Constraint Satisfaction Problems over the Integers with Successor

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Let σ be a relational signature.
A σ-sentence ϕ is primitive positive if it is of the form

$$\exists x_1, \ldots, x_n \bigwedge_{i=1}^{m} R_i(x_{i_1}, \ldots, x_{i_k}), \quad R_i \in (\sigma \cup \{=\}).$$

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- $\{+, \times\}$: undecidable (Hilbert's Tenth Problem)

Definition (CSP(Γ))

Let Γ be a relational structure with a finite signature. The constraint satisfaction problem of Γ is the following decision problem:
INPUT: a primitive positive sentence ϕ in the language of Γ,
QUESTION: is ϕ true in Γ?

The structure Γ is called the template of CSP(Γ).

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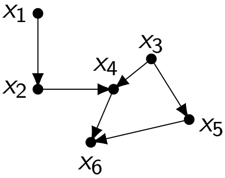
$$\exists x_1, \ldots, x_6 (x_2 = \mathrm{succ}(x_1) \wedge \\ x_4 = \mathrm{succ}(x_2) \wedge \ldots) \quad \widehat{=}$$



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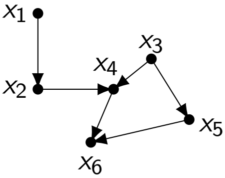

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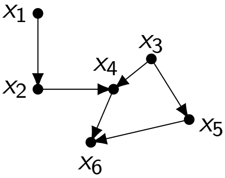

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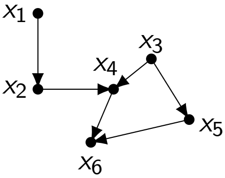

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Let Γ, Δ be structures over the same domain. We say that Γ is a reduct of Δ when all the relations of Γ are (fo-)definable in Δ.

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Problem (Complexity classification project for $(\mathbb{Z}; \mathrm{succ})$)

Give a complete classification of the complexity of distance CSPs.

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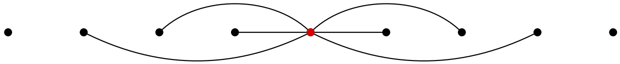

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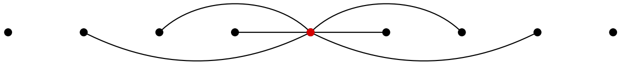

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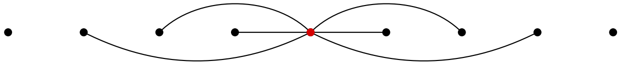

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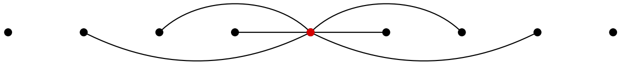

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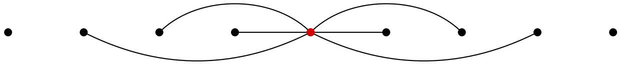

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- Complete classification of the complexity of distance CSPs.
- Systematic approach using universal algebraic methods.

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Let Γ be a relational structure, and let R be a relation that has a primitive positive definition in Γ. Then $\mathrm{CSP}(\Gamma)$ and $\mathrm{CSP}(\Gamma, R)$ are polynomial-time equivalent.

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- The previous lemma generalizes to infinite structures which have many automorphisms.
- In general, a reduct of $(\mathbb{Z}; \mathrm{succ})$ does not satisfy this condition.
- Solution: we can recover a part of the connection if Γ has enough elements.

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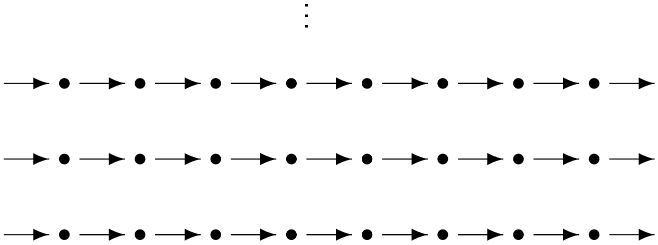

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Example: succ consists of 1 orbit under Aut($\omega.\Gamma$). By the lemma, succ is pp-definable in $\omega.\Gamma$ iff it is preserved by all the endomorphisms of $\omega.\Gamma$.

Theorem (Bodirsky, Martin, AM '15)

Let Γ *be a reduct of* $(\mathbb{Z}; \mathrm{succ})$ *with a finite signature. There exists a structure* Δ *with* $\mathsf{CSP}(\Gamma) = \mathsf{CSP}(\Delta)$ *and such that one of the following cases applies.*

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3. Δ *is a reduct of* $(\mathbb{Z}; \mathrm{succ})$ *whose endomorphisms are all isometries. In this case,* $\mathsf{CSP}(\Gamma)$ *is in P or NP-complete. Moreover, the tractability of* $\mathsf{CSP}(\Gamma)$ *is characterized by the existence of certain polymorphisms of finite arity.*

- We obtain a characterization of those reducts of $(\mathbb{Z}; \mathrm{succ})$ that are not homomorphically equivalent to a finite structure or to a reduct of $(\mathbb{Z}; =)$.

Idea of the Proof

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- This characterization tells us that the endomorphisms of $\omega.\Gamma$ have infinite range; and that there exists $t > 0$ such that for every endomorphism e of $\omega.\Gamma$ and all $x \in \omega.\mathbb{Z}$, we have

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- We inductively reduce t to 1 by replacing Γ, which finally gives us that all the endomorphisms of $\omega.\Gamma$ satisfy

$$|e(x+1) - e(x)| = 1.$$

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- More ambitious project: classify the complexity of reducts of $(\mathbb{Z}; <, +)$, i.e., reducts of Presburger Arithmetic.