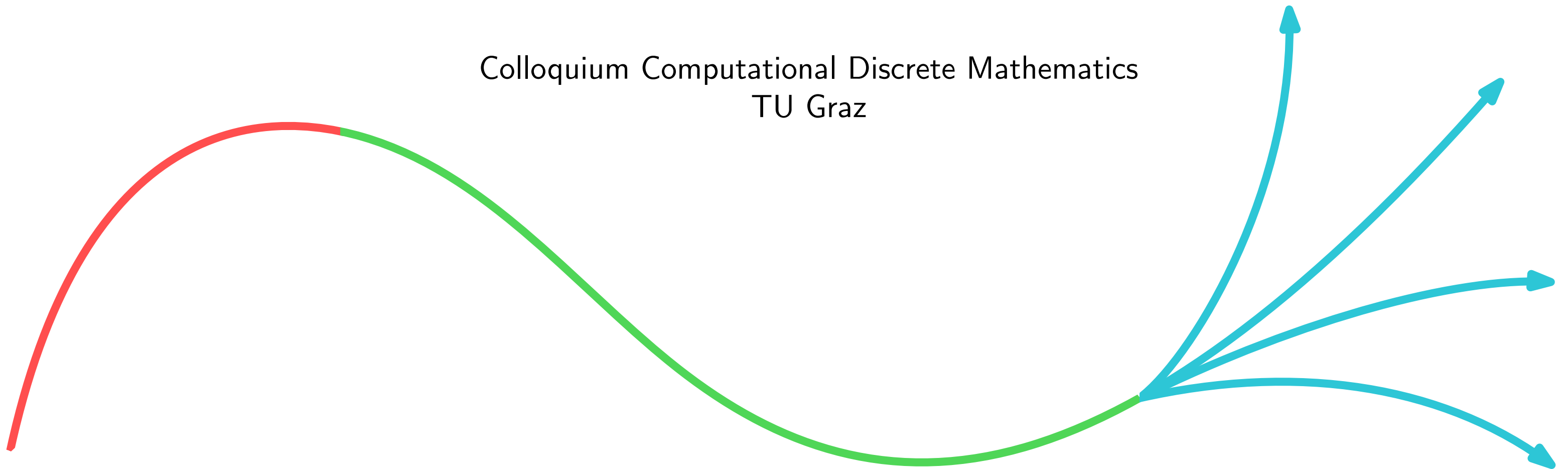


Theory of Constraint Satisfaction Problems

Antoine Mottet Wiehe

Colloquium Computational Discrete Mathematics
TU Graz



Instance:

- a **domain** A
- variables x, y, z, \dots
- constraints: $x \vee y \vee z, z \leq x, 2x + y = 0, \dots$

Question: does there exist $h: \text{Vars} \rightarrow A$ satisfying every constraint?

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Is $(x \vee y \vee z) \wedge (\neg z \vee t \vee p) \wedge \dots$ satisfiable?

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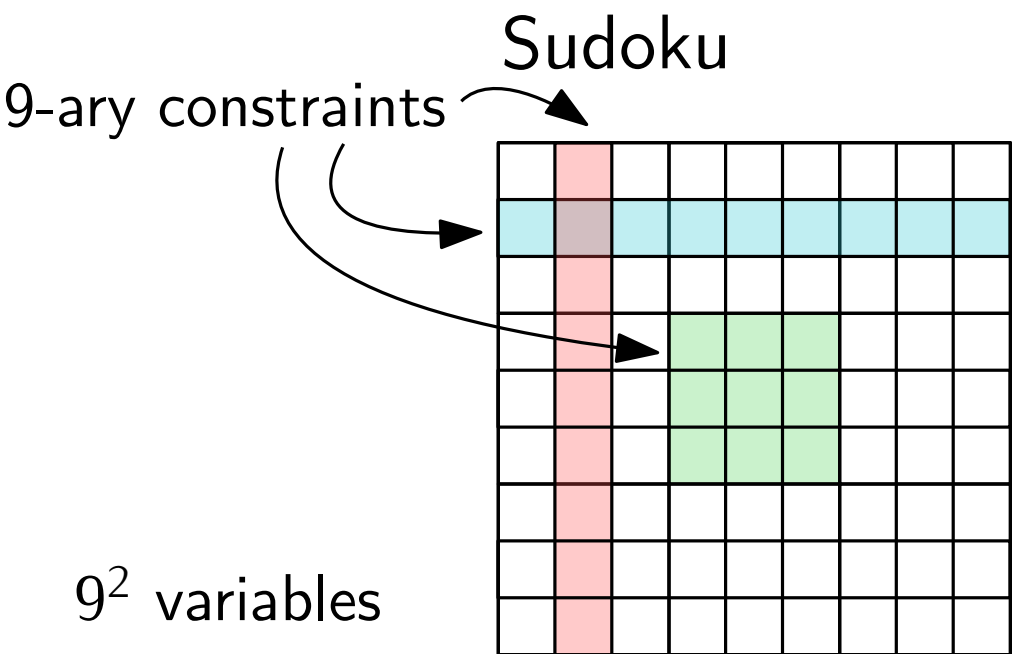
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Is $\mathbf{Ax} = \mathbf{b}$ satisfiable in $\mathbb{R}_{\geq 0}$?



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- **Non-uniform** CSP: domain is **fixed** and only certain types of constraints are allowed

Constraint with r variables \leftrightarrow relation $R \subseteq A^r$ listing all valid assignments

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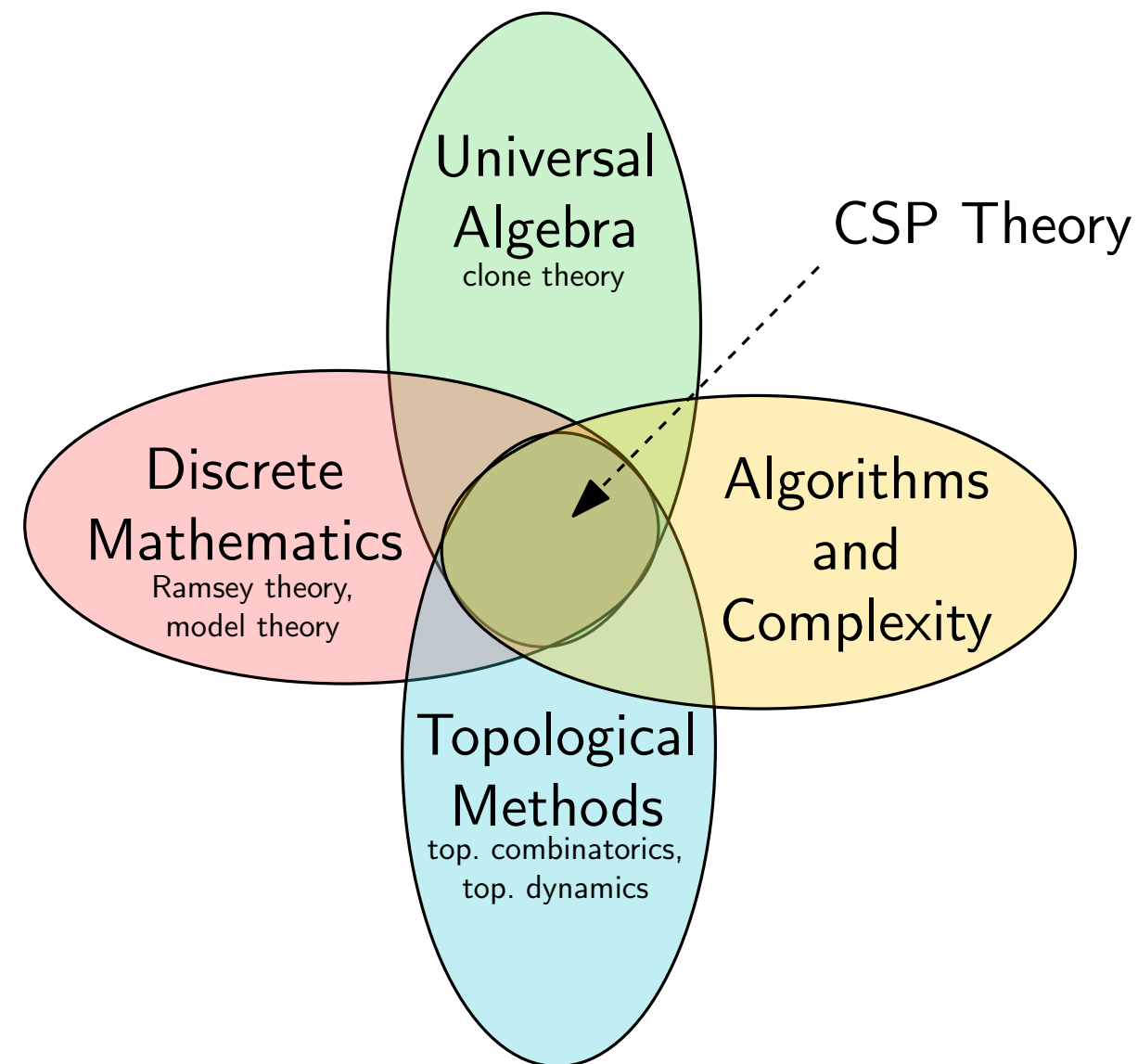
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Theorem (Bulatov // Zhuk '17). For every template \mathbb{A} with a **finite** domain, $\text{CSP}(\mathbb{A})$ is solvable in polynomial time or NP-complete.

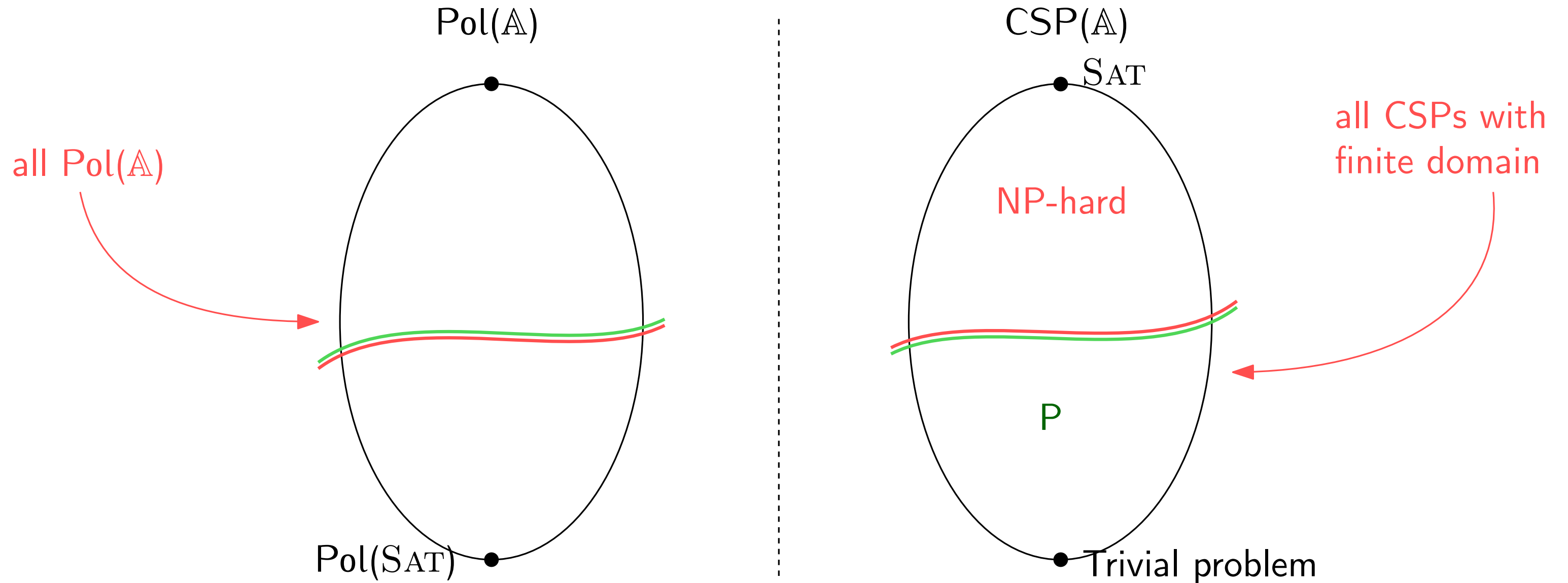
Finite-domain dichotomy result built on **polymorphisms** $\text{Pol}(\mathbb{A}) = \{f: \mathbb{A}^n \rightarrow \mathbb{A}, n \geq 1\}$

- **Clones**: closed under composition, contain **projections**
- Standard object in **universal algebra**
- Clone **actions** of $\text{Pol}(\mathbb{A})$ tell us about the complexity of $\text{CSP}(\mathbb{A})$



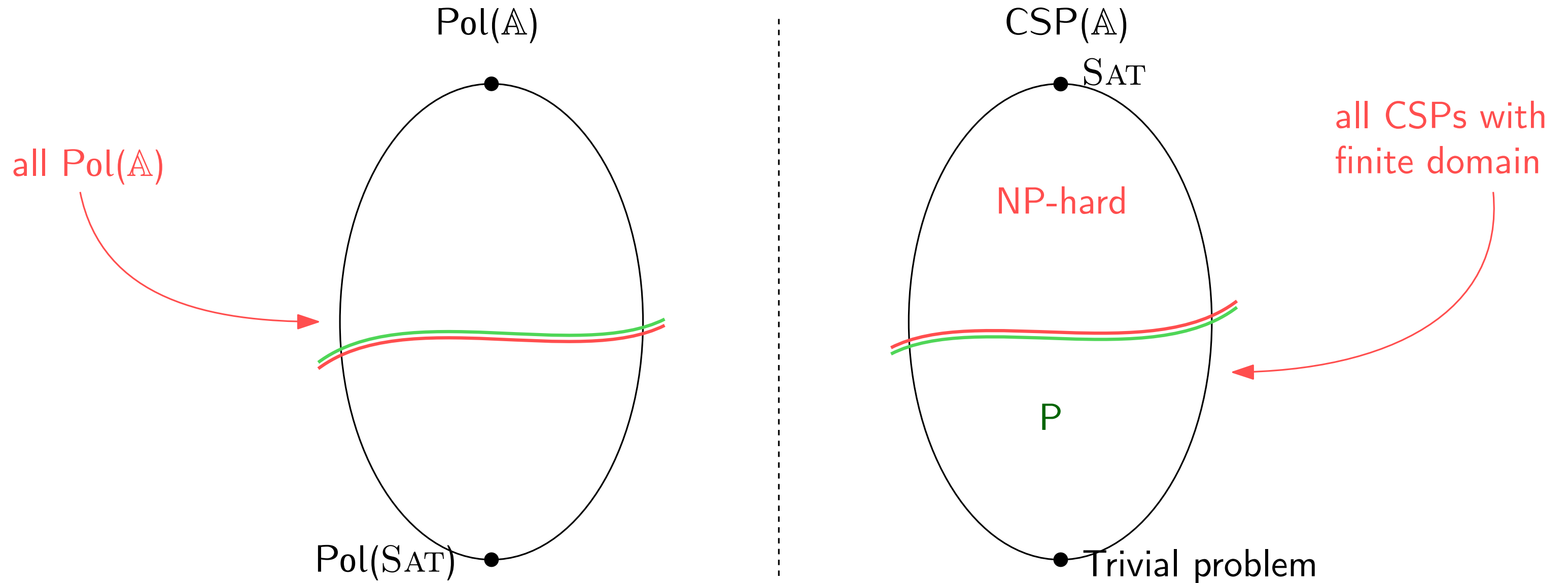
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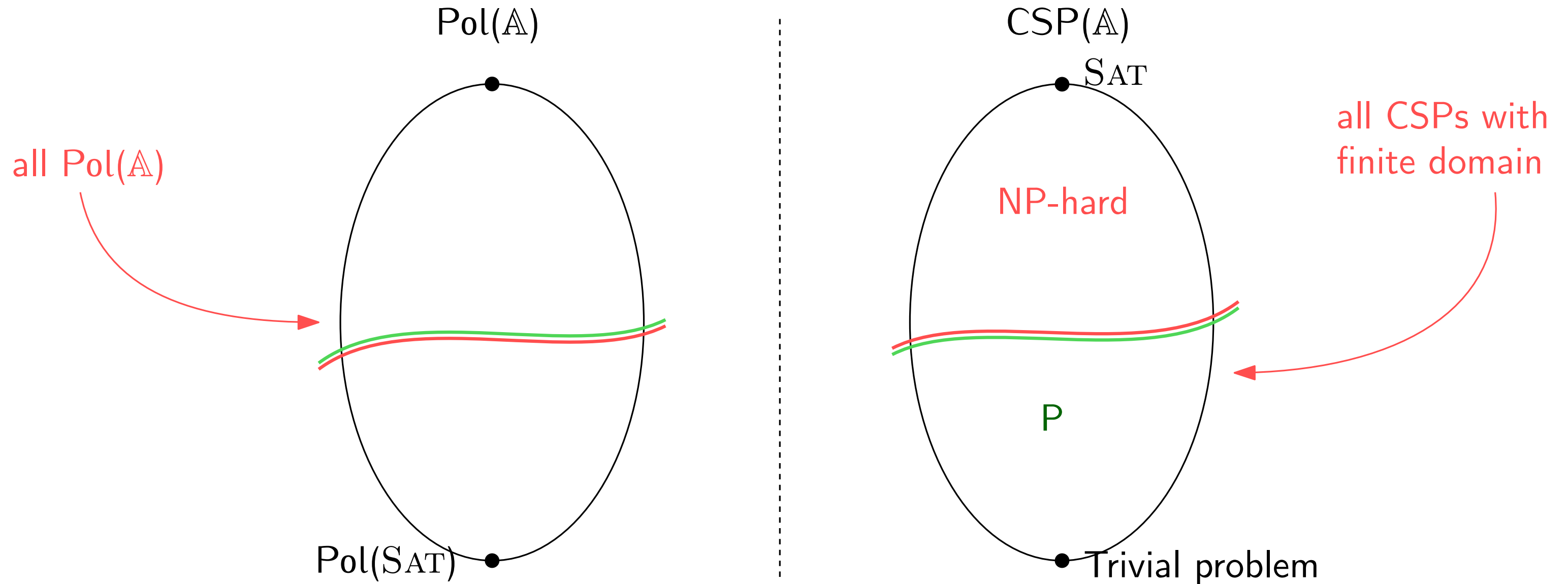


- $\text{Pol}(\mathbb{A}) = \text{Pol}(\mathbb{B})$ implies $\text{CSP}(\mathbb{A}) \equiv \text{CSP}(\mathbb{B})$

(Bulatov, Jeavons, Krokhin)

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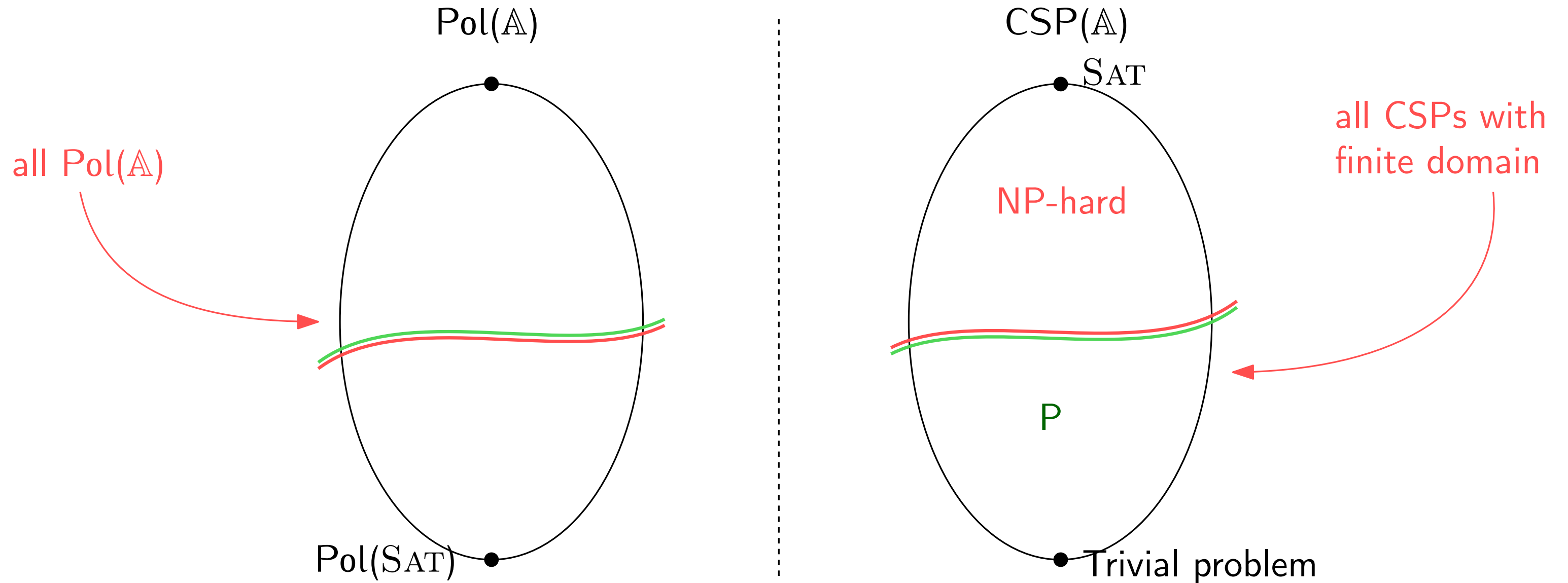


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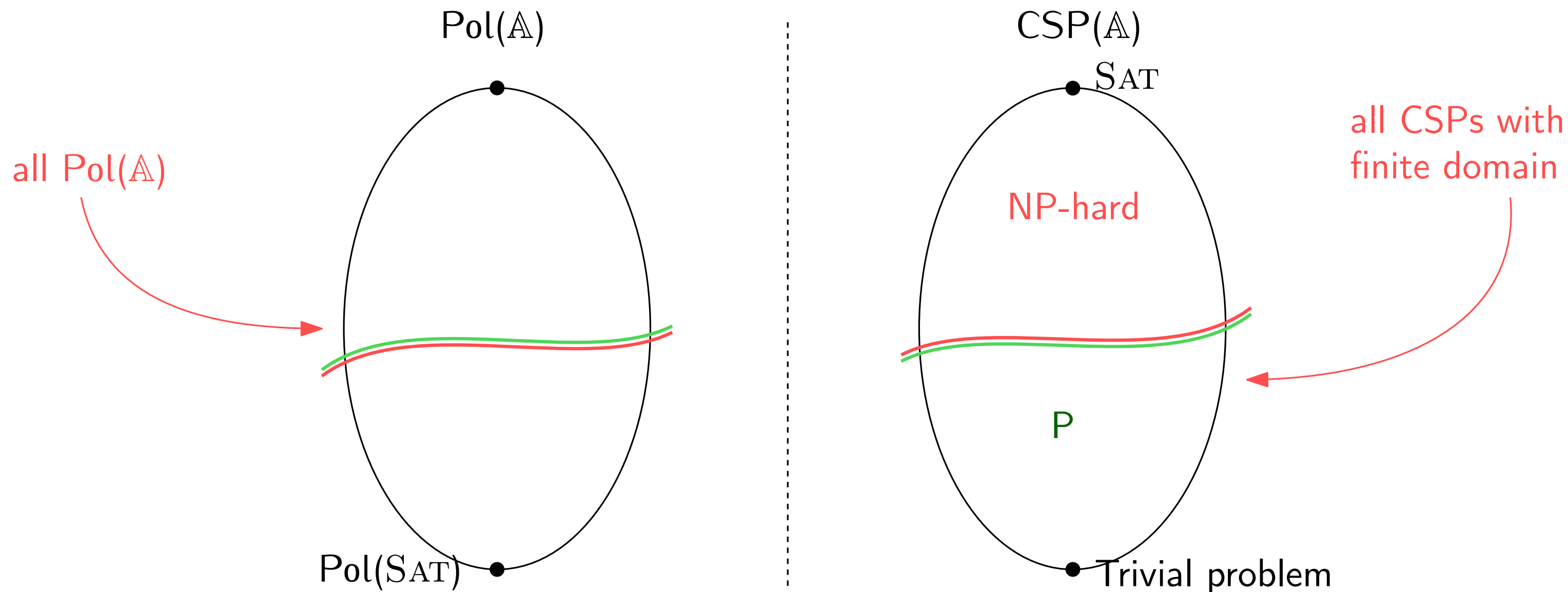


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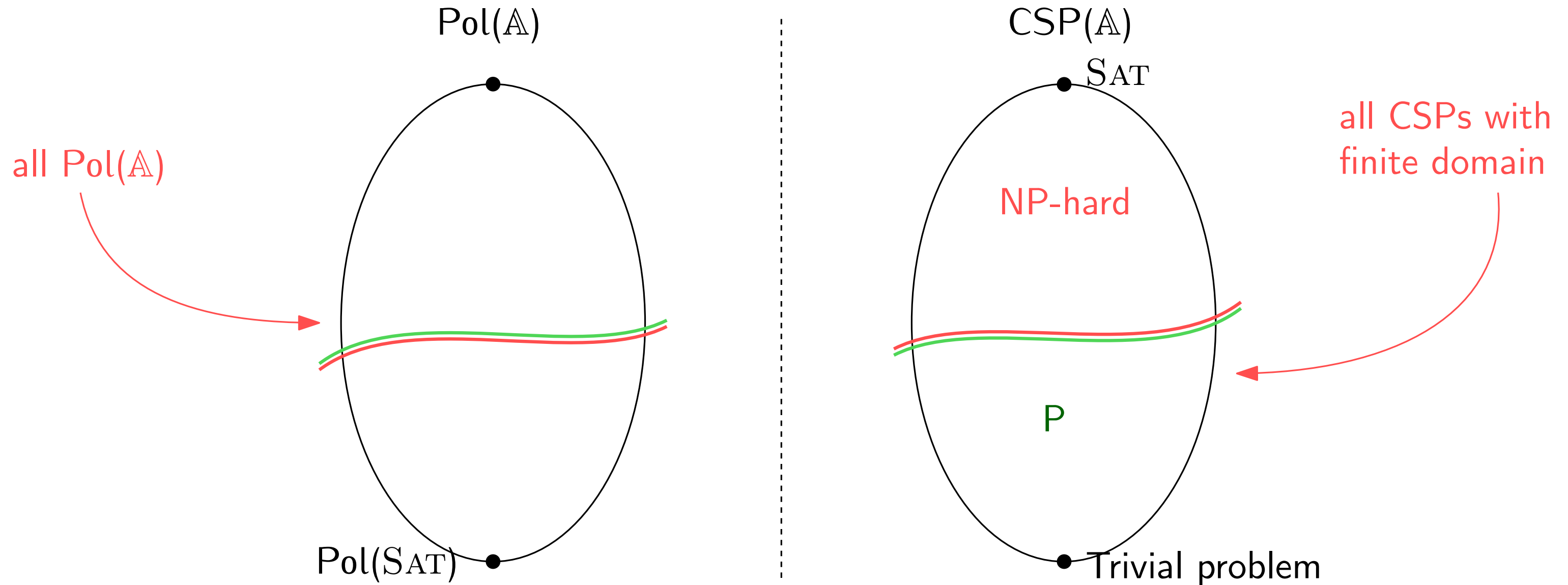
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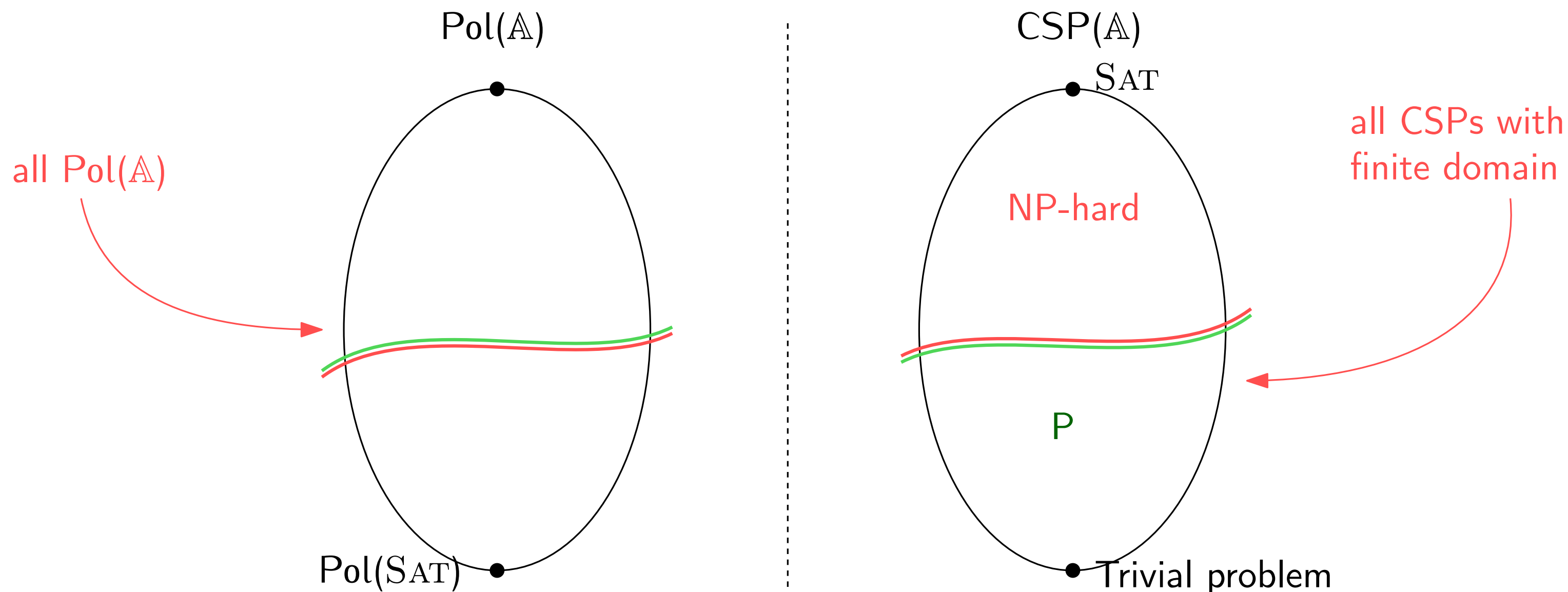
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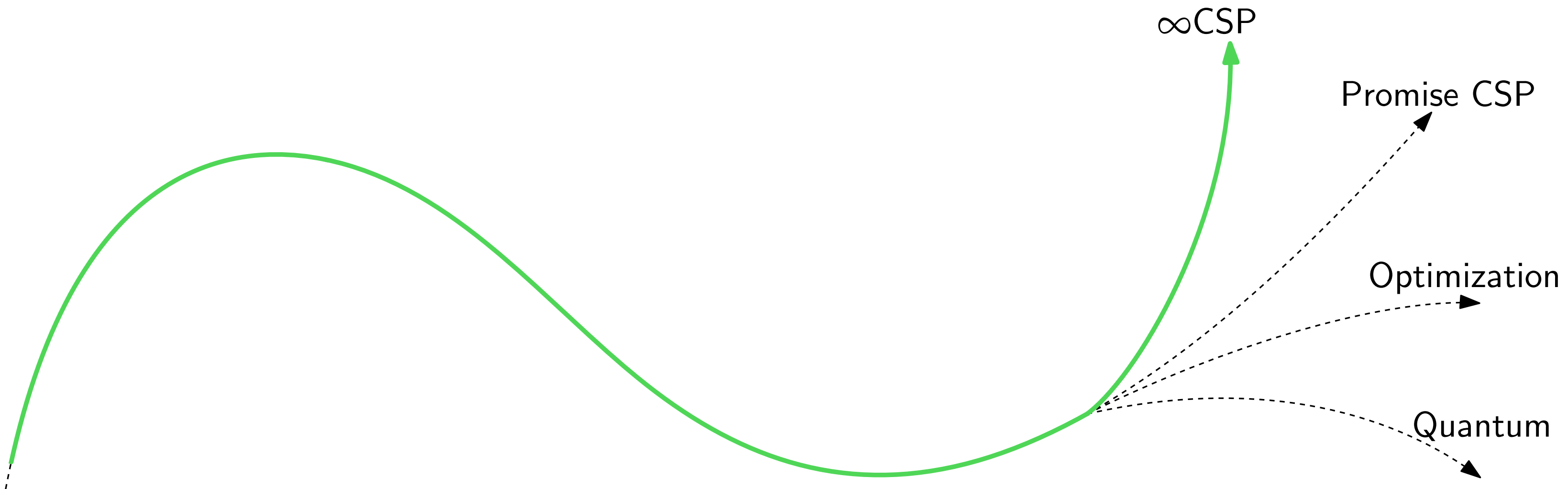
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- If $\text{Pol}(\mathbb{A}) \not\rightarrow \text{Pol}(\text{SAT})$, then $\text{CSP}(\mathbb{A})$ is solvable in polynomial time (Bulatov // Zhuk)
- Can the **algebraic approach** be leveraged for other computational problems?

Research in infinite-domain CSPs



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AND/OR scheduling

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Theorem (Bodirsky-Martin-M., *J. ACM*'18). Let \mathbb{A} be a temporal template. Then $\text{CSP}(\mathbb{A})$ is in P or NP-complete. Given \mathbb{A} , it is **decidable** which case applies.

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Project (with D. Perinti). Understand the complexity of edge partitioning problems.

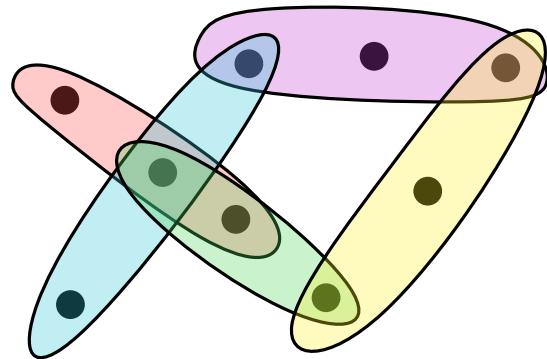
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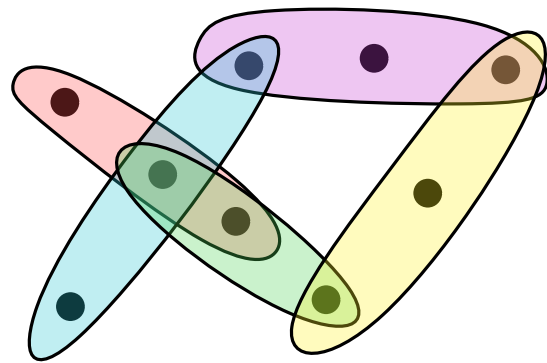
Instance



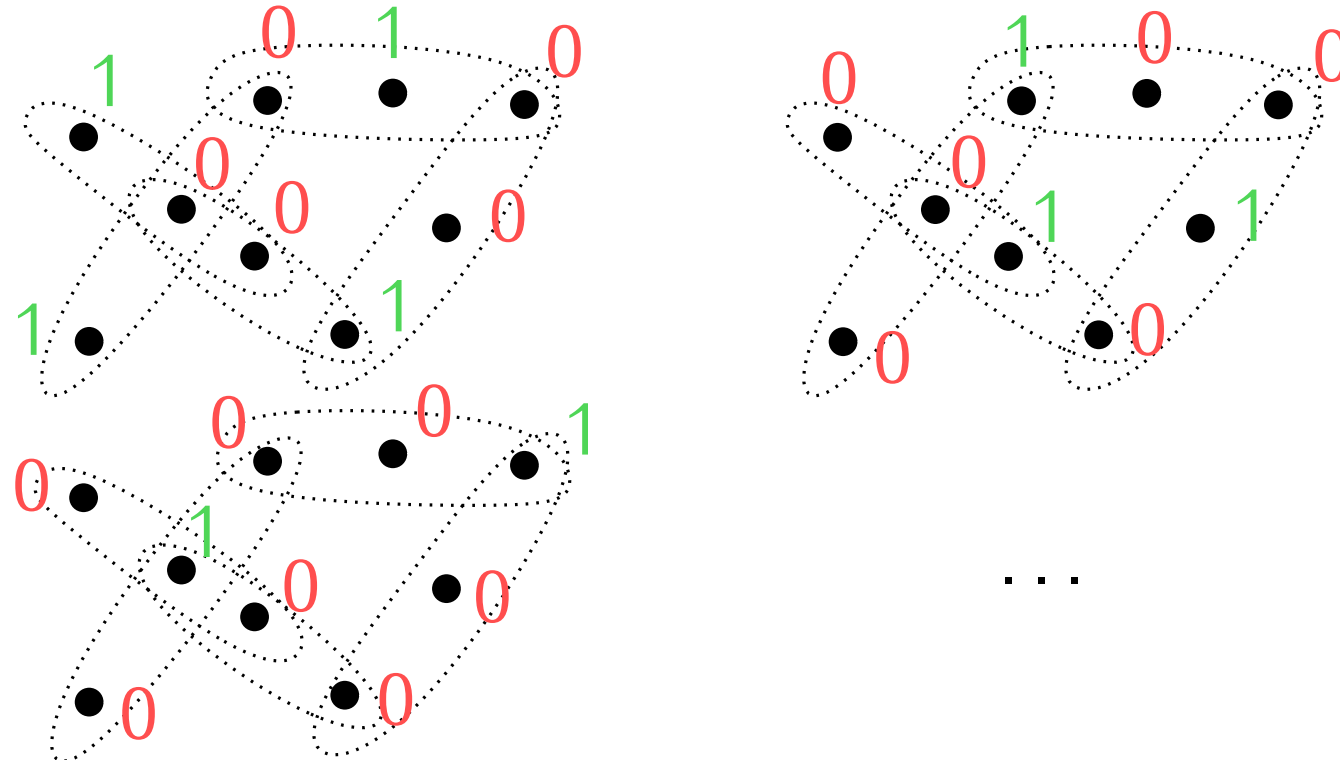
Algebraic approach works whenever $\text{CSP}(\mathbb{A})$ has **combinatorial certificates**.

Example (1-in-3 Satisfiability). Given a set of constraints $(x, y, z), (y, z, t), \dots$, decide if there is a map $h: V \rightarrow \{0, 1\}$ such that for every constraint (a, b, c) , exactly one of $h(a), h(b), h(c)$ is 1.

Instance



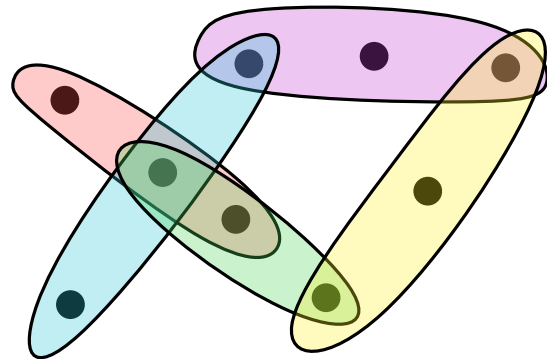
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Example (Betweenness). Given a set of constraints $(x, y, z), (y, z, t), \dots$, decide if there is a linear order on the variables such that for every constraint (a, b, c) , either $a < b < c$ or $c < b < a$.

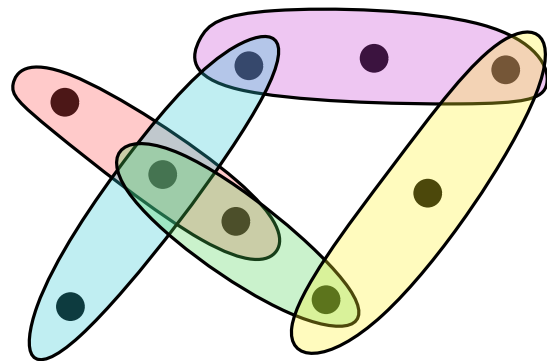
Instance



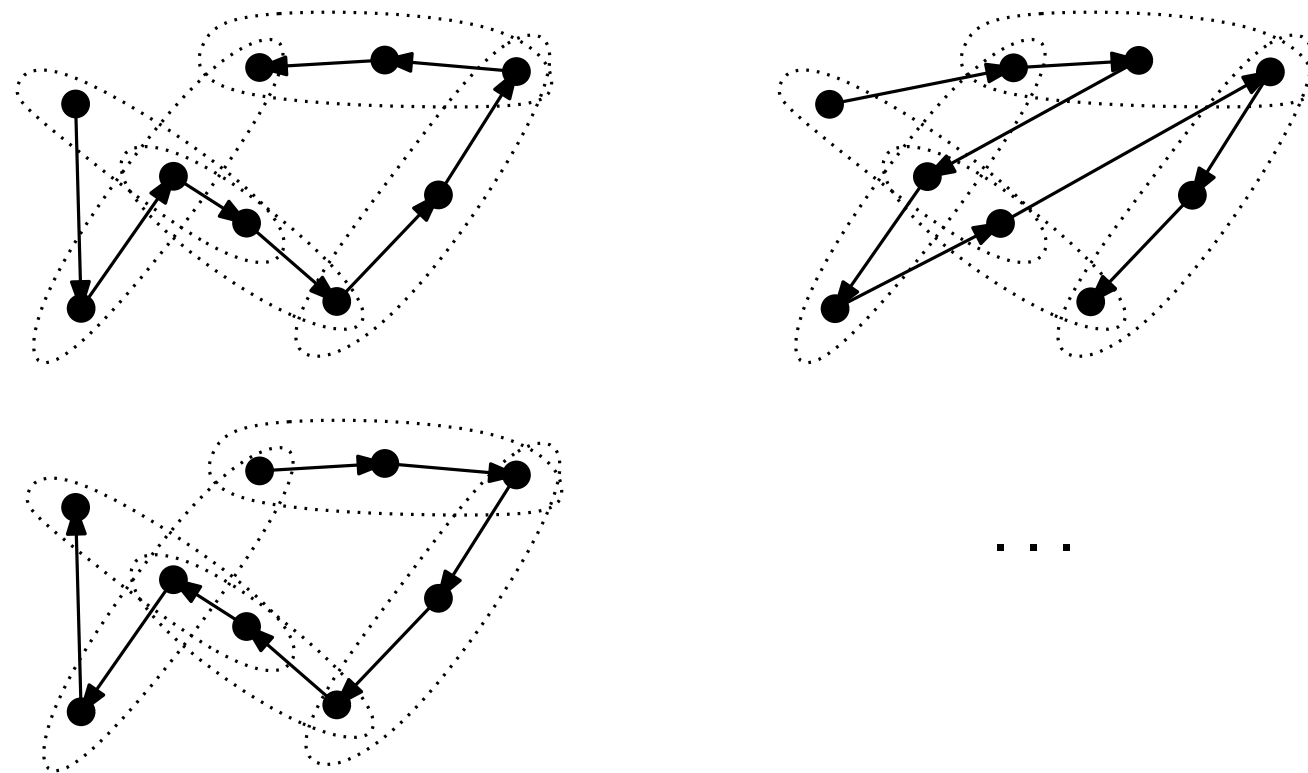
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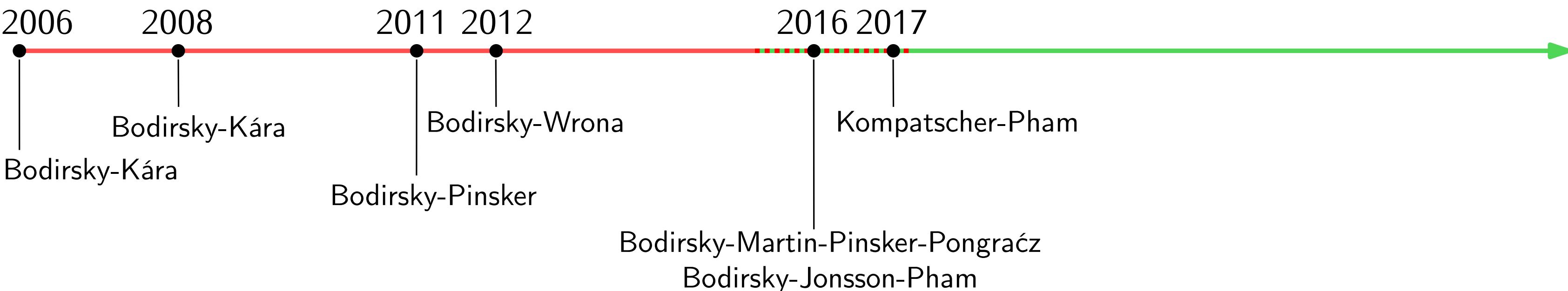
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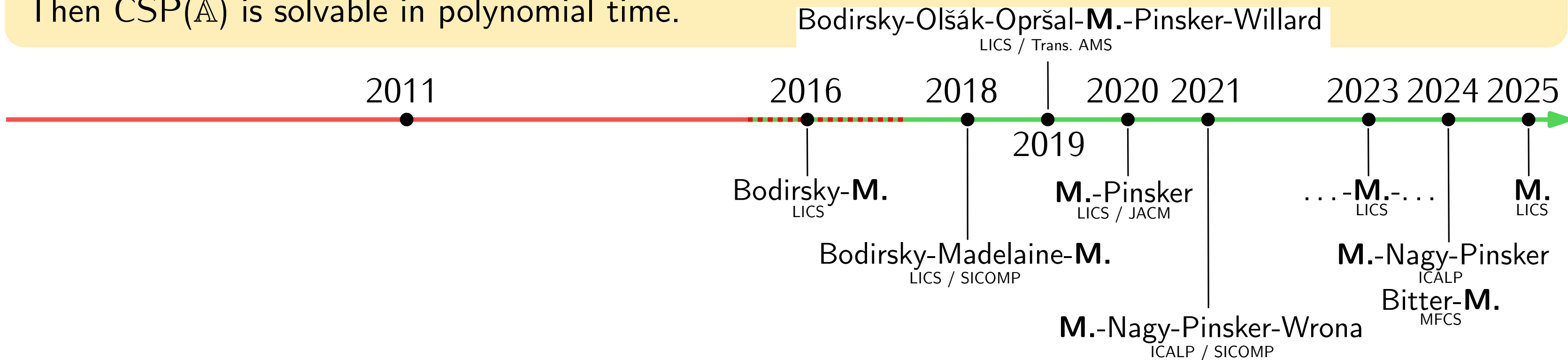


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Modern proofs:

- Unifying algebraic theory
- No combinatorial explosion
- Generic reduction to finite-domain CSPs
- Generic uniform algorithms
- Finer understanding than P vs. NP

Theorem (Bodirsky-M. LICS'16). For every \mathbb{A} with combinatorial certificates, there exists an equivalence relation \equiv such that if $\text{Pol}(\mathbb{A}, \equiv) \not\rightarrow \text{Pol}(\text{SAT})$, then $\text{CSP}(\mathbb{A})$ is solvable in polynomial time.

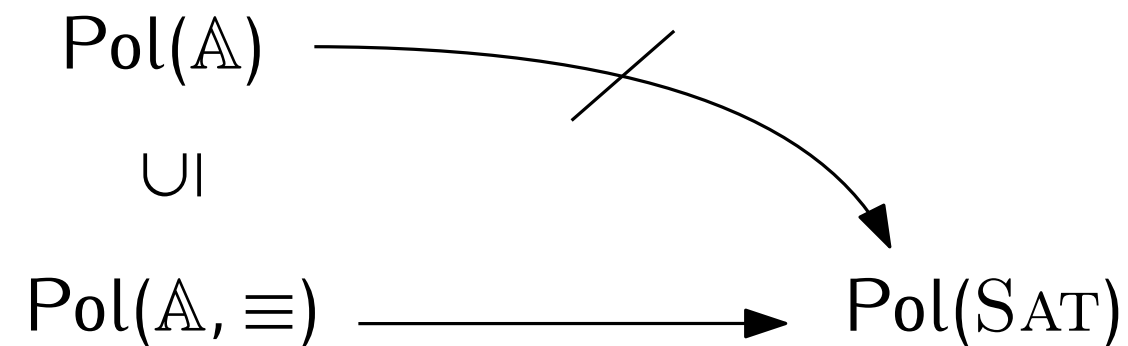
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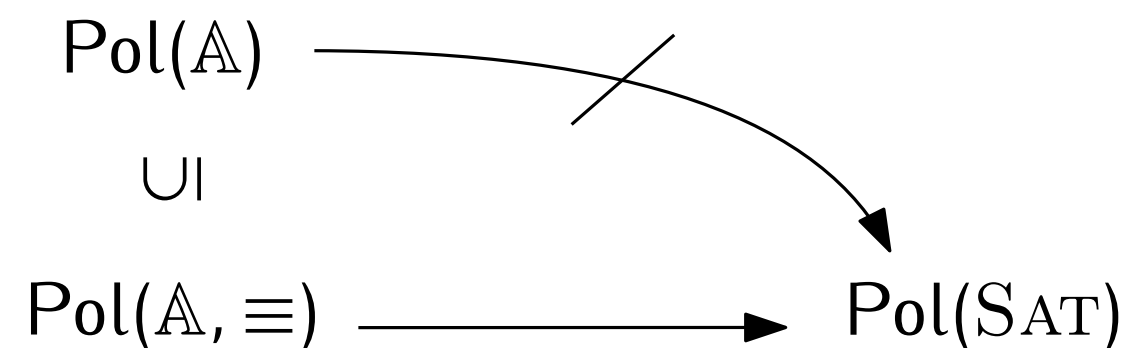
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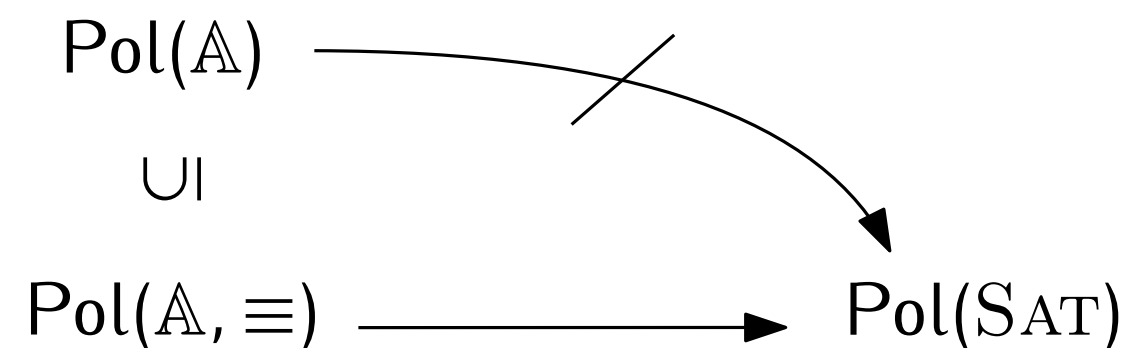


2020–2025: **algebraic theory** to handle this situation in generality

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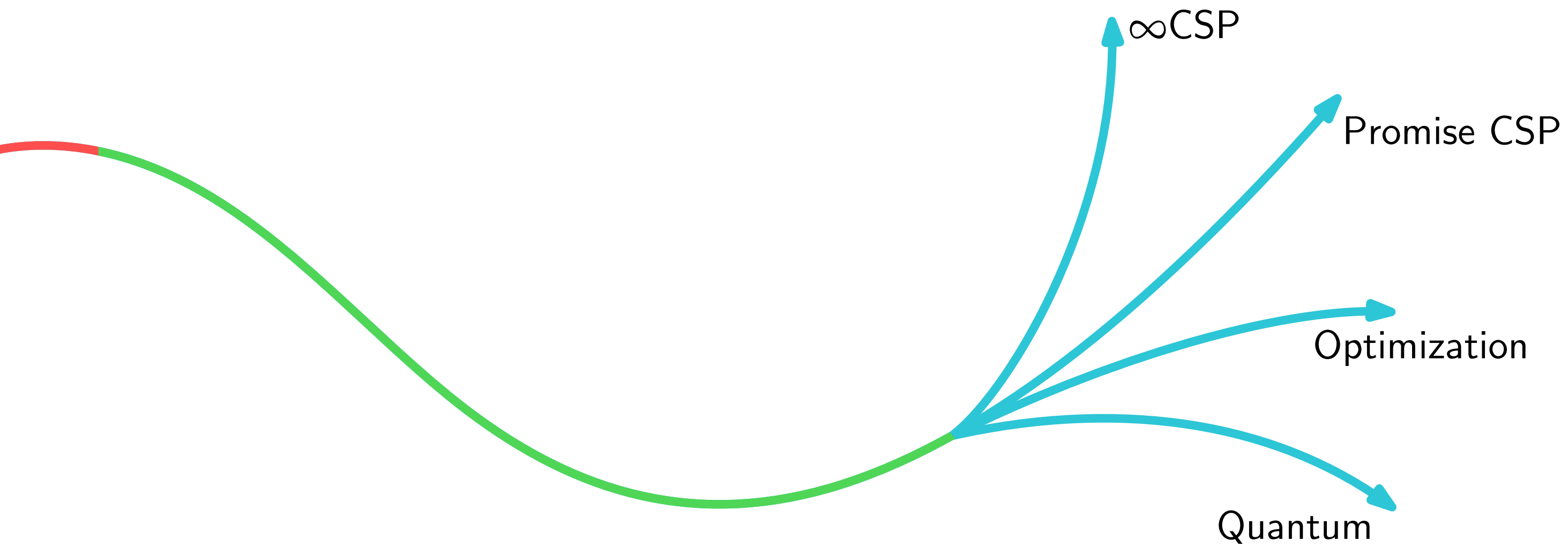
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Invited tutorial from
2025 Dagstuhl seminar:

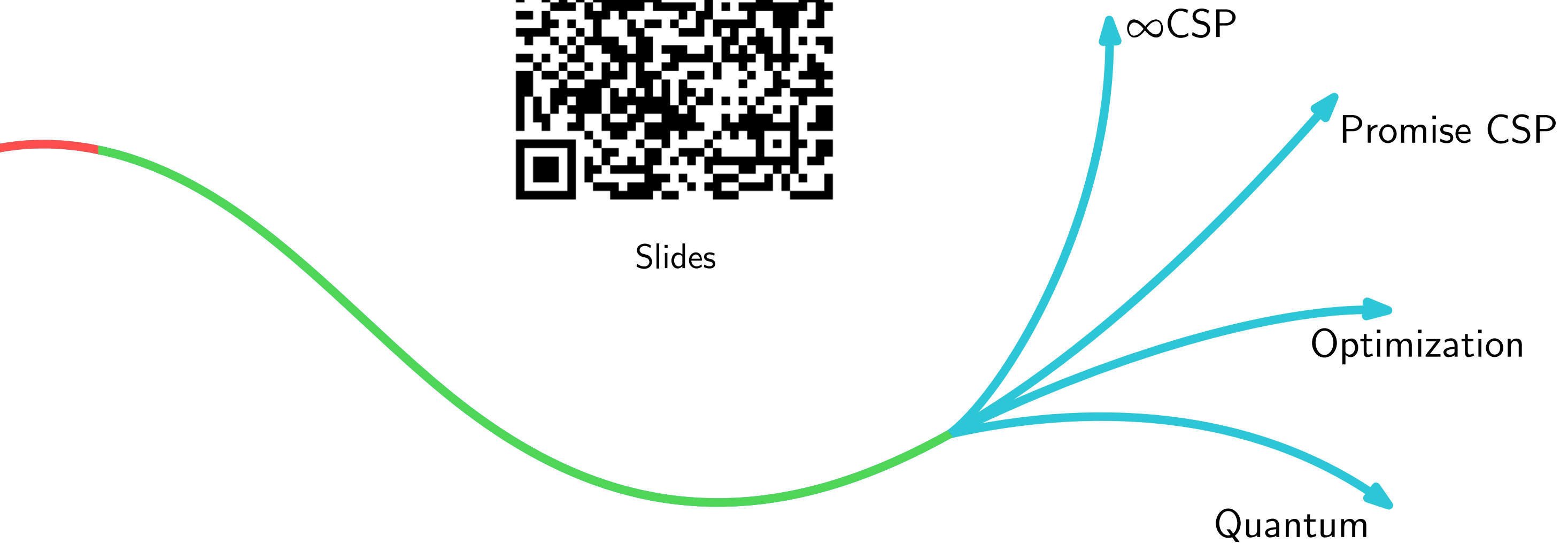




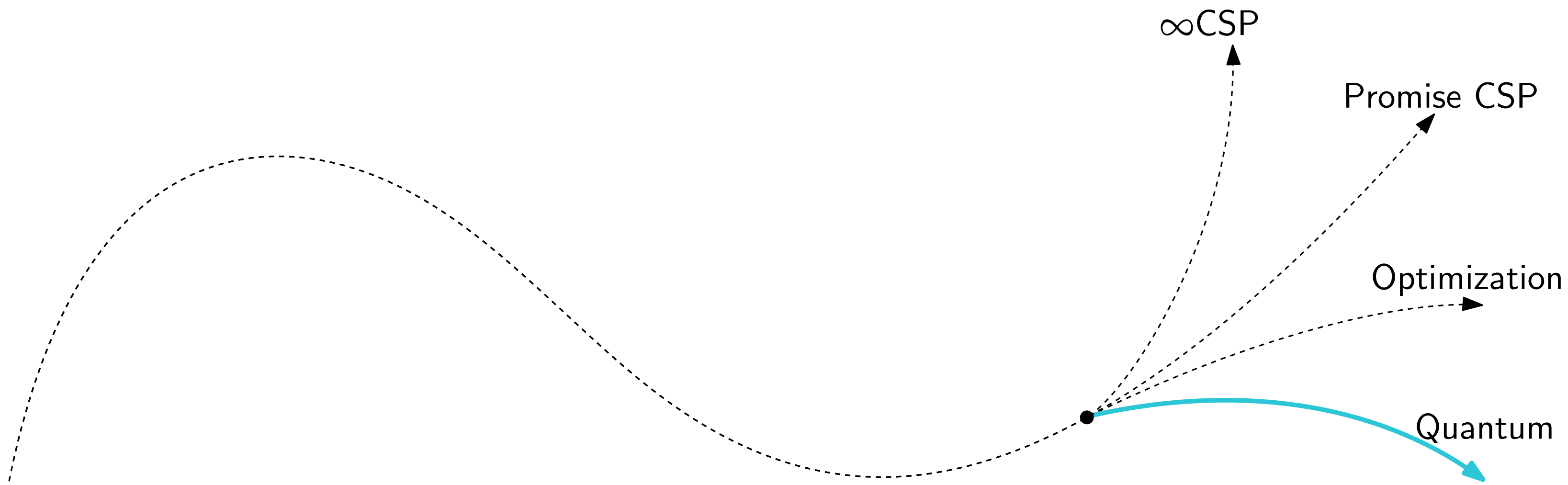
Thank you for your attention!



Slides

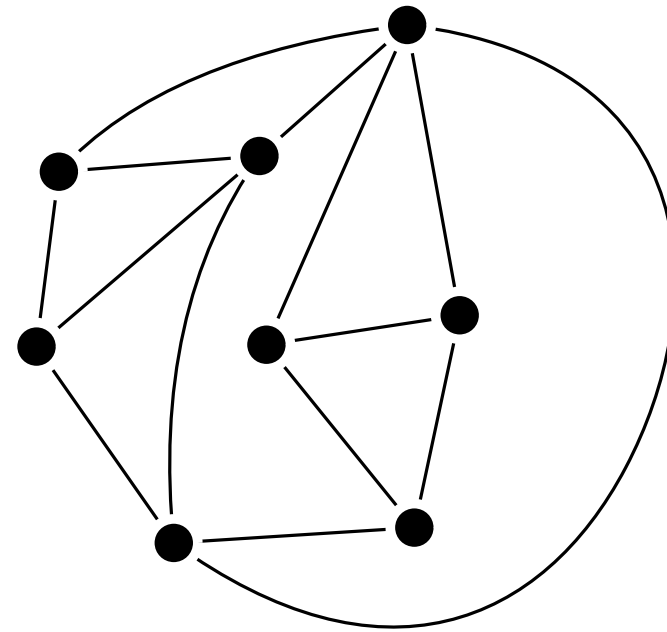


Research in Quantum Constraint Satisfaction



Verifier

Alice

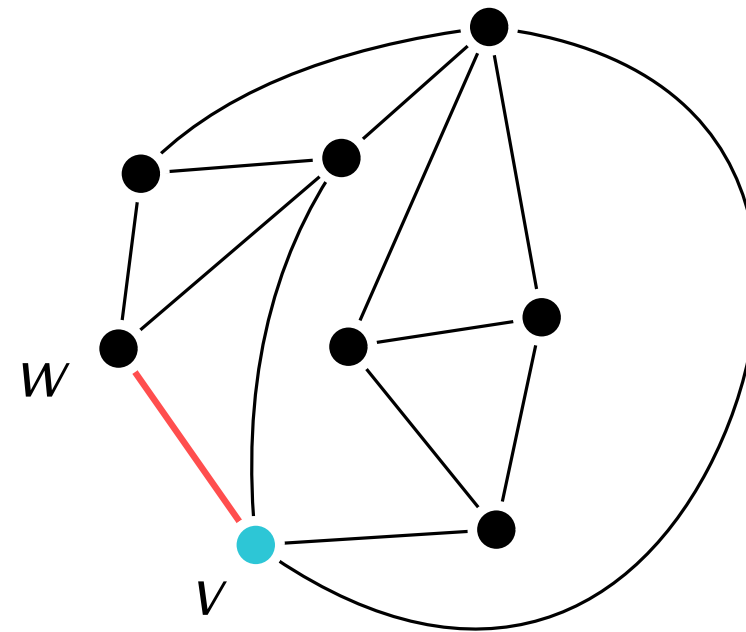


Bob

Verifier

1. selects $e = (v, w)$ following distribution π

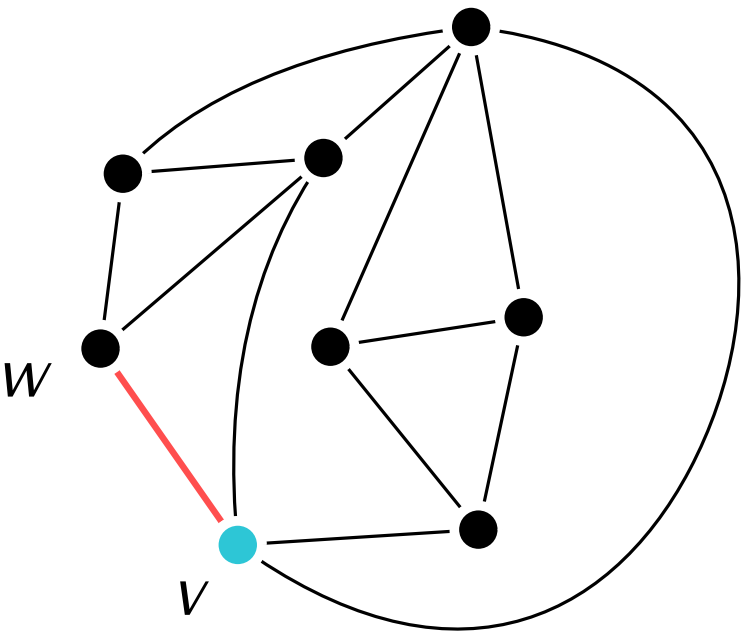
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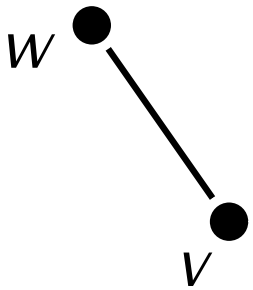
Verifier

- 1. selects $e = (v, w)$ following distribution π
- 2. sends e to Alice
- 3. sends v to Bob



Alice

- 4. Answers $(a, b) \in [k]^2$ with $a \neq b$ following distribution p_e



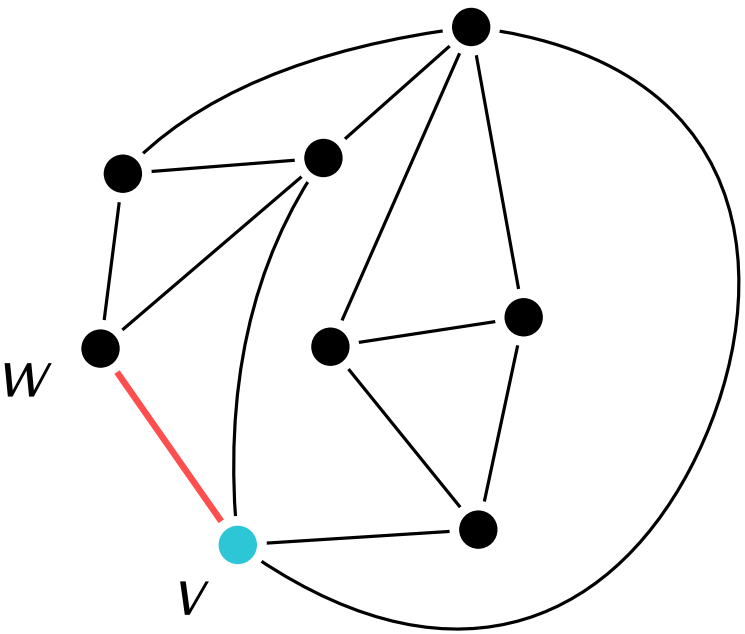
Bob



- 4. Answers $c \in [k]$ following distribution p_v

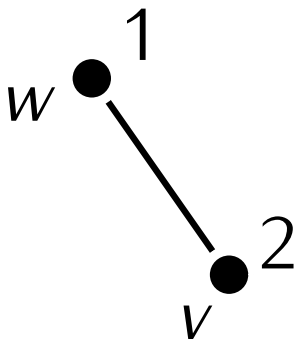
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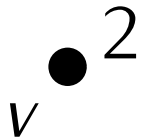
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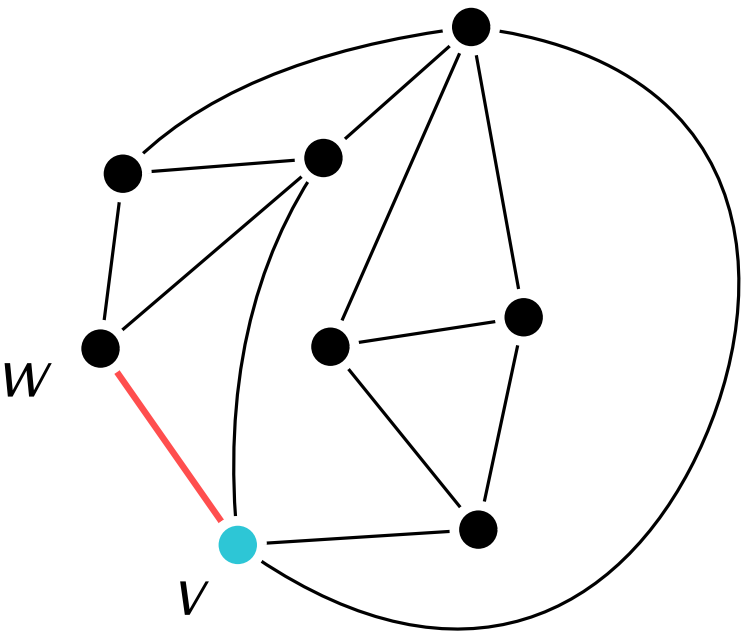
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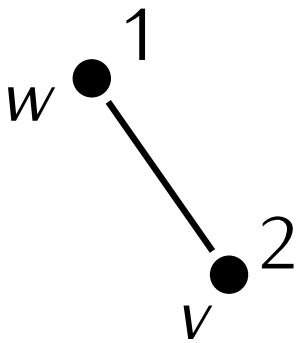
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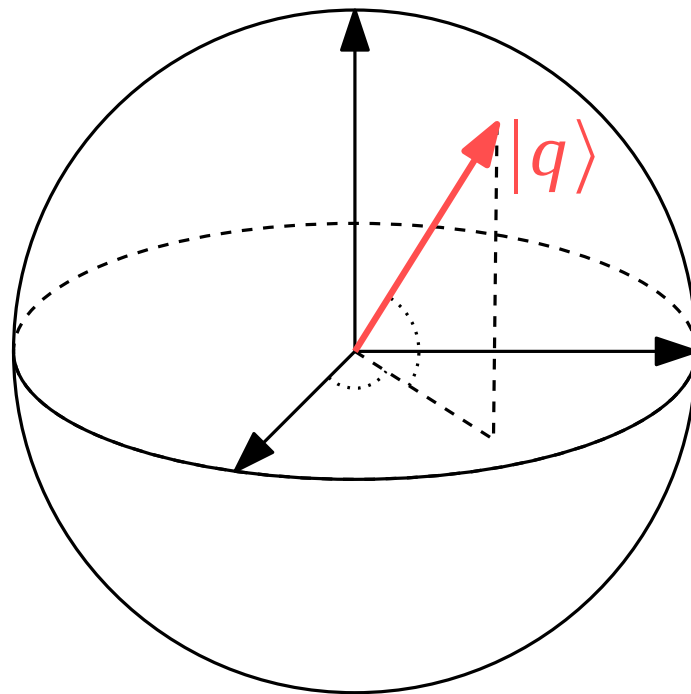
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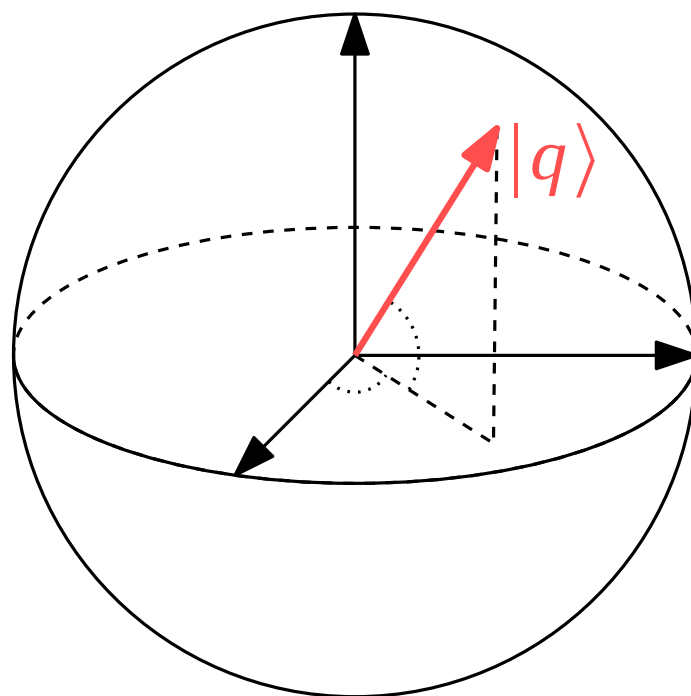
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How Alice performs a measurement:

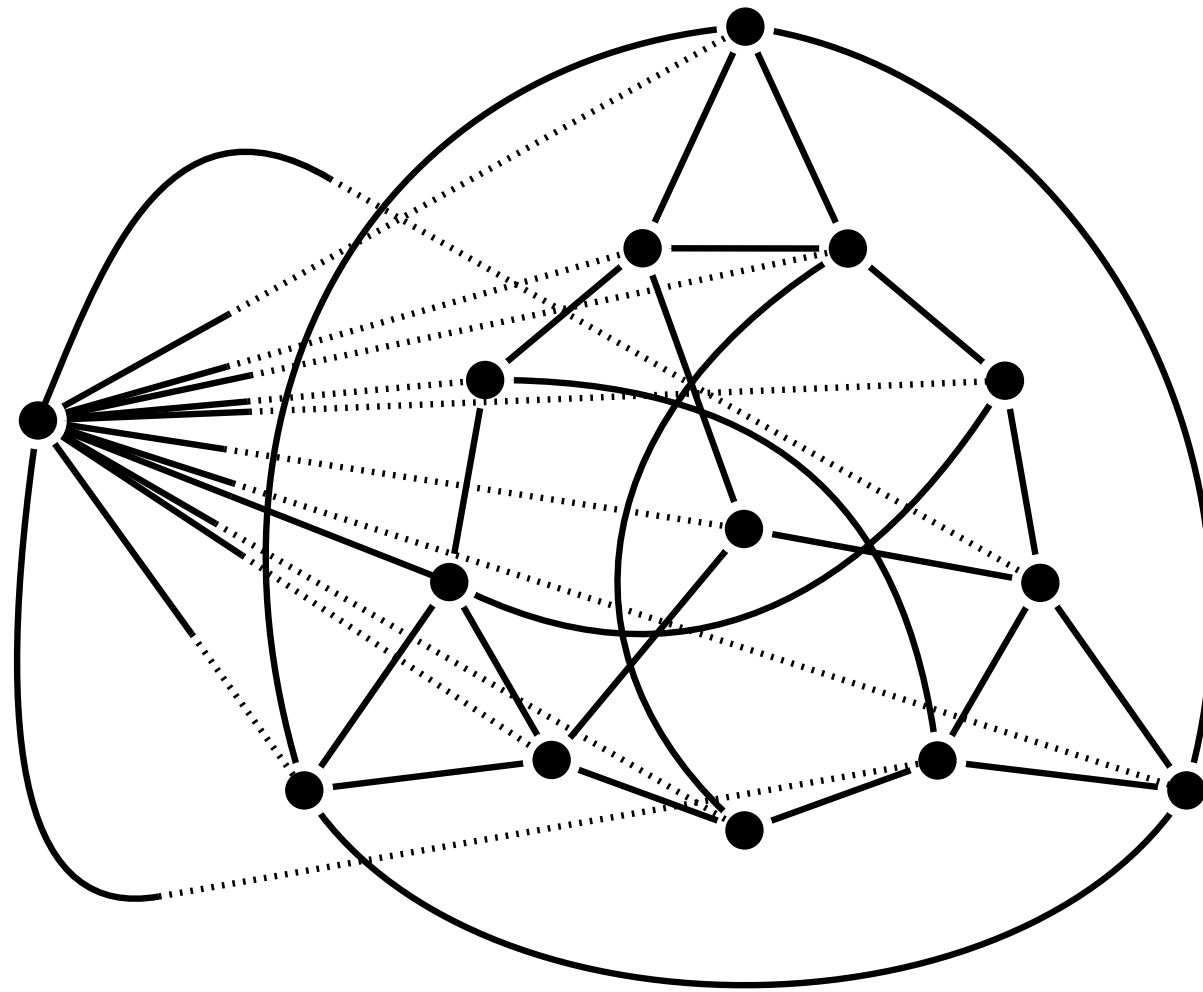
- Choose projectors $Q_{a,b}$ summing to id
- Output of measurement is (a, b) with probability $\langle q|Q_{a,b}|q\rangle$

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Quantum 4-colorable graph with chromatic number 5
[Mańćinska-Roberson '18]

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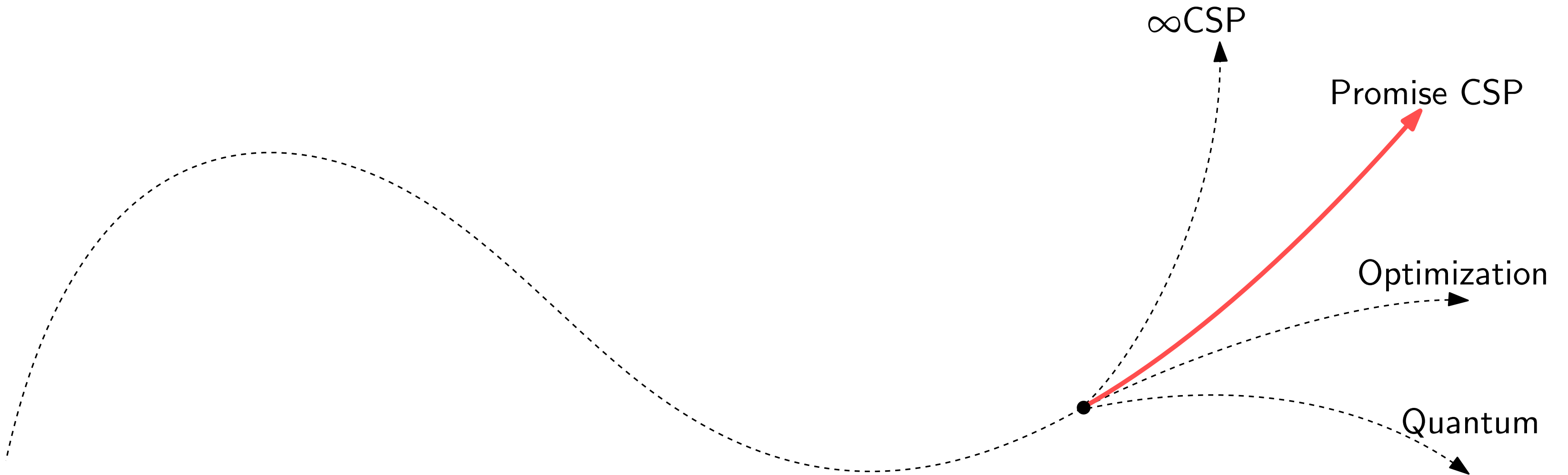
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Project (with L. Ciardo and G. Joubert). Understand the complexity landscape for entangled CSPs.

Research in Promise Constraint Satisfaction



Solving CSPs under structural assumptions on the **instances**:

Definition. $\text{PCSP}(\mathbb{A}, \mathbb{B})$ is the problem to find a satisfying assignment in \mathbb{B} , under the **promise** that there exists one in \mathbb{A} .

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Given an undirected **3-colorable** graph G , find a 6-coloring.

Unknown complexity, related to the existence of a **2-approximation algorithm** for the chromatic number.

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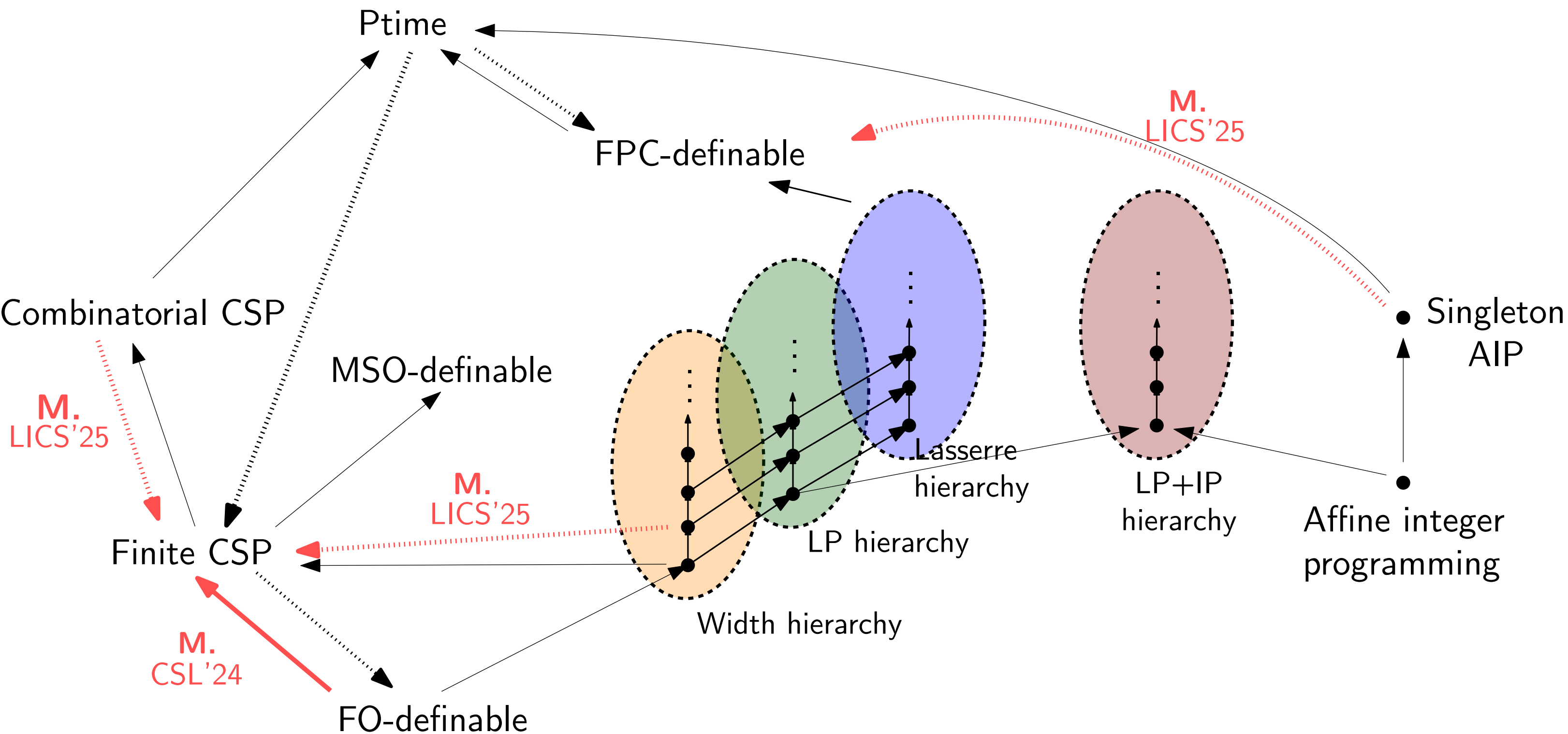
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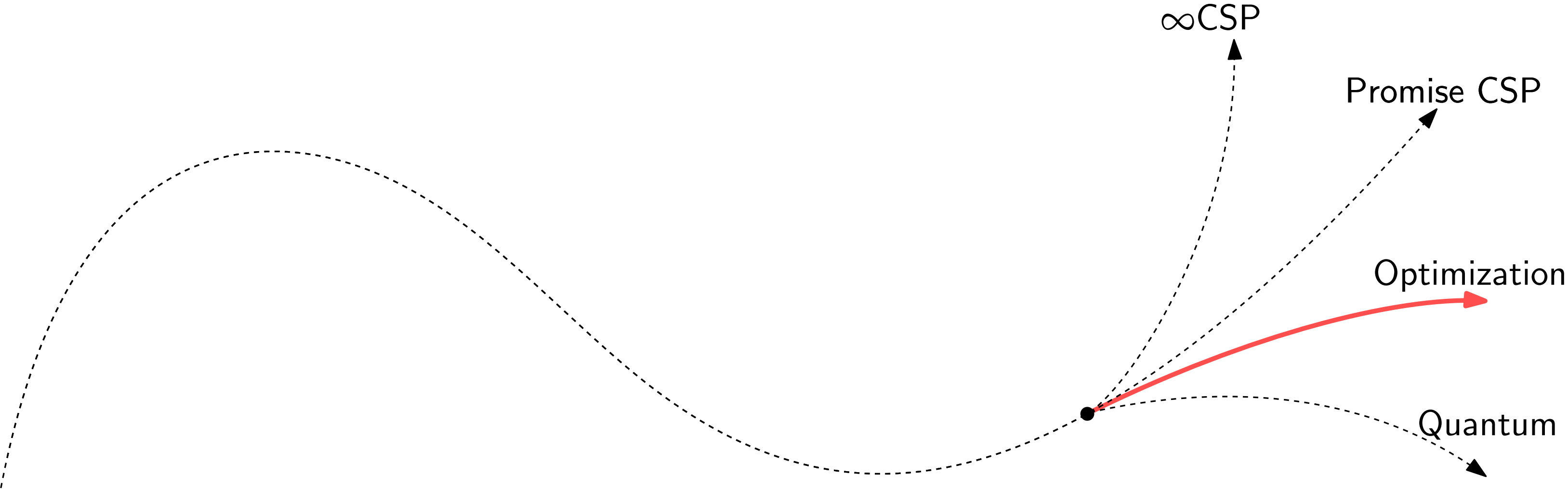
- Massive framework for the study of approximation problems
- Existence of a suitable generalization to the algebraic approach
- Surprising connection between **finite-domain** PCSPs and **combinatorial** CSPs:

Theorem (M. LICS'25). The following hold:

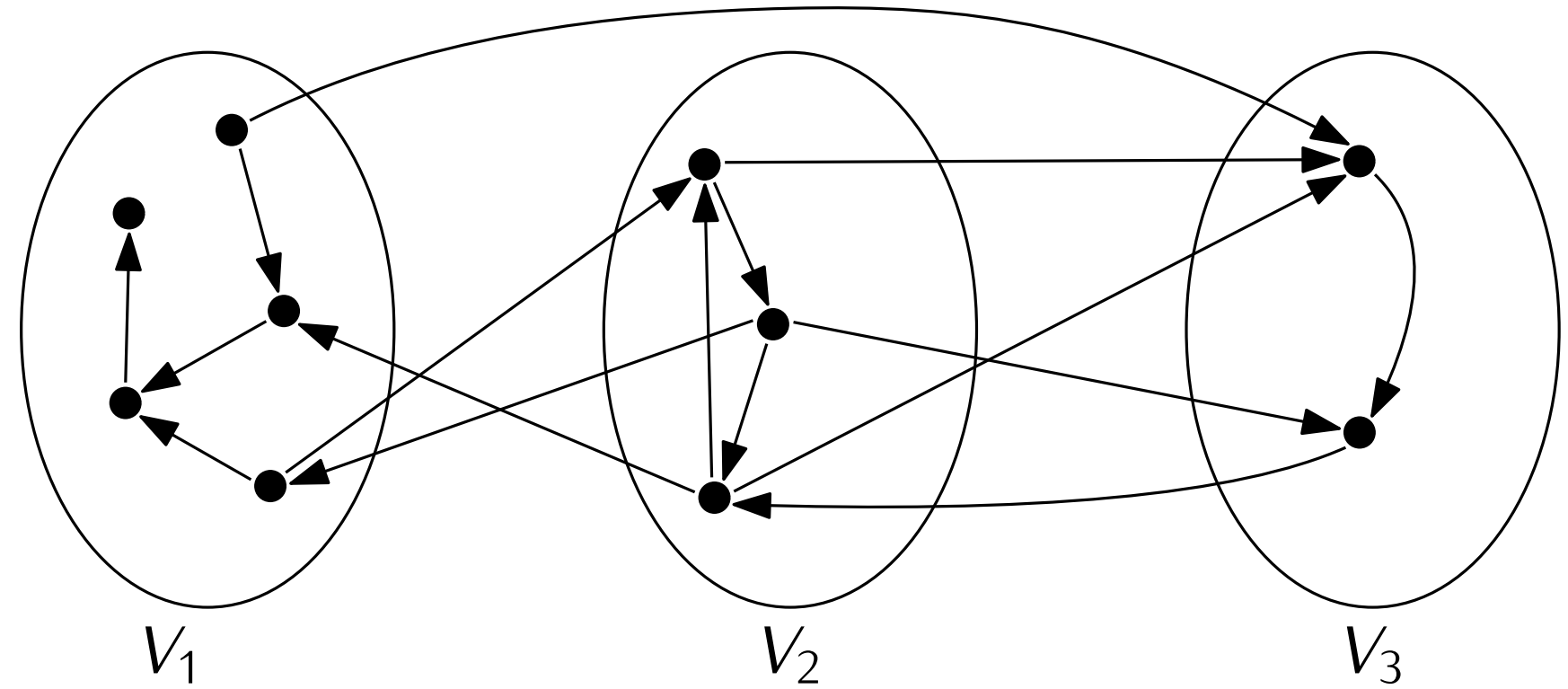
- Combinatorial CSP $\simeq \text{PCSP}(\text{combinatorial}, \text{finite})$
- There are finite PCSPs whose tractability can be shown by combinatorial CSPs and not by finite CSPs.
- There exists a uniform algorithm for temporal CSPs obtained by a reduction to finite PCSPs.



Research in Optimization

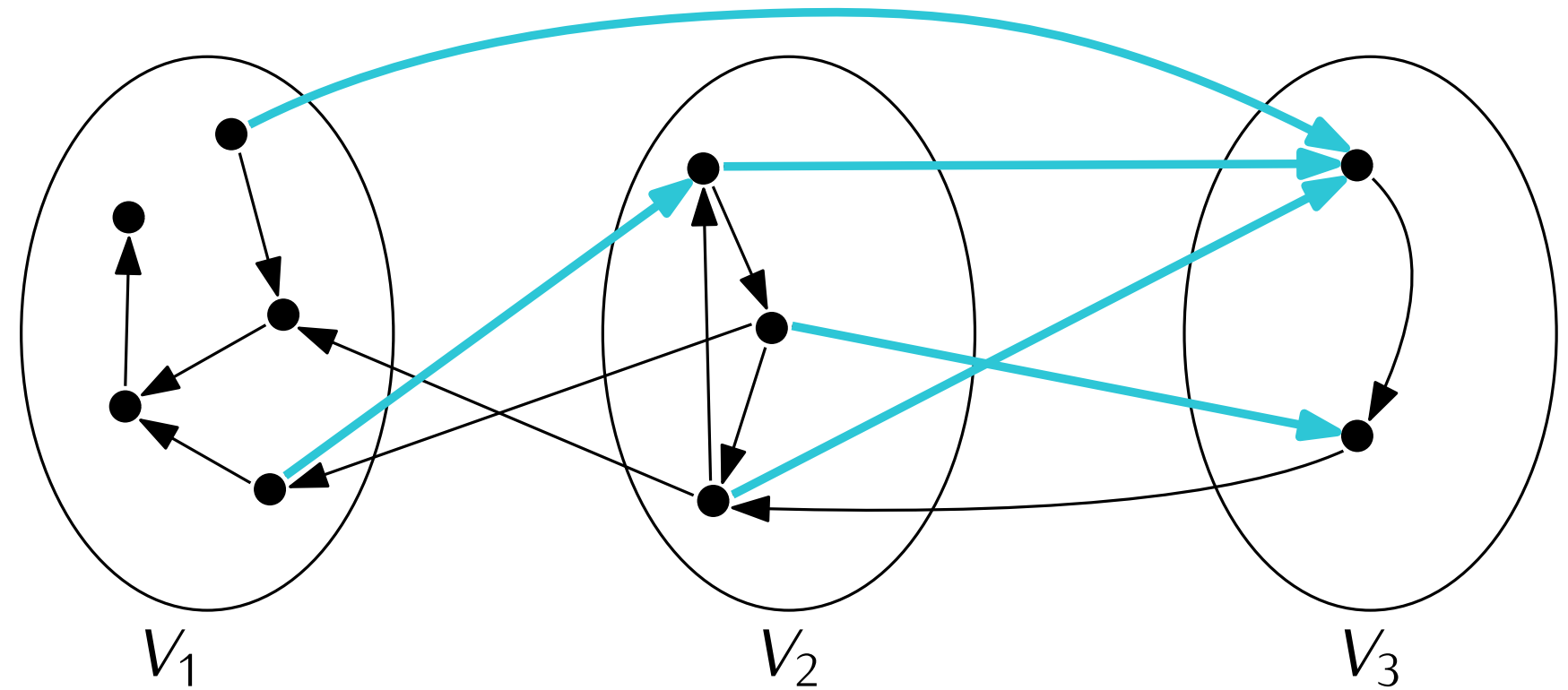


k -layering of a digraph: map $\ell: V \rightarrow \{1, \dots, k\}$



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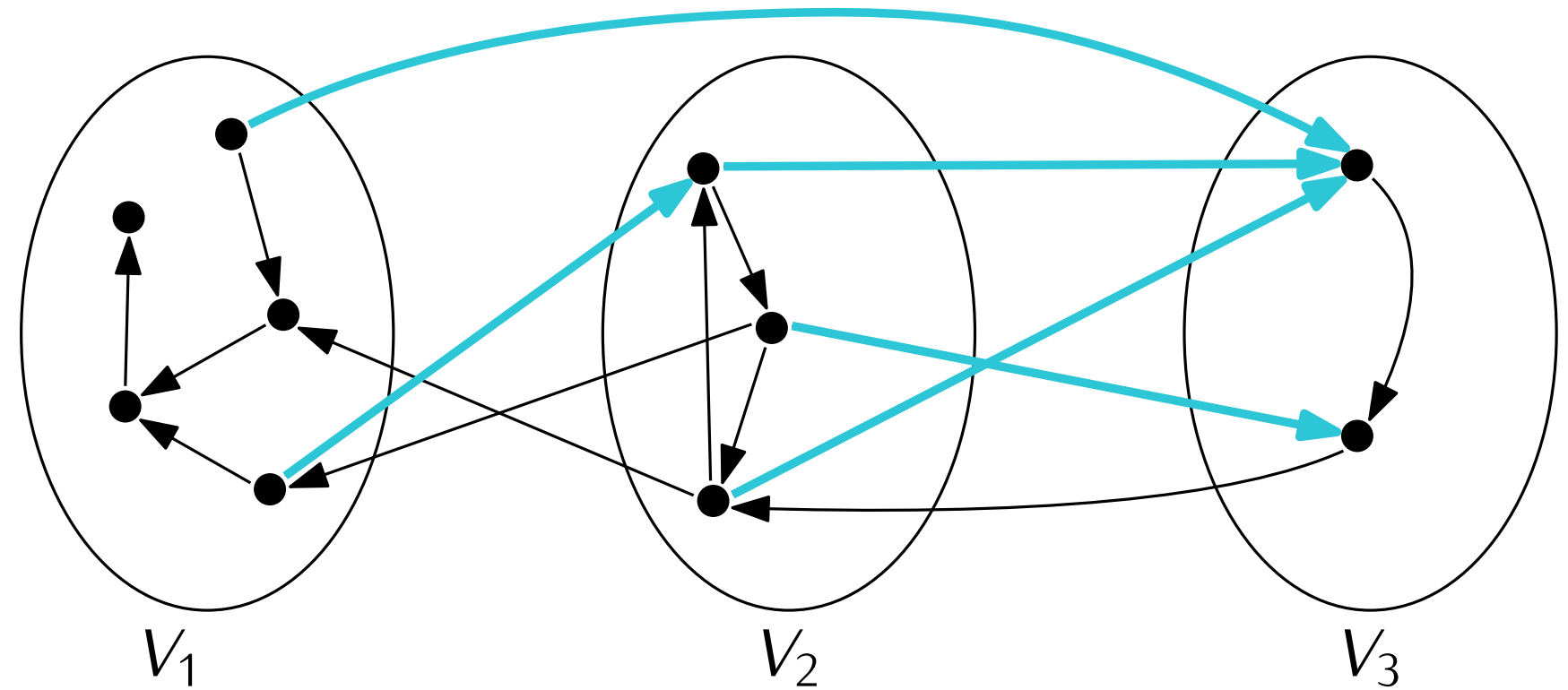
Value of ℓ : $\#\{(u, v) \in E \mid \ell(u) < \ell(v)\}$



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Note: every layering with value ρ is an acyclic subgraph with ρ edges

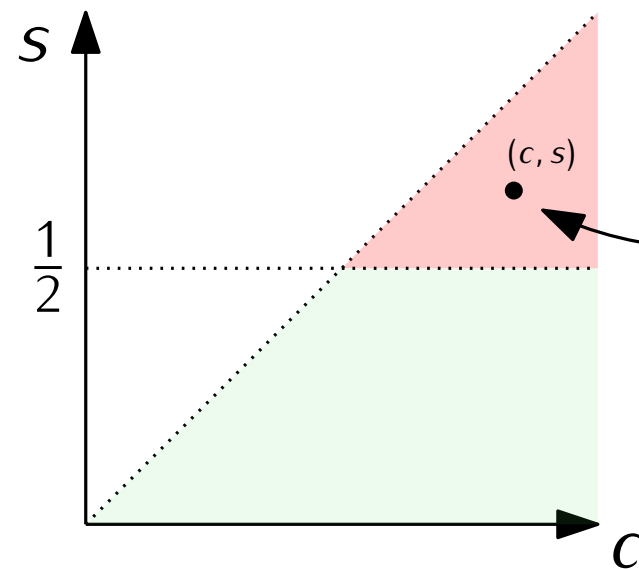


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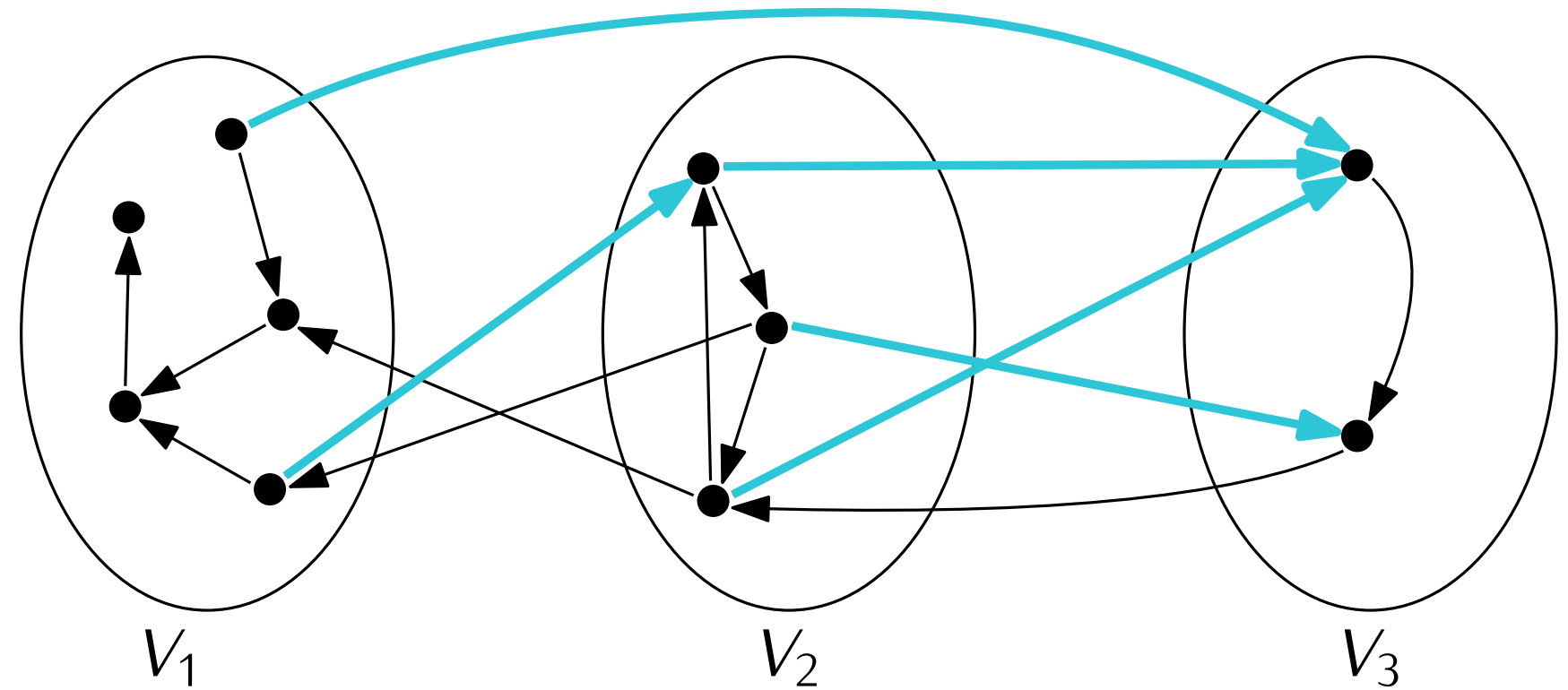
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Approximation diagram for
Maximum Acyclic Subgraph



Given a digraph with an acyclic subgraph of weight $\geq c|E|$,
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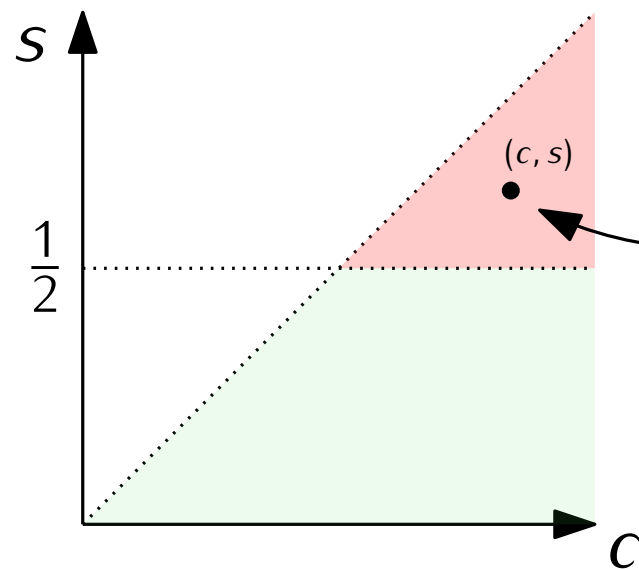


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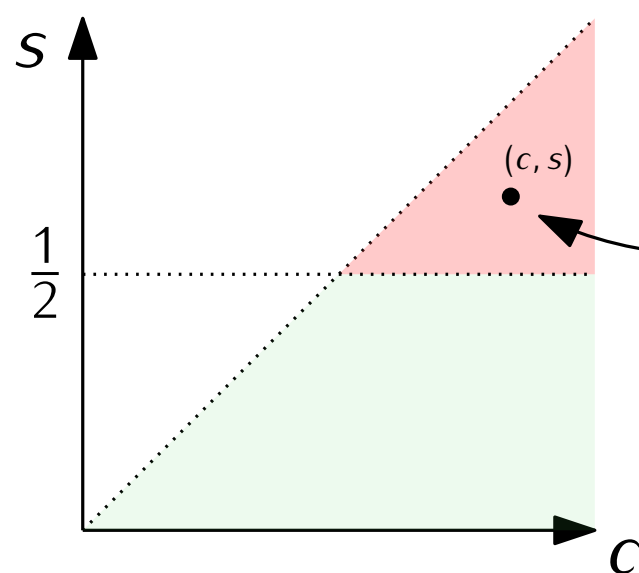
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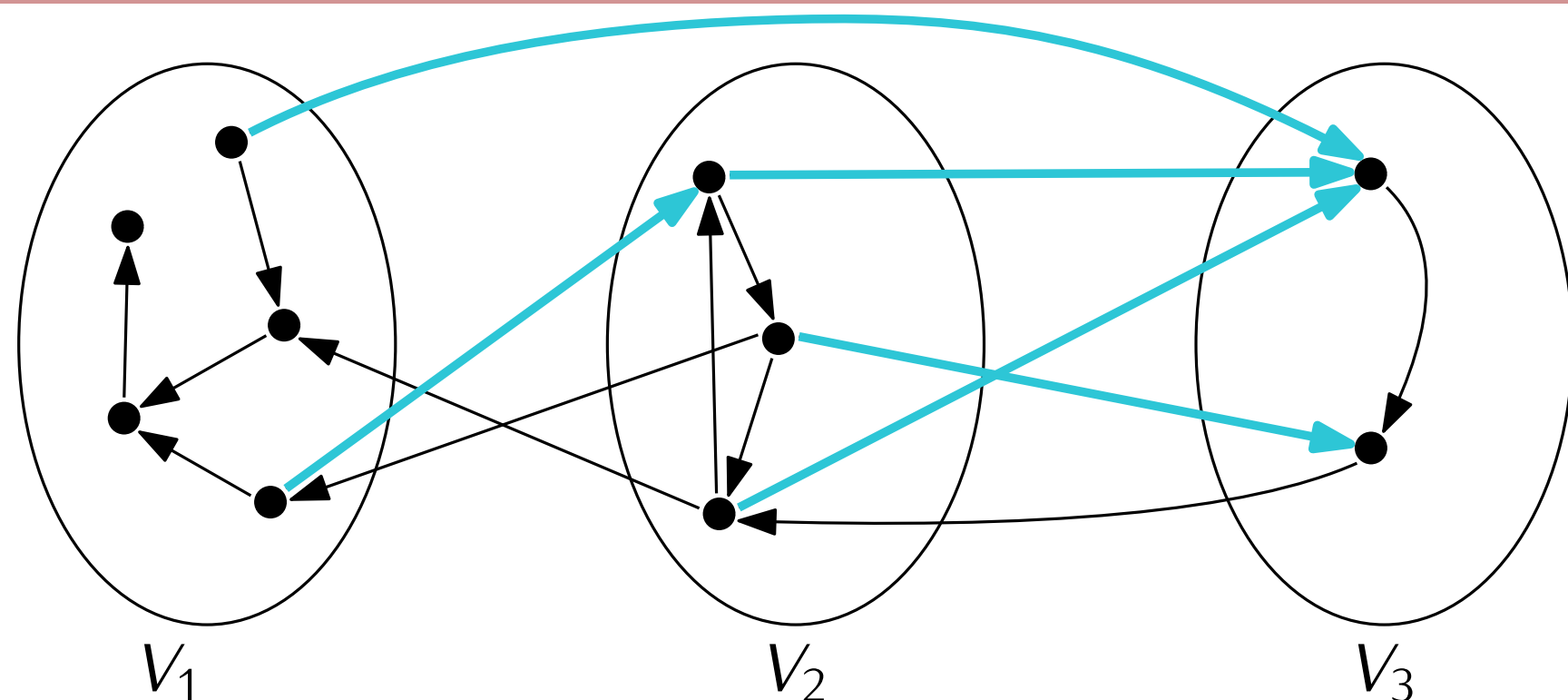
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PCSP($\text{LO}_k, \text{LO}_\infty$) (maximization version): given a directed graph G admitting a k -layering of value ρ , find an acyclic subgraph of G containing at least ρ edges.

- Solvable in polynomial time for $k = 2$ (Nakajima-Živný, **M.** '25)
- NP-hard for $k \geq 4$, but unknown tractability boundary in approximation diagram
- Open complexity for $k = 3$