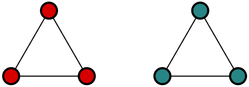
# MMSNP: An algebraic proof of the dichotomy

Manuel Bodirsky, Florent Madelaine, **Antoine Mottet**

September 25, 2018

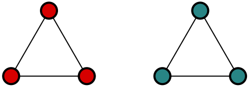
- Input: undirected graph $G$,
- Question: can one colour the vertices of $G$ in a way to avoid the following patterns:



- Complexity:

## Two problems

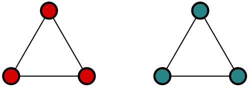
- Input: undirected graph $G$,
- Question: can one colour the vertices of $G$ in a way to avoid the following patterns:



- Complexity: NP-complete.

# Two problems

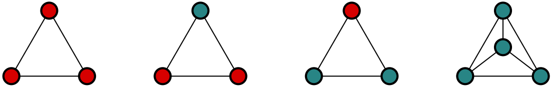
- Input: undirected graph $G$,
- Question: can one colour the vertices of $G$ in a way to avoid the following patterns:



- Complexity: NP-complete.

---

- Input: undirected graph $G$
- Question: can one colour the vertices of $G$ in a way to avoid the following patterns:



- Complexity: in P.

## FPP and MMSNP

In general, for some fixed set $\mathcal{F}$ of vertex-coloured graphs, the problem FPP($\mathcal{F}$) is:

- Input: finite graph $G$,
- Question: can one colour the vertices of $G$ and avoid $\mathcal{F}$?

(i.e., find $G^*$ such that $\forall F^* \in \mathcal{F}, F^* \not\to G^*$)

In general, for some fixed set $\mathcal{F}$ of vertex-coloured graphs, the problem FPP($\mathcal{F}$) is:

- Input: finite graph $G$,
- Question: can one colour the vertices of $G$ and avoid $\mathcal{F}$?

(i.e., find $G^*$ such that $\forall F^* \in \mathcal{F}, F^* \not\to G^*$)

MMSNP: formulas of the form

$$\exists M_1 \cdots \exists M_n$$

where:

- $M_1, \ldots, M_n$ are unary predicates,

In general, for some fixed set $\mathcal{F}$ of vertex-coloured graphs, the problem FPP($\mathcal{F}$) is:

- Input: finite graph $G$,
- Question: can one colour the vertices of $G$ and avoid $\mathcal{F}$?

  (i.e., find $G^*$ such that $\forall F^* \in \mathcal{F}, F^* \not\to G^*$)

MMSNP: formulas of the form

$$\exists M_1 \cdots \exists M_n \forall \overline{x} \bigwedge_i \neg( \qquad\qquad\qquad\qquad )$$

where:

- $M_1, \ldots, M_n$ are unary predicates,

In general, for some fixed set $\mathcal{F}$ of vertex-coloured graphs, the problem FPP($\mathcal{F}$) is:

- Input: finite graph $G$,
- Question: can one colour the vertices of $G$ and avoid $\mathcal{F}$?

  (i.e., find $G^*$ such that $\forall F^* \in \mathcal{F}, F^* \not\to G^*$)

MMSNP: formulas of the form

$$\exists M_1 \cdots \exists M_n \forall \overline{x} \bigwedge_i \neg(\alpha_i(\overline{x}) \qquad\qquad )$$

where:

- $M_1, \ldots, M_n$ are unary predicates,
- $\alpha_i$ is a conjunction of positive literals $E(x_j, x_k)$,

In general, for some fixed set $\mathcal{F}$ of vertex-coloured graphs, the problem FPP($\mathcal{F}$) is:

- Input: finite graph $G$,
- Question: can one colour the vertices of $G$ and avoid $\mathcal{F}$?

  (i.e., find $G^*$ such that $\forall F^* \in \mathcal{F}, F^* \not\to G^*$)

MMSNP: formulas of the form

$$\exists M_1 \cdots \exists M_n \forall \overline{x} \bigwedge_i \neg(\alpha_i(\overline{x}) \wedge \beta_i(\overline{x}, M_1, \ldots, M_n))$$

where:

- $M_1, \ldots, M_n$ are unary predicates,
- $\alpha_i$ is a conjunction of positive literals $E(x_j, x_k)$,
- $\beta_i$ is a conjunction of literals.

In general, for some fixed set $\mathcal{F}$ of vertex-coloured graphs, the problem FPP($\mathcal{F}$) is:

- Input: finite graph $G$,
- Question: can one colour the vertices of $G$ and avoid $\mathcal{F}$?

  (i.e., find $G^*$ such that $\forall F^* \in \mathcal{F}, F^* \not\to G^*$)

MMSNP: formulas of the form

$$\exists M_1 \cdots \exists M_n \forall \overline{x} \bigwedge_i \neg(\alpha_i(\overline{x}) \wedge \beta_i(\overline{x}, M_1, \ldots, M_n))$$

where:

- $M_1, \ldots, M_n$ are unary predicates,
- $\alpha_i$ is a conjunction of positive literals $E(x_j, x_k)$,
- $\beta_i$ is a conjunction of literals.

$\rightsquigarrow$ MMSNP is the logic of FPP.

Dichotomy: “every problem is easy or hard.”

# Dichotomies: MMSNP and CSP

Dichotomy: "every problem is easy or hard."

**Theorem (Bulatov, Zhuk '17)**

*Every finite-domain CSP is in P or NP-complete.*

## Dichotomies: MMSNP and CSP

Dichotomy: "every problem is easy or hard."

**Theorem (Bulatov, Zhuk '17)**

*Every finite-domain CSP is in P or NP-complete.*

**Theorem (Feder-Vardi '93, Kun '13)**

*MMSNP has a complexity dichotomy if and only if finite-domain CSPs have a complexity dichotomy.*

Dichotomy: "every problem is easy or hard."

**Theorem (Bulatov, Zhuk '17)**

*Every finite-domain CSP is in P or NP-complete.*

**Theorem (Feder-Vardi '93, Kun '13)**

*MMSNP has a complexity dichotomy if and only if finite-domain CSPs have a complexity dichotomy.*

# The End.

Dichotomy: "every problem is easy or hard."

**Theorem (Bulatov, Zhuk '17)**

*Every finite-domain CSP is in P or NP-complete.*

**Theorem (Feder-Vardi '93, Kun '13)**

*MMSNP has a complexity dichotomy if and only if finite-domain CSPs have a complexity dichotomy.*

# The End?

## Algebraic Dichotomy?

- Finite-domain dichotomy is universal-algebraic.
  Feder-Vardi-Kun's reduction is not.

## Algebraic Dichotomy?

- Finite-domain dichotomy is universal-algebraic.
  Feder-Vardi-Kun's reduction is not.
- MMSNP can describe some infinite-domain CSPs.
  The conjecture for infinite domains is still open.

## Algebraic Dichotomy?

- Finite-domain dichotomy is universal-algebraic.
  Feder-Vardi-Kun's reduction is not.
- MMSNP can describe some infinite-domain CSPs.
  The conjecture for infinite domains is still open.

### Theorem

*Let $\mathcal{A}$ be such that* $\mathrm{CSP}(\mathcal{A})$ *is in MMSNP.*
*Exactly one of the following holds:*

- *There is a uniformly continuous homomorphism* $\mathrm{Pol}(\mathcal{A}) \to \mathrm{Pol}(SAT)$, *and* $\mathrm{CSP}(\mathcal{A})$ *is NP-complete,*
- $\mathrm{Pol}(\mathcal{A})$ *contains a cyclic operation and* $\mathrm{CSP}(\mathcal{A})$ *is in P.*

- Finite-domain dichotomy is universal-algebraic.
  Feder-Vardi-Kun's reduction is not.
- MMSNP can describe some infinite-domain CSPs.
  The conjecture for infinite domains is still open.

**Theorem**

*Let $\mathcal{A}$ be such that $\mathrm{CSP}(\mathcal{A})$ is in MMSNP.*
*Exactly one of the following holds:*

- *There is a uniformly continuous homomorphism $\mathrm{Pol}(\mathcal{A}) \to \mathrm{Pol}(SAT)$, and $\mathrm{CSP}(\mathcal{A})$ is NP-complete,*
- *$\mathrm{Pol}(\mathcal{A})$ contains a cyclic operation and $\mathrm{CSP}(\mathcal{A})$ is in P.*

$\rightsquigarrow$ new proof of the Feder-Vardi-Kun theorem,

- Finite-domain dichotomy is universal-algebraic.
  Feder-Vardi-Kun's reduction is not.
- MMSNP can describe some infinite-domain CSPs.
  The conjecture for infinite domains is still open.

**Theorem**

*Let $\mathcal{A}$ be such that $\mathrm{CSP}(\mathcal{A})$ is in MMSNP.*
*Exactly one of the following holds:*

- *There is a uniformly continuous homomorphism* $\mathrm{Pol}(\mathcal{A}) \to \mathrm{Pol}(SAT)$, *and* $\mathrm{CSP}(\mathcal{A})$ *is NP-complete,*
- $\mathrm{Pol}(\mathcal{A})$ *contains a cyclic operation and* $\mathrm{CSP}(\mathcal{A})$ *is in P.*

$\rightsquigarrow$ new proof of the Feder-Vardi-Kun theorem,
$\rightsquigarrow$ the infinite-domain conjecture for CSPs in MMSNP is true.

## Definition

$\mathcal{B} = (B; E)$ a graph.
CSP($\mathcal{B}$) is the problem:

- **Input:** a finite graph $\mathcal{A}$,
- **Question:** is there a homomorphism $\mathcal{A} \to \mathcal{B}$?

## Definition

$\mathcal{B} = (B; E)$ a graph.
CSP($\mathcal{B}$) is the problem:

- **Input:** a finite graph $\mathcal{A}$,
- **Question:** is there a homomorphism $\mathcal{A} \to \mathcal{B}$?

Examples:

- CSP($K_3$): 3-colouring problem,
- CSP($\mathbb{Q}, <$): digraph acyclicity,
- In general, we don't restrict to graphs.
  CSP($\mathbb{Z}; +, \times, 1$): satisfiability of diophantine equations.

## Definition

$\mathcal{B} = (B; E)$ a graph.
CSP($\mathcal{B}$) is the problem:

- **Input:** a finite graph $\mathcal{A}$,
- **Question:** is there a homomorphism $\mathcal{A} \to \mathcal{B}$?

Examples:

- CSP($K_3$): 3-colouring problem,
- CSP($\mathbb{Q}, <$): digraph acyclicity,
- In general, we don't restrict to graphs.
  CSP($\mathbb{Z}; +, \times, 1$): satisfiability of diophantine equations.

If $\mathcal{B}$ is finite, CSP($\mathcal{B}$) is in P or NP-complete.

- For a nice structure $\mathcal{B}$,

$$\text{complexity of } \mathrm{CSP}(\mathcal{B}) \leftrightarrow \text{algebraic properties of } \mathrm{Pol}(\mathcal{B})$$

- For a nice structure $\mathcal{B}$,

$$\text{complexity of } \mathrm{CSP}(\mathcal{B}) \leftrightarrow \text{algebraic properties of } \mathrm{Pol}(\mathcal{B})$$

- One measures how "trivial" $\mathrm{Pol}(\mathcal{B})$ is by the equations satisfiable in it.

- For a nice structure $\mathcal{B}$,

$$\text{complexity of } \mathrm{CSP}(\mathcal{B}) \leftrightarrow \text{algebraic properties of } \mathrm{Pol}(\mathcal{B})$$

- One measures how "trivial" Pol($\mathcal{B}$) is by the equations satisfiable in it.
- Trivial-est of all: Pol($SAT$), which only contains projections.

- For a nice structure $\mathcal{B}$,

$$\text{complexity of CSP}(\mathcal{B}) \leftrightarrow \text{algebraic properties of Pol}(\mathcal{B})$$

- One measures how "trivial" Pol($\mathcal{B}$) is by the equations satisfiable in it.
- Trivial-est of all: Pol($SAT$), which only contains projections.
- A homomorphism Pol($\mathcal{B}$) $\to$ Pol($\mathcal{A}$) induces a log-space reduction CSP($\mathcal{A}$) $\leq$ CSP($\mathcal{B}$).

- For a nice structure $\mathcal{B}$,

$$\text{complexity of CSP}(\mathcal{B}) \leftrightarrow \text{algebraic properties of Pol}(\mathcal{B})$$

- One measures how "trivial" Pol($\mathcal{B}$) is by the equations satisfiable in it.
- Trivial-est of all: Pol($SAT$), which only contains projections.
- A homomorphism Pol($\mathcal{B}$) $\to$ Pol($\mathcal{A}$) induces a log-space reduction CSP($\mathcal{A}$) $\leq$ CSP($\mathcal{B}$).
- Algebraic hardness: Pol($\mathcal{A}$) $\to$ Pol($SAT$) implies that CSP($\mathcal{A}$) is NP-hard.

# MMSNP is essentially a subset of CSP

# MMSNP is essentially a subset of CSP

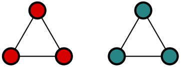
## Theorem (Cherlin-Shelah-Shi, '99)

*For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists a countable universal $\mathcal{F}$-free graph.*

# MMSNP is essentially a subset of CSP

## Theorem (Cherlin-Shelah-Shi, '99)

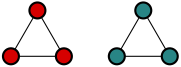
*For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists a countable universal $\mathcal{F}$-free graph.*

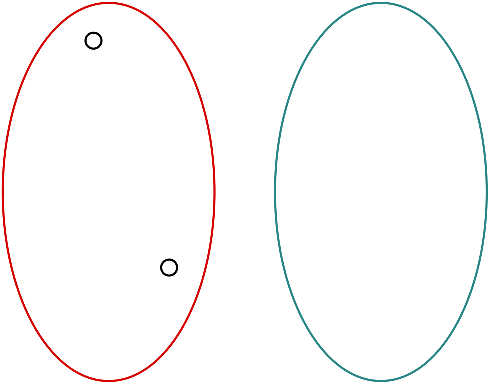
Obstructions $\mathcal{F}$: 

# MMSNP is essentially a subset of CSP

## Theorem (Cherlin-Shelah-Shi, '99)

*For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists a countable universal $\mathcal{F}$-free graph.*

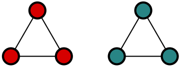
Obstructions $\mathcal{F}$: 

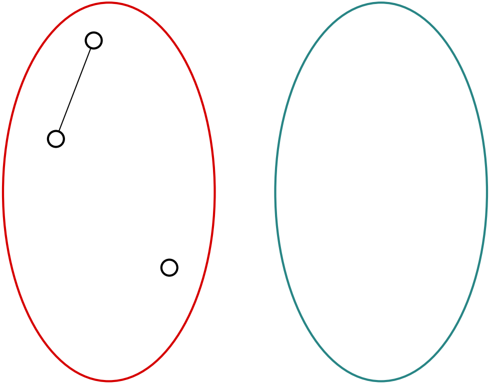


# MMSNP is essentially a subset of CSP

## Theorem (Cherlin-Shelah-Shi, '99)

*For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists a countable universal $\mathcal{F}$-free graph.*

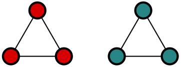
Obstructions $\mathcal{F}$: 

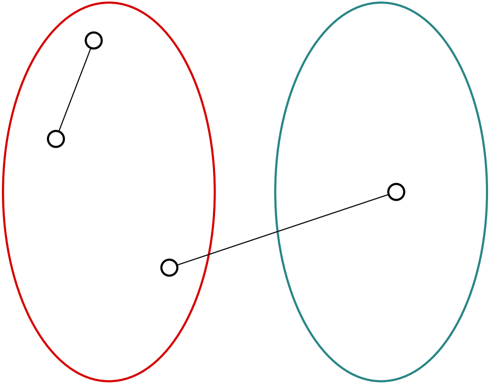


# MMSNP is essentially a subset of CSP

## Theorem (Cherlin-Shelah-Shi, '99)

*For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists a countable universal $\mathcal{F}$-free graph.*

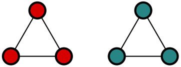
Obstructions $\mathcal{F}$: 

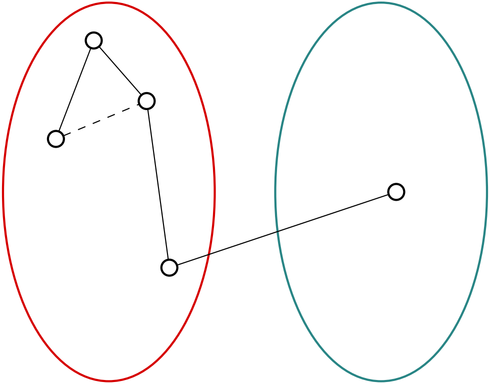


## MMSNP is essentially a subset of CSP

### Theorem (Cherlin-Shelah-Shi, '99)

*For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists a countable universal $\mathcal{F}$-free graph.*

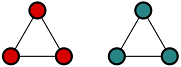
Obstructions $\mathcal{F}$: 

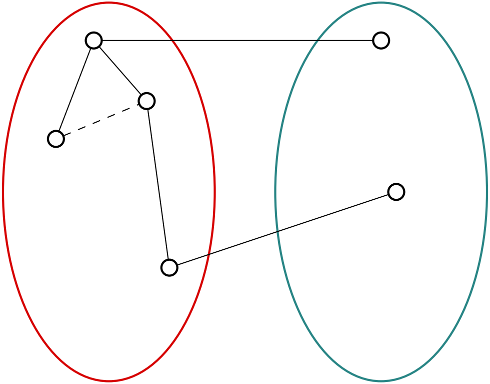


## MMSNP is essentially a subset of CSP

**Theorem (Cherlin-Shelah-Shi, '99)**

*For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists a countable universal $\mathcal{F}$-free graph.*

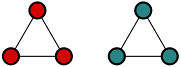
Obstructions $\mathcal{F}$: 

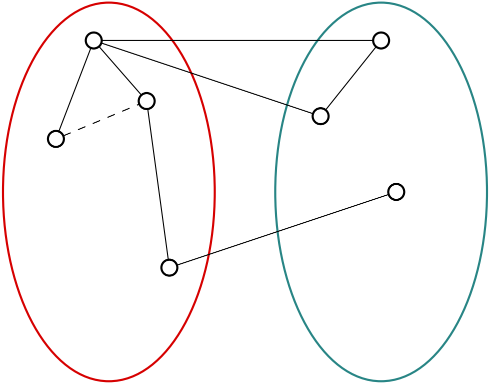


## MMSNP is essentially a subset of CSP

**Theorem (Cherlin-Shelah-Shi, '99)**

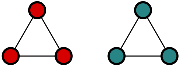
*For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists a countable universal $\mathcal{F}$-free graph.*

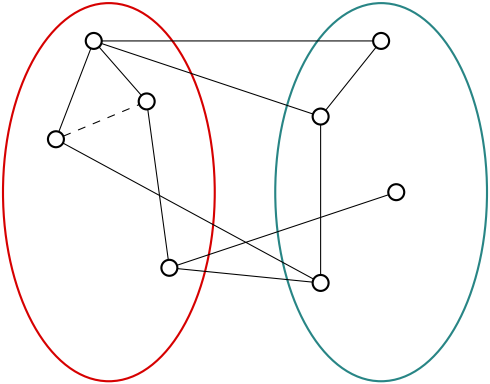
Obstructions $\mathcal{F}$: 



**Theorem (Cherlin-Shelah-Shi, '99)**

*For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists a countable universal $\mathcal{F}$-free graph.*

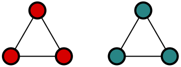
Obstructions $\mathcal{F}$: 

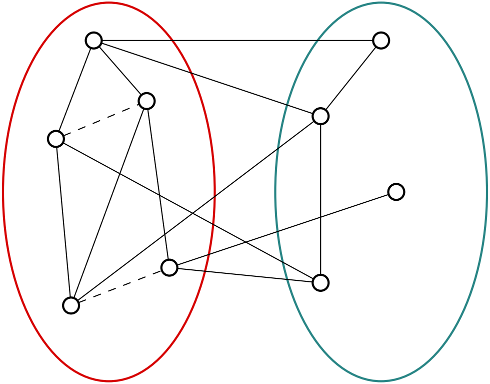


## MMSNP is essentially a subset of CSP

### Theorem (Cherlin-Shelah-Shi, '99)

*For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists a countable universal $\mathcal{F}$-free graph.*

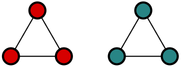
Obstructions $\mathcal{F}$: 

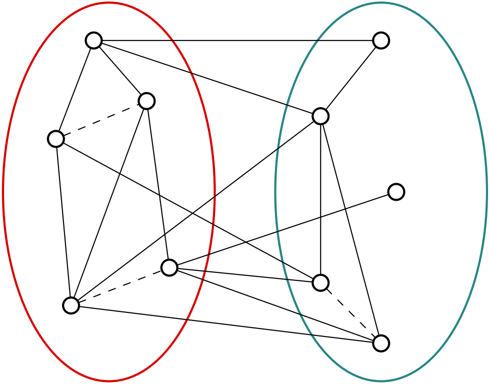


# MMSNP is essentially a subset of CSP

## Theorem (Cherlin-Shelah-Shi, '99)

*For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists a countable universal $\mathcal{F}$-free graph.*

Obstructions $\mathcal{F}$: 



"Algebraic hardness is the only source of hardness":

**Conjecture (Bodirsky-Pinsker, '11 (rephrased))**

*Let $\mathcal{B}$ be a reduct of a finitely bounded homogeneous structure. If there is no uniformly continuous homomorphism* $\mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$, *then* $\mathrm{CSP}(\mathcal{B})$ *is in P.*

"Algebraic hardness is the only source of hardness":

### Conjecture (Bodirsky-Pinsker, '11 (rephrased))

*Let $\mathcal{B}$ be a reduct of a finitely bounded homogeneous structure. If there is no uniformly continuous homomorphism* $\mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$, *then* $\mathrm{CSP}(\mathcal{B})$ *is in P.*

### Theorem

*The conjecture is true for $\mathcal{B}$ such that* $\mathrm{CSP}(\mathcal{B})$ *is in MMSNP.*

## Definition

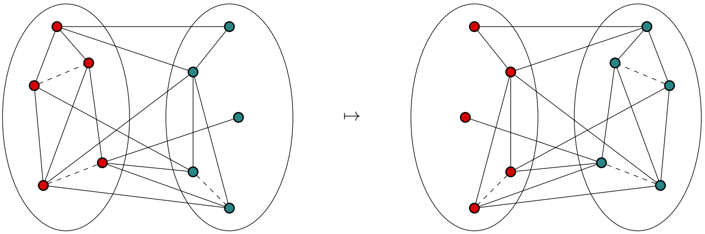
$f : B^k \to B$. f is canonical if

"the colour of the output only depends on the colours of the inputs".

## Definition

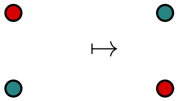
$f\colon B^k \to B$. f is canonical if

"the colour of the output only depends on the colours of the inputs".



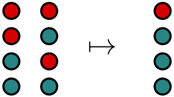
# Behaviours

Every canonical function determines a behaviour on colours.

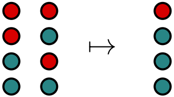


# Behaviours

Every canonical function determines a behaviour on colours.



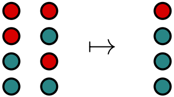
Every canonical function determines a behaviour on colours.



We view this behaviour as an operation on a finite set.
Only finitely many behaviours of a given arity.

## Behaviours

Every canonical function determines a behaviour on colours.



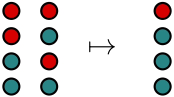
We view this behaviour as an operation on a finite set.
Only finitely many behaviours of a given arity.

**Theorem (Bodirsky-M, '16)**

*Let $\mathcal{B}$ be in the class (reduct of...).*
*If there is no homomorphism* $\mathrm{Pol}(\mathcal{B})_{can} \to \mathrm{Pol}(SAT)$, *then* $\mathrm{CSP}(\mathcal{B})$ *is in P.*

## Behaviours

Every canonical function determines a behaviour on colours.



We view this behaviour as an operation on a finite set.
Only finitely many behaviours of a given arity.

### Theorem (Bodirsky-M, '16)

*Let $\mathcal{B}$ be in the class (reduct of...).*
*If there is no homomorphism* $\mathrm{Pol}(\mathcal{B})_{can} \to \mathrm{Pol}(SAT)$*, then* $\mathrm{CSP}(\mathcal{B})$ *is in P.*

$\rightsquigarrow$ the BP conjecture is true when $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}} = \mathrm{Pol}(\mathcal{B})$.

- $\mathrm{Pol}(\mathcal{B})_{\mathsf{can}}$ is not sufficient in general.

- $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}}$ is not sufficient in general.
- Challenge: extending $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathrm{Pol}(SAT)$ to $\mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$.

- $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}}$ is not sufficient in general.
- Challenge: extending $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathrm{Pol}(SAT)$ to $\mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$.

1. We suppose that there is a clone homomorphism $\xi\colon \mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathrm{Pol}(SAT)$.

## Bridging the gap

- $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}}$ is not sufficient in general.
- Challenge: extending $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathrm{Pol}(SAT)$ to $\mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$.

1. We suppose that there is a clone homomorphism $\xi \colon \mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathrm{Pol}(SAT)$.
2. Prove that $\mathrm{CSP}(\mathcal{B})$ is algebraically hard iff $\mathrm{CSP}(\mathcal{B}, \bullet, \bullet)$ is.

- $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}}$ is not sufficient in general.
- Challenge: extending $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathrm{Pol}(SAT)$ to $\mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$.

1. We suppose that there is a clone homomorphism $\xi\colon \mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathrm{Pol}(SAT)$.
2. Prove that $\mathrm{CSP}(\mathcal{B})$ is algebraically hard iff $\mathrm{CSP}(\mathcal{B}, {\color{red}\bullet}, {\color{teal}\bullet})$ is.
3. $\mathrm{Pol}(\mathcal{B}, {\color{red}\bullet}, {\color{teal}\bullet})_{\mathrm{can}}$ is an idempotent clone.
   $\rightsquigarrow$ subfactor of $\mathrm{Pol}(\mathcal{B}, {\color{red}\bullet}, {\color{teal}\bullet})_{\mathrm{can}}$ isomorphic to $\mathrm{Pol}(SAT)$.

## Bridging the gap

- $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}}$ is not sufficient in general.
- Challenge: extending $\mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathrm{Pol}(SAT)$ to $\mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$.

1. We suppose that there is a clone homomorphism $\xi\colon \mathrm{Pol}(\mathcal{B})_{\mathrm{can}} \to \mathrm{Pol}(SAT)$.
2. Prove that $\mathrm{CSP}(\mathcal{B})$ is algebraically hard iff $\mathrm{CSP}(\mathcal{B}, \textcolor{red}{\bullet}, \textcolor{teal}{\bullet})$ is.
3. $\mathrm{Pol}(\mathcal{B}, \textcolor{red}{\bullet}, \textcolor{teal}{\bullet})_{\mathrm{can}}$ is an idempotent clone.
   $\rightsquigarrow$ subfactor of $\mathrm{Pol}(\mathcal{B}, \textcolor{red}{\bullet}, \textcolor{teal}{\bullet})_{\mathrm{can}}$ isomorphic to $\mathrm{Pol}(SAT)$.
4. Define a homomorphism $\mathrm{Pol}(\mathcal{B}, \textcolor{red}{\bullet}, \textcolor{teal}{\bullet}) \to \mathrm{Pol}(SAT)$ by canonizing and composing with $\xi$.

Summing up:

**Theorem**

*Let $\mathcal{B}$ be an MMSNP structure.*
*Then either the following equivalent statements hold:*

1. *there is no uniformly continuous homomorphism* $\mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$,

*and* $\mathrm{CSP}(\mathcal{B})$ *is in P, or* $\mathrm{CSP}(\mathcal{B})$ *is NP-complete.*

Summing up:

**Theorem**

*Let $\mathcal{B}$ be an MMSNP structure.*
*Then either the following equivalent statements hold:*

1. *there is no uniformly continuous homomorphism* $\mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$,
2. *there is no uniformly continuous homomorphism* $\mathrm{Pol}(\mathcal{B})_{can} \to \mathrm{Pol}(SAT)$,

*and* $\mathrm{CSP}(\mathcal{B})$ *is in P, or* $\mathrm{CSP}(\mathcal{B})$ *is NP-complete.*

Summing up:

**Theorem**

*Let $\mathcal{B}$ be an MMSNP structure.*
*Then either the following equivalent statements hold:*

1. *there is no uniformly continuous homomorphism* $\mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$,
2. *there is no uniformly continuous homomorphism* $\mathrm{Pol}(\mathcal{B})_{can} \to \mathrm{Pol}(SAT)$,

*and* $\mathrm{CSP}(\mathcal{B})$ *is in P, or* $\mathrm{CSP}(\mathcal{B})$ *is NP-complete.*

Item 2. can be checked effectively given an MMSNP sentence.

Summing up:

**Theorem**

*Let $\mathcal{B}$ be an MMSNP structure.*
*Then either the following equivalent statements hold:*

1. *there is no uniformly continuous homomorphism* $\mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$,
2. *there is no uniformly continuous homomorphism* $\mathrm{Pol}(\mathcal{B})_{can} \to \mathrm{Pol}(SAT)$,

*and* $\mathrm{CSP}(\mathcal{B})$ *is in P, or* $\mathrm{CSP}(\mathcal{B})$ *is NP-complete.*

Item 2. can be checked effectively given an MMSNP sentence.

# The End.

## Another question about MMSNP

A precoloured forbidden patterns problem is an FPP where the input can be partially coloured.

# Another question about MMSNP

A precoloured forbidden patterns problem is an FPP where the input can be partially coloured.

## Question (Lutz-Wolter, '15)

*$\mathcal{F}$ finite family of finite connected coloured graphs. Do* $\mathrm{FPP}(\mathcal{F})$ *and its precoloured version have the same complexity?*

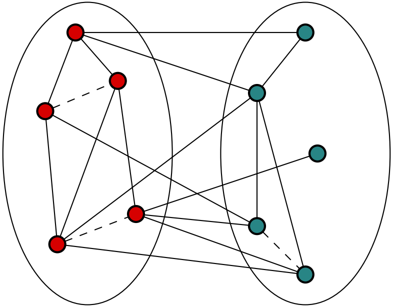
## Another question about MMSNP

A precoloured forbidden patterns problem is an FPP where the input can be partially coloured.

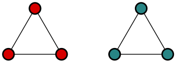
### Question (Lutz-Wolter, '15)

*$\mathcal{F}$ finite family of finite connected coloured graphs. Do* FPP($\mathcal{F}$) *and its precoloured version have the same complexity?*

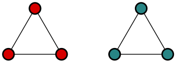
Rephrased: do CSP($\mathcal{B}$, ●, ●) and CSP($\mathcal{B}$) have same complexity?

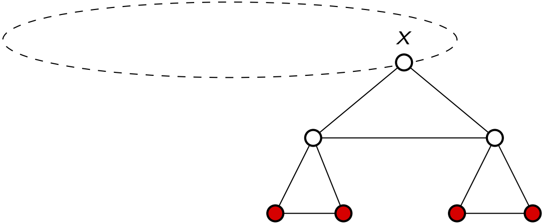


# Precolouring gadget

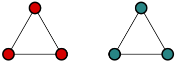
- Obstructions $\mathcal{F}$: 
- $x$ precoloured in the instance of CSP$(\mathcal{B}, \bullet, \bullet)$:

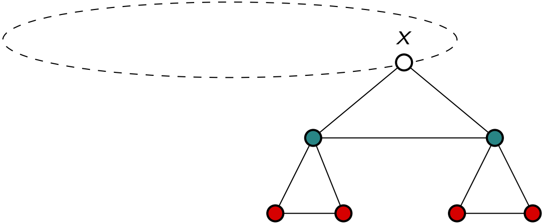
## Precolouring gadget

- Obstructions $\mathcal{F}$: 
- $x$ precoloured in the instance of CSP($\mathcal{B}$, , ):

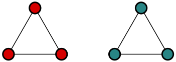


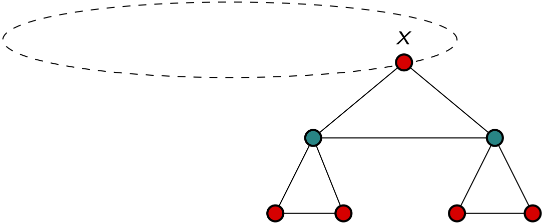
# Precolouring gadget

- Obstructions $\mathcal{F}$: 
- $x$ precoloured in the instance of CSP($\mathcal{B}$, , ):

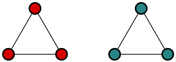


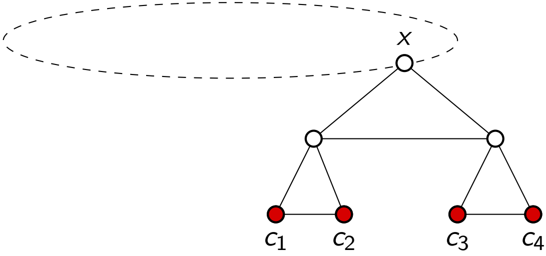
# Precolouring gadget

- Obstructions $\mathcal{F}$: 
- $x$ precoloured in the instance of CSP($\mathcal{B}$, , ):

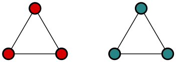


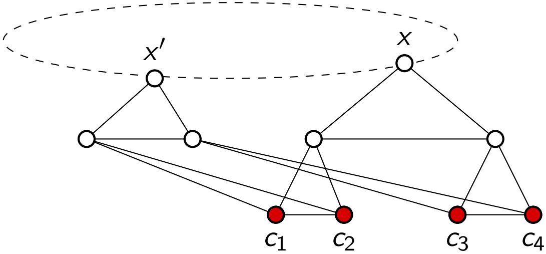
## Precolouring gadget

- Obstructions $\mathcal{F}$: 
- $x$ precoloured in the instance of CSP($\mathcal{B}$, ●, ●):

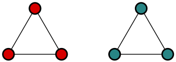


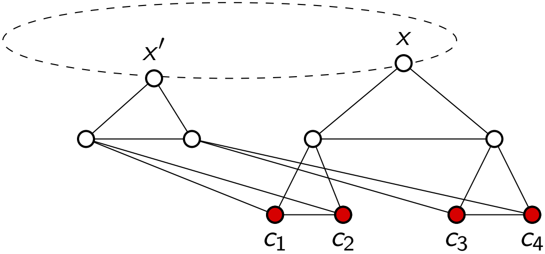
## Precolouring gadget

- Obstructions $\mathcal{F}$: 
- $x$ precoloured in the instance of CSP($\mathcal{B}$, ●, ●):



## Precolouring gadget

- Obstructions $\mathcal{F}$: 
- $x$ precoloured in the instance of $\mathrm{CSP}(\mathcal{B}, \bullet, \bullet)$:




- The input precoloured graph is colourable iff the graph obtained by adding the gadgets is colourable.

- $\sigma$: set of colour symbols.
- A trivial subfactor of $\mathscr{C}$ is a partition $S \uplus T \subseteq \sigma$ such that $\mathscr{C}/\!\sim$ is isomorphic to $\mathrm{Pol}(SAT)$.

## Proposition

*$S, T$ trivial subfactor of $\mathscr{C}$. $\exists(\sigma, E)$ bipartite undirected graph s.t.:*

- *$(\sigma, E)$ contains an edge from $S$ to $T$;*
- *$(\sigma, E)$ has no path of even length from $S$ to $T$;*
- *$E$ is preserved by $\mathrm{Pol}(\mathcal{B})$.*

## Getting canonical functions

**Theorem (Hubička-Nešetřil, 2016)**

*Let $\mathcal{B}$ be an MMSNP structure. Then there is a linear order $<$ on $B$ such that $(\mathcal{B}, <)$ is $\omega$-categorical and Ramsey.*

**Theorem (Hubička-Nešetřil, 2016)**

*Let $\mathcal{B}$ be an MMSNP structure. Then there is a linear order $<$ on $B$ such that $(\mathcal{B}, <)$ is $\omega$-categorical and Ramsey.*

**Theorem (Bodirsky-Pinsker-Tsankov, 2010)**

*Suppose that $\mathcal{G}$ is the automorphism group of an $\omega$-categorical ordered Ramsey structure. For every $f\colon B^k \to B$, there exists a function $g \in \overline{\mathcal{G} f \mathcal{G}}$ that is canonical with respect to $\mathcal{G}$.*

## Theorem (Hubička-Nešetřil, 2016)

*Let $\mathcal{B}$ be an MMSNP structure. Then there is a linear order $<$ on $B$ such that $(\mathcal{B}, <)$ is $\omega$-categorical and Ramsey.*

## Theorem (Bodirsky-Pinsker-Tsankov, 2010)

*Suppose that $\mathcal{G}$ is the automorphism group of an $\omega$-categorical ordered Ramsey structure. For every $f : B^k \to B$, there exists a function $g \in \overline{\mathcal{G} f \mathcal{G}}$ that is canonical with respect to $\mathcal{G}$.*

Given $f \in \mathrm{Pol}(\mathcal{B})$, and $\xi \colon \mathscr{C} \to \mathrm{Pol}(SAT)$ given by subfactor $\{S, T\}$, define $\phi(f) := \xi(g)$ where $g$ is an arbitrary function in $\mathscr{C} \cap \overline{\mathrm{Aut}(\mathcal{B}, <) f \, \mathrm{Aut}(\mathcal{B}, <)}$.

## Proposition

*$\phi\colon \mathrm{Pol}(\mathcal{B}) \to \mathrm{Pol}(SAT)$ is a well-defined uniformly continuous homomorphism.*