The Containment Problem for Unambiguous Register Automata

Antoine Mottet, Karin Quaas

QuantLA Workshop 2018

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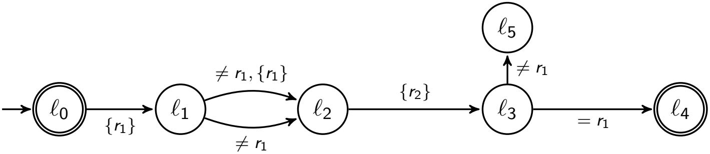

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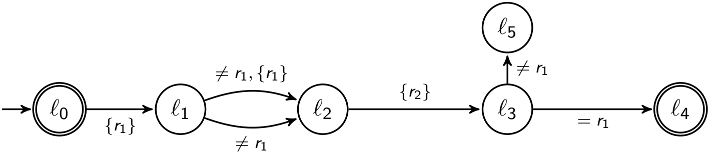

$$(\ell_0, \bot, \bot) \xrightarrow{\binom{a}{0}} (\ell_0, 0, \bot) \xrightarrow{\binom{a}{1}} (\ell_2, 0, \bot) \xrightarrow{\binom{a}{3}} (\ell_3, 0, 3) \xrightarrow{\binom{a}{0}} (\ell_4, 0, 3)$$

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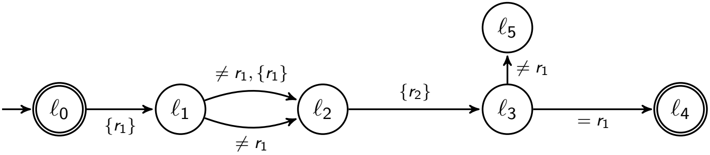

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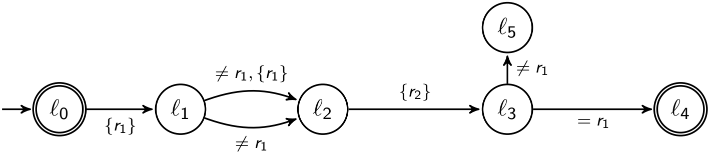

$$\begin{array}{ccccccccc} (\ell_0, \perp, \perp) & \xrightarrow{\binom{a}{0}} & (\ell_0, 0, \perp) & \xrightarrow{\binom{a}{1}} & (\ell_2, 0, \perp) & \xrightarrow{\binom{a}{3}} & (\ell_3, 0, 3) & \xrightarrow{\binom{a}{0}} & (\ell_4, 0, 3) \\ & & & \longrightarrow & (\ell_2, 1, \perp) & \longrightarrow & (\ell_3, 1, 3) & \longrightarrow & (\ell_5, 1, 3) \end{array}$$

- Recognizers of orbits:

$$\binom{a}{0}\binom{a}{1}\binom{a}{3}\binom{a}{0} \sim \binom{a}{4}\binom{a}{3}\binom{a}{1}\binom{a}{4}.$$

- Projection of $L \subseteq (\Sigma \times \mathbb{N})^*$ onto Σ^*: set of words $w \in \Sigma^*$ such that $(w_1, d_1) \dots (w_n, d_n) \in L$ for some $d_1, \dots, d_n \in \mathbb{N}$.

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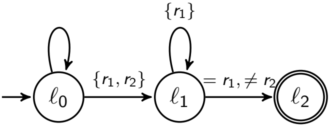

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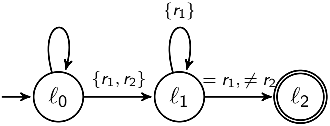

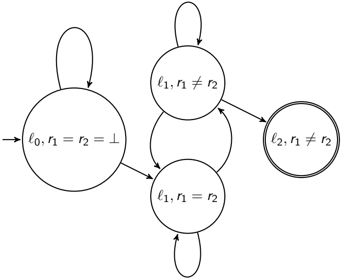

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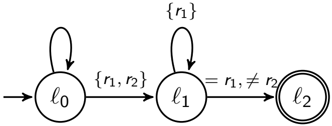
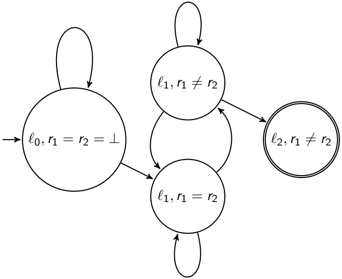
- Emptiness of L is decidable: $L = \emptyset \Leftrightarrow$ its projection is empty.





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- Succinctness,
- Important problems related to unambiguity (parity games in $UP \setminus P$?).

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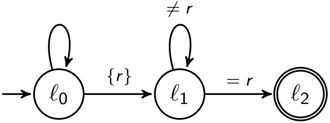

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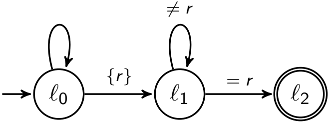

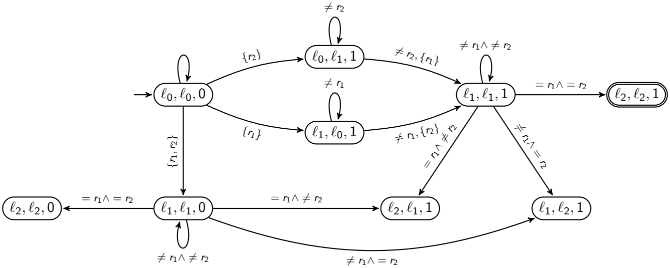

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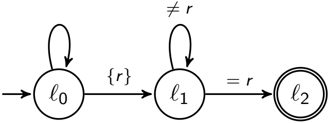

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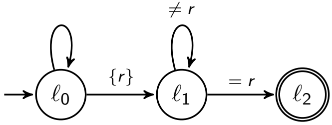

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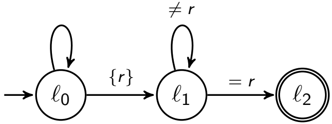

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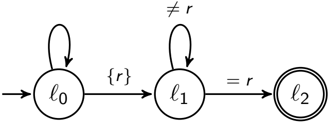

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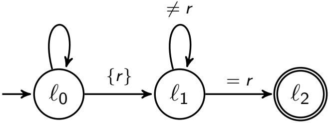

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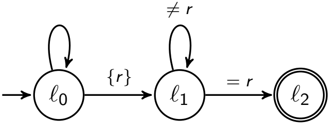

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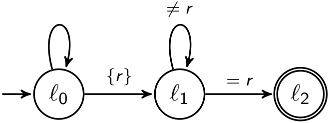

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- In particular L not recognizable by deterministic RA.

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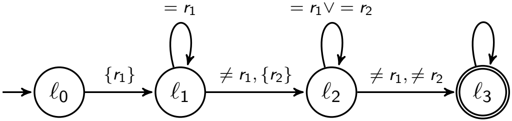

"Succinctness"

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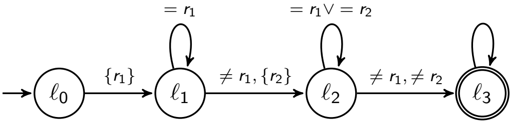

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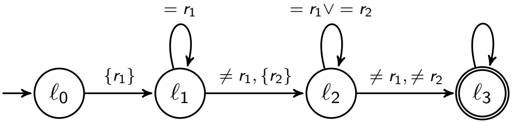

- Needs 2 registers.
- Exists a 1-register unambiguous RA.

- Universality: Given $\mathcal{B}$, determine if $L(\mathcal{B}) = (\Sigma \times \mathbb{N})^*$.

$\mathcal{B}$	DRA	URA	NRA
1 register	NL-complete	?	Ackermann-complete
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  $\rightsquigarrow$ "on-the-fly" complementation.

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Synchronized configurations

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The unambiguous case: try to bound size of configurations.

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- $\mathrm{tp}(d_1, \dots, d_n) = \mathrm{tp}(e_1, \dots, e_n) \Leftrightarrow$ $\exists$ permutation α of $\mathbb{N}$ s.t. $\alpha(d_i) = e_i$.

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Collapsing

Proposition (M-Quaas '18)

C reachable. $\overline{a}, \overline{b}$ such that $\overline{a} \equiv_C \overline{b}$.
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C reaches a bad configuration in k steps iff
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$\mathcal{B}$	DRA	URA	NRA
1 register	PSPACE-comp.	EXPSPACE	Ackermann-comp.
*	PSPACE-comp.	2-EXPSPACE	Undecidable

Open problems

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- For RAs over ordered domain: decidability for ≥ 2 registers?
- Timed automata: decidability for ≥ 2 clocks?