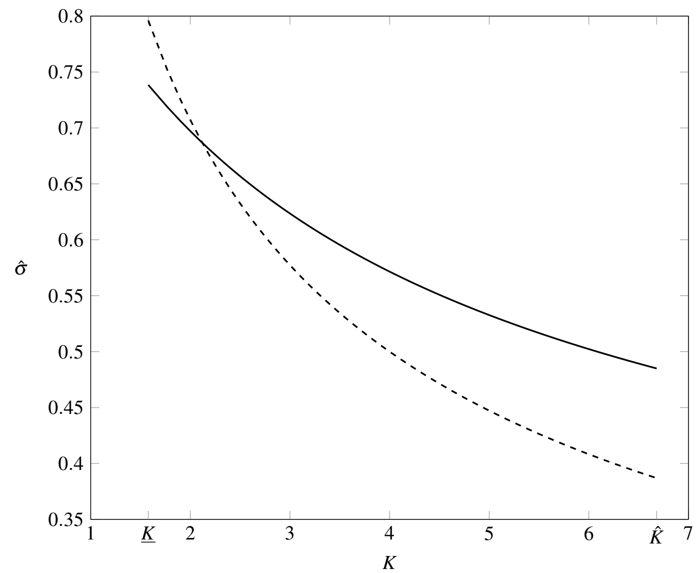


Figure 9: The solid line is the implied volatility as a function of $K$ with $x_0 = 2$ and $r = 0.05$ and the dashed line is the volatility function $\sigma(x) = x^{-1/2}$.

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