MMSNP: An algebraic proof of the dichotomy

Manuel Bodirsky, Florent Madelaine, **Antoine Mottet**

September 25, 2018

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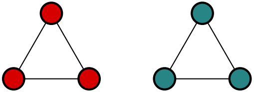

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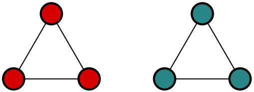

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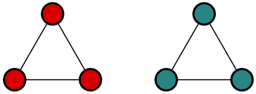

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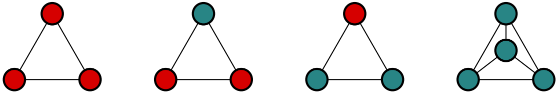

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MMSNP and FPP are computationally equivalent.

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Finite-domain CSPs have a complexity dichotomy.

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Let $\mathcal{A}$ be ω-categorical and such that $\mathrm{CSP}(\mathcal{A})$ is in MMSNP. Then one of the following holds:

- *there is a uniformly continuous clonoid homomorphism $\mathrm{Pol}(\mathcal{A}) \to \mathscr{P}$, and $\mathrm{CSP}(\mathcal{A})$ is NP-complete,*
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In particular, this confirms the infinite-domain conjecture for CSPs in MMSNP.

Outline

Definition

$\mathcal{B} = (B; E)$ a graph.
CSP($\mathcal{B}$) is the problem:

- **Input:** a finite graph $\mathcal{A}$,
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In general, the forbidden patterns problem (FPP) for $\mathcal{F}$ is not a CSP, but a finite union of CSPs.

Proposition

Every FPP reduces in polynomial-time to a finite number of FPP of connected structures.

MMSNP is essentially a subset of CSP

Theorem (Cherlin-Shelah-Shi, '99)

For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists an ω-categorical partially coloured graph $\mathcal{B}^*$ such that $\mathcal{A}^* \to \mathcal{B}^*$ iff $\mathcal{A}^*$ avoids $\mathcal{F}$.

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Construct a new $\mathcal{B}$ by:

- deleting the uncoloured elements in $\mathcal{B}^*$,
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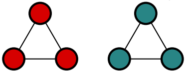
Proposition (Bodirsky-Dalmau, '06)

$\mathrm{CSP}(\mathcal{B}) = \mathrm{FPP}(\mathcal{F})$.

Moreover, $\mathcal{B}$ belongs to the class of reducts of finitely bounded homogeneous structures.

Example

Obstructions $\mathcal{F}$:



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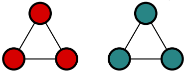

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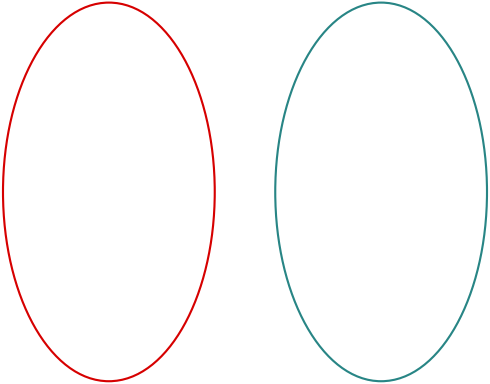

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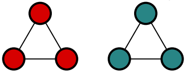

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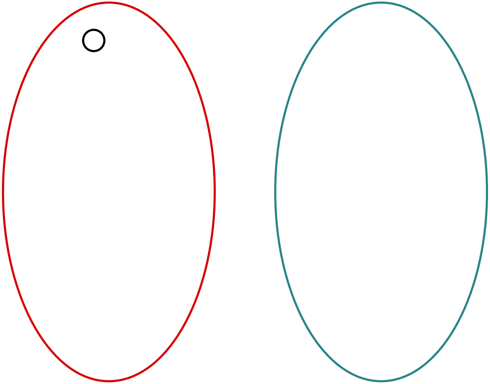

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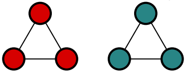

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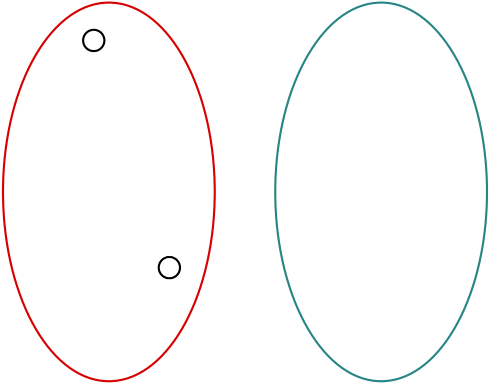

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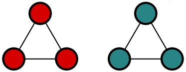

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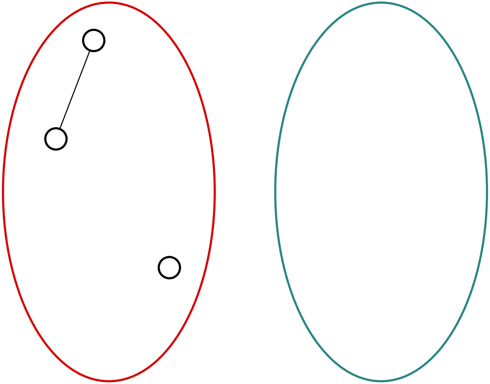

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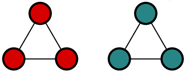

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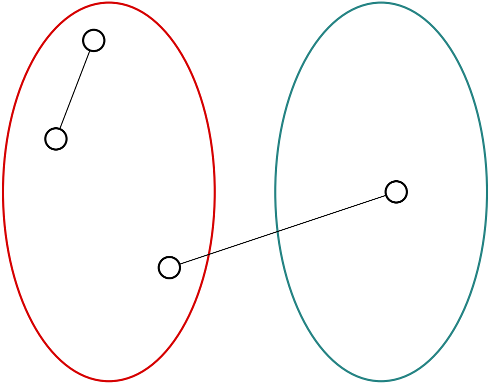

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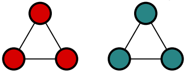

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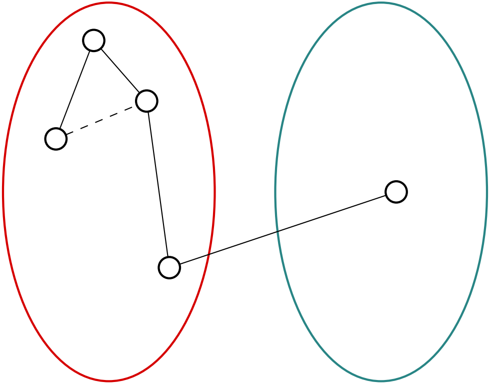

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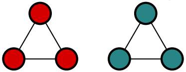

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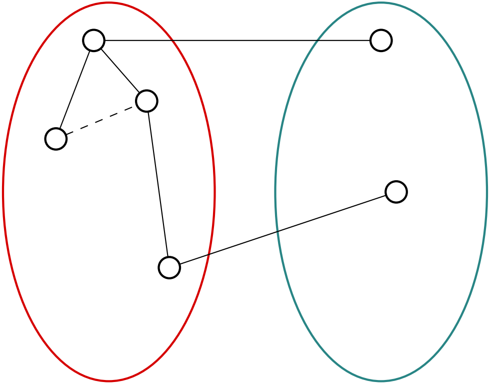

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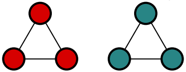

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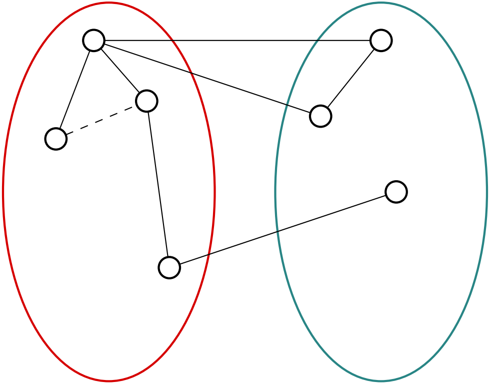

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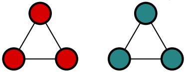

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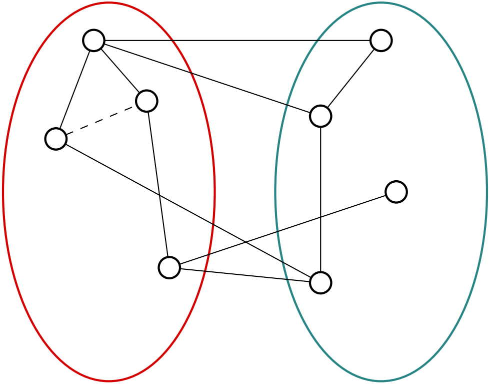

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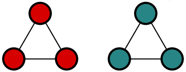

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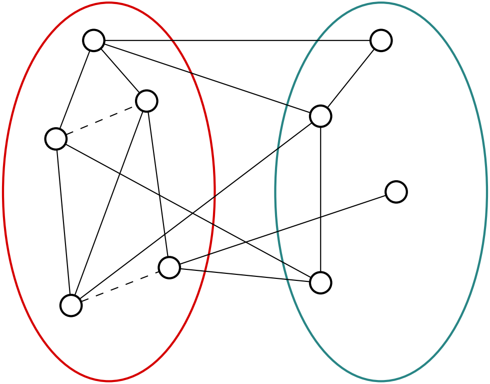

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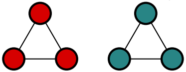

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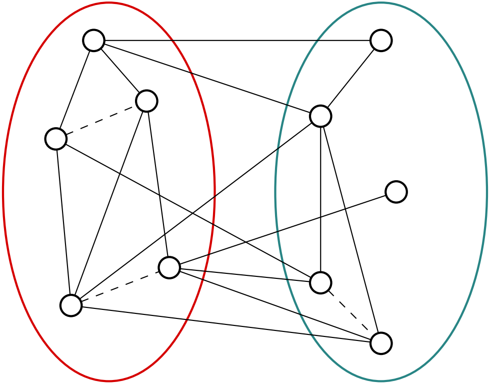

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Theorem (Bodirsky-Madelaine-M, '18)

Let $\mathcal{B}$ be ω-categorical such that $\mathrm{CSP}(\mathcal{B})$ *is in MMSNP. If there is no uniformly continuous clonoid homomorphism* $\mathrm{Pol}(\mathcal{B}) \to \mathscr{P}$, *then* $\mathrm{CSP}(\mathcal{B})$ *is in P.*

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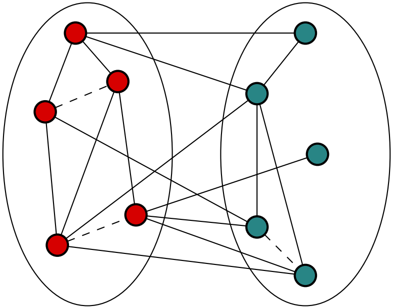
Rephrased: do CSP($\mathcal{B}$, ●, ●) and CSP($\mathcal{B}$) have same complexity?



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Good news: we can choose the MMSNP structure $\mathcal{B}$ so that $(\mathcal{B}, \neq)$ is an ω-categorical model-complete core.

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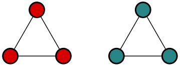
- Obstructions $\mathcal{F}$: 
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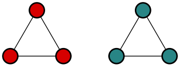
- Obstructions $\mathcal{F}$: 
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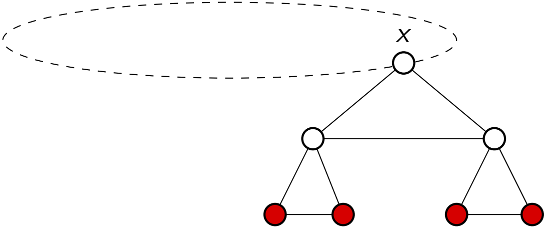

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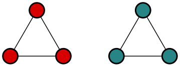
- Obstructions $\mathcal{F}$: 
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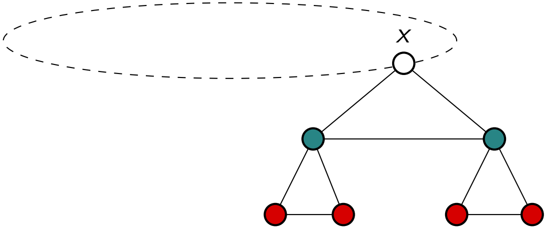

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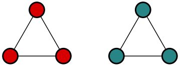
- Obstructions $\mathcal{F}$: 
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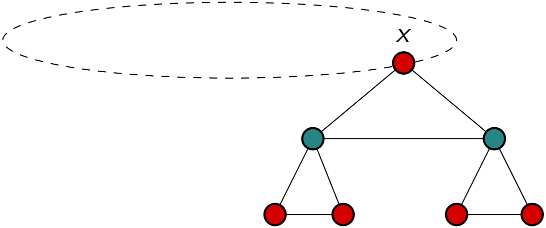

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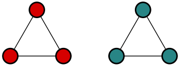
- Obstructions $\mathcal{F}$: 
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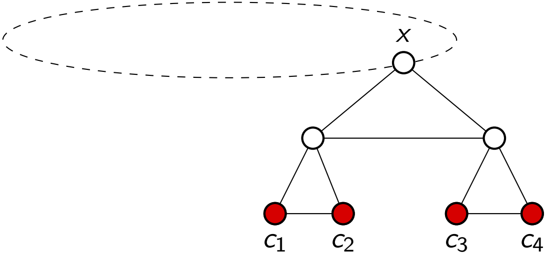

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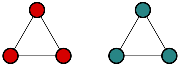
- Obstructions $\mathcal{F}$: 
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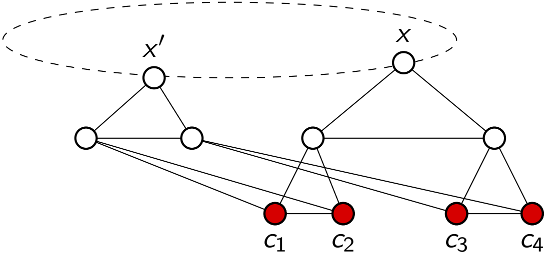

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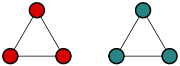
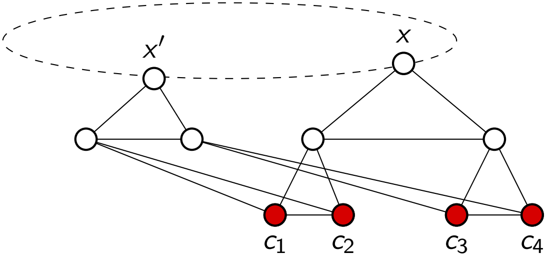
- Obstructions $\mathcal{F}$: 
- x precoloured in the instance of CSP($\mathcal{B}$, ●, ●):



Proposition

The input precoloured graph is colourable iff the graph obtained by adding the gadgets is colourable.

Outline

Definition

$f\colon B^k \to B$, a group $\mathcal{G}$ acting on B. f is canonical (wrt $\mathcal{G}$) if for every finite subset $S \subseteq B$ of B and $\alpha_1, \ldots, \alpha_k \in \mathcal{G}$, there exists $\beta \in \mathcal{G}$ such that $\beta \circ f|_S = f \circ (\alpha_1, \ldots, \alpha_k)|_S$.

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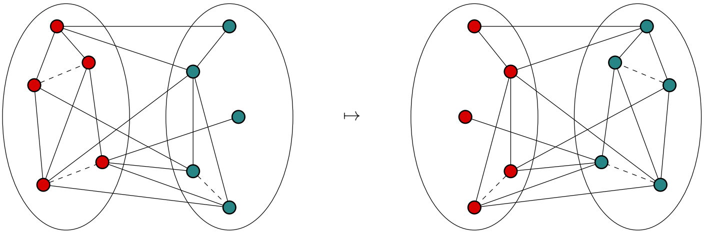

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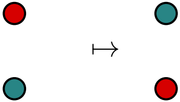

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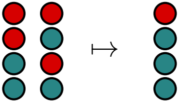

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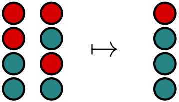

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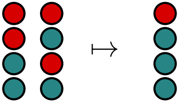

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Let $\mathcal{B}$ be in the class (reduct of...). If $\mathrm{Pol}(\mathcal{B})$ *contains a pseudo-Siggers operation modulo* $\overline{\mathrm{Aut}(\mathcal{B})}$ *that is canonical with respect to* $\mathrm{Aut}(\mathcal{B})$*, then* $\mathrm{CSP}(\mathcal{B})$ *is in P.*

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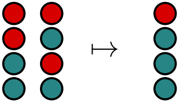

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$\Rightarrow$ If there is no clone homomorphism $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathscr{P}$, then $\mathrm{CSP}(\mathcal{B})$ is in P.

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4. Define a clonoid homomorphism $\mathrm{Pol}(\mathcal{B}) \to \mathscr{P}$ by canonizing and composing with ξ.

- σ: set of colour symbols.
- A trivial subfactor of $\mathscr{C}$ is a partition $S \uplus T \subseteq \sigma$ such that $\mathscr{C}/\sim$ is isomorphic to $\mathscr{P}$.

Proposition

S, T trivial subfactor of $\mathscr{C}$. $\exists(B, E)$ undirected graph s.t.:

- *(B, E) contains an edge from S to T but does not contain pseudo-loops;*
- *the connected components of N are included in S, included in T, or bipartite with the bipartition induced by S and T;*
- *E is preserved by $\mathrm{Pol}(\mathcal{B})$.*

Getting canonical functions

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Given $f \in \mathrm{Pol}(\mathcal{B})$, and $\xi \colon \mathscr{C} \to \mathscr{P}$ given by subfactor $\{S, T\}$, define $\phi(f) := \xi(g)$ where g is an arbitrary function in $\mathscr{C} \cap \overline{\mathrm{Aut}(\mathcal{B}, <) f \, \mathrm{Aut}(\mathcal{B}, <)}$.

Proposition

$\phi \colon \mathrm{Pol}(\mathcal{B}) \to \mathscr{P}$ is a well-defined uniformly continuous height 1 homomorphism.

Summing up:

Theorem

Let $\mathcal{B}$ be a MMSNP structure.
Then either the following equivalent statements hold:

1. *there is no uniformly continuous height 1 homomorphism* $\mathrm{Pol}(\mathcal{B}) \to \mathscr{P}$,

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Items 3. and 4. can be checked effectively.