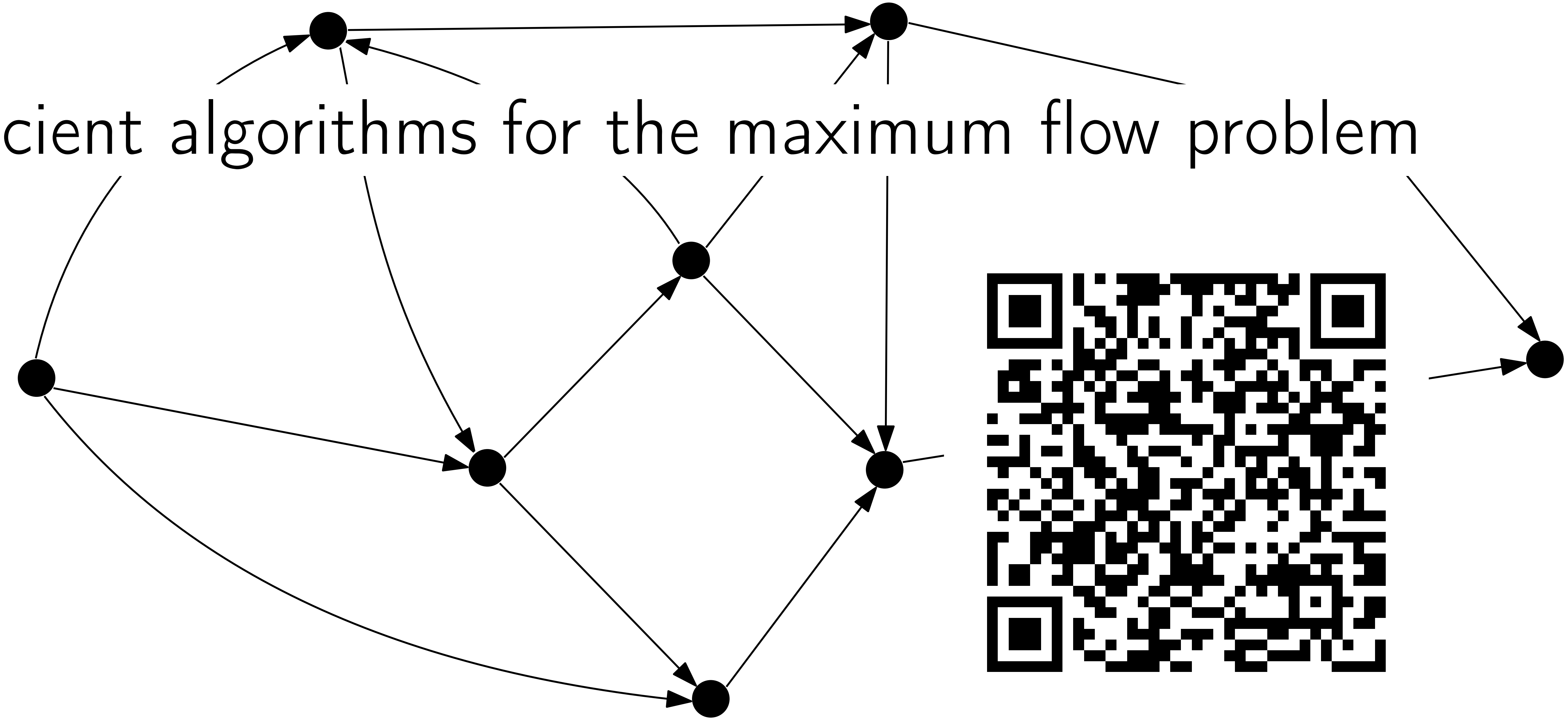
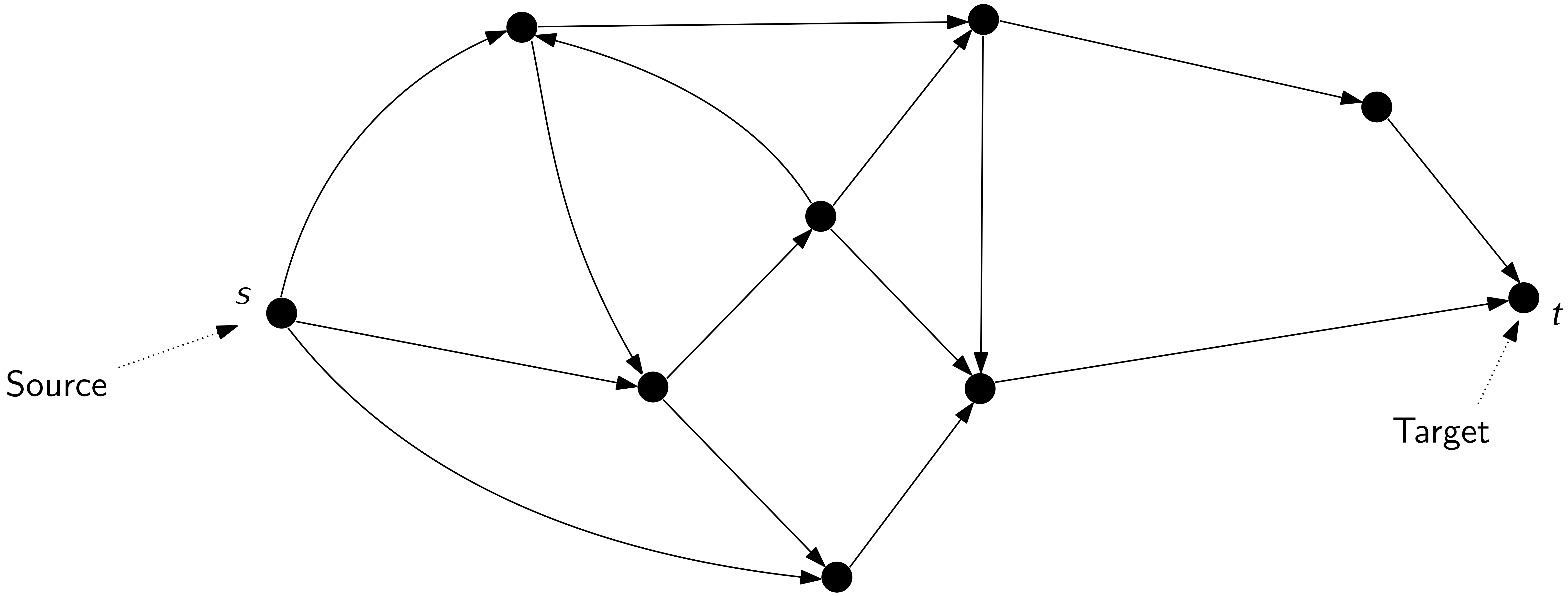
Efficient algorithms for the maximum flow problem



A flow network is a directed graph without isolated vertices and:

- a source s
- a target t



A flow network is a directed graph without isolated vertices and:

- a source s
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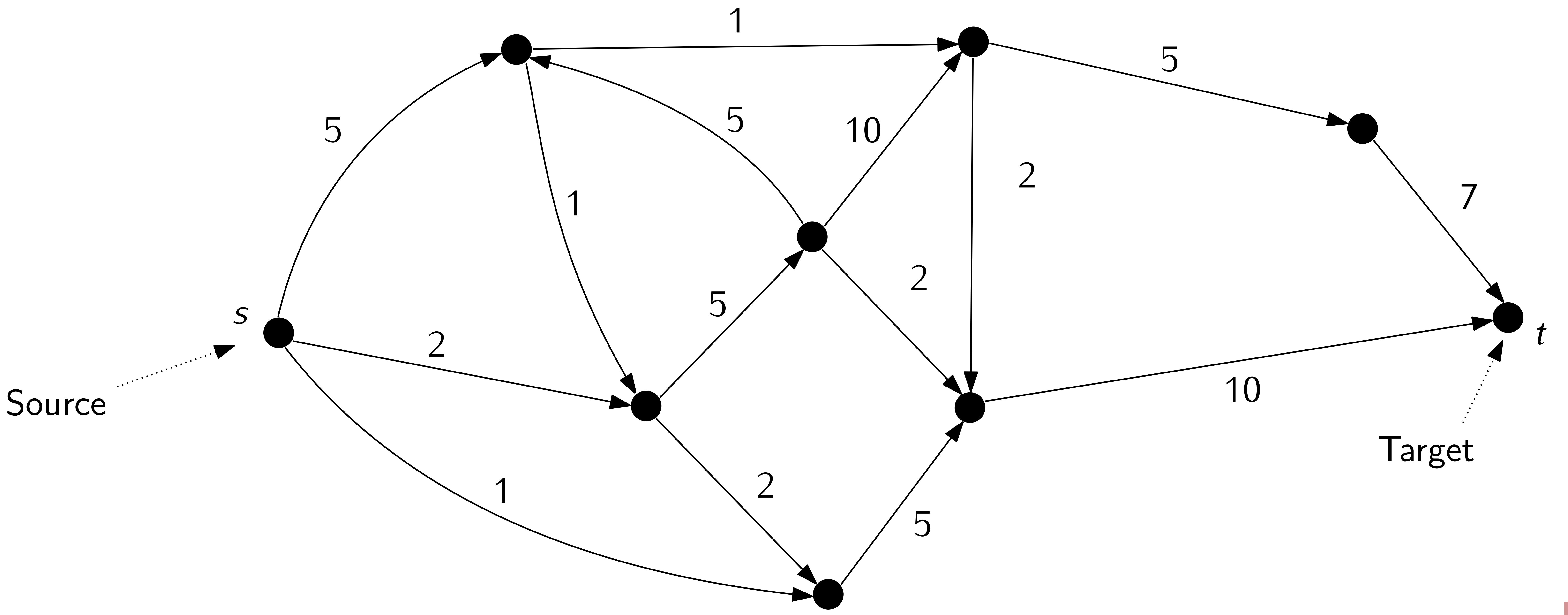
- capacities: a function $c\colon E \to \mathbb{N}$



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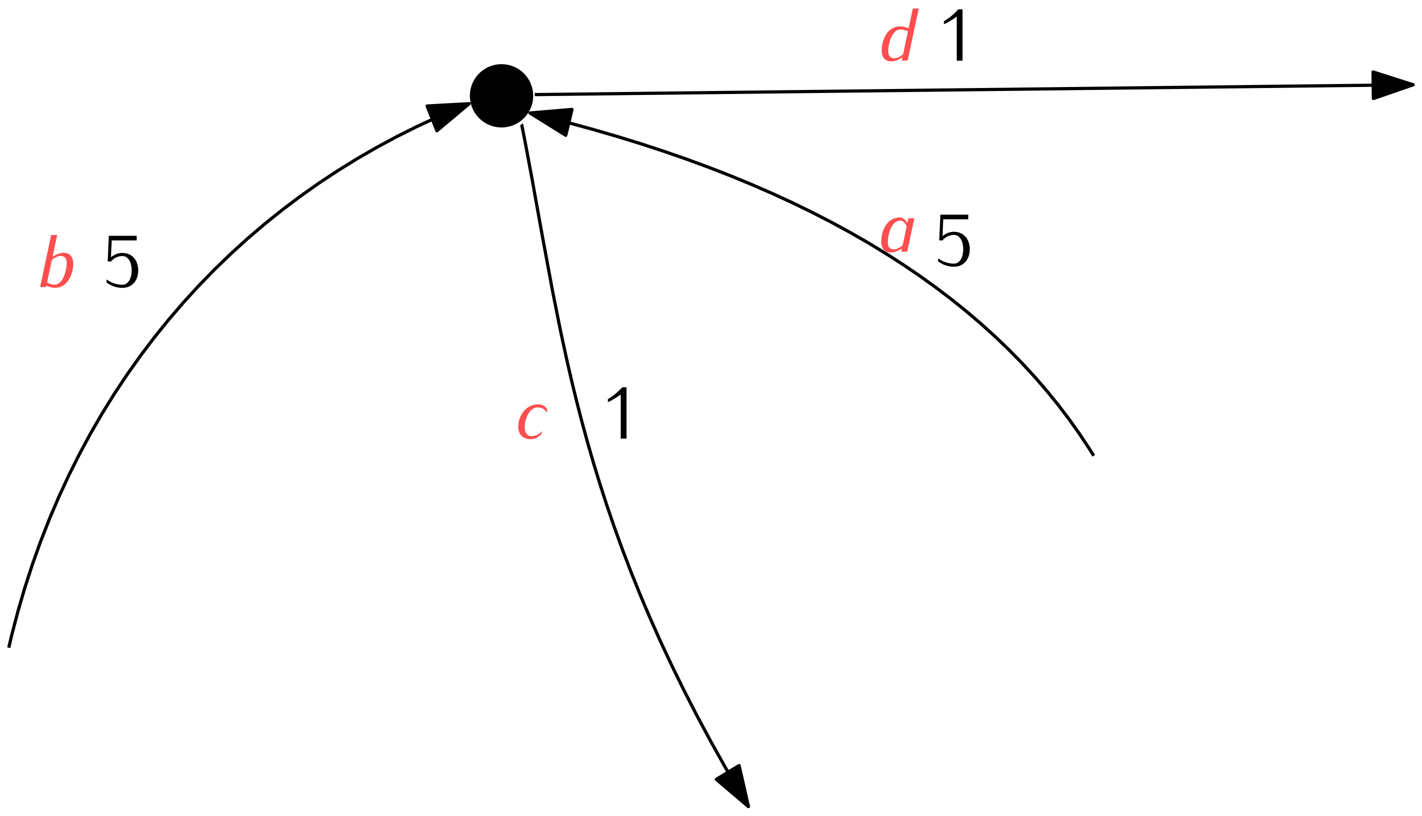
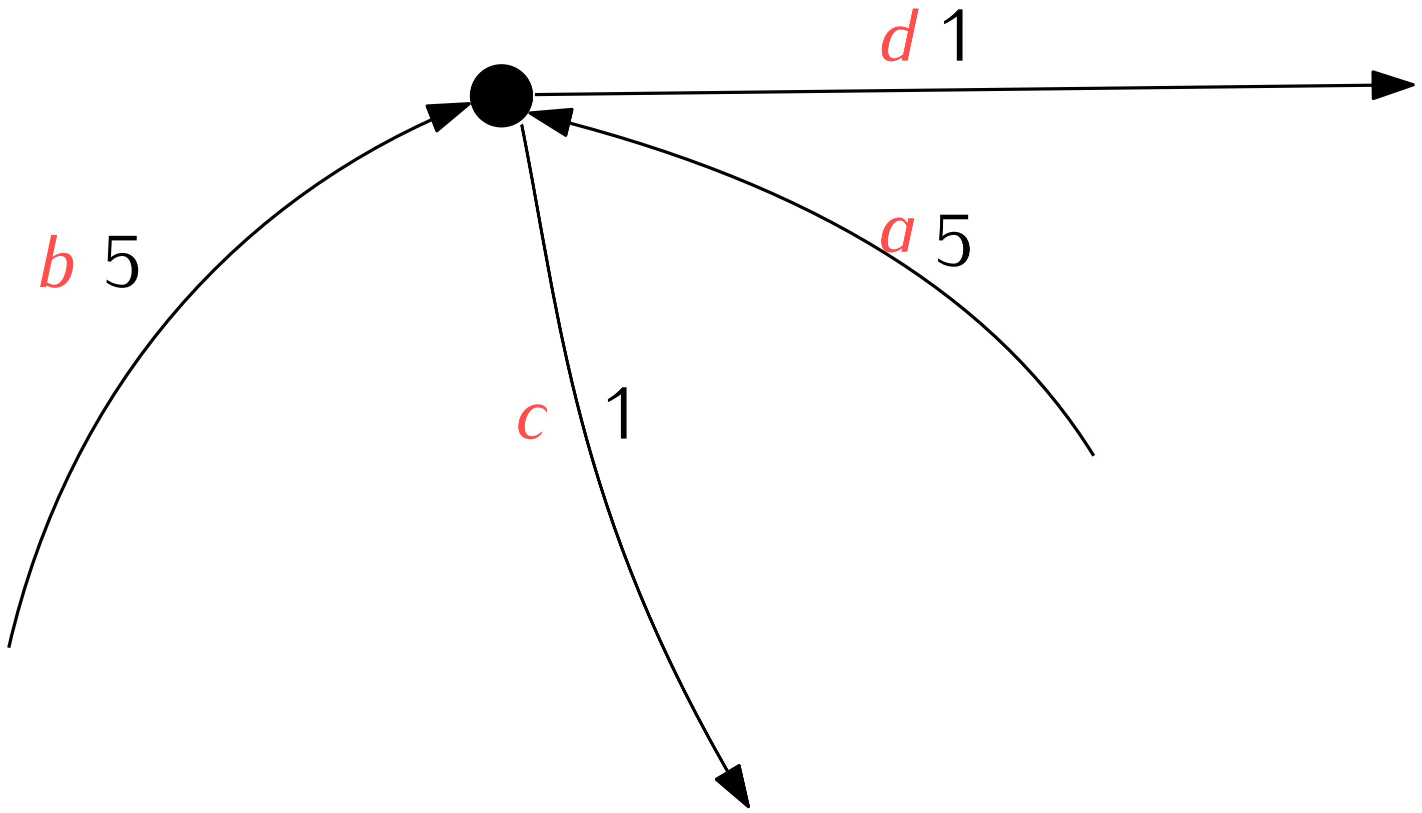


A flow is a function $f\colon E \to \mathbb{R}_{\geq 0}$ satisfying:

- For all $e \in E$, $0 \leq f(e) \leq c(e)$
- For all $v \in V \setminus \{s, t\}$, $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$.



$f(e)$ $c(e)$

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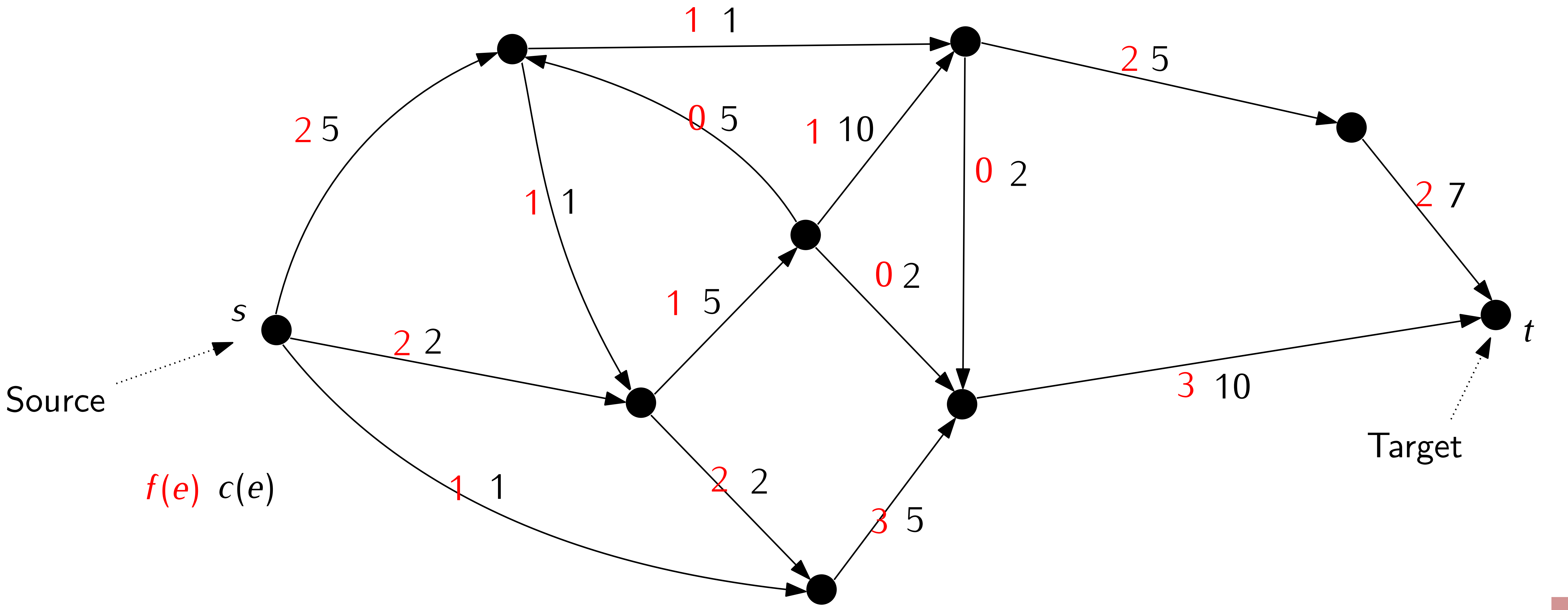

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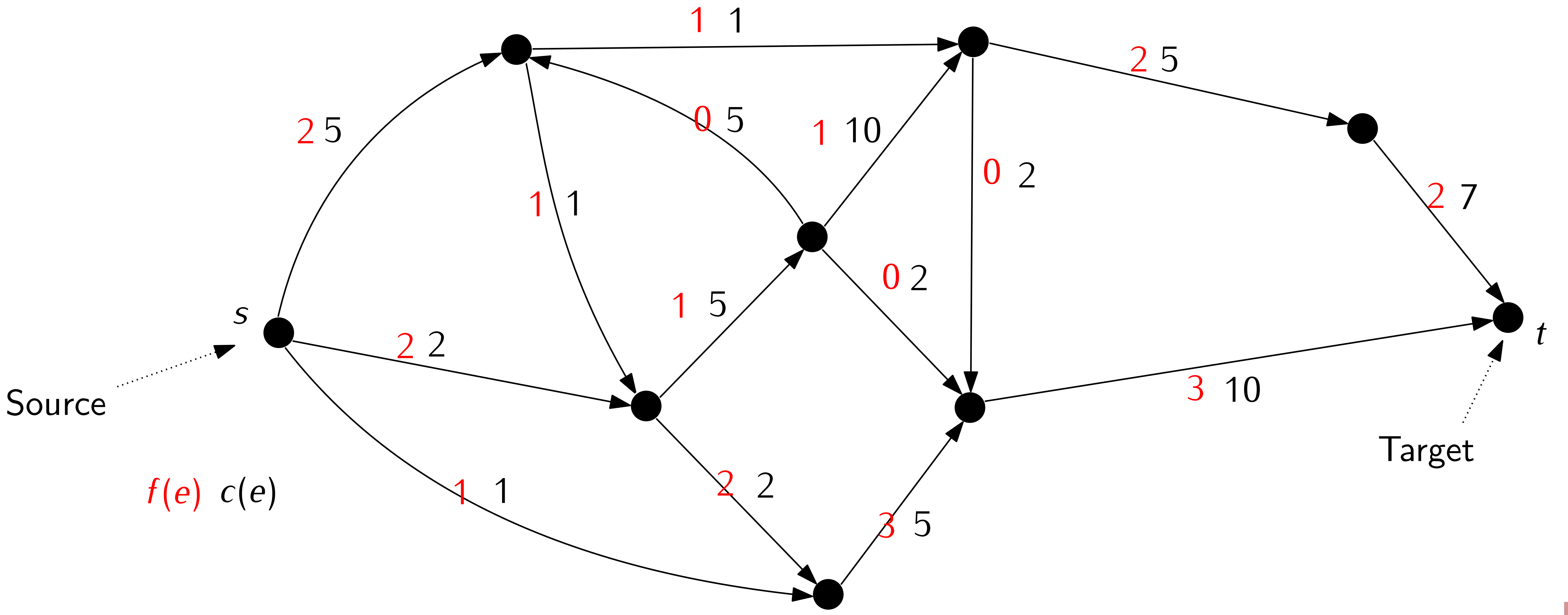

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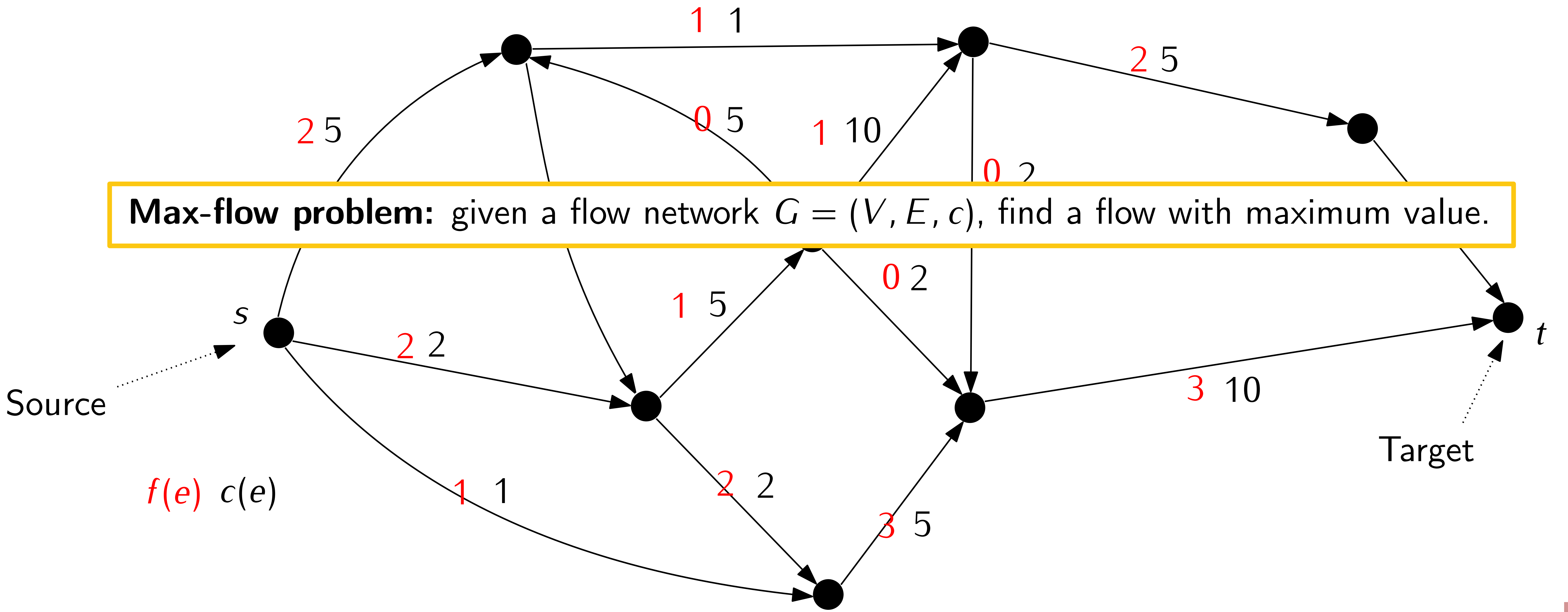
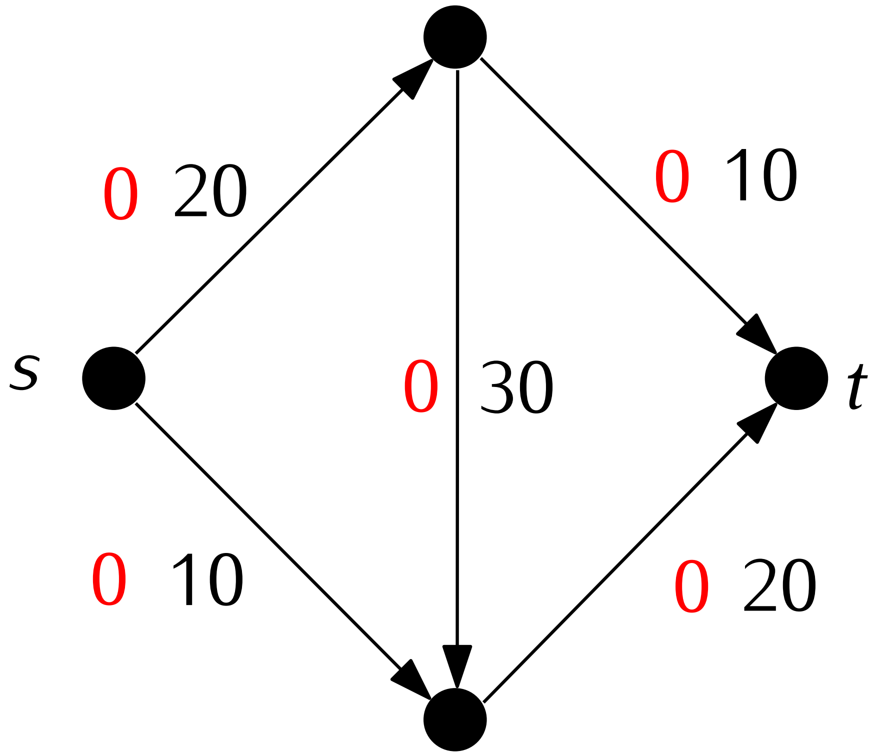


Max-flow problem: given a flow network $G = (V, E, c)$, find a flow with maximum value.

- Start with 0 flow: $f(e) = 0$ for all $e \in E$



f_0

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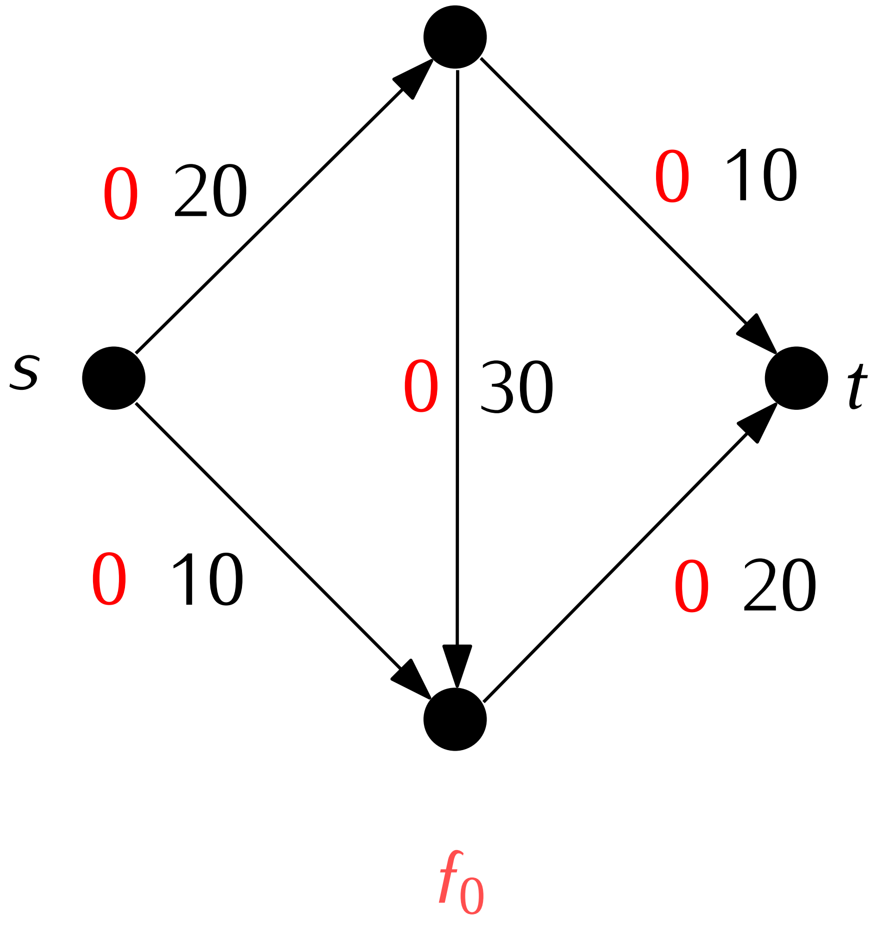
- Find a way to improve the flow...



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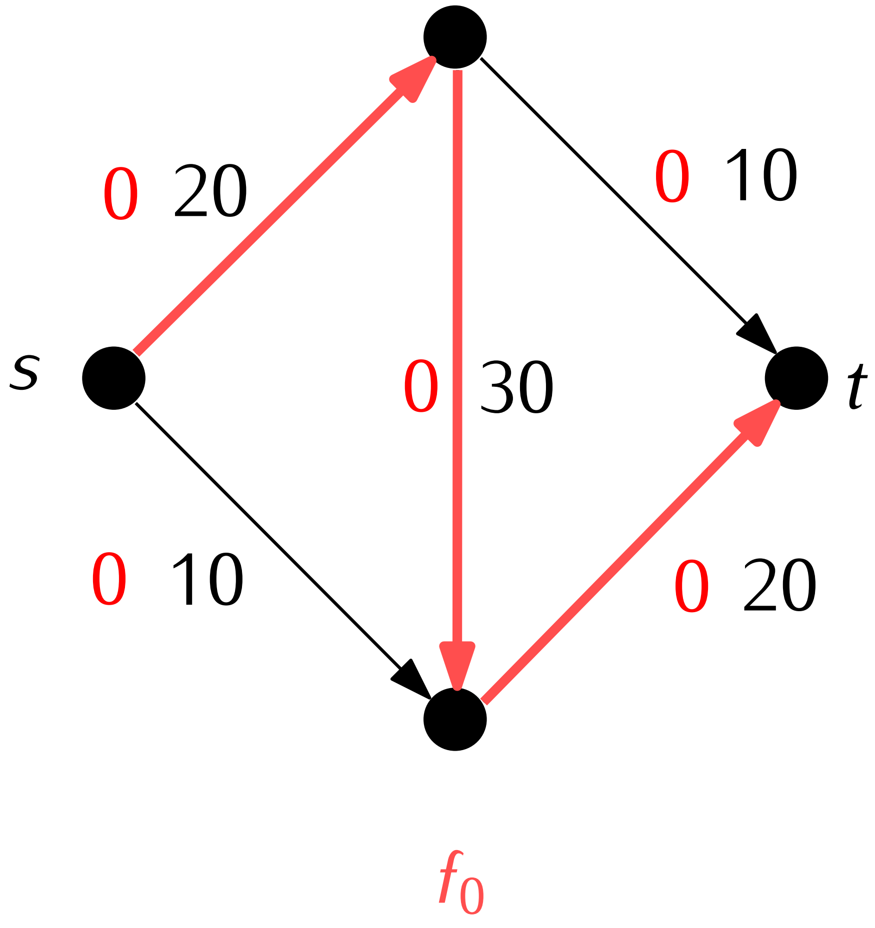

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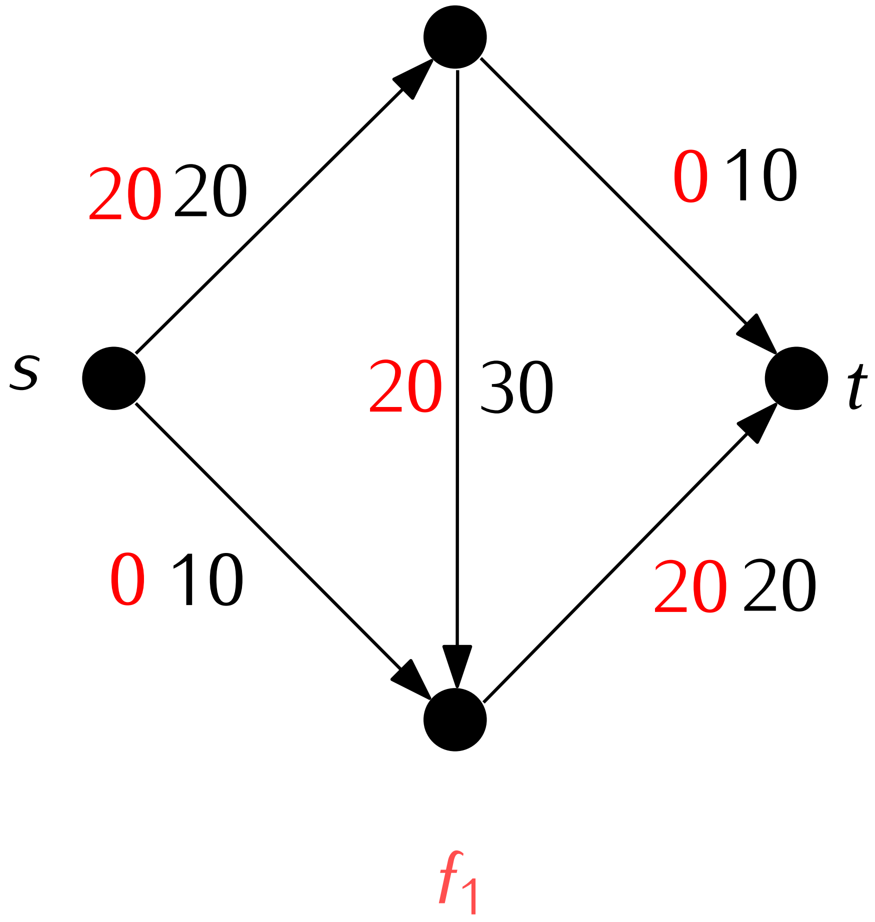

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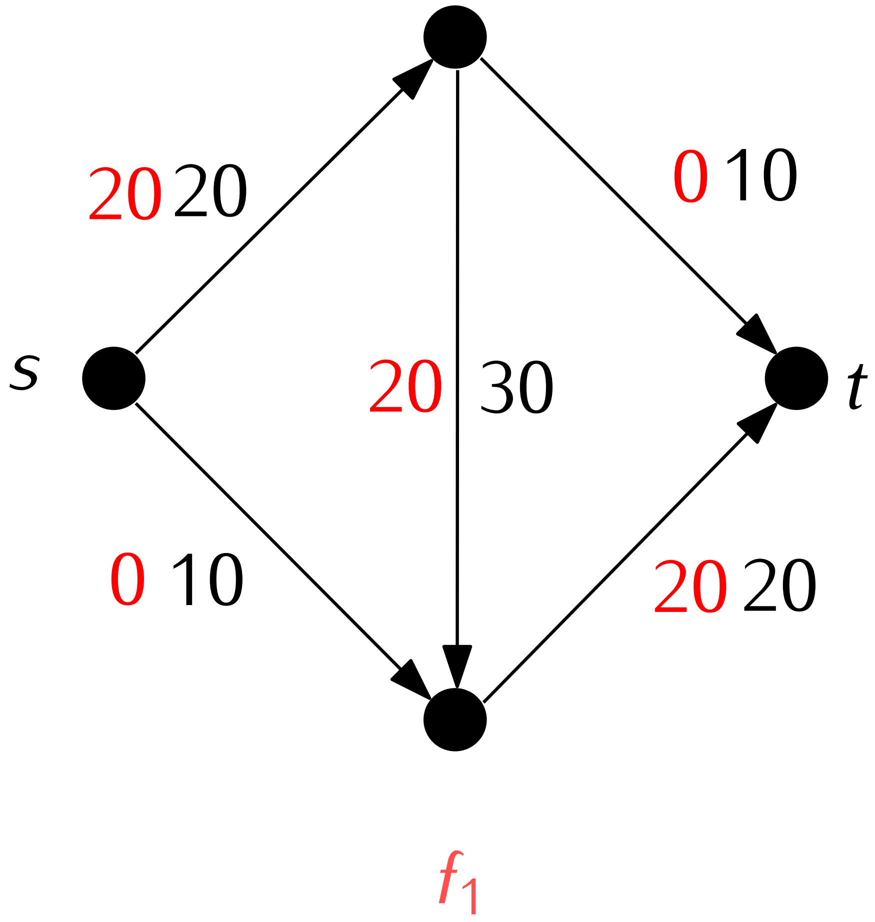

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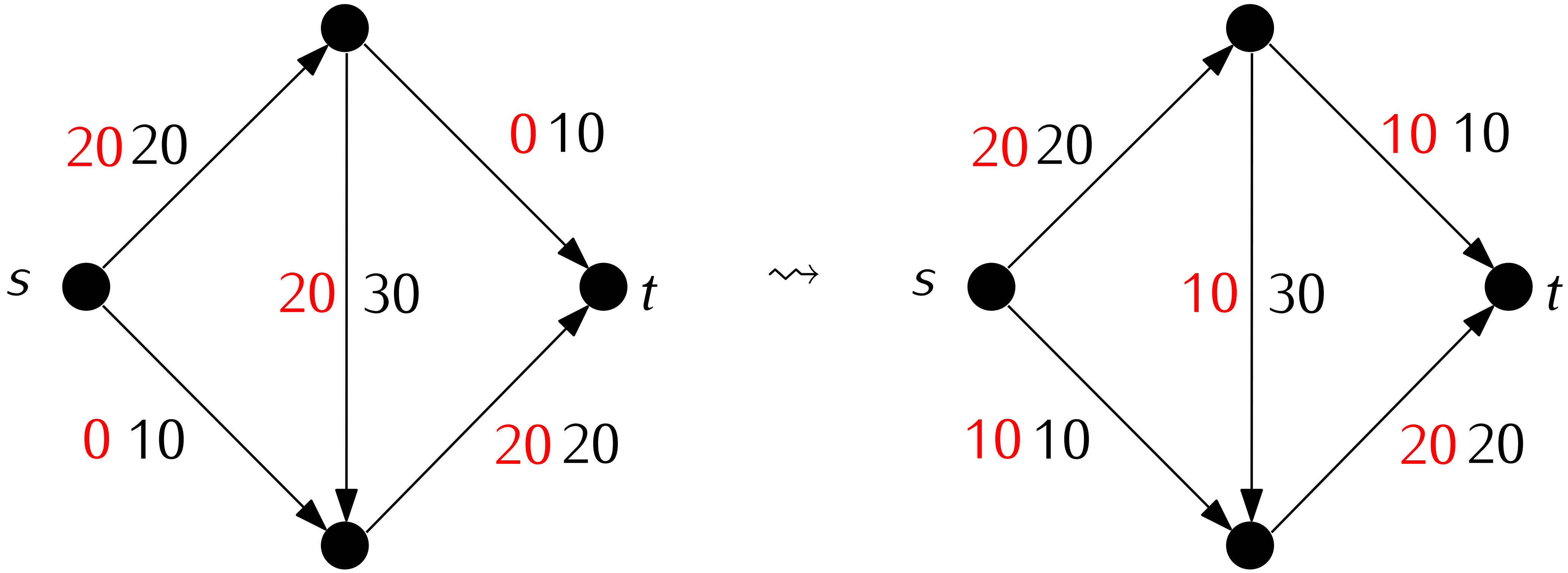

f_1 f_2

- Start with 0 flow: $f(e) = 0$ for all $e \in E$
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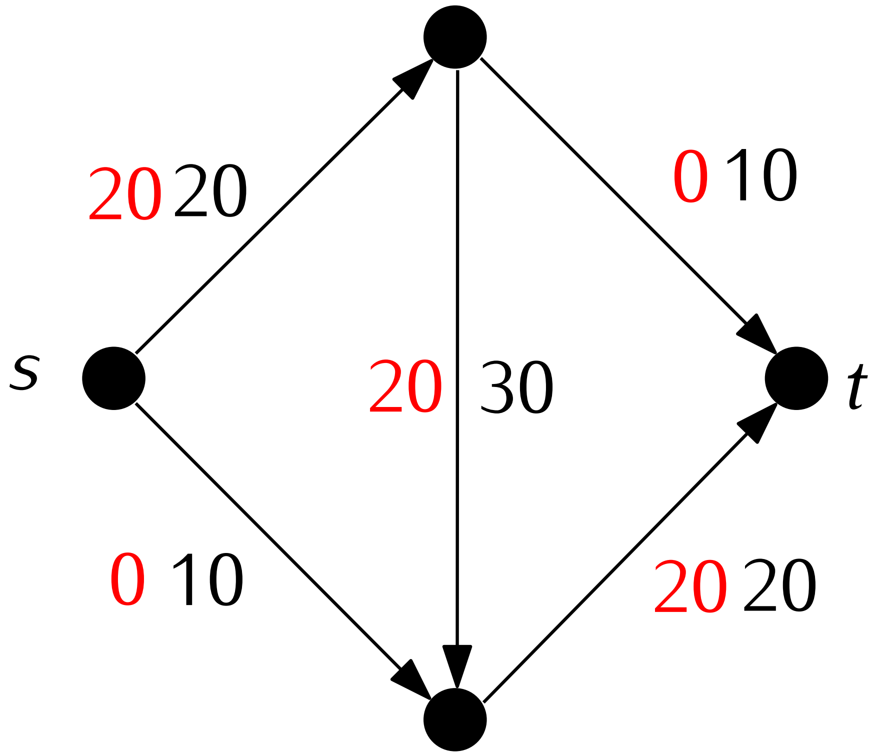
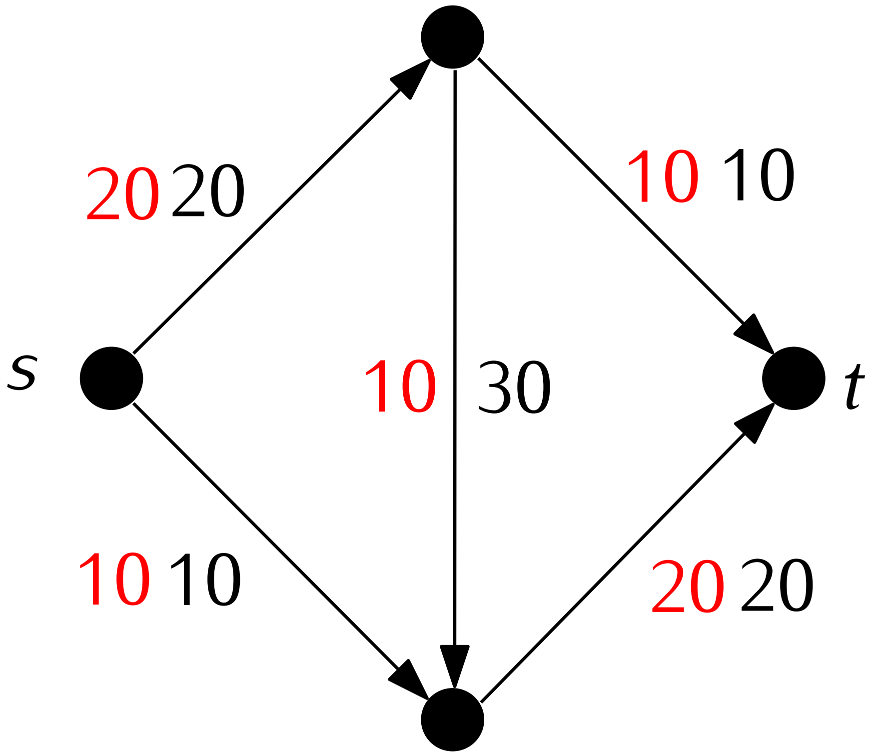
- **Want:** data structure to find those steps efficiently



f_1

$\rightsquigarrow$



f_2

Definition (Residual network). $G = (V, E, c)$ a flow network, $f \colon E \to \mathbb{R}_{\geq 0}$ a flow
Define $G_f = (V, E_f, c_f)$ by:

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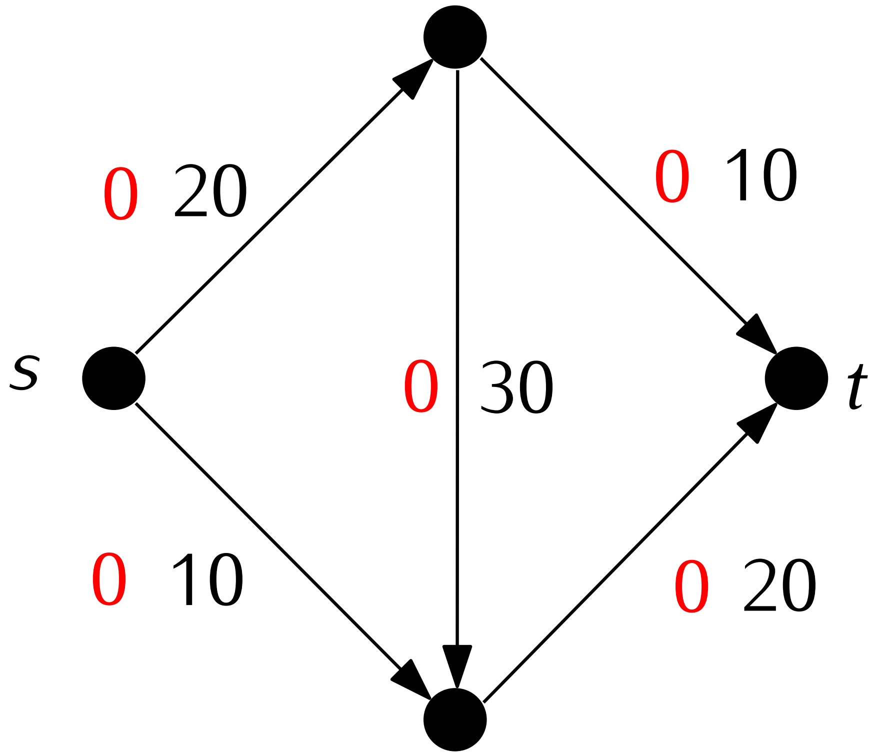

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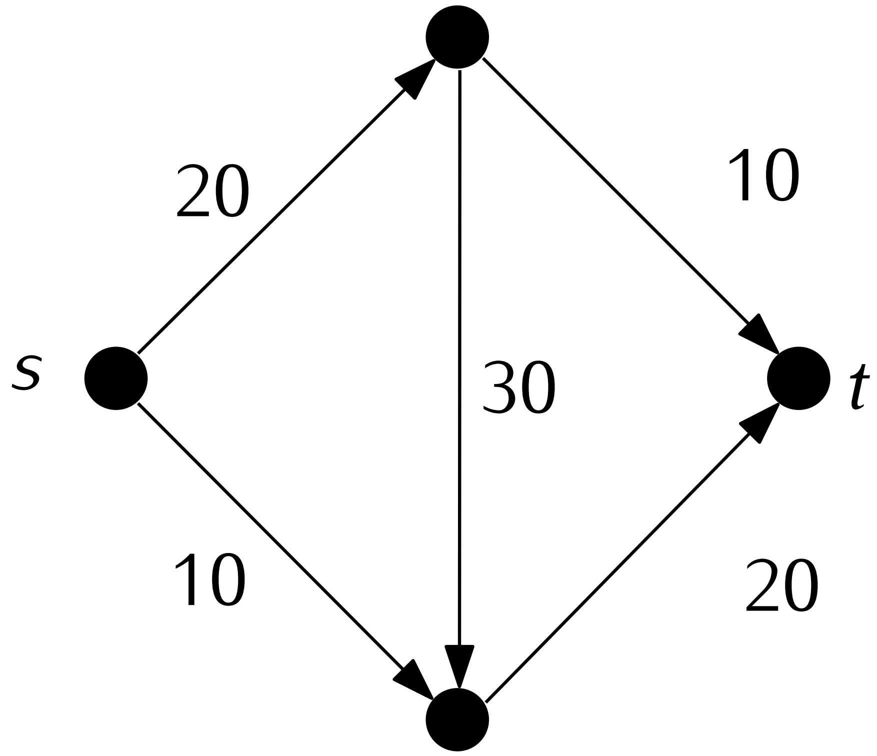

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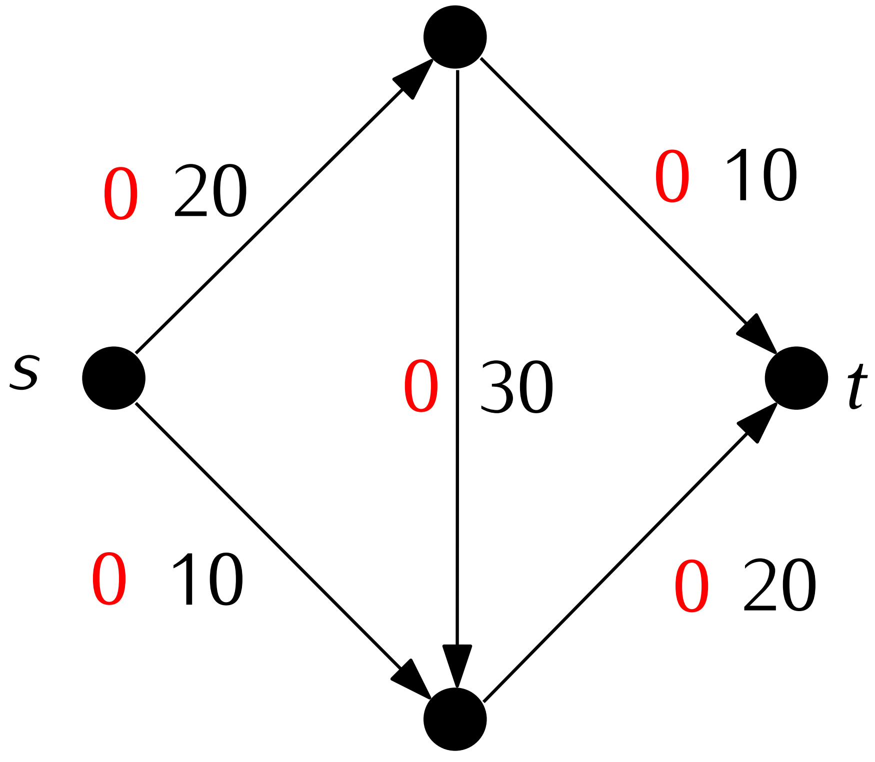

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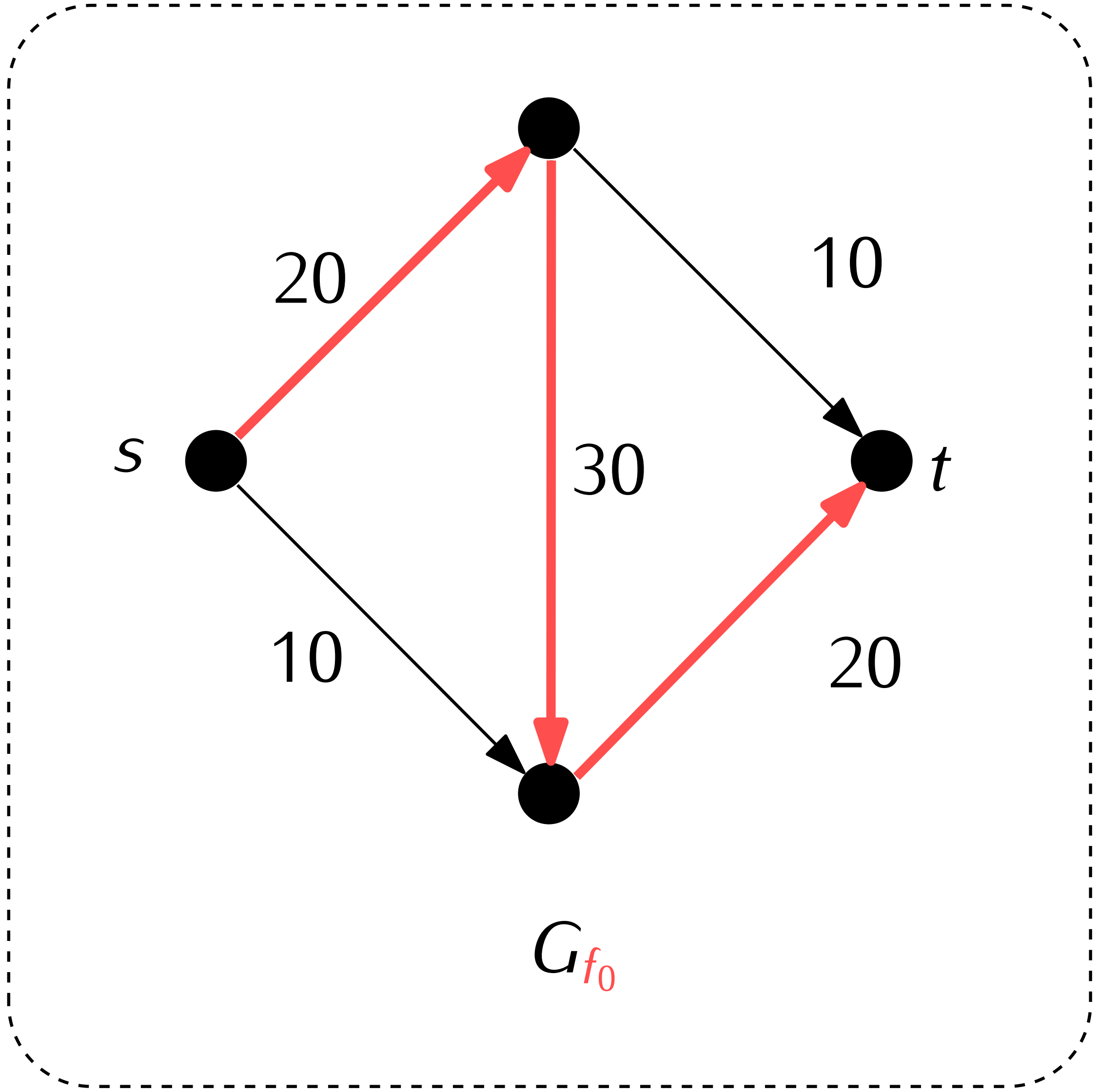

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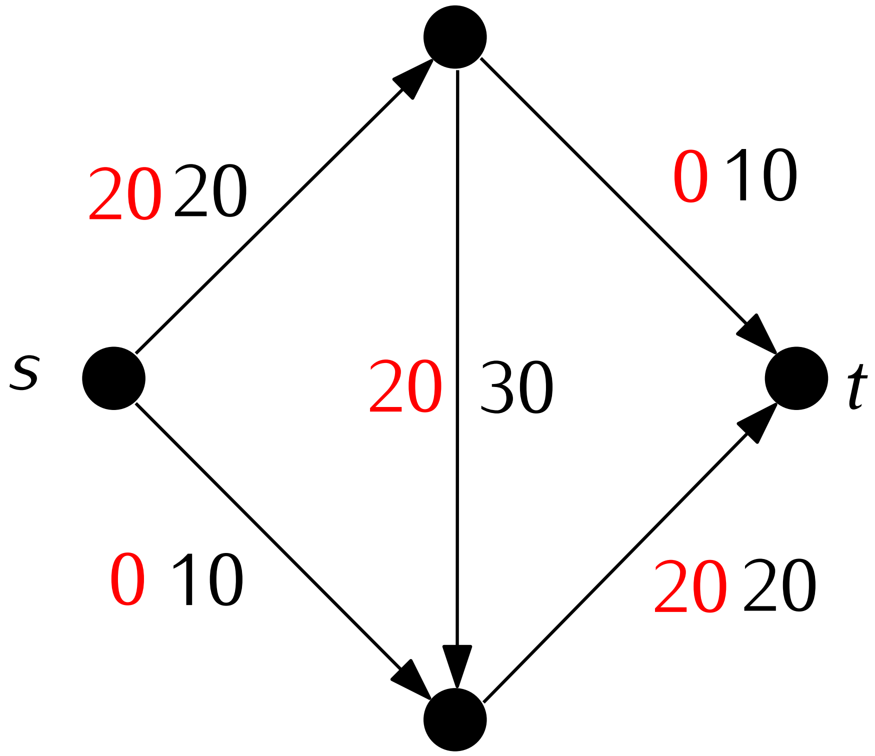
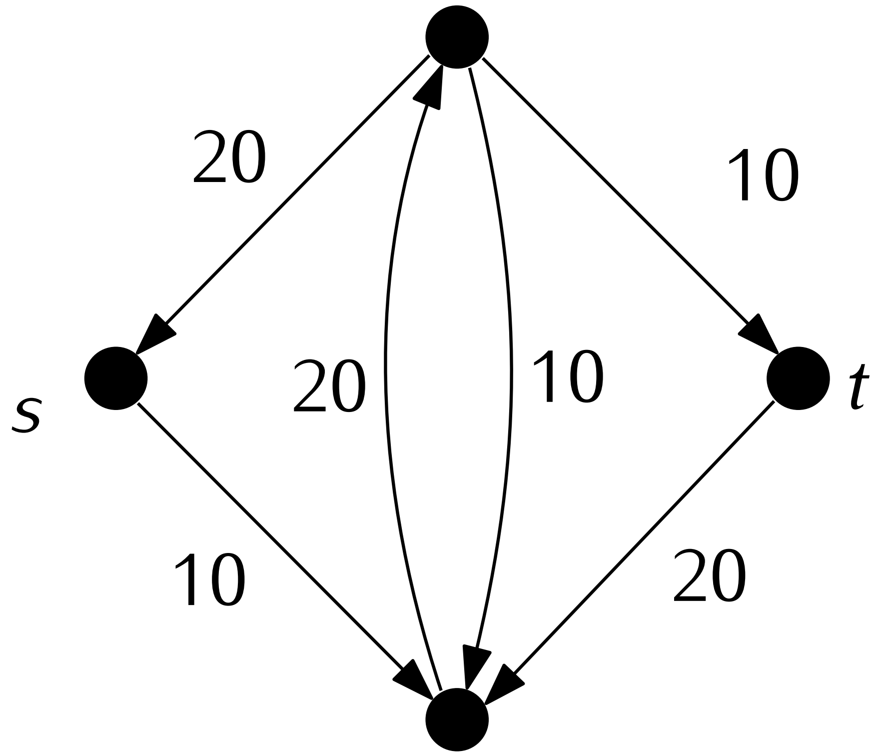


f_1



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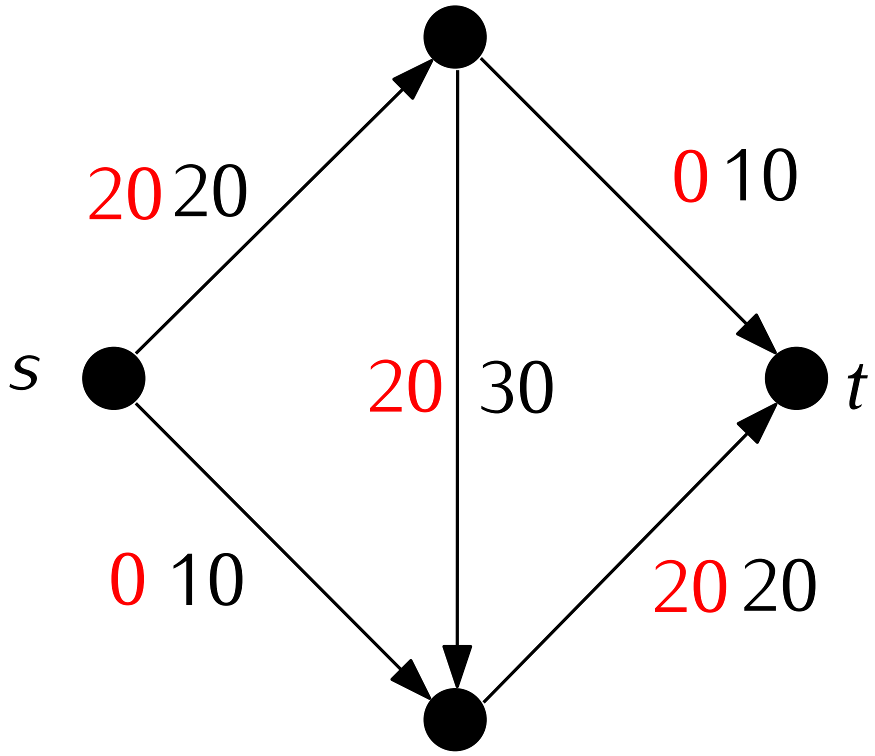

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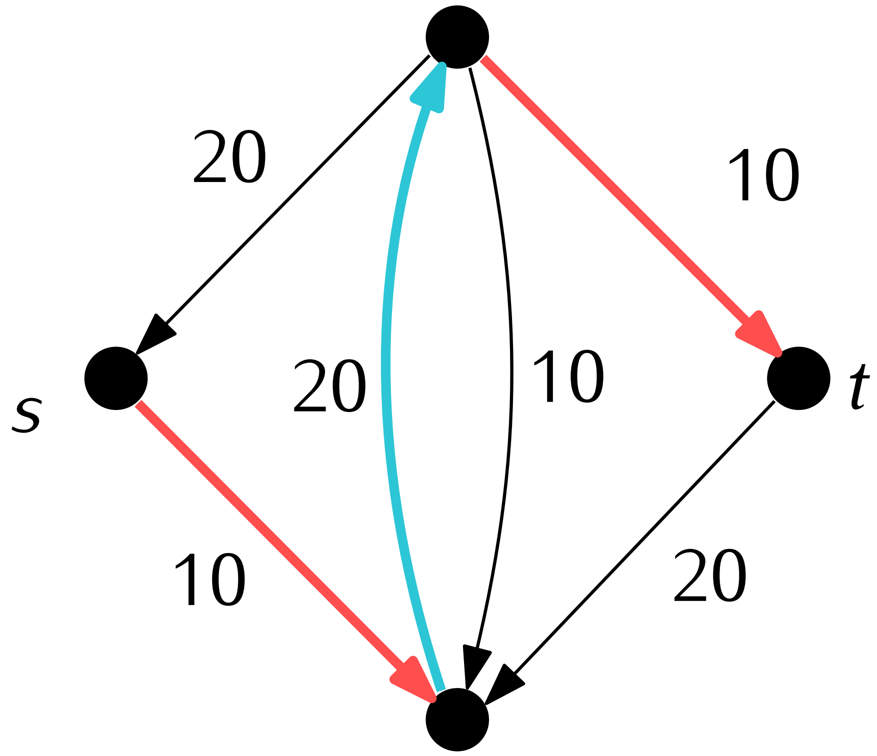

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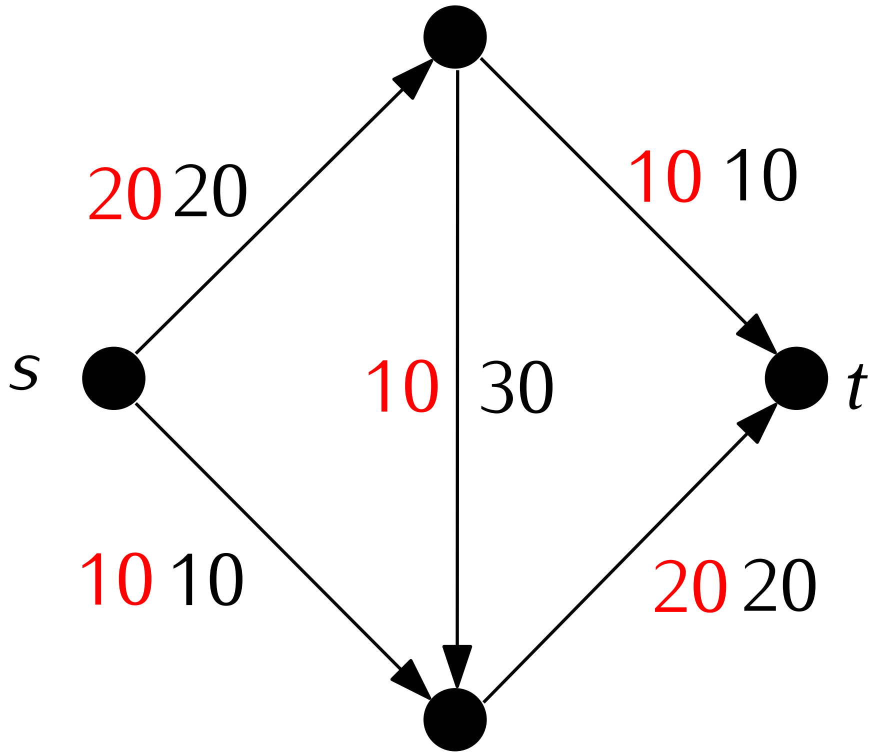
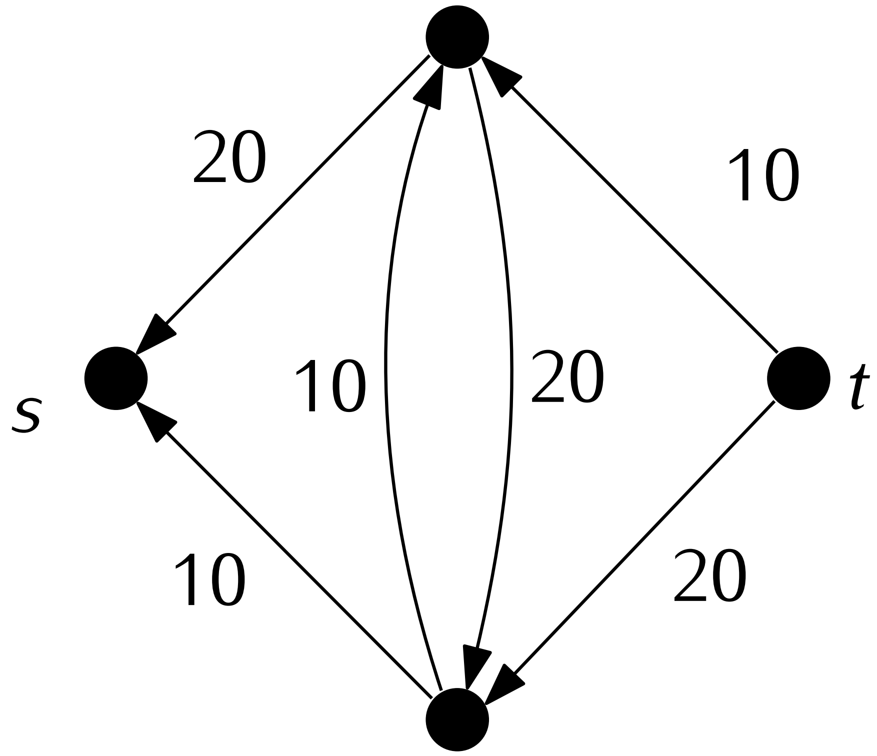


f_2



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1. Start with zero flow: $f(e) = 0$ for all $e \in E$
2. While there exists a simple s-t path $e_1, \ldots, e_k$ in G_f:
   (a) Find $m = \min\{c_f(e_i) \mid i \in \{1, \ldots, k\}\}$
   (b) Define:
      - $f'(e_i) := f(e_i) + m$ for all $i \in \{0, \ldots, k\}$ if $e_i \in E$
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Questions:

- Does it terminate? 2. could loop forever...
- How fast does it terminate?
- Is f a flow?
- Does f always have maximal value?

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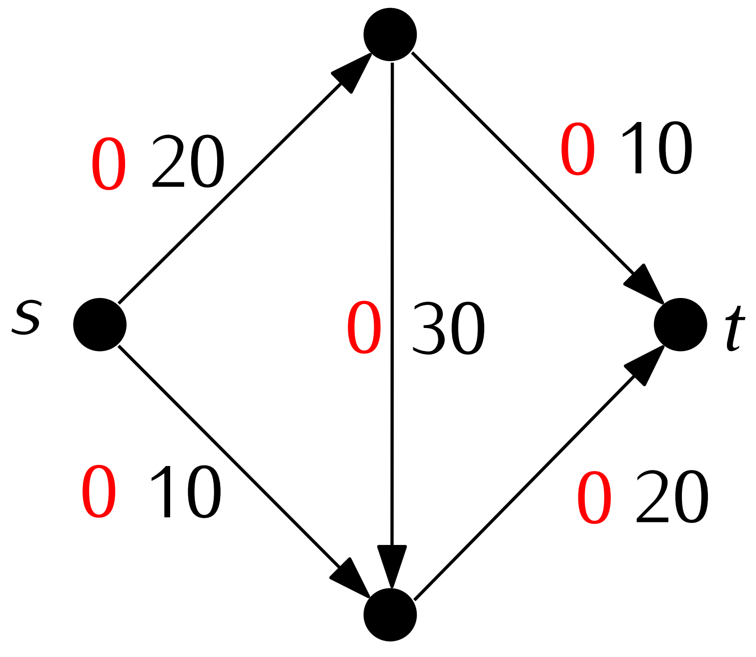
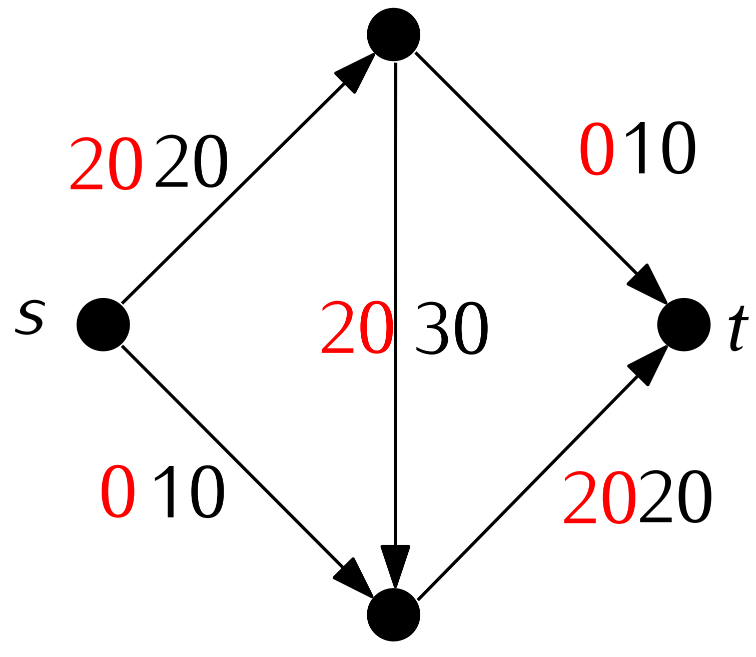


f_0



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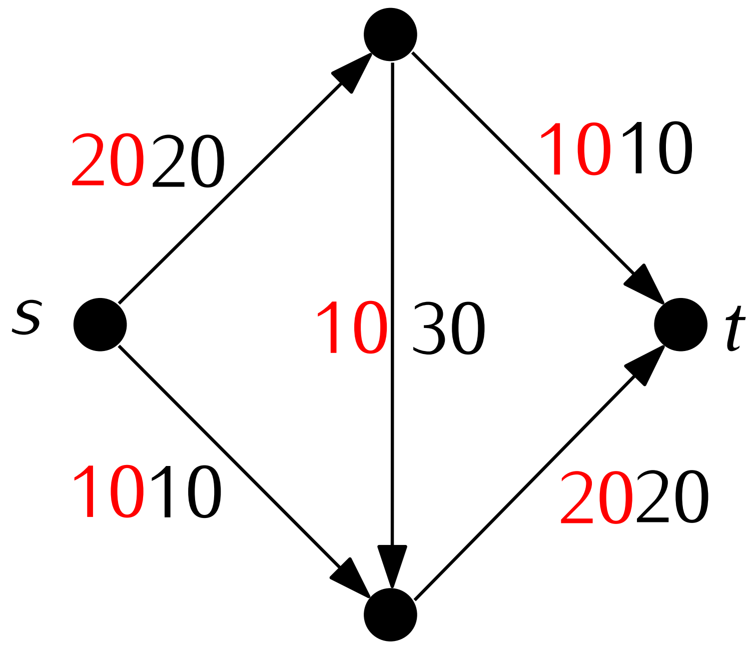

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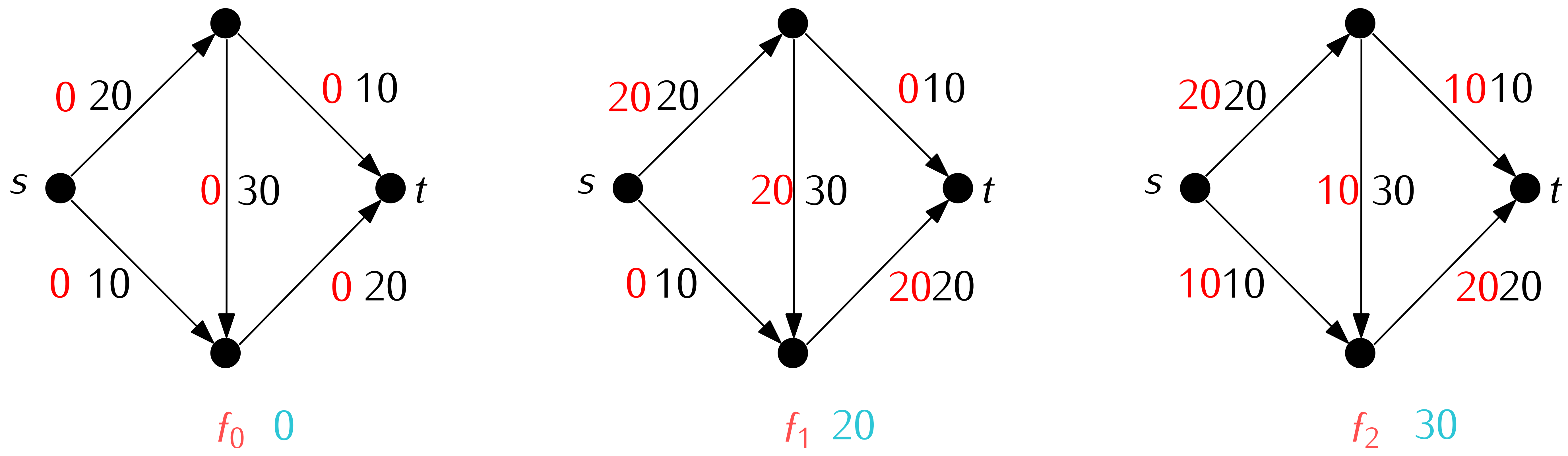

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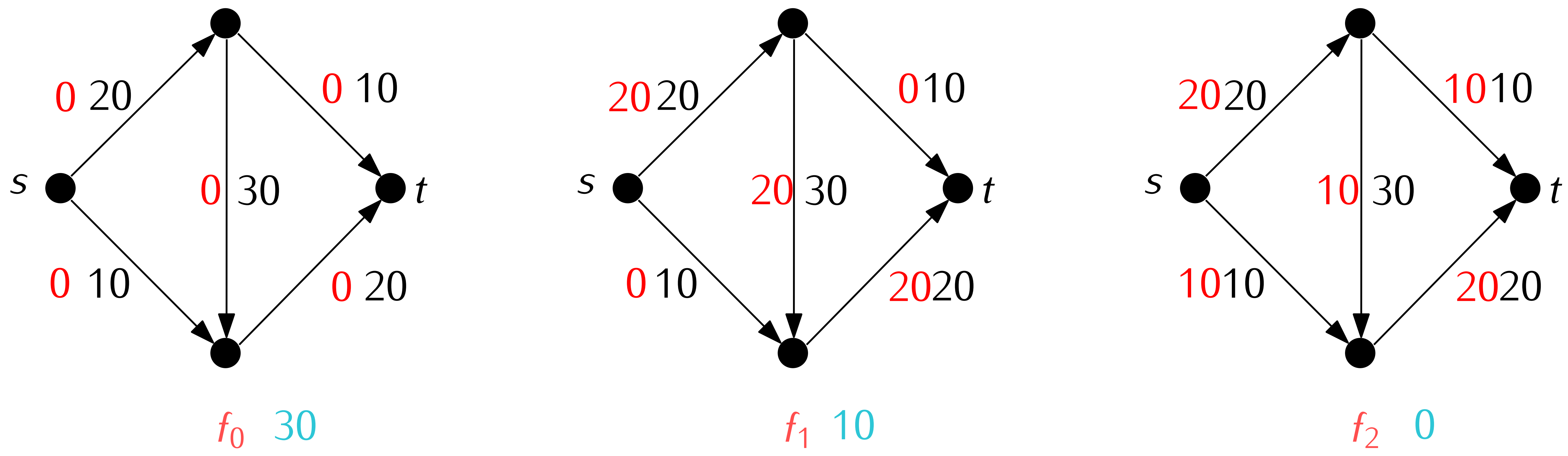

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Attempt: $m(f) = \sum_{e \text{ out of } s}(c(e) - f(e))$ ✅

Lemma. $m(f) \geq 0$ and strictly decreases at every iteration.

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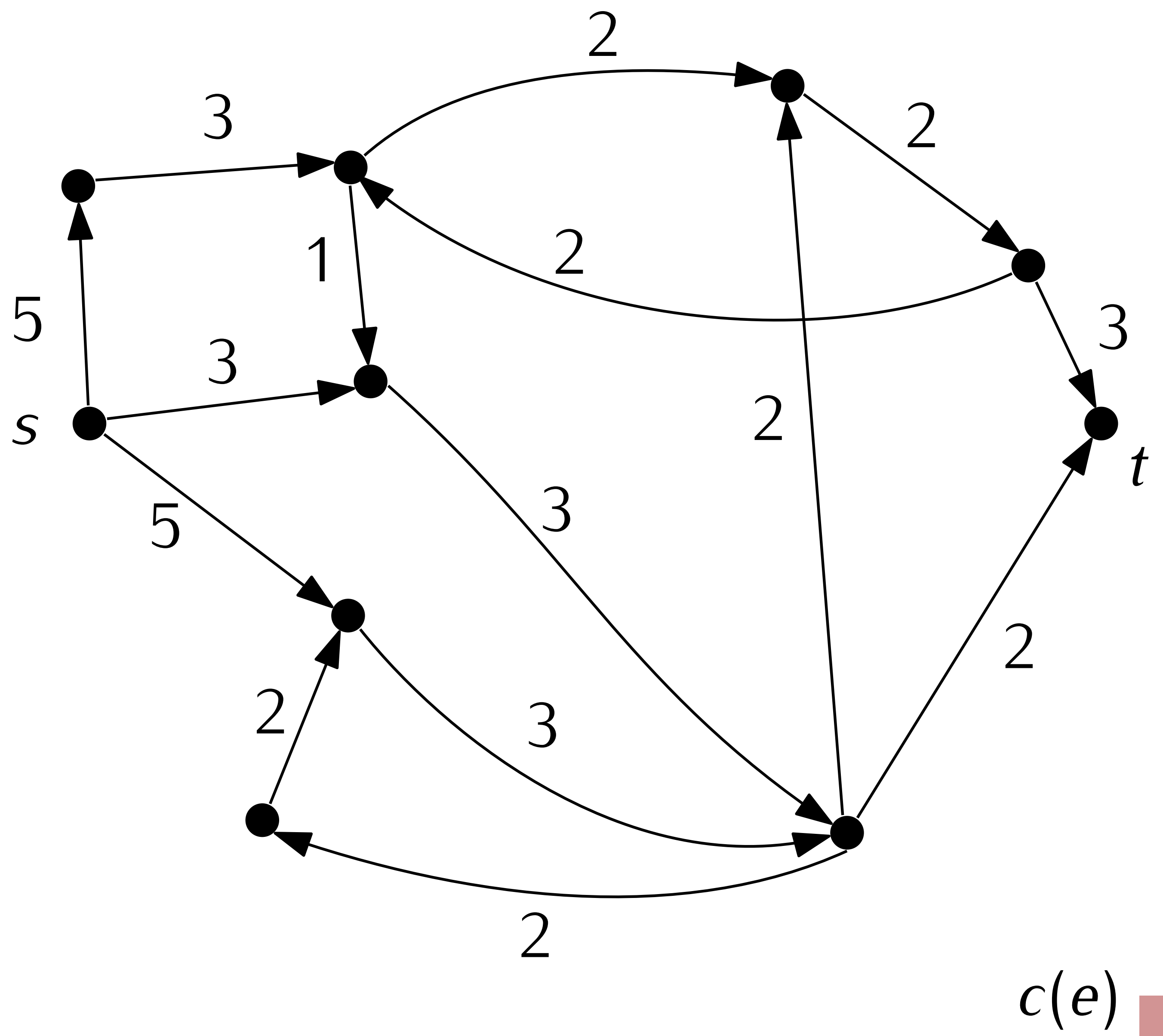
Lemma. $m(f) \geq 0$ and strictly decreases at every iteration.

- Algorithm terminates in at most $C := \sum_{e \text{ out of } s} c(e)$ iterations.
- $O(C|E|)$ running time
- We later see that there are at most $|V||E|$ iterations if we choose shortest path at 2.

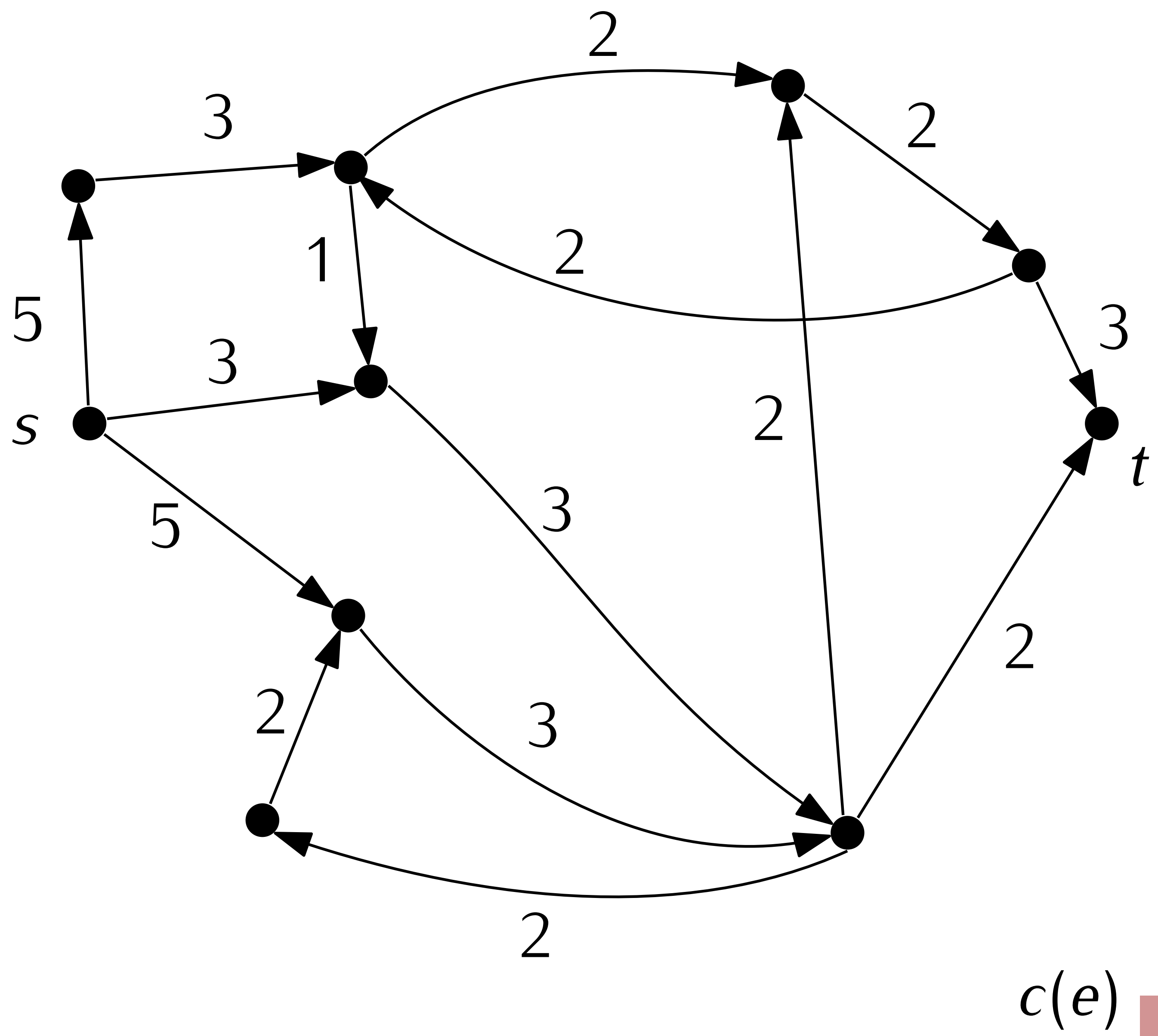
  $\rightsquigarrow$ Edmonds-Karp algorithm

What are values that can in principle be achieved?



What are values that can in principle be achieved?

- Value is $\leq \sum_{e \text{ out of } s} c(e) = 13$



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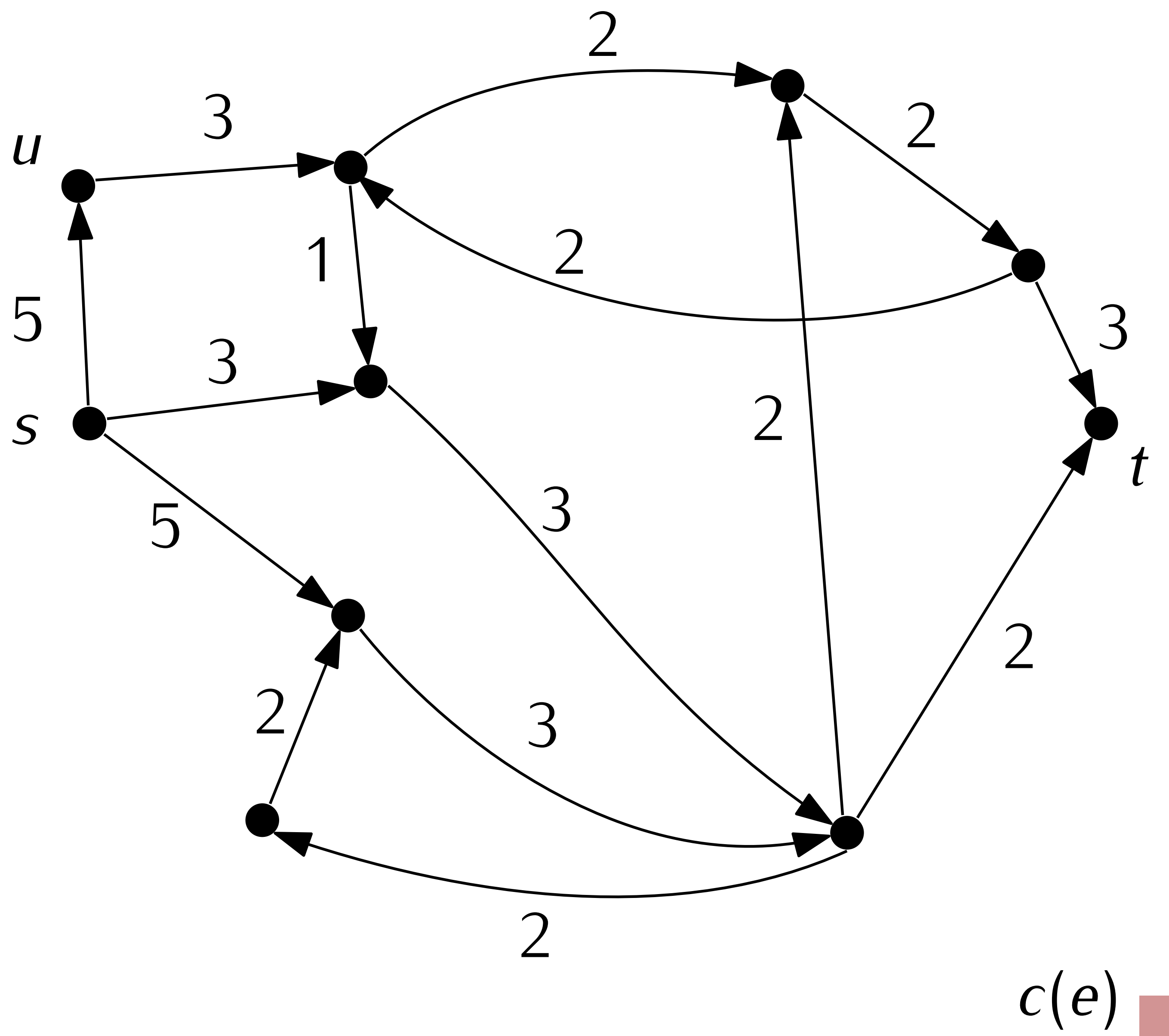

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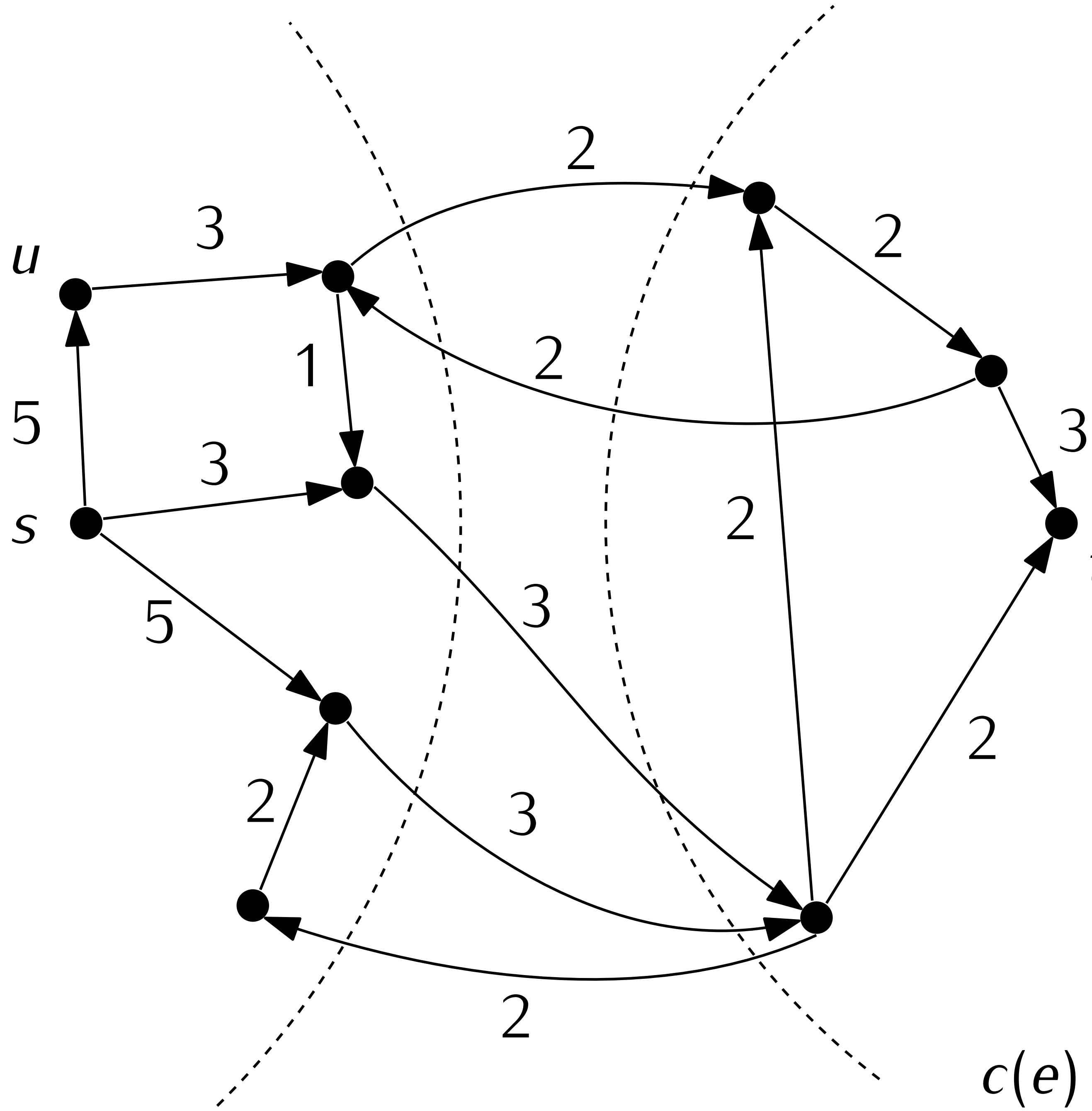

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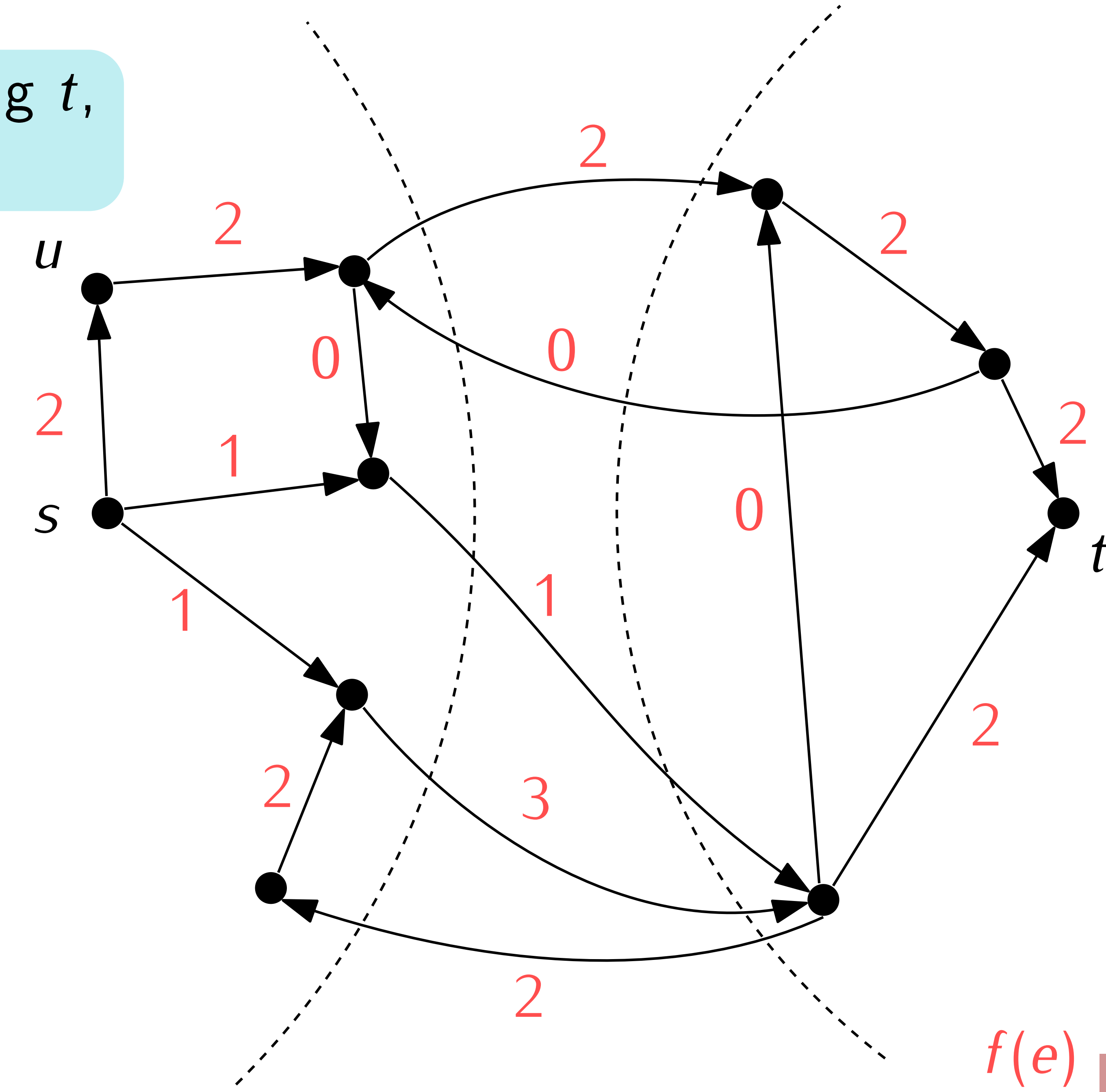

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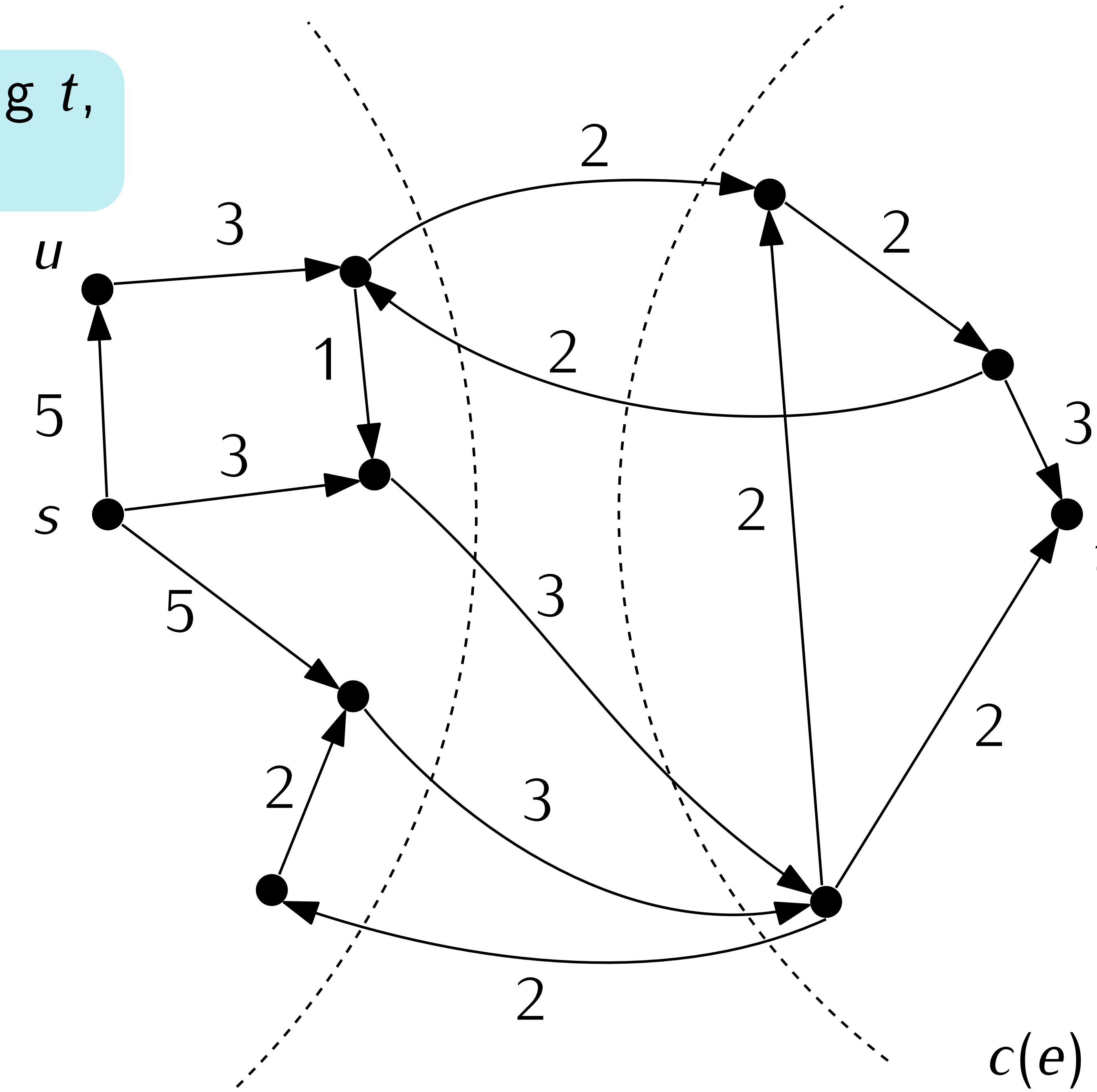

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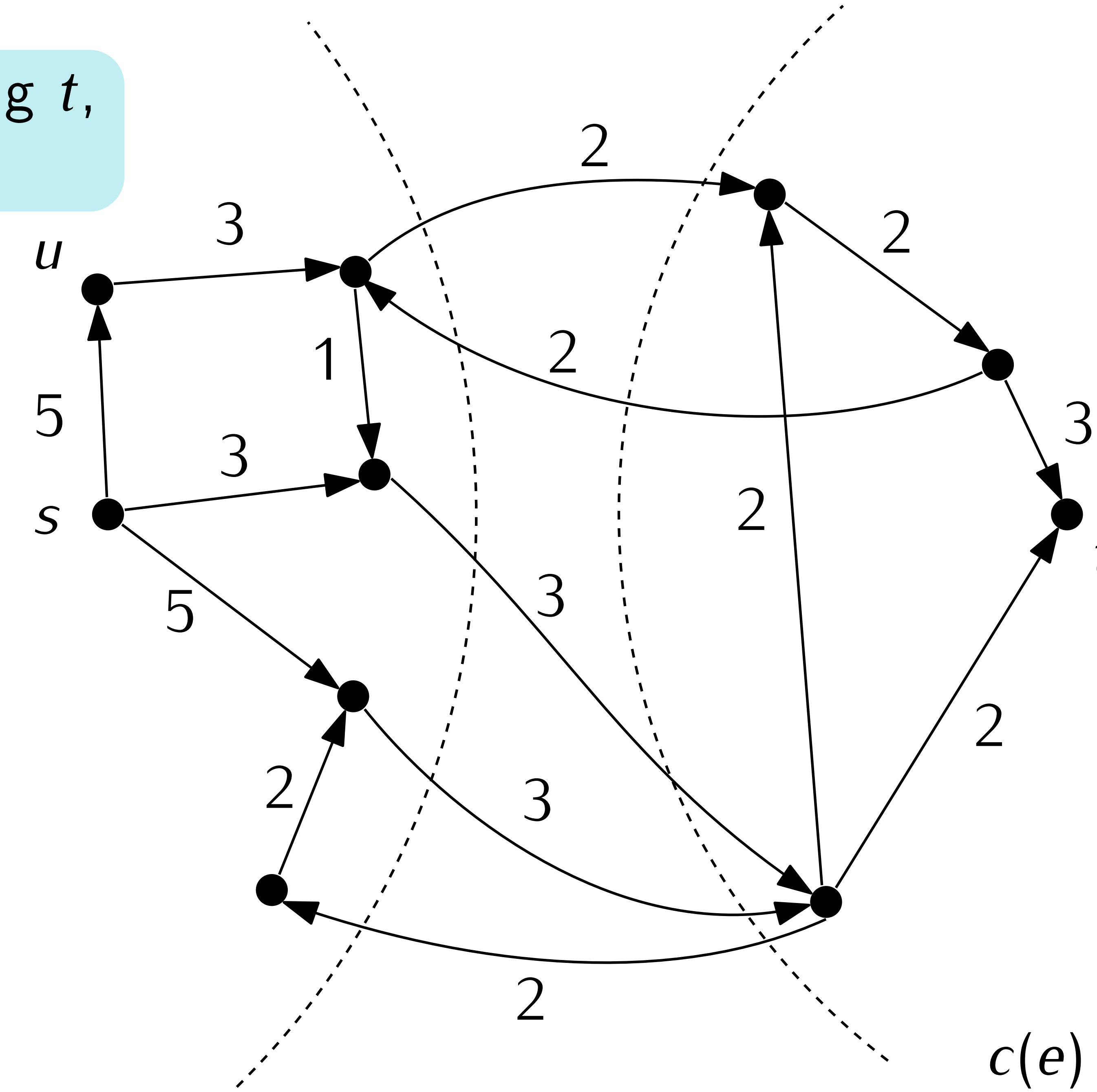
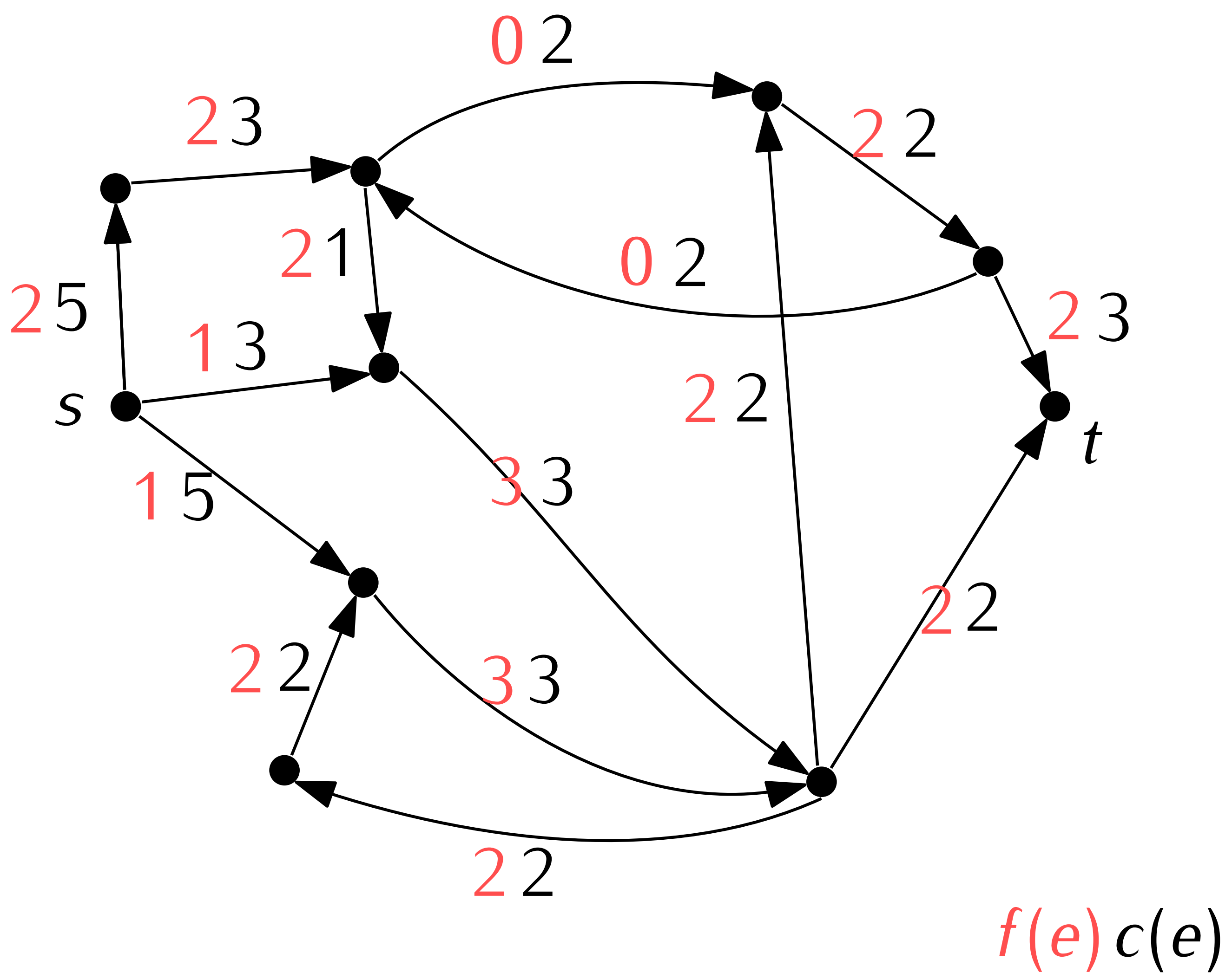
Since this holds for all s-t cuts (A, B):

Theorem. For all G, we have

$$\mathrm{maxflow}(G) \leq \min_{(A,B)\ s\text{-}t \text{ cut}} \sum_{e \text{ out of } A} c(e)$$



How can we know that a flow f is maximal?



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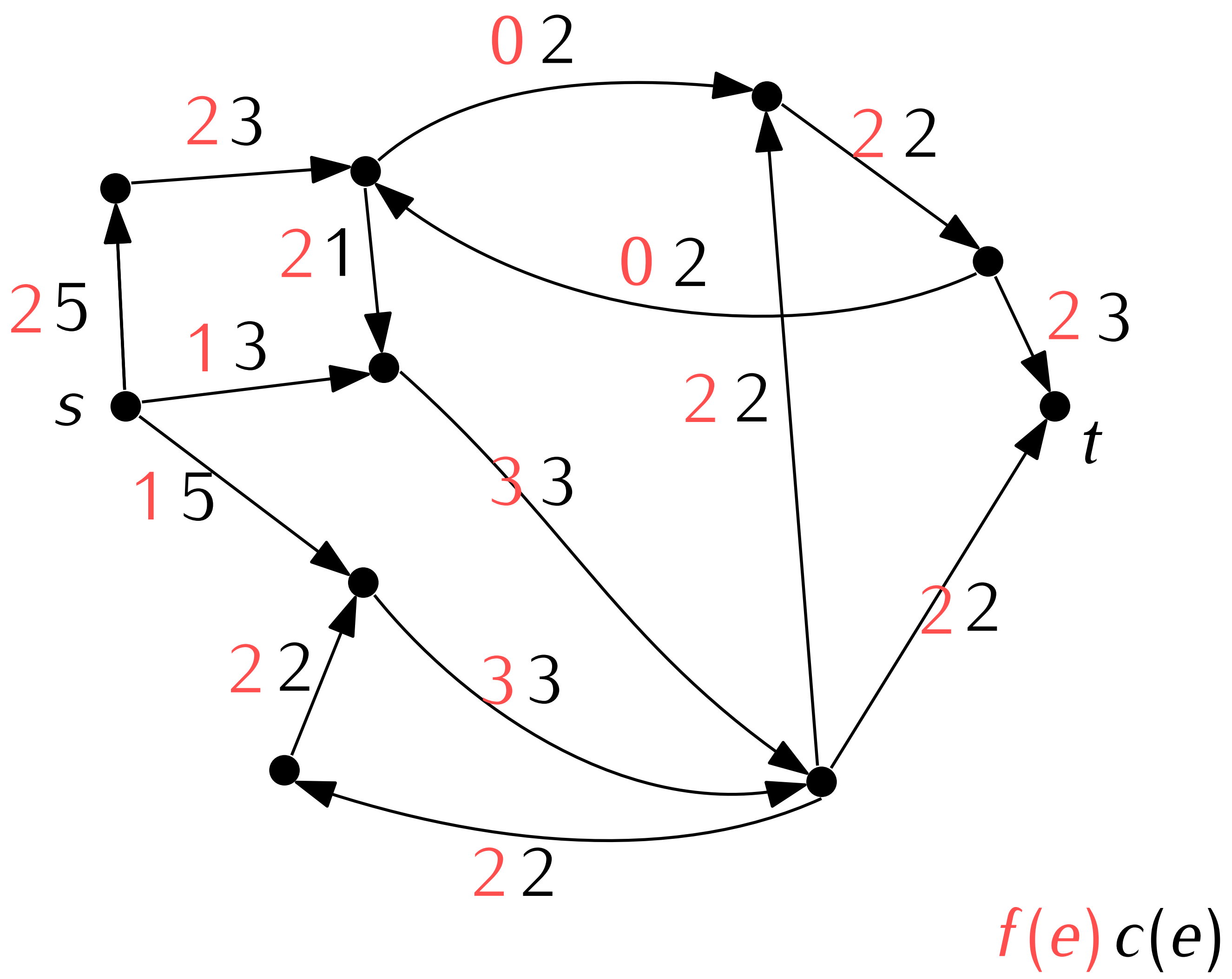

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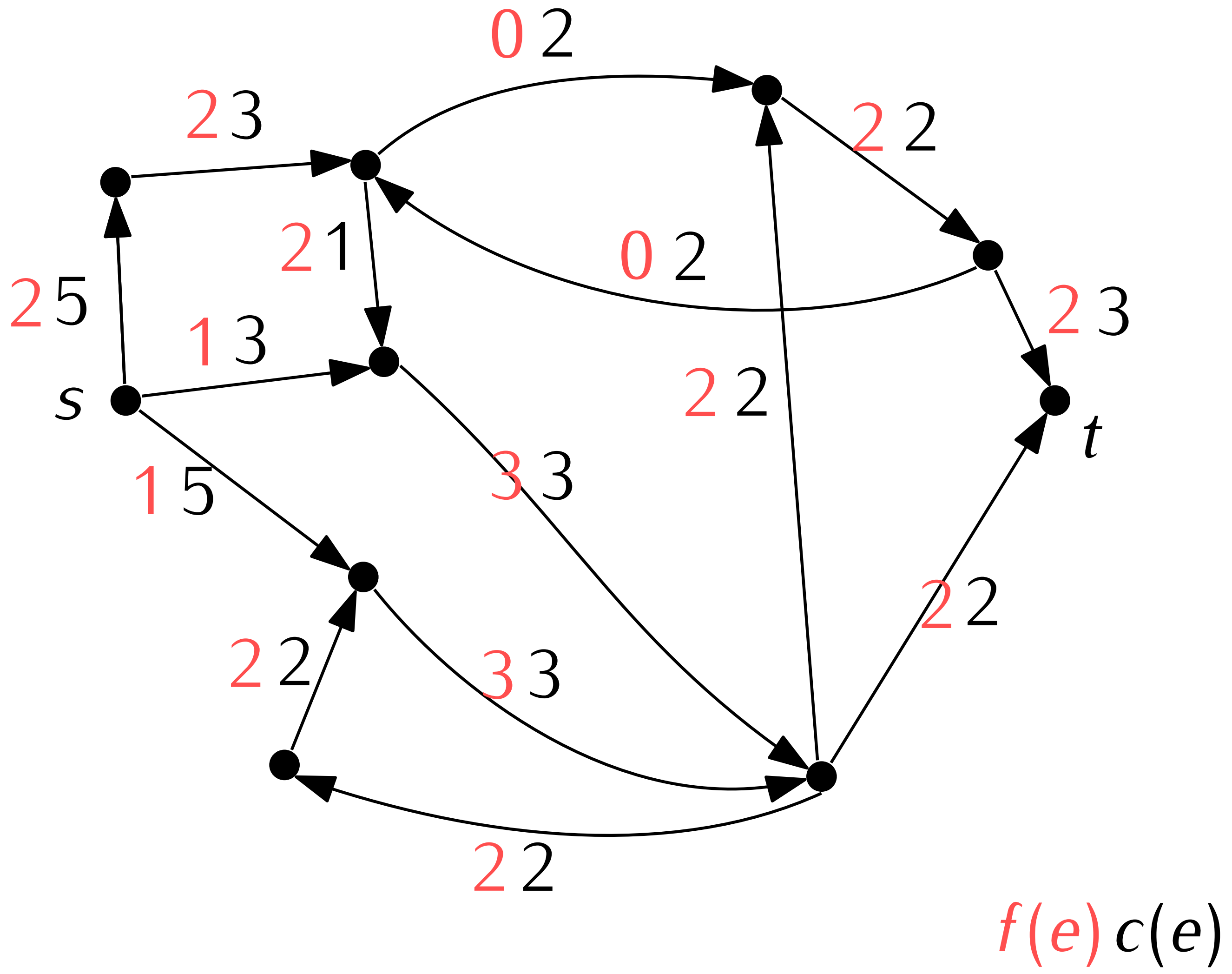

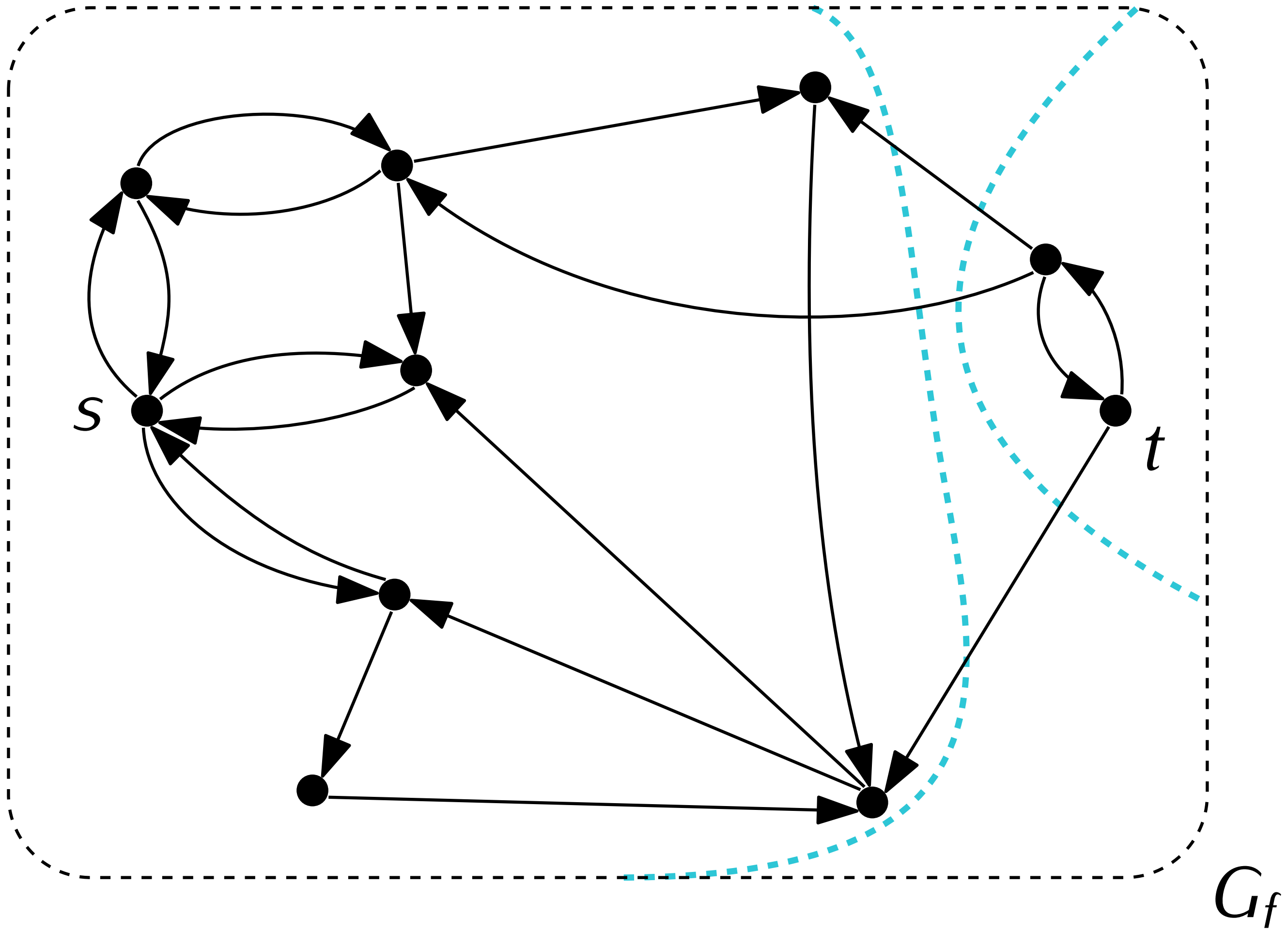

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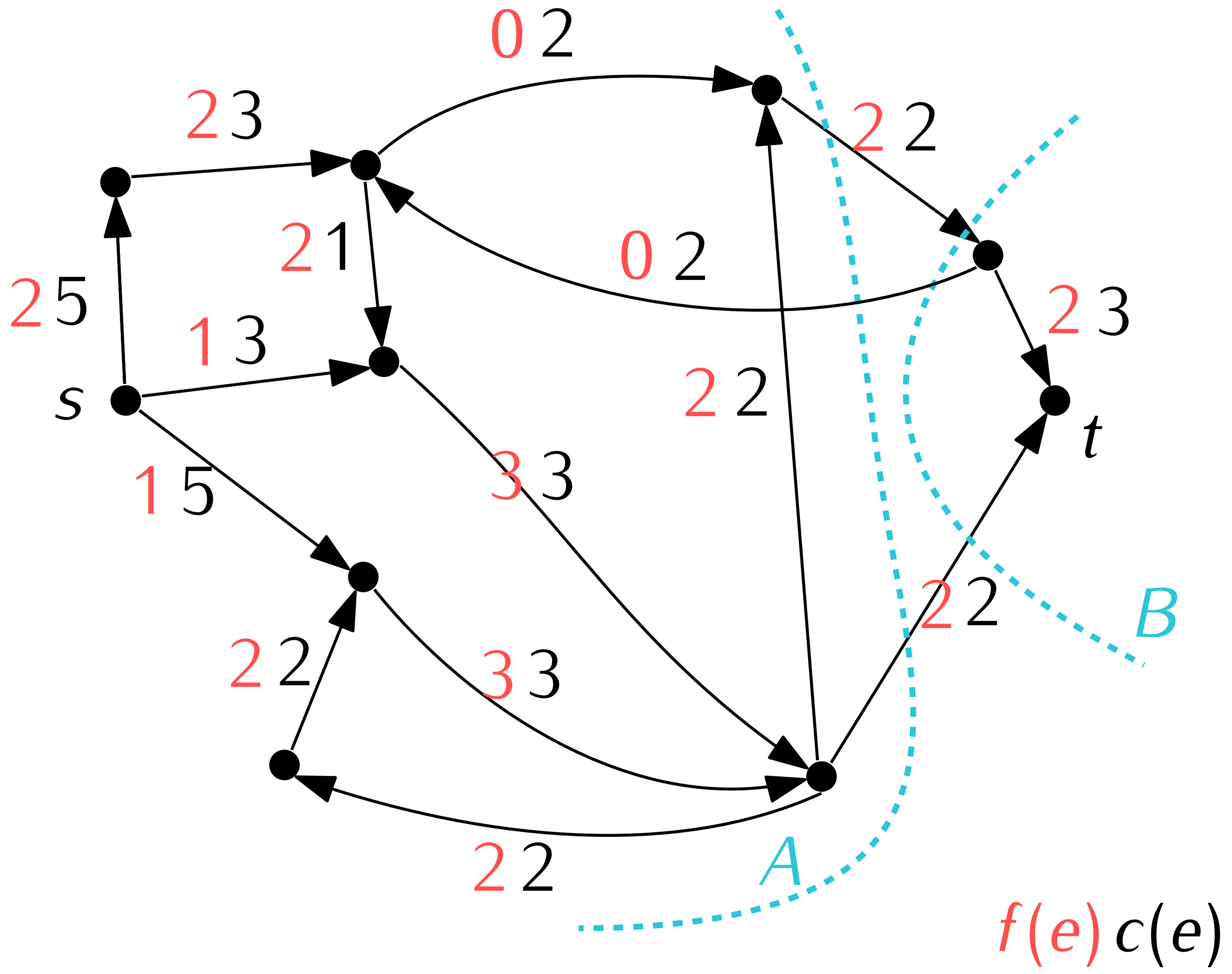

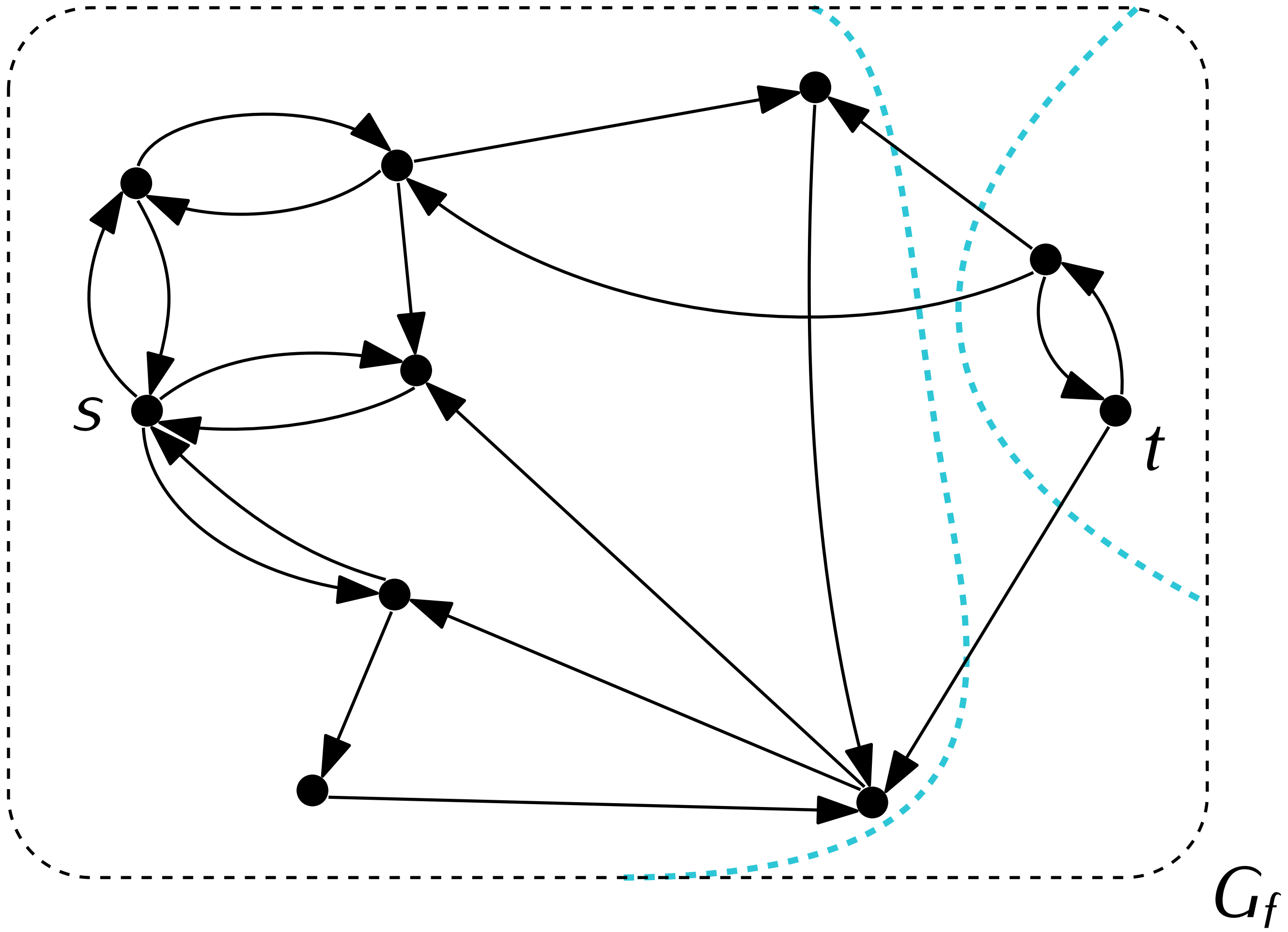

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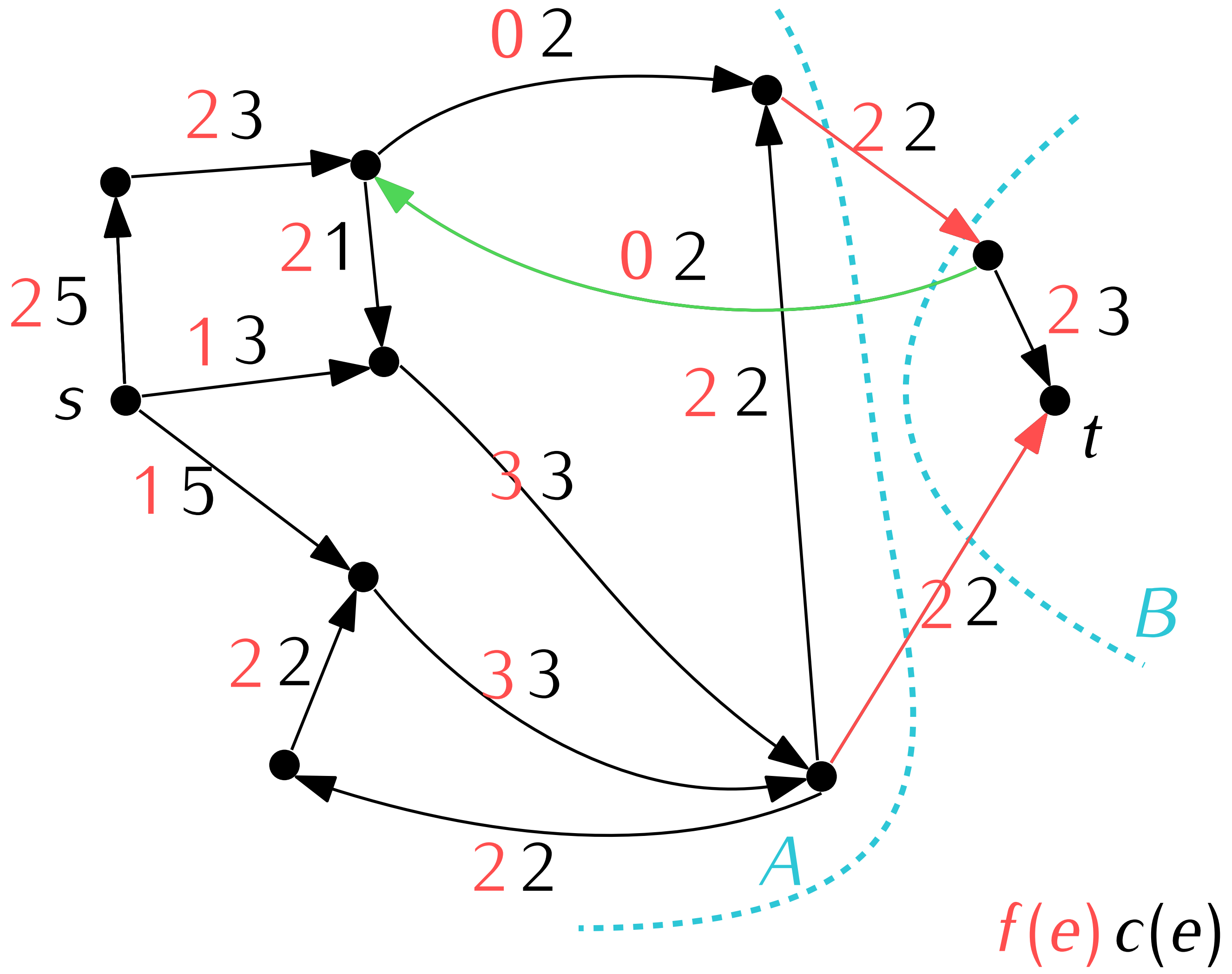
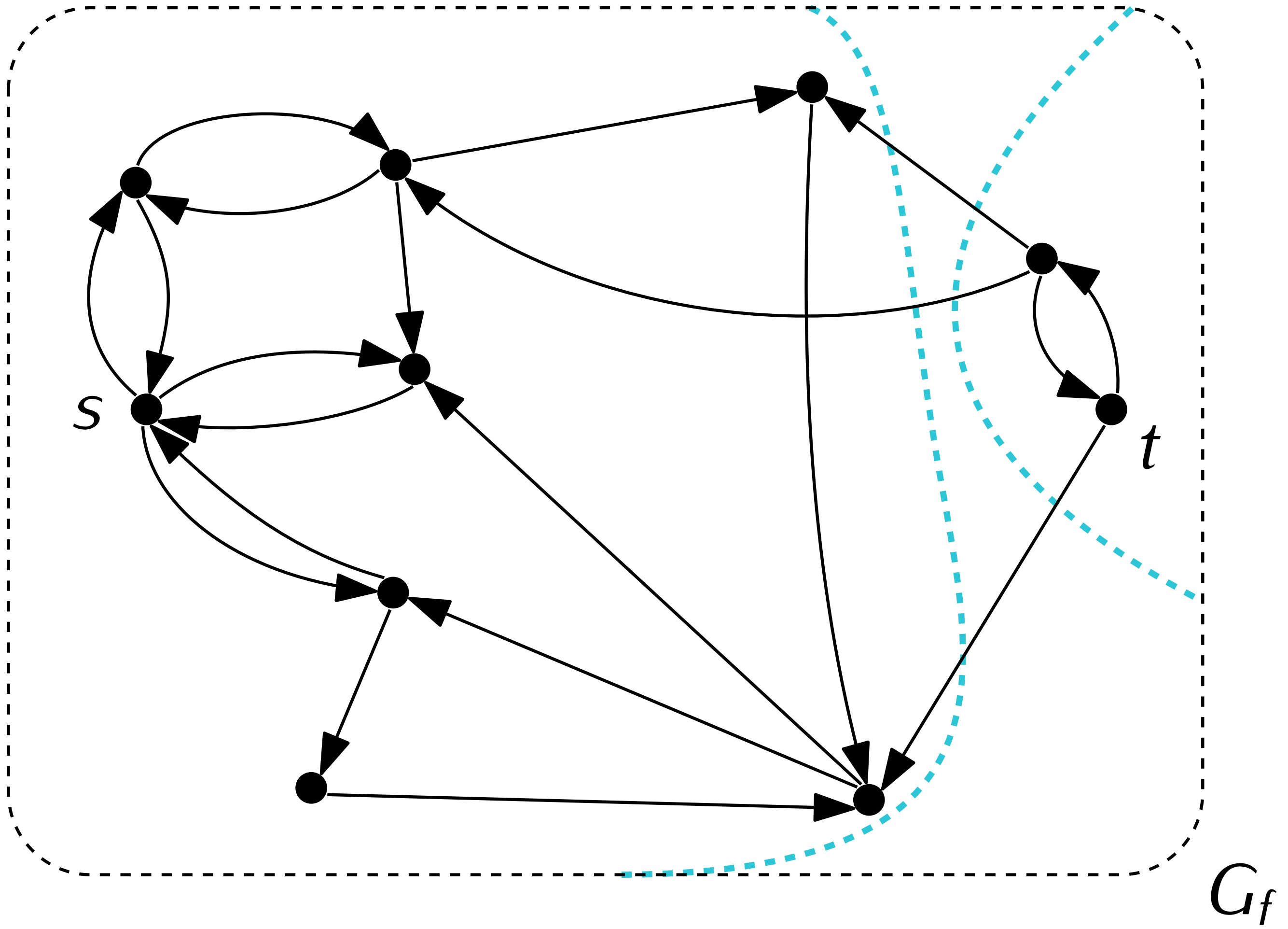
Lemma. All crossing edges from A to B are saturated, all crossing edges from B to A have 0 flow.





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We get: value of $f = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = \sum_{e \text{ out of } A} c(e) \geq \text{maxflow}(G)$

Our algorithm in fact proves an important result:

Theorem. Every flow network has a maximal flow f such that $f(e) \in \mathbb{N}$ for all $e \in E$.

Interesting applications of the algorithm for max-flow and the integrality theorem:

- Efficient algorithm for maximum bipartite matching
- Hall's perfect matching theorem
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Other algorithms are possible:

$$\text{maximize } \sum_{e \text{ out of } s} c(e)$$
$$\text{subject to } \left\{ \begin{array}{rcll} \sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) & = & 0 & \text{for all } v \in V \\ c(e) - f(e) & \geq & 0 & \text{for all } e \in E \\ f(e) & \geq & 0 & \text{for all } e \in E \end{array} \right.$$

Such linear programs can be solved in polynomial time: area of linear and convex optimization