MMSNP: An algebraic proof of the dichotomy

Manuel Bodirsky, **Antoine Mottet**

October 12, 2018

Two problems

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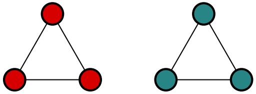

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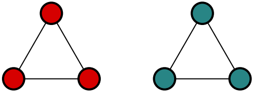

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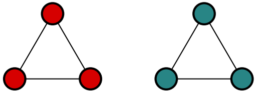

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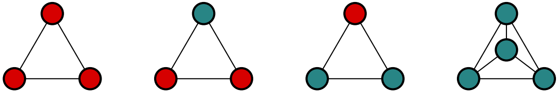

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In general, for some fixed set $\mathcal{F}$ of vertex-coloured graphs, the problem FPP($\mathcal{F}$) is:

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(**S**trict **NP**)

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MMSNP and CSP

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MMSNP has a complexity dichotomy if and only if finite-domain CSPs have a complexity dichotomy.

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Proof.

- finite CSP $\subseteq$ MMSNP,
- randomized reduction (Feder-Vardi) from MMSNP to CSP,
- derandomization by Kun (expander structures).

□

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Let $\mathcal{A}$ be ω-categorical and such that $\mathrm{CSP}(\mathcal{A})$ *is in MMSNP. Then one of the following holds:*

- *there is a uniformly continuous clonoid homomorphism* $\mathrm{Pol}(\mathcal{A}) \to \mathscr{P}$*, and* $\mathrm{CSP}(\mathcal{A})$ *is NP-complete,*
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In particular, this confirms the infinite-domain conjecture for CSPs in MMSNP.

Outline

Definition

$\mathcal{B} = (B; E)$ a graph.
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In general, the forbidden patterns problem (FPP) for $\mathcal{F}$ is not a CSP, but a finite union of CSPs.

Proposition

Every FPP reduces in polynomial-time to a finite number of FPP of connected structures.

MMSNP is essentially a subset of CSP

Theorem (Cherlin-Shelah-Shi, '99)

For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists an ω-categorical partially coloured graph $\mathcal{B}^*$ such that $\mathcal{A}^* \to \mathcal{B}^*$ iff $\mathcal{A}^*$ avoids $\mathcal{F}$.

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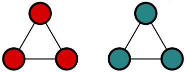
Proposition (Bodirsky-Dalmau, '06)

$\mathrm{CSP}(\mathcal{B}) = \mathrm{FPP}(\mathcal{F})$.

Moreover, $\mathcal{B}$ belongs to the class of reducts of finitely bounded homogeneous structures.

Example

Obstructions $\mathcal{F}$:



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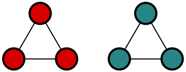

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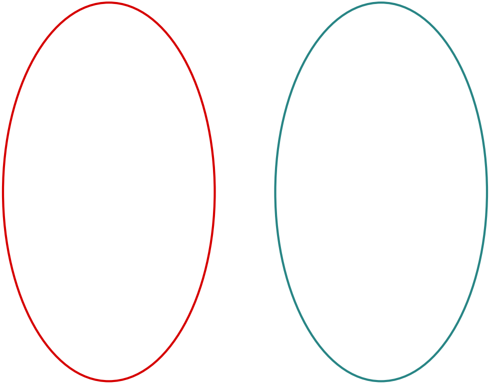

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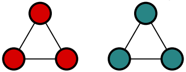

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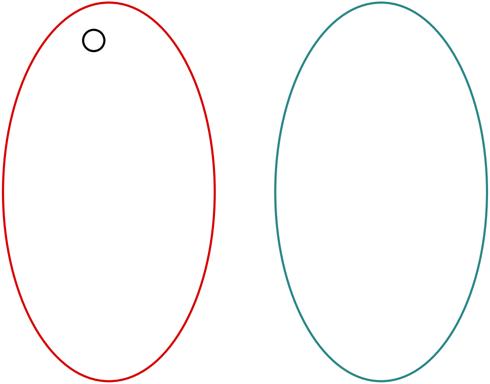

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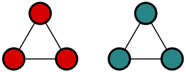

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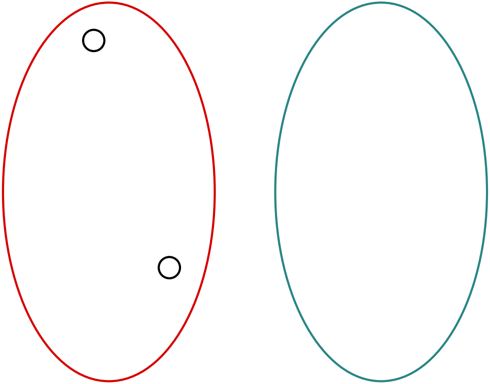

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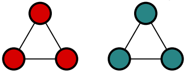

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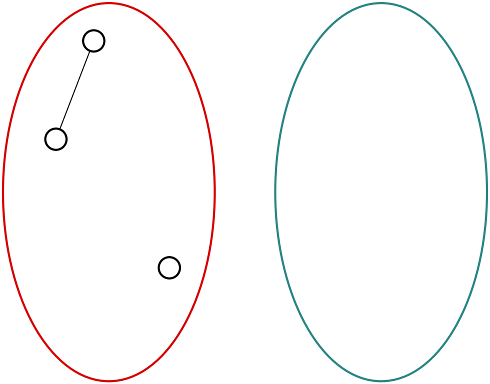

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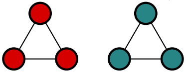

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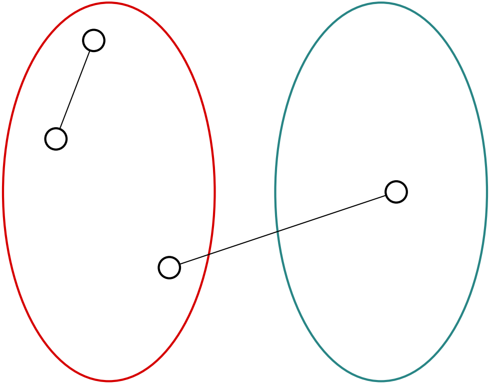

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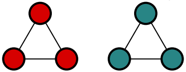

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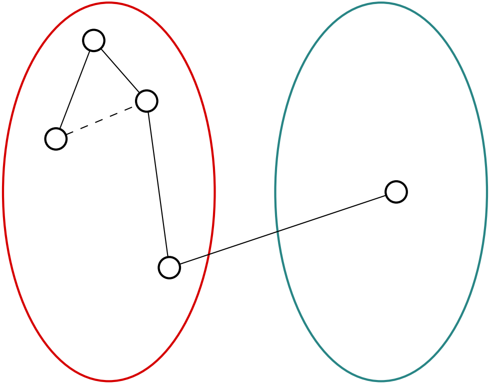

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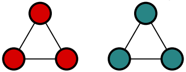

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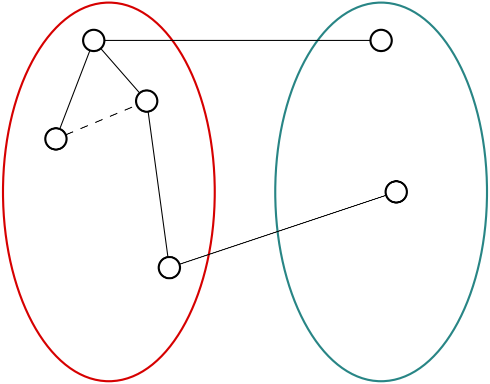

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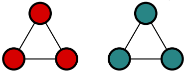

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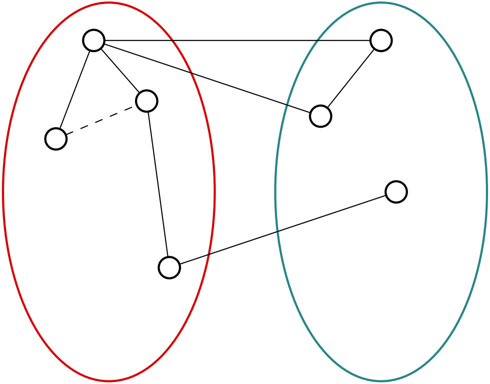

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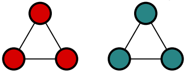

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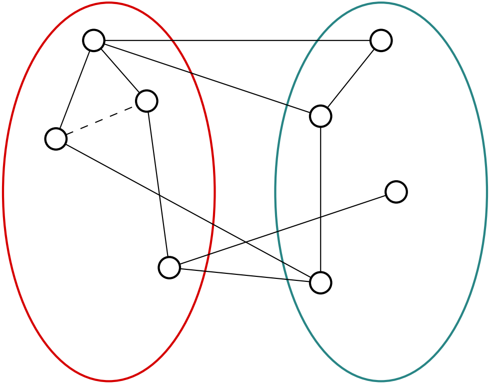

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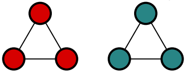

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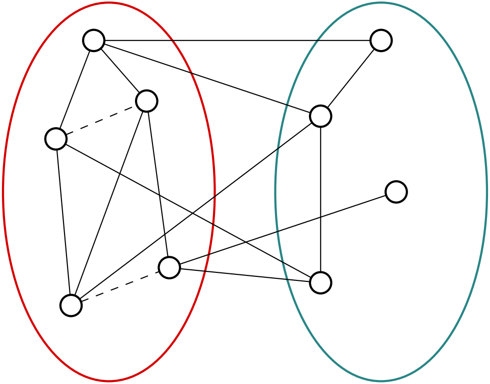

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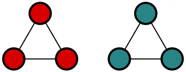

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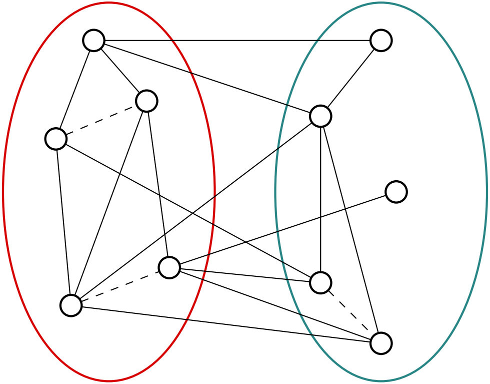

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Interesting?

- statement and its consequences: ★★☆☆☆
- proofs: ★★★★★

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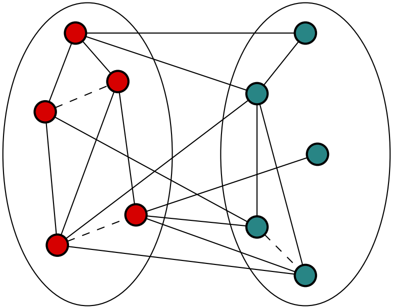
Rephrased: do CSP($\mathcal{B}$, ●, ●) and CSP($\mathcal{B}$) have same complexity?



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Good news: we can choose the MMSNP structure $\mathcal{B}$ so that $(\mathcal{B}, \neq)$ is an ω-categorical model-complete core.

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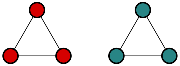
- Obstructions $\mathcal{F}$: 
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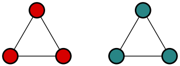
- Obstructions $\mathcal{F}$: 
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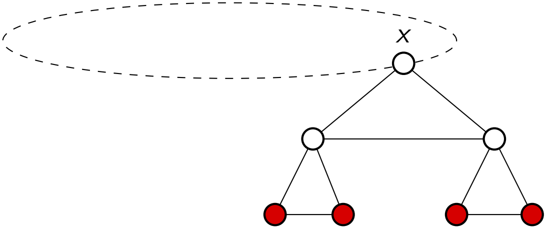

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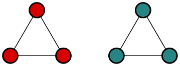
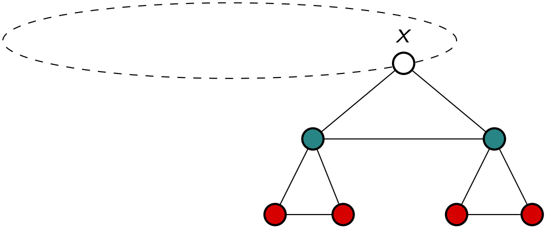
- Obstructions $\mathcal{F}$: 
- x precoloured in the instance of CSP($\mathcal{B}$, ●, ●):



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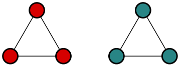
- Obstructions $\mathcal{F}$: 
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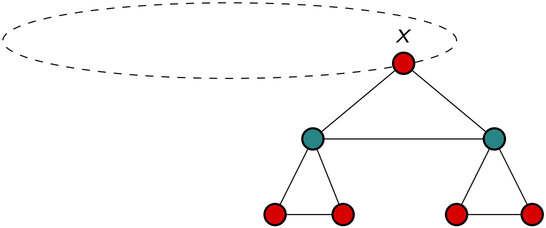

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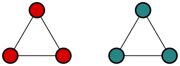
- Obstructions $\mathcal{F}$: 
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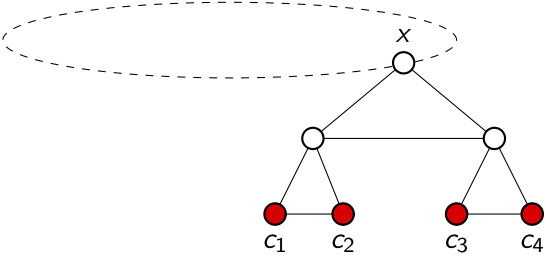

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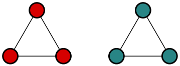
- Obstructions $\mathcal{F}$: 
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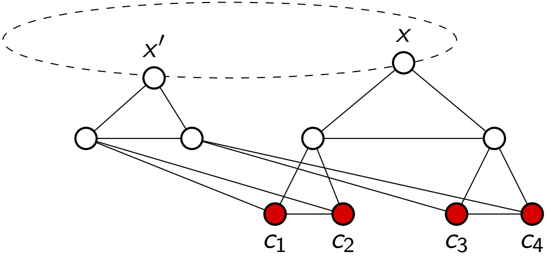

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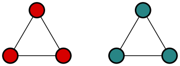
- Obstructions $\mathcal{F}$: 
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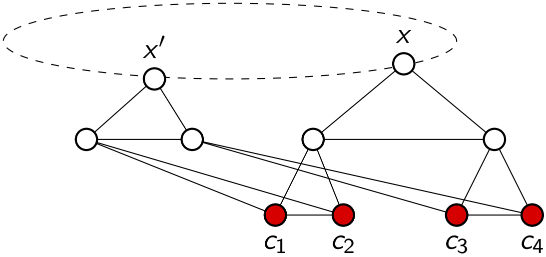

Proposition

There is a unif. cont. clonoid homomorphism $\mathrm{Pol}(\mathcal{B}) \to \mathscr{P}$ *iff there is one* $\mathrm{Pol}(\mathcal{B}, ●, ●) \to \mathscr{P}$.

Outline

Definition

$f\colon B^k \to B$, a group $\mathcal{G}$ acting on B. f is canonical (wrt $\mathcal{G}$) if for every finite subset $S \subseteq B$ of B and $\alpha_1, \ldots, \alpha_k \in \mathcal{G}$, there exists $\beta \in \mathcal{G}$ such that $\beta \circ f|_S = f \circ (\alpha_1, \ldots, \alpha_k)|_S$.

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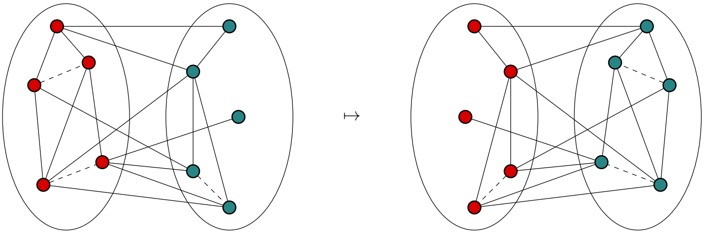

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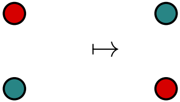

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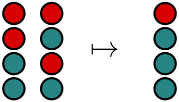

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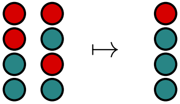

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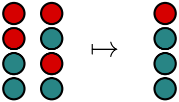

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Let $\mathcal{B}$ be in the class (reduct of...). If $\mathrm{Pol}(\mathcal{B})$ *contains a pseudo-Siggers operation modulo* $\overline{\mathrm{Aut}(\mathcal{B})}$ *that is canonical with respect to* $\mathrm{Aut}(\mathcal{B})$*, then* $\mathrm{CSP}(\mathcal{B})$ *is in P.*

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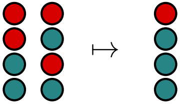

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$\Rightarrow$ If there is no clone homomorphism $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathscr{P}$, then $\mathrm{CSP}(\mathcal{B})$ is in P.

Three cases:

1. No clone homomorphism $\mathrm{Pol}(\mathcal{B})_{\mathsf{can}} \to \mathscr{P}$: done (in P),
2. Clone homomorphism $\mathrm{Pol}(\mathcal{B}) \to \mathscr{P}$: done (NP-hard),
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Proposition

For $\mathcal{B}$ an MMSNP structure, case 3 does not happen.

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- Define extension of ξ in natural way.

$\square$

- σ: set of colour symbols.
- A trivial subfactor of $\mathscr{C}$ is a partition $S \uplus T \subseteq \sigma$ such that $\mathscr{C}/\sim$ is isomorphic to $\mathscr{P}$.

Proposition

Suppose there is $\mathrm{Pol}(\mathcal{B})_{can} \to \mathscr{P}$. *There exists* S, T *trivial subfactor of* $\mathscr{C}$ *and* E *undirected graph on* σ *s.t.:*

- *E is preserved by* $\mathrm{Pol}(\mathcal{B})$,
- *E contains an edge from S to T but does not contain pseudo-loops;*
- *there is no E-path of even length between S and T.*

$\rightsquigarrow$ clone homomorphism $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathscr{P}$ that is constant on every set of canonical friends.

Getting canonical functions

Theorem (Hubička-Nešetřil, 2016)

Let $\mathcal{B}$ be an MMSNP structure. Then there is a linear order $<$ on B such that $(\mathcal{B}, <)$ is ω-categorical and Ramsey.

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Theorem (Bodirsky-Pinsker-Tsankov, 2010)

Suppose that $\mathcal{G}$ is the automorphism group of an ω-categorical ordered Ramsey structure. For every $f : B^k \to B$, there exists a function $g \in \overline{\mathcal{G} f \mathcal{G}}$ that is canonical with respect to $\mathcal{G}$.

Summing up:

Theorem

Let $\mathcal{B}$ be an MMSNP structure.
Then either the following equivalent statements hold:

1. *there is no uniformly continuous height 1 homomorphism* $\mathrm{Pol}(\mathcal{B}) \to \mathscr{P}$,

and $\mathrm{CSP}(\mathcal{B})$ *is in P, or* $\mathrm{CSP}(\mathcal{B})$ *is NP-complete.*

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Algebraic Dichotomy

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Items 3. and 4. are decidable.