Infinite-domain CSPs: part 1

Why? What?

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1. Hope for a general understanding of a class of problems
   Want to understand why problems are easy/hard
2. Connection between tractability and closure properties
3. Induction
4. Natural class due to connection with Monotone Monadic SNP without inequalities (MMSNP)

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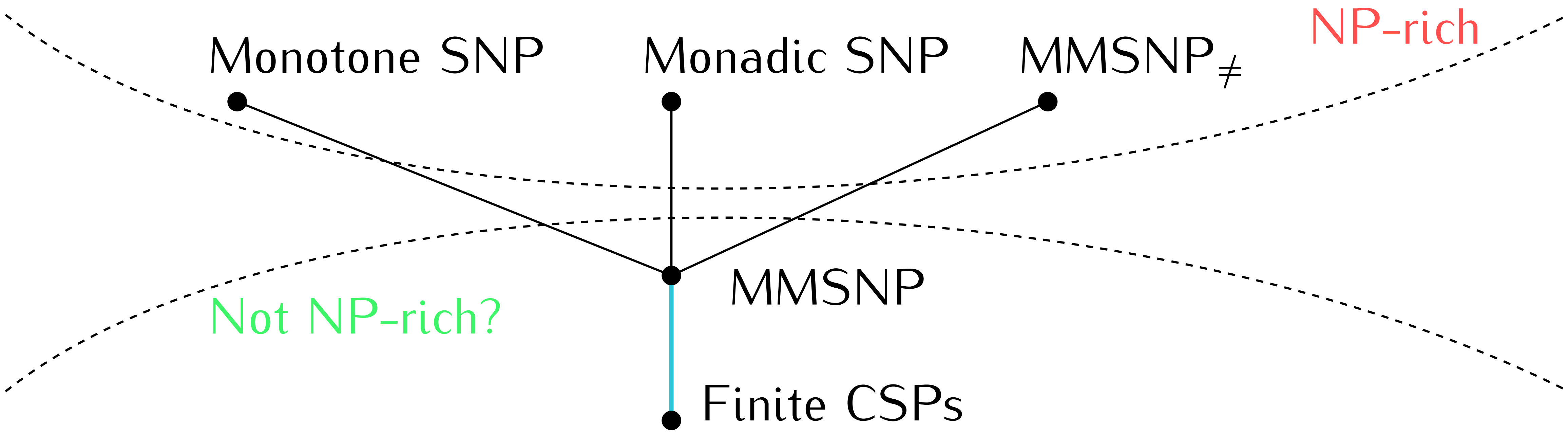

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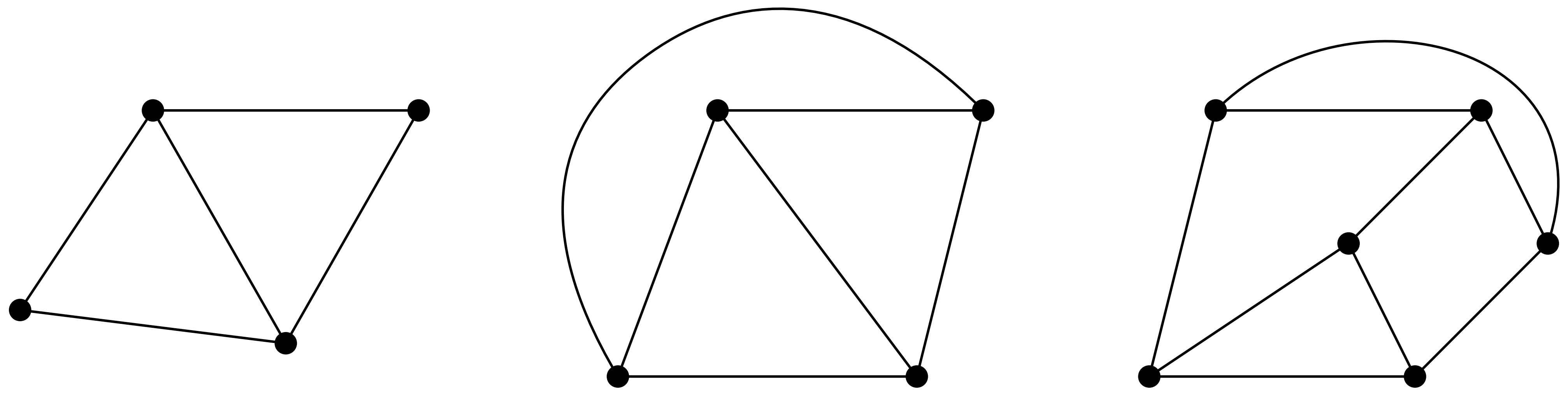
- Take $\mathbb{A} = \bigsqcup_{\mathbb{X} \in \mathcal{C}} \mathbb{X}$
- Can be a useful point of view, but most likely not

Goal: understand finite 3-colorable graphs



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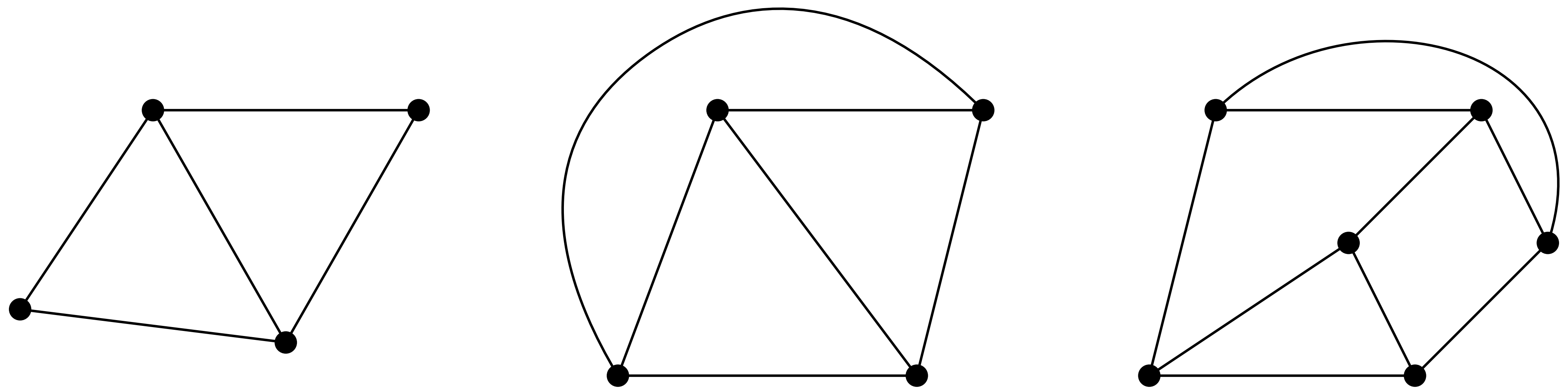

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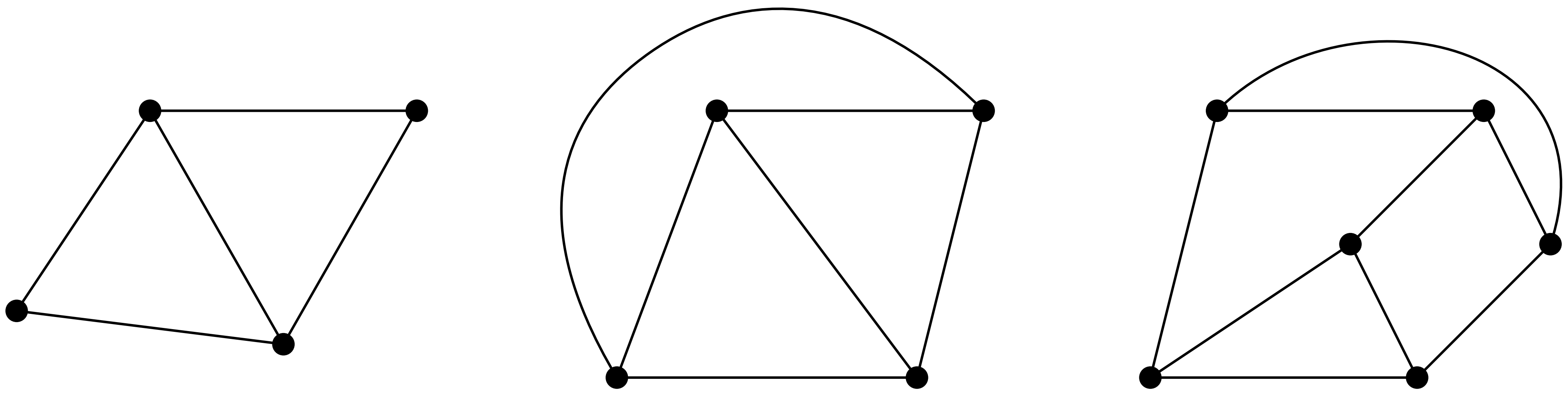

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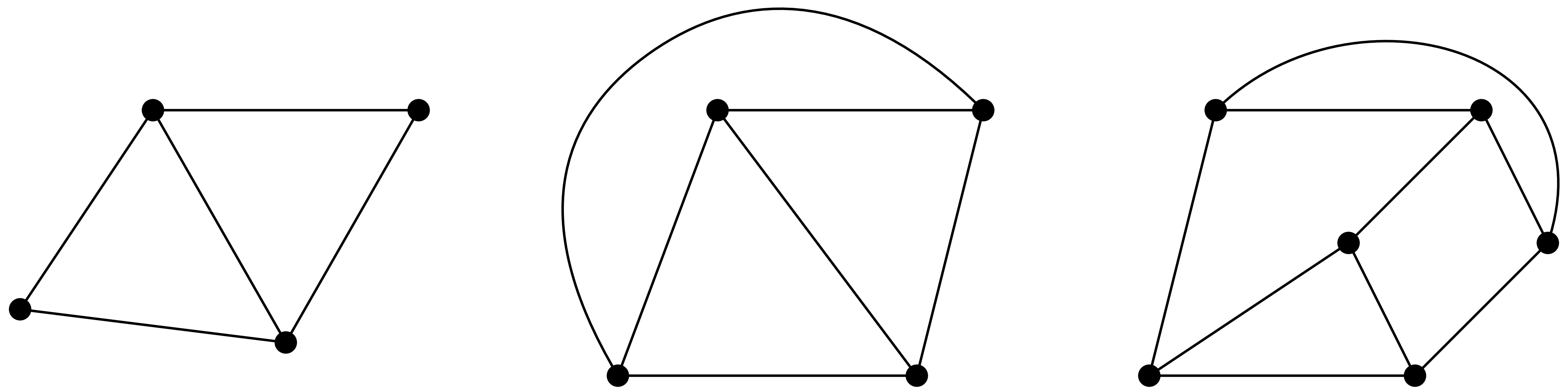

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Template-free CSPs: 3-coloring

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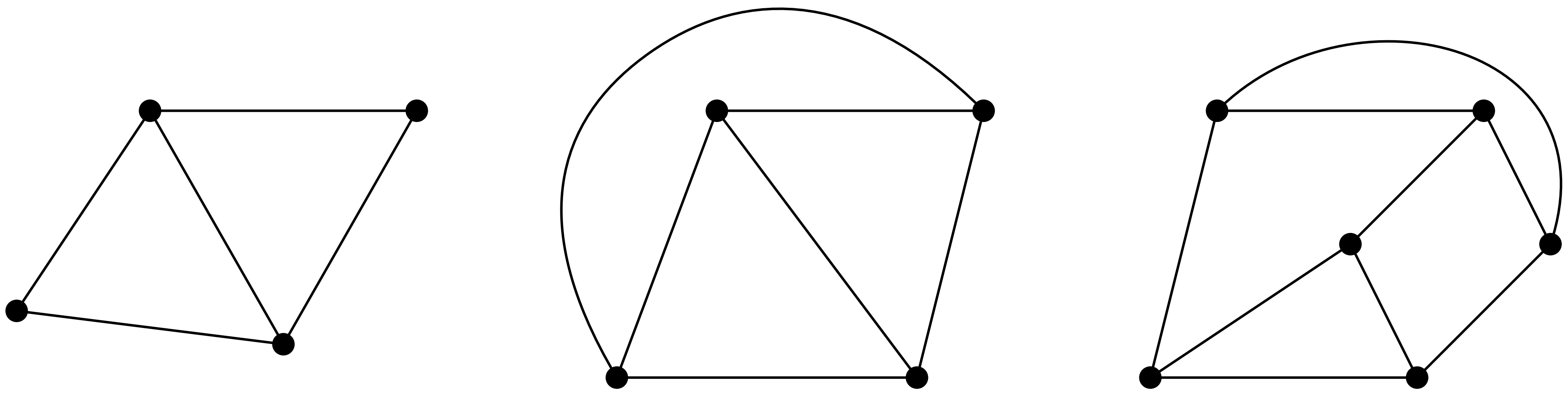

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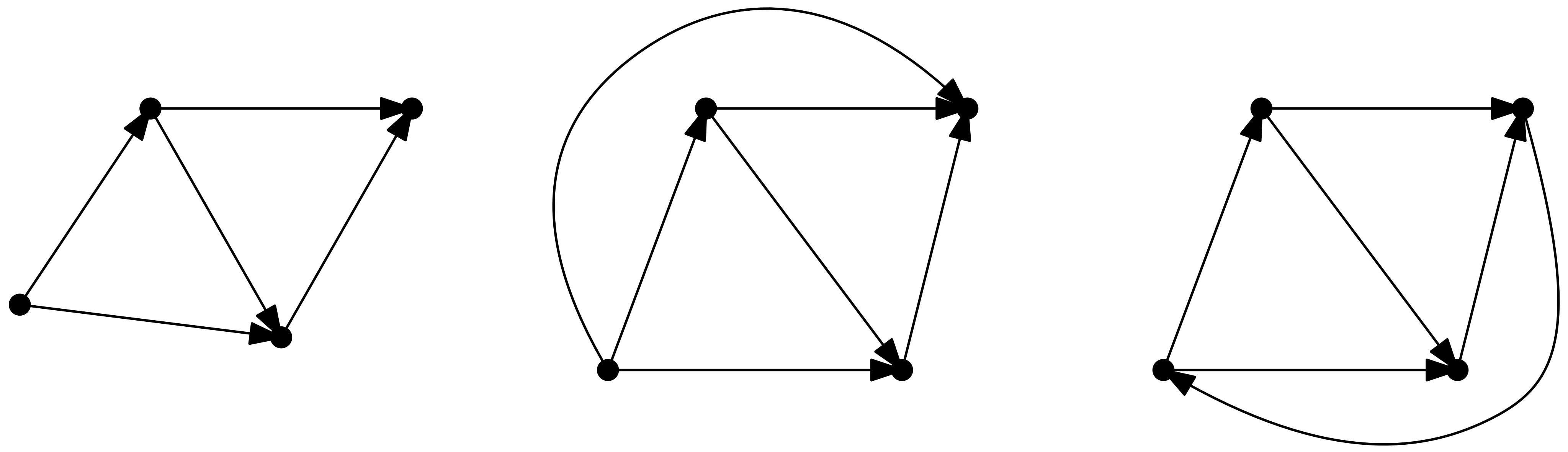

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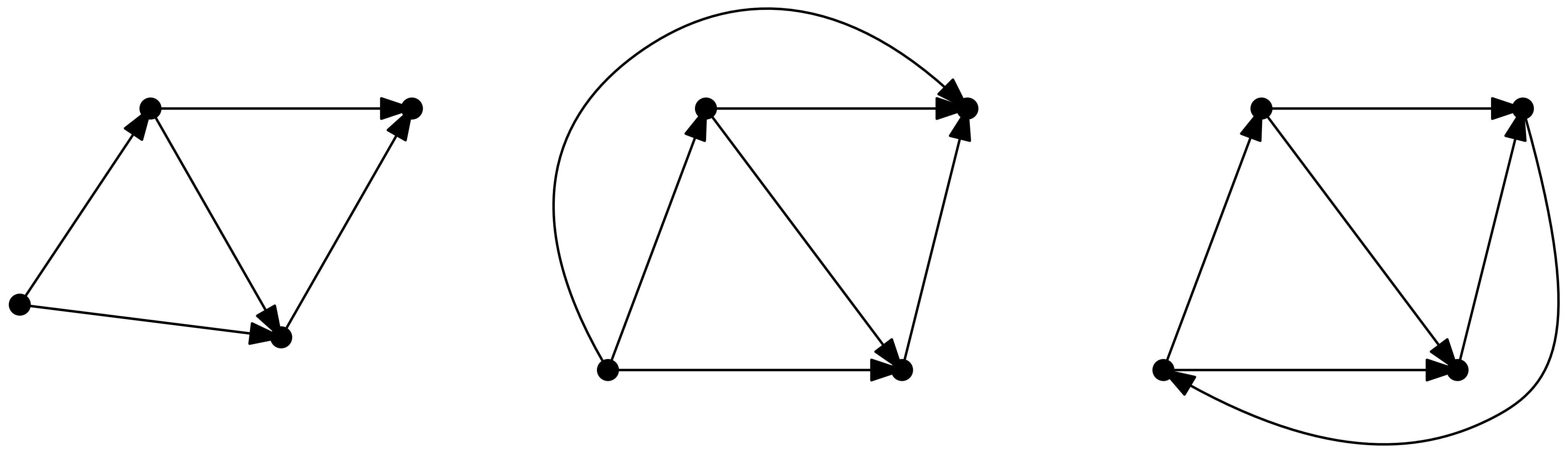

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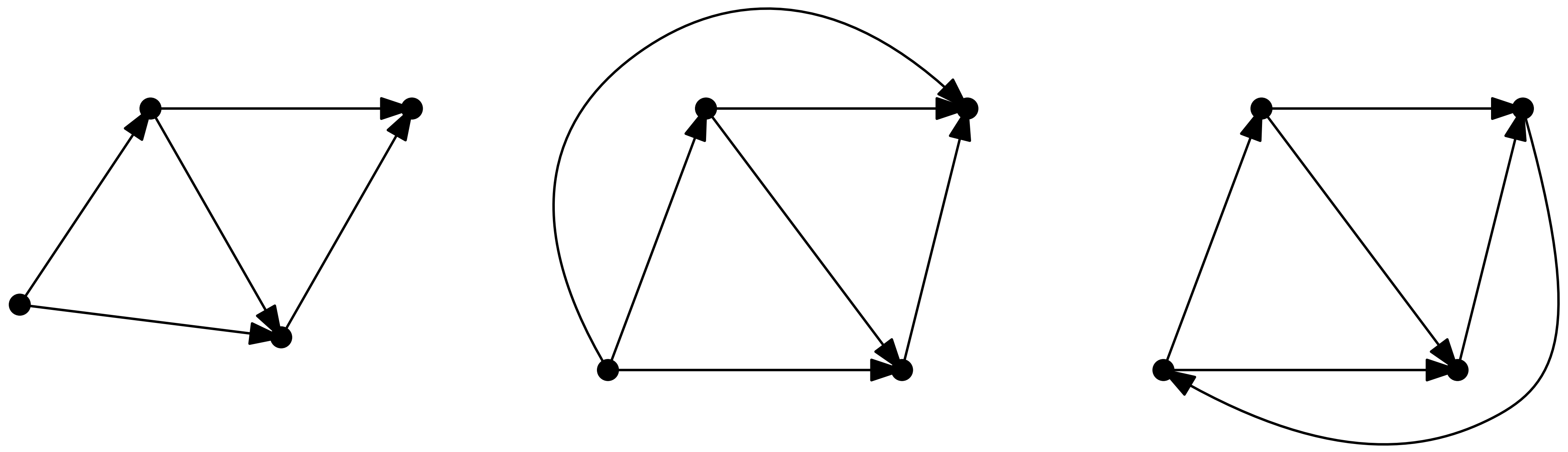

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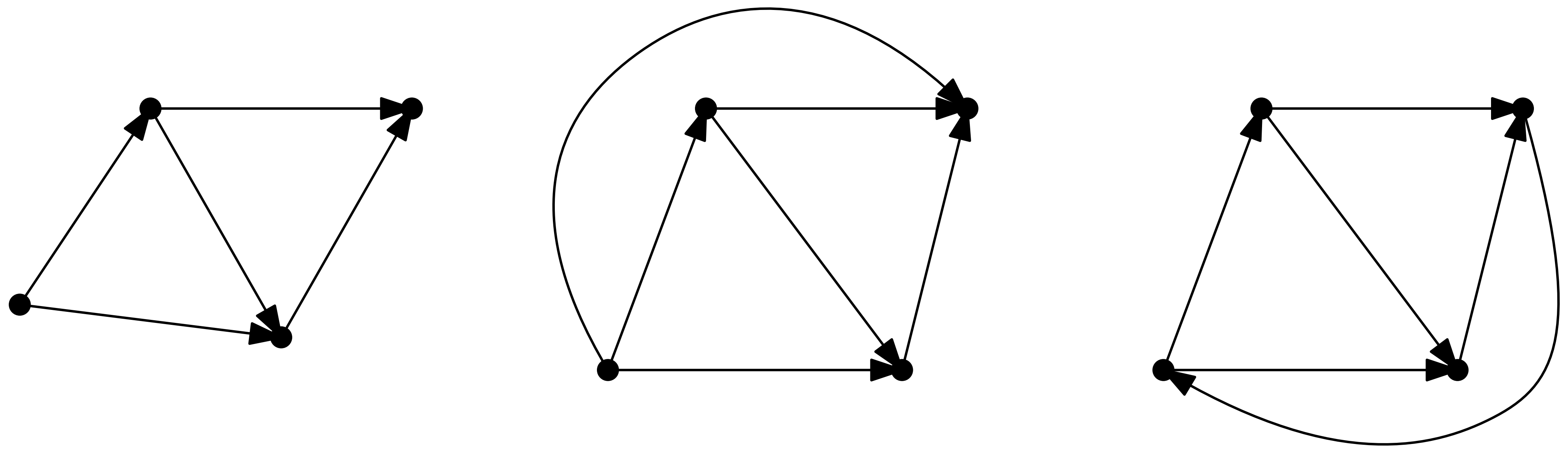

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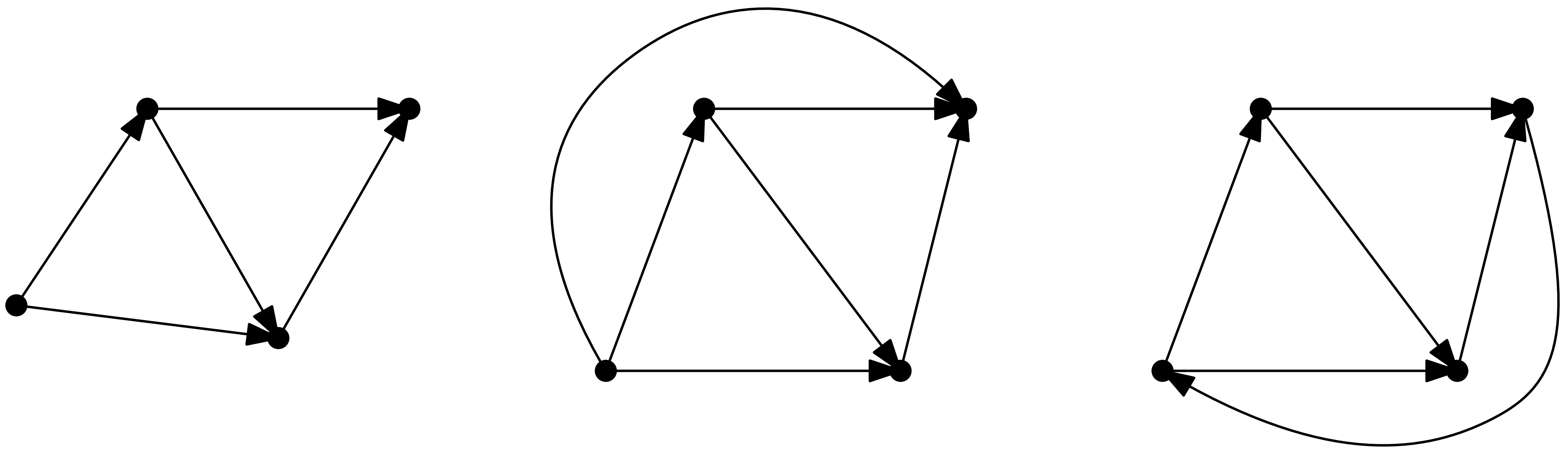

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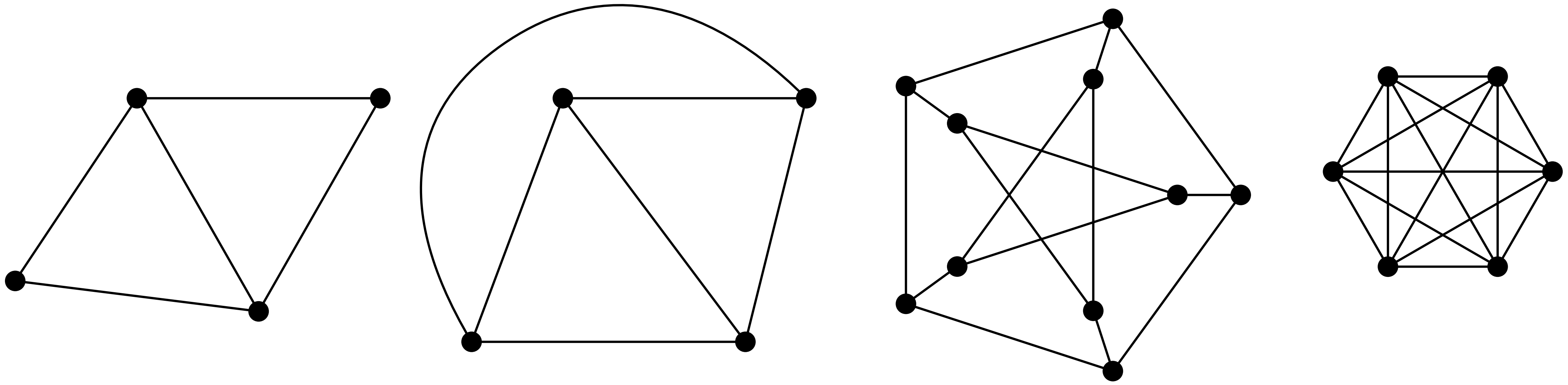
Step 4: $\mathrm{CSP}(G^*) = \mathrm{CSP}(\mathbb{N}; <)$. Acyclic digraphs are those with a topological ordering.



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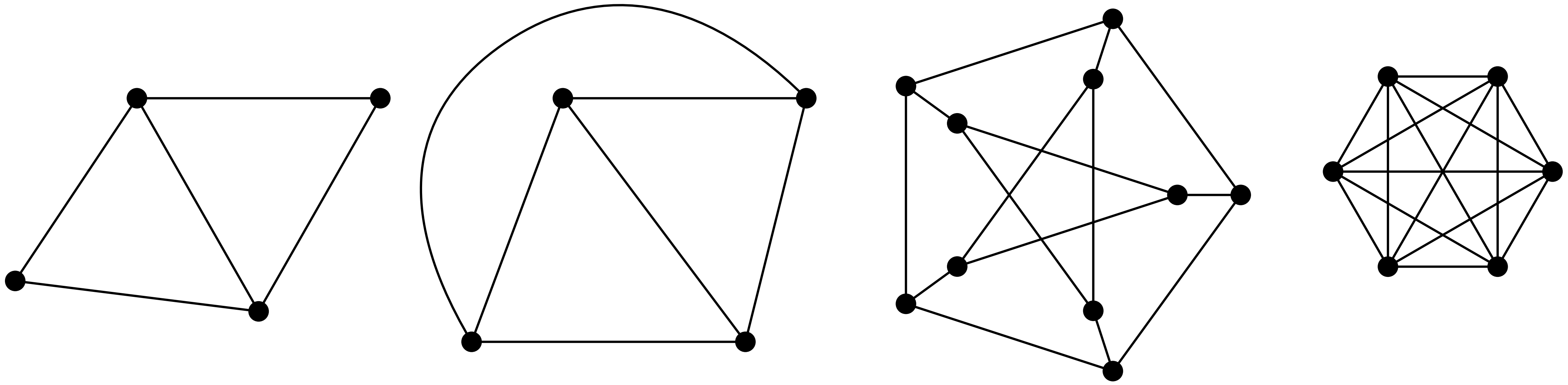

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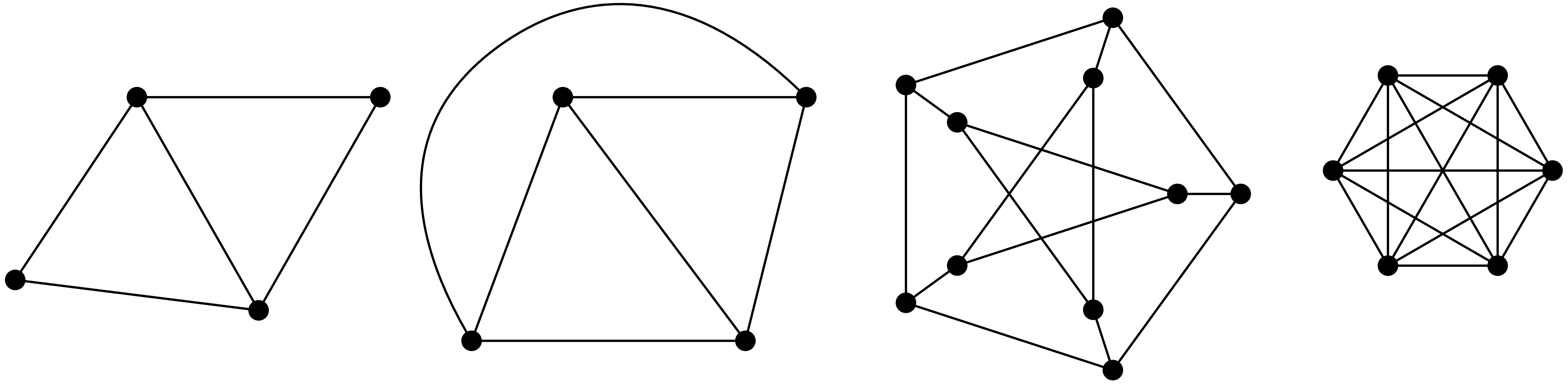
Template-free CSPs: Monochromatic triangle



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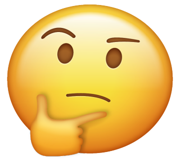

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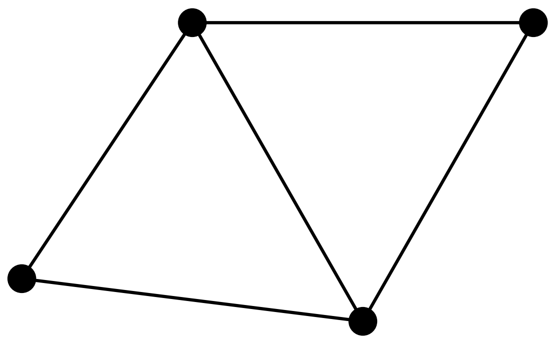
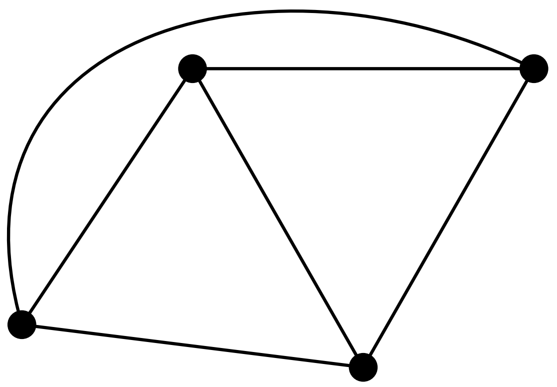
Observation. $\forall n \geq 1 \exists G$ triangle-free that cannot be partitioned into n independent sets.





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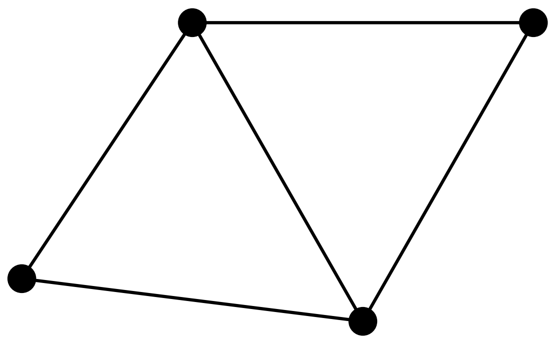

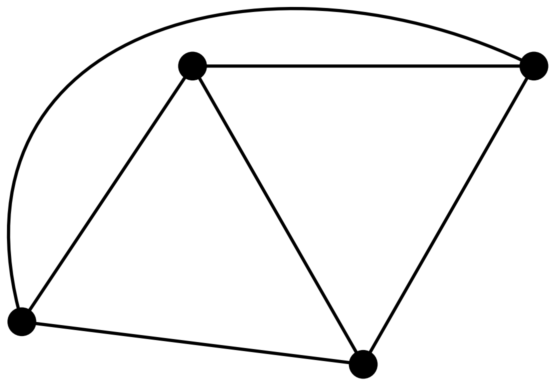

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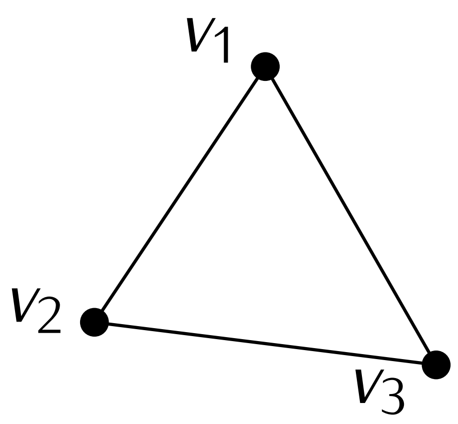

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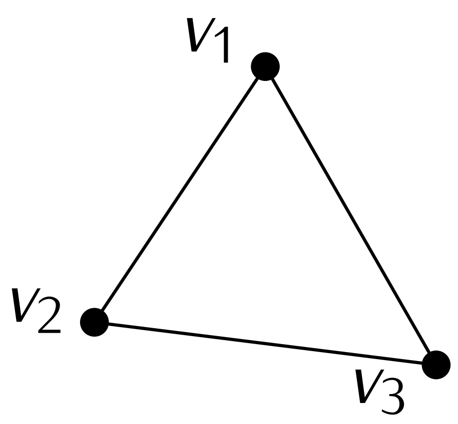

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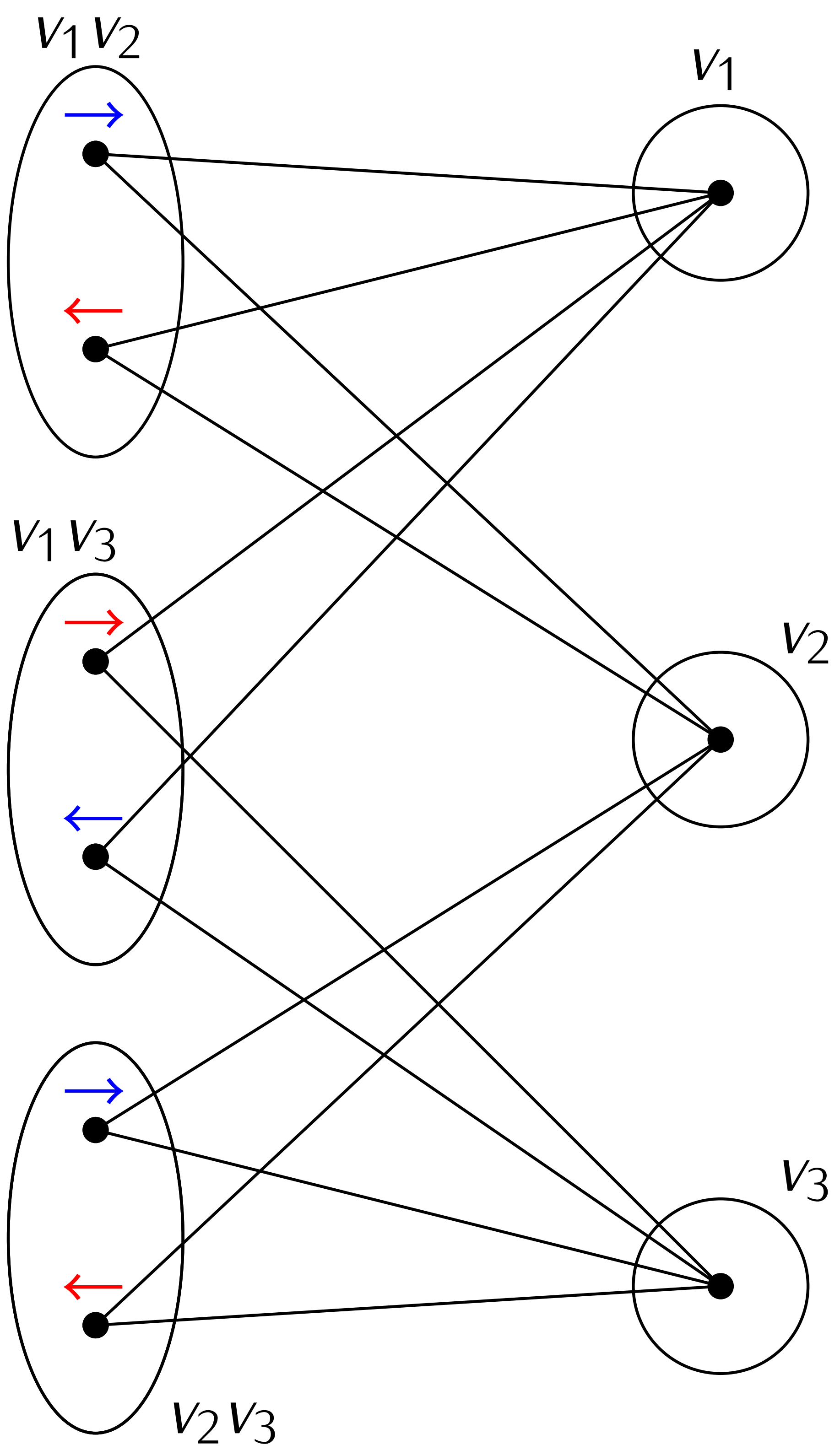

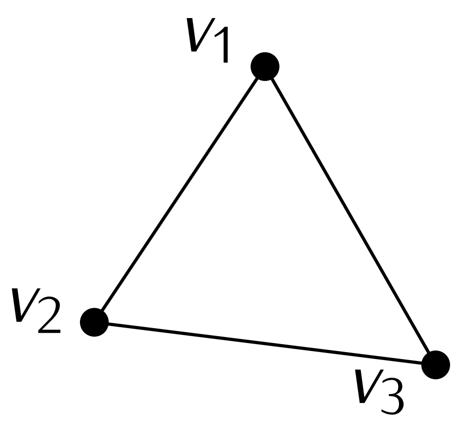

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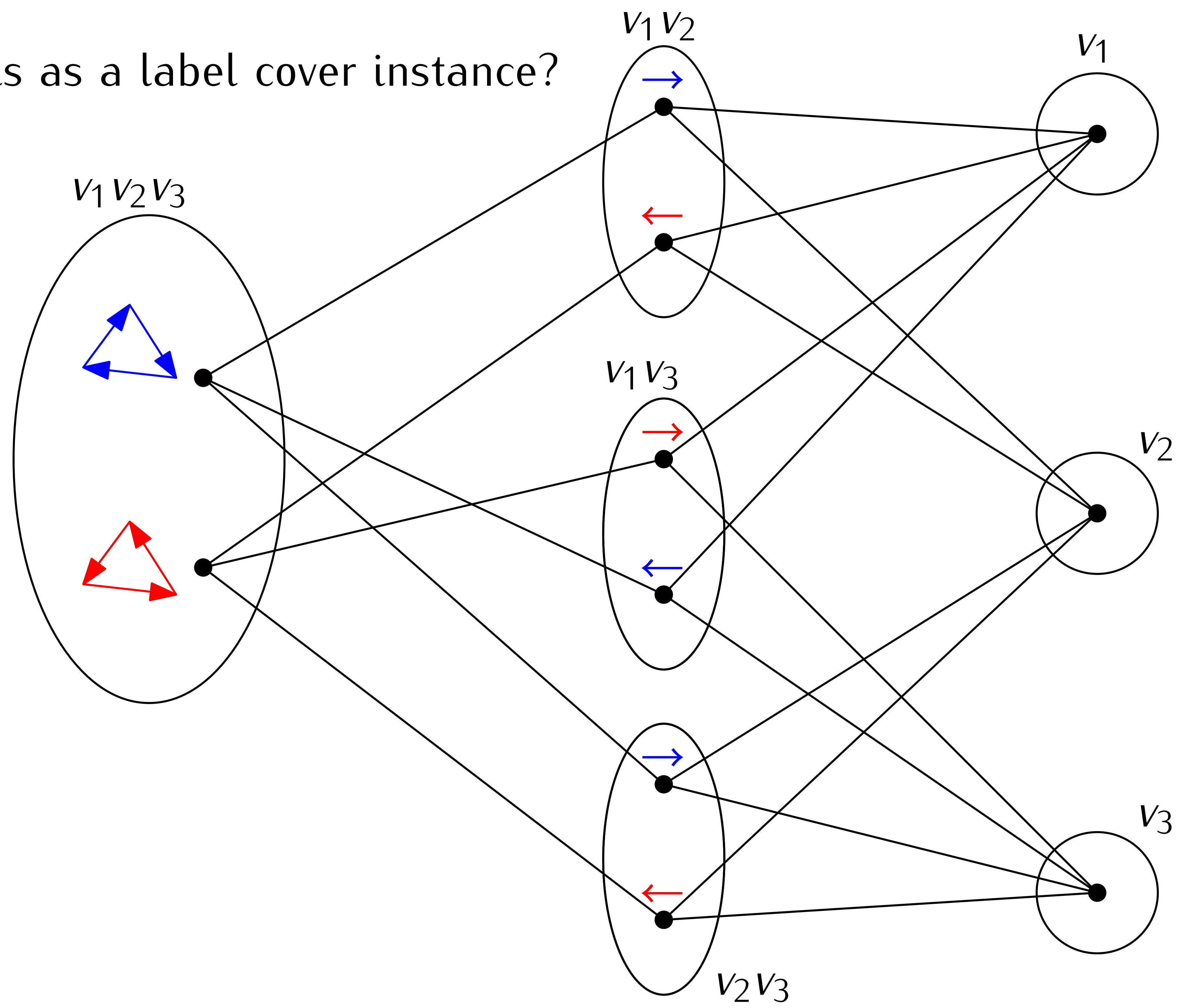

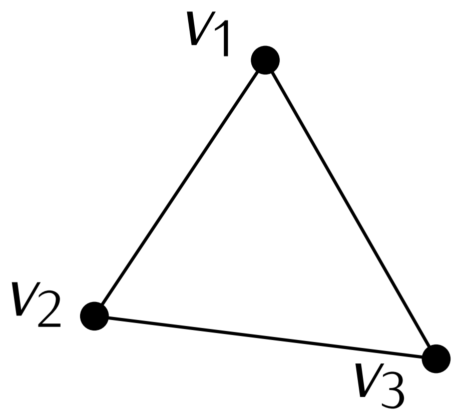

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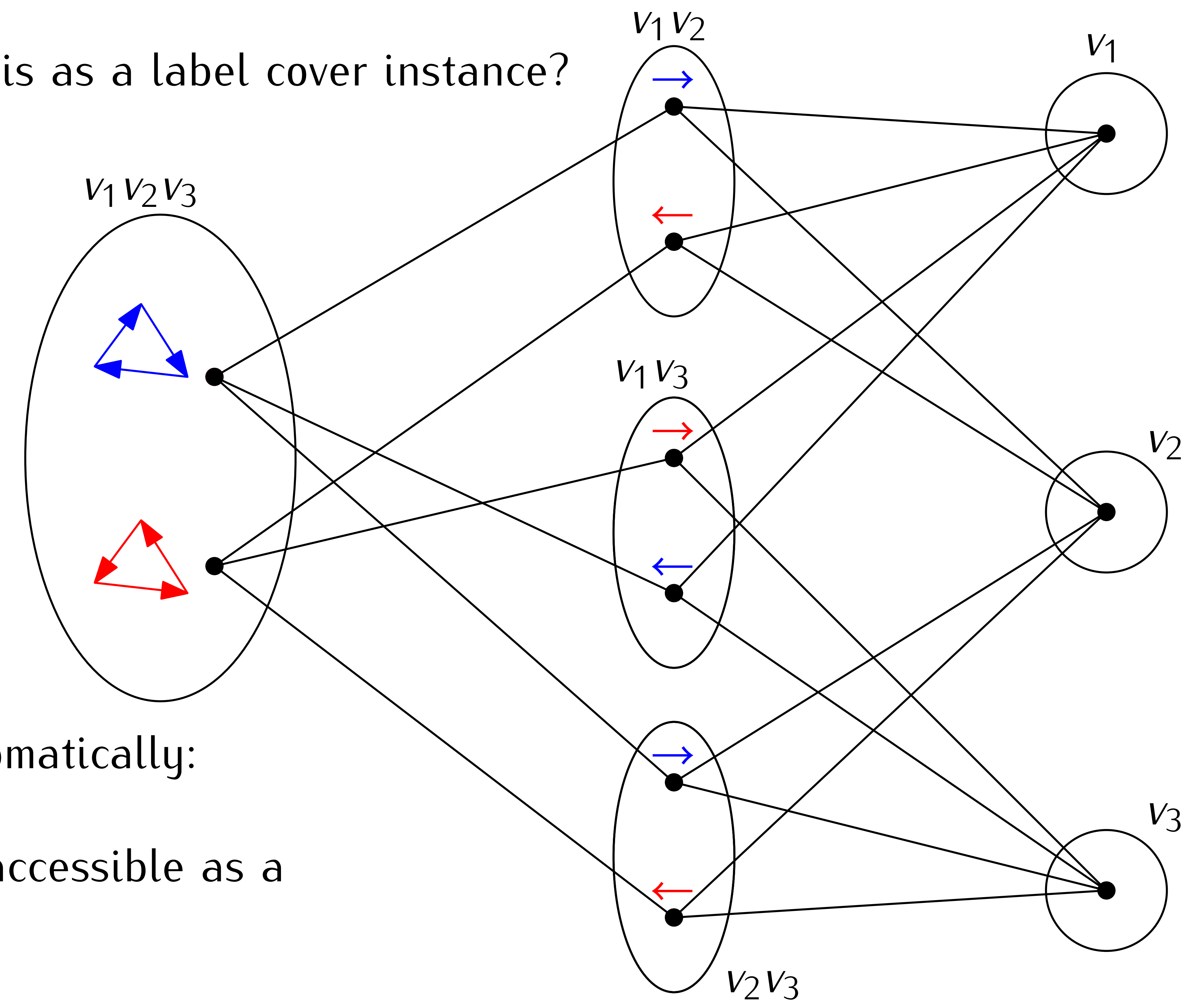

This potato is implied, added automatically:

- 1 constraint per triangle
- This type of constraint is not accessible as a "real" ternary constraint

All problems so far:

- label cover over a finite set
- some structural constraints are "automatically" added
- structural constraints are not part of the "normal" constraints
- scope of structural constraints is of bounded size
- Except for 3-coloring, not CSP($\mathbb{A}$) for a finite $\mathbb{A}$, yet CSPs of bad structures (disjoint unions)

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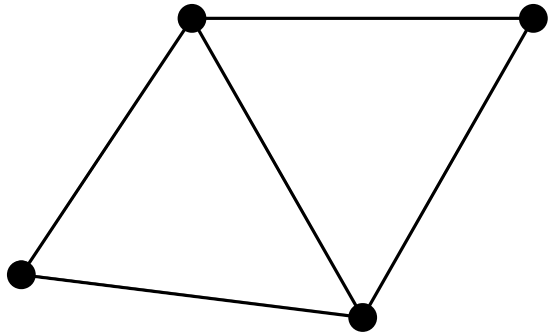
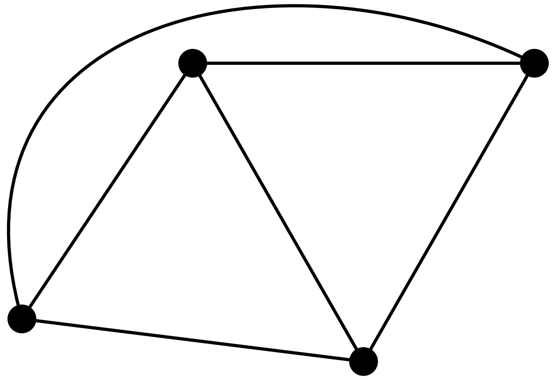
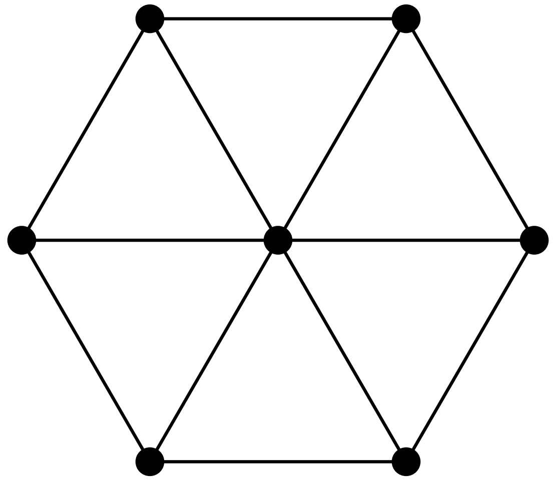
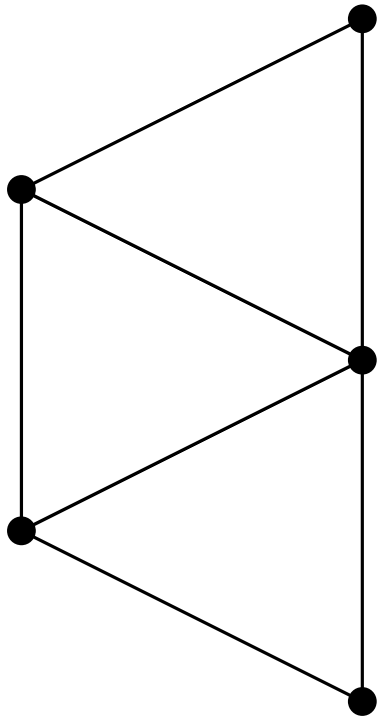
What templates to study these problems?
How to study the complexity of the CSPs using these templates?

Orbit-finiteness,
reducts of finitely bounded homogeneous structures









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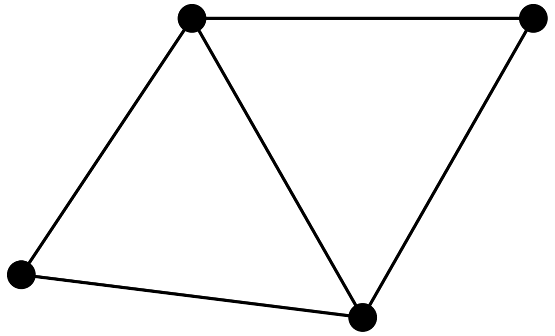
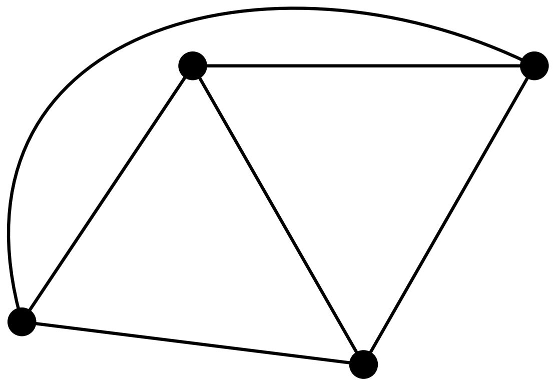
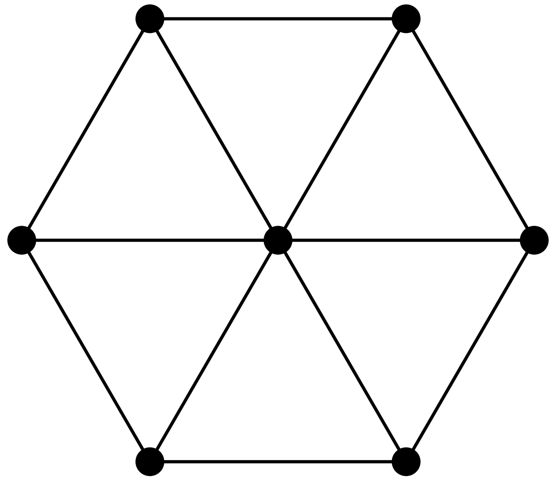
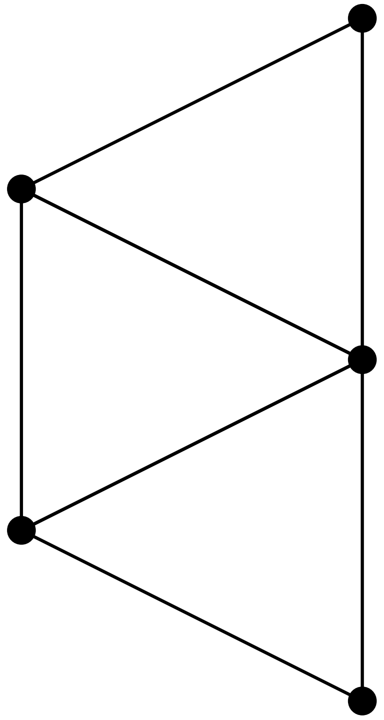
This step is wasteful in terms of vertices and edges

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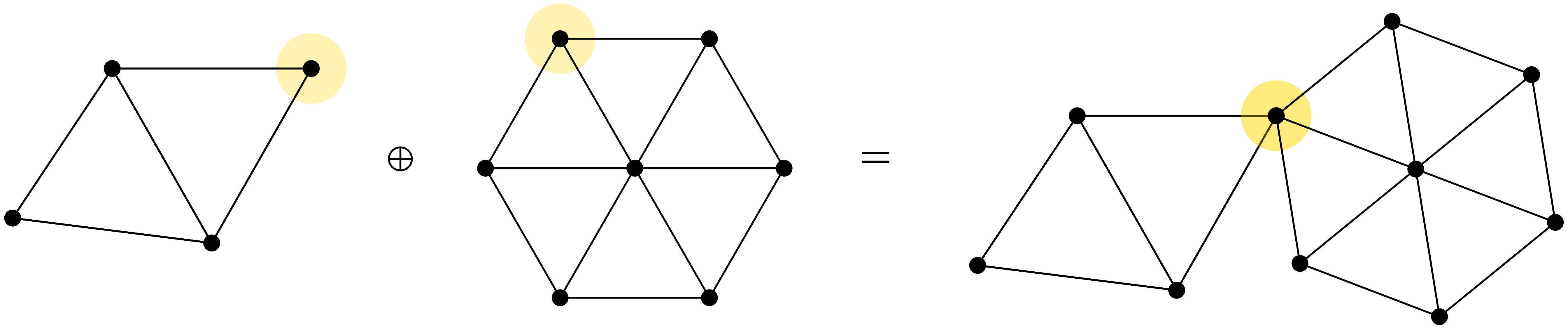

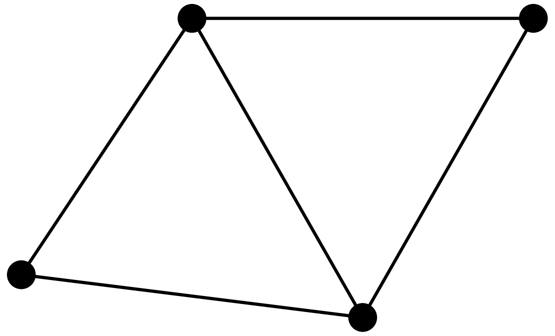
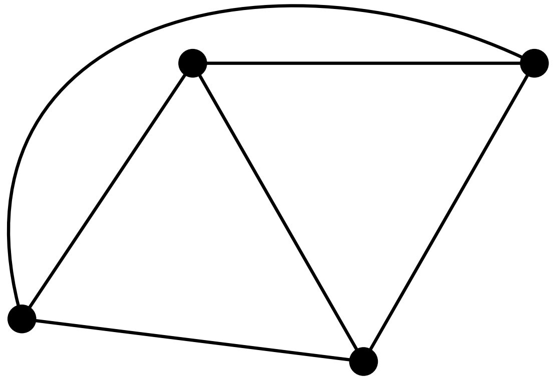
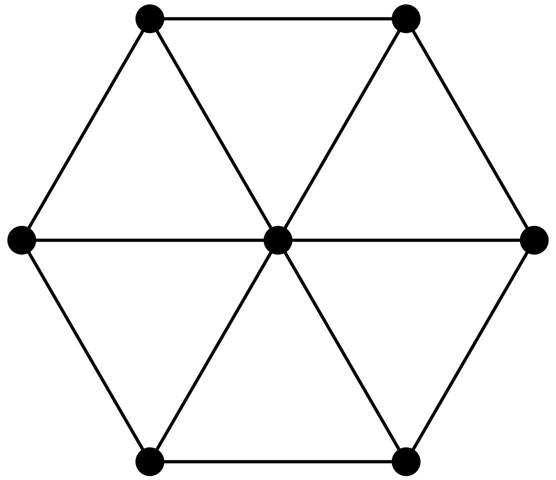
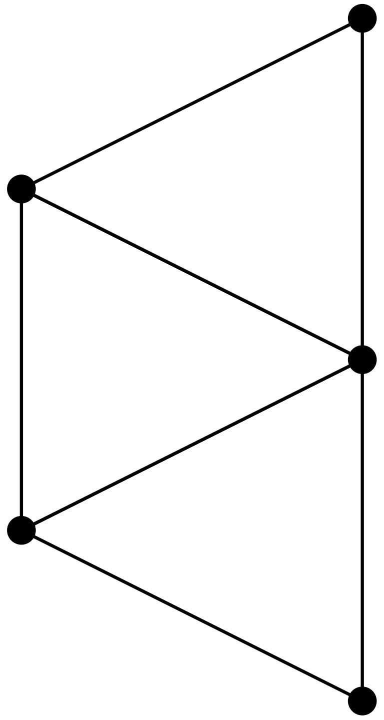

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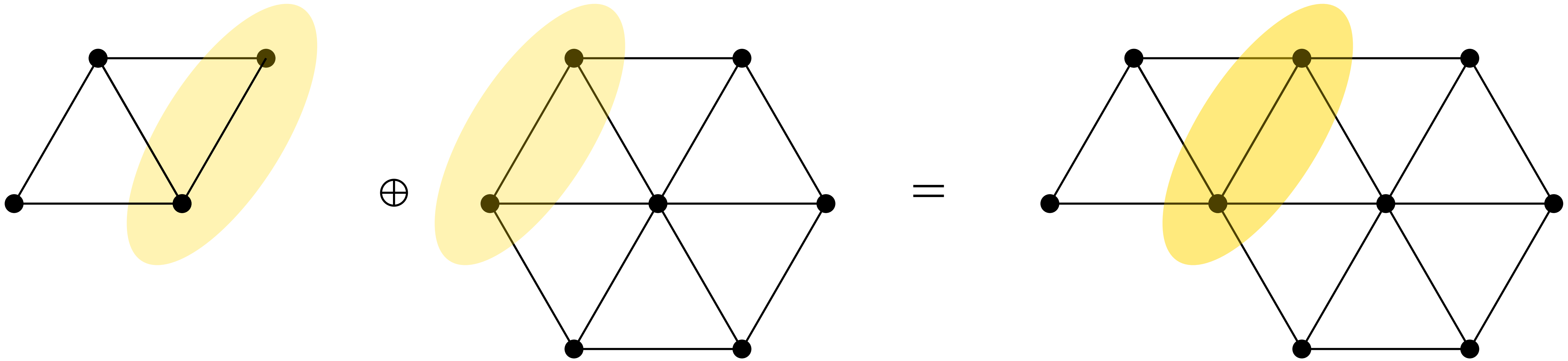

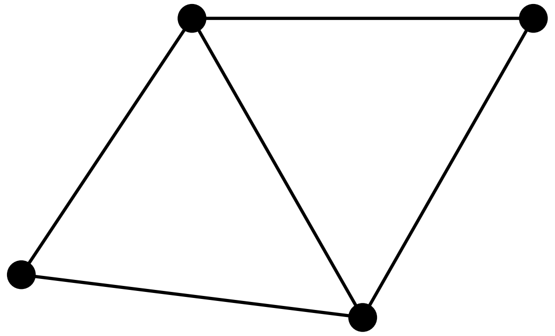

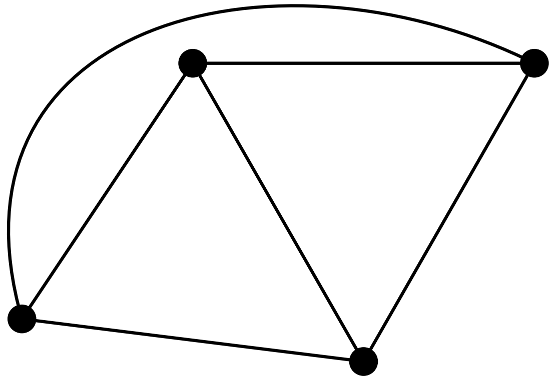

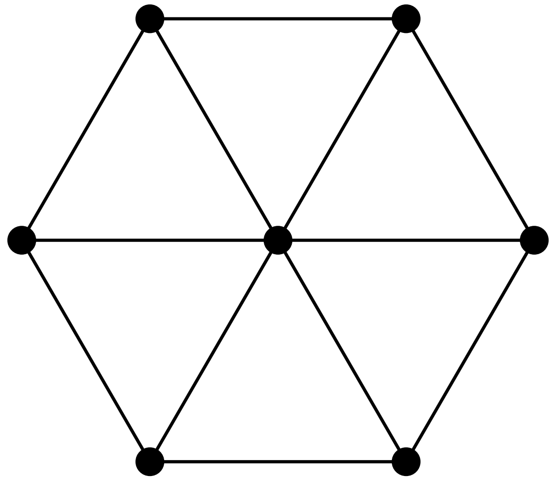

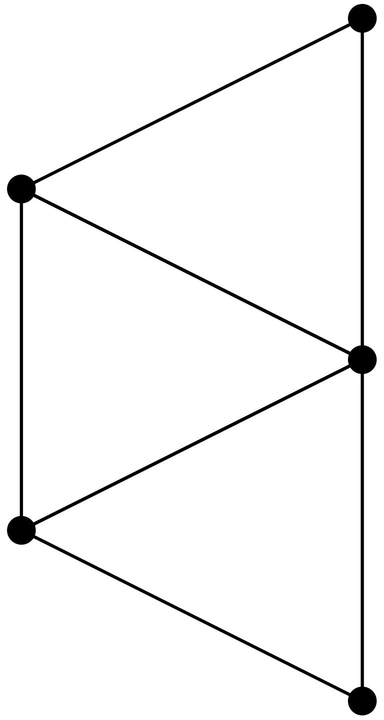

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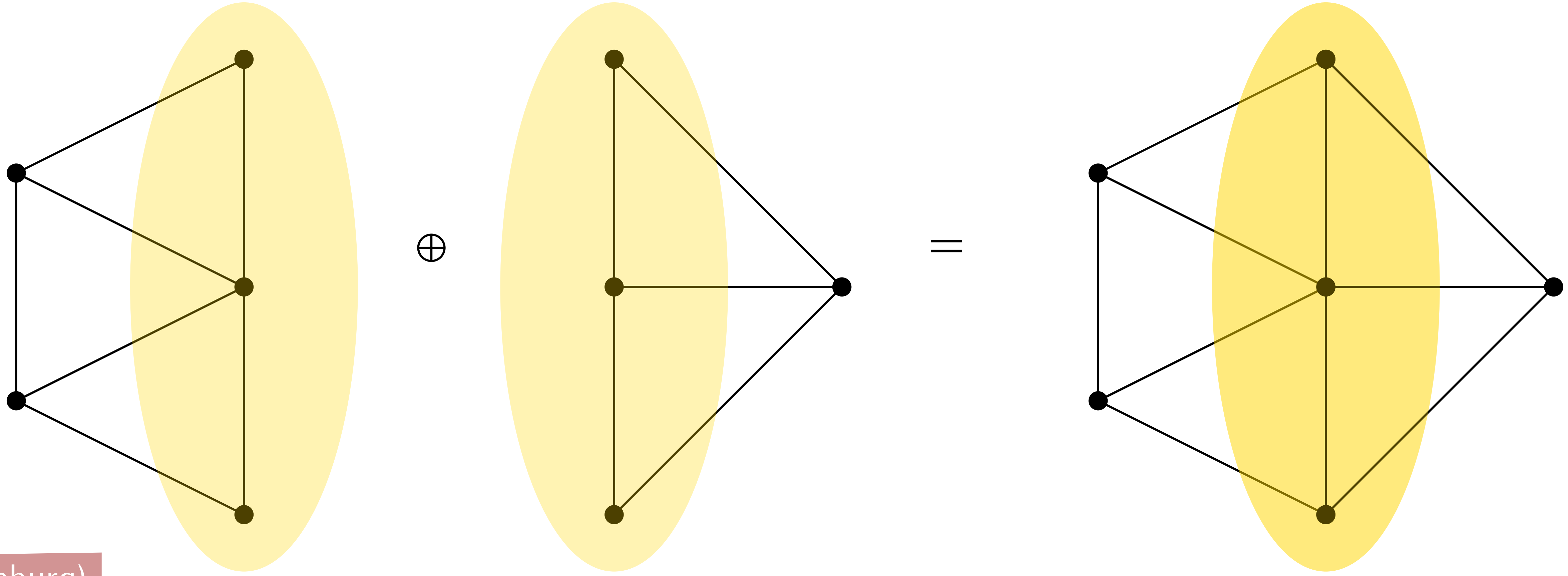

structure with two edge relations

Define $\mathcal{V} = \{G * \vec{G} \mid \vec{G}$ cyclic orientation of $G\}$ (where $(V;E) * (V;A) = (V;E,A)$)

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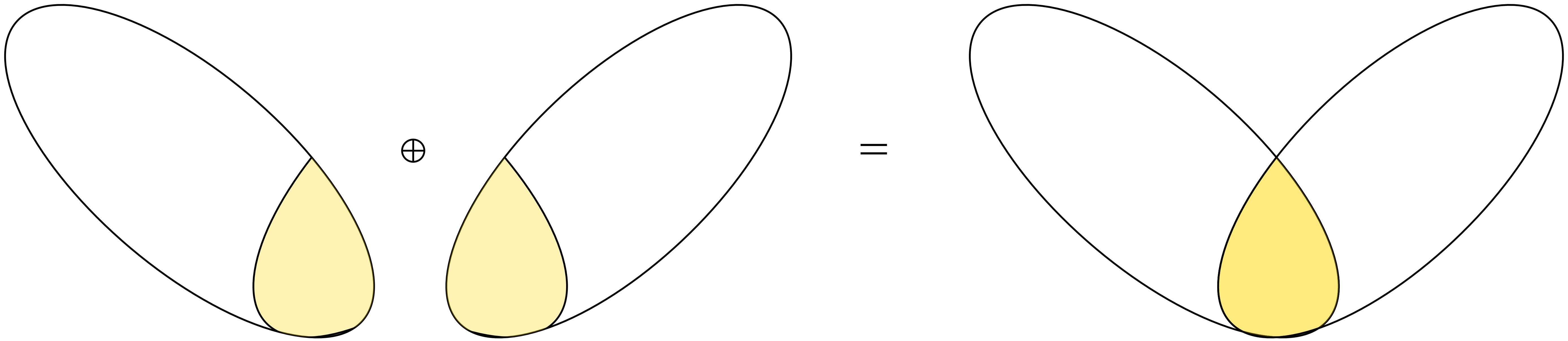

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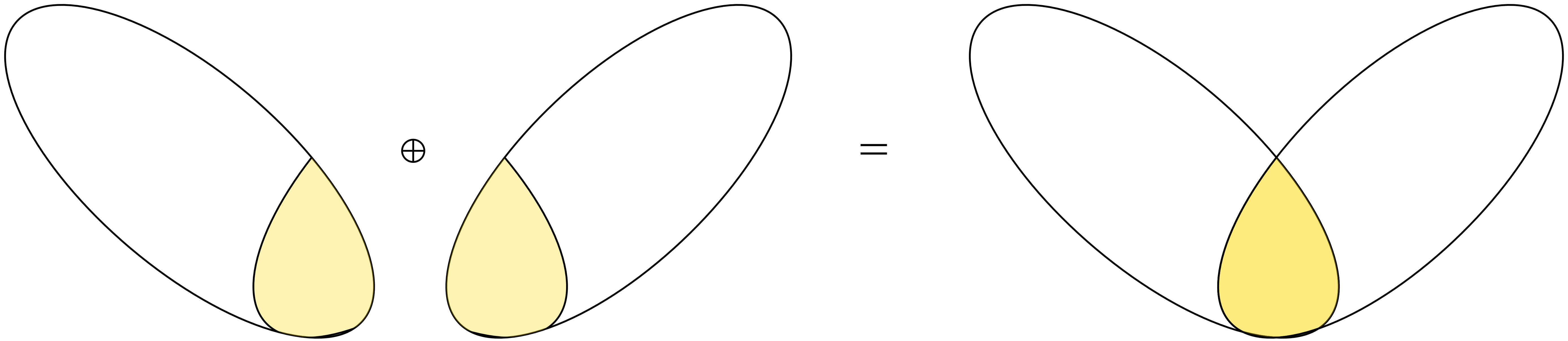

Theorem (Fraïssé). If $\mathcal{V}$ is closed under substructures and has the amalgamation property, there exists G^* such that:

- The finite substructures of G^* are exactly those in $\mathcal{V}$
- For any isomorphism $f\colon G \to G'$ between finite substructures of G^*, there exists $\alpha \in \mathrm{Aut}(G^*)$ s.t. $\alpha|_G = f$.

G^* is unique up to isomorphism: it is the Fraïssé limit of $\mathcal{V}$.

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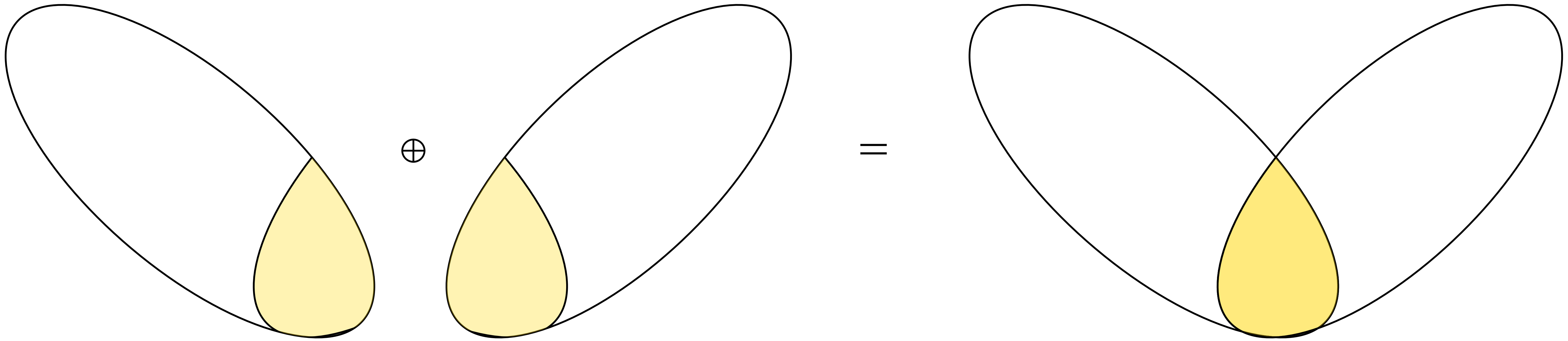

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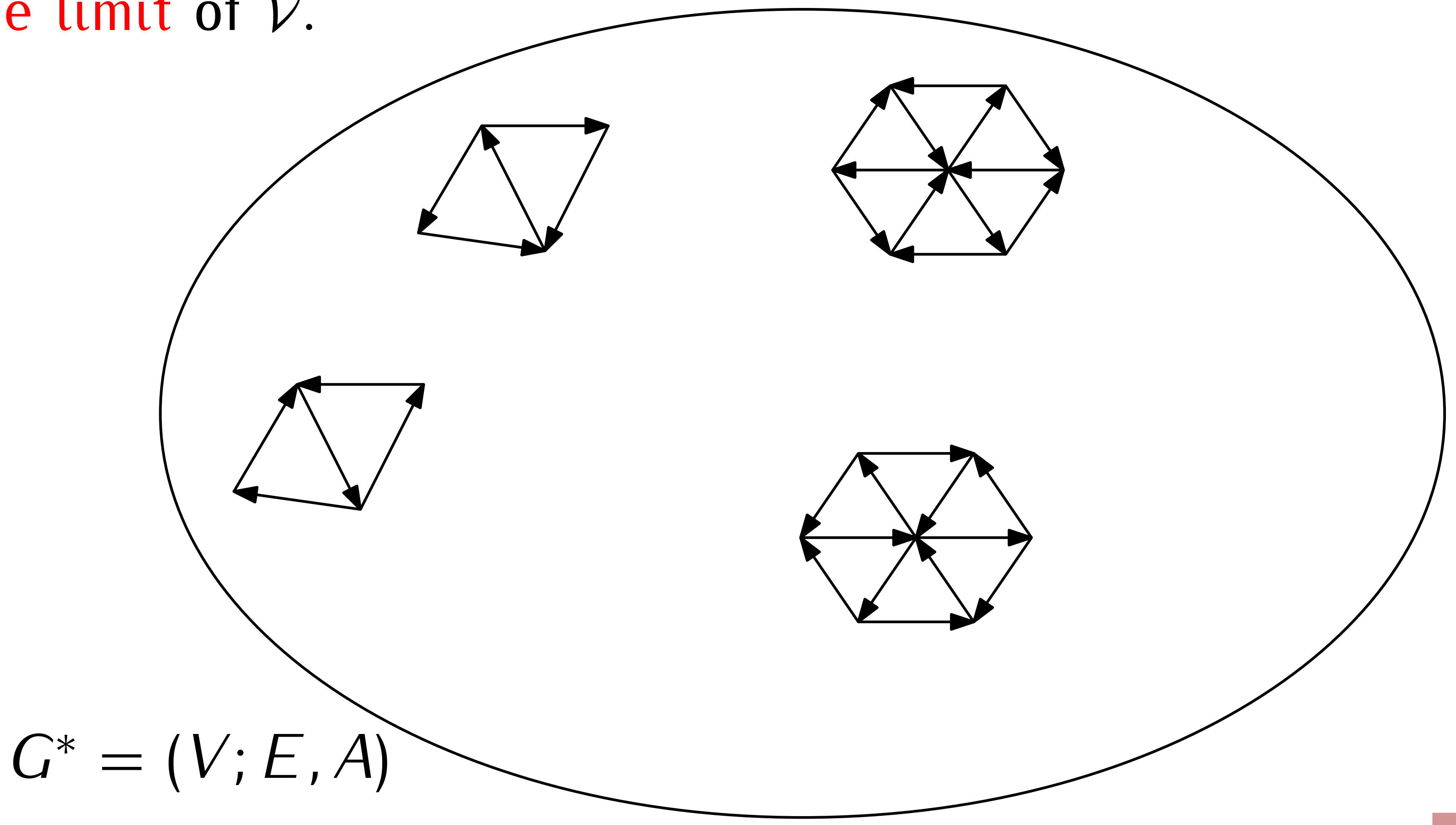

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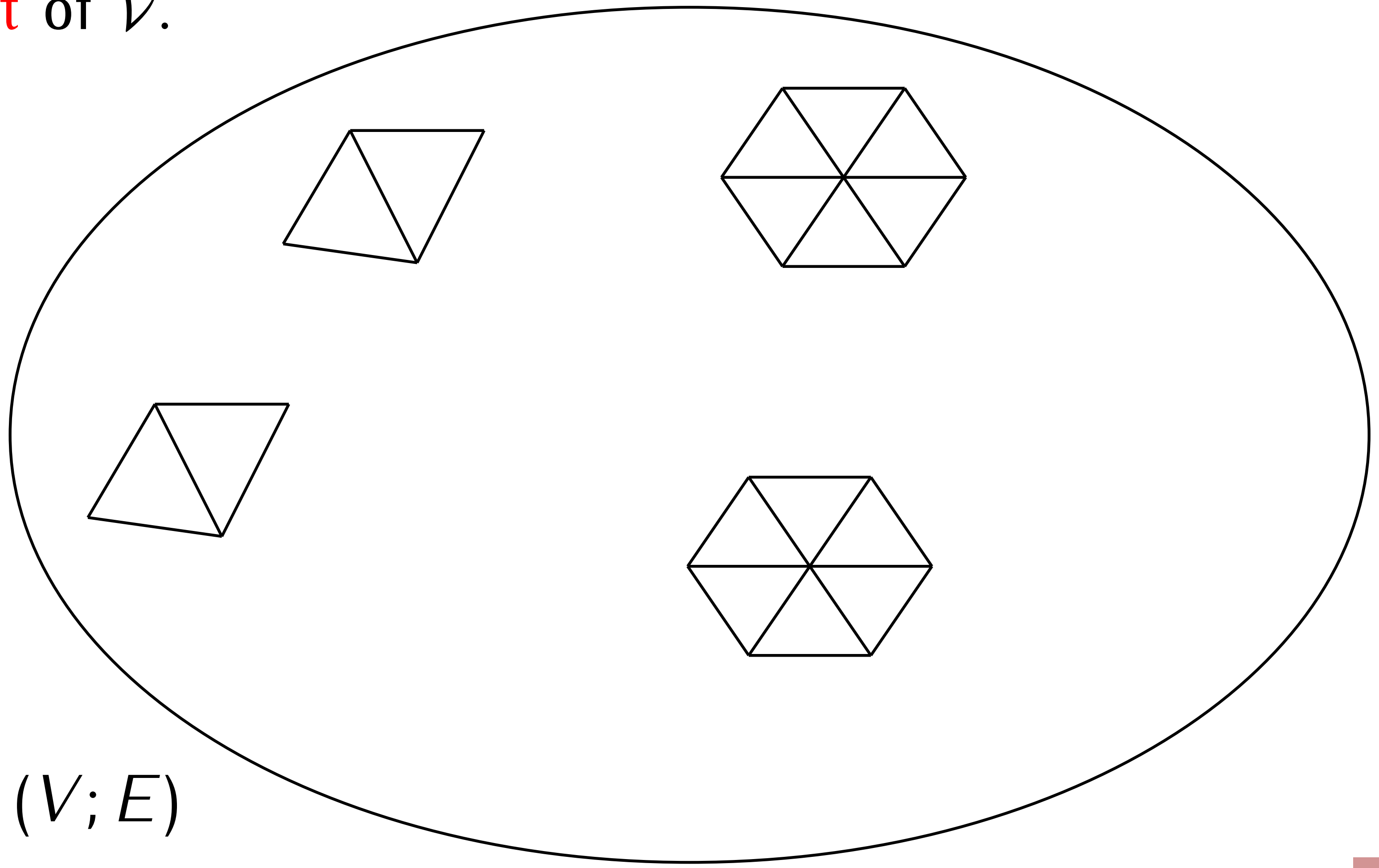

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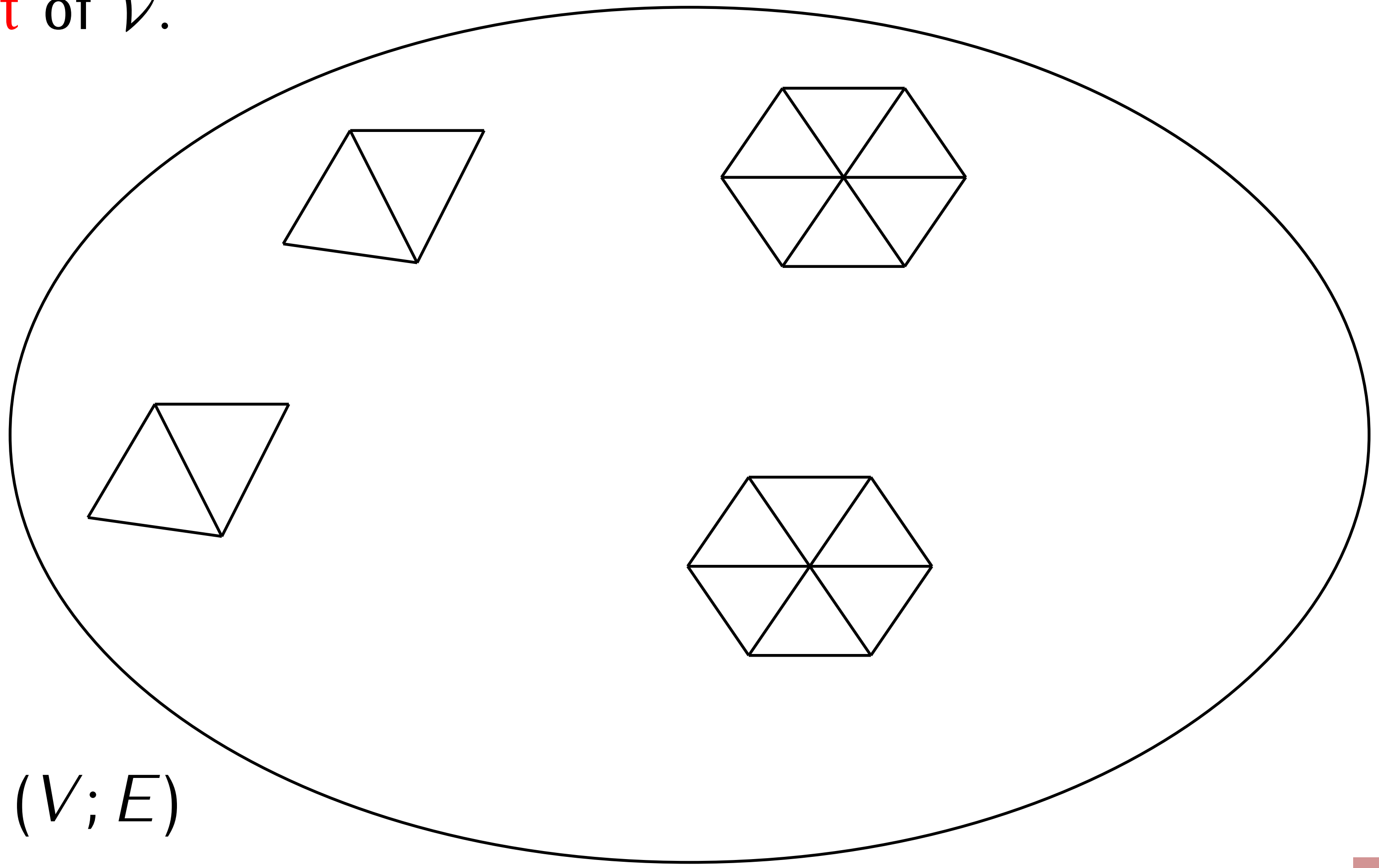

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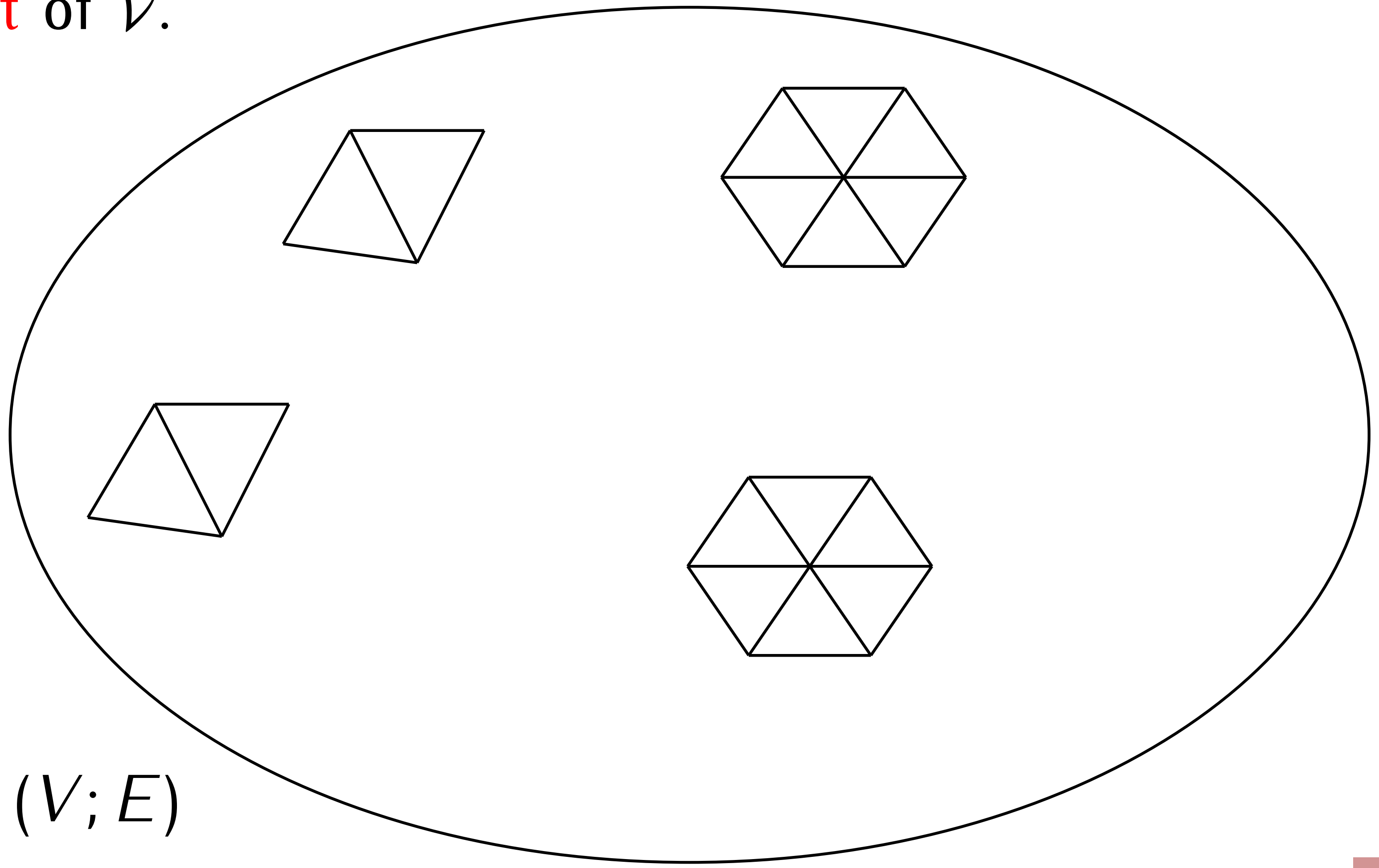

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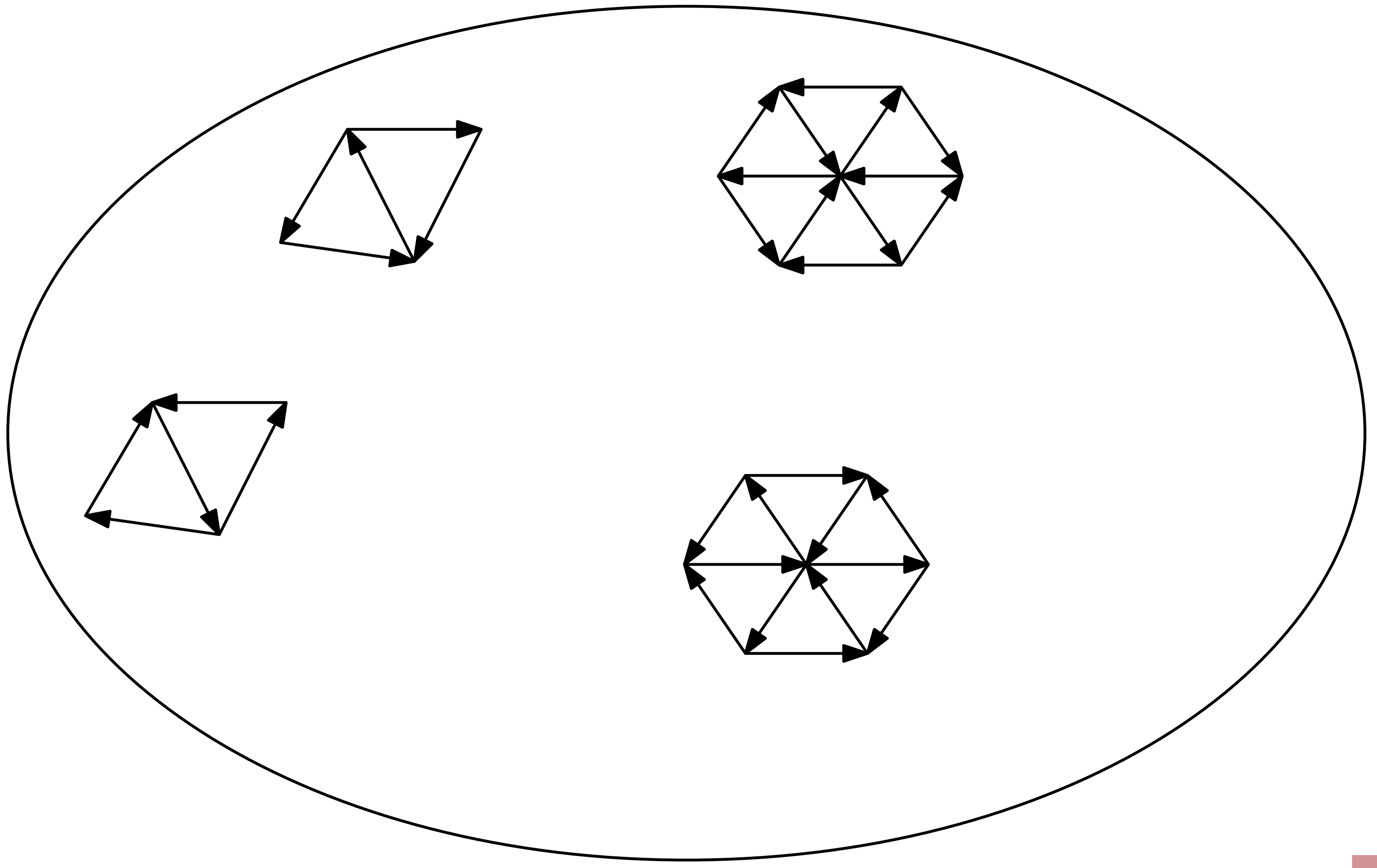
Why is G^* "better" than $\bigsqcup_{G\in\mathcal{C}} G$?



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Homogeneity

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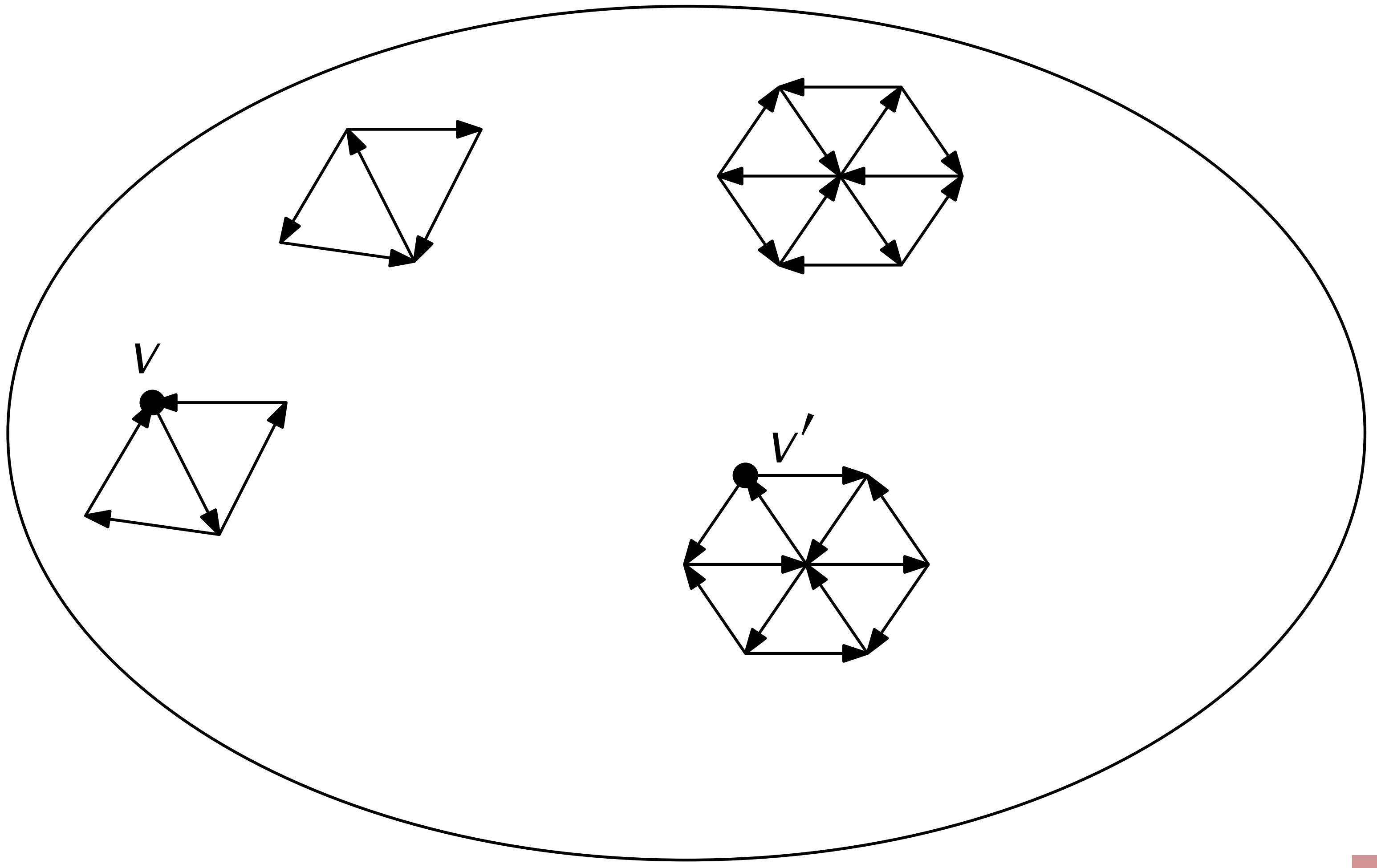

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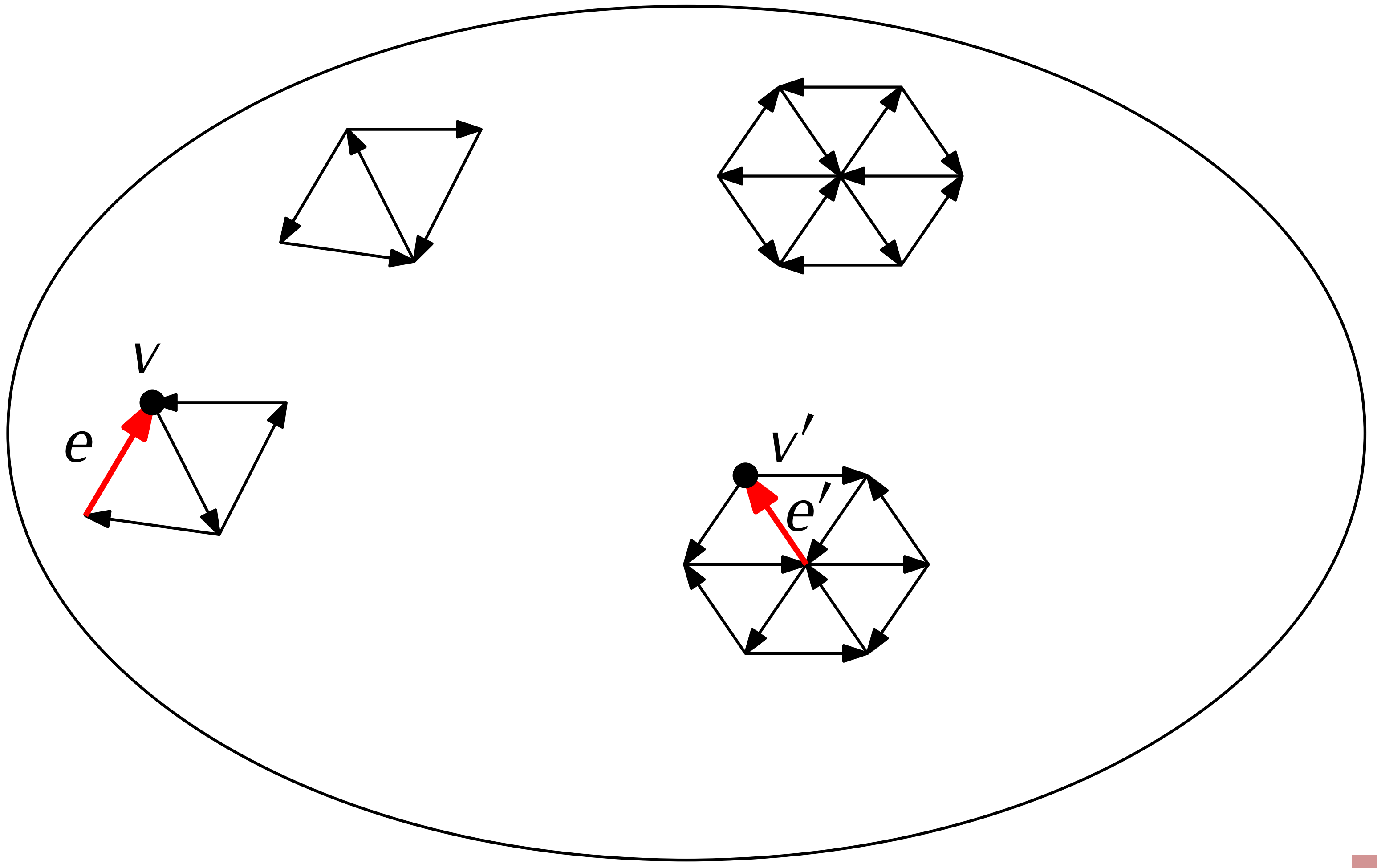

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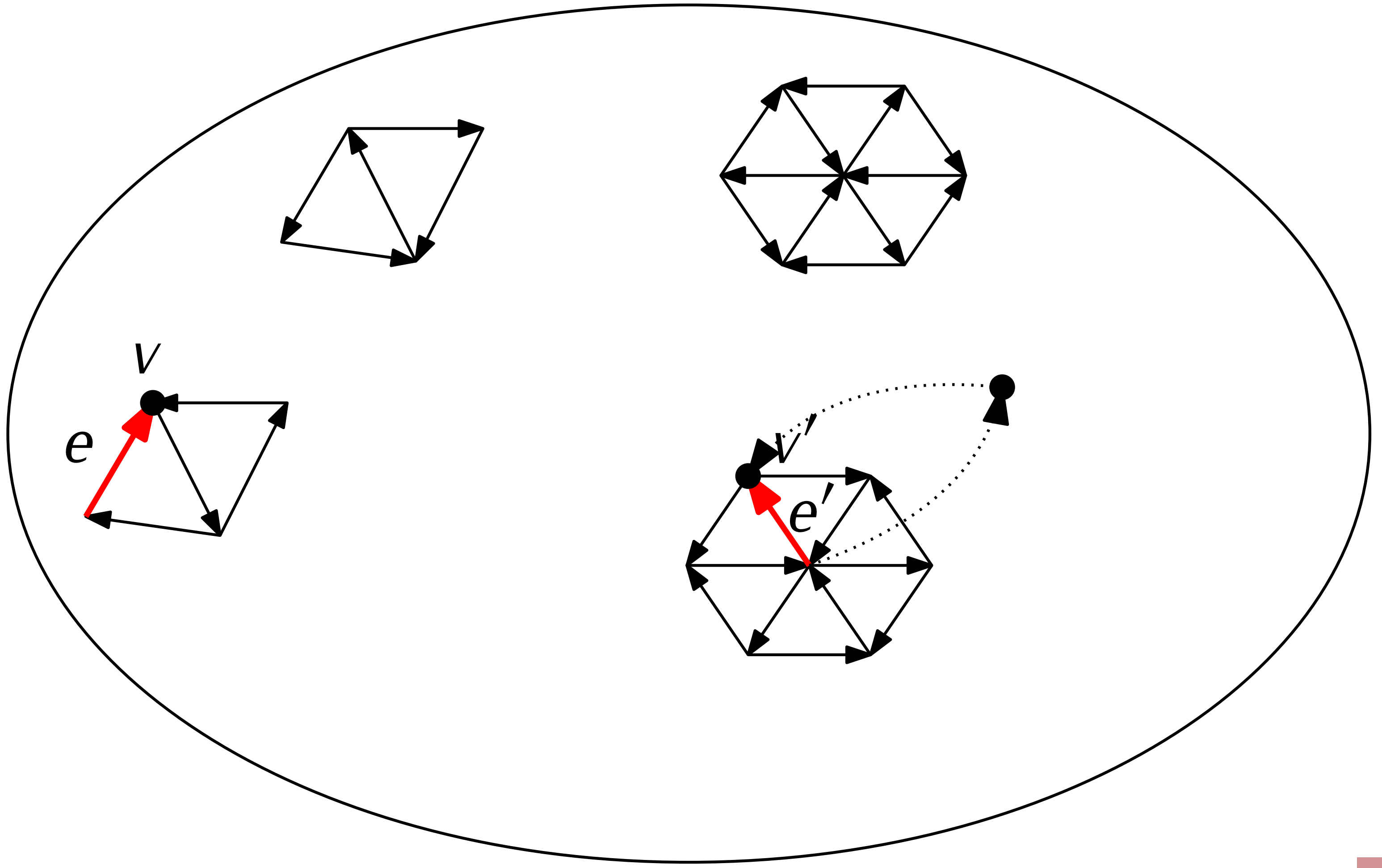

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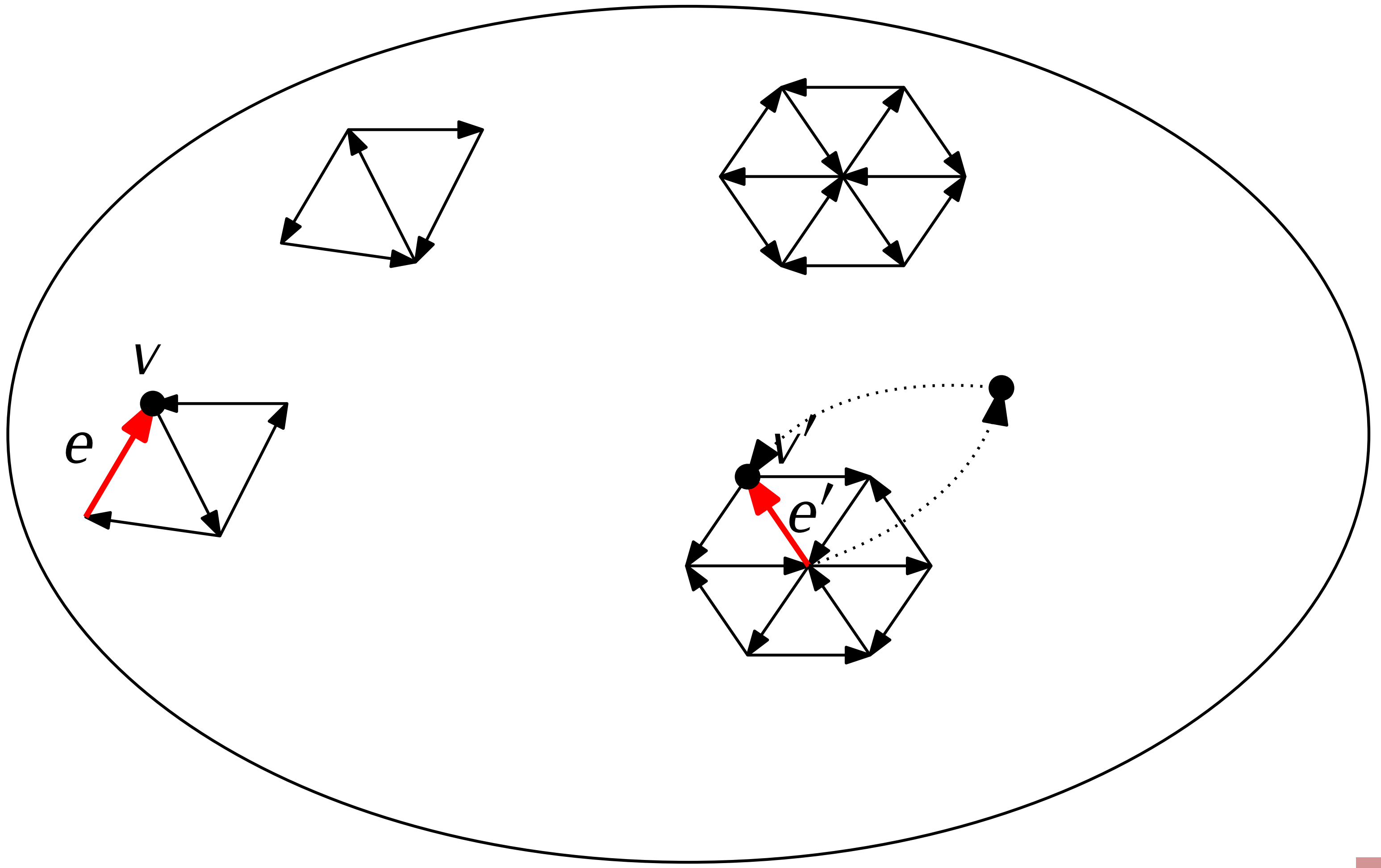
Finitely many directed graphs on n vertices
$\rightsquigarrow \mathrm{Aut}(G^*) \curvearrowright V^n$ has finitely many orbits

$\mathrm{Aut}(G^*)$ is oligomorphic

G^* is ω-categorical



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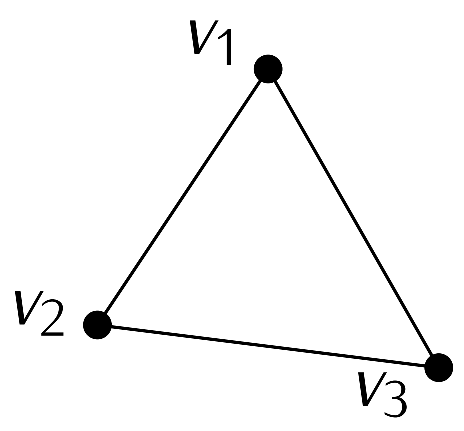
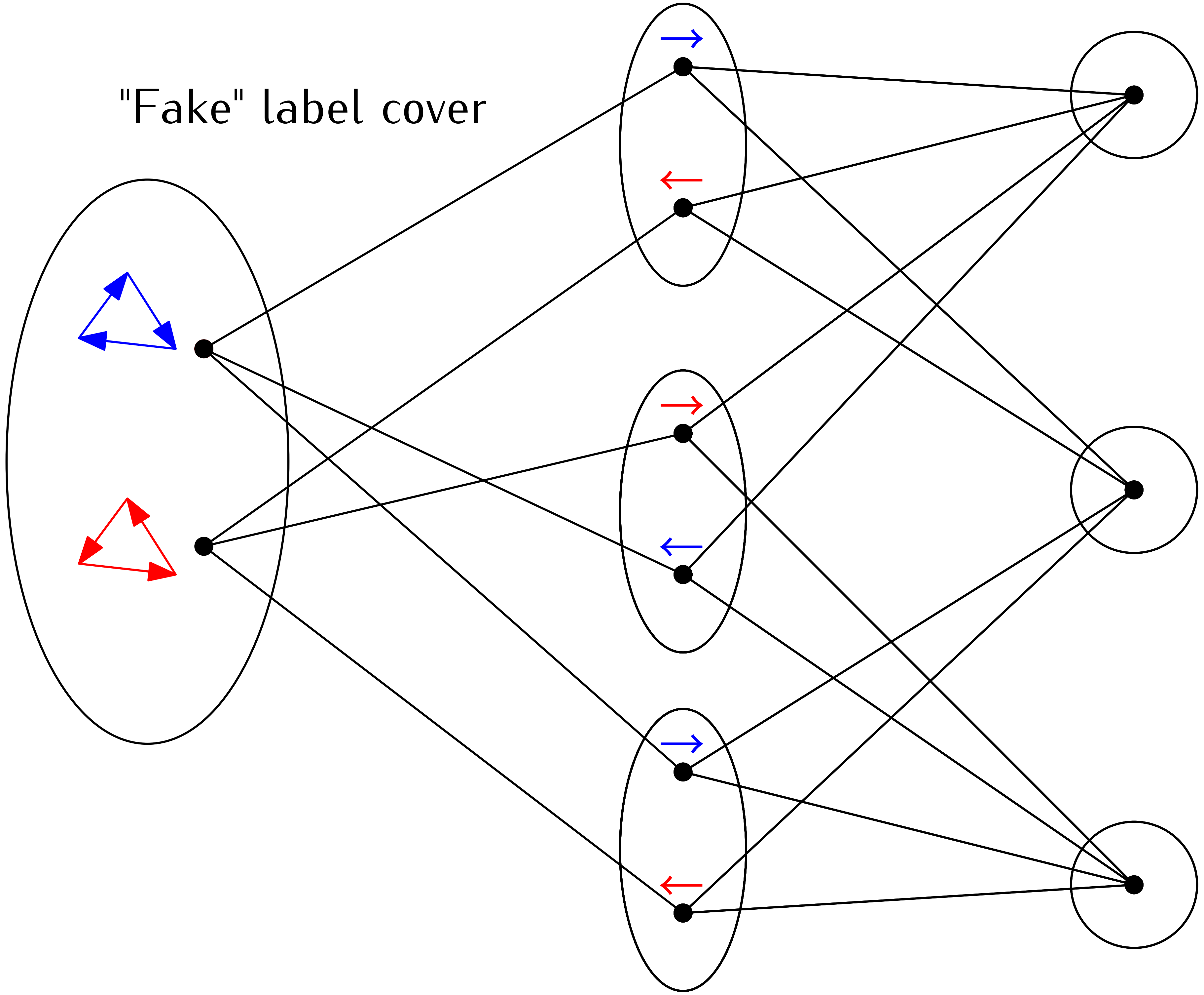
CSP$(V; E)$ as label cover



"Fake" label cover



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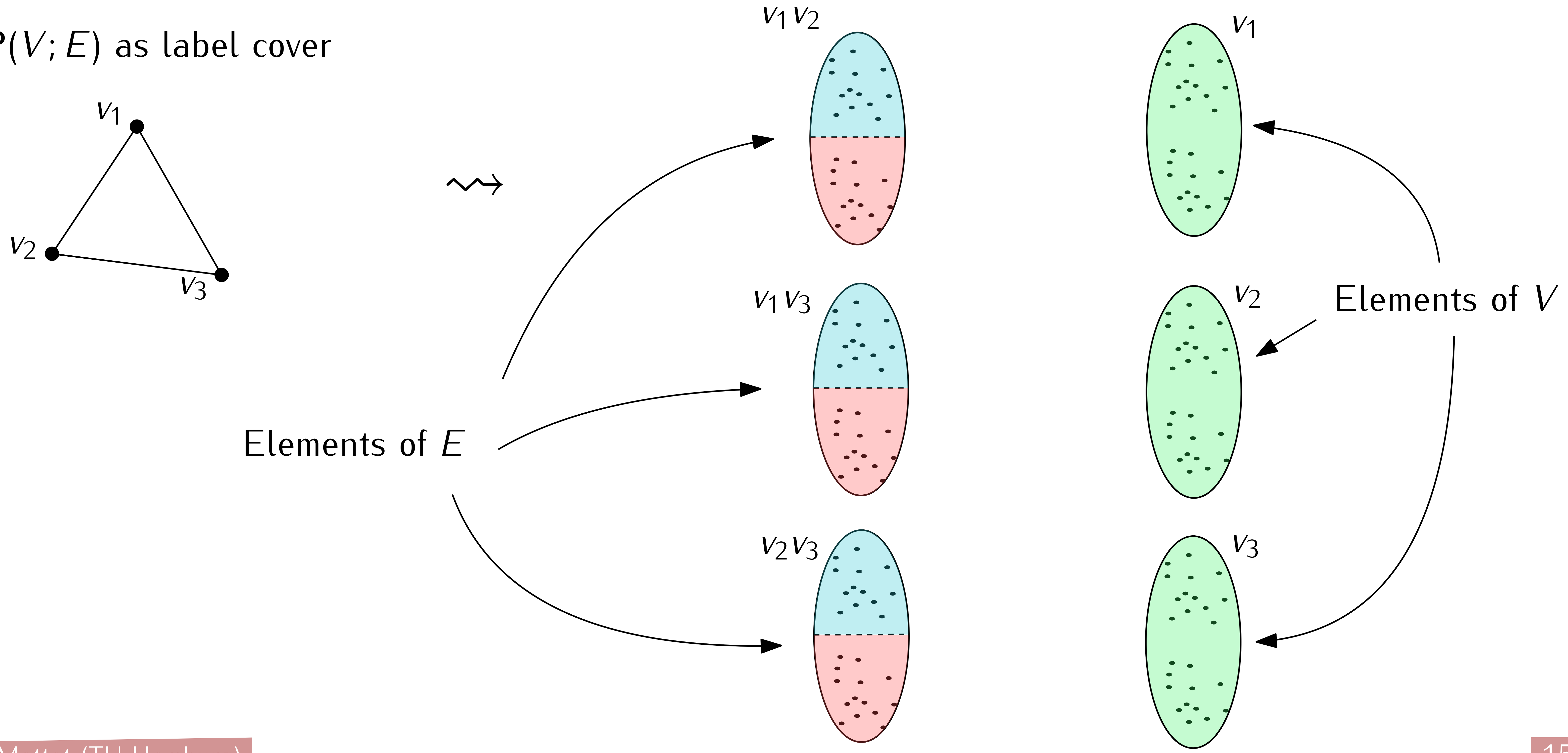

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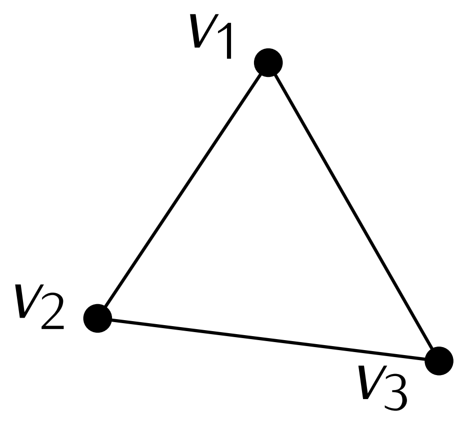
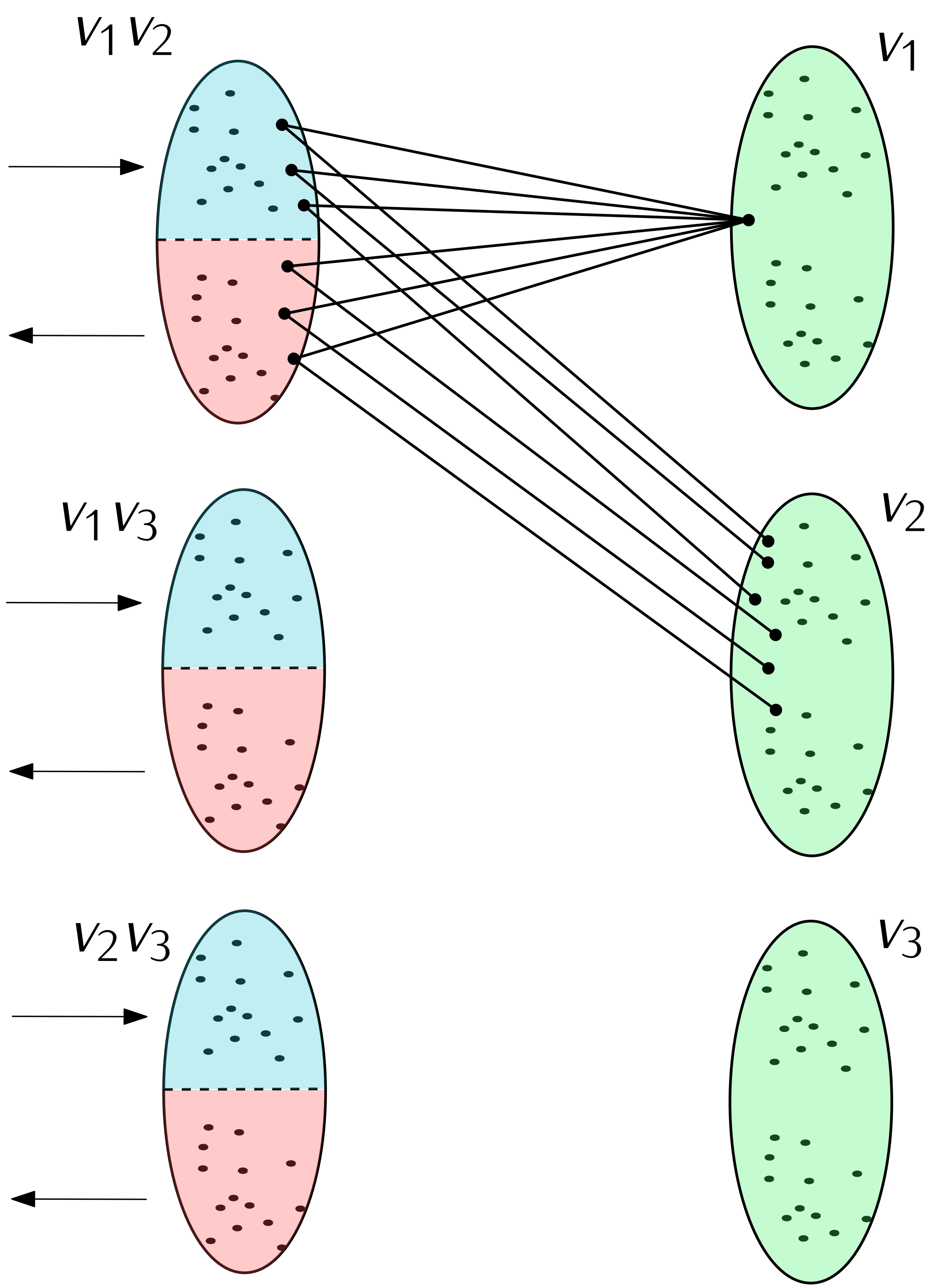


$\rightsquigarrow$



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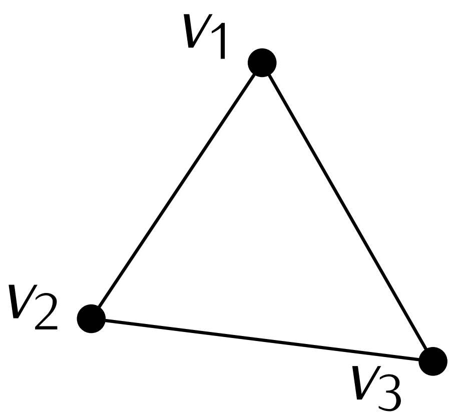

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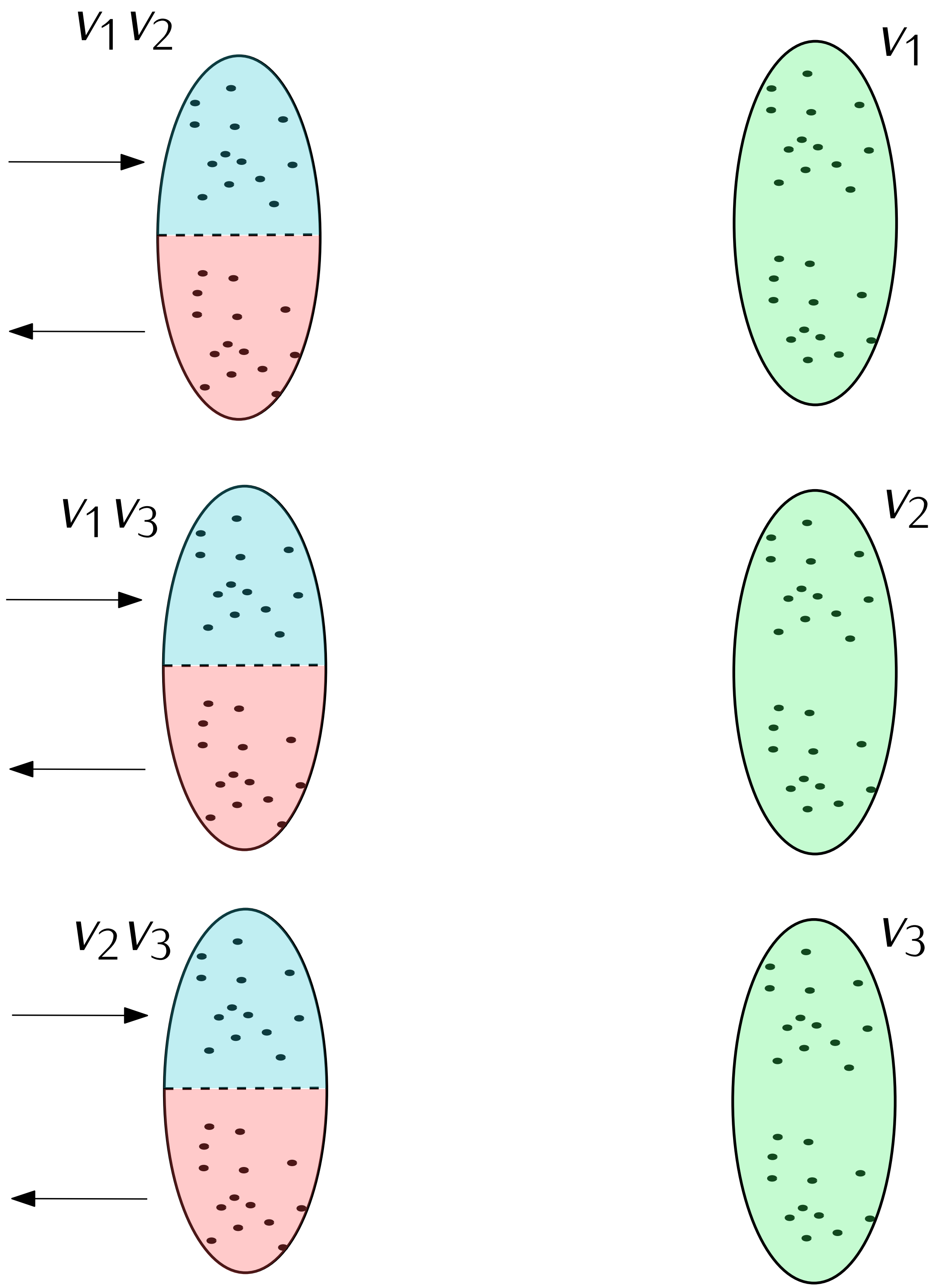

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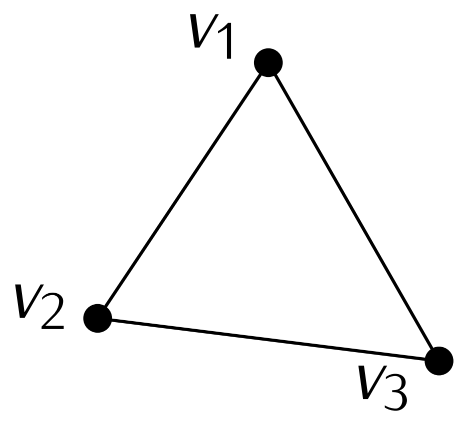

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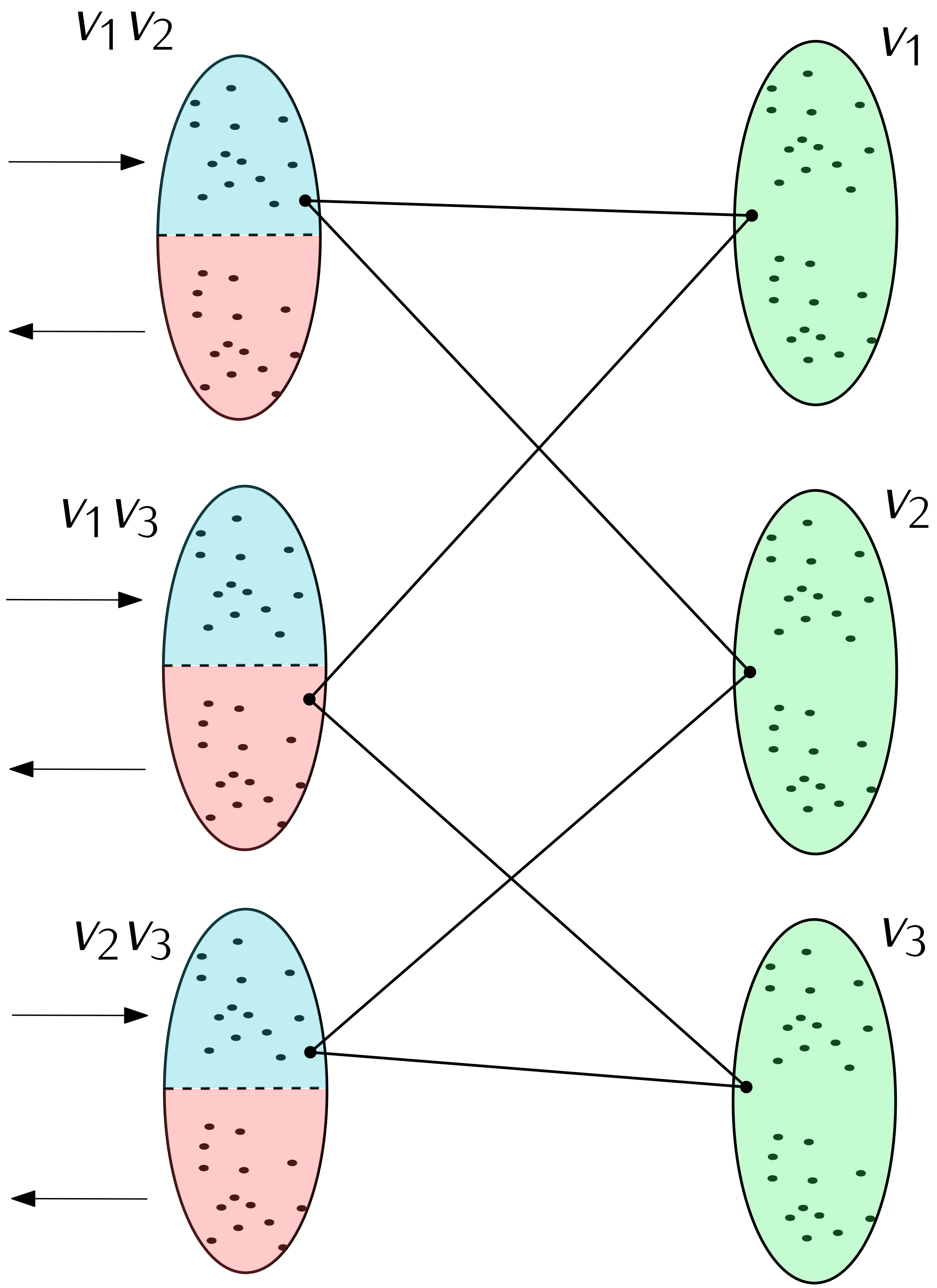

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Template for cyclic orientability

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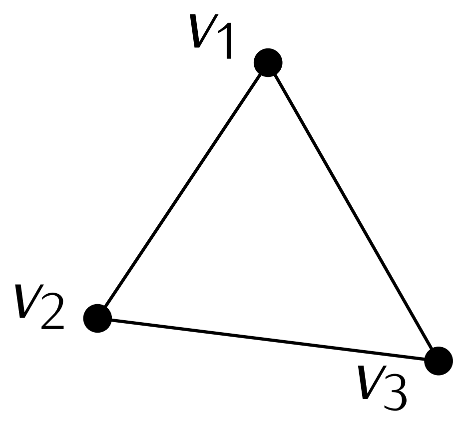

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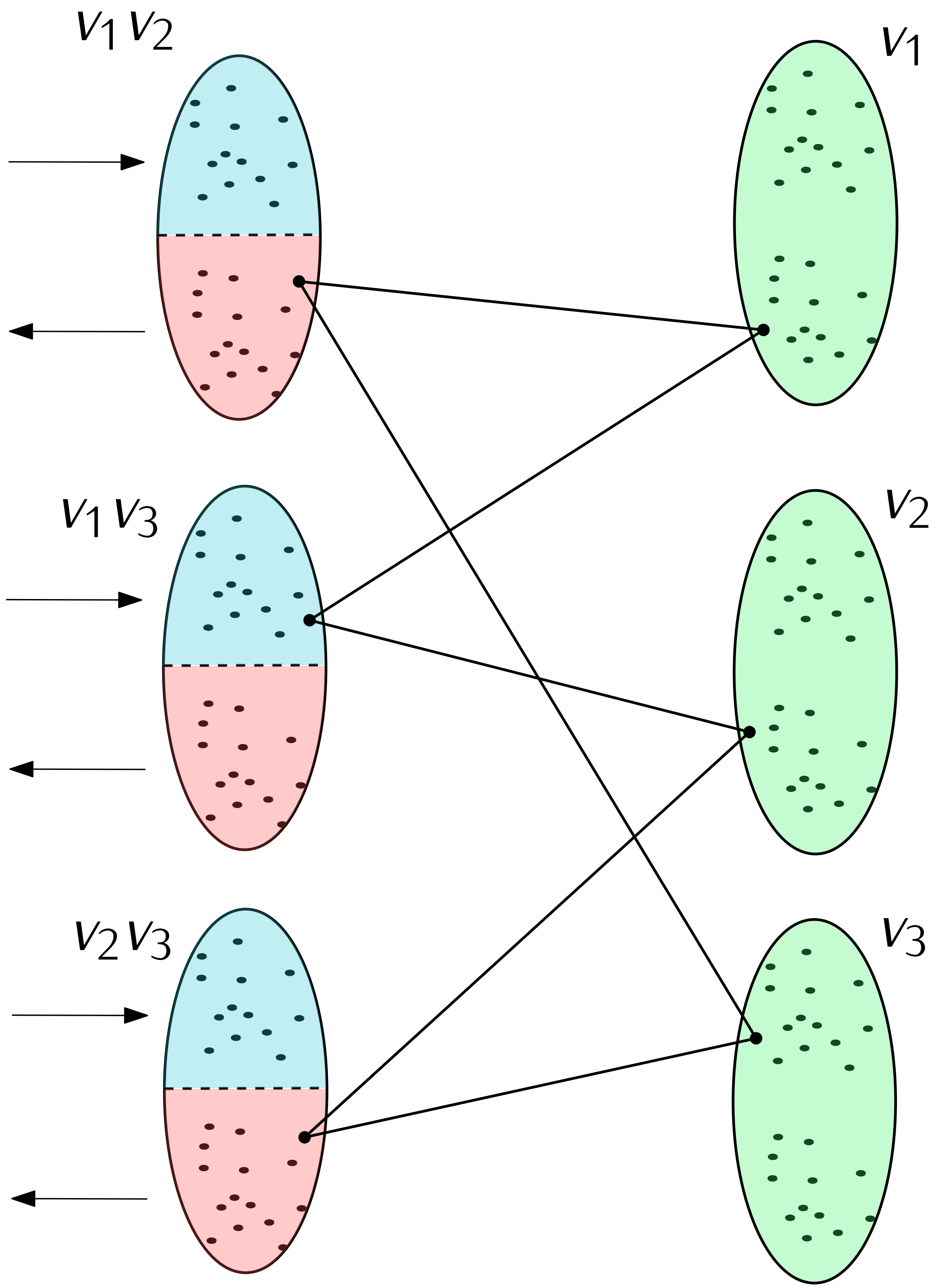

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Conjecture. Let $\mathcal{C}$ be a CSP

Then $\mathcal{C}$ is in P or NP-complete.

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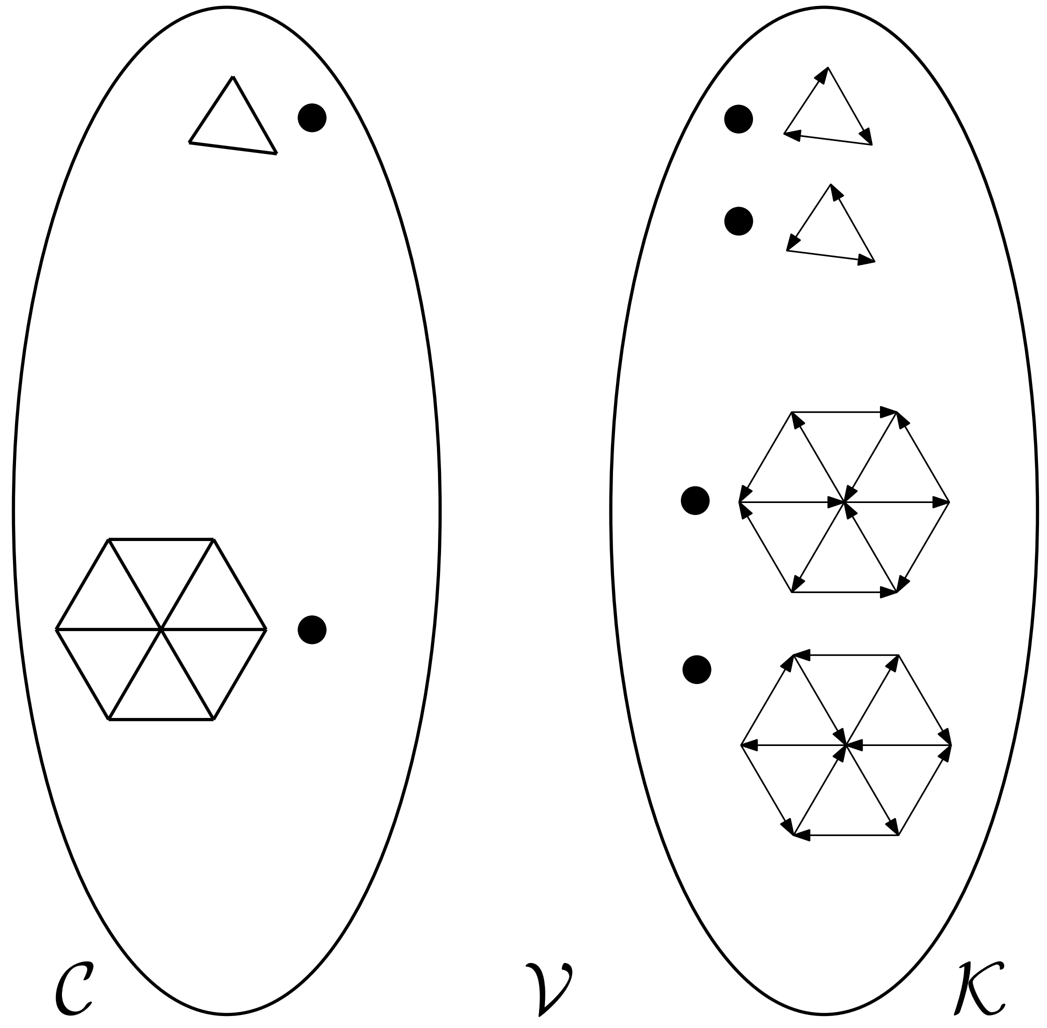

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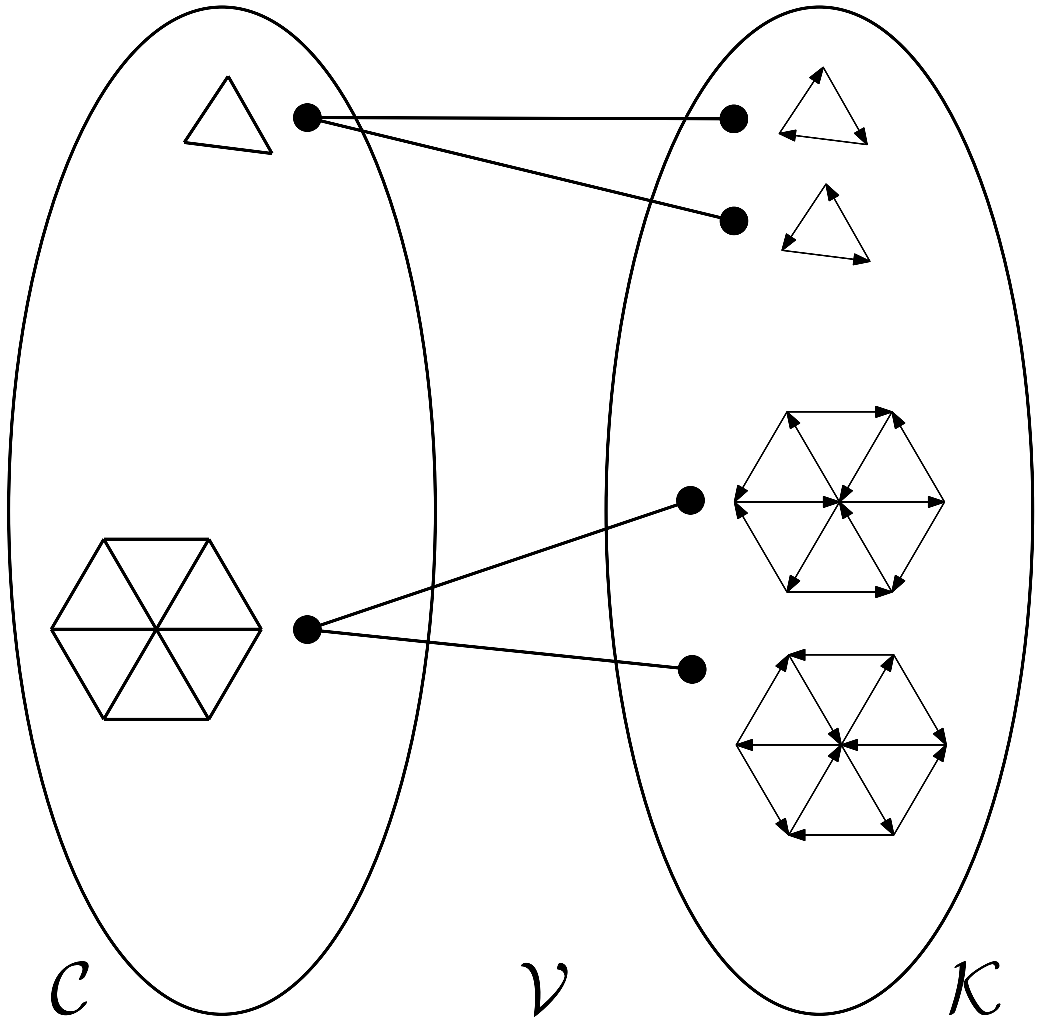

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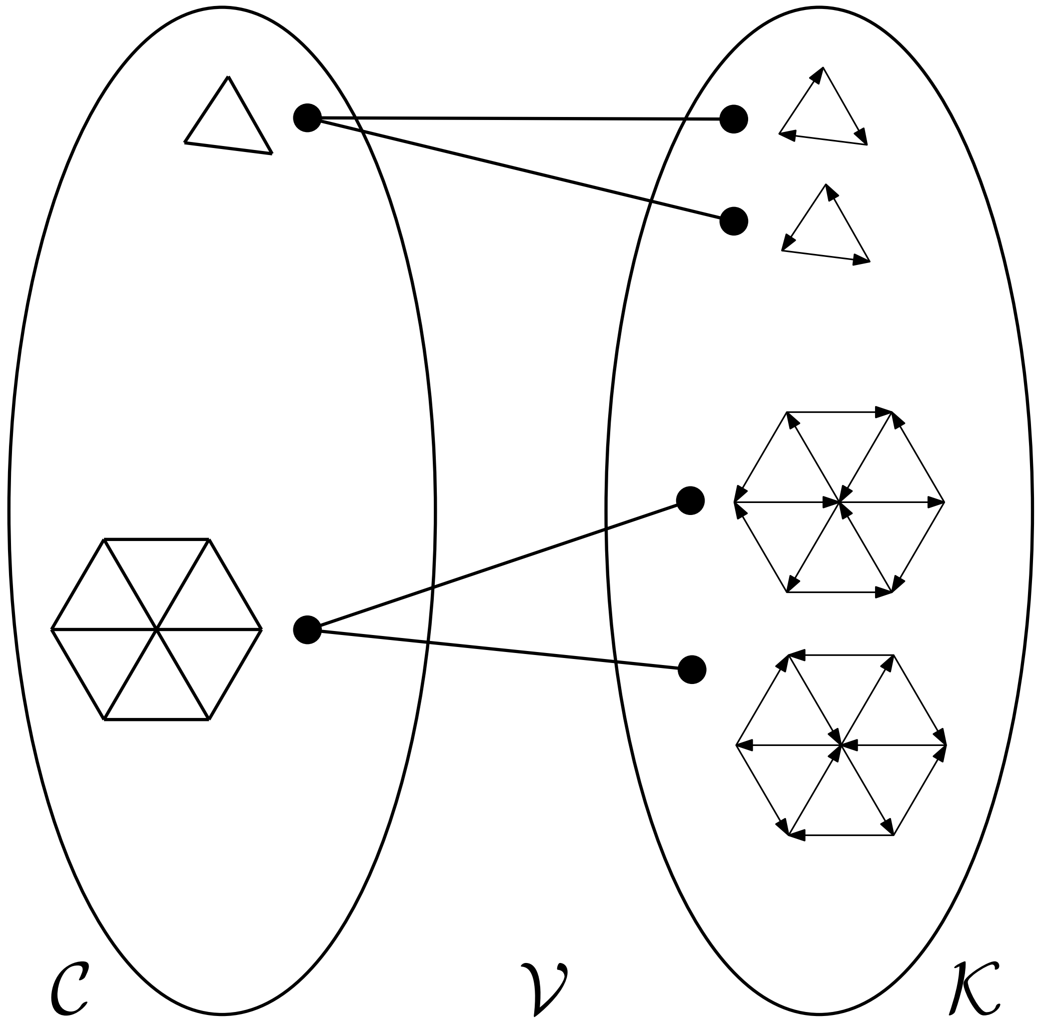

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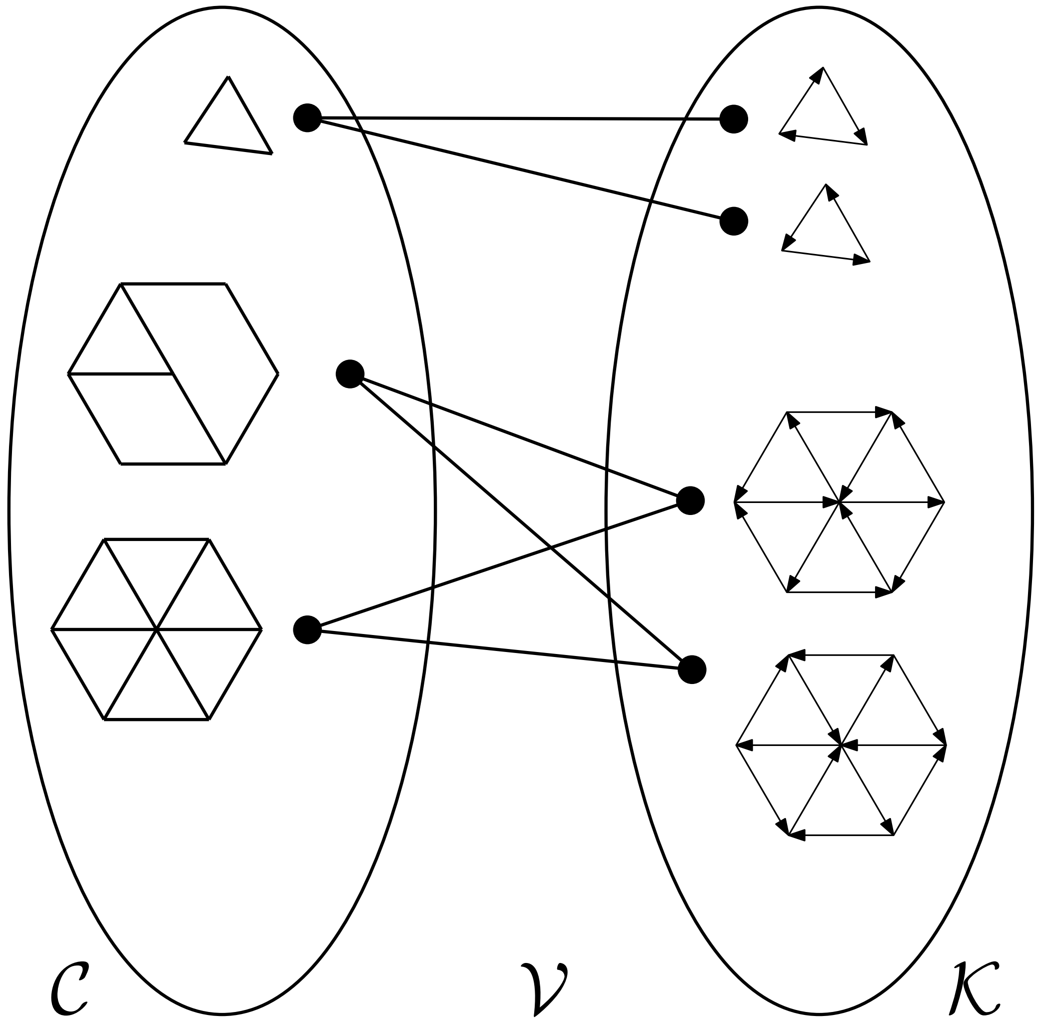

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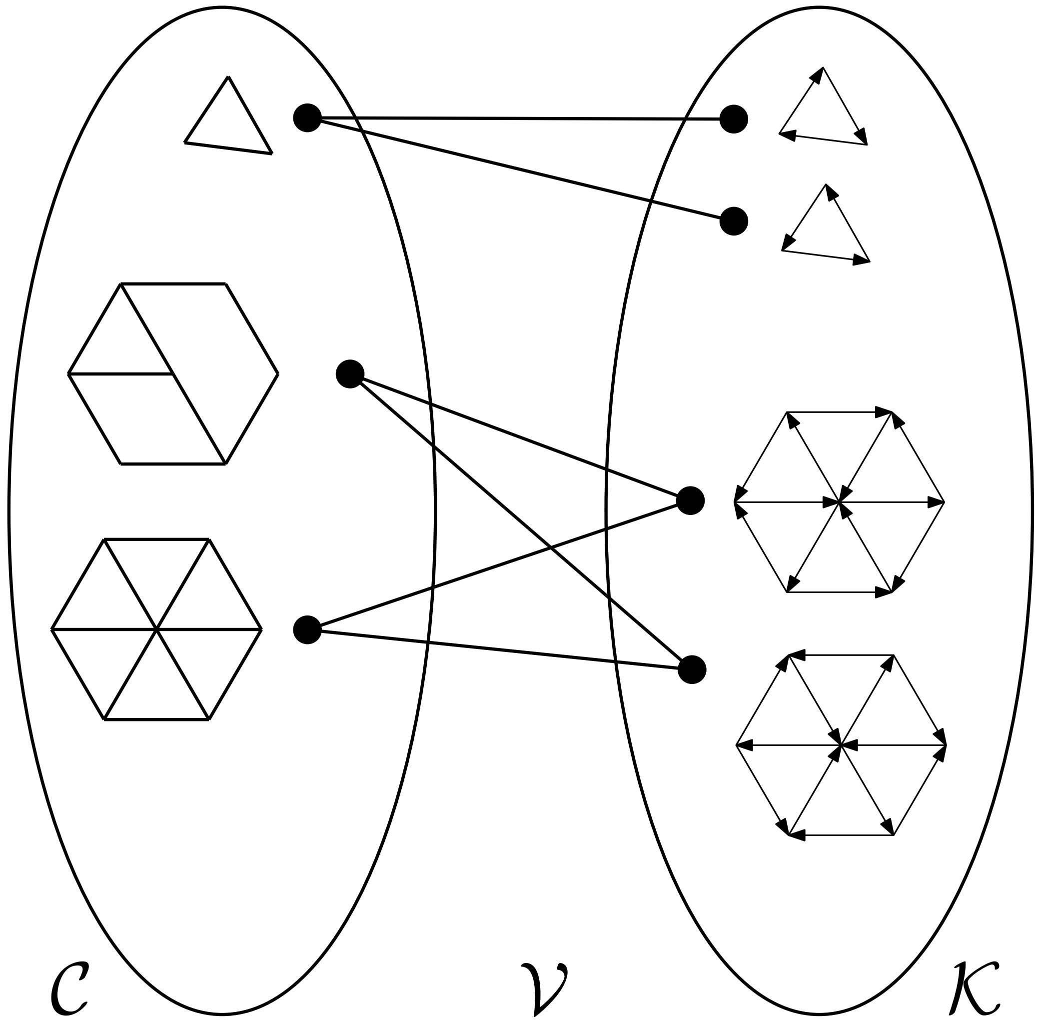

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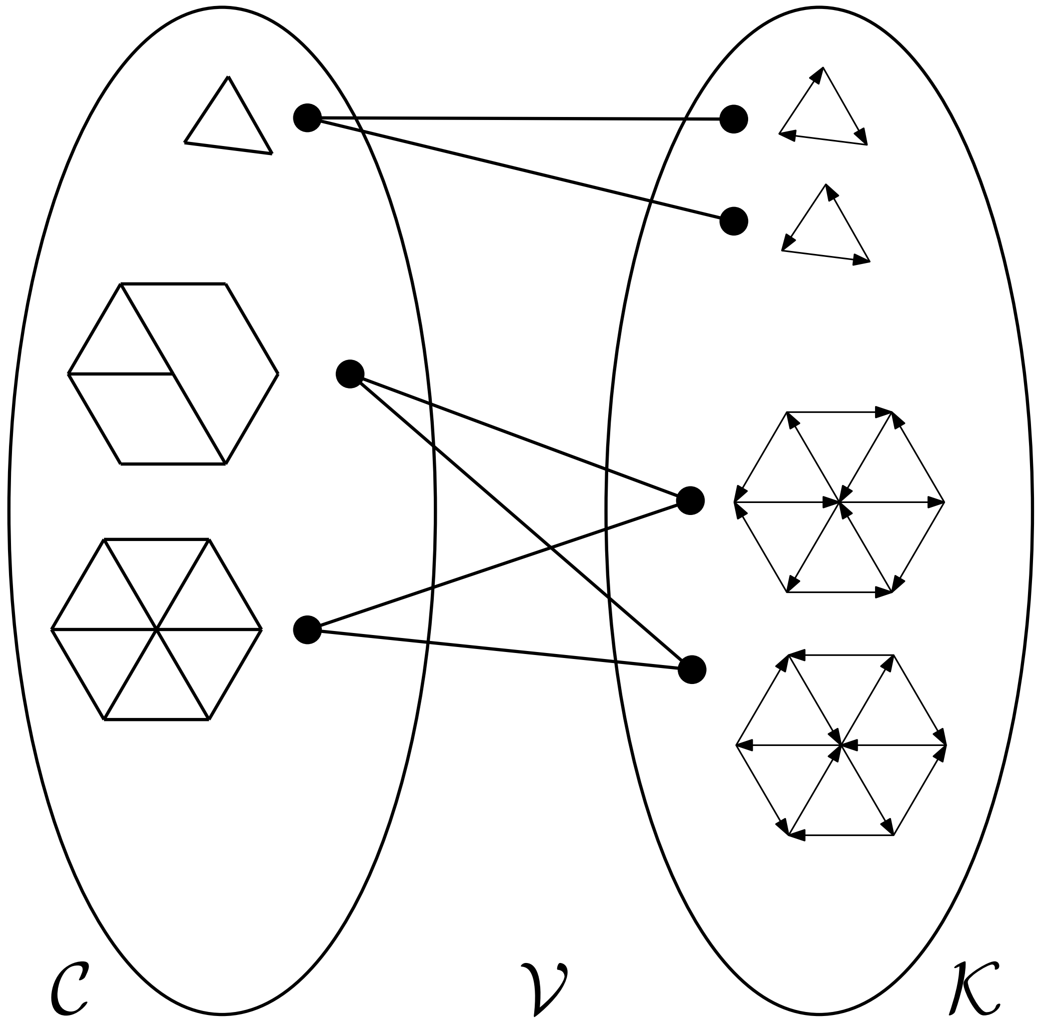
Under those conditions:

- $\mathcal{C} = \mathrm{CSP}(\mathbb{A})$, where $\mathbb{A}$ is a reduct of a finitely bounded homogeneous structure $\mathbb{A}^*$.
- $\mathrm{Aut}(\mathbb{A})$ is oligomorphic



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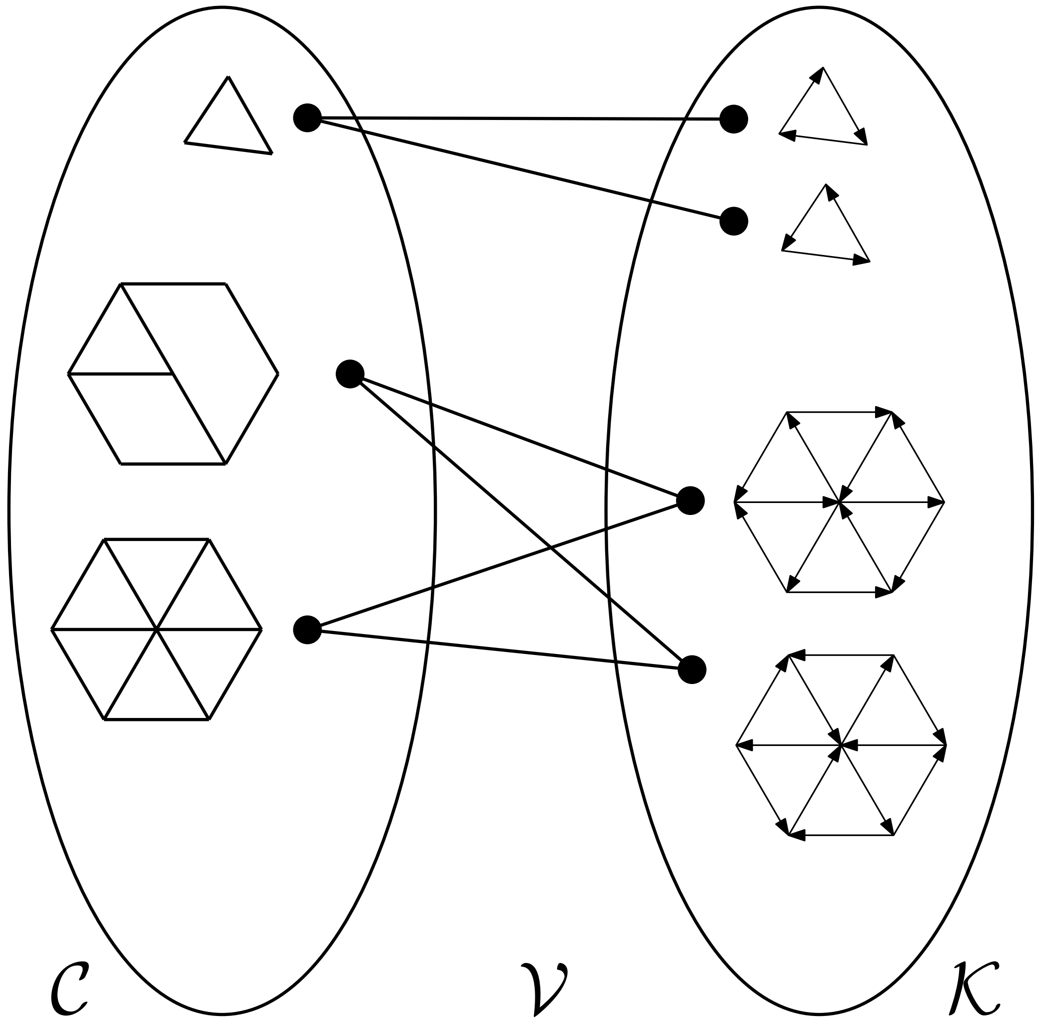
Examples:

- order CSPs, phylogenetic CSPs
- GMSNP: problems of coloring edges of a graph while avoiding forbidden configurations



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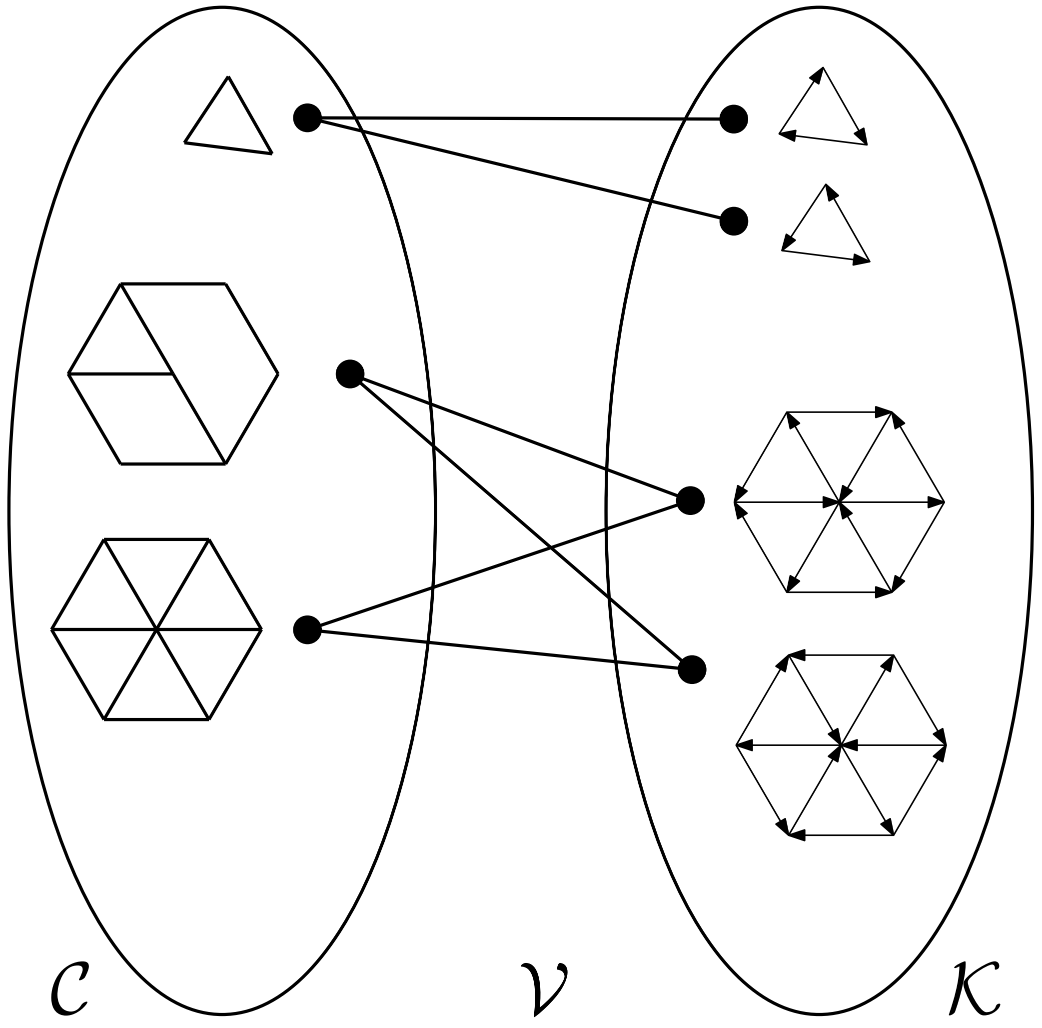
Then $\mathcal{C}$ is in P or NP-complete.

Non-examples:

- $\mathcal{C}$ = satisfiable systems of affine equations over $\mathbb{Z}$
- …feasible linear programs
- …feasible SDPs



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- phylogenetic trees (Bodirsky, Jonsson, Pham)
- graphs (Bodirsky, Pinsker)
- K_n-free graphs for some $n \geq 3$ (Bodirsky, Martin, Pinsker, Pongrácz)
- equivalence relations (with c classes of size s)
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... and in several other cases where also $\mathcal{V}$ is restricted:

- MMSNP (Feder, Vardi) (Bodirsky, Madelaine, M.)
- Orientability with forbidden tournaments (Bodirsky, Guzmán-Pro) (Feller, Pinsker) (Bitter, M.)

Algebra

&

"Topology"

Definition. A polymorphism of $\mathbb{A}$ is a function $f\colon \mathbb{A}^n \to \mathbb{A}$ such that for every relation $R^{\mathbb{A}}$ we have

$$\mathbf{a^1}, \ldots, \mathbf{a^n} \in R^{\mathbb{A}} \implies \underbrace{f(\mathbf{a^1}, \ldots, \mathbf{a^n})}_{\parallel \atop (f(a_1^1, \ldots, a_1^n), \ldots, f(a_r^1, \ldots, a_r^n))} \in R^{\mathbb{A}}$$

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Even if $\mathbb{A}$ is infinite:

- If $h_1, \dots, h_n \colon \mathbb{X} \to \mathbb{A}$ are solutions, $f \circ (h_1, \dots, h_n)$ is a solution
- $\mathrm{Pol}(\mathbb{A})$ is a clone (contains projections, closed under composition)
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Definition. A polymorphism of $\mathbb{A}$ is a function $f\colon \mathbb{A}^n \to \mathbb{A}$ such that for every relation $R^{\mathbb{A}}$ we have

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projection of the solution set of a CSP instance

Lemma. Let $\mathbb{X}$ be a finite structure, $x_1, \ldots, x_n \in X$.
There exists a pp-formula $\varphi_{\mathbb{X}}(x_1, \ldots, x_n)$ such that for all $\mathbb{A}$:

$$\mathbb{A} \models \varphi_{\mathbb{X}}(a_1, \ldots, a_n) \iff \exists h\colon \mathbb{X} \to \mathbb{A} \forall i : h(x_i) = a_i$$

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$$\begin{array}{rcccccc}
 & & \mathbf{a}^1 & \mathbf{a}^2 & & \mathbf{a}^n & \\
 & & \| & \| & & \| & \\
\mathbf{b}^1 = & (& a^1_1 & a^2_1 & \cdots & a^n_1 &) \\
 & & \vdots & \vdots & & \vdots & \\
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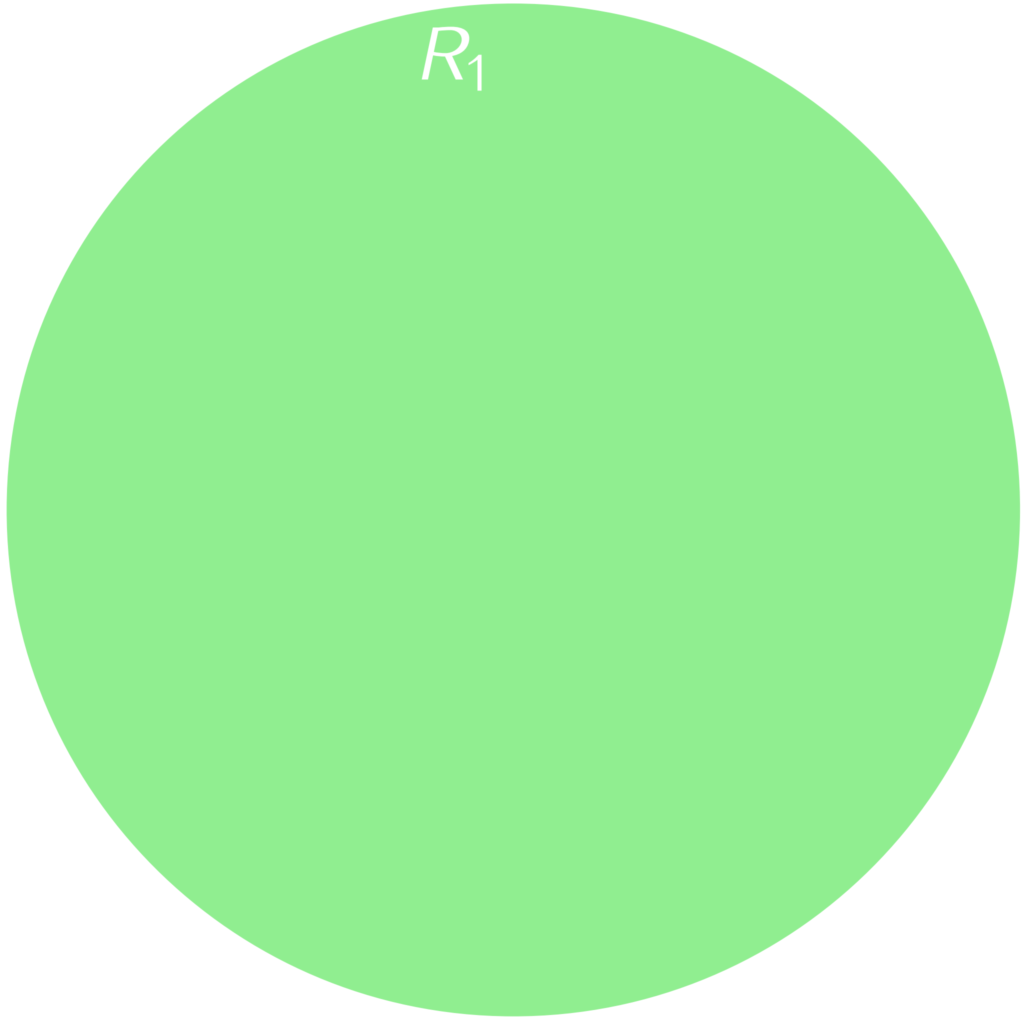

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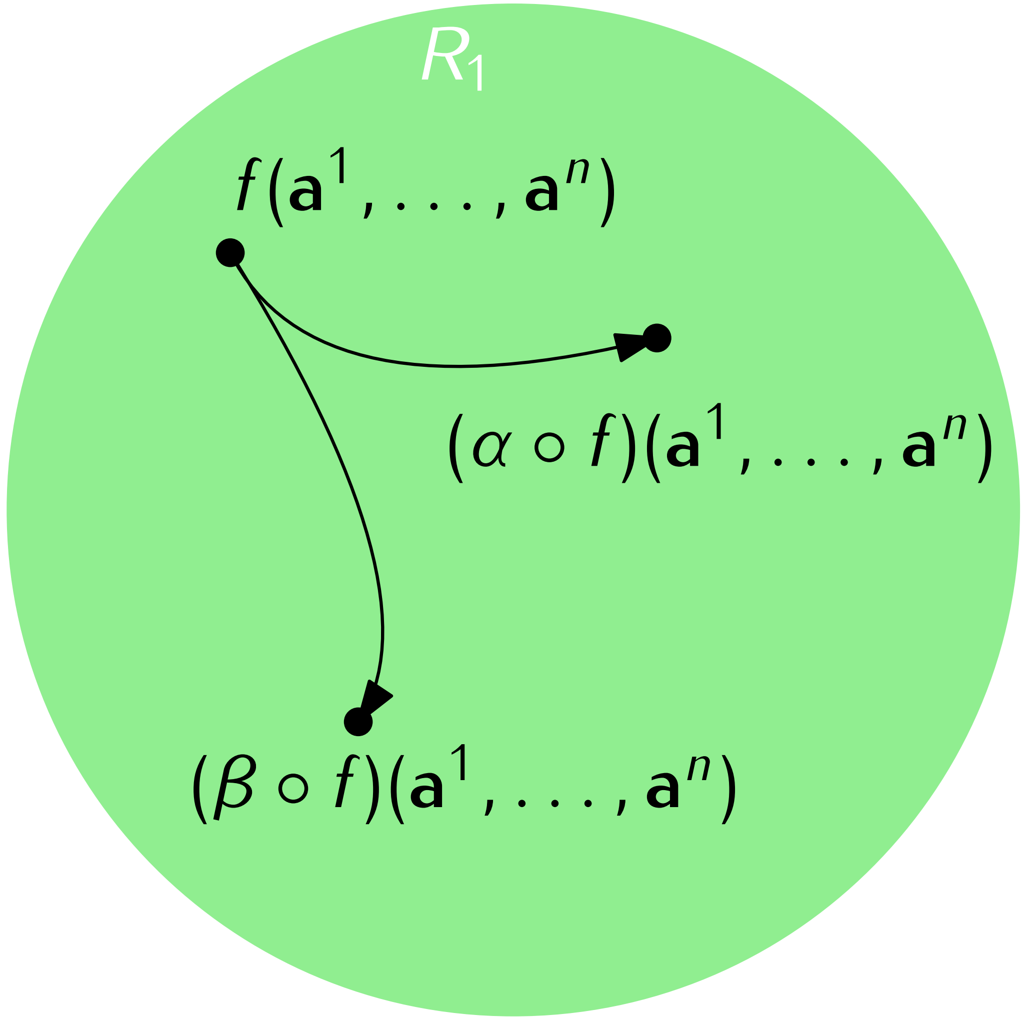

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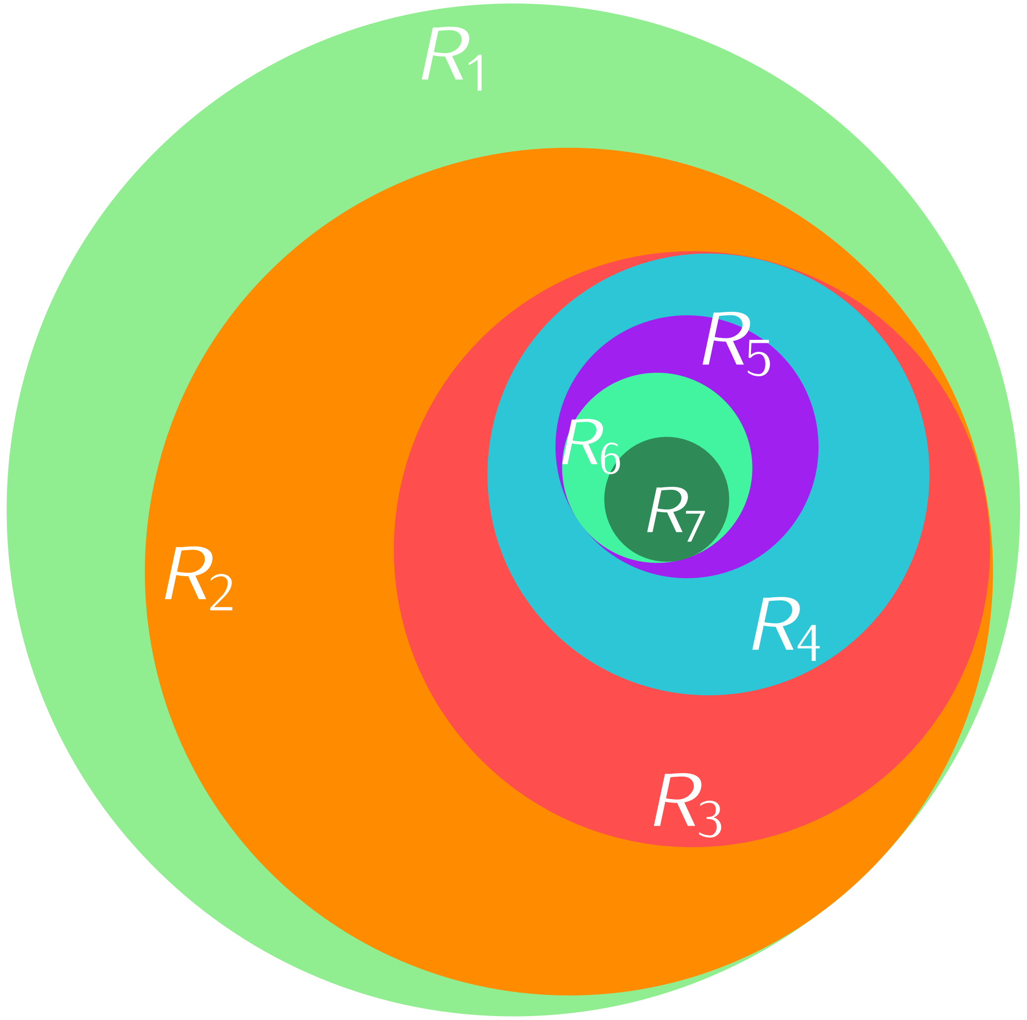

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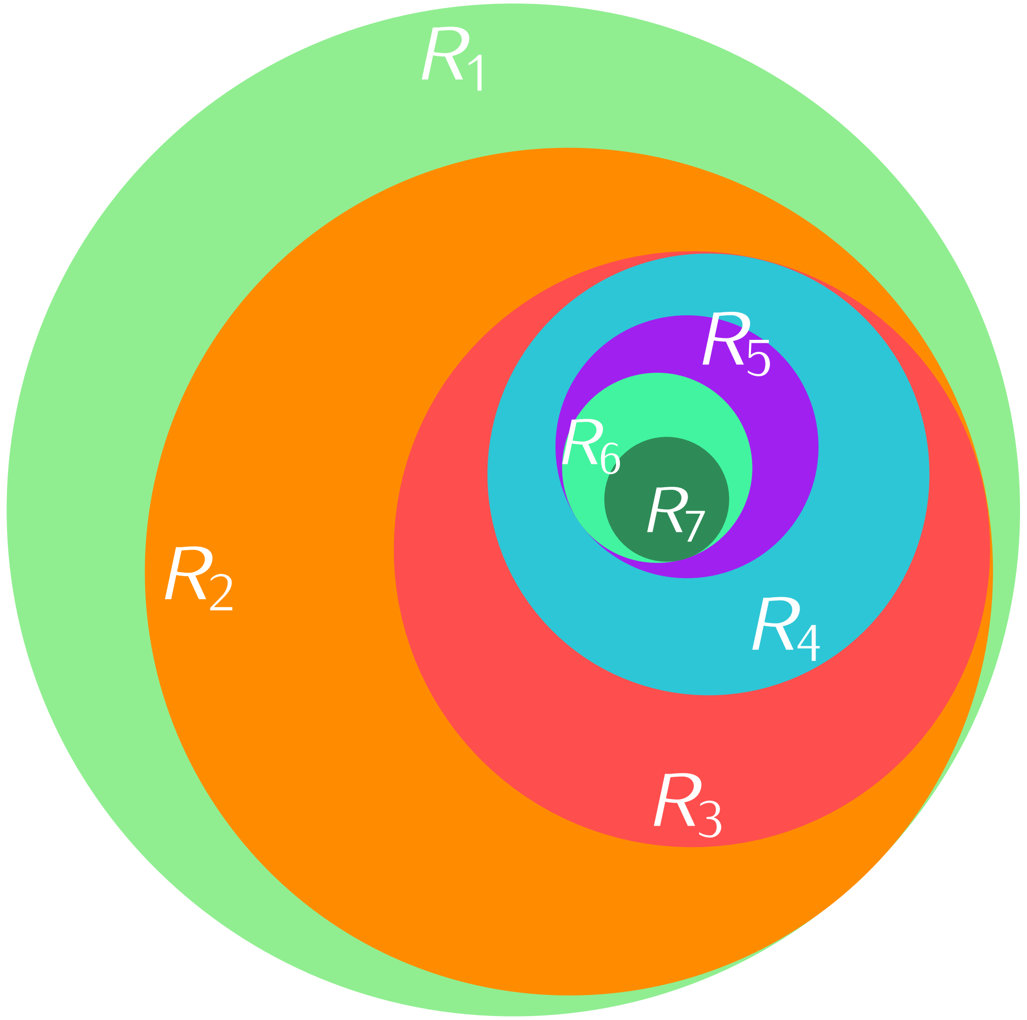

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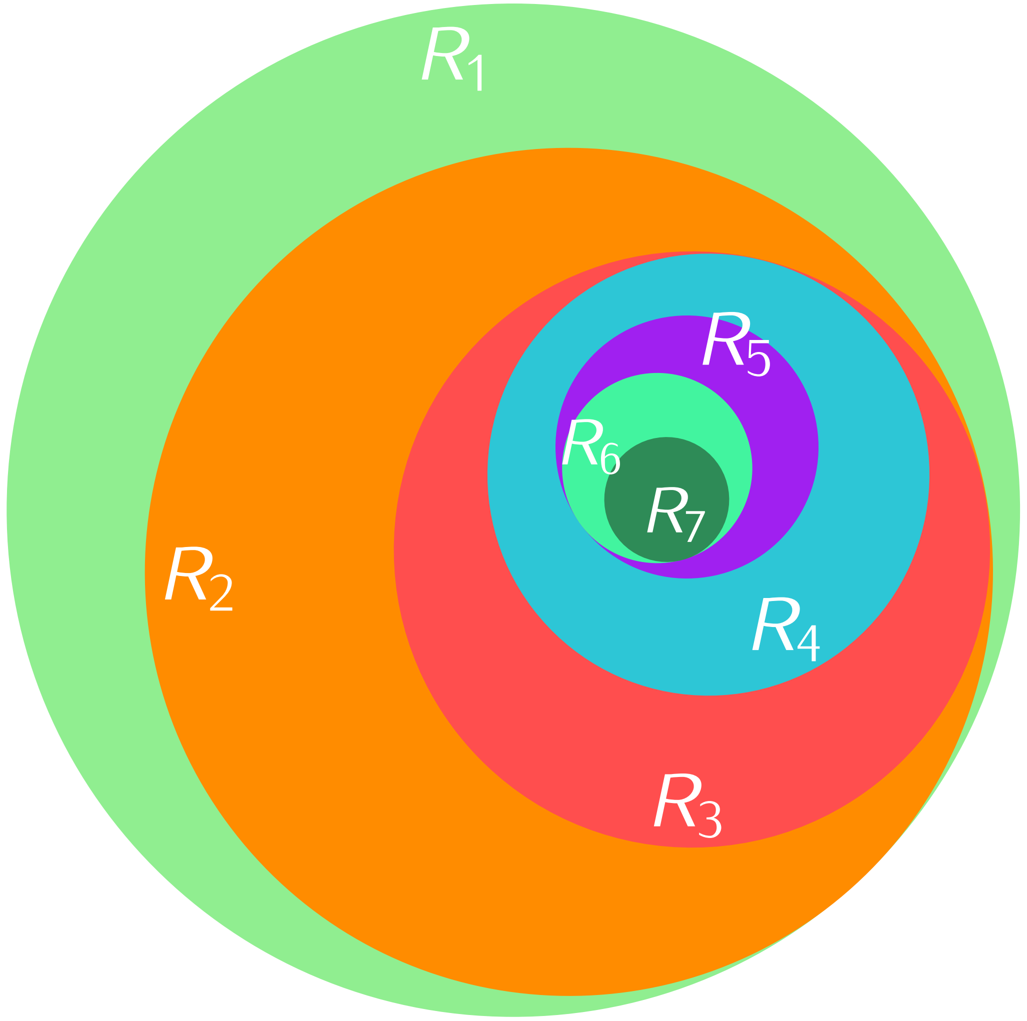

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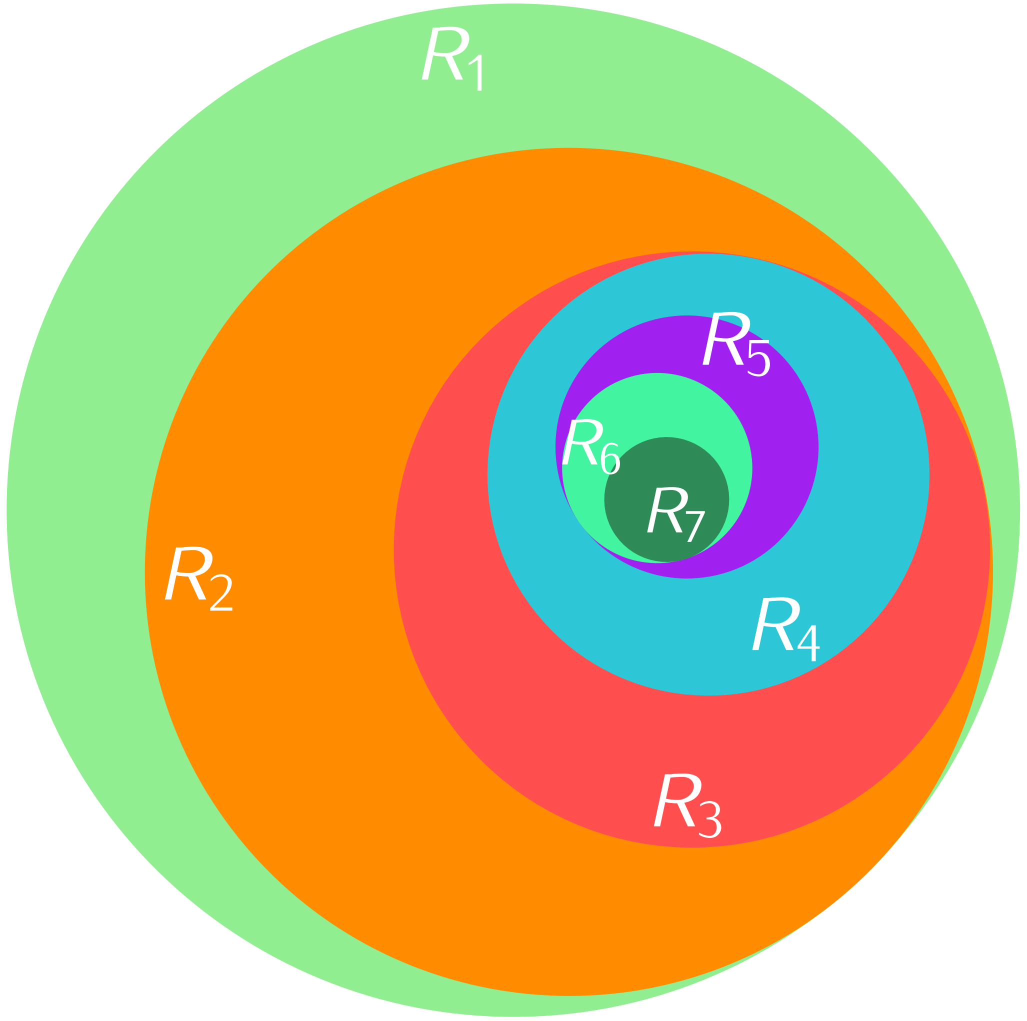

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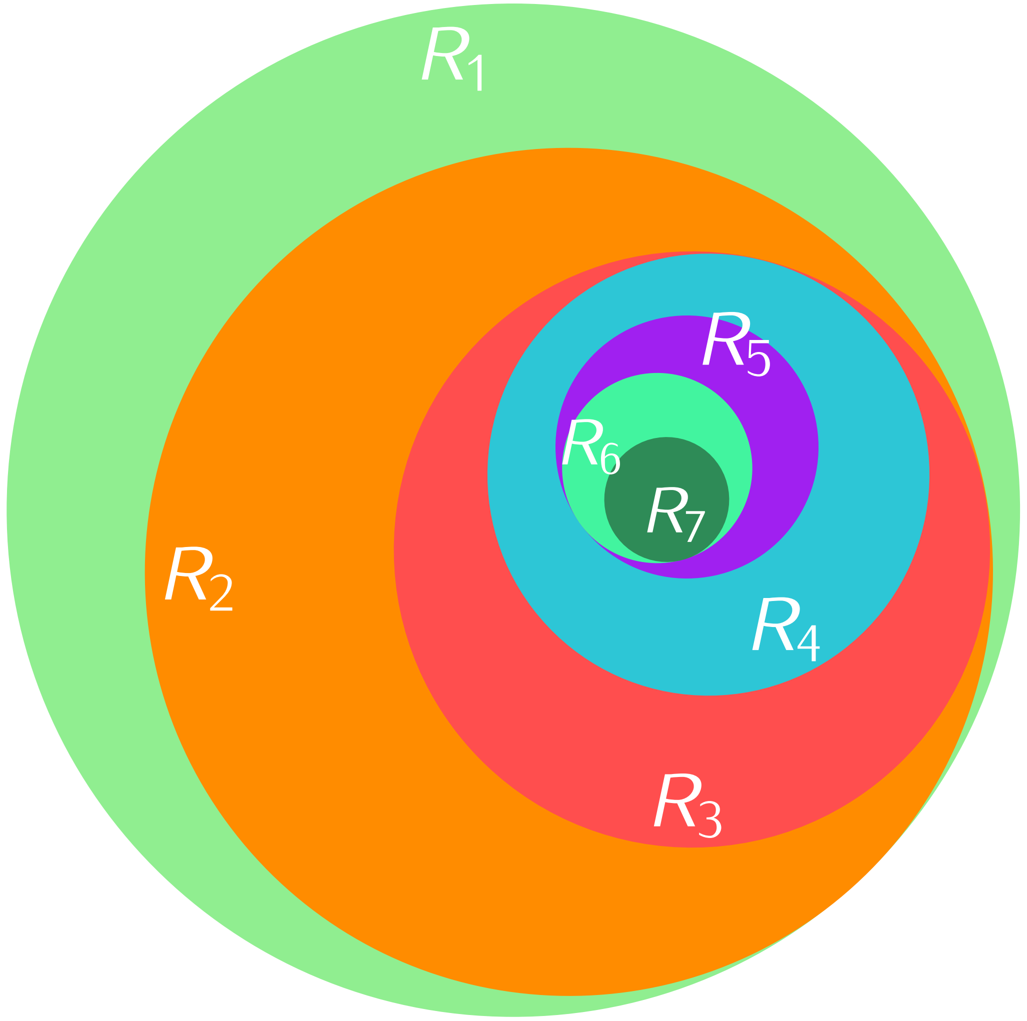
- We can always find finitely many generators, by ω-categoricity



Theorem (Bodirsky, Nešetřil). Let $\mathbb{A}$ be ω-categorical and $R \subseteq A^r$.
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Corollary. If $\mathbb{A}$ is ω-categorical, the complexity of $\mathrm{CSP}(\mathbb{A})$ only depends on $\mathrm{Pol}(\mathbb{A})$.

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This order is way too fine, no hope of understanding it all (already continuum many clones on $\{0, 1, 2\}$)

Definition (Minion/clone homomorphism). Arity-preserving map $\xi\colon \mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$ such that:

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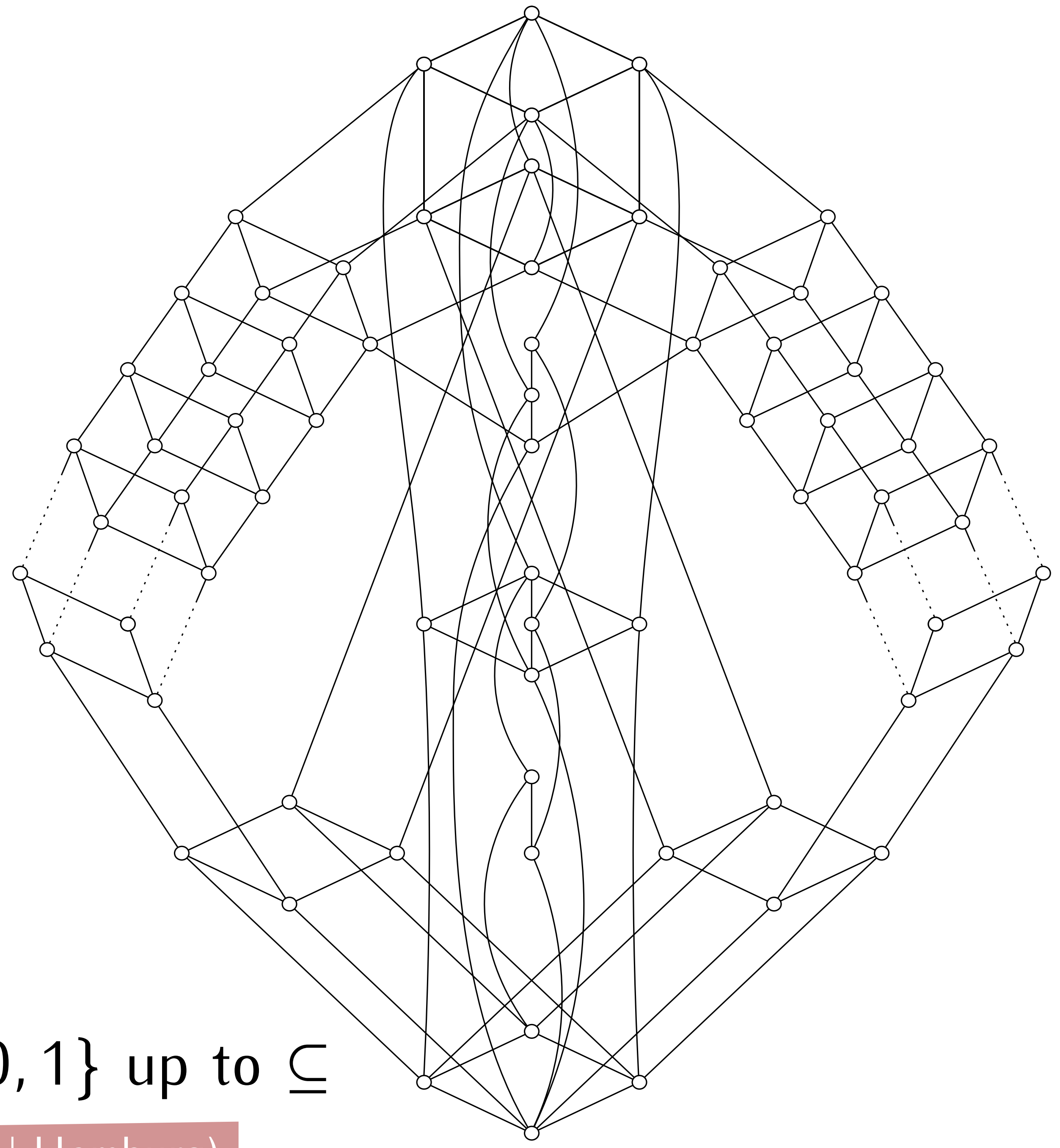

$\rightsquigarrow$

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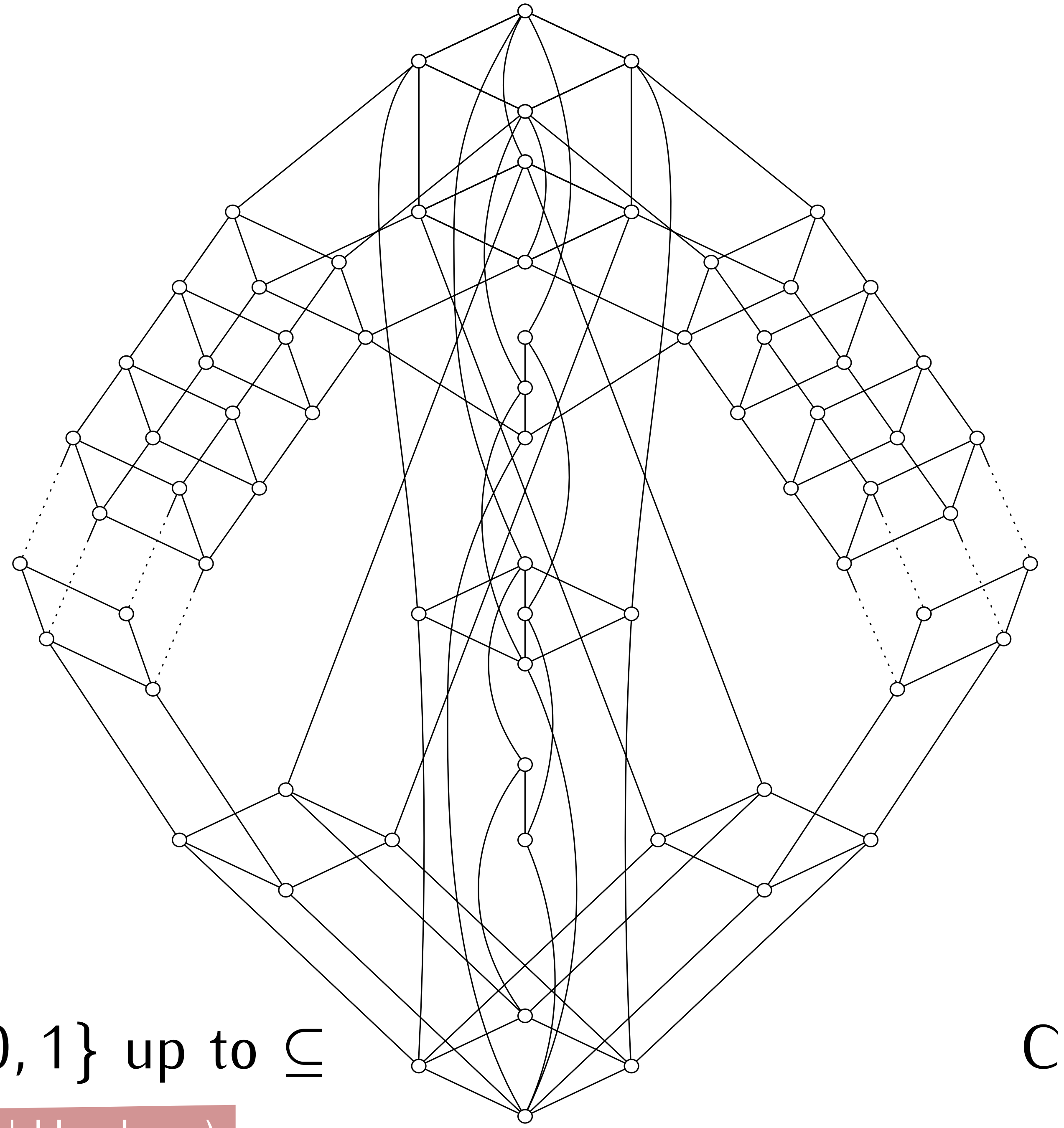

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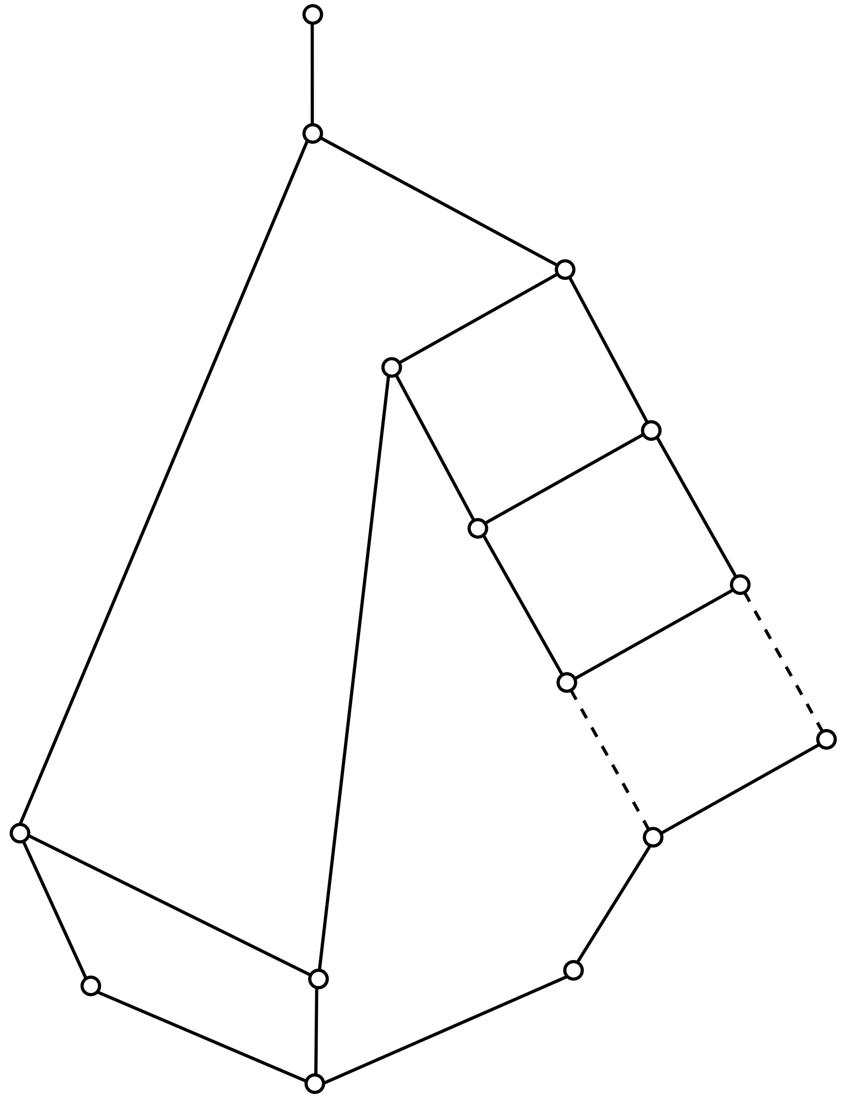

Clones on $\{0, 1\}$ up to $\subseteq$

Clones on $\{0, 1\}$ up to minion homomorphisms (Bodirsky, Vucaj)

Definition. $f_n \to f$ if for all finite $S \subseteq A$, $\exists N \forall n \geq N : f_n|_S = f|_S$

This turns $\mathrm{Pol}(\mathbb{A})$ into a topological clone with nice properties:

- It is closed: if $f_n \to f$ and $\{f_n : n \geq 1\} \subseteq \mathrm{Pol}(\mathbb{A})$, then $f \in \mathrm{Pol}(\mathbb{A})$
- If $\mathbb{A}$ is ω-categorical, $\mathrm{Pol}(\mathbb{A})^{(n)}/\mathrm{Aut}(\mathbb{A})$ is compact (Bodirsky, Pinsker)

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The following are equivalent:

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$$\begin{tikzcd} \mathrm{Pol}(\mathbb{A}) \arrow[rr, "\xi"] \arrow[dr, "\mathrm{restr}_F"'] & & \mathrm{Pol}(\mathbb{B}) \\ & \mathrm{Pol}(\mathbb{F}, \mathbb{A}) \arrow[ur, dashed] & \end{tikzcd}$$

Conjecture (Bodirsky, Pinsker)**.** Let $\mathbb{A}$ be a reduct of a finitely bounded homogeneous structure. Suppose

- $\nexists$ uniformly continuous minion homomorphism $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(K_3)$, or (equivalently)
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Those are (finite, infinite)-type minions

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Why not directly maps $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{K}_3)$?

Theorem (Barto, Pinsker). Let $\mathbb{B}$ be an ω-categorical structure that is a core.
The following are equivalent:

- For every finite tuple $\mathbf{c}$, $\mathrm{Pol}(\mathbb{B}, \mathbf{c})$ has no uniformly continuous clone homomorphism to $\mathrm{Pol}(K_3)$
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$$u \circ s(x, y, x, z, y, z) \approx v \circ s(y, x, z, x, z, y) \qquad \text{(pseudo-Siggers modulo } \mathrm{Aut}(\mathbb{B}))$$

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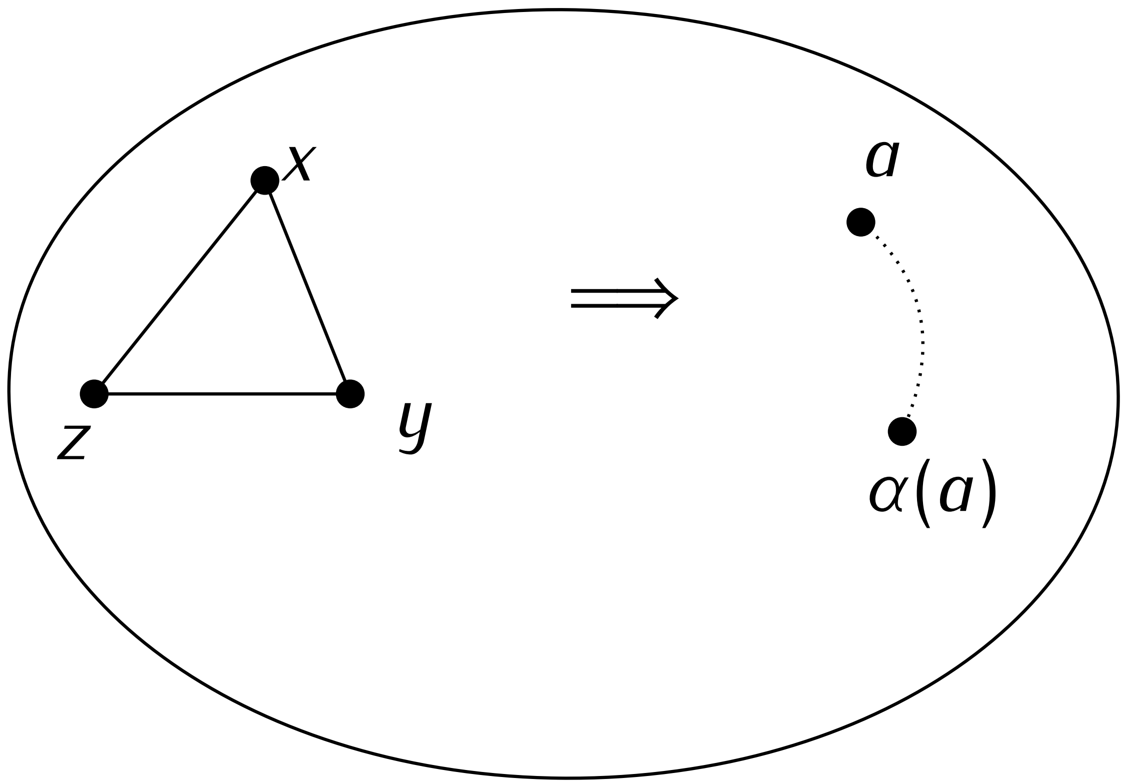

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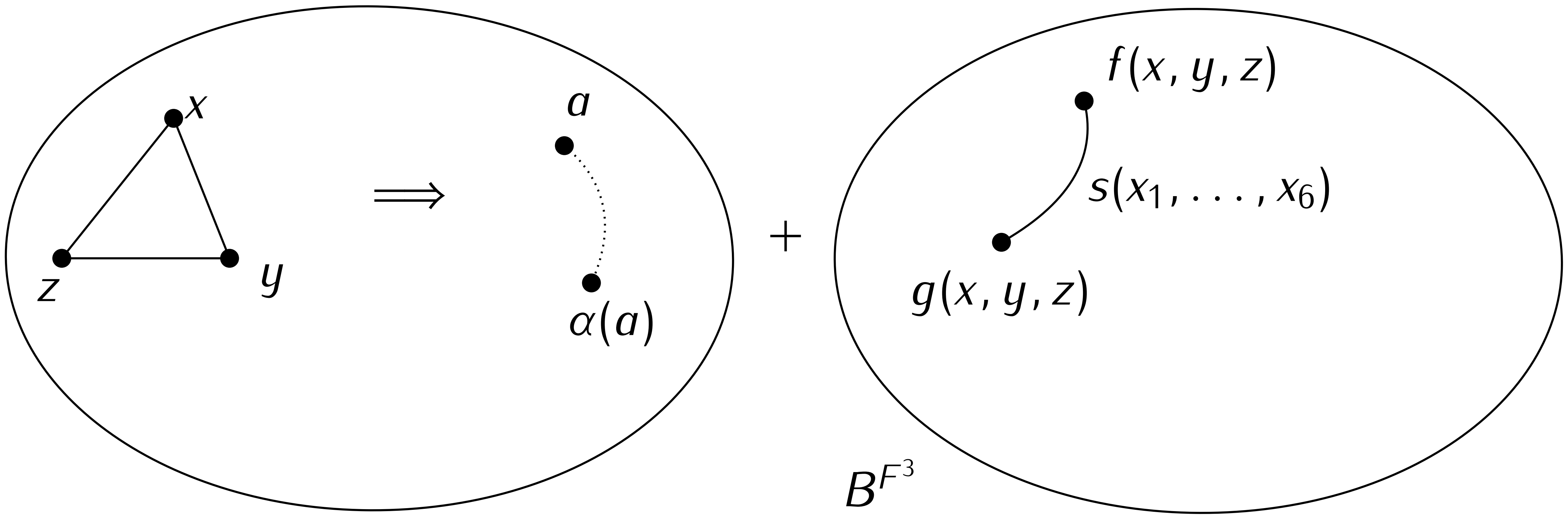

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Free algebra

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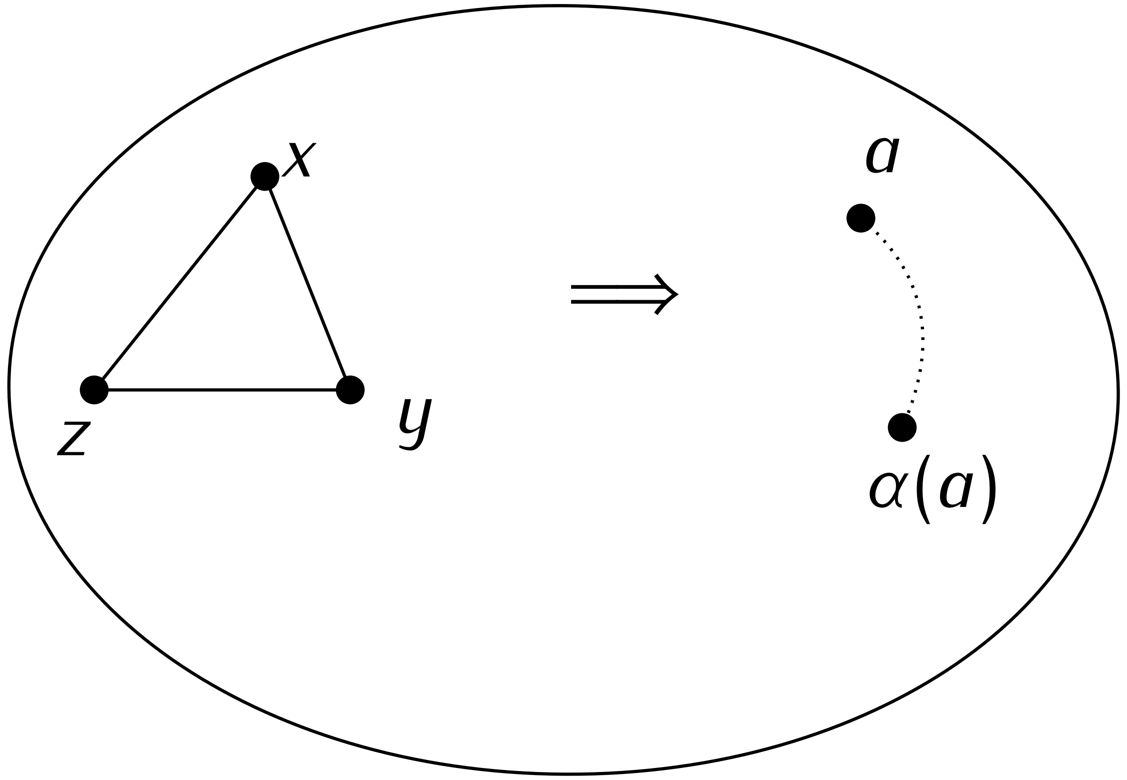
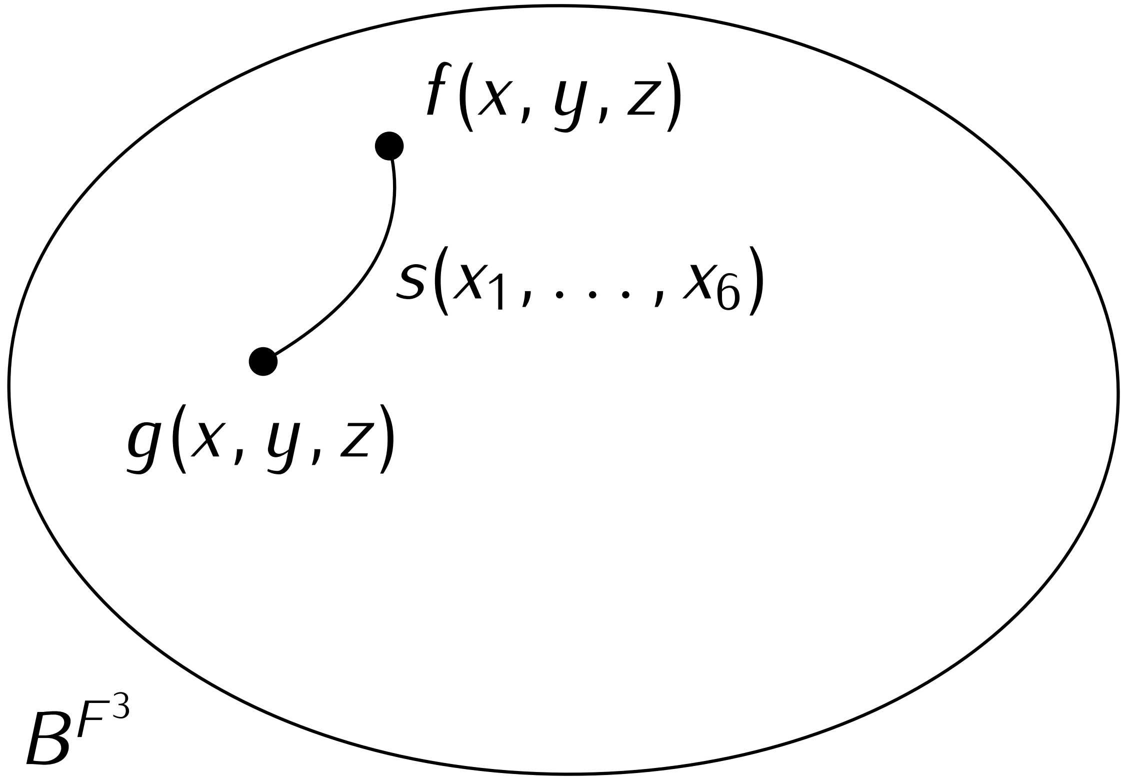


Loop lemma

\+



Free algebra

\+ compactness

Theorem (Barto, Pinsker). Let $\mathbb{B}$ be an ω-categorical structure that is a core.
The following are equivalent:

- For every finite tuple $\mathbf{c}$, $\mathrm{Pol}(\mathbb{B}, \mathbf{c})$ has no uniformly continuous clone homomorphism to $\mathrm{Pol}(K_3)$
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Several improvements recently:

- Loop lemma without the constants (Barto, Bodor, Kozik, M., Pinsker)
  (also interesting for finite structures!)
- Loop lemma for directed graphs (Brunar, Kozik, Nagy, Pinsker)
  $\rightsquigarrow$ 4-ary pseudo-Siggers

Answer 2: Topology is relevant (for minion homomorphisms to $\mathrm{Pol}(\mathbb{K}_3)$)

Theorem (Gillibert, Jonušas, Kompatscher, M., Pinsker).
There exists an ω-categorical $\mathbb{A}$ with finite signature such that:

- $\mathrm{Pol}(\mathbb{A})$ does not satisfy any minor condition
  $\rightsquigarrow$ $\exists$ minion homomorphism $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(K_3)$
- $\mathrm{Pol}(\mathbb{B}, \mathbb{A})$ satisfies a non-trivial minor condition for every finite $\mathbb{B} \subseteq \mathbb{A}$
  $\rightsquigarrow$ $\nexists$ uniformly continuous minion homomorphism $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(K_3)$

Note: $\mathbb{A}$ not subject to the dichotomy conjecture

$$
\begin{tikzcd}
 & \mathrm{Pol}(\mathbb{B}_3, \mathbb{A}) \arrow[d] \arrow[dr, dashed] & \\
\mathrm{Pol}(\mathbb{A}) \arrow[ur] \arrow[r] \arrow[dr] \arrow[rr, bend left=60] & \mathrm{Pol}(\mathbb{B}_2, \mathbb{A}) \arrow[d] \arrow[r, dashed] & \mathrm{Pol}(K_3) \\
 & \mathrm{Pol}(\mathbb{B}_1, \mathbb{A}) \arrow[ur, dashed] &
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Conjecture. No uniformly continuous minion/clone homomorphism $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(K_3)$ implies $\mathrm{CSP}(\mathbb{A})$ in P.

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Theorem. For a finite $\mathbb{A}$, the following are equivalent:

- $\mathrm{CSP}(\mathbb{A})$ is in P
- $\nexists$ minion homomorphism $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(K_3)$
- $\mathrm{Pol}(\mathbb{A})$ has a 6-ary operation (Siggers)

$$s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y)$$

- $\mathrm{Pol}(\mathbb{A})$ has for some $k \geq 3$ a weak near unanimity operation (Maróti, McKenzie)

$$w(x, \ldots, x, x, y) \approx w(x, \ldots, x, y, x) \approx \cdots \approx w(y, x, \ldots, x, x)$$

- $\mathrm{Pol}(\mathbb{A})$ has for all large prime $n \geq 3$ a cyclic operation (Barto, Kozik)

$$c(x_1, \ldots, x_n) \approx c(x_2, \ldots, x_n, x_1)$$

Theorem (Bodirsky, M., Olšák, Opršal, Pinsker, Willard).
$\forall$ minor condition Σ, $\exists$ reduct $\mathbb{A}$ of a finitely bounded homogeneous structure such that:

- Pol($\mathbb{A}$) satisfies some non-trivial minor condition
- Pol($\mathbb{A}$) does not satisfy Σ
- CSP($\mathbb{A}$) is in P (even in AC^0)

Easy CSPs without given identities
(within the conjecture)

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$\forall$ equational condition Σ true for all CSPs in AC^0,
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Hard CSPs with given identities
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Hard CSPs with given identities (outside the conjecture)

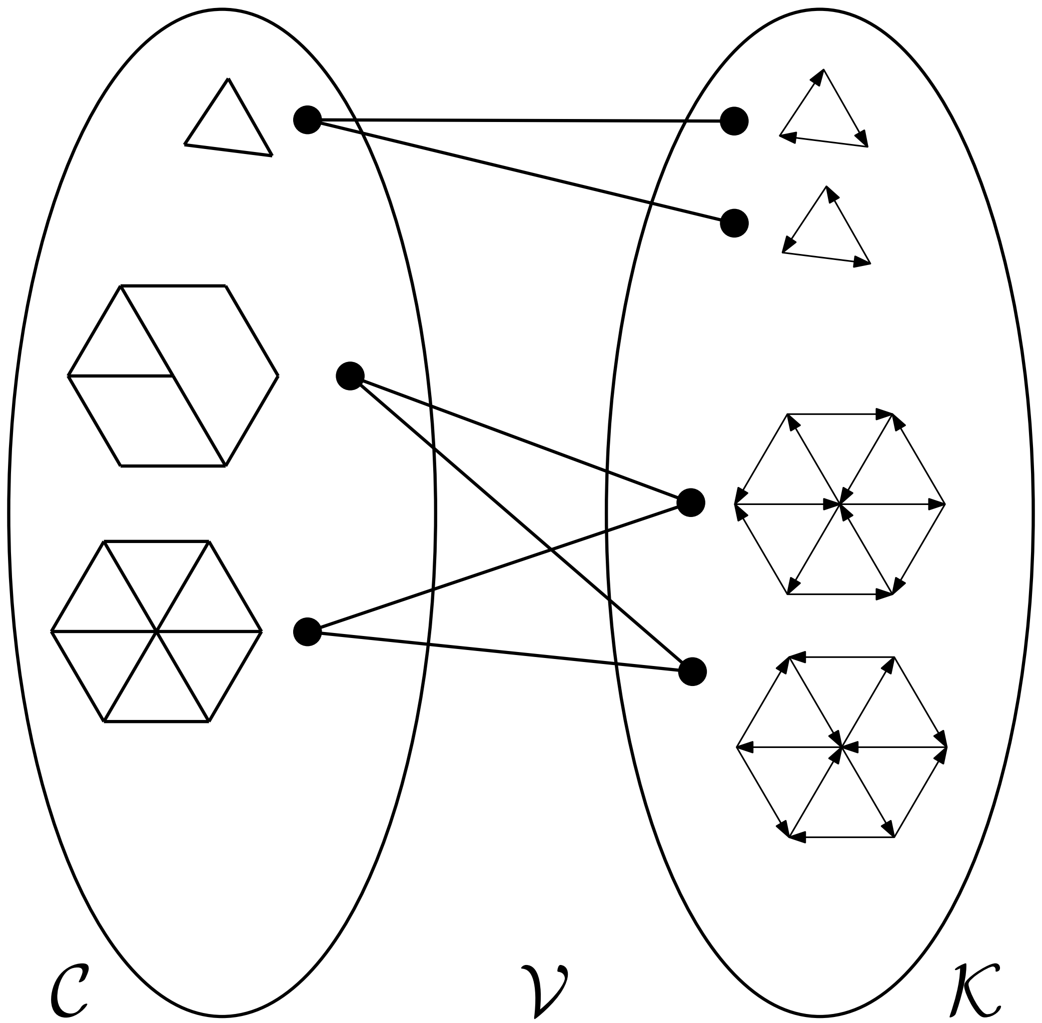
Other bad news:

- Bounded width cannot be characterized by identities (within the conjecture) (Bodirsky, Rydval)
- Non-trivial Pol($\mathbb{A}$) w/o pseudo cyclic polymorphisms (within the conjecture) (Bodirsky, Kára)
- Non-trivial Pol($\mathbb{A}$) w/o pseudo weak near unanimity operations (outside the conjecture) (Barto, Bodor, Kozik, M., Pinsker)

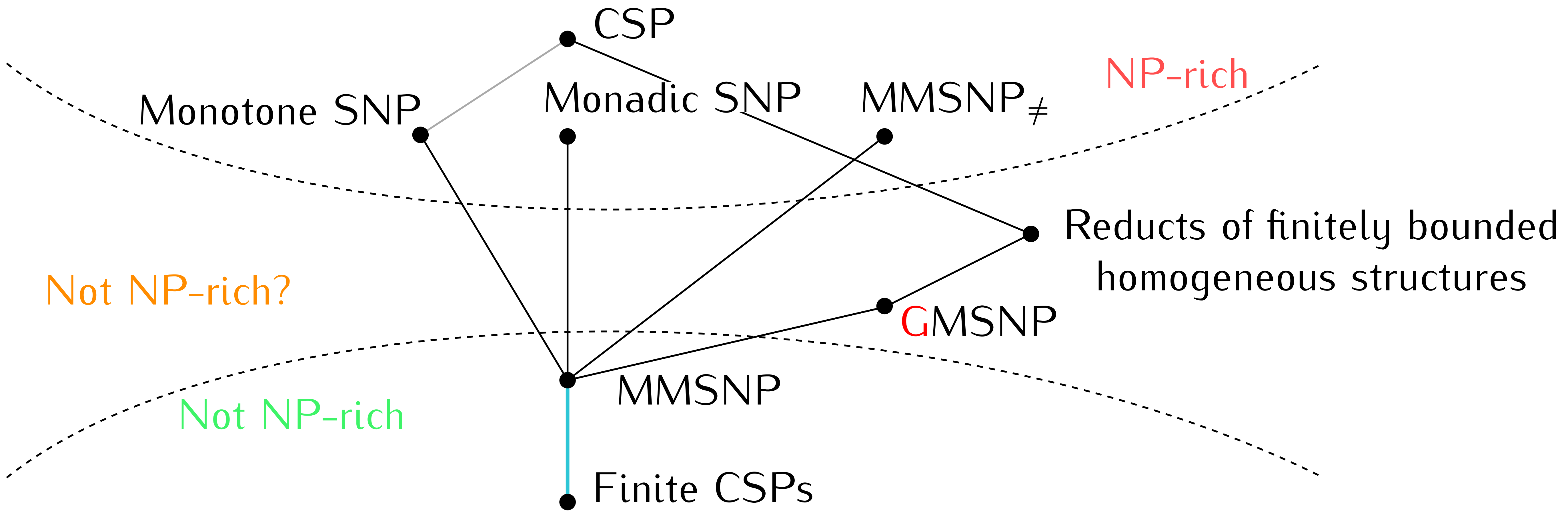
Conjecture. Let $\mathcal{C}$ be a CSP such that there exists a finitely bounded Fraïssé class $\mathcal{K}$ (the certificates) and a verifier $\mathcal{V} \subseteq \mathcal{C} * \mathcal{K}$ such that:

- For all $\mathbb{A} \in \mathcal{C}$, there exists $\mathbb{K} \in \mathcal{K}$ such that $\mathbb{A} * \mathbb{K} \in \mathcal{V}$ (yes-instances have a certificate)
- $\forall \mathbb{A} * \mathbb{K} \in \mathcal{V}$ and $\mathbb{K} \hookrightarrow \mathbb{L} \in \mathcal{K}$, $\exists \mathbb{A} \subseteq \mathbb{B} \in \mathcal{K}$ such that $\mathbb{B} * \mathbb{L} \in \mathcal{V}$ (bigger certificates are fine)
- $\forall \mathbb{B} * \mathbb{K} \in \mathcal{V}$ and $\mathbb{A} \subseteq \mathbb{B}$, $\mathbb{A} * \mathbb{K}|_A \in \mathcal{V}$ (smaller instances are fine)
- $\forall \mathbb{K} \in \mathcal{K}$, $\{\mathbb{A} \in \mathcal{C} \mid \mathbb{A} * \mathbb{K} \in \mathcal{V}\}$ has a maximal element

Then $\mathcal{C}$ is in P or NP-complete.



1. Hope for a general understanding of a class of problems
2. Connection between tractability and closure properties
3. Induction + Compactness
4. Naturally defined class



Smooth Approximations

How to prove tractability/hardness of such CSPs

Smooth Approximations 1	Smooth Approximations 2	Smooth Approximations 3
2022	2021	2024
M., Pinsker	M., Nagy, Pinsker, Wrona	M., Nagy, Pinsker

∞CSPs and Finite PCSPs

Descriptive complexity: bounded width, first-order solvability ...

New algorithms for order CSPs and hypergraph CSPs

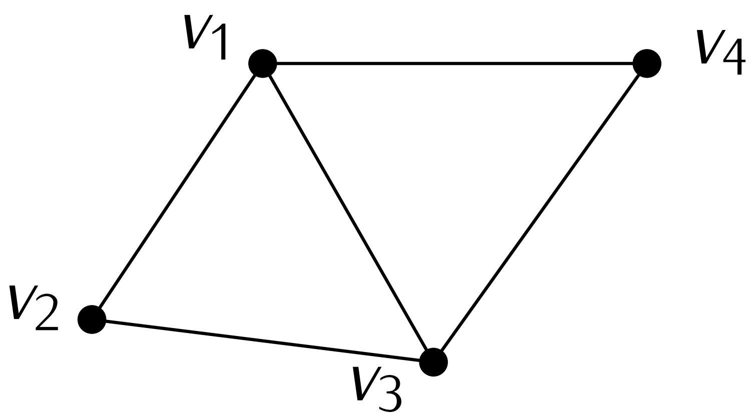
Conjecture (Bodirsky, Pinsker). Let $\mathbb{A}$ be a reduct of a finitely bounded homogeneous structure with certificates $\mathcal{K}$. Suppose

- $\nexists$ uniformly continuous minion homomorphism $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(K_3)$, or (equivalently)
- if $\mathbb{B}$ is the core of $\mathbb{A}$, $\nexists \mathbf{c}, \xi : \mathrm{Pol}(\mathbb{B}, \mathbf{c}) \to \mathrm{Pol}(\mathbb{K}_3)$ uniformly continuous clone homomorphism.

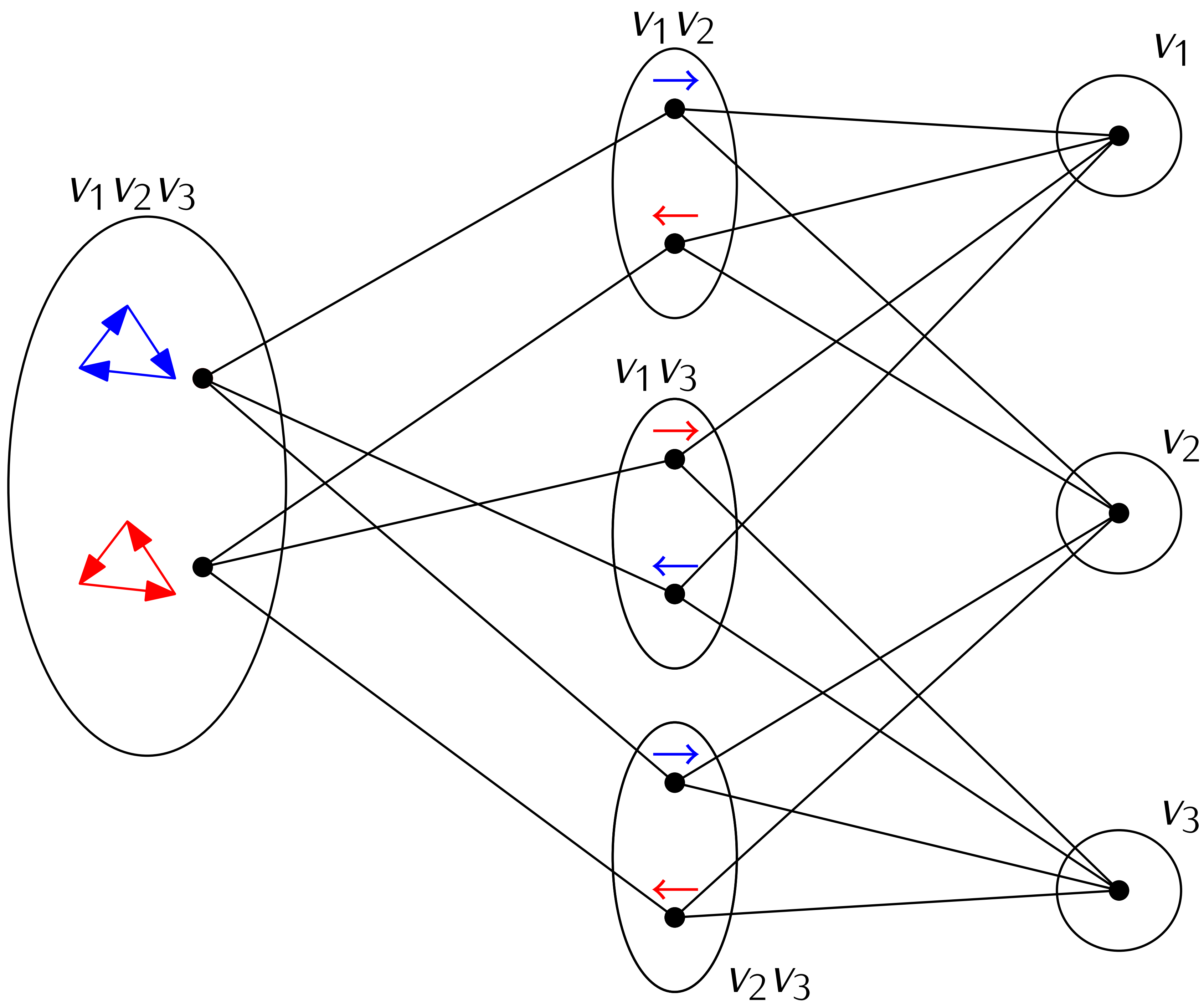
Then $\mathrm{CSP}(\mathbb{A})$ is in P.

The conjecture is proved when $\mathcal{K}$ is the class of:

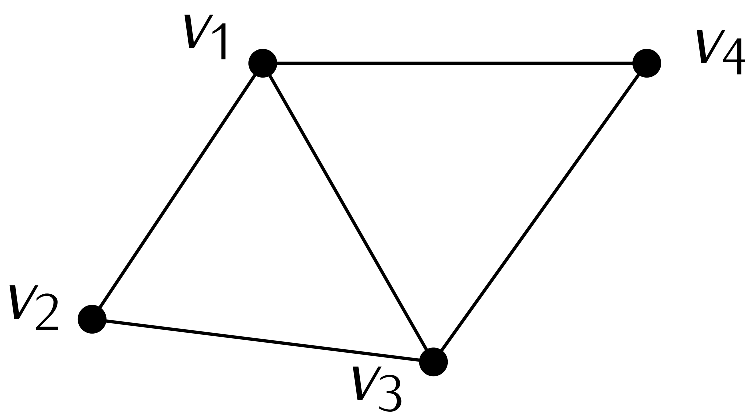
- maps to a finite set (Bulatov, Zhuk)
- linear orders (Bodirsky, Kára)
- phylogenetic trees (Bodirsky, Jonsson, Pham)
- graphs (Bodirsky, Pinsker)
- K_n-free graphs for some $n \geq 3$ (Bodirsky, Martin, Pinsker, Pongrácz)
- equivalence relations (with c classes of size s)
- partial orders (Kompatscher, Pham)
- unary structures (Bodirsky, M.)
- tournaments (M., Pinsker)
- (K_n^r-free) r-uniform hypergraphs (M., Nagy, Pinsker)



$\rightsquigarrow$



G orientable iff the LC instance has a solution



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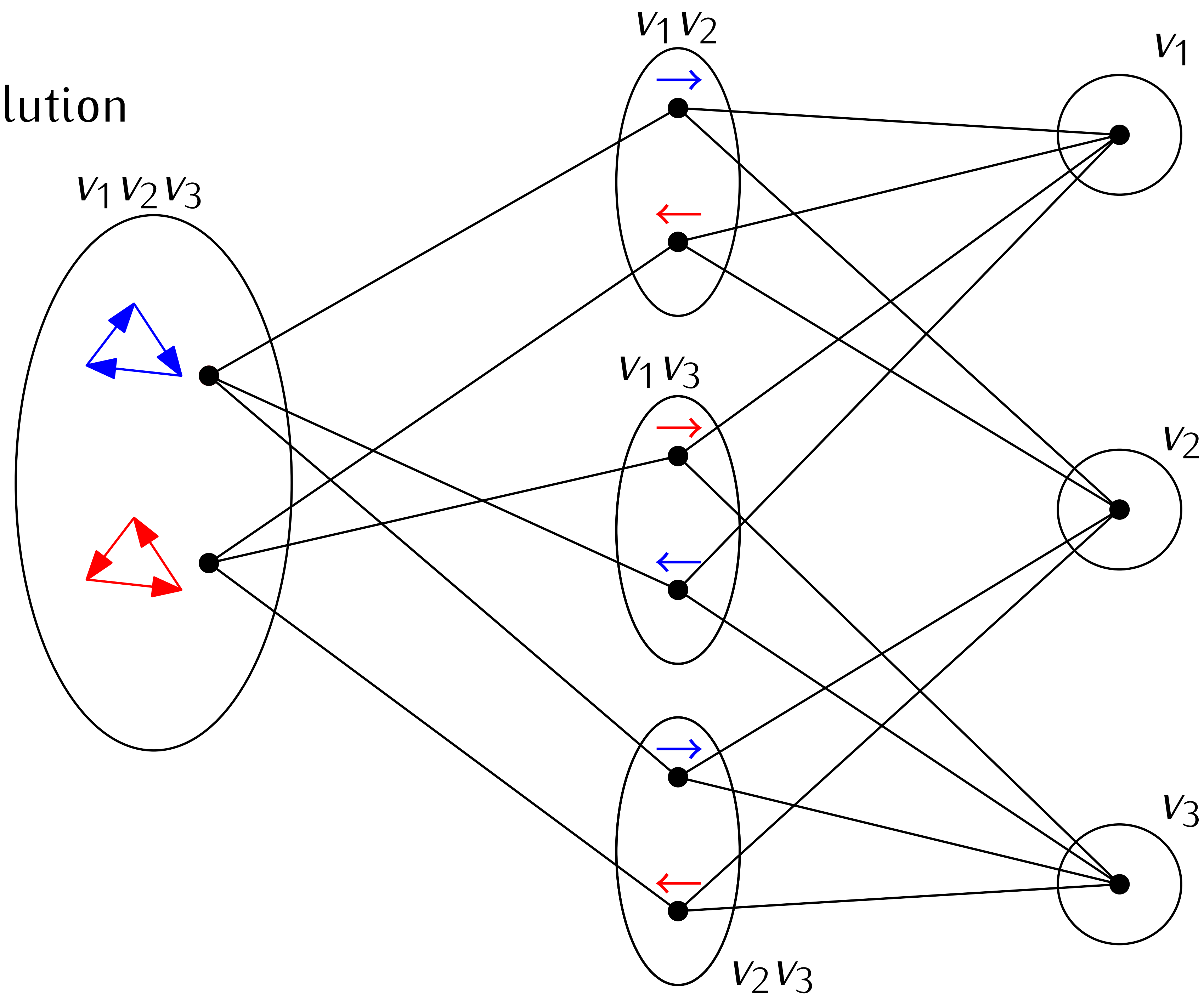

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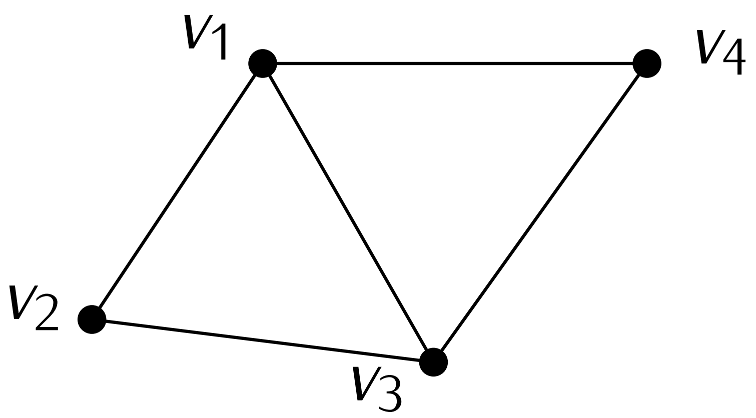

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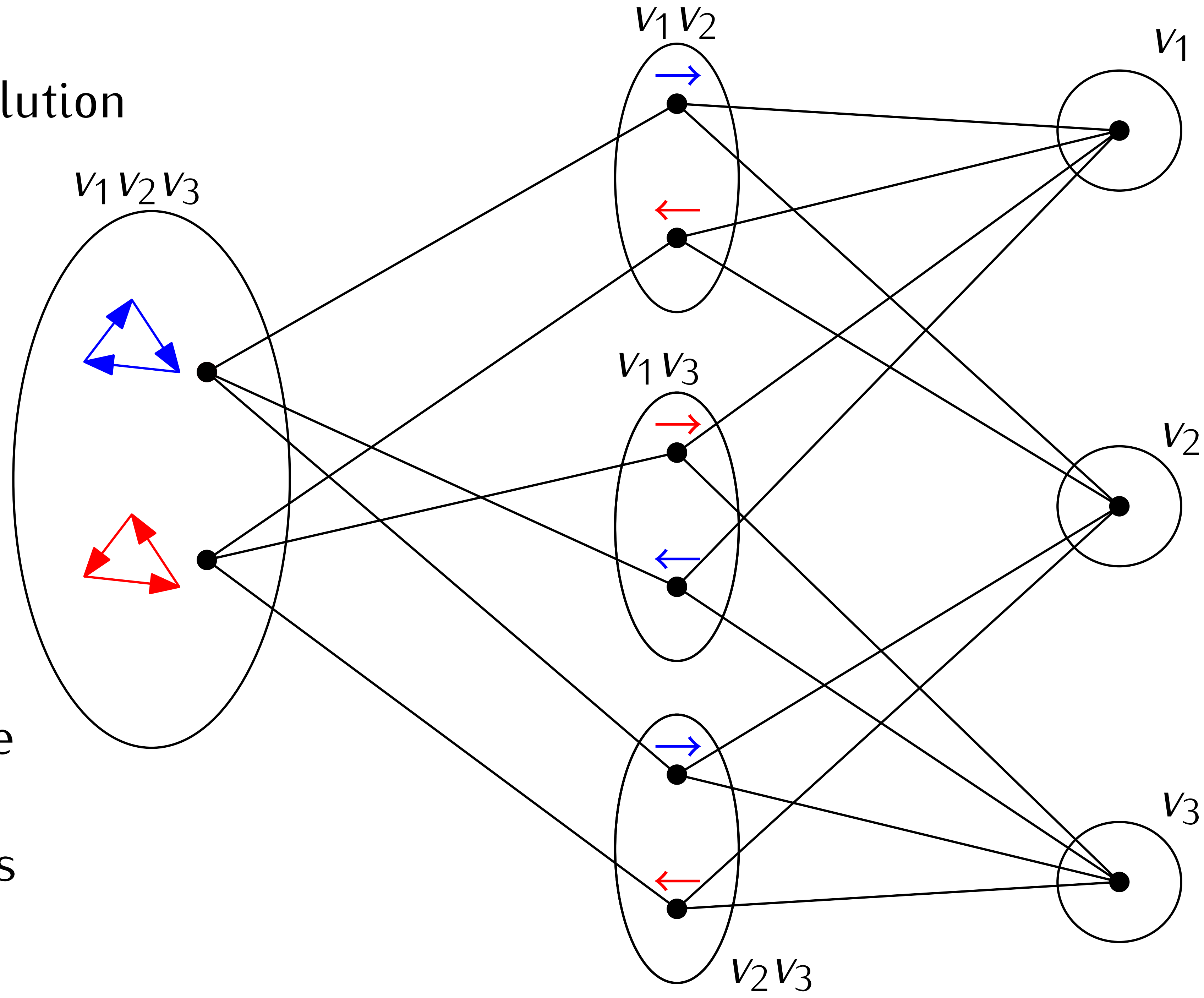

- Domain: structures in $\mathcal{K}$ of bounded size
- Unary relations: original constraints
- Binary relations: matching substructures

The “reduction to the finite”: cyclic orientability

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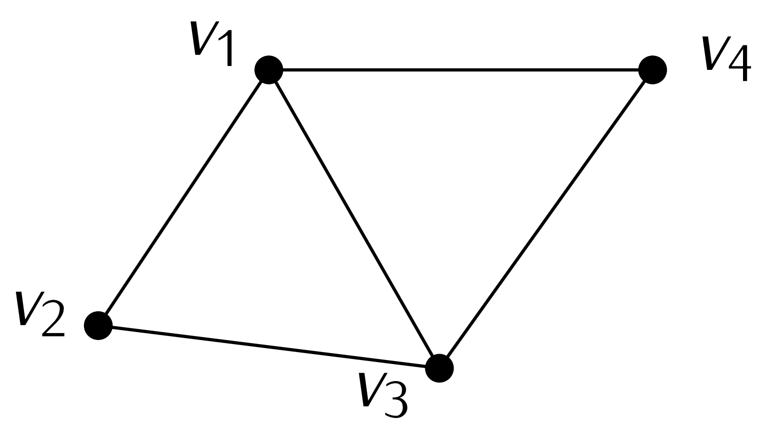

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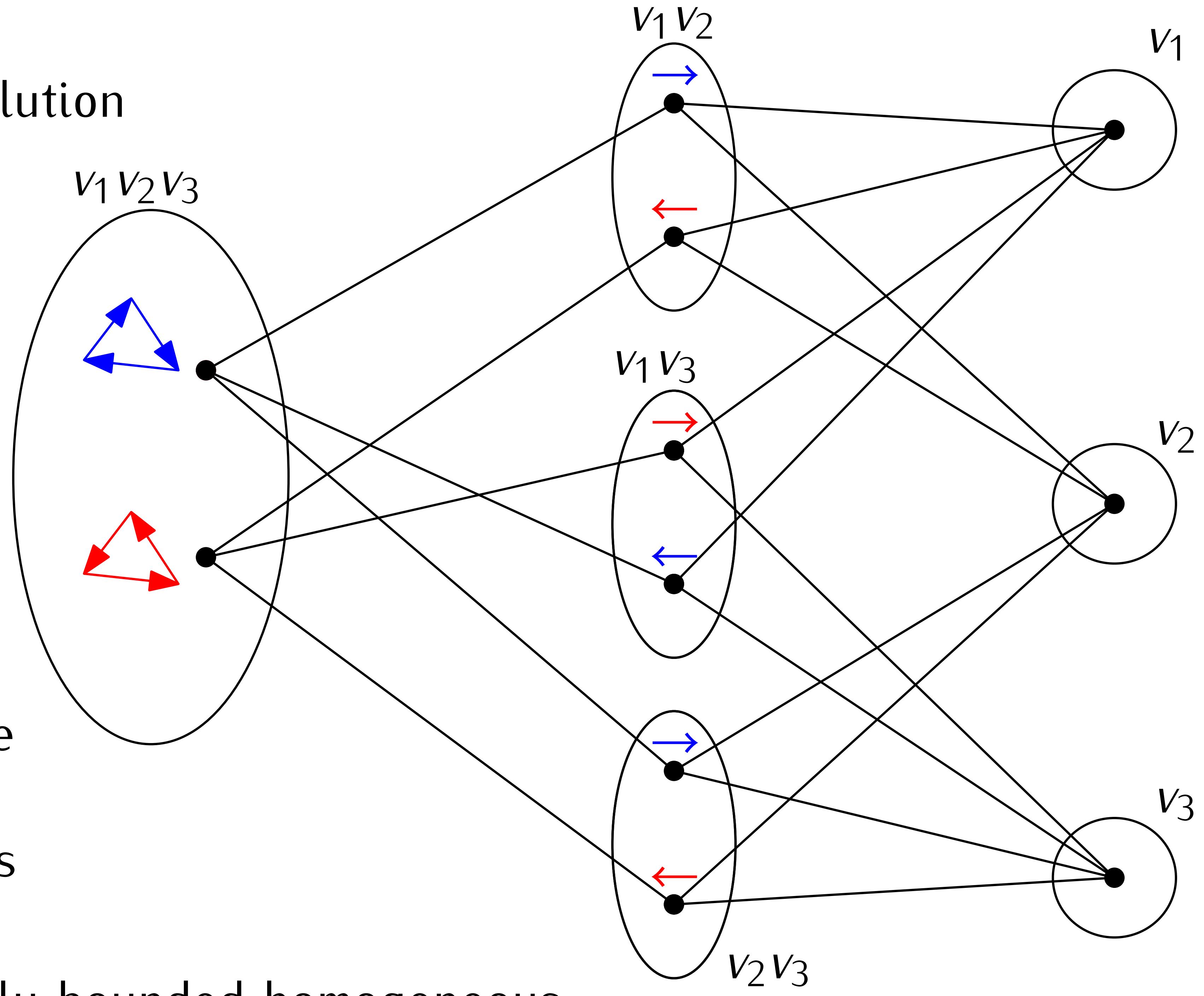

- Domain: structures in $\mathcal{K}$ of bounded size
- Unary relations: original constraints
- Binary relations: matching substructures

Observation. For every reduct $\mathbb{A}$ of a finitely bounded homogeneous structure with certificates $\mathcal{K}$, CSP($\mathbb{A}$) $\leq_P$ CSP($\mathbb{A}_{fin}$).

Define $\equiv_{\mathcal{K}}$ as a binary relation on n-tuples such that

$$\mathbf{a} \equiv_{\mathcal{K}} \mathbf{b} \iff \mathbf{a} \text{ and } \mathbf{b} \text{ induce the same certificate } \mathbb{K} \in \mathcal{K}$$

This relation induces equality on the finite label cover instance

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Question. Suppose that $\mathrm{CSP}(\mathbb{A}, \equiv_{\mathcal{K}})$ is NP-hard.

- In which cases can we prove that $\mathrm{CSP}(\mathbb{A})$ is NP-hard?
- What can we say otherwise?

$$\begin{tikzcd} \mathrm{Pol}(\mathbb{A}) \arrow[drr, dashed, bend left, "?"] \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] & & \\ \mathrm{Pol}(\mathbb{A}, \equiv_{\mathcal{K}}) \arrow[rr] & & \text{hardness clone} \end{tikzcd}$$

Polymorphisms of $\mathbb{A}_{fin}$: Canonical Functions

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Similar questions are interesting for "hardnesses" other than NP-hardness:

- (un)bounded width,
- (un)bounded linear width,
- (in)solvability in particular logics

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\end{tikzcd}$$

Problem (Monochromatic triangle)**.** Following decision problem:

Input A finite undirected graph $G = (V, E)$

Question Does there exist a coloring $\chi\colon E \to \{R, B\}$ such that $\chi^{-1}(c)$ is triangle-free for every $c \in \{R, B\}$?

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Garey & Johnson:

[GT6] **MONOCHROMATIC TRIANGLE**

INSTANCE: Graph $G=(V,E)$.

QUESTION: Is there a partition of E into two disjoint sets E_1,E_2 such that neither $G_1=(V,E_1)$ nor $G_2=(V,E_2)$ contains a triangle?

Reference: [Burr, 1976]. Transformation from 3SAT.

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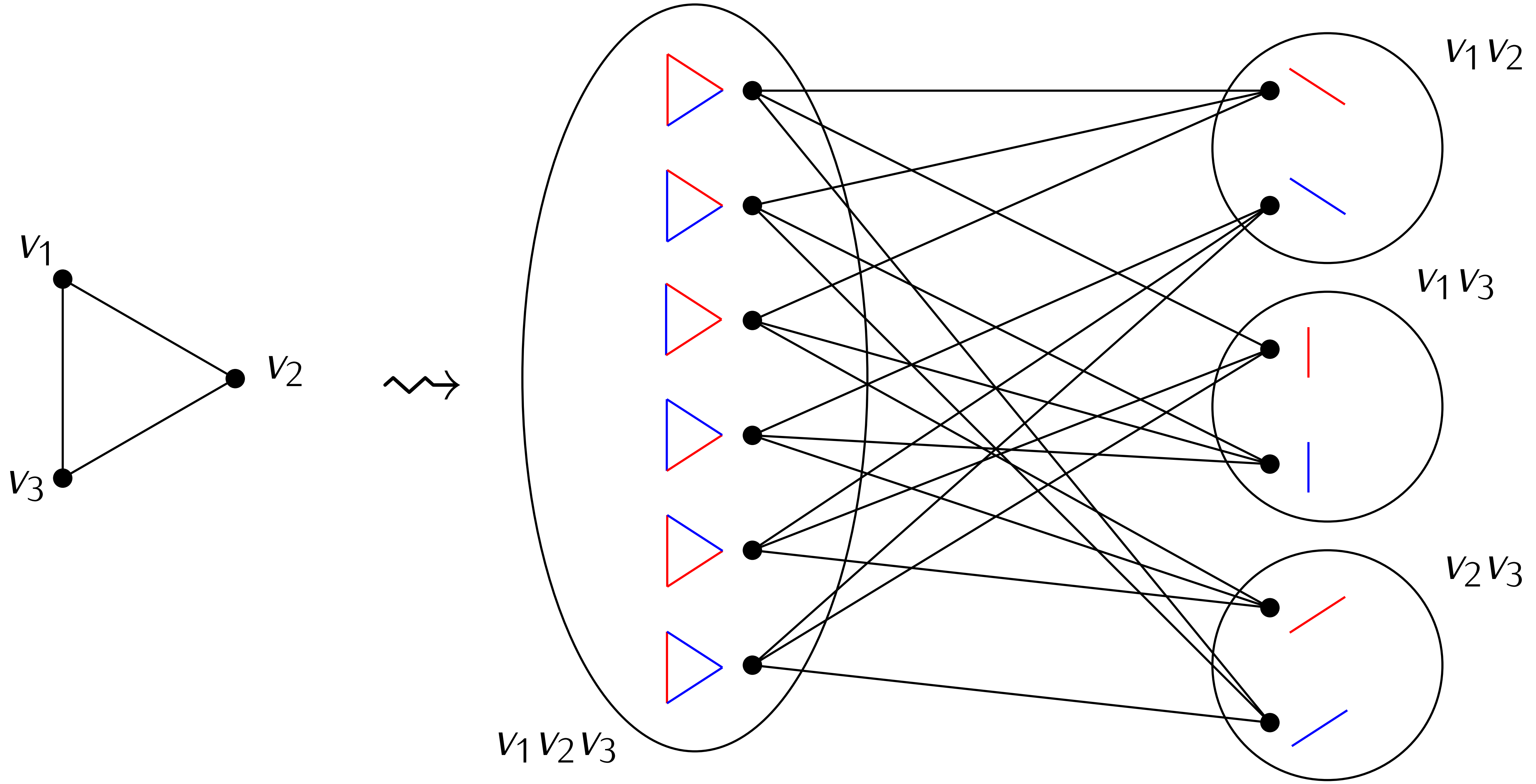

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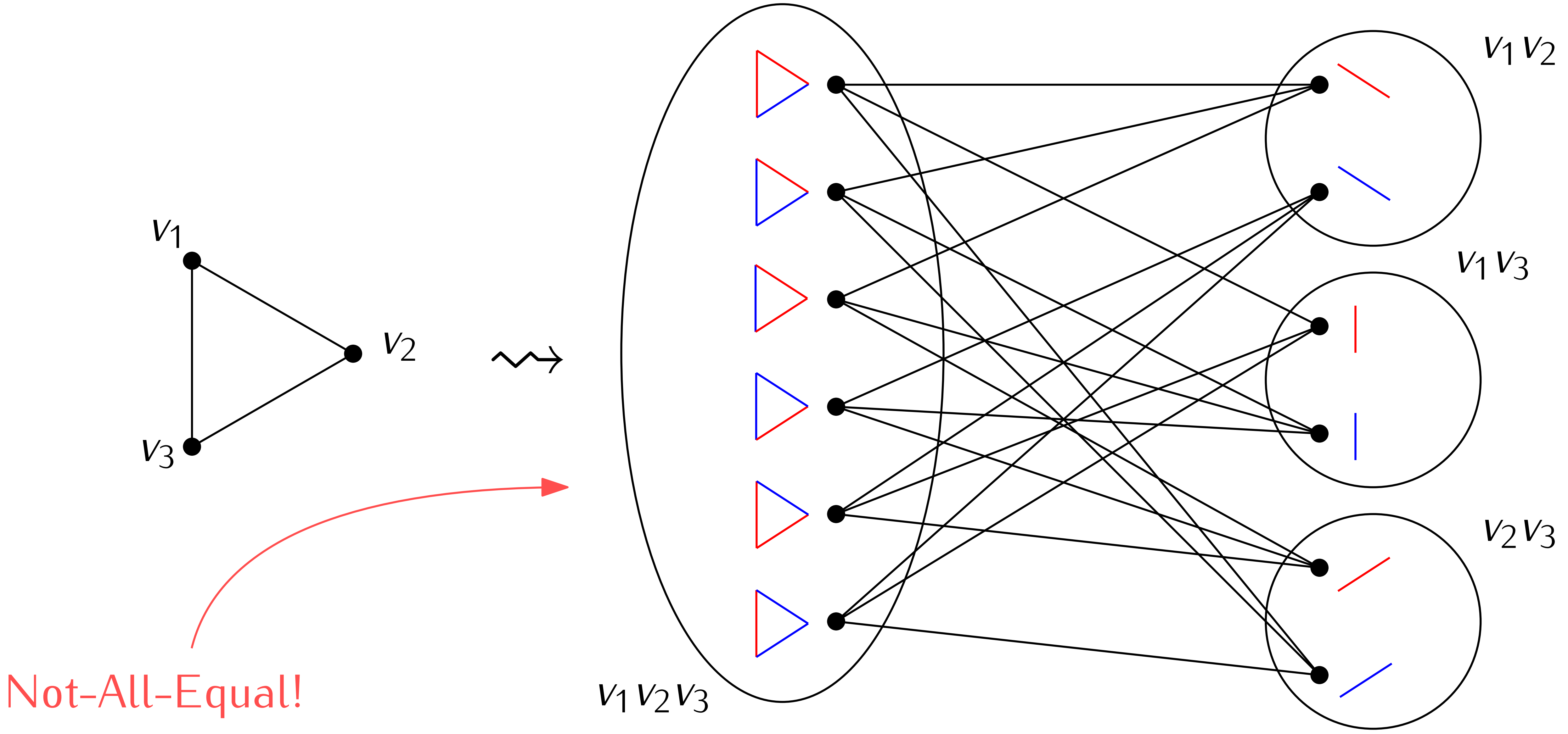

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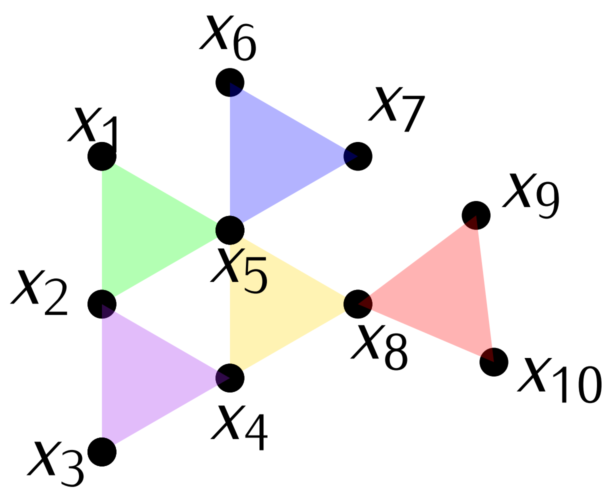

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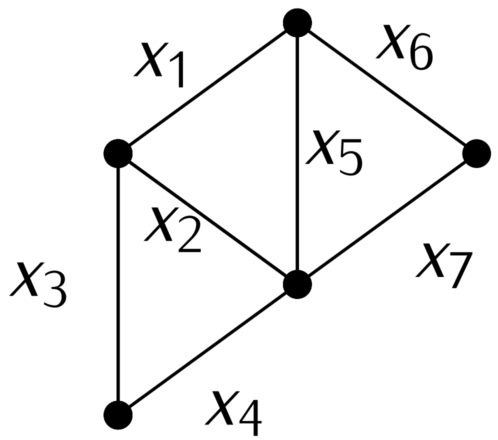

Instance of Not-All-Equal

Instance of monochromatic triangle

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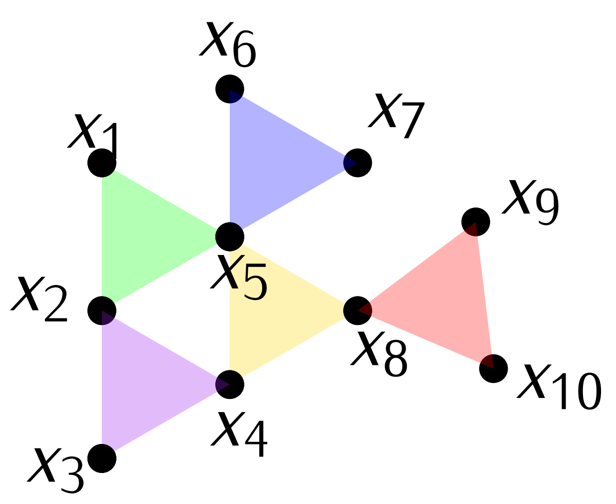

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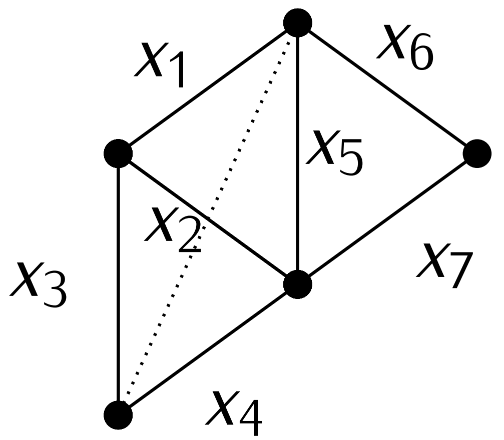

Where to put the edge for x_8?

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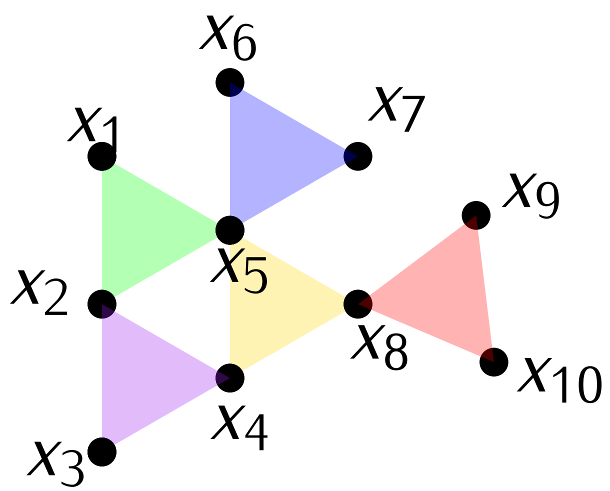

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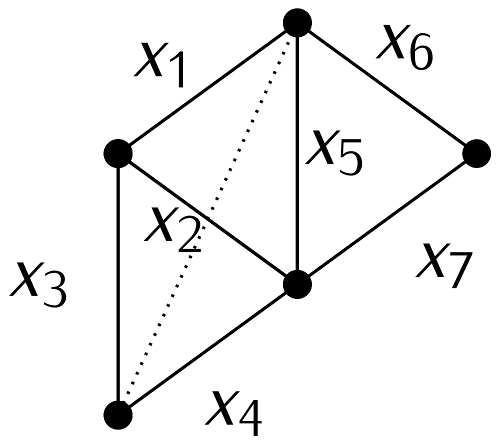

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The difference between NAE and Monochromatic Triangle:

- In NAE: unrestricted use of ternary constraints
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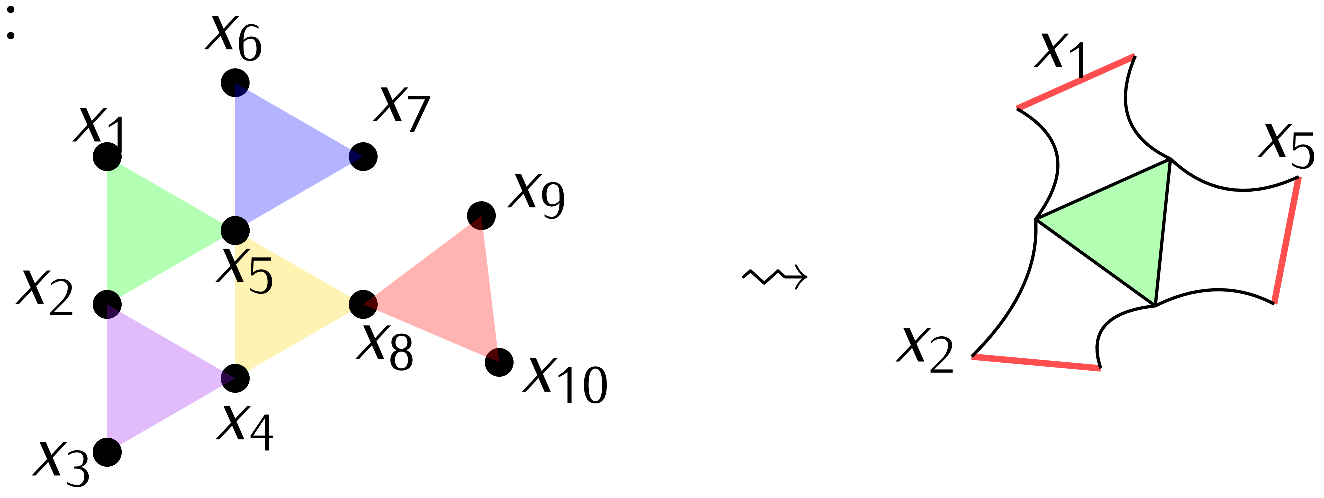

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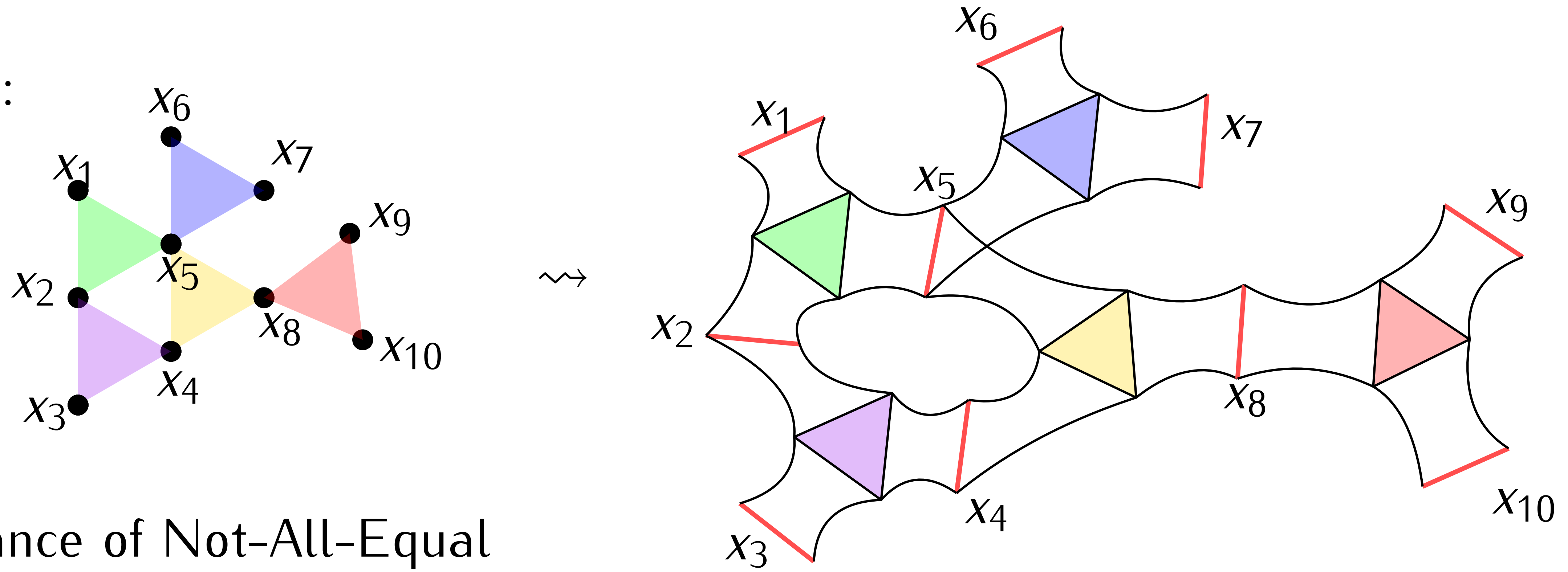

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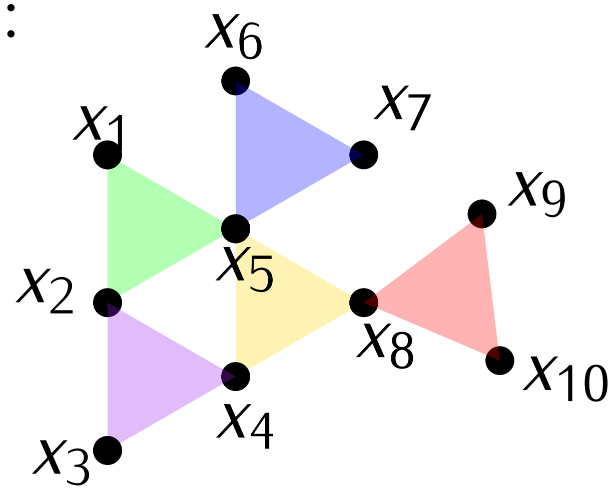

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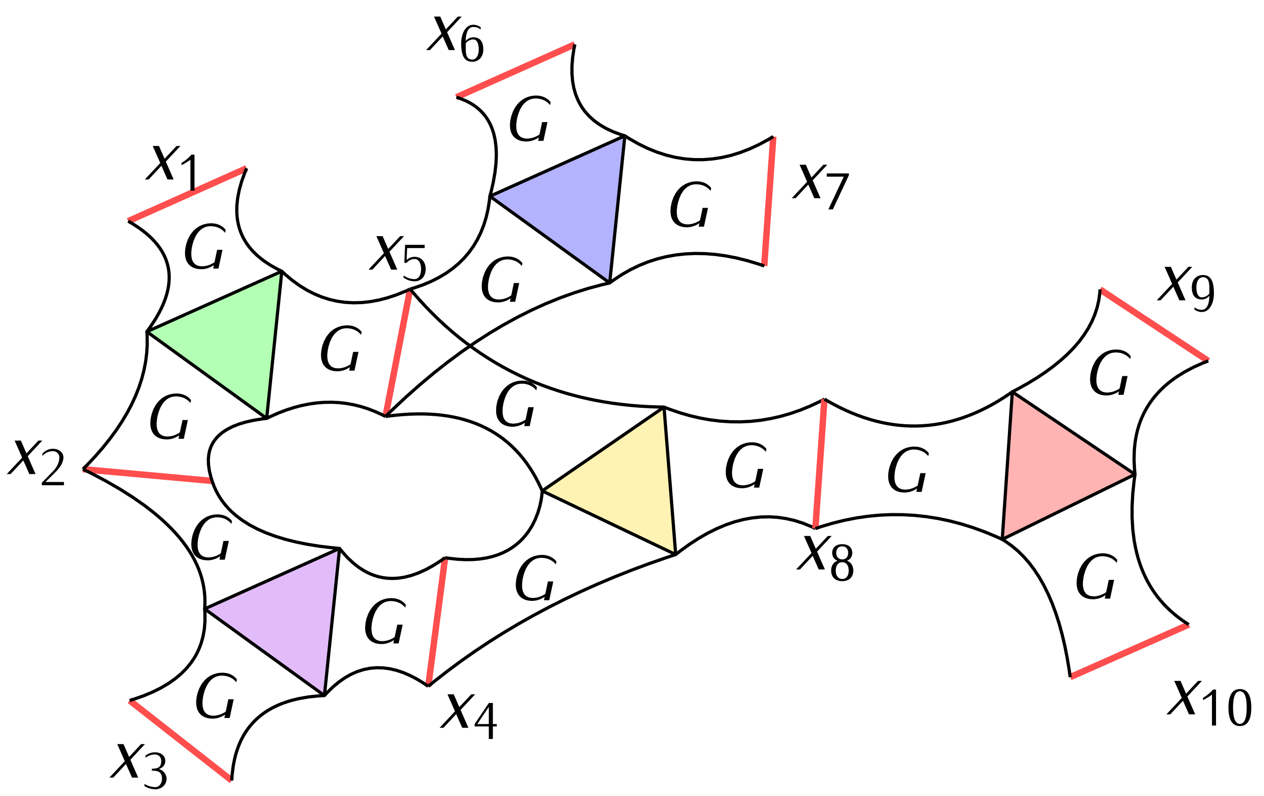

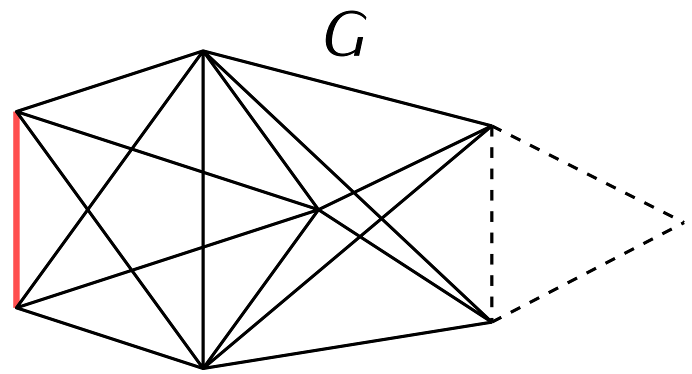

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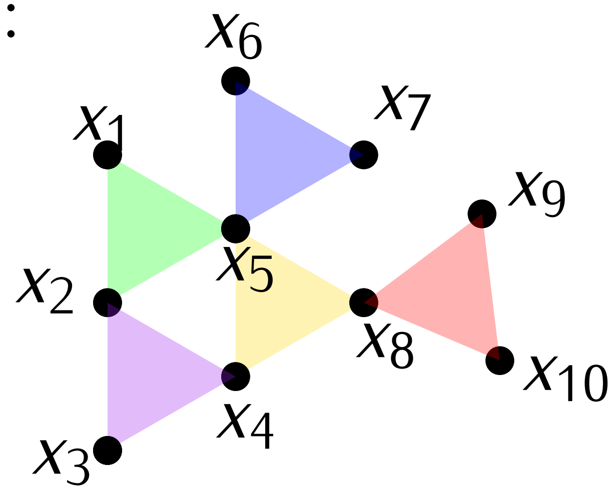
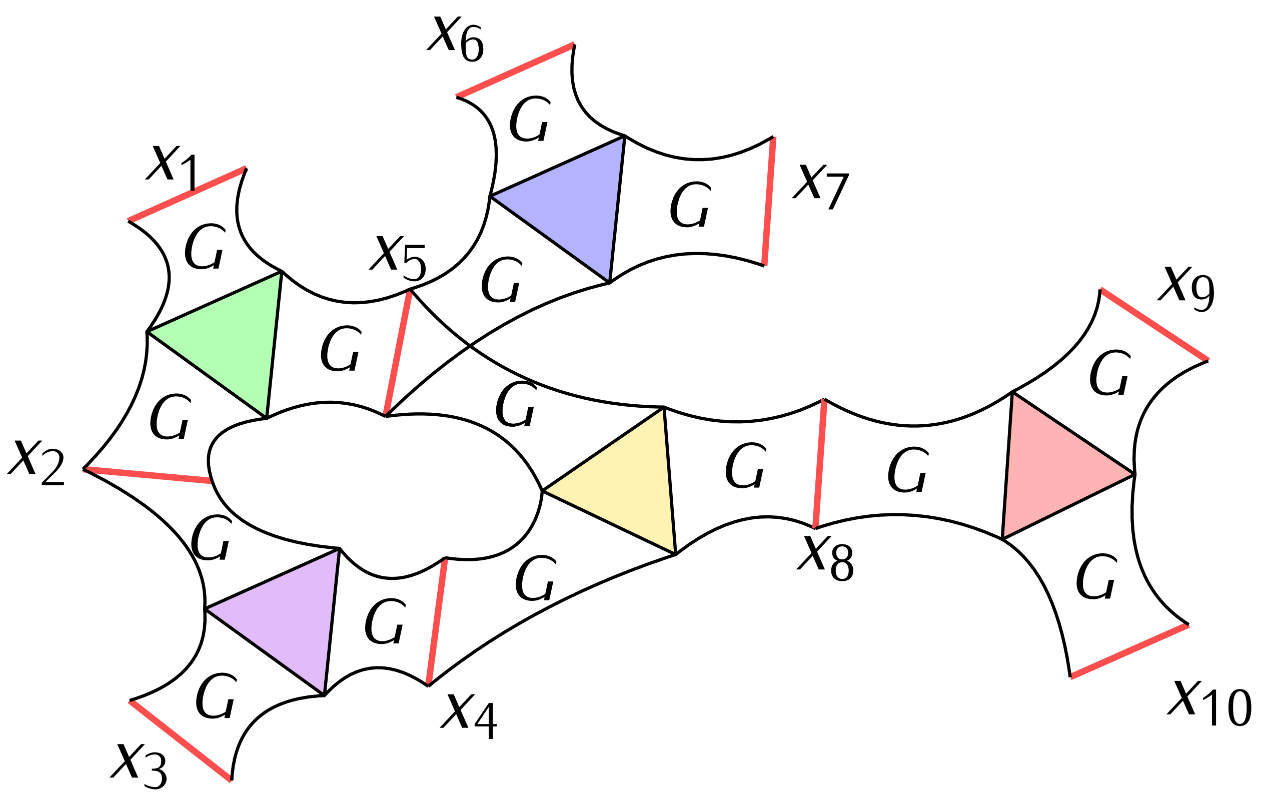


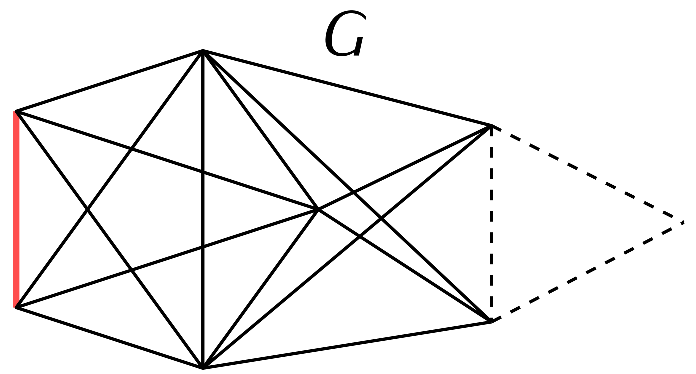
Instance of Not-All-Equal

$\rightsquigarrow$



Intuition:

- In the finite template, equality comes for free
- equality allows one to use "structural" constraints as "normal" constraints

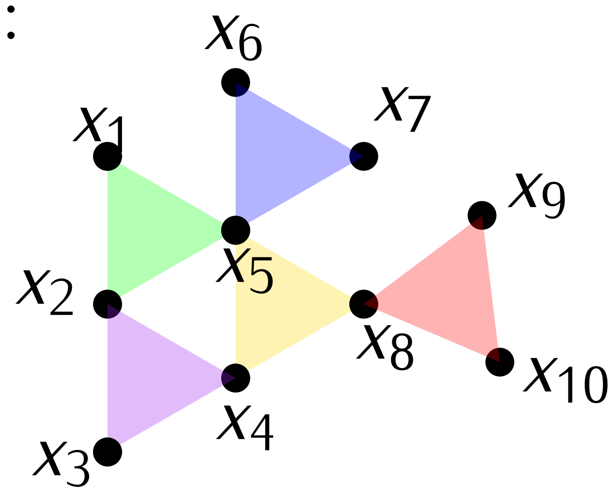


Problem (Monochromatic triangle). Following decision problem:

Input A finite undirected graph $G = (V, E)$

Question Does there exist a coloring $\chi\colon E \to \{R, B\}$ such that $\chi^{-1}(c)$ is triangle-free for every $c \in \{R, B\}$?

Second attempt:



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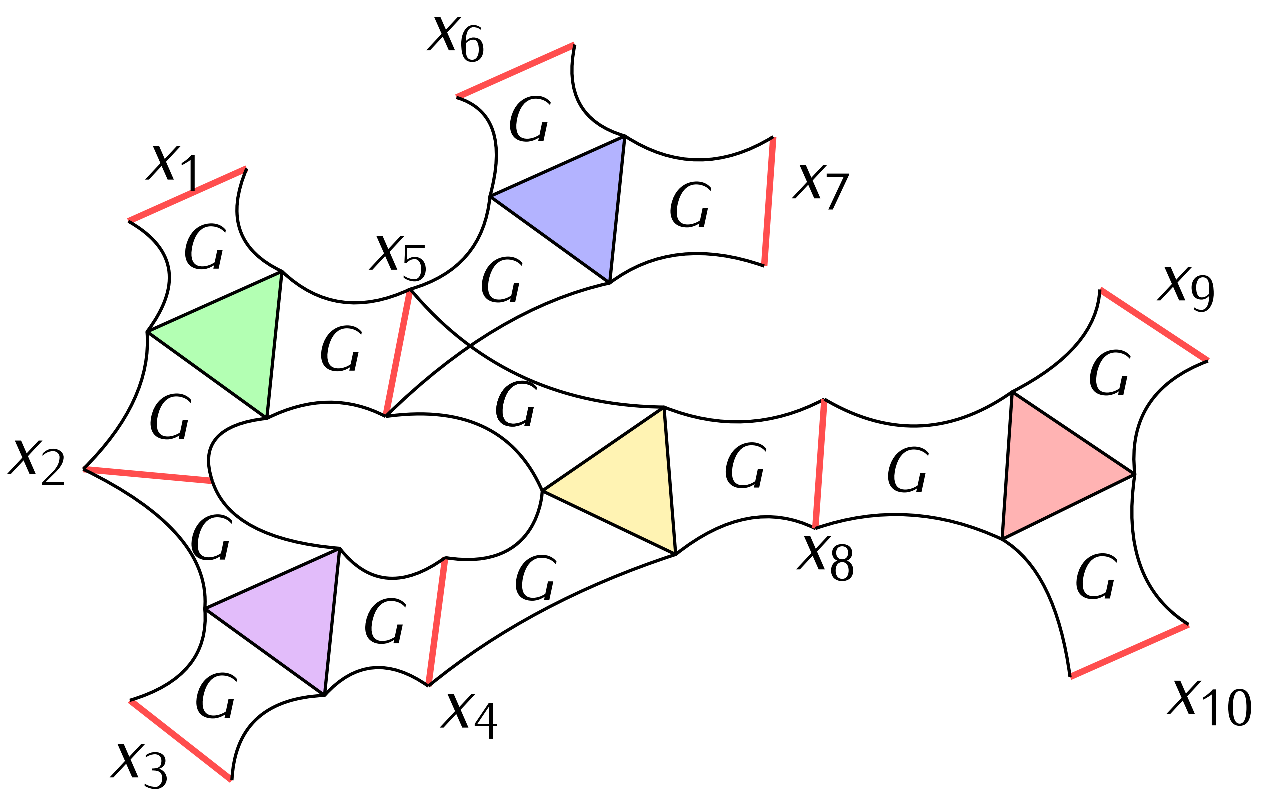
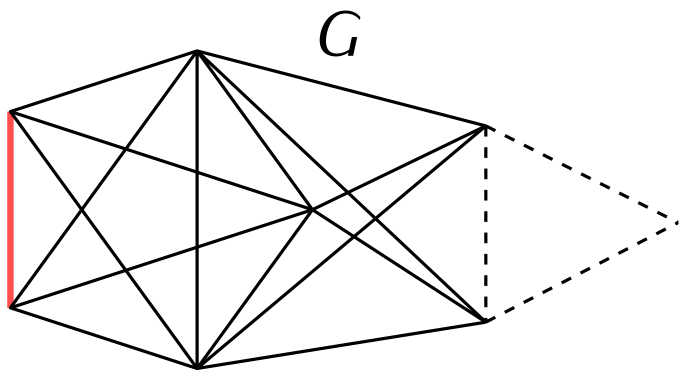


Proposition (Barsukov, M., Perinti). Monochromatic Triangle remains hard on K_4-free graphs

(The gadget is beautiful but too large to fit on this slide)



Observation. $\mathrm{CSP}(\mathbb{A}_{fin})$ is NP-hard whenever $\mathcal{K}$ is the class of linear orders.

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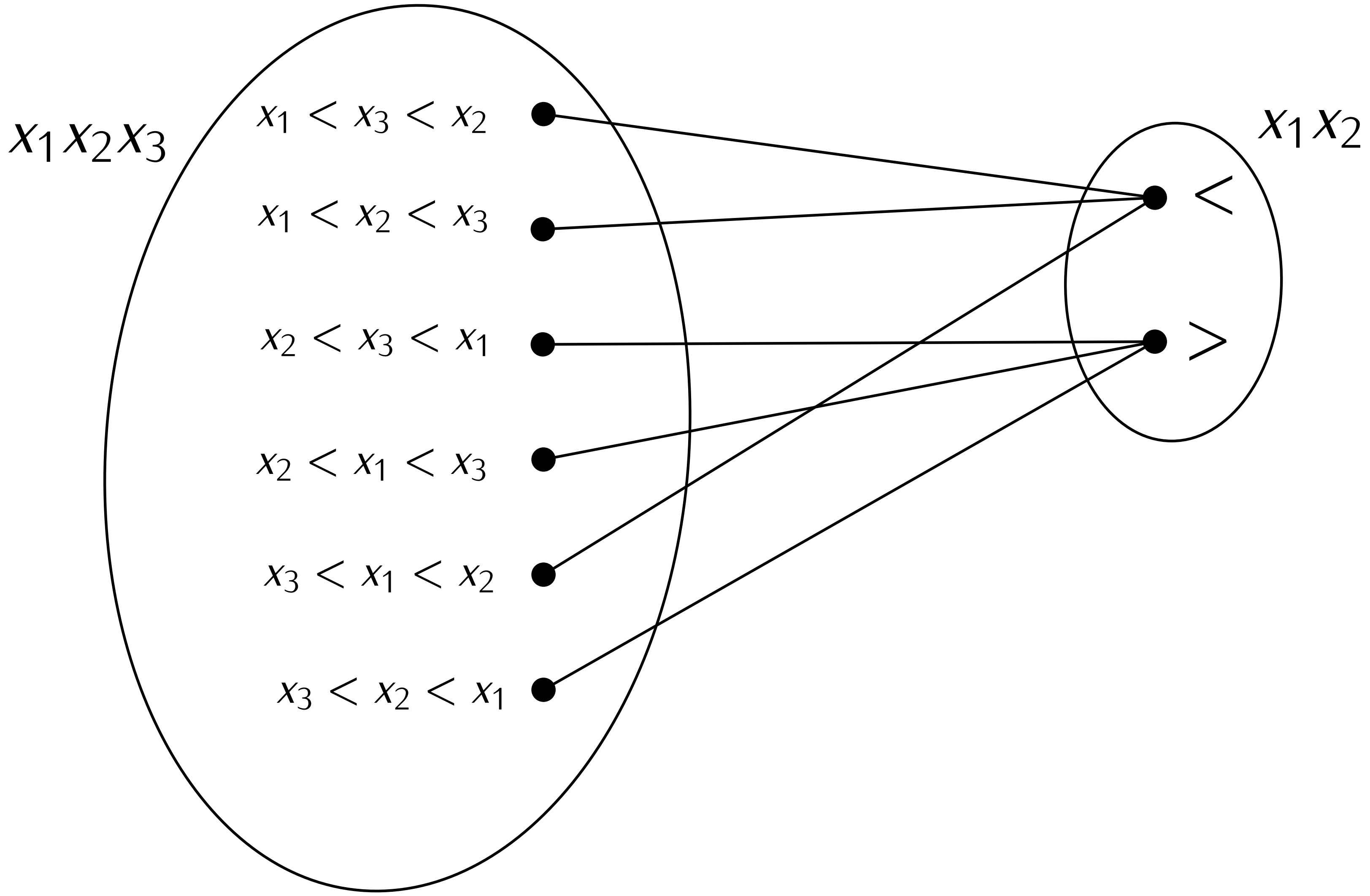

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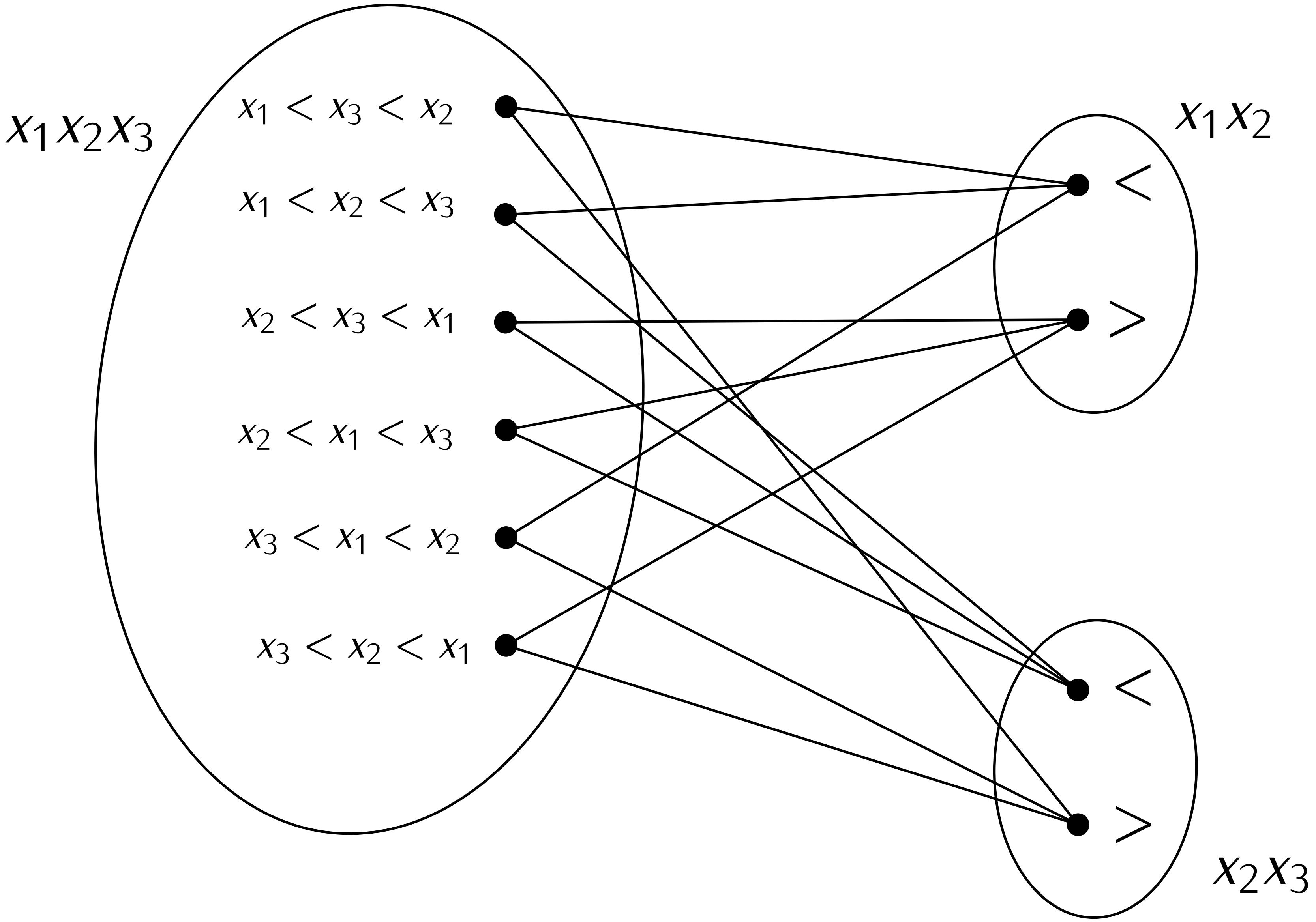

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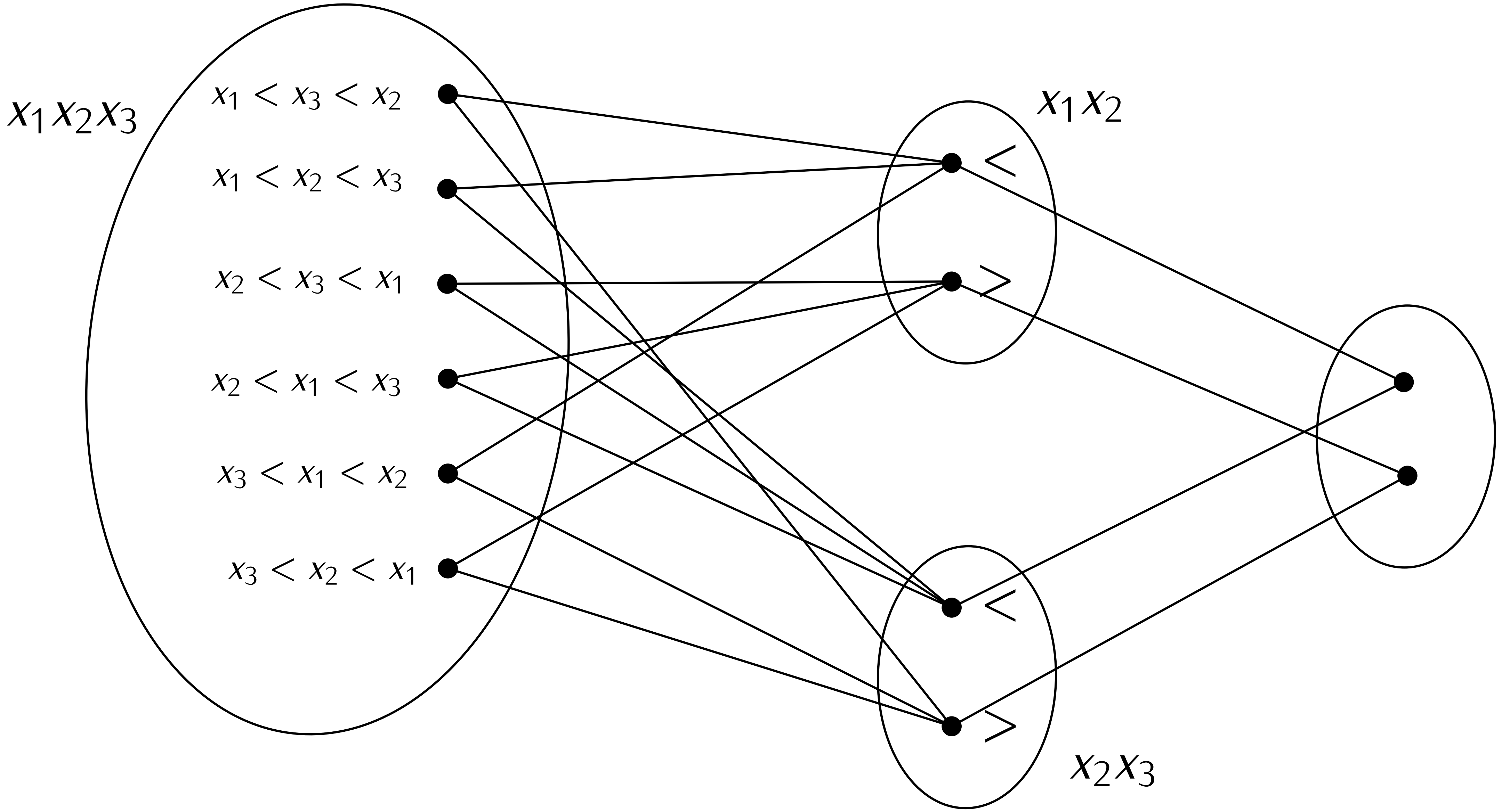

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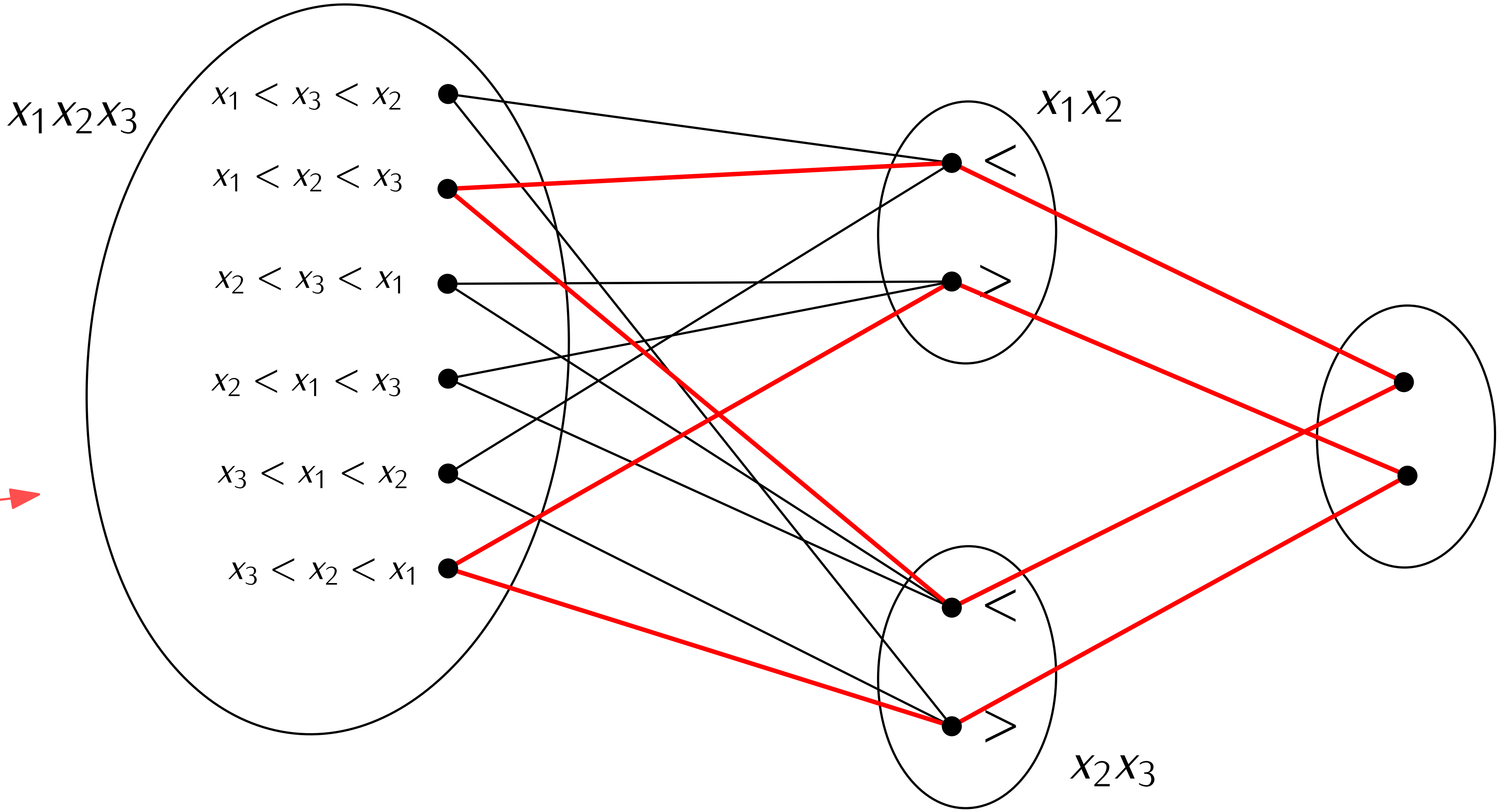
This is Betweenness



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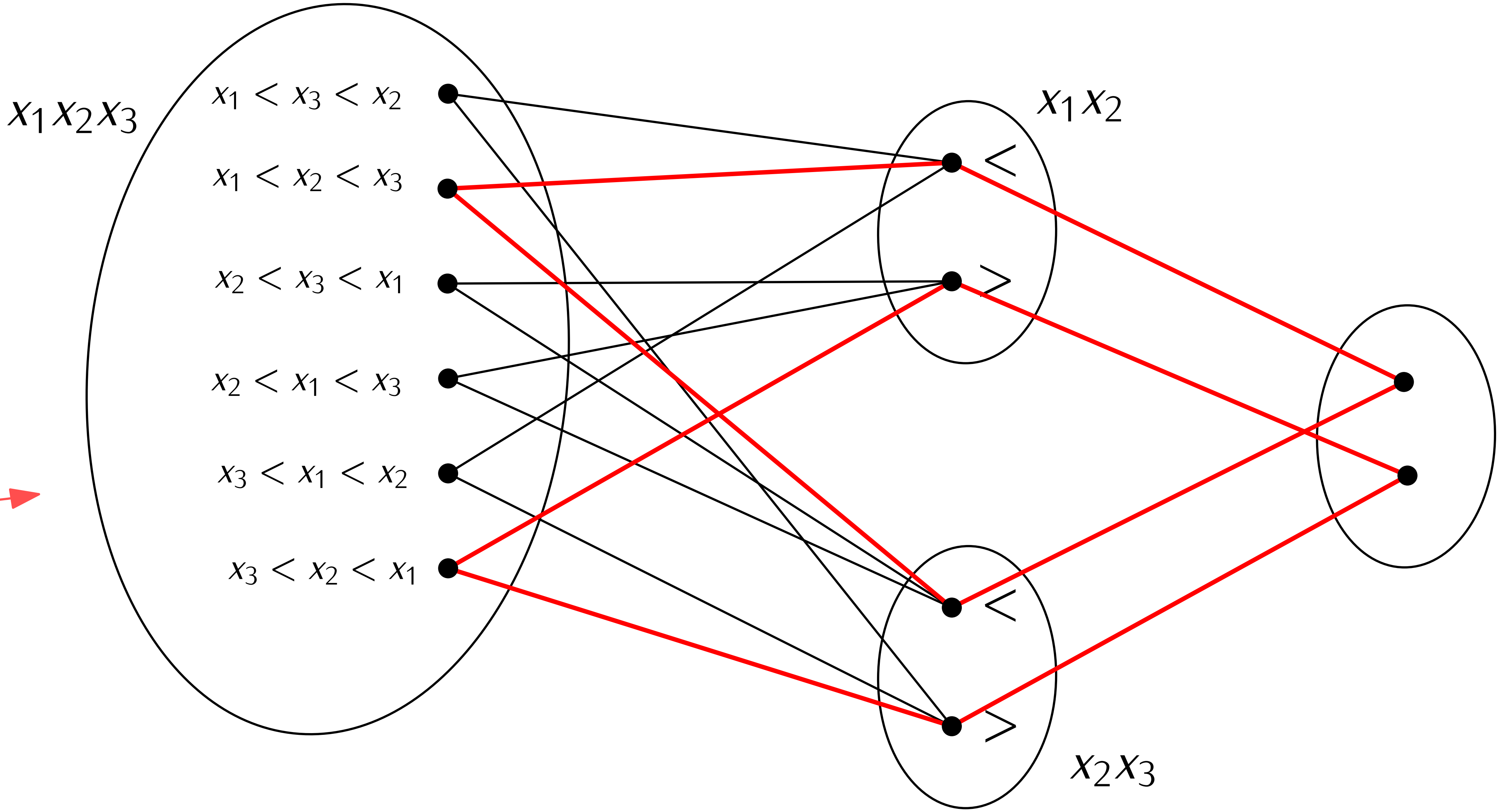
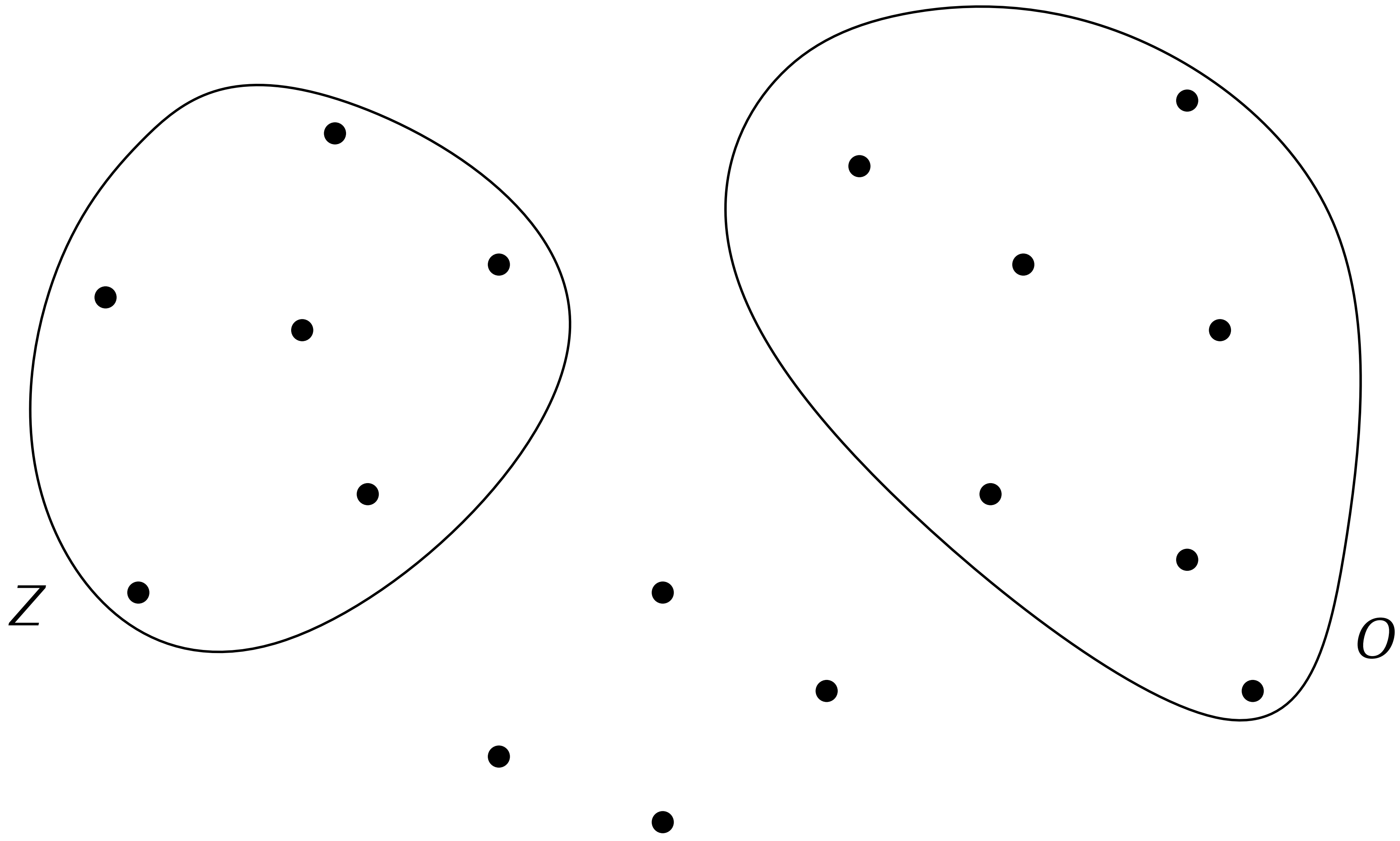


Proposition (Pinsker). CSP($\mathbb{A}_{fin}$) is NP-hard whenever $\mathcal{K}$ is a class of linearly ordered structures.

- CSP($\mathbb{A}_{fin}$) is NP-hard if there exist pp-definable $O, Z \subseteq \mathbb{A}_{fin}$ and $N \subseteq (O \cup Z)^3$ representing NAE.



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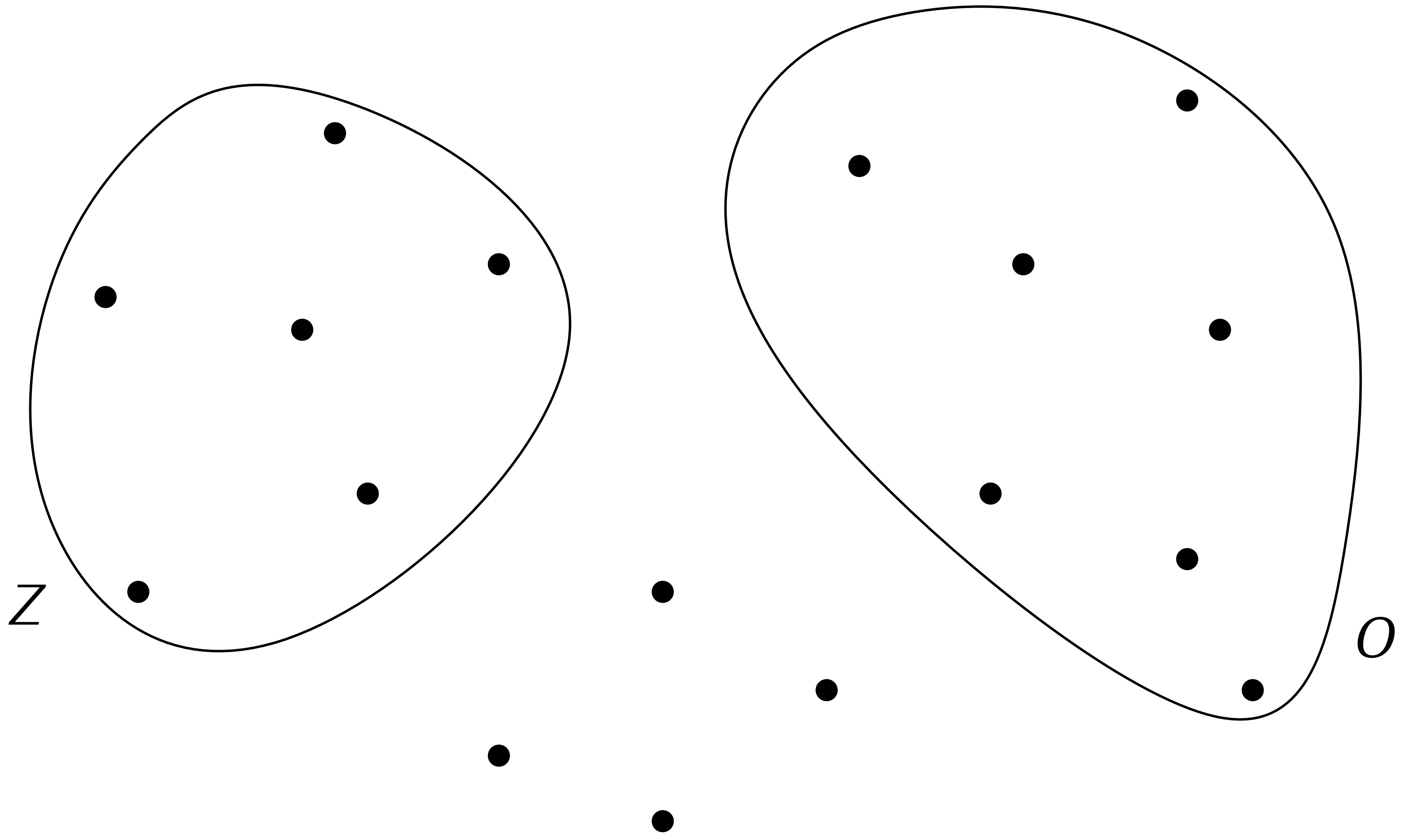
- If we can define those relations in $\mathbb{A}$, then CSP($\mathbb{A}$) is NP-hard



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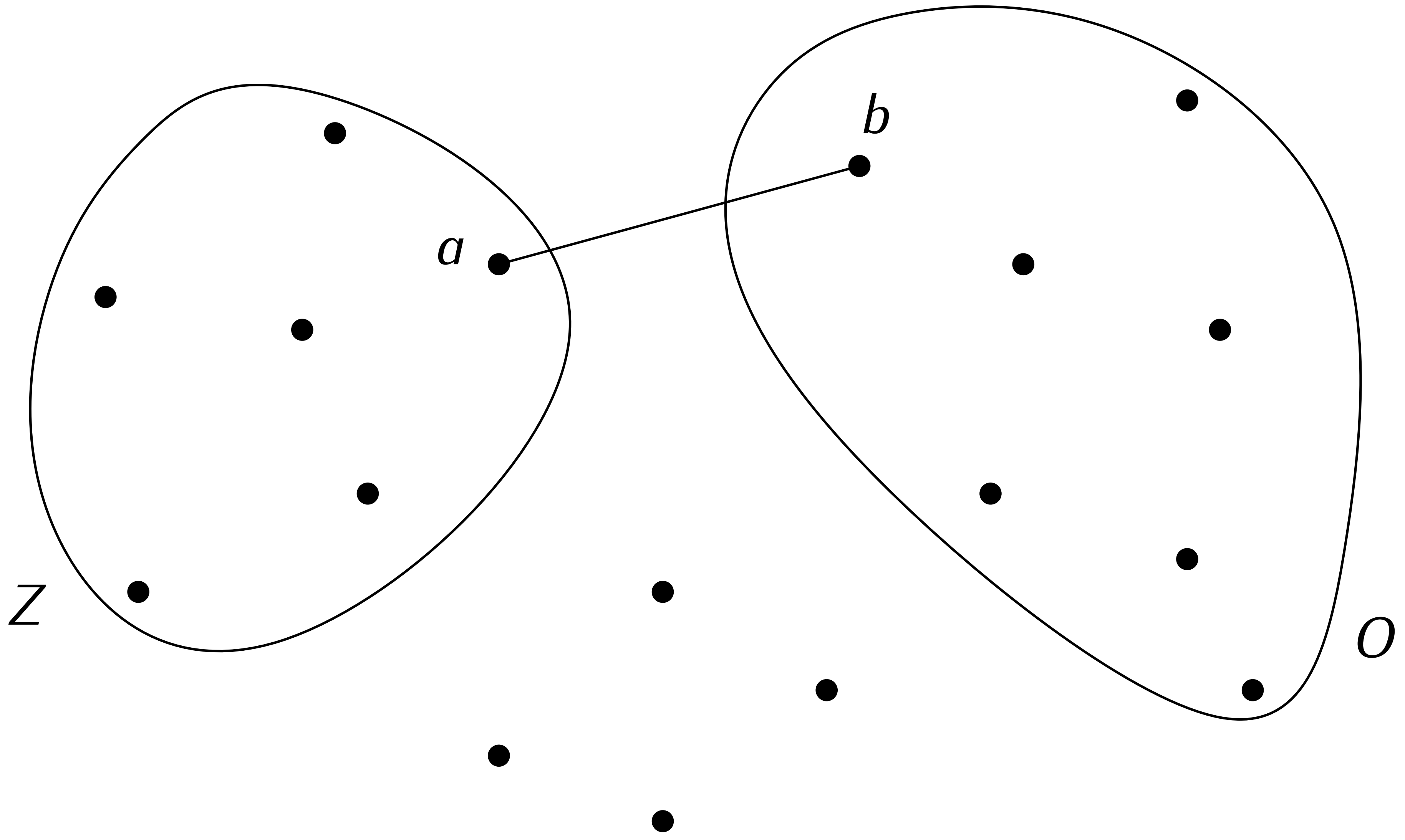

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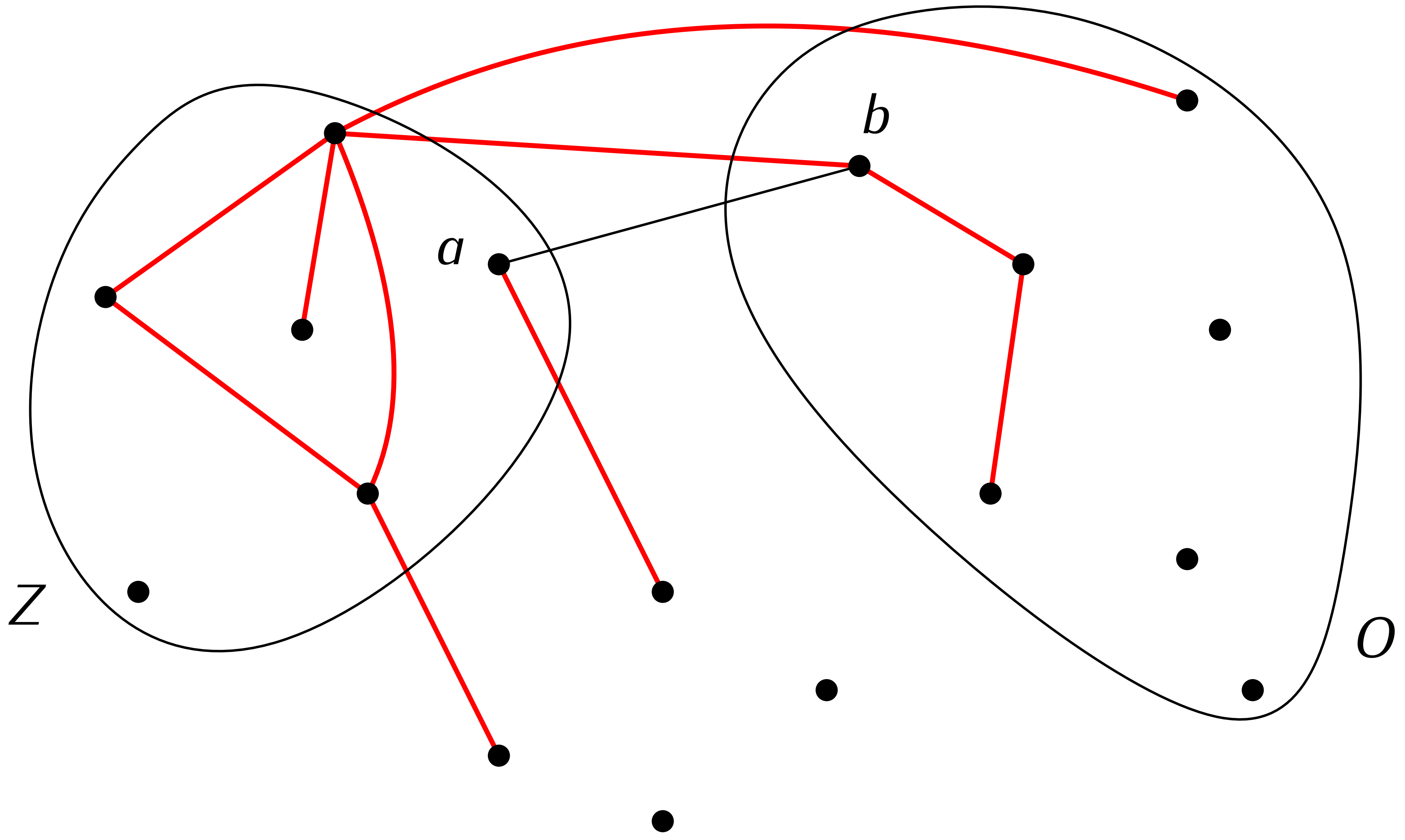

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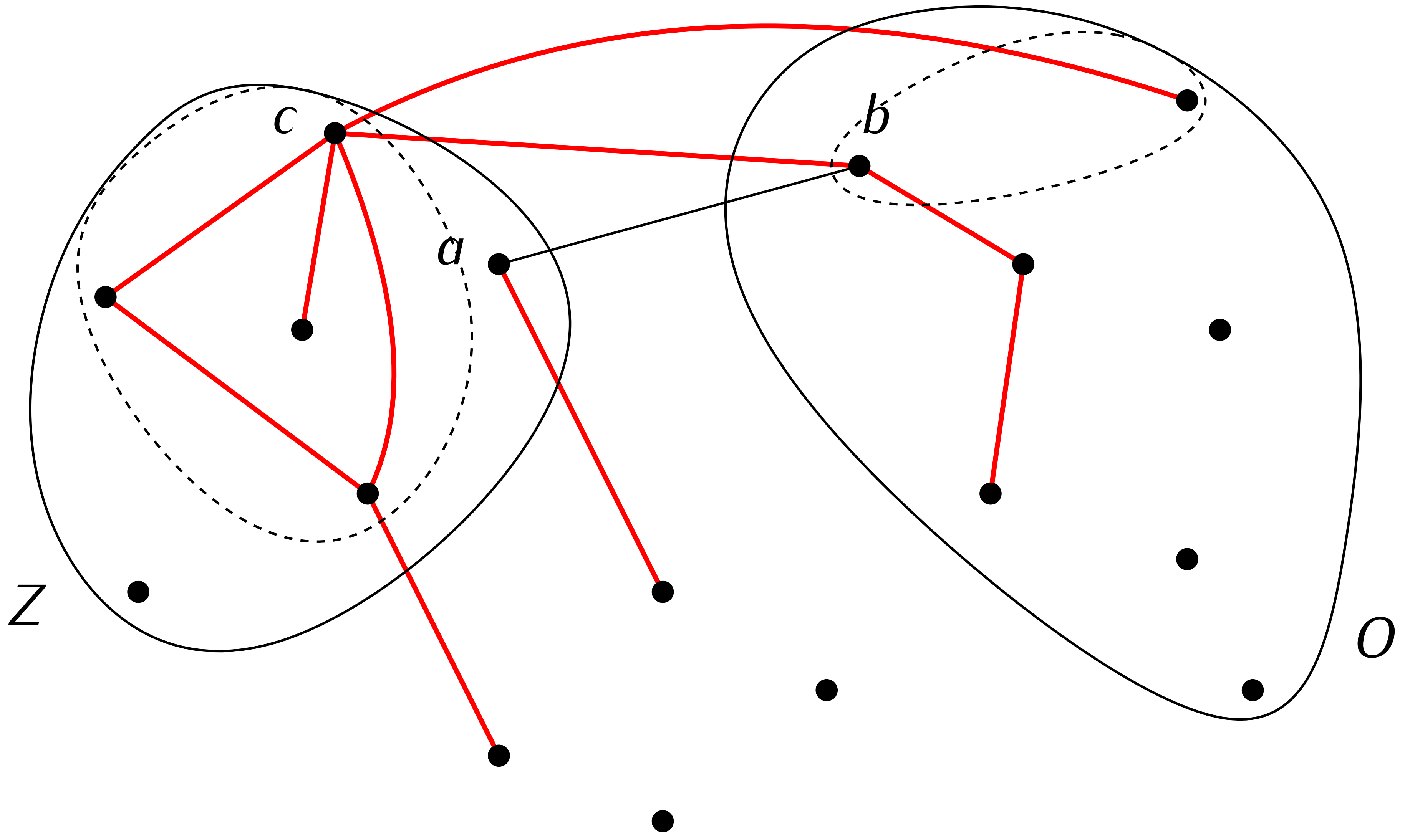

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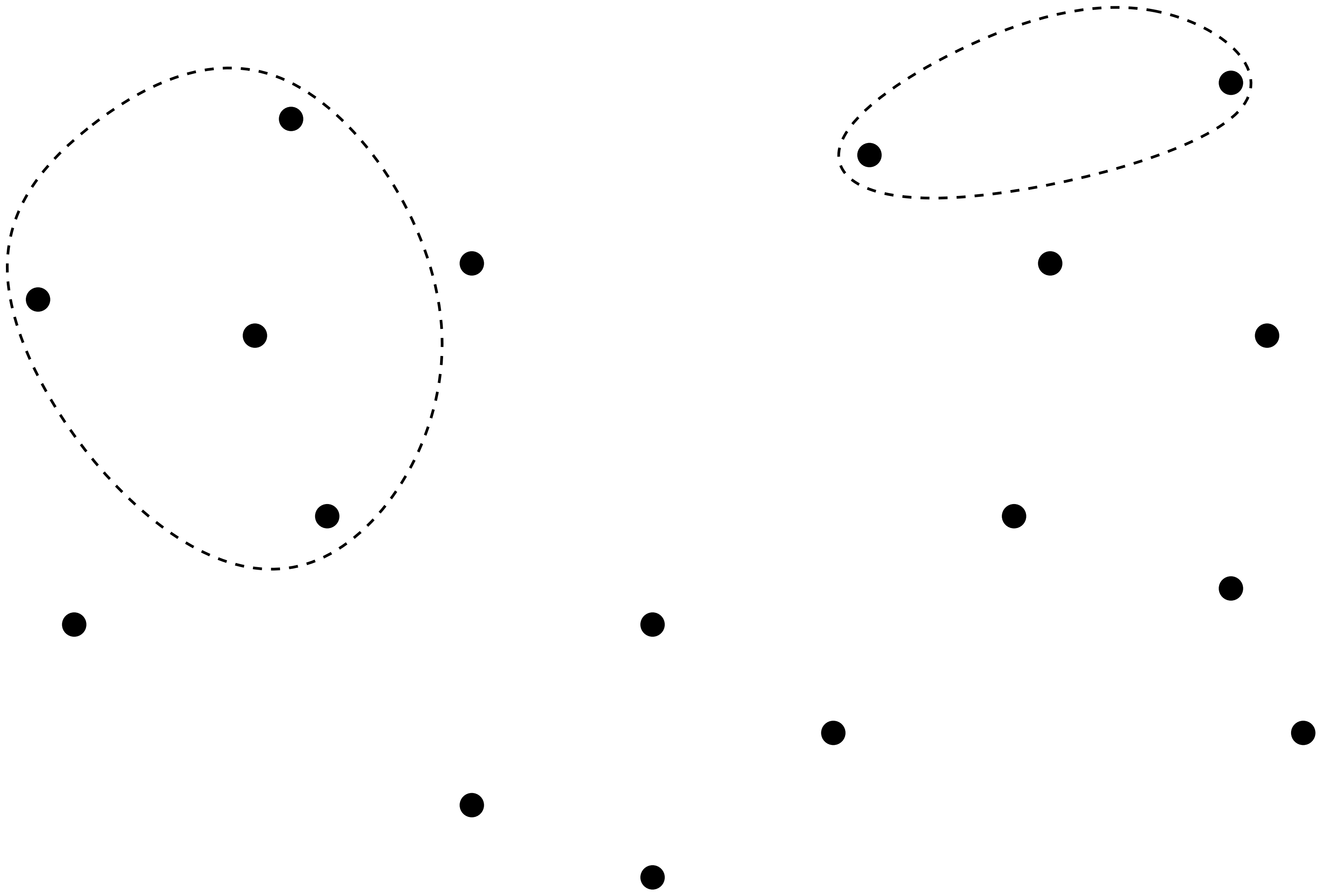
- Finite CSP must be hard on those new subsets $\rightsquigarrow$ existence of a new "NAE" like relation



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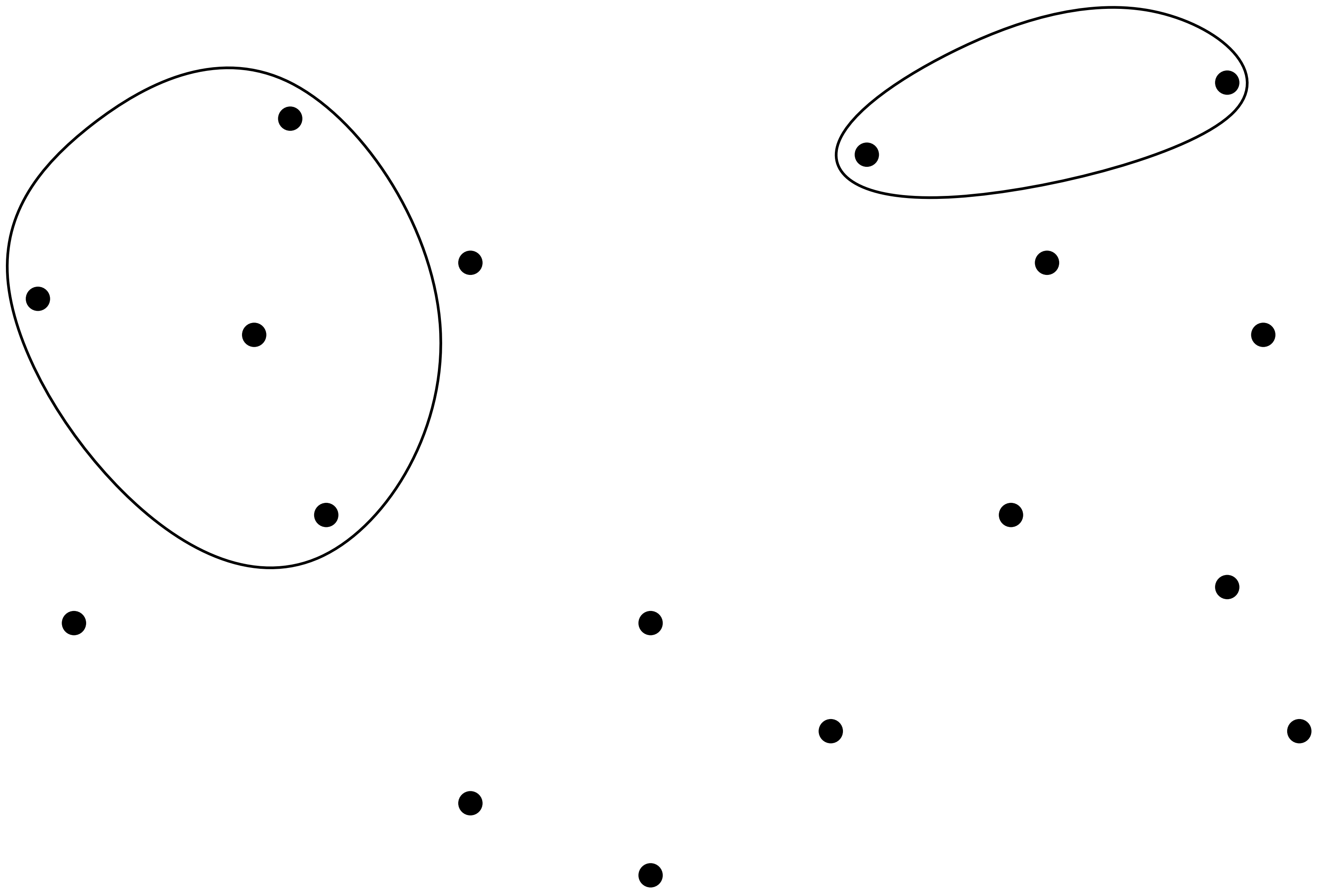

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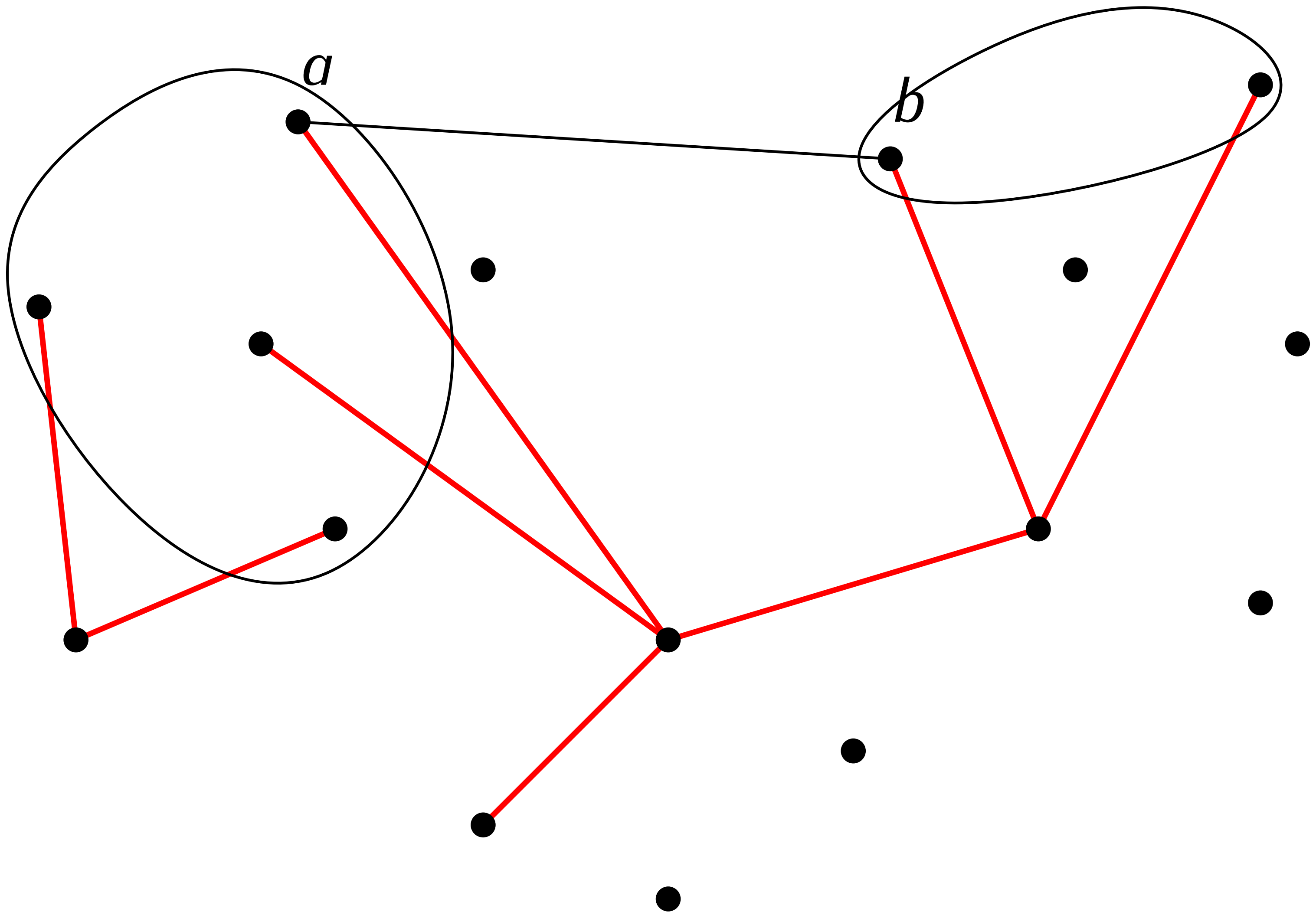
- No more paths of even length between the potatoes



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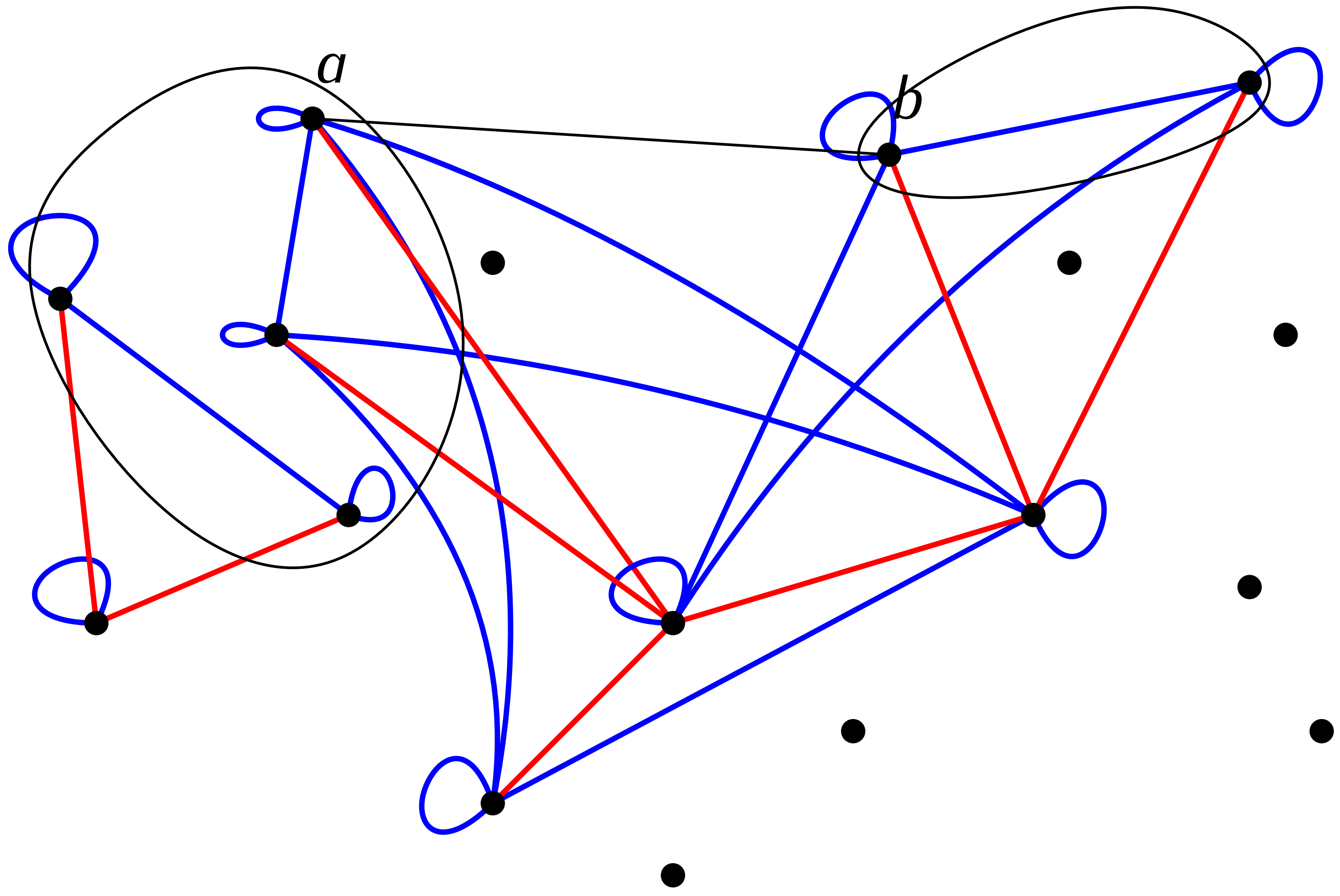

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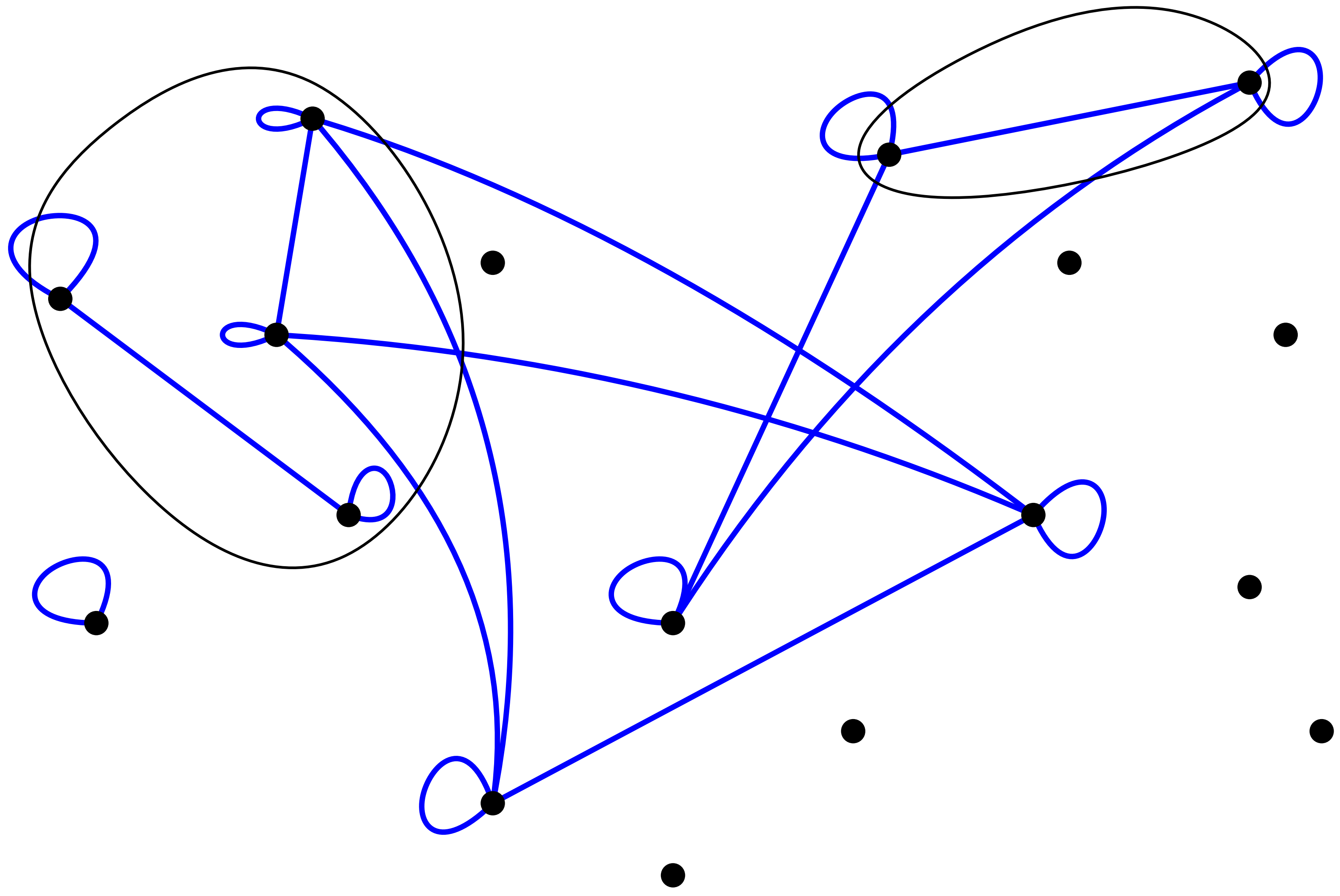

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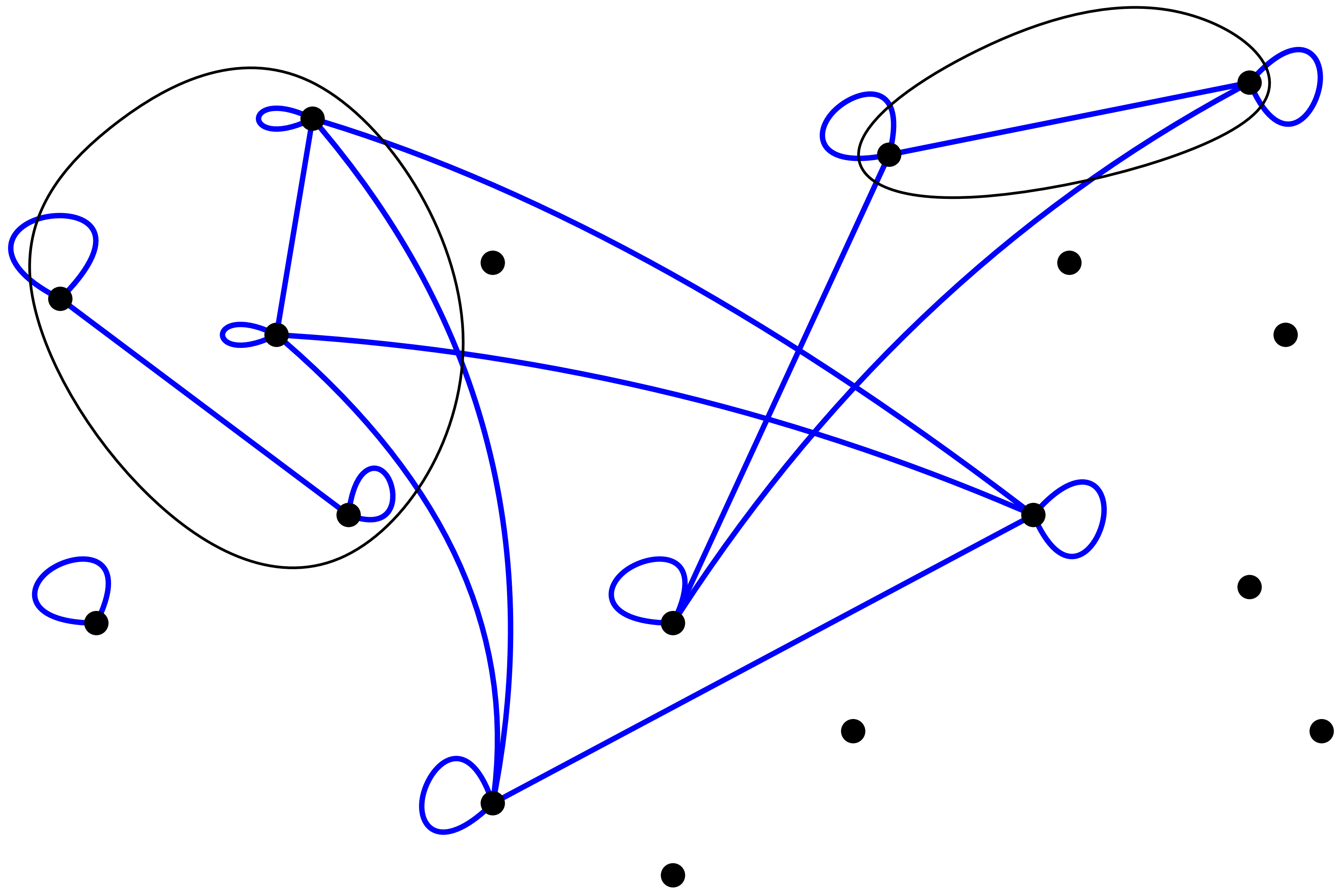
- This defines (in $\mathbb{A}$) an equivalence relation that refines the original partition



Theorem (M., Pinsker). Suppose CSP($\mathbb{A}_{fin}$) is NP-hard.
There exists an NP-hardness "certificate" $\{Z, O\}$ such that one of the following holds:

- $\exists$ pp-definable $\sim$ that refines $\{Z, O\}$
  $\sim$ is an approximation of $\{Z, O\}$.
- Every pp-definable binary relation with an edge between Z and O contains a pair $(a, \alpha(a))$ for some $\alpha \in \mathrm{Aut}(\mathbb{A})$

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Theorem (M., Nagy, Pinsker, Wrona)**.** Suppose CSP($\mathbb{A}_{fin}$) is not NP-hard.
For every minimal subfactor of Pol($\mathbb{A}_{fin}$), one of the following holds:

- $\exists$ pp-definable $\sim$ that approximates the subfactor
- Every pp-definable cyclic relation on the subfactor contains a pseudoloop.

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Theorem (M., Pinsker). Suppose

- $\{Z, O\}$ has a smooth approximation $\sim$ definable in $\mathbb{A}$
- $\mathrm{Pol}(\mathbb{A})$ locally interpolates $\mathrm{Pol}(\mathbb{A}, \equiv_{\mathcal{K}})$.

Then $\exists$ uniformly continuous minion homomorphism $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{K}_3)$.
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- Need some assumptions on Aut($\mathbb{A}$) to go from "normal" approximation to "smooth"
  No algebraicity and primitivity
- "local interpolation" means every $f \in$ Pol($\mathbb{A}$) "looks like" some $\tilde{f} \in$ Pol($\mathbb{A}, \equiv_{\mathcal{K}}$) on every finite set

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Implies $\exists f \in \mathrm{Pol}(\mathbb{A})$ such that if $f(a,b), f(b,a) \in Z \cup O$

$$f(a,b) \in Z \Leftrightarrow f(b,a) \in Z$$

for all a, b disjoint

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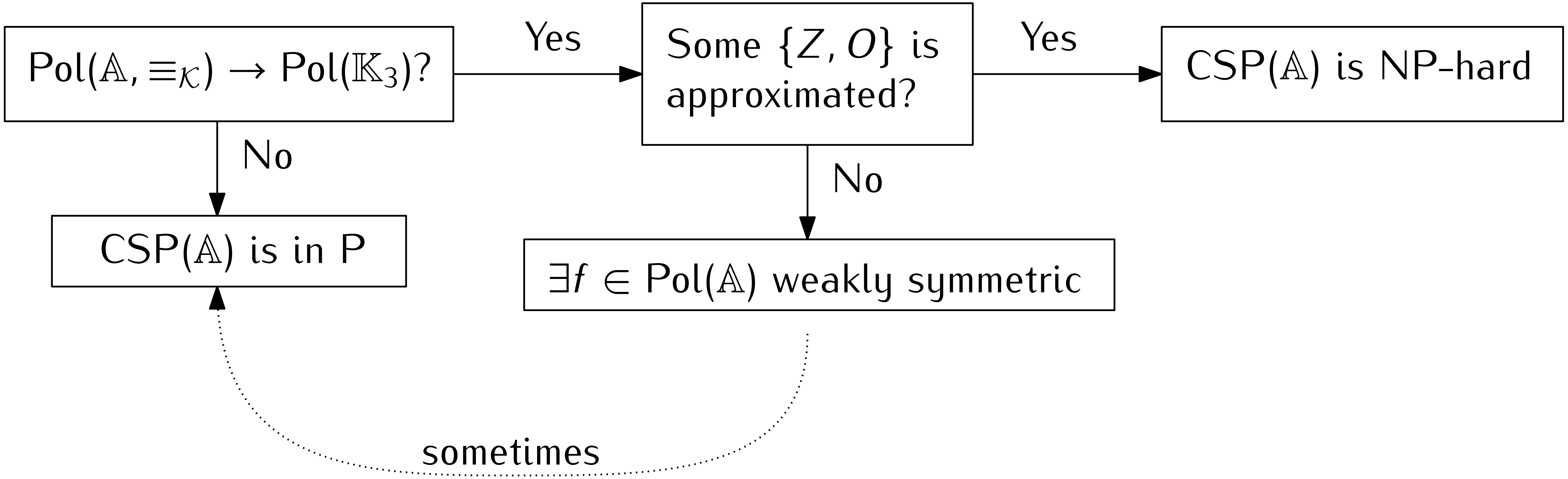

Connections with Promise CSPs

$$\begin{tikzcd} & \text{Infinite} \arrow[d] & \\ \mathbb{A} \arrow[r] & \mathbb{C} \arrow[r] & \mathbb{B} \\ & \text{Finite} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Theorem (M.). Let $\mathbb{B}$ be a reduct of a finitely bounded homogeneous structure $\mathbb{B}^*$.
There exists $\mathbb{A}$ (infinite) s.t. $\mathrm{CSP}(\mathbb{B})$ is ptime equivalent to $\mathrm{PCSP}(\mathbb{A}, \mathbb{B}_{fin})$.

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$$(\forall \mathbb{A}' \subset \mathbb{A} \text{ finite} : \mathrm{Pol}(\mathbb{A}', \mathbb{B}_{fin}) \models \Sigma) \iff \mathrm{Pol}(\mathbb{B}) \models \Sigma \text{ mod } \mathrm{Aut}(\mathbb{B}^*)$$

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \mathrm{Pol}(\mathbb{A}_4, \mathbb{B}_{fin}) \arrow[d] \\
\mathrm{Pol}(\mathbb{B}) \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \mathrm{Pol}(\mathbb{A}_3, \mathbb{B}_{fin}) \arrow[d] \\
 & \mathrm{Pol}(\mathbb{A}_2, \mathbb{B}_{fin}) \arrow[d] \\
 & \mathrm{Pol}(\mathbb{A}_1, \mathbb{B}_{fin})
\end{tikzcd}$$

Finite minions!

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Corollary (Barto, Bulín, Krokhin, Opršal). If Pol($\mathbb{B}$) does not contain pseudo-Olšák operations, then approximate hypergraph coloring reduces to CSP($\mathbb{B}$).

$$e_1 f(x,x,y,y,y,x) \approx e_2 f(x,y,x,y,x,y) \approx e_3 f(y,x,x,x,y,y)$$

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
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\end{tikzcd}$$

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Corollary (Atserias, Dalmau). If Pol($\mathbb{B}$) does not contain a pseudo-WNU of some arity $k \geq 3$, then $\mathbb{B}$ does not have bounded width.

$$e_1 w(x, \ldots, x, x, y) \approx e_2 w(x, \ldots, x, y, x) \approx \cdots \approx e_k w(y, x, \ldots, x)$$

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Theorem (M.). Let $\mathbb{B}$ be a reduct of a finitely bounded homogeneous structure $\mathbb{B}^*$. There exists $\mathbb{A}$ (infinite) s.t. CSP($\mathbb{B}$) is ptime equivalent to PCSP($\mathbb{A}, \mathbb{B}_{fin}$).

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$$u \circ s(x, y, x, z, y, z) \approx v \circ s(y, x, z, x, z, y)$$

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Theorem (M.). Let $\mathbb{B}$ be a maximally tractable temporal CSP, $\mathbb{A}$ as above.
There exists $\mathbb{A}' \subseteq \mathbb{A}$ such that $\mathrm{PCSP}(\mathbb{A}', \mathbb{B}_{fin})$ is not finitely tractable:

$$\mathbb{A}' \to \mathbb{C} \to \mathbb{B}_{fin} \text{ and } \mathbb{C} \text{ finite} \implies \mathrm{CSP}(\mathbb{C}) \text{ is NP-hard}$$

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
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\end{tikzcd}$$

Finite minions!

Theorem (M.). Let $\mathbb{B}$ be a reduct of a finitely bounded homogeneous structure $\mathbb{B}^*$. There exists $\mathbb{A}$ (infinite) s.t. CSP($\mathbb{B}$) is ptime equivalent to PCSP$(\mathbb{A}, \mathbb{B}_{fin})$.

$$(\forall \mathbb{A}' \subset \mathbb{A} \text{ finite} : \mathrm{Pol}(\mathbb{A}', \mathbb{B}_{fin}) \models \Sigma) \iff \mathrm{Pol}(\mathbb{B}) \models \Sigma \bmod \mathrm{Aut}(\mathbb{B}^*)$$

Theorem (M.). Let $\mathbb{B}$ be a maximally tractable temporal CSP, $\mathbb{A}$ as above.
There exists $\mathbb{A}' \subseteq \mathbb{A}$ such that PCSP$(\mathbb{A}', \mathbb{B}_{fin})$ is not finitely tractable:

$$\mathbb{A}' \to \mathbb{C} \to \mathbb{B}_{fin} \text{ and } \mathbb{C} \text{ finite} \implies \mathrm{CSP}(\mathbb{C}) \text{ is NP-hard}$$

Question. Do these finite and tractable PCSP templates need new algorithms?

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
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Temporal CSPs / ordering CSPs:

- certificate class $\mathcal{K}$: linear orders
- $\mathrm{CSP}(\mathbb{Q}; R_1, \ldots, R_k)$ where $R_1, \ldots, R_k$ definable using $<$

Theorem (Bodirsky, Kára). For every temporal $\mathrm{CSP}(\mathbb{B})$, if $\nexists\, \mathrm{Pol}(\mathbb{B}, \mathbf{c}) \to \mathrm{Pol}(\mathbb{K}_3)$, then $\mathrm{CSP}(\mathbb{B})$ is in P.

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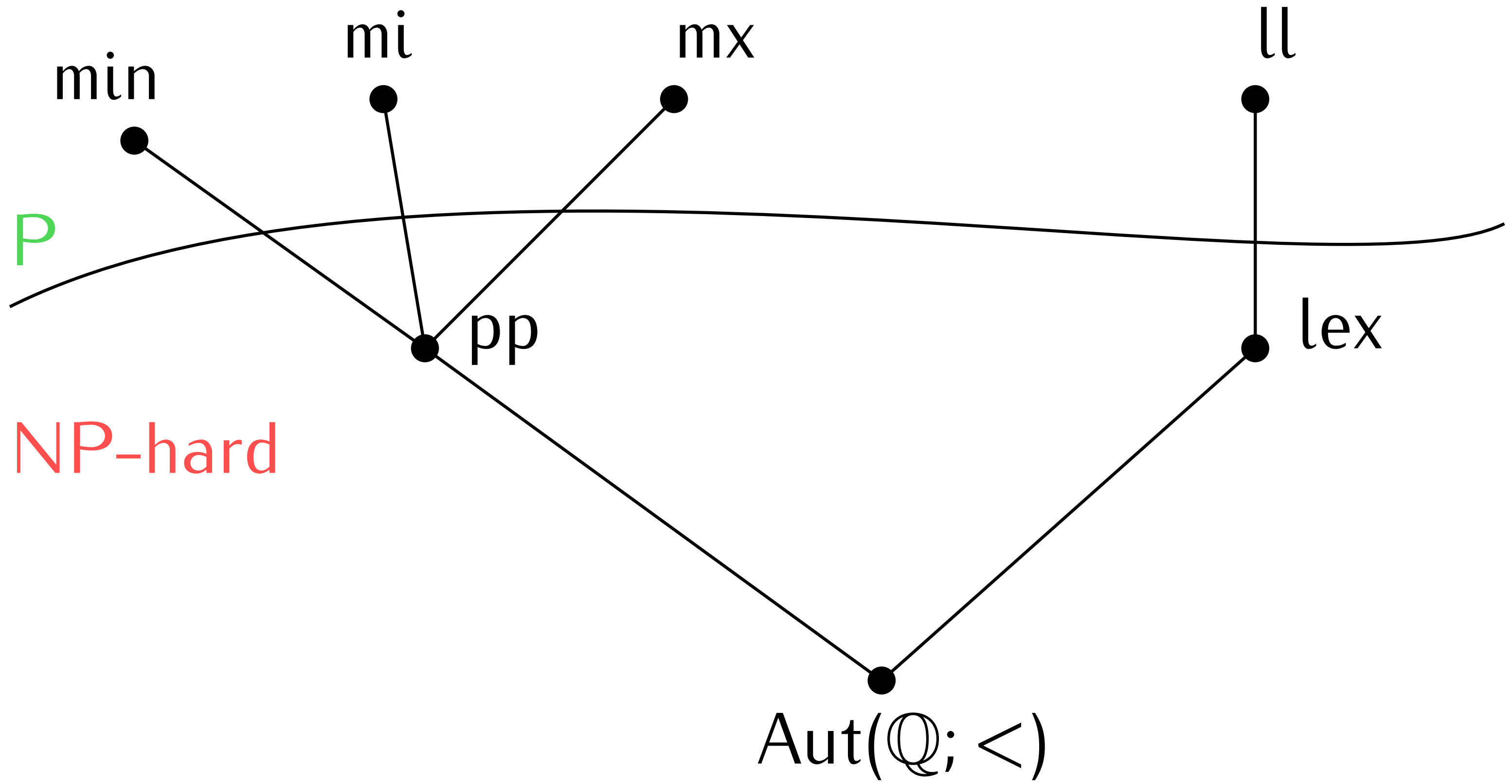

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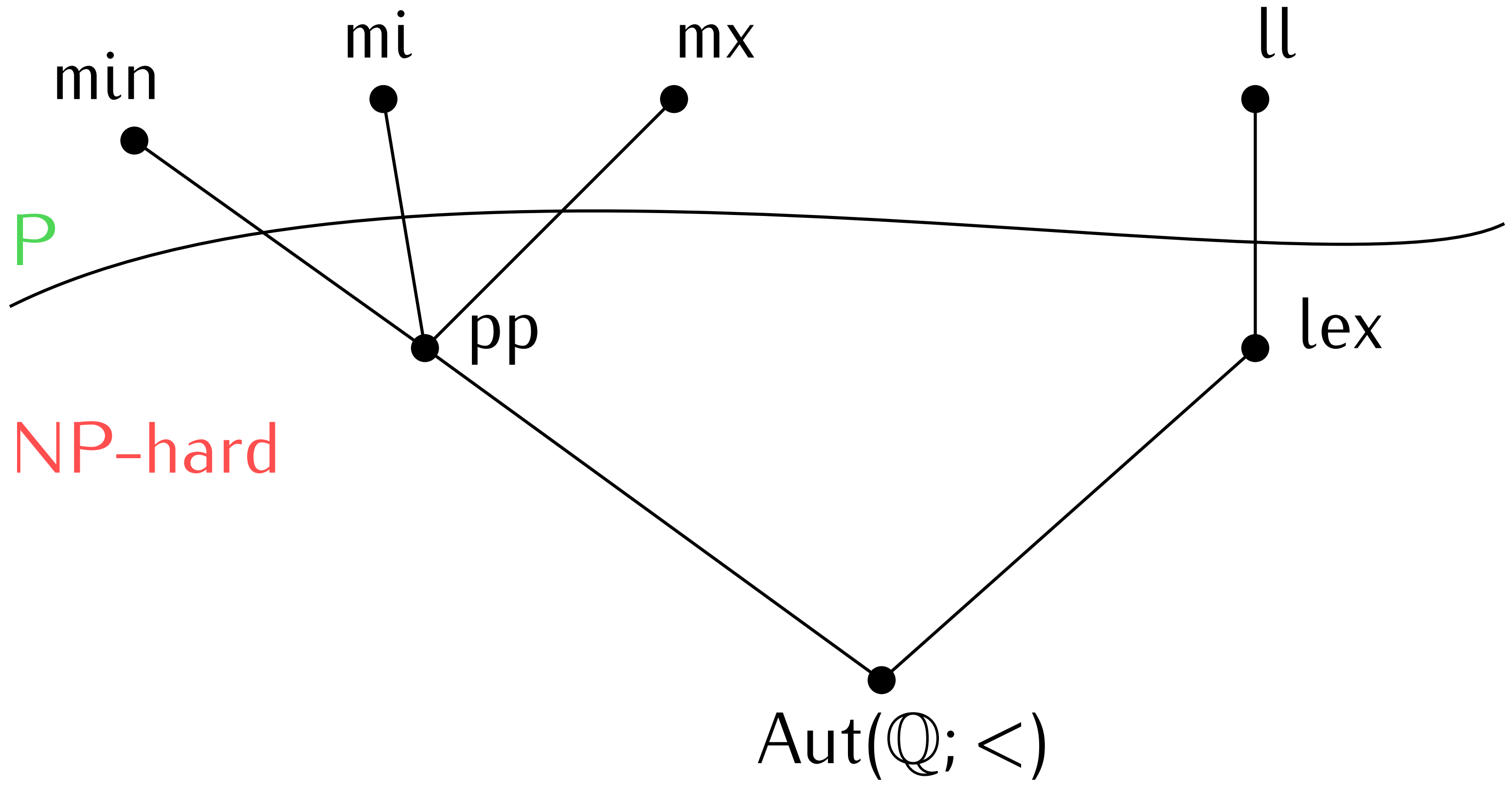
Non-satisfactory parts of the proof:

- 4 ad-hoc algorithms, yet with a "finite flavor"
- Algorithms need polymorphisms to work
- Any work building on this must essentially follow the same case distinction
- Hard to use as a blackbox



- Original idea: iteratively decompose instance by finding a non-empty set of variables than can be minimal in a solution: free set of variables (Bodirsky, Kára)
- Deciding the existence of such a free set is a boolean CSP with domain $\{\text{min}, \text{not min}\}$ (M., Pinsker)
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Observation. On input with n variables, CSP($\mathbb{B}$) reduces (trivially) to PCSP($\mathbb{A}, \mathbb{B}$) where $\mathbb{A}$ has domain $\{1, \ldots, n\}$.

Theorem (M.). Let $\mathbb{B}$ be a temporal structure with tractable CSP, $\mathbb{A} \subset \mathbb{B}$ finite.
$k \geq 1$ large enough (depending on arity of relations).
Let $\mathbb{X}$ be an instance that has a (kth level) singleton BLP/AIP solution w.r.t. $\mathbb{A}$.
Then $\mathbb{X}$ has a non-empty free set F s.t. $\mathbb{X} \wedge$ "F is minimal" still has a (kth level) singleton BLP/AIP solution.

Theorem (M.). $\mathbb{B}$ temporal structure such that $\nexists\, \mathrm{Pol}(\mathbb{B}, \mathbf{c}) \to \mathrm{Pol}(K_3)$, $\mathbb{A}$ finite structure.
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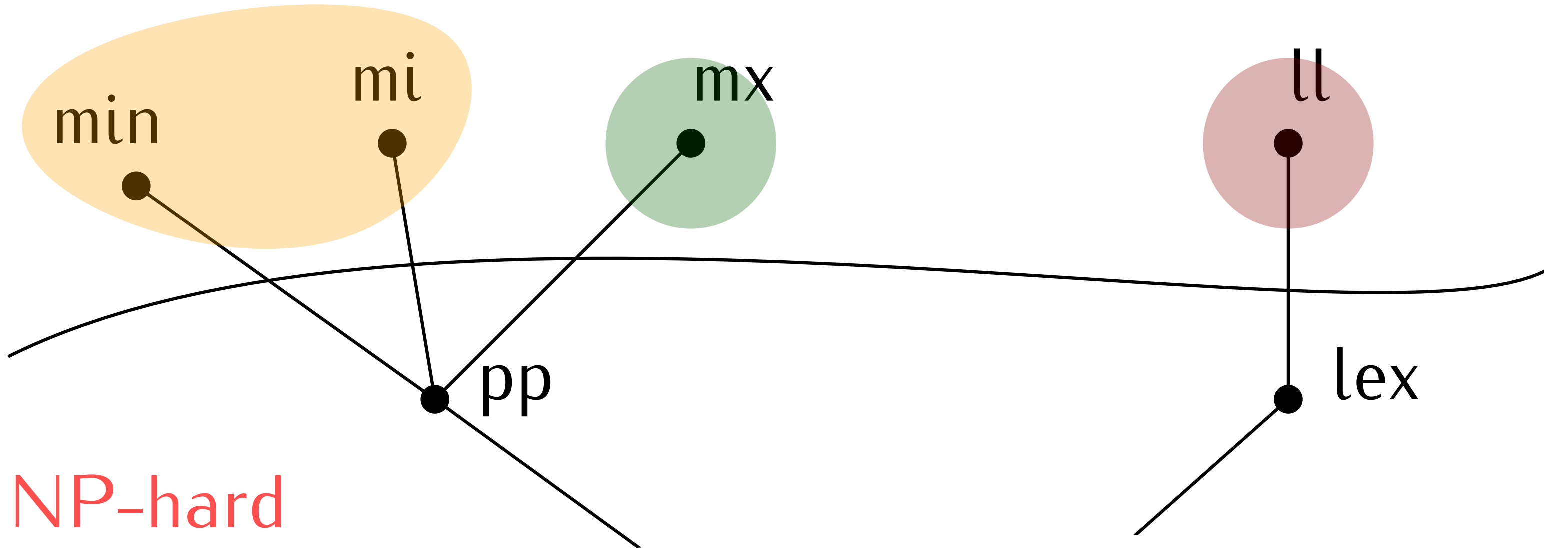
Singleton Arc Consistency

Singleton AIP

Width $\leq r+1$



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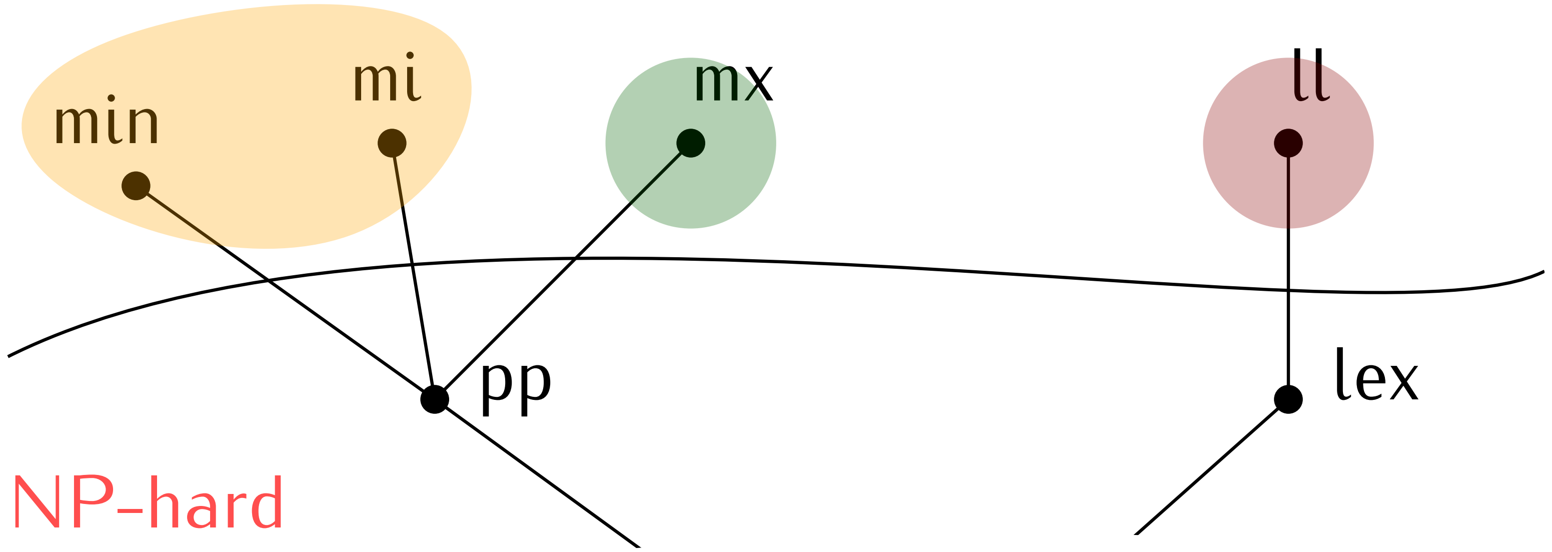

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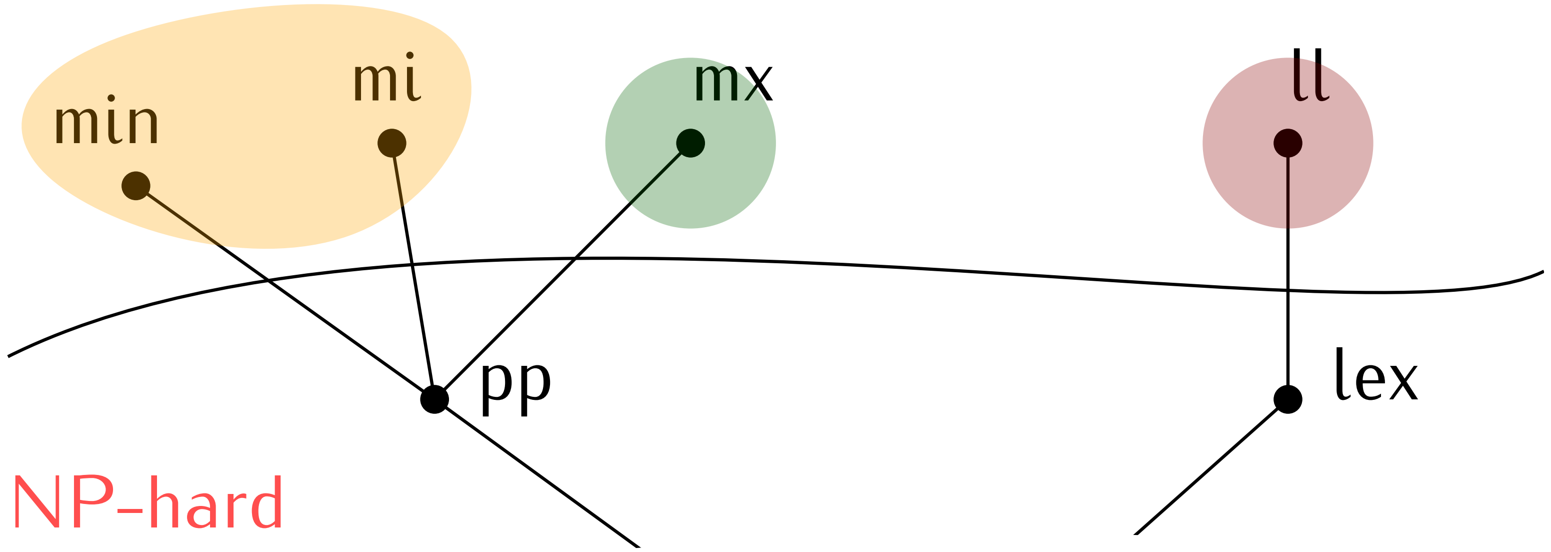

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Corollary (M.). There exist finite PCSP templates $(\mathbb{A}, \mathbb{B})$ such that:

- $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ has bounded width,
- $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ not solvable by BLP+AIP,
- Not finitely tractable: every finite $\mathbb{C}$ such that $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ has an NP-hard CSP. In particular, no finite $\mathbb{C}$ with bounded width.

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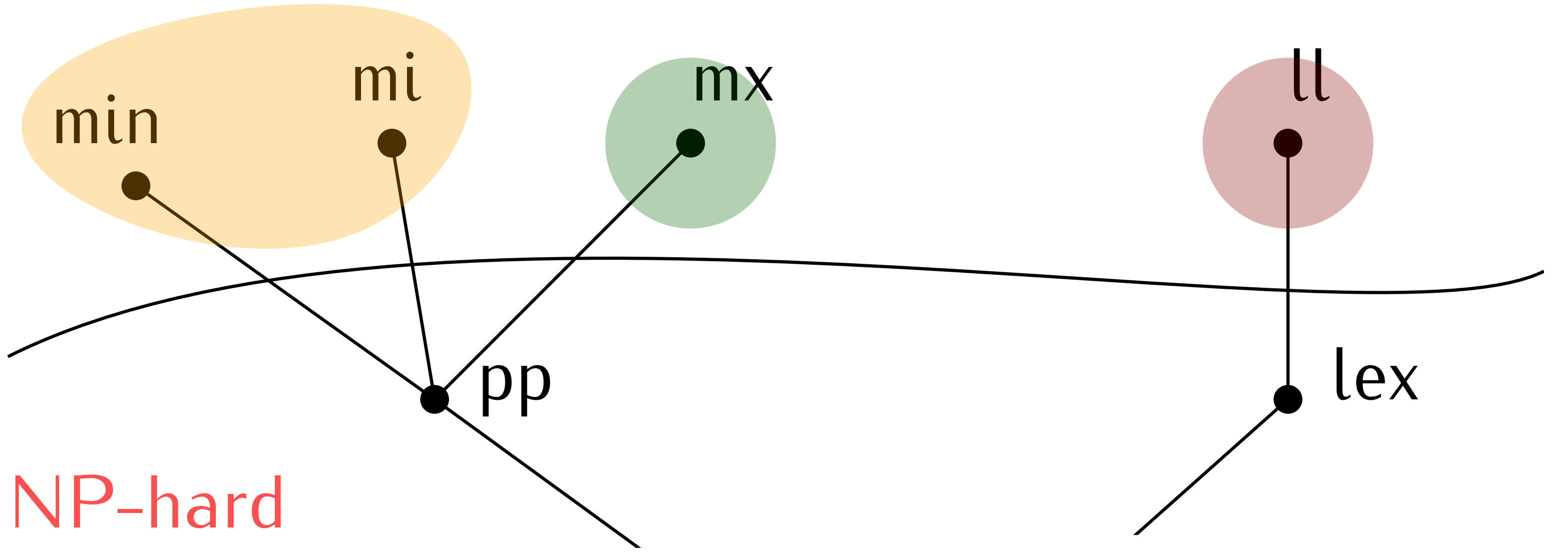

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Corollary (M.). There exist finite PCSP templates $(\mathbb{A}, \mathbb{B})$ such that:

- $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ solvable by singleton AIP,
- $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ not FPC-solvable,
- Not finitely tractable: every finite $\mathbb{C}$ such that $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ has an NP-hard CSP. In particular, no finite $\mathbb{C}$ with bounded width.

CSPs over Hypergraphs:

- Certificate class $\mathcal{K}$: all hypergraphs, or all K_n^r-free hypergraphs ($n > r$)
- $\mathrm{CSP}(V; R_1, \ldots, R_k)$ where $R_1, \ldots, R_k$ definable over certain homogeneous hypergraphs

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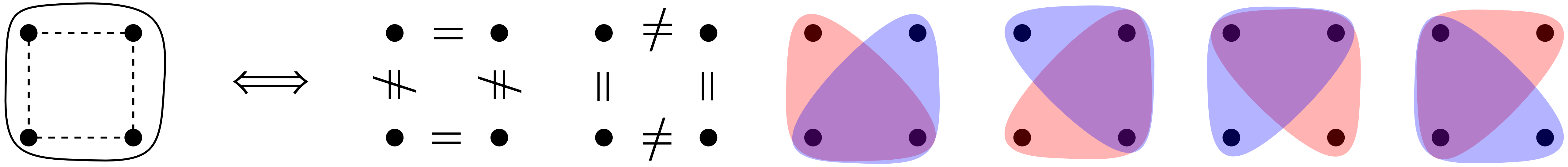
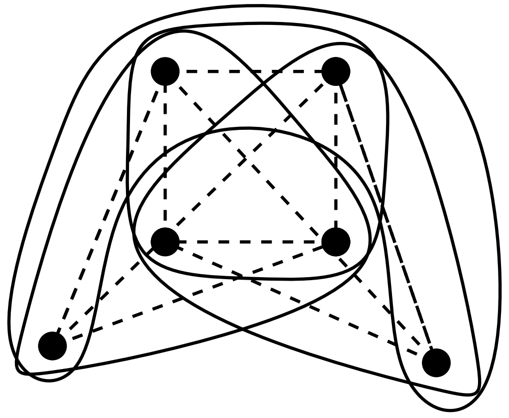
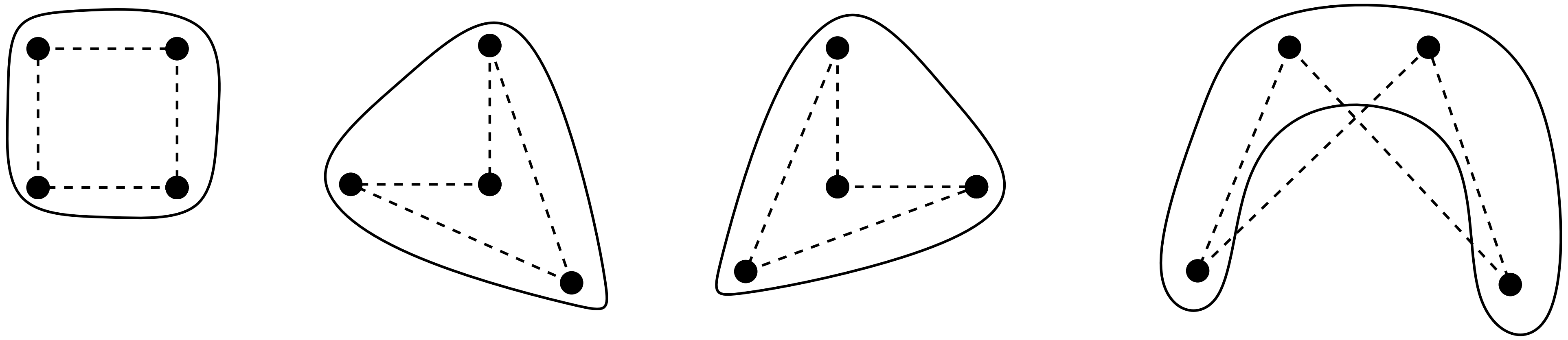




instance
with 4
constraints



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Theorem (M., Nagy, Pinsker)**.** The conjecture is true for such CSPs.

Our proof has interesting features:

- It is not true that "good polymorphisms" preserve $\equiv_{\mathcal{K}}$ (like for ordered structures)
- It is not true that "good polymorphisms" are completely wild (unlike for ordered structures)
- Algorithm loosely based on Zhuk's approach (binary absorption into affine algebra)

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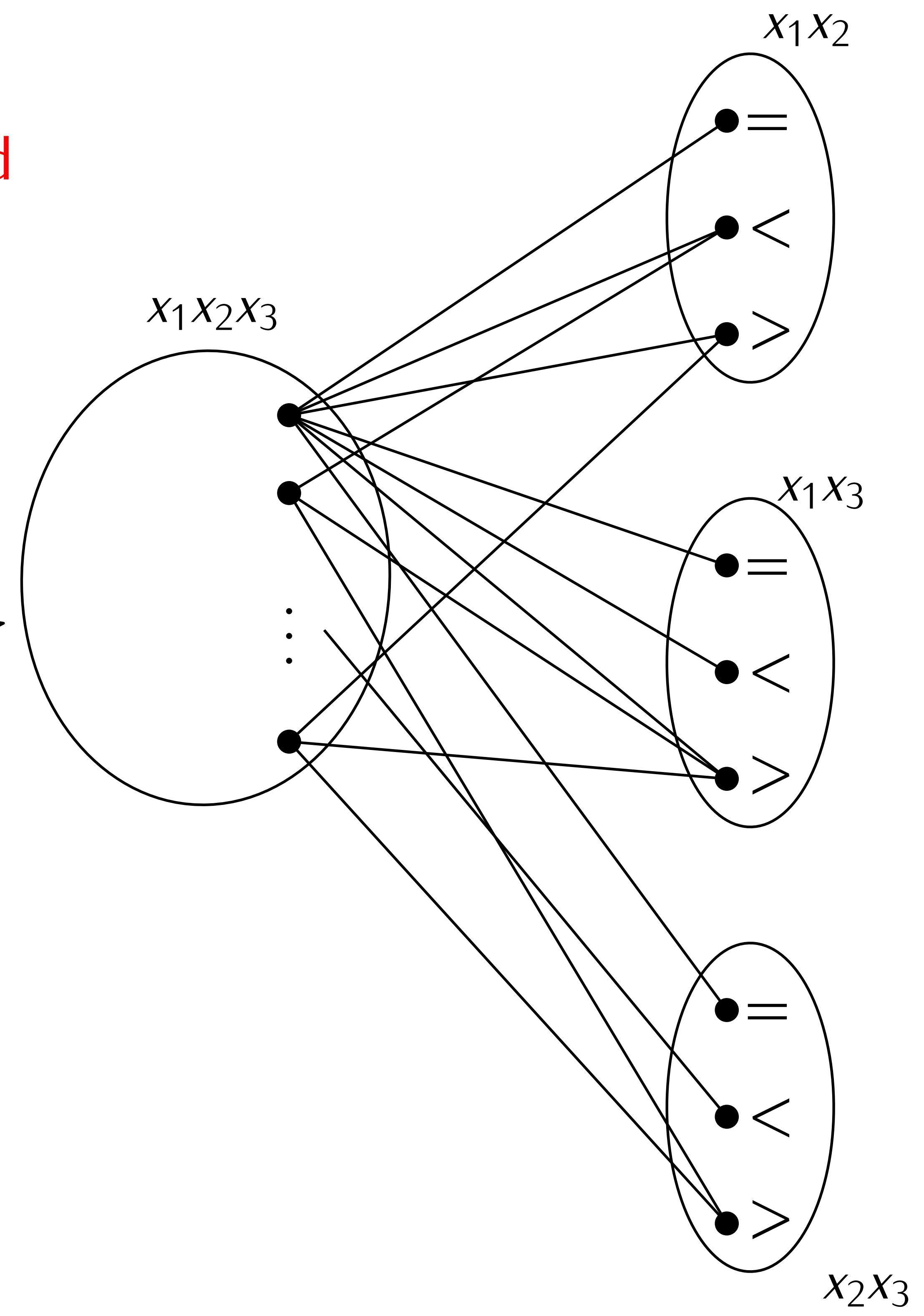
- Label cover instance with ordered hypergraphs

all 27 types of ordered 3-hypergraphs on ≤ 3 vertices



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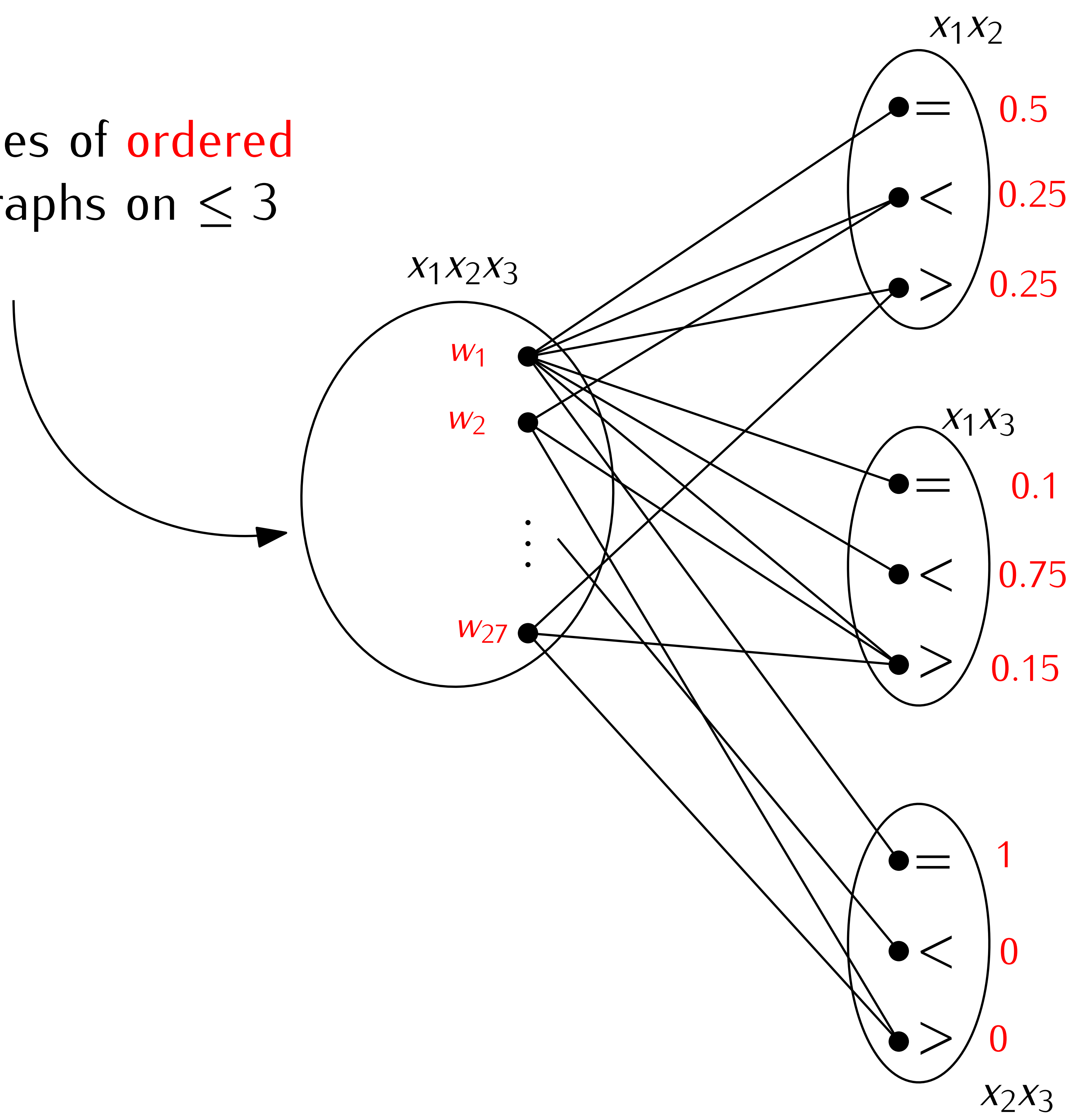

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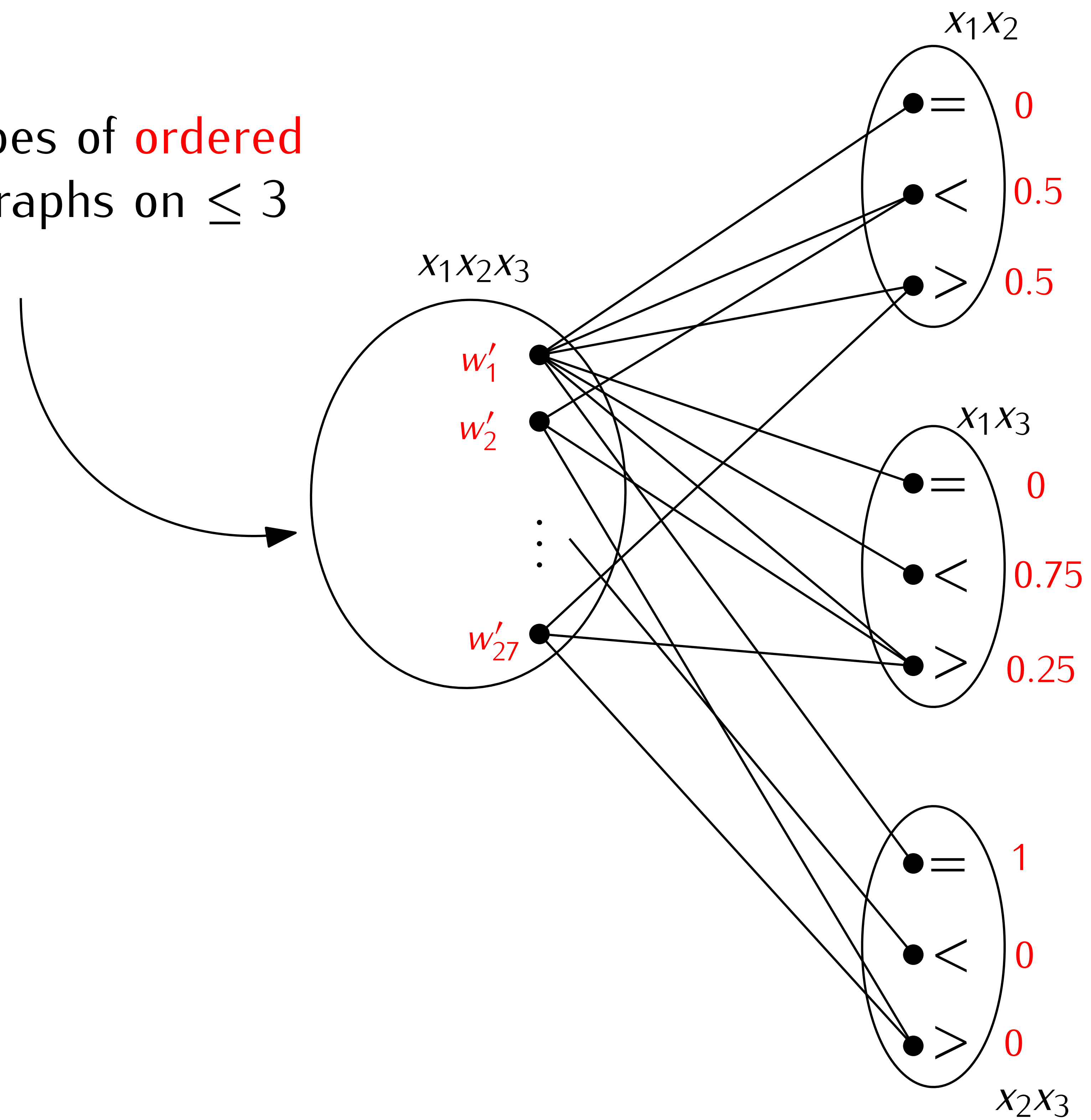

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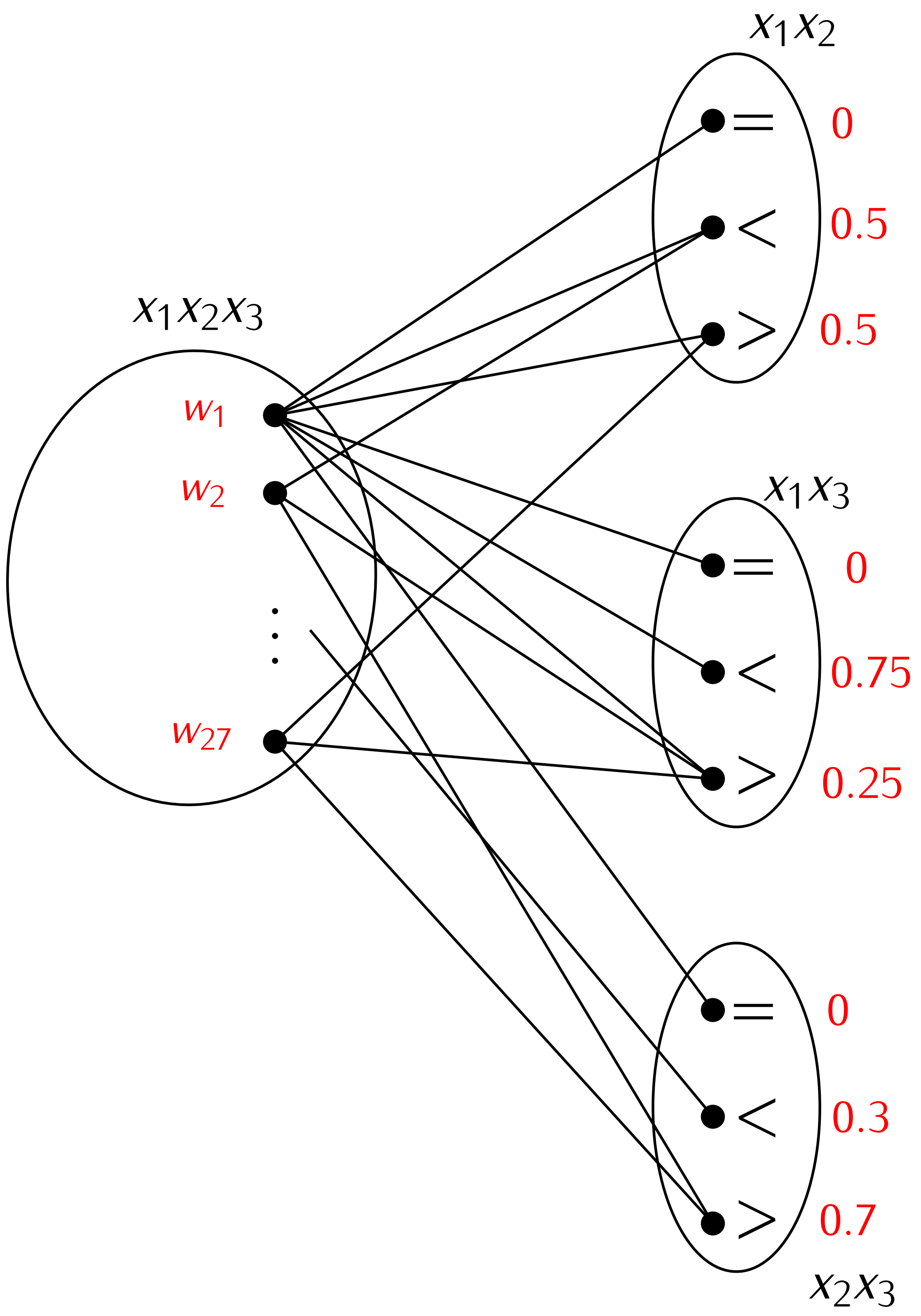

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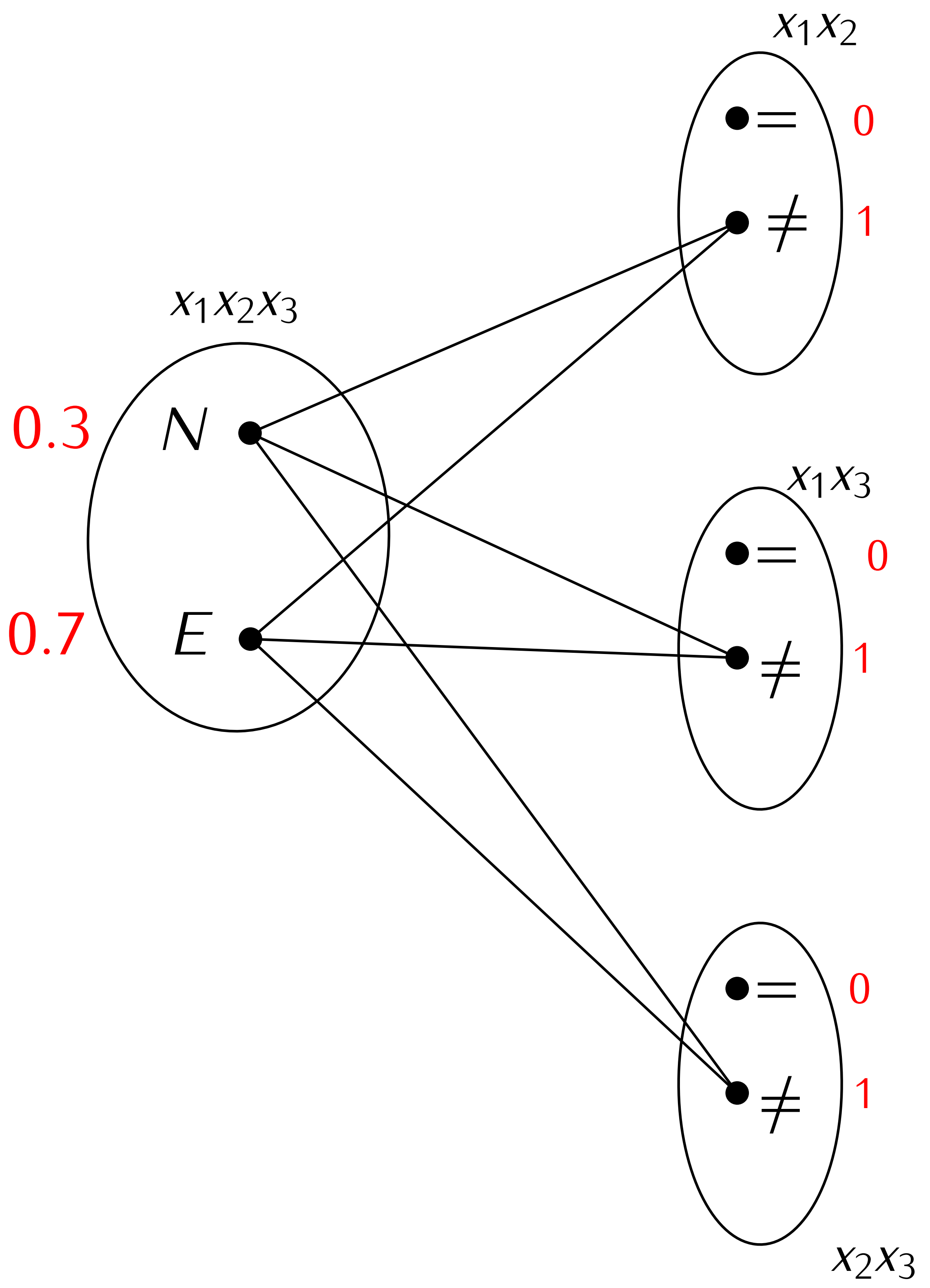

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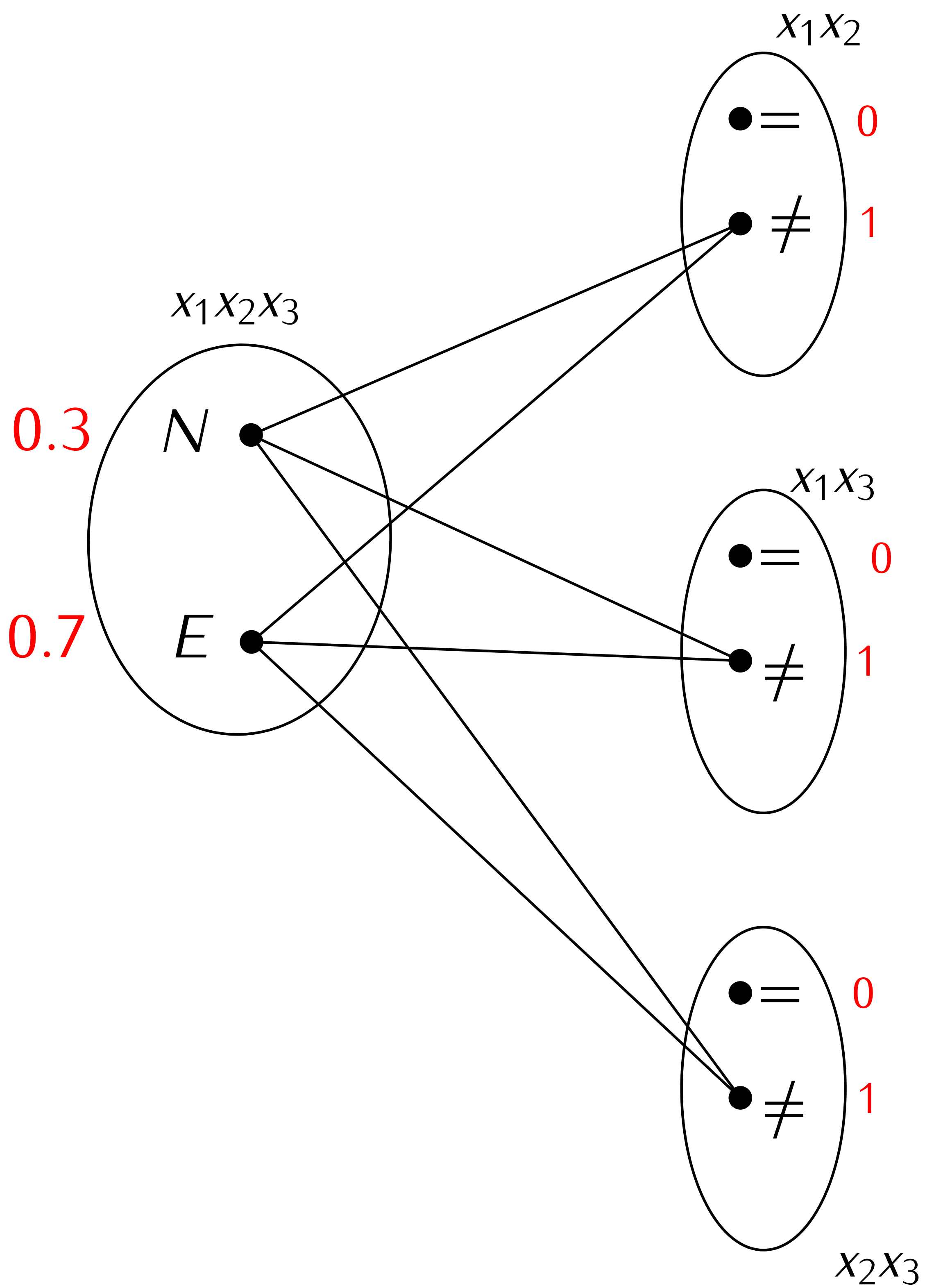

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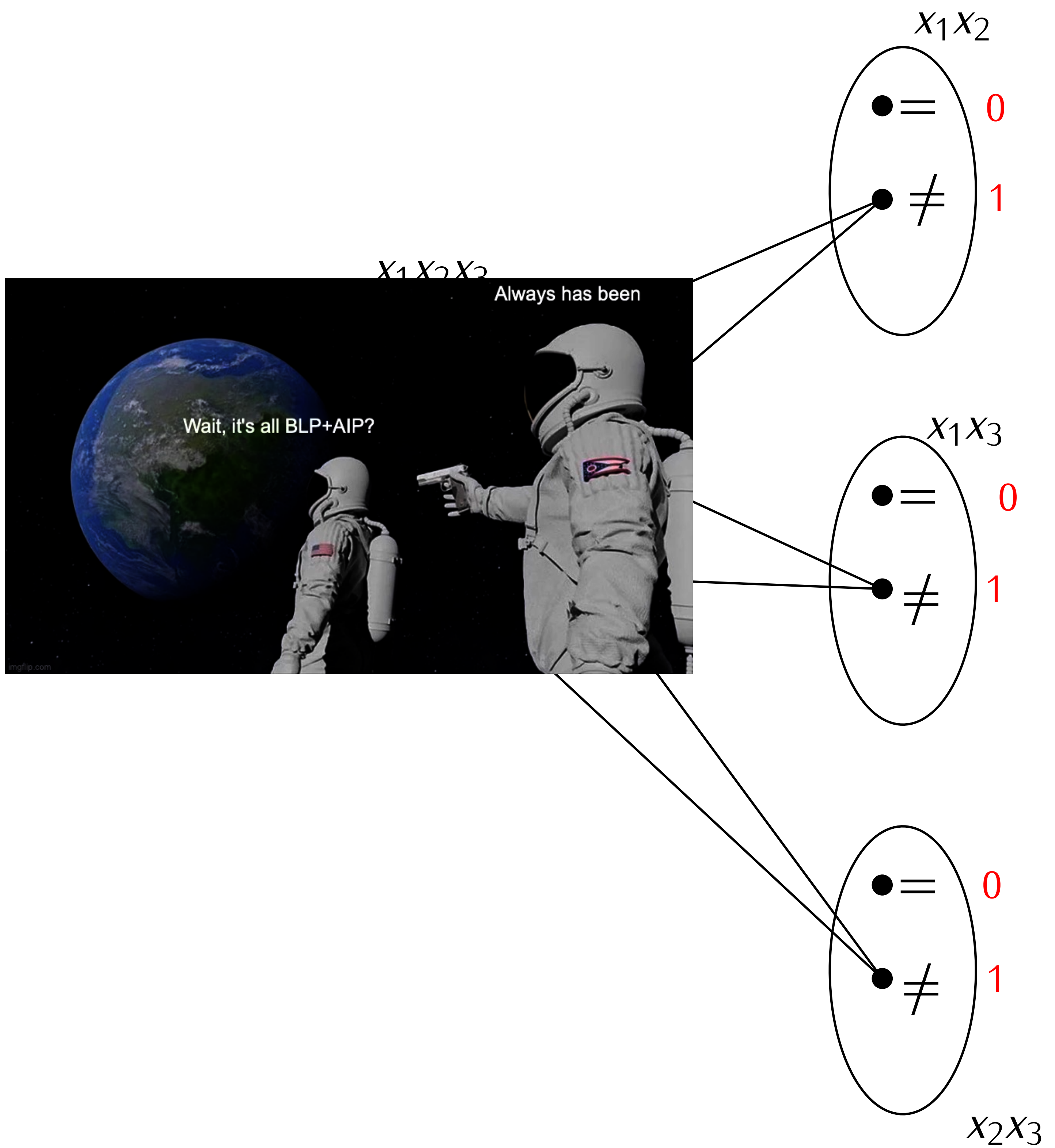

- Decision: basically done with the "Schaefer" case
  Next: progress on uniform algorithm for finite CSPs could help solving the infinite conjecture as well
- Optimisation: valued CSPs classified for order CSPs (Bodirsky, Bonnet, Semanišinová)
- Approximation resistance:
  - known for order CSPs (Guruswami, Håstad, Manokaran, Raghavendra, Charikar)
  - phylogenetic CSPs (Chatziafratis, Makarychev)
- Parametrized versions (Jonsson, Pilipczuk, Osipov, Wahlström, ...)