

# Promises and Infinite-Domain Constraint Satisfaction

Antoine Mottet  
TU Hamburg

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Currently hiring a PhD student!

[antoine.mottet@tuhh.de](mailto:antoine.mottet@tuhh.de)

# Constraint Satisfaction Problems

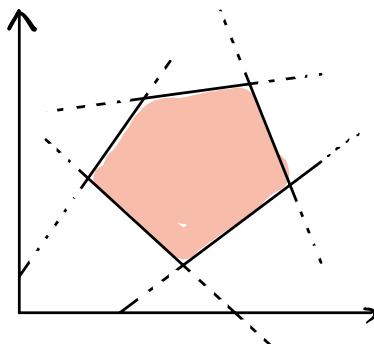
variables  $\rightsquigarrow$  domain satisfying some constraints

# Constraint Satisfaction Problems

Solve over  $\mathbb{Z}$ :

$$\begin{cases} 5x + y - z = 1 \\ x - y + z = 2 \end{cases}$$

Solve  $Ax \geq b$  over  $\mathbb{R}_{\geq 0}$ :



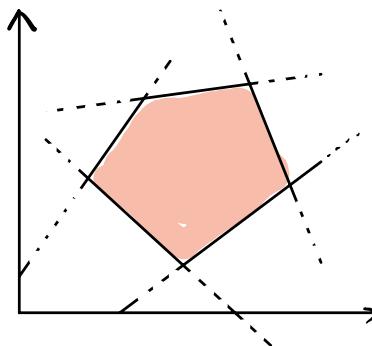
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Solve over {true, false}:

$$(p \vee q \vee \bar{r}) \wedge (\bar{r} \vee \bar{p}) \wedge (\bar{r} \vee \bar{q})$$

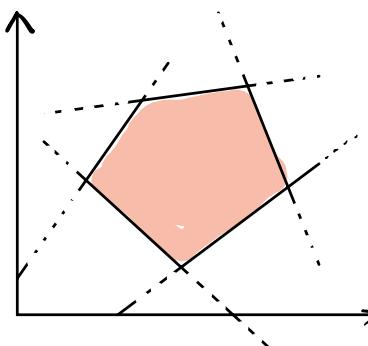
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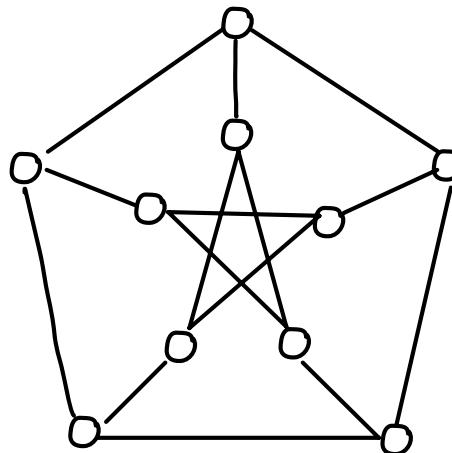
$$(p \vee q \vee \bar{r}) \wedge (\bar{r} \vee \bar{p}) \wedge (\bar{r} \vee \bar{q})$$

Solve over  $\{1, \dots, 9\}$ :

3			8	1		2
2		1		3	6	4
			2	4		
8	9				1	6
	6					5
7	2			4	9	
9		5	9			
	4		8	7	5	
6		1	7			3

variables  $\rightsquigarrow$  domain satisfying some constraints

"Solve" over {○, ○, ○}:



# Constraint Satisfaction Problems

Input:  $D \dots$  domain       $V \dots$  variables

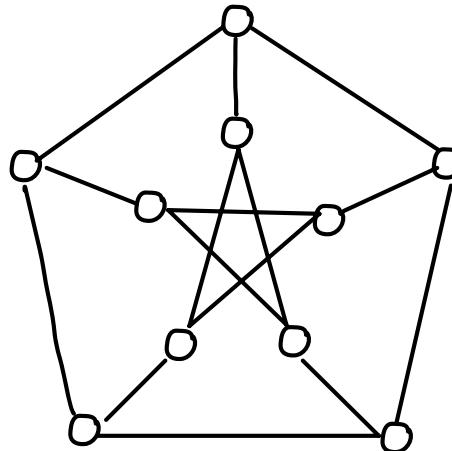
$C \dots$  constraints:  $(v_1, \dots, v_r) \rightarrow \{ \text{list of allowed assignments} \}$

Question:  $\exists h: V \rightarrow D$  satisfying all constraints?

Solve over  $\{1, \dots, 9\}$ :

3			8	1		2
2		1		3	6	4
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8	9				1	6
	6					5
7	2			4	9	
9		5	9			
	4		8	7	5	
6		1	7		3	

"Solve" over  $\{\textcolor{brown}{0}, \textcolor{blue}{0}, \textcolor{green}{0}\}$ :



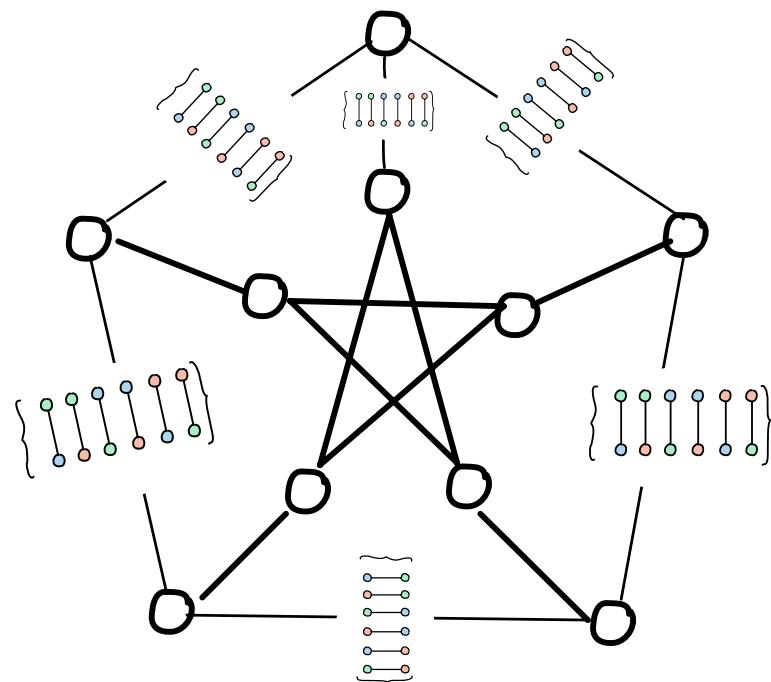
# Constraint Satisfaction Problems

Input:  $D \dots \text{domain}$        $V \dots \text{variables}$

$C \dots \text{constraints}: (v_1, \dots, v_r) \rightarrow \{\text{list of allowed assignments}\}$

Question:  $\exists h: V \rightarrow D$  satisfying all constraints?

$$D = \{ \textcolor{brown}{o}, \textcolor{blue}{o}, \textcolor{green}{o} \}$$



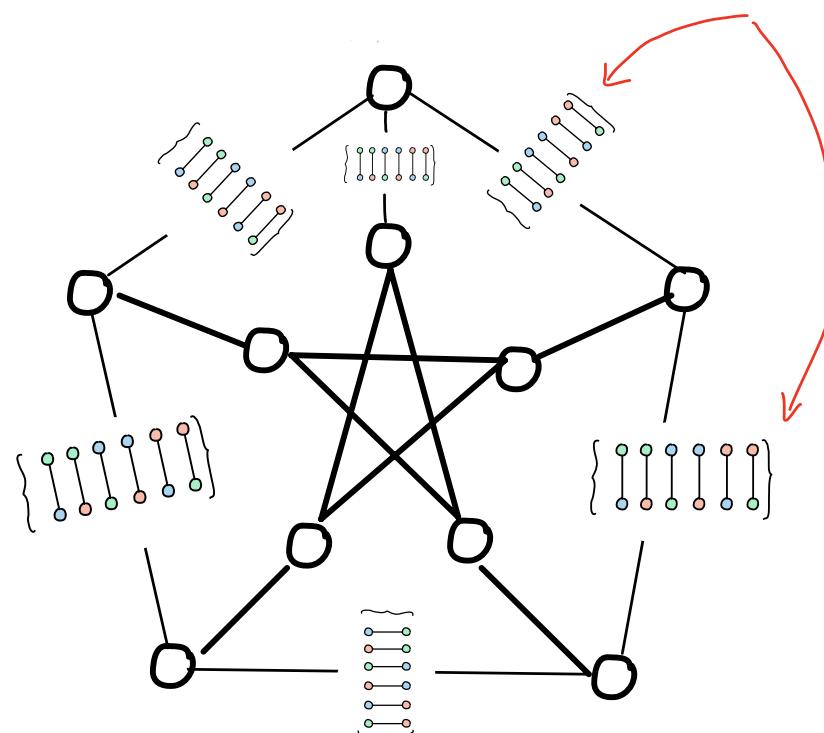
# Constraint Satisfaction Problems

Input of CSP( $D$ ):  $V \dots$  variables

$C \dots$  constraints:  $(v_1, \dots, v_r) \rightarrow E$

Question:  $\exists h: V \rightarrow D$  satisfying all constraints?

$$D = \{ \textcolor{brown}{\circ}, \textcolor{blue}{\circ}, \textcolor{green}{\circ} \}$$



all instances of 3-coloring  
use the same constraint relation  
 $E \subseteq D^2$

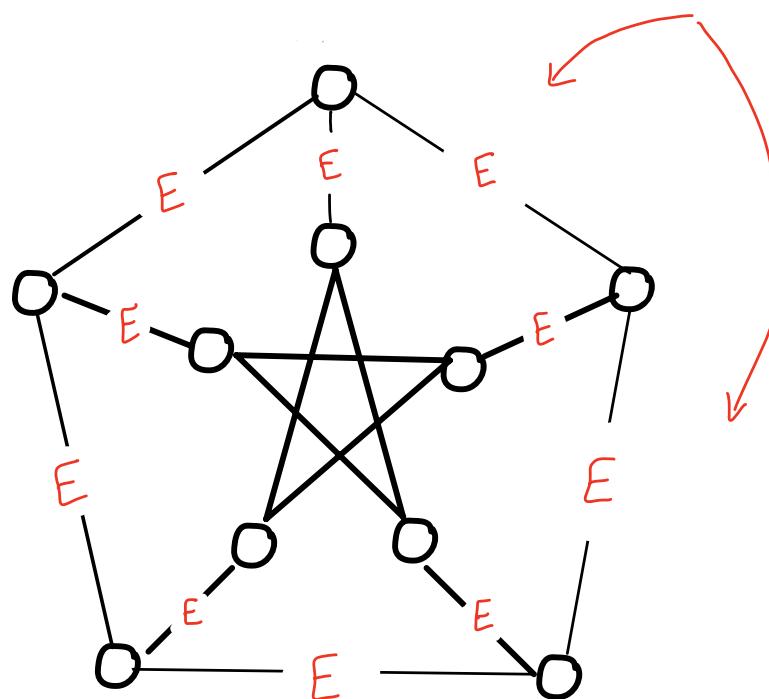
$$D = (D, E)$$

# Constraint Satisfaction Problems

Input of CSP( $D$ ):  $V \dots$  variables  
 $C \dots$  constraints:  $(v_1, \dots, v_r) \rightarrow E$  } itself a relational structure  $\times$

Question:  $\exists h: V \rightarrow D$  satisfying all constraints? }  $\exists$  homomorphism  $X \rightarrow D$ ?

$$D = \{ \textcolor{brown}{0}, \textcolor{blue}{0}, \textcolor{green}{0} \}$$



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# Constraint Satisfaction Problems

Input of  $CSP(D)$ :  $V \dots$  variables      }  
 C ... constraints:  $(v_1, \dots, v_r) \rightarrow E$       } itself a relational  
 structure  $\times$

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3-COL

$$(\{\bullet, \circ, \textcolor{green}{\circ}\}; \{\bullet\bullet\bullet, \bullet\circ\circ, \circ\circ\bullet\})$$

1 in 3-SAT

$$\left( \{\text{True, False}\}; \begin{pmatrix} \text{True} \\ \text{False} \end{pmatrix}, \begin{pmatrix} \text{False} \\ \text{True} \end{pmatrix}, \begin{pmatrix} \text{True} \\ \text{False} \end{pmatrix}, \begin{pmatrix} \text{False} \\ \text{True} \end{pmatrix} \right)$$

Lin-Eq ( $\mathbb{F}_2$ )

$$(\{0, 1\}; \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}, \{0\})$$

NOT-ALL-EQUAL-SAT

$$\left( \{\text{True, False}\}; \begin{pmatrix} \text{True} \\ \text{False} \end{pmatrix}, \begin{pmatrix} \text{False} \\ \text{True} \end{pmatrix}, \begin{pmatrix} \text{False} \\ \text{False} \end{pmatrix}, \begin{pmatrix} \text{True} \\ \text{True} \end{pmatrix}, \begin{pmatrix} \text{True} \\ \text{False} \end{pmatrix}, \begin{pmatrix} \text{False} \\ \text{True} \end{pmatrix} \right)$$

Horn-SAT

$$\left( \{\text{True, False}\}; \text{all except } \begin{pmatrix} \text{True} \\ \text{False} \end{pmatrix} \right)$$

Theorem: (Bulatov - Zhuk '17)

For every finite  $D$ ,  $CSP(D)$  is in P or NP-complete.

# Extensions

High-dimensional  
constraints

Infinite  
Domains

Approximations  
(qualitative/quantitative)

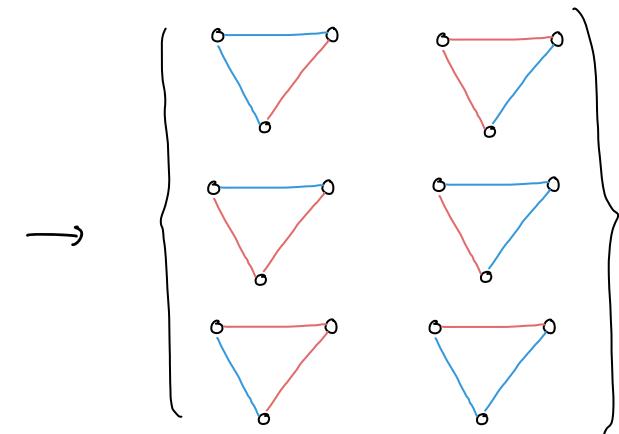
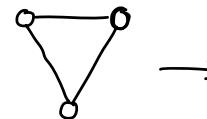
# Extensions

High-dimensional  
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Infinite  
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Find assignment edges  $\rightarrow \left\{ \begin{matrix} \text{o---o} \\ \text{o---o} \end{matrix} \right\}$  s.t.



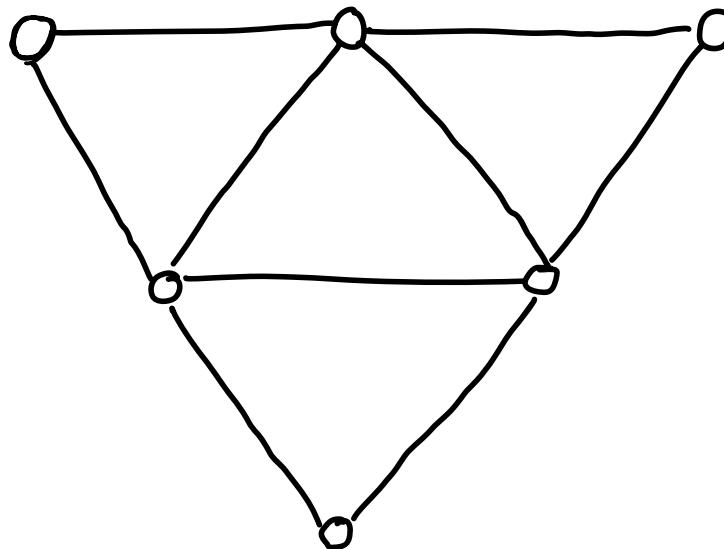
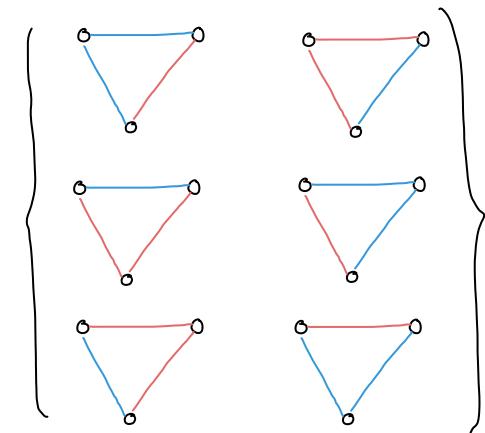
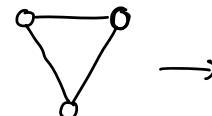
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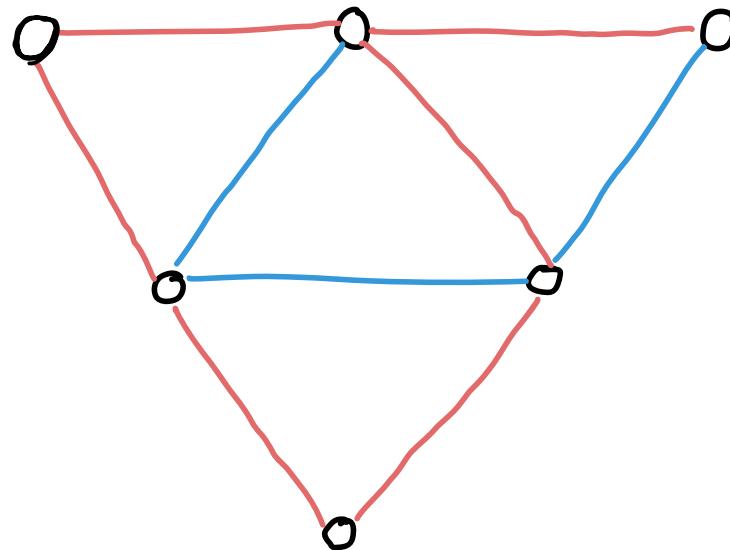
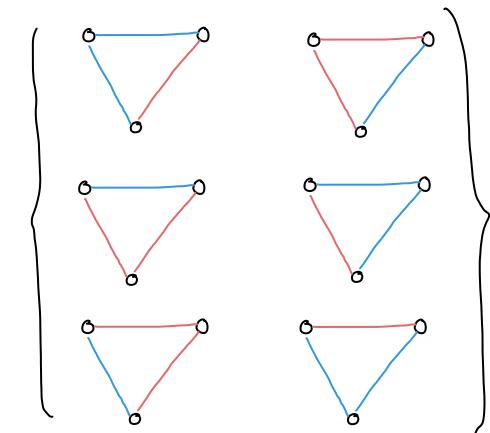
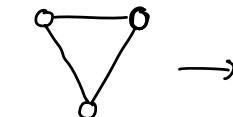
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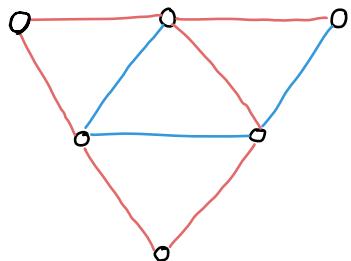
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Infinite  
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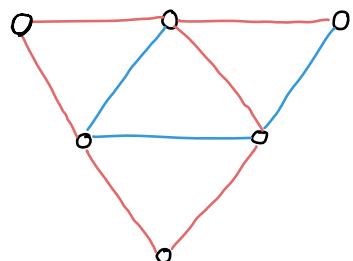
Domains and constraints are not part of the input  
~ they can be infinite!

Example:

$\exists h: V \rightarrow \mathbb{Q}$  satisfying some constraints  
 $(u, v, w) \rightarrow \{(a, b, c) \mid a < \max(b, c)\} ?$

# Extensions

High-dimensional  
constraints



Infinite  
Domains

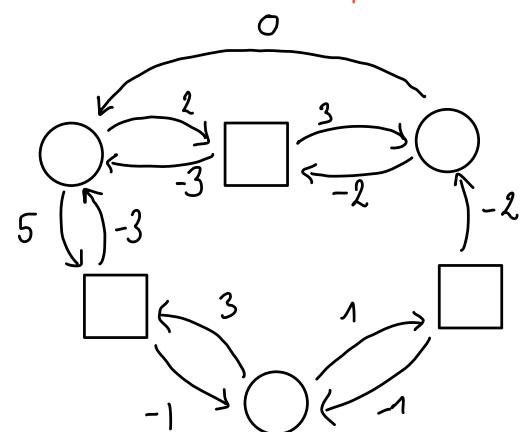
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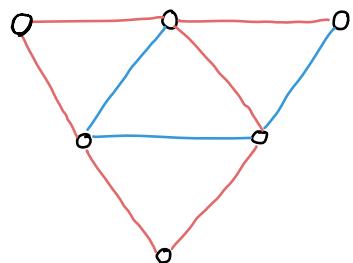
$\exists h: V \rightarrow \mathbb{Q}$  satisfying some constraints  
 $(u, v, w) \rightarrow \{(a, b, c) \mid a < \max(b, c)\} ?$

closely related to  
mean-payoff games, tropical linear programming

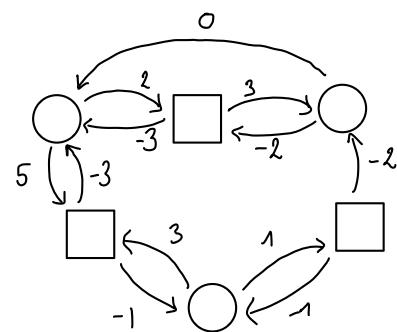


# Extensions

High-dimensional  
constraints



Infinite  
Domains

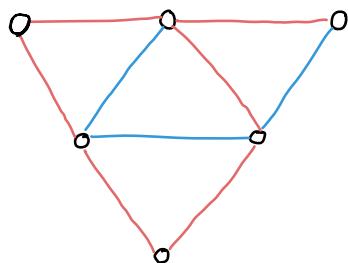


Approximations  
(qualitative/quantitative)

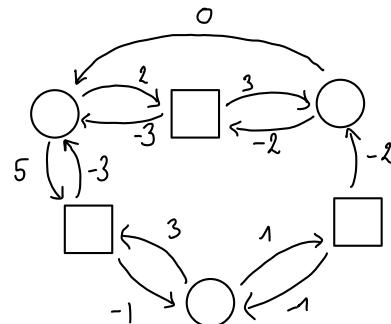
- satisfy as many constraints as possible
- satisfy all constraints in a weakened form

# Extensions

High-dimensional constraints



Infinite Domains

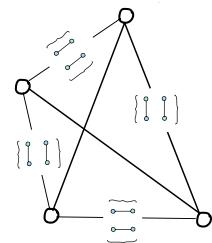


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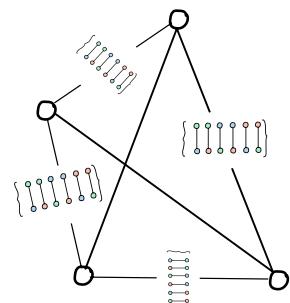
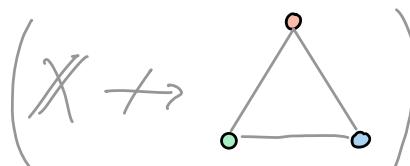
satisfy as many constraints as possible

satisfy all constraints in a weakened form

Yes-instances: strongly satisfiable

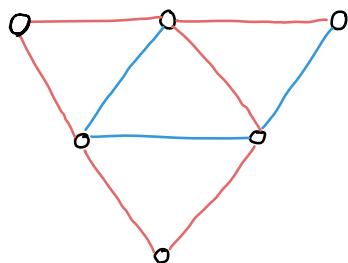


No-instances: not weakly satisfiable

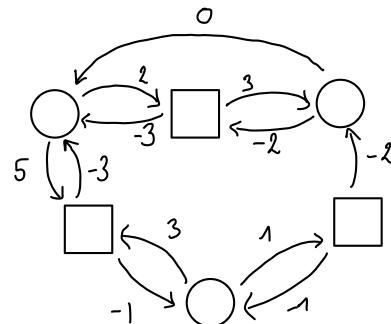


# Extensions

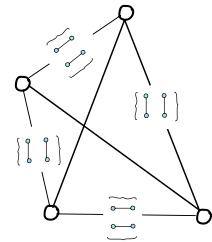
High-dimensional constraints



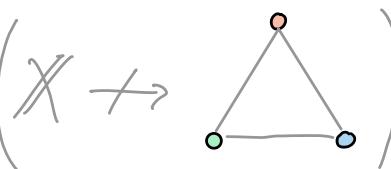
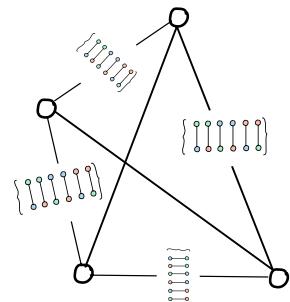
Infinite Domains



Yes-instances: strongly satisfiable



No-instances: not weakly satisfiable



Approximations  
(qualitative/quantitative)

satisfy as many constraints as possible

satisfy all constraints in a weakened form

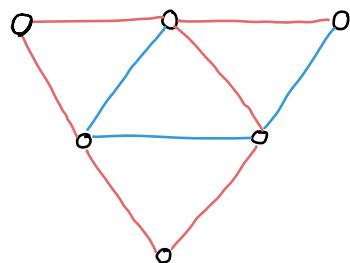
Denoted by

$\text{PCSP}(\quad, \quad)$

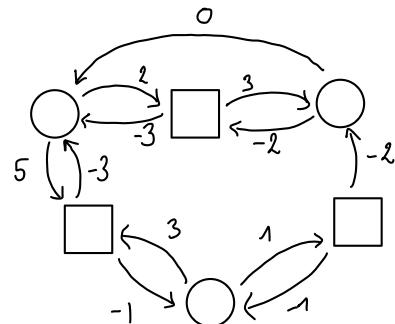
Promise

# Extensions

High-dimensional  
constraints



Infinite  
Domains

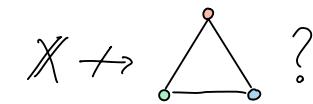
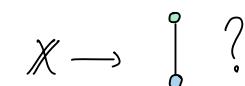


Very similar: in many cases, high-dimensional constraints are 1-dimensional constraints on an infinite set



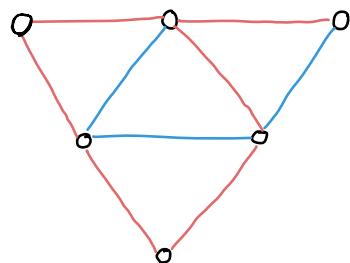
Fraïssé Theory of  
Generic structures

Approximations  
(qualitative/quantitative)

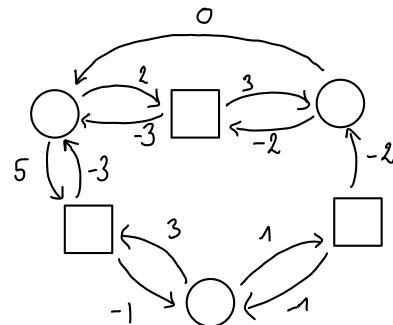


# Extensions

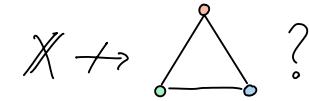
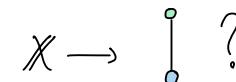
High-dimensional  
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Infinite  
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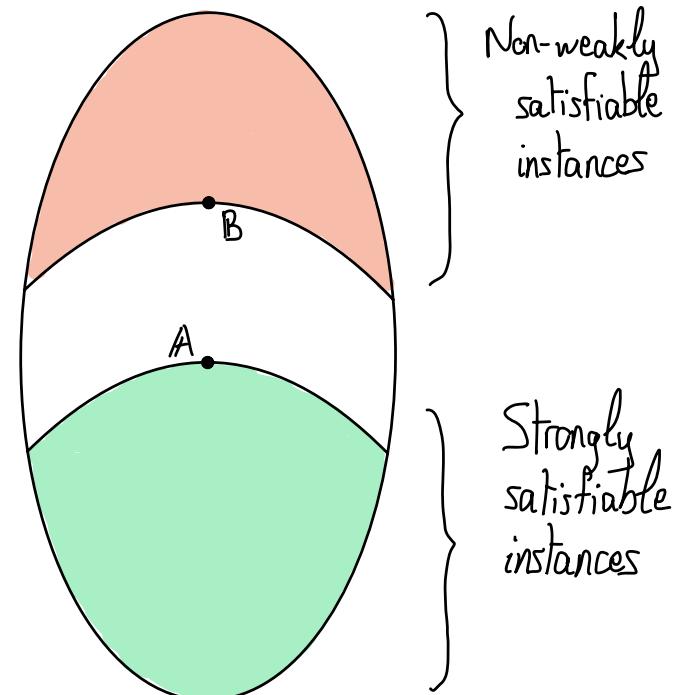
Connections?

Fraïssé Theory of  
Generic structures

# Logical Solvability of PCSPs

PCSP( $A, B$ ): Given  $X$ , decide if

- Yes:  $X$  is strongly satisfiable ( $X \rightarrow A$ )
- No:  $X$  is not weakly satisfiable ( $X \not\rightarrow B$ )



# Logical Solvability of PCSPs

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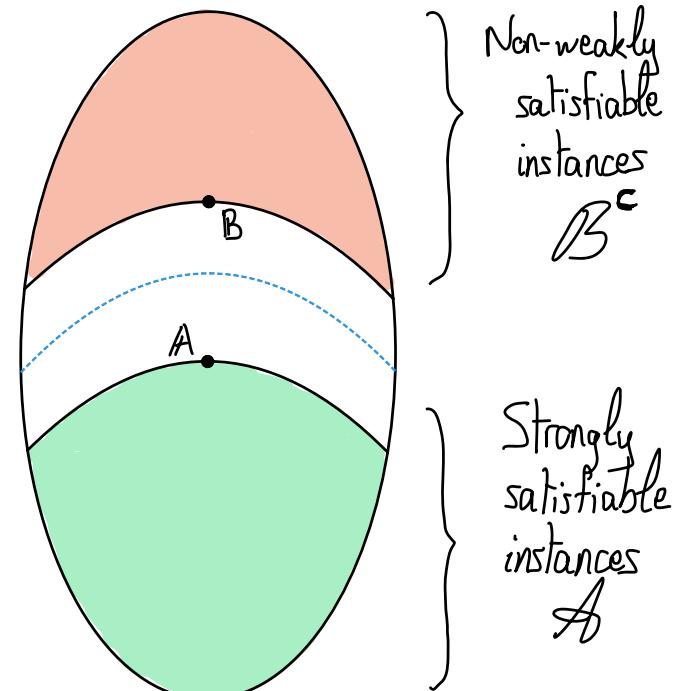
- Yes:  $X$  is strongly satisfiable ( $X \rightarrow A$ )
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An algorithm solving PCSP( $A, B$ ) realizes  
a separation between

$$\mathcal{A} = \{X \mid X \rightarrow A\}$$

and

$$\mathcal{B}^c = \{X \mid X \not\rightarrow B\}$$



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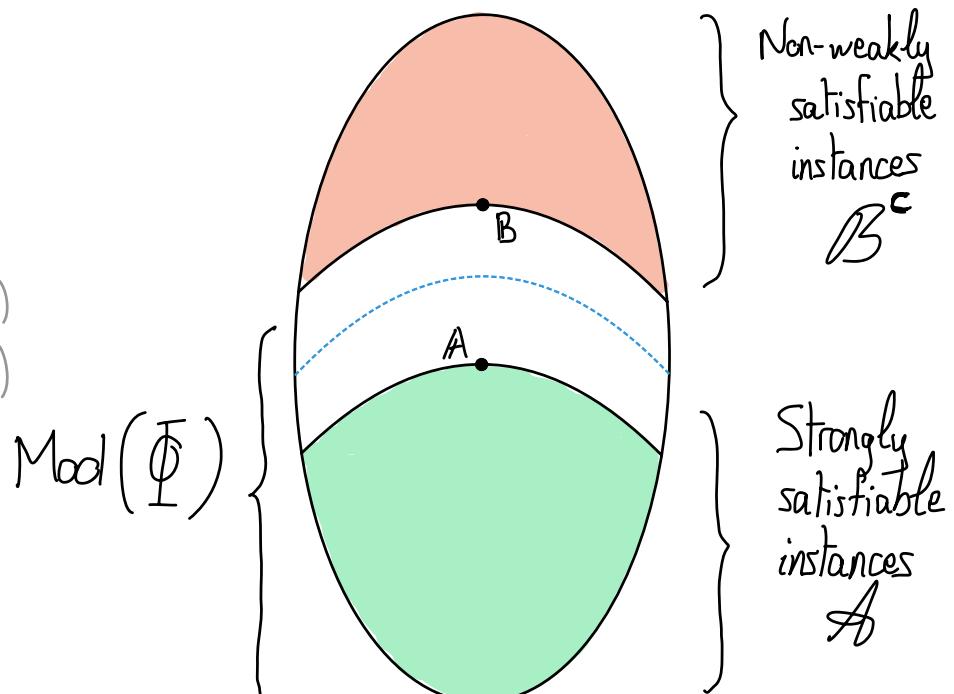
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Questions: For a logic  $\mathcal{L}$  and fixed problem  
PCSP( $A, B$ ):

- is there a formula  $\Phi \in \mathcal{L}$  s.t.

$$\mathcal{A} \subseteq \text{Mod}(\emptyset) \subseteq \mathcal{B} ?$$

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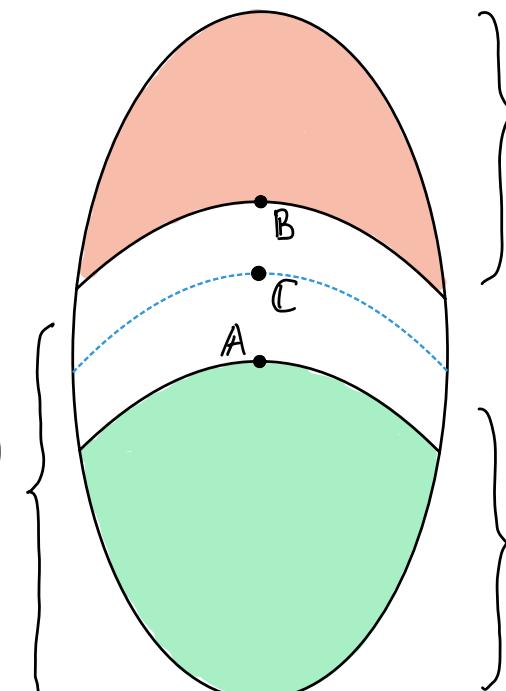
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$$\text{Mod}(\emptyset)$$



Questions: For a logic  $\mathcal{L}$  and fixed problem PCSP( $A, B$ ):

- is there a formula  $\Phi \in \mathcal{L}$  s.t.

$$A \subseteq \text{Mod}(\emptyset) \subseteq B ?$$

- is there a (finite) structure  $C$  s.t.

$$A \subseteq C \subseteq B \text{ and } \text{CSP}(C) \text{ definable in } \mathcal{L} ?$$

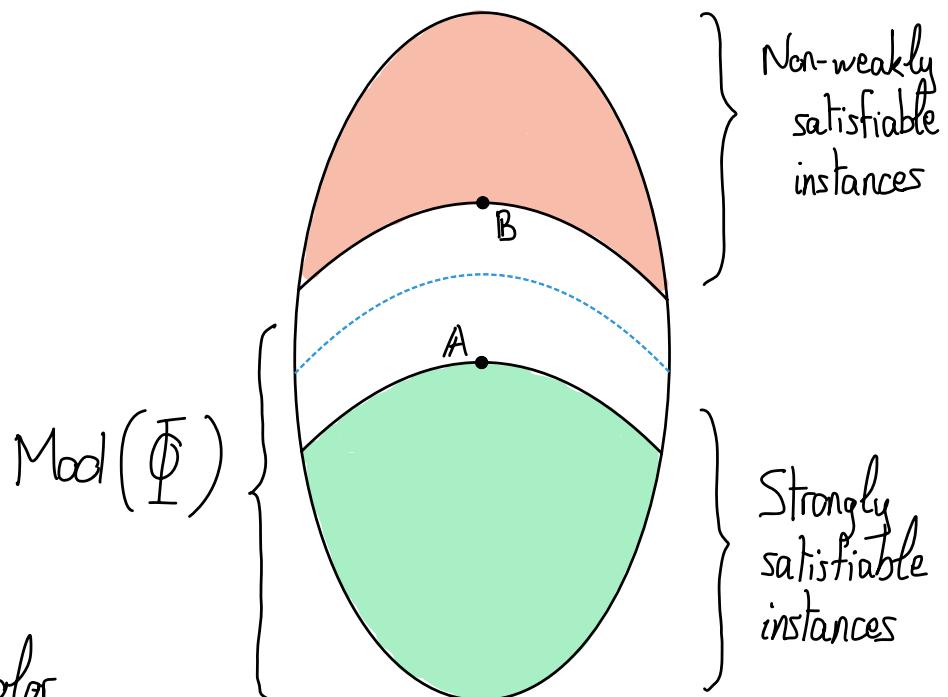
# Logical Solvability of PCSPs

PCSP(A, B): Given  $\mathbb{X}$ , decide if

- Yes:  $\mathbb{X}$  is strongly satisfiable ( $\mathbb{X} \rightarrow A$ )
- No:  $\mathbb{X}$  is not weakly satisfiable ( $\mathbb{X} \not\rightarrow B$ )

Example:  $A = \textcircled{o} \xrightarrow{\text{red}} \textcircled{o} \xrightarrow{\text{blue}} \textcircled{o} \xrightarrow{\text{red}} \textcircled{o} \xrightarrow{\text{blue}} \textcircled{o}$   
 $B = \textcircled{o} \xrightarrow{\text{red}} \textcircled{o} \xrightarrow{\text{blue}} \textcircled{o} \xleftarrow{\text{red}} \textcircled{o}$

- $\Phi$ :
- no two consecutive edges of the same color
  - no edges of different colors arriving at a vertex
  - no edges of different colors leaving a vertex

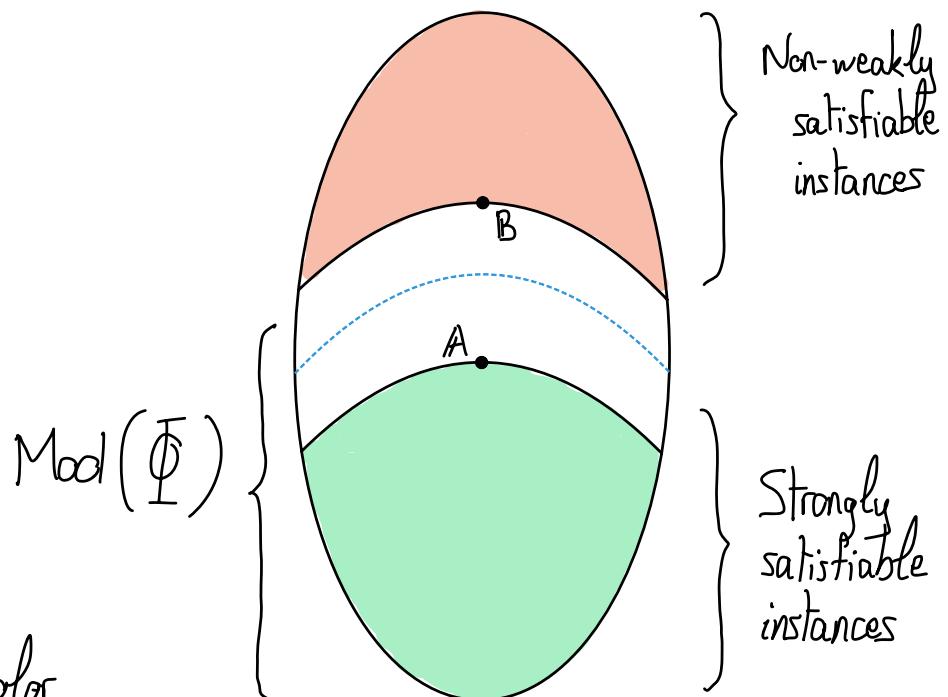


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Example:  $A = \circ \xrightarrow{\text{red}} \circ \xrightarrow{\text{blue}} \circ \xrightarrow{\text{red}} \circ \xrightarrow{\text{blue}} \circ$

 $B = \circ \xrightarrow{\text{red}} \circ \xrightarrow{\text{red}} \circ \xleftarrow{\text{blue}} \circ$ 


- $\bar{\Phi}$ :
- no two consecutive edges of the same color
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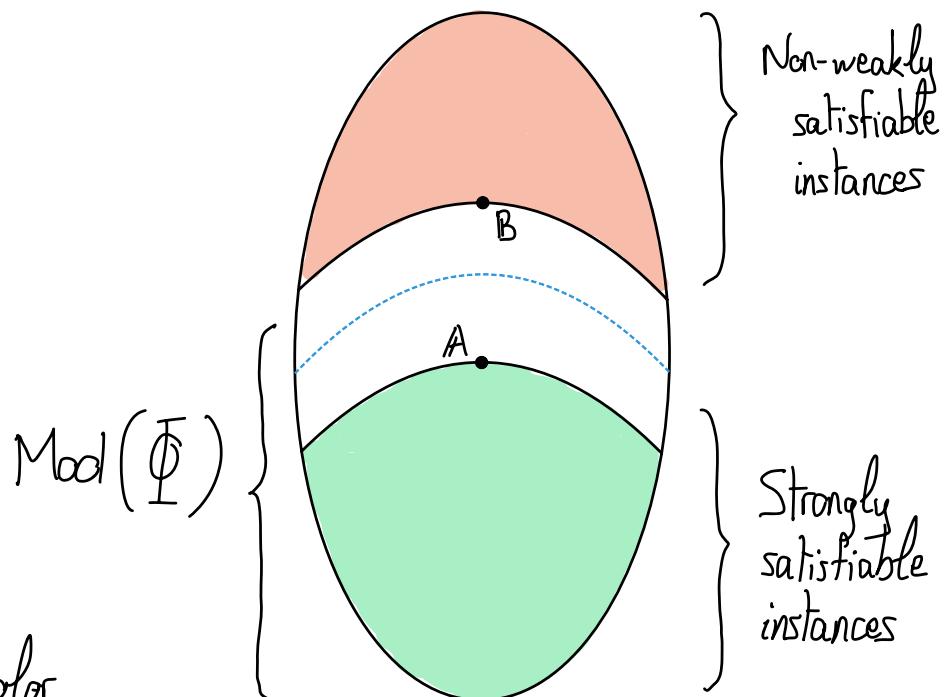
- If  $\mathbb{X}$  strongly satisfiable ( $\mathbb{X} \rightarrow A$ ) then  $\mathbb{X} \models \bar{\Phi}$
- If  $\mathbb{X} \models \bar{\Phi}$  then  $\mathbb{X}$  weakly satisfiable ( $\mathbb{X} \not\rightarrow B$ )

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- $\bar{\Phi}$ :
- no two consecutive edges of the same color
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- If  $X$  strongly satisfiable ( $X \rightarrow A$ ) then  $X \models \bar{\Phi}$
- If  $X \models \bar{\Phi}$  then  $X$  weakly satisfiable ( $X \not\rightarrow B$ )

(Exercise: no first-order formula characterizes strongly satisfiable instances )

" \_\_\_\_\_ " \_\_\_\_\_ weakly \_\_\_\_\_ - \_\_\_\_\_ )

## The case of first-order logic

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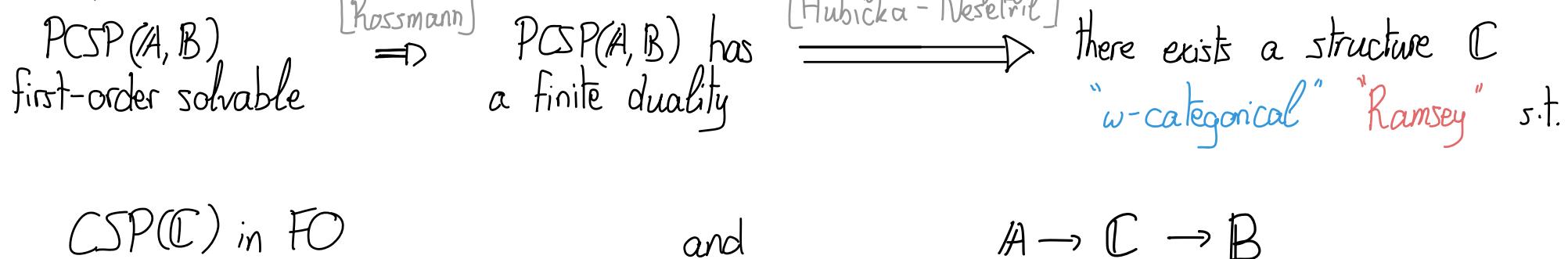
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$\xrightarrow{\text{[Larose-Loten-Tardieu]}}$

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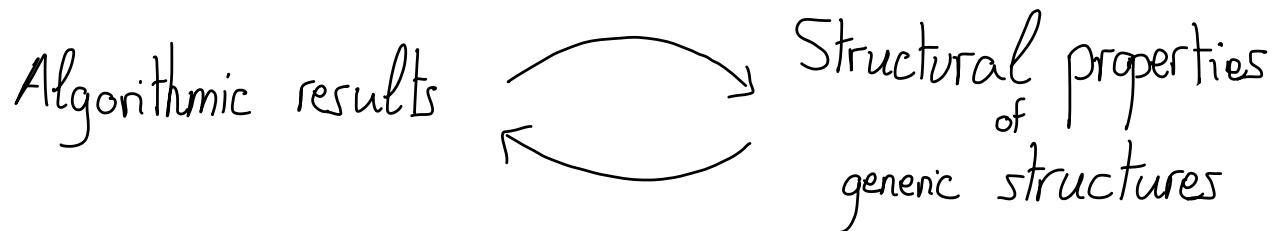
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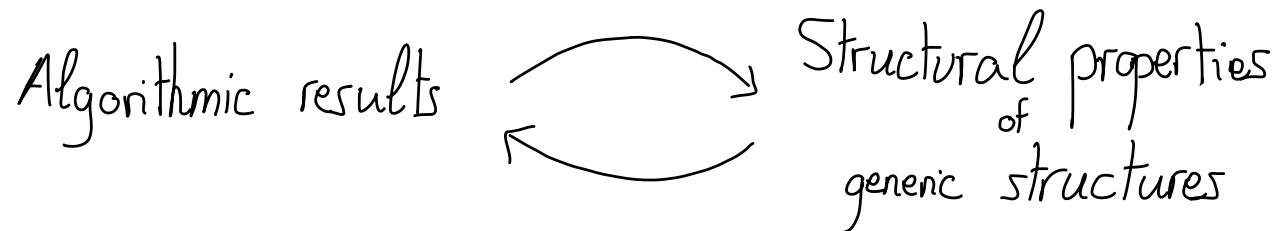
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Open: · Consistency, Sherali-Adams, and Lasserre hierarchies  
· Fix point theorems

- Fixpoint logics

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- Decidability of "solvable by ⟨insert algorithm here⟩?"

## Take away

- Connections between finite promise CSPs and infinite classical CSPs
- Meta result: partial recipe for characterizing the power of algorithmic techniques to solve PCSPs.
- Extending this partial recipe could solve important problems in promise constraint satisfaction
- Hiring a PhD student!