Theory of Constraint Satisfaction Problems

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Colloquium Computational Discrete Mathematics

TU Graz

Instance:

- a domain A
- variables $x, y, z, \ldots$
- constraints: $x \vee y \vee z, z \leq x, 2x + y = 0, \ldots$

Question: does there exist $h\colon \mathrm{Vars} \to A$ satisfying every constraint?

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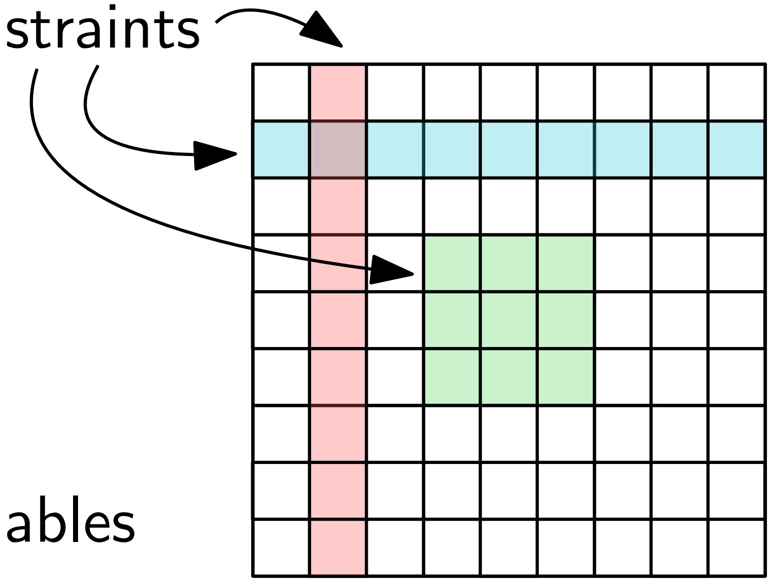
Sudoku

9-ary constraints



9^2 variables

Linear Programming

Is $\mathbf{A}x = \mathbf{b}$ satisfiable in $\mathbb{R}_{\geq 0}$?

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- Non-uniform CSP: domain is fixed and only certain types of constraints are allowed

  Constraint with r variables $\leftrightarrow$ relation $R \subseteq A^r$ listing all valid assignments

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$\textsc{1-In-3-Sat} : (\{0, 1\}; \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\})$
$n\text{-}\textsc{Coloring} : (\{1, \ldots, n\}, \neq)$
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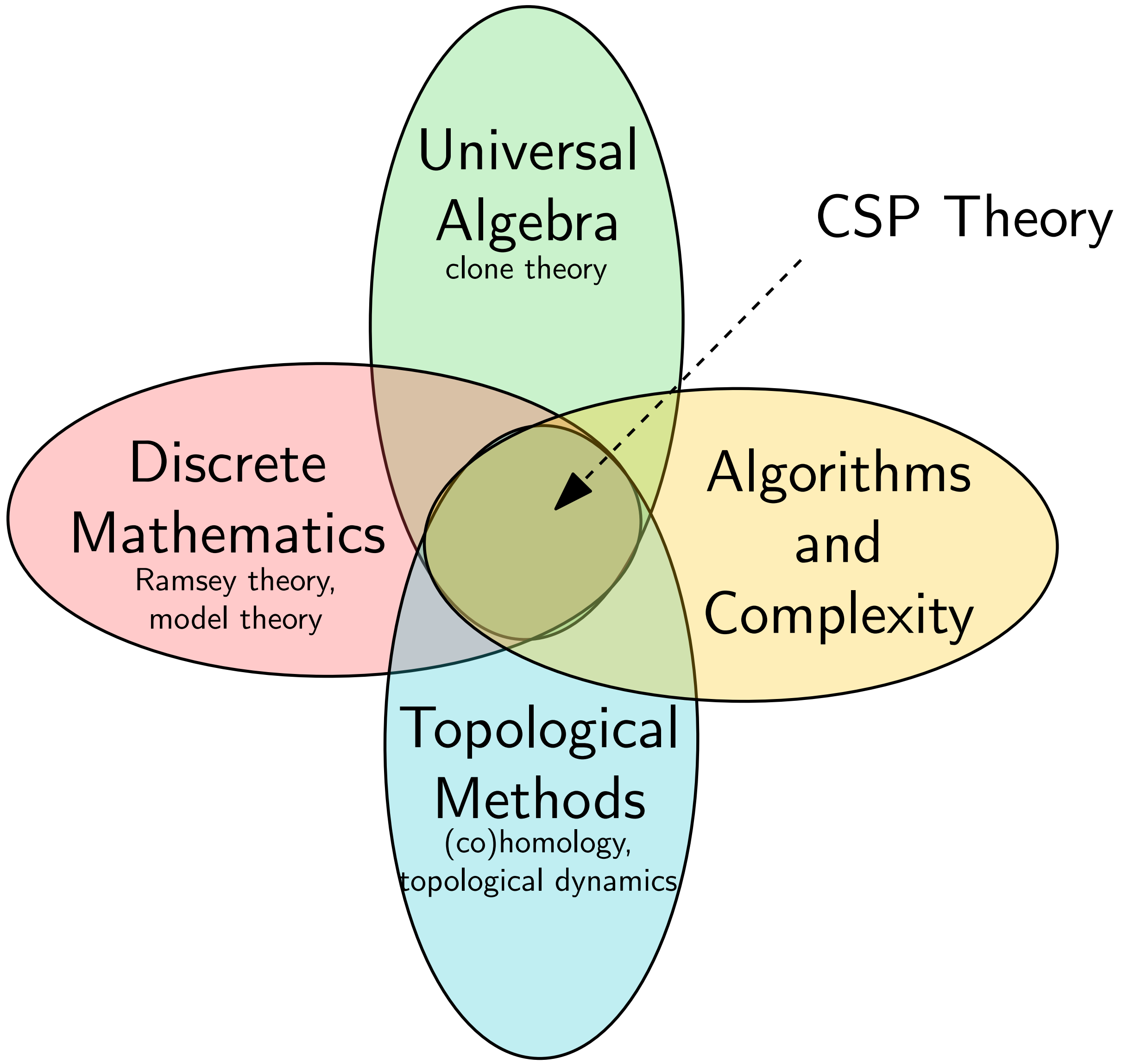
Theorem (Bulatov // Zhuk '17)**.** For every template $\mathbb{A}$ with a finite domain, CSP($\mathbb{A}$) is solvable in polynomial time or NP-complete.

Finite-domain dichotomy result built on polymorphisms $\mathrm{Pol}(\mathbb{A}) = \{f \colon \mathbb{A}^n \to \mathbb{A}, n \geq 1\}$

- Clones: closed under composition, contain projections
- Standard object in universal algebra
- Clone actions of $\mathrm{Pol}(\mathbb{A})$ tell us about the complexity of $\mathrm{CSP}(\mathbb{A})$



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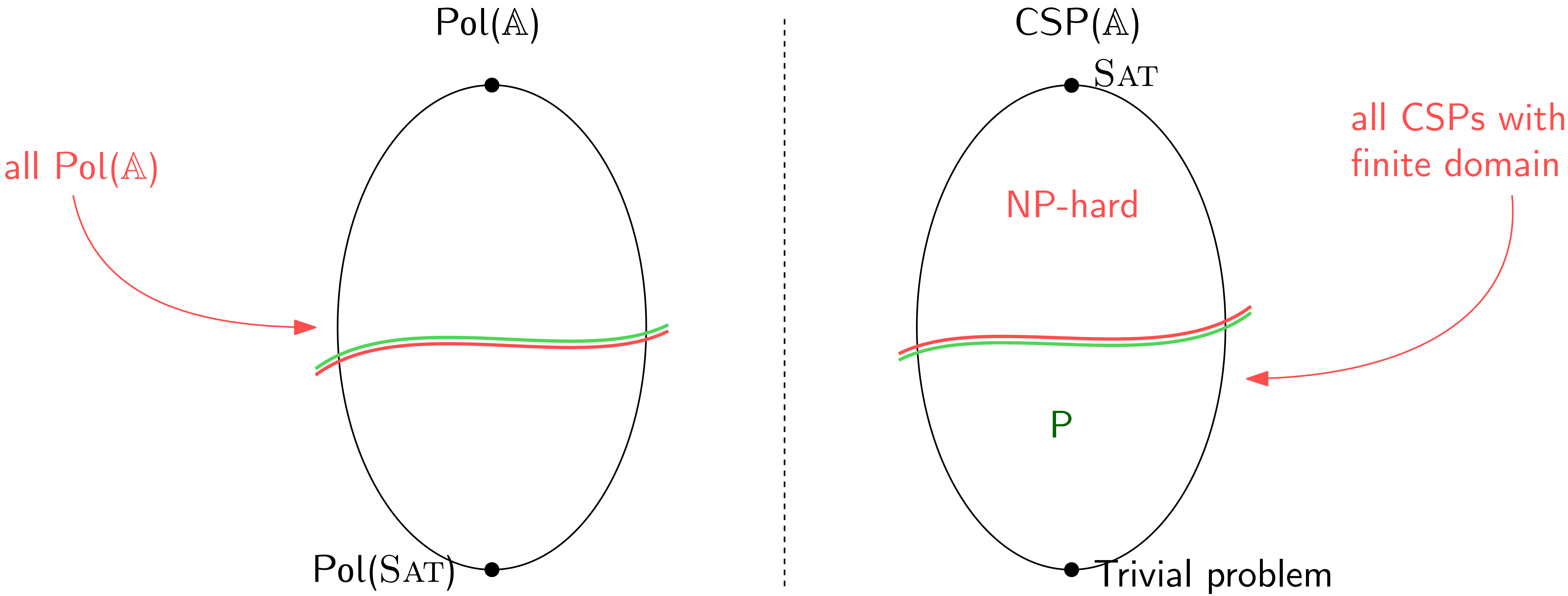

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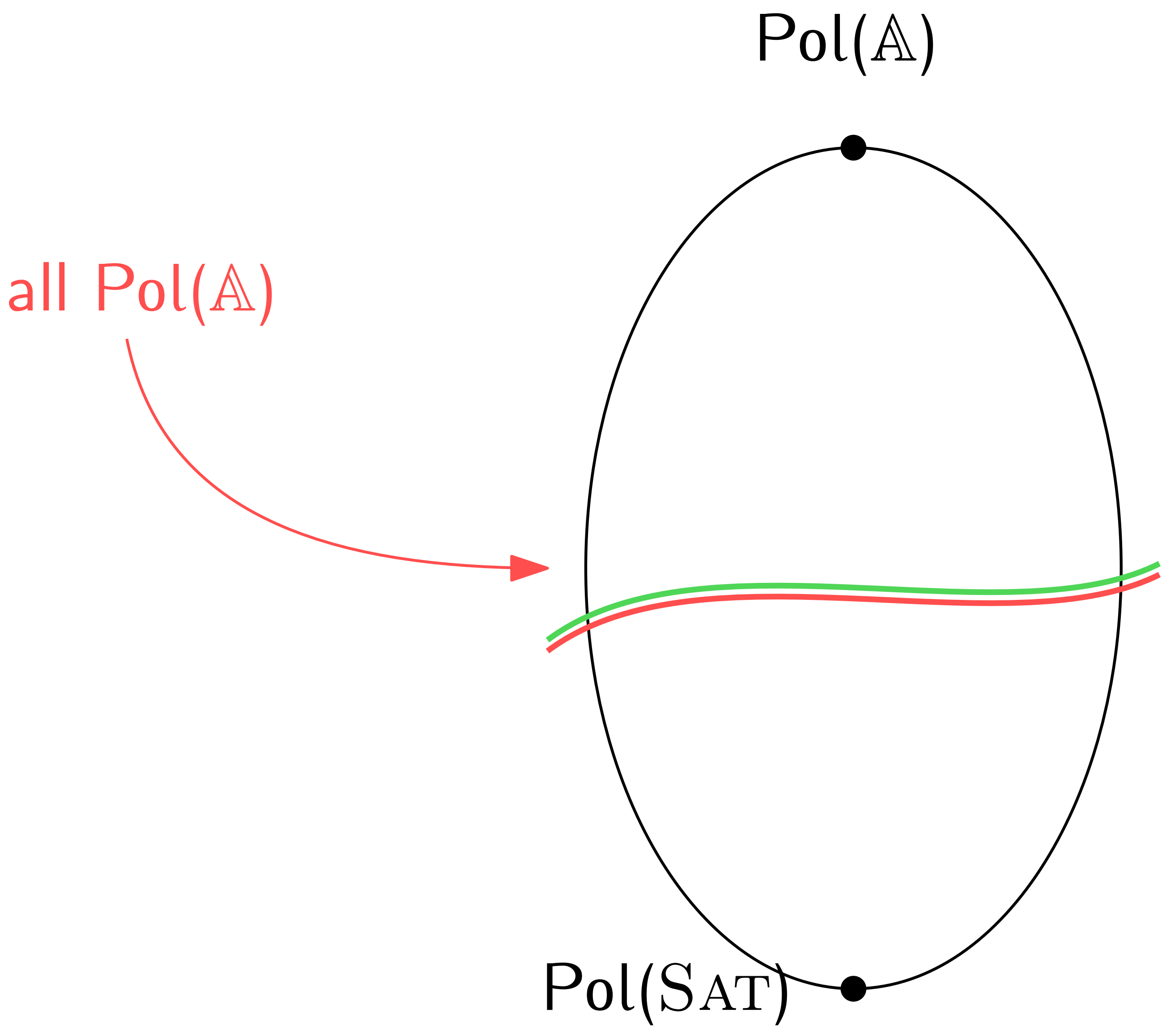

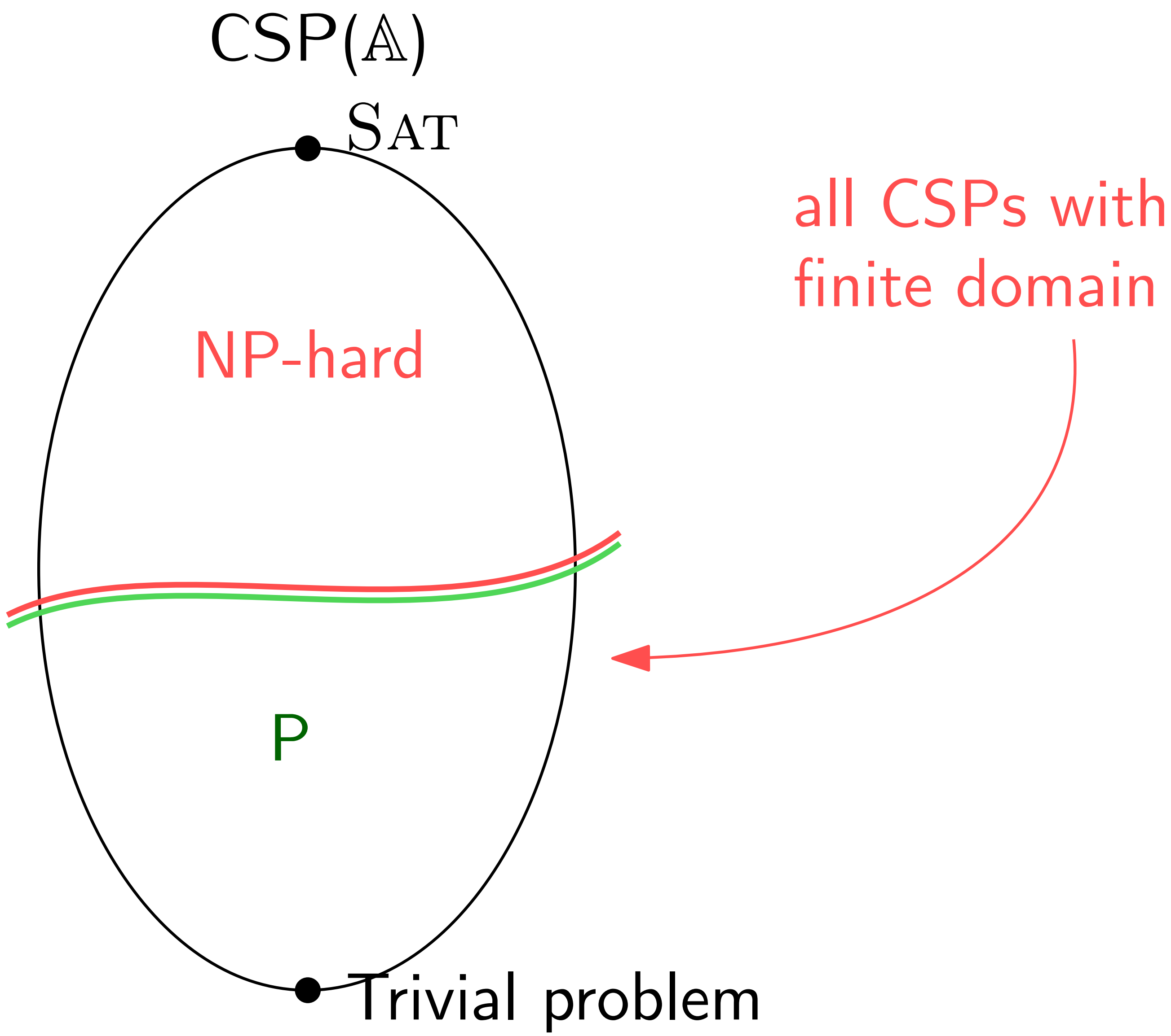

- $\mathrm{Pol}(\mathbb{A}) = \mathrm{Pol}(\mathbb{B})$ implies $\mathrm{CSP}(\mathbb{A}) \equiv \mathrm{CSP}(\mathbb{B})$ (Bulatov, Jeavons, Krokhin)

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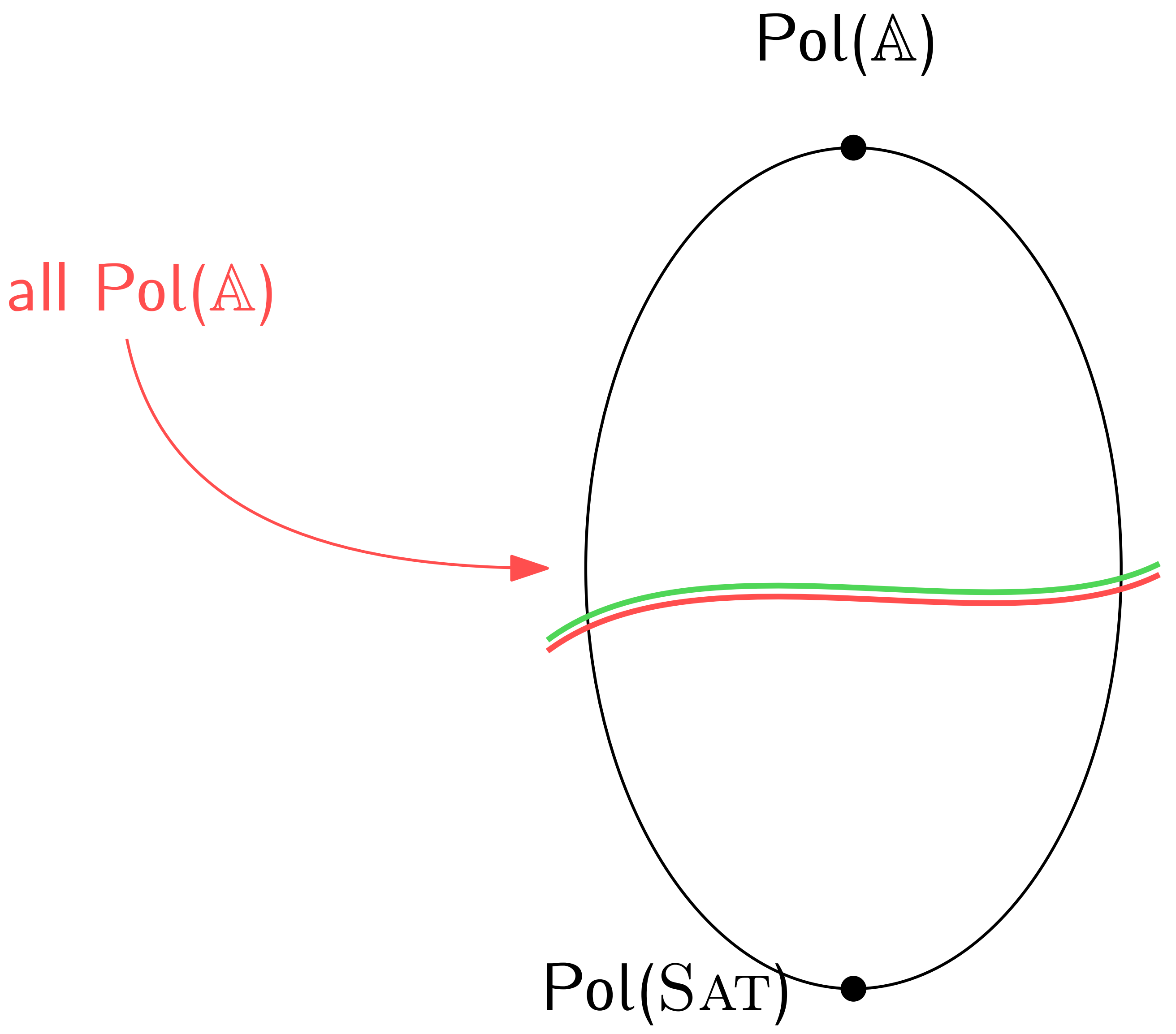

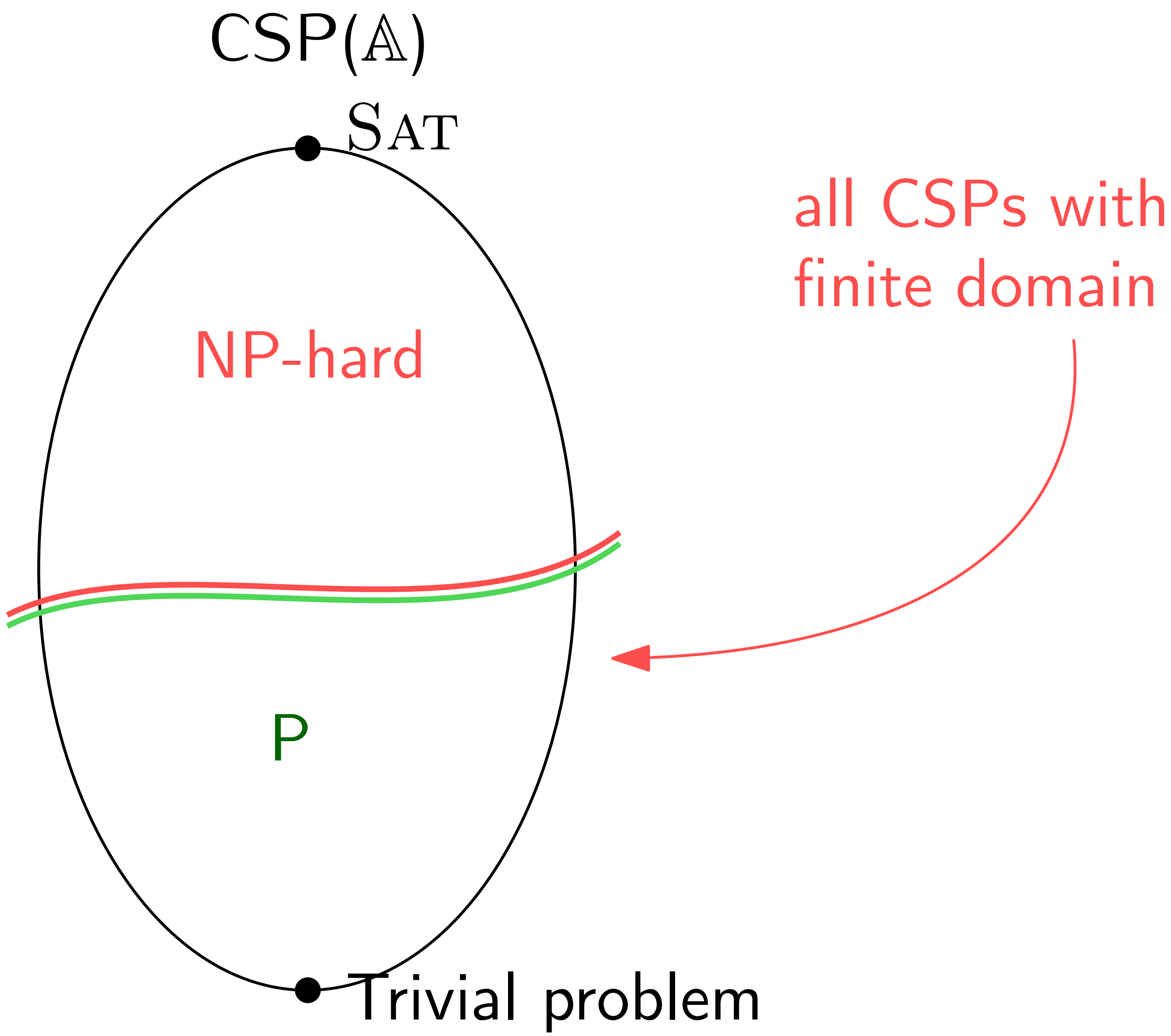

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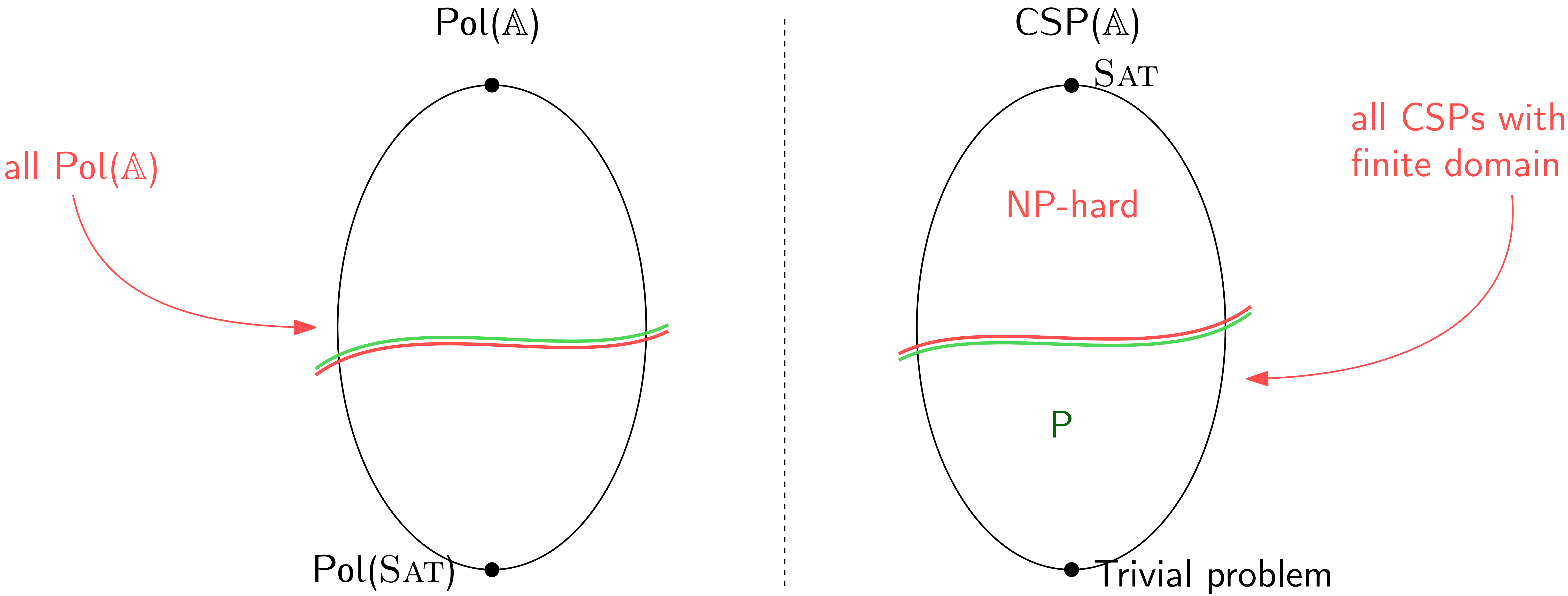

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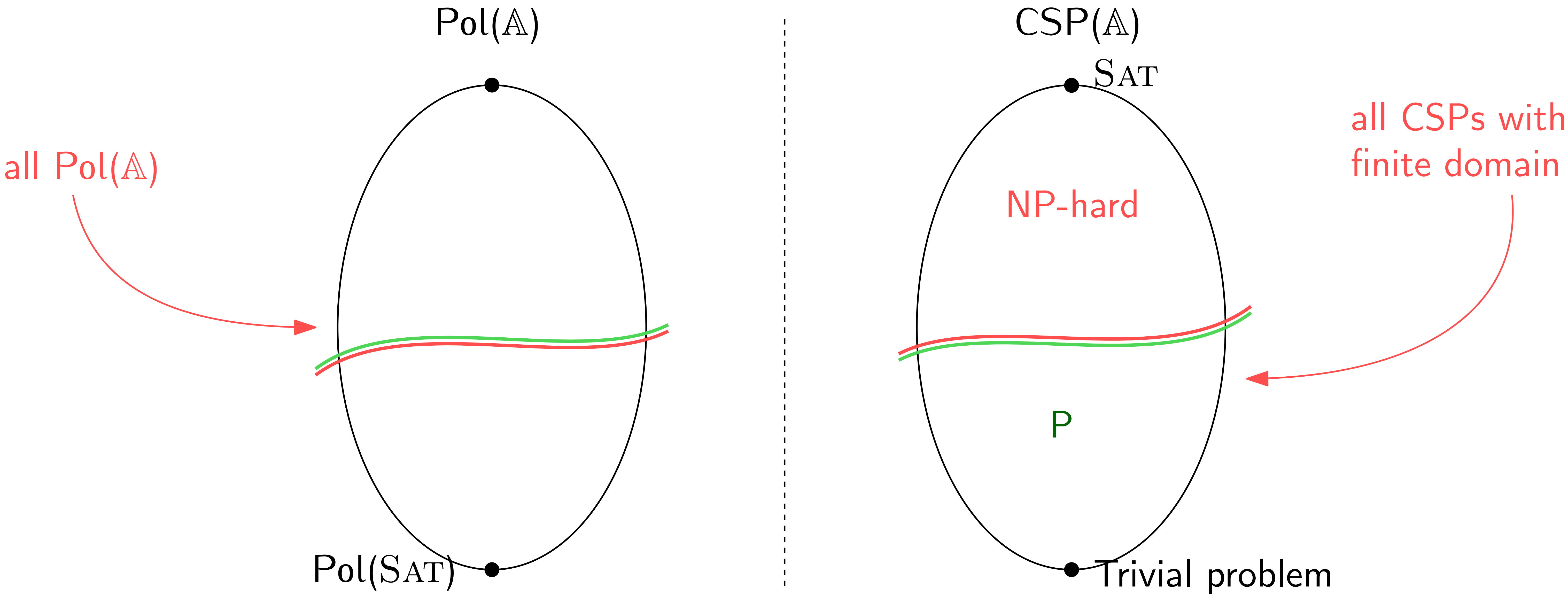

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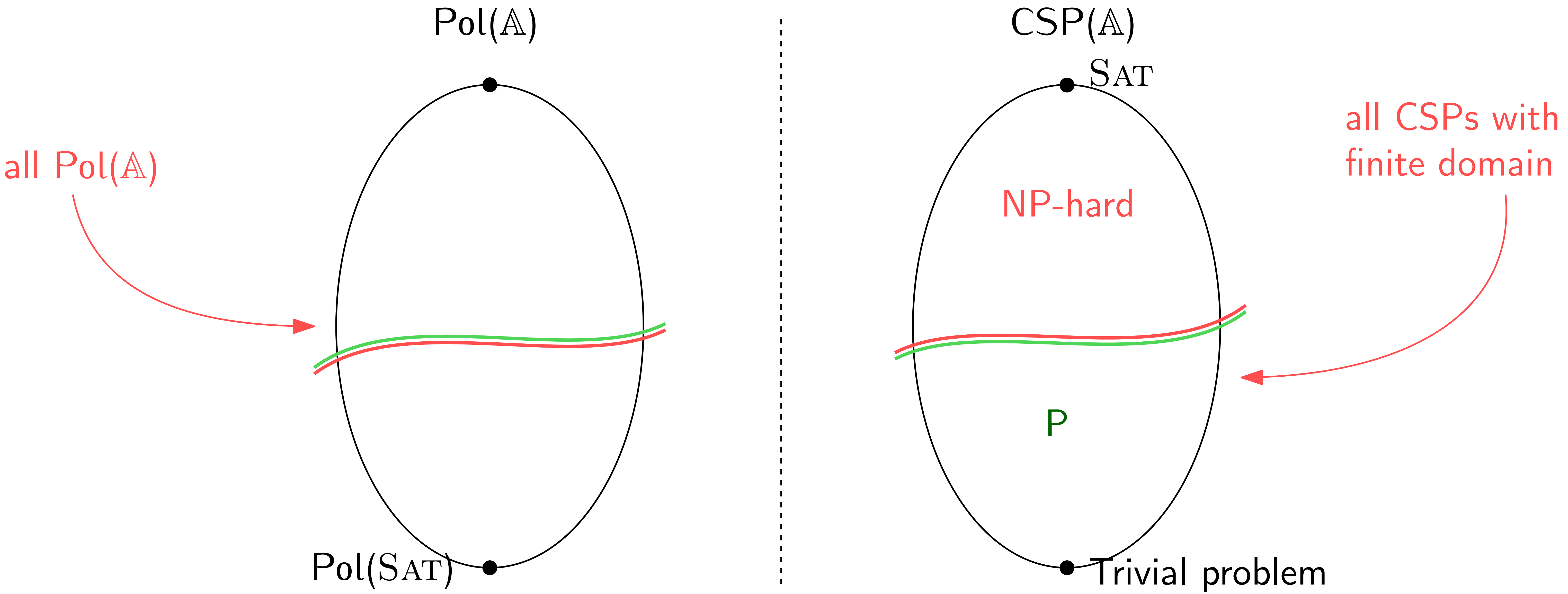

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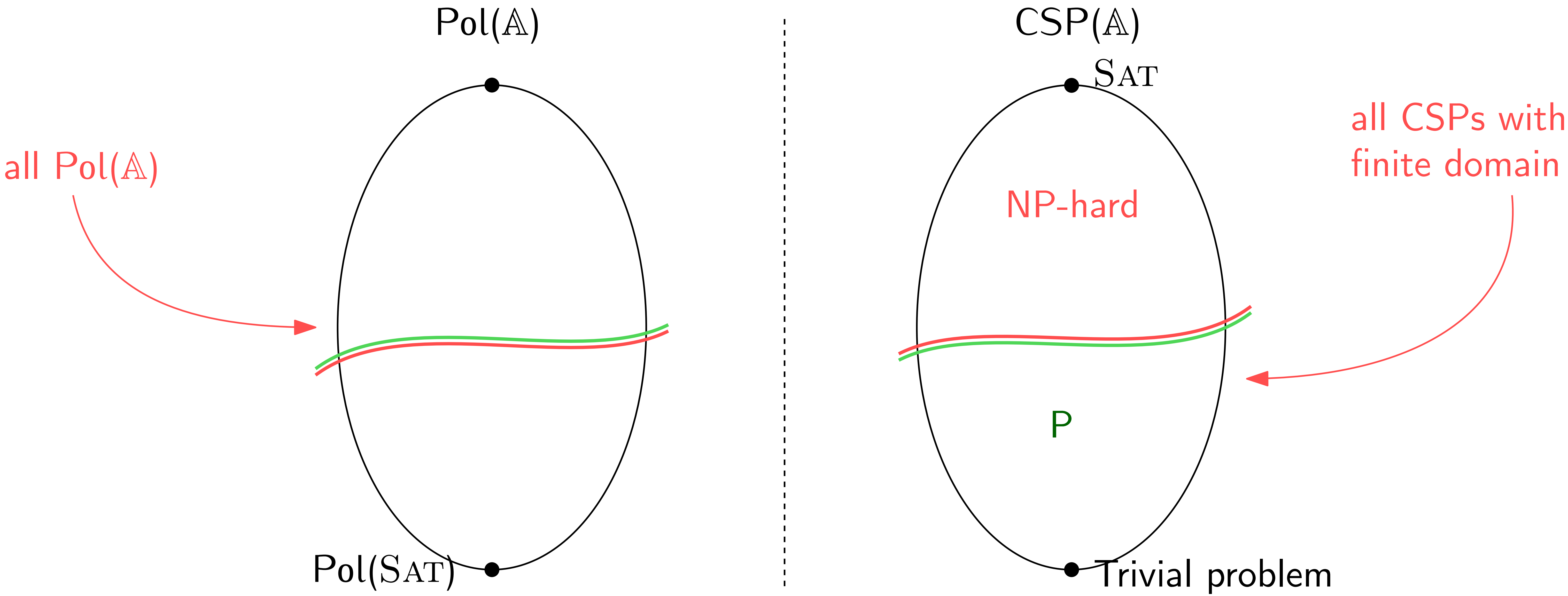

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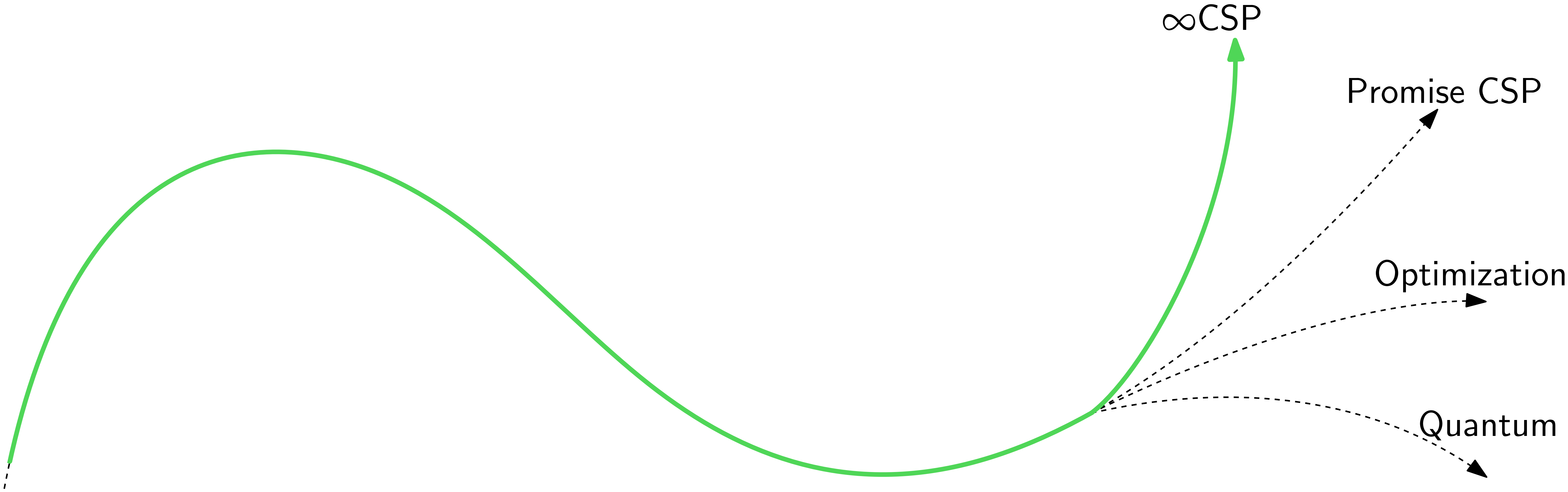
- If $\mathrm{Pol}(\mathbb{A}) \not\to \mathrm{Pol}(\textsc{Sat})$, then $\mathrm{CSP}(\mathbb{A})$ is solvable in polynomial time (Bulatov // Zhuk)
- Can the algebraic approach be leveraged for other computational problems?

Research in infinite-domain CSPs



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Theorem (Bodirsky-Martin-**M.**, *J. ACM*'18)**.** Let $\mathbb{A}$ be a temporal template.
Then CSP($\mathbb{A}$) is in P or NP-complete. Given $\mathbb{A}$, it is decidable which case applies.

Definition. Fix a set $\mathcal{F}$ of vertex-colored graphs. The vertex partitioning problem for $\mathcal{F}$ is:
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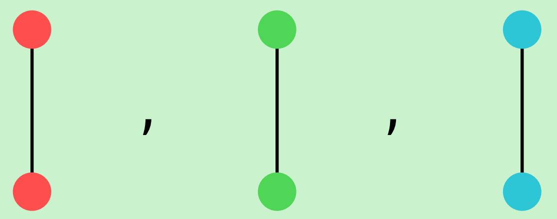
Examples. $\mathcal{F} = \Big\{$  $\Big\}$

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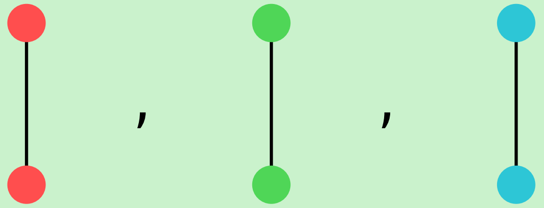
Examples. $\mathcal{F} = \Big\{$  $\Big\}$ $\rightsquigarrow$ 3-coloring problem

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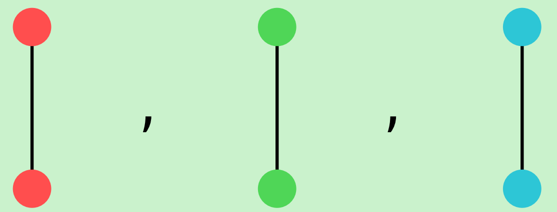
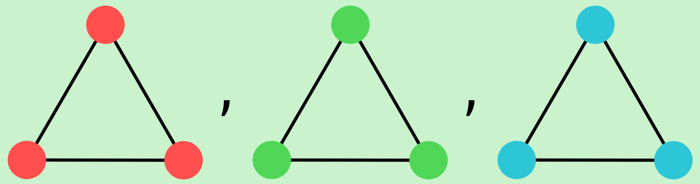
Examples. $\mathcal{F} = \{$  $\}$ $\rightsquigarrow$ 3-coloring problem

$\mathcal{F} = \{$  $\}$

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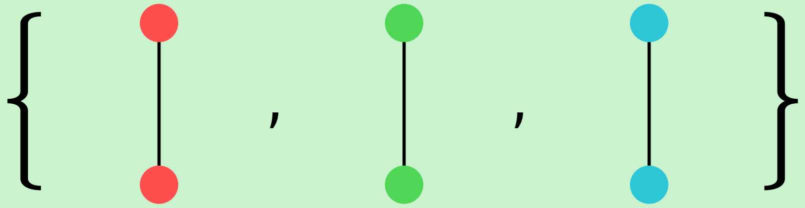
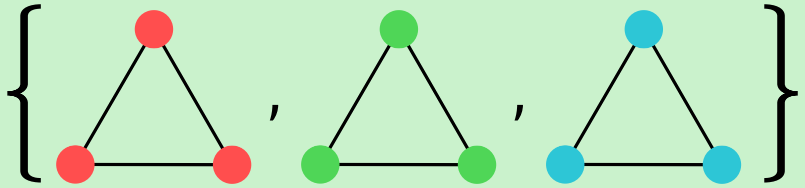
Examples. $\mathcal{F} =$  $\rightsquigarrow$ 3-coloring problem

$\mathcal{F} =$  $\rightsquigarrow$ no-monochromatic-triangle problem

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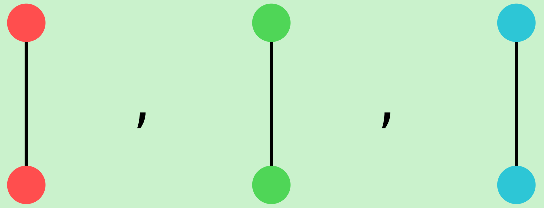
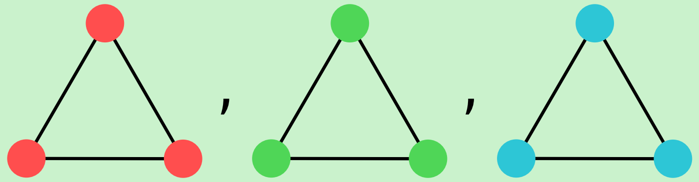
Examples. $\mathcal{F} = \{$  $\}$ $\rightsquigarrow$ 3-coloring problem

$\mathcal{F} = \{$  $\}$ $\rightsquigarrow$ no-monochromatic-triangle problem

Observation. There is $\mathbb{A}$ such that $\mathrm{CSP}(\mathbb{A})$ is the no-mono-triangle problem, but no finite $\mathbb{A}$.

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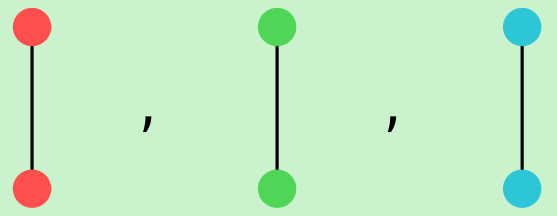
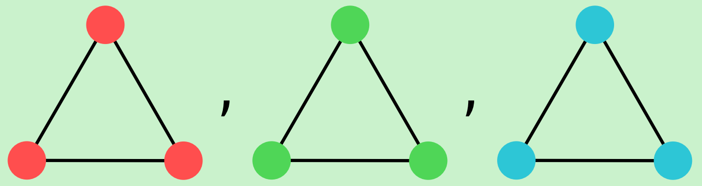
Examples. $\mathcal{F} = \{$  $\}$ $\rightsquigarrow$ 3-coloring problem

$\mathcal{F} = \{$  $\}$ $\rightsquigarrow$ no-monochromatic-triangle problem

Observation. There is $\mathbb{A}$ such that CSP($\mathbb{A}$) is the no-mono-triangle problem, but no finite $\mathbb{A}$.

Theorem (Bodirsky-Madelaine-**M.**, *SIAM J. Comp.* '21)**.** Every vertex partitioning problem is in P or NP-complete. Given $\mathcal{F}$, it is decidable which case applies.

The proof uses techniques from infinite-domain constraint satisfaction.

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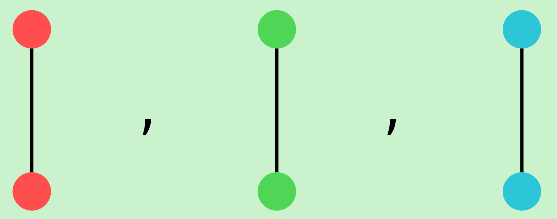
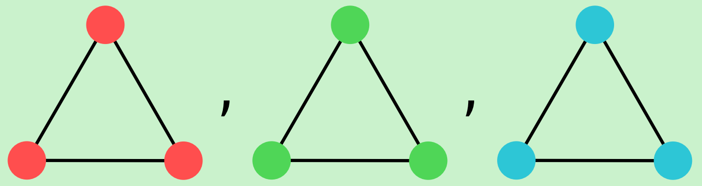
Examples. $\mathcal{F} = \{$  $\}$ $\rightsquigarrow$ 3-coloring problem

$\mathcal{F} = \{$  $\}$ $\rightsquigarrow$ no-monochromatic-triangle problem

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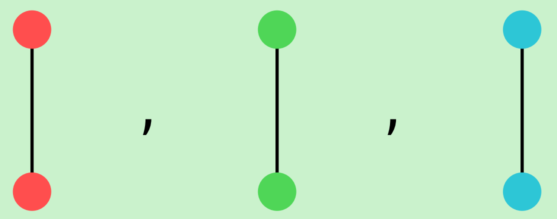
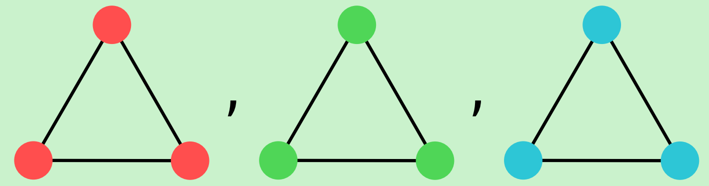
Examples. $\mathcal{F} = \{$  $\}$ $\rightsquigarrow$ 3-coloring problem

$\mathcal{F} = \{$  $\}$ $\rightsquigarrow$ no-monochromatic-triangle problem

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Project (with D. Perinti)**.** Understand the complexity of edge partitioning problems.

Algebraic approach works whenever $\mathrm{CSP}(\mathbb{A})$ has combinatorial certificates.

Example (1-in-3 Satisfiability). Given a set of constraints $(x, y, z), (y, z, t), \ldots,$ decide if there is a map $h\colon V \to \{0, 1\}$ such that for every constraint (a, b, c), exactly one of $h(a), h(b), h(c)$ is 1.

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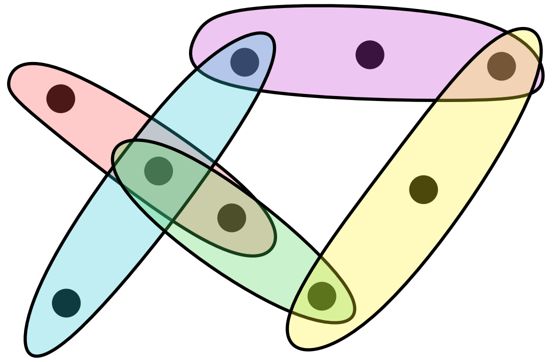

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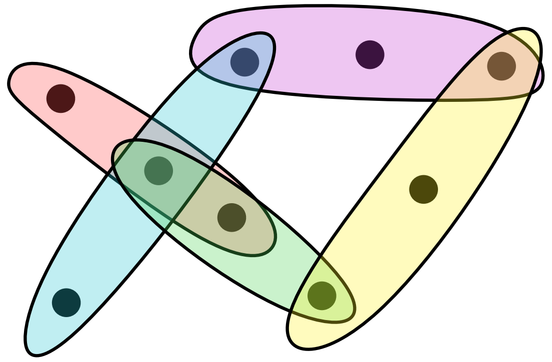

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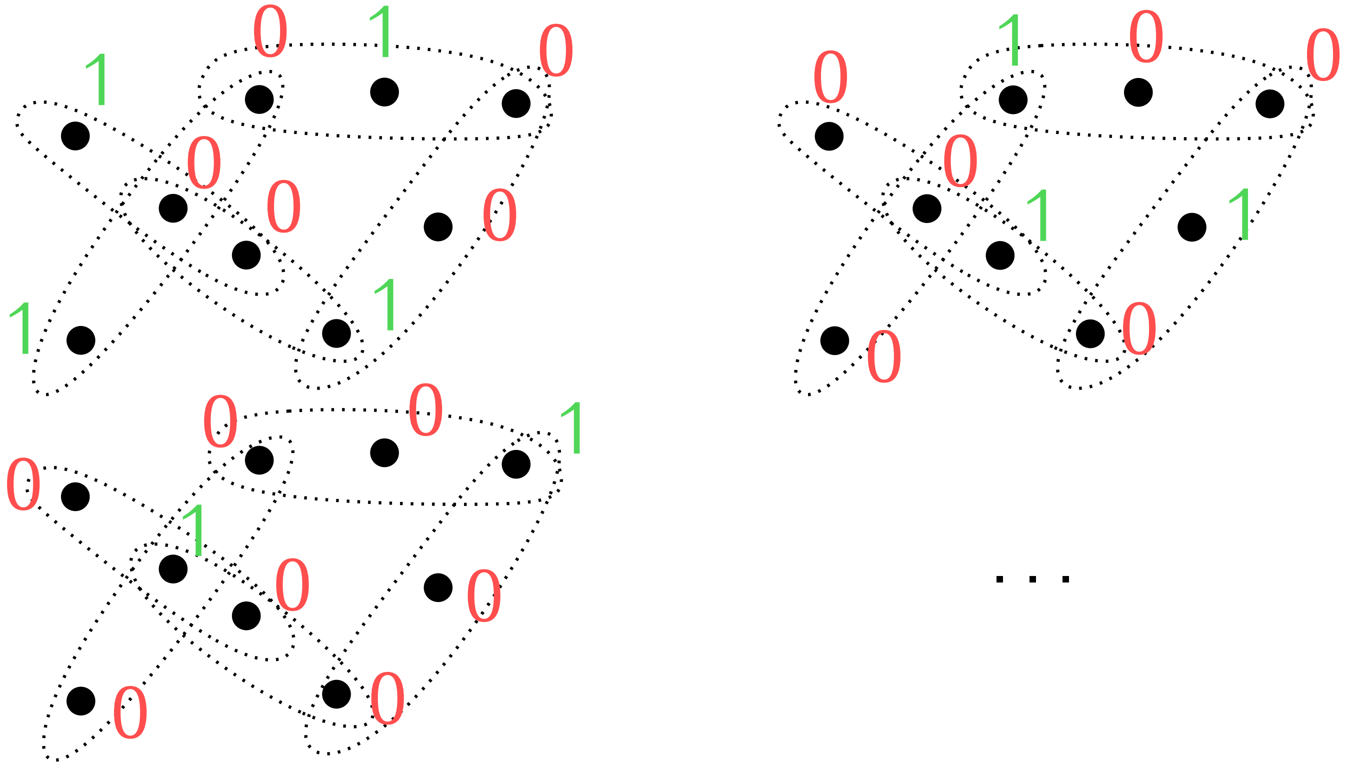

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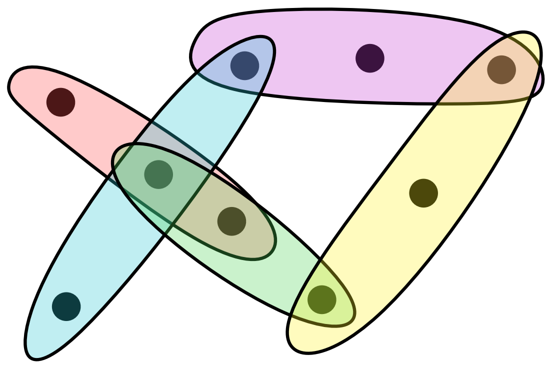

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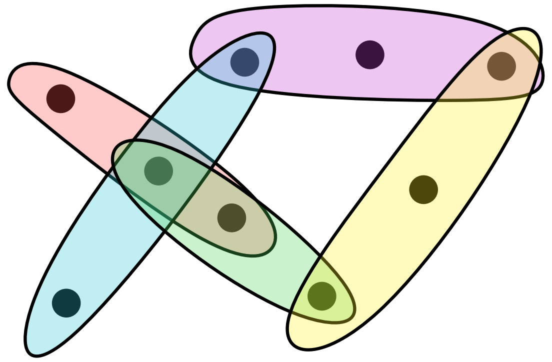

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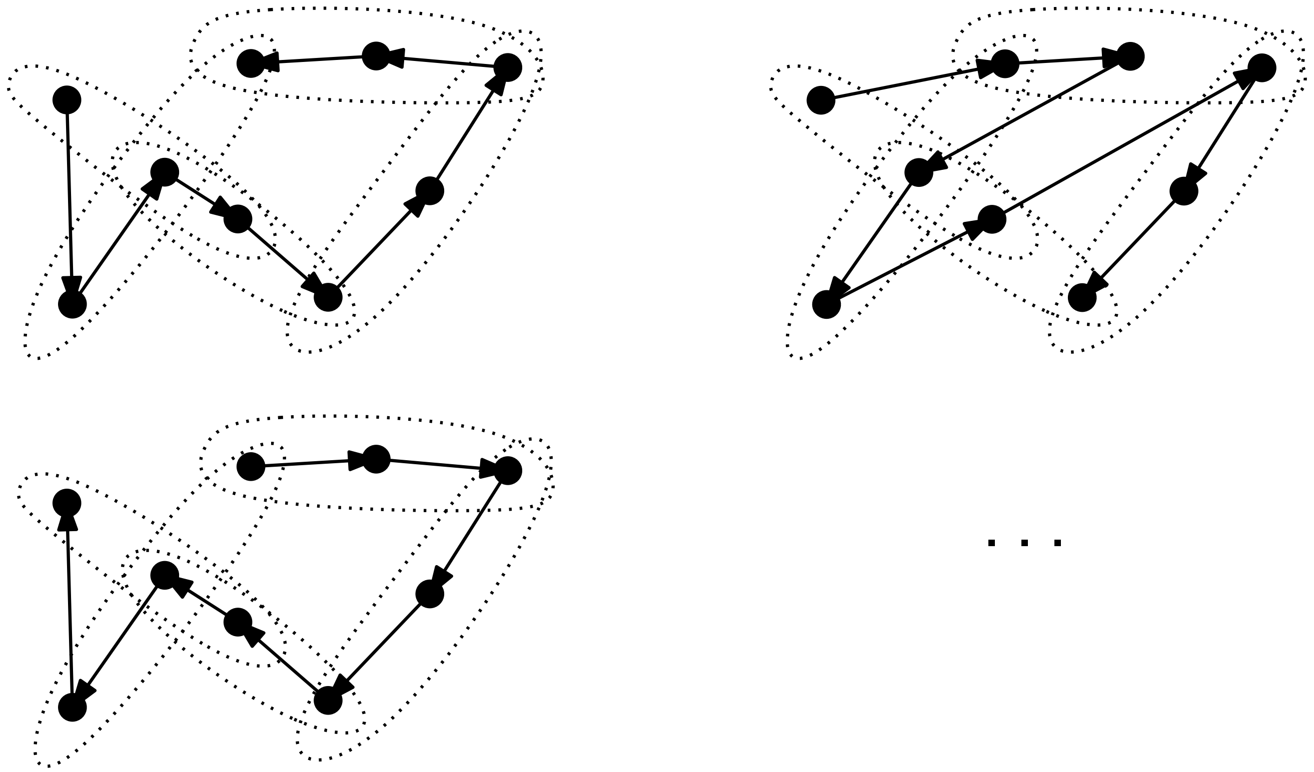

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Conjecture (2011). Let $\mathbb{A}$ be s.t. $\mathrm{CSP}(\mathbb{A})$ has combinatorial certificates and $\mathrm{Pol}(\mathbb{A}) \not\to \mathrm{Pol}(\mathrm{S}\small{\mathrm{AT}})$. Then $\mathrm{CSP}(\mathbb{A})$ is solvable in polynomial time.

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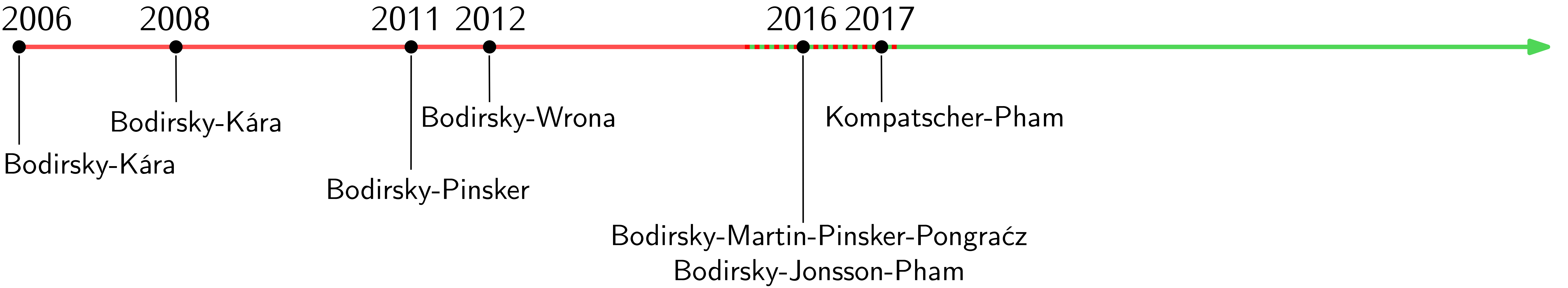

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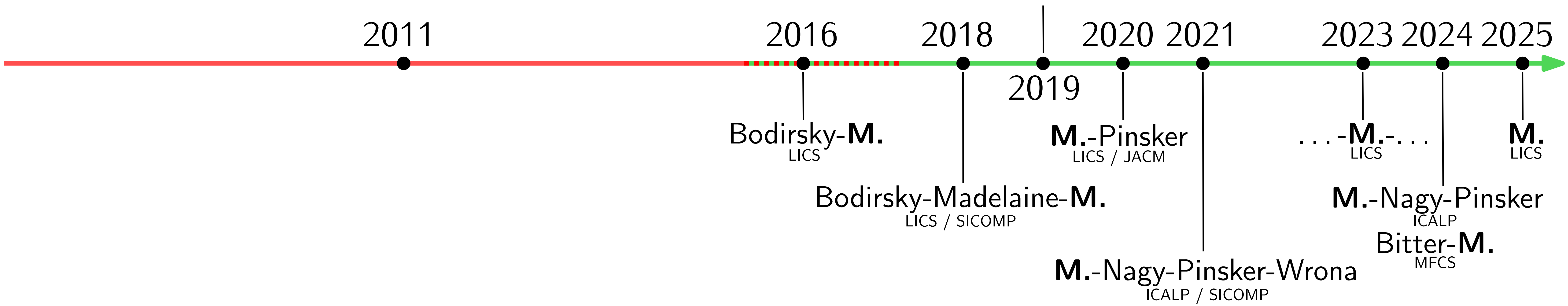

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Modern proofs:

- Unifying algebraic theory
- No combinatorial explosion
- Generic reduction to finite-domain CSPs
- Generic uniform algorithms
- Finer understanding than P vs. NP

Theorem (Bodirsky-**M.** LICS'16)**.** For every $\mathbb{A}$ with combinatorial certificates, there exists an equivalence relation $\equiv$ such that if $\mathrm{Pol}(\mathbb{A}, \equiv) \not\to \mathrm{Pol}(\textsc{Sat})$, then $\mathrm{CSP}(\mathbb{A})$ is solvable in polynomial time.

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\cup\!\mid & & \\
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2020–2025: algebraic theory to handle this situation in generality

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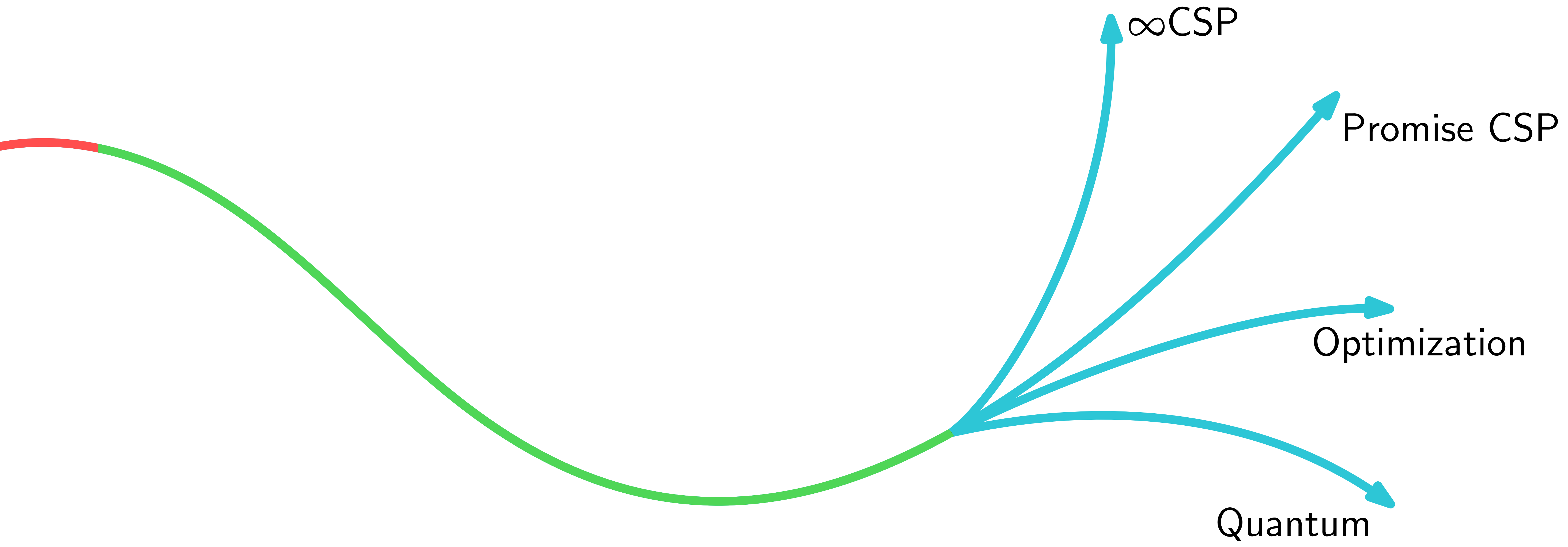
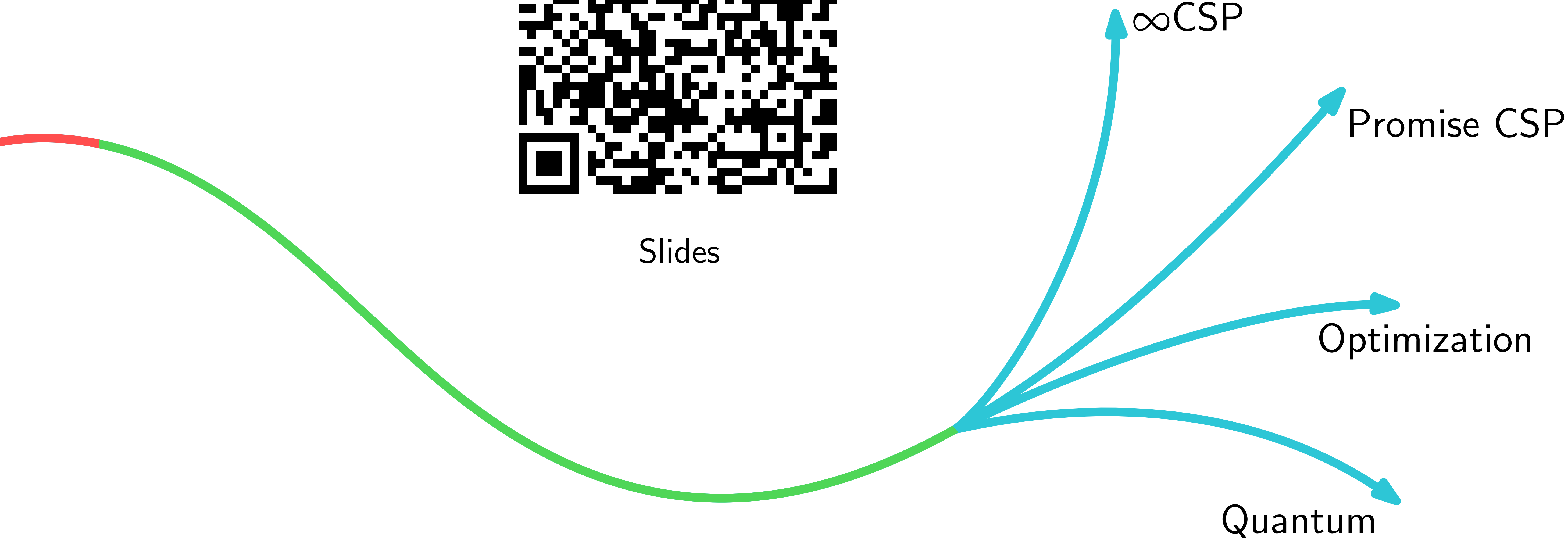
2020–2025: algebraic theory to handle this situation in generality

Invited tutorial from 2025 Dagstuhl seminar:

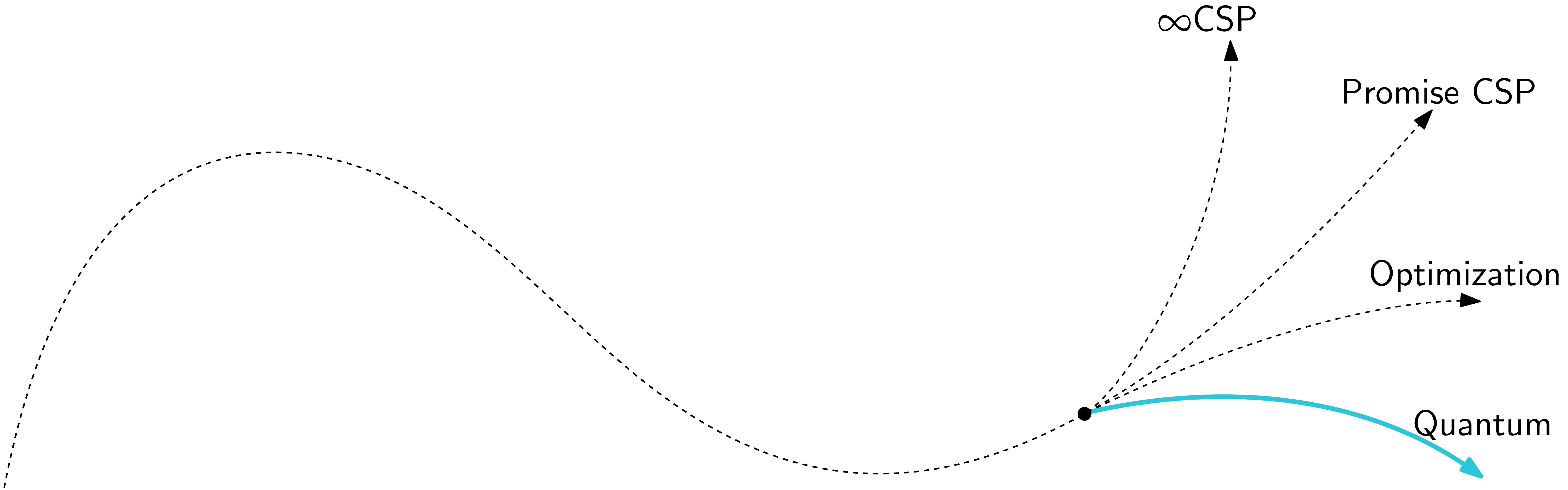

∞CSP
Promise CSP
Optimization
Quantum

Thank you for your attention!

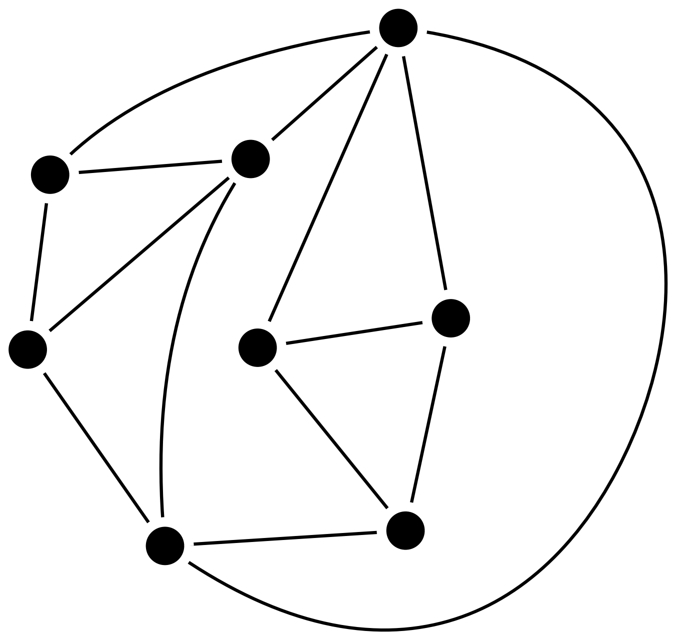
Slides



Research in Quantum Constraint Satisfaction



Verifier

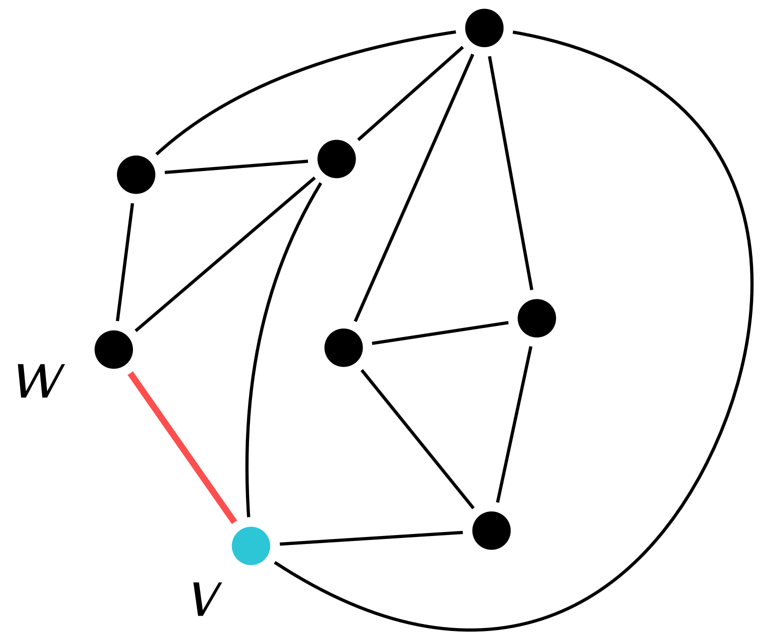


Alice

Bob

Verifier

1. selects $e = (v, w)$ following distribution π



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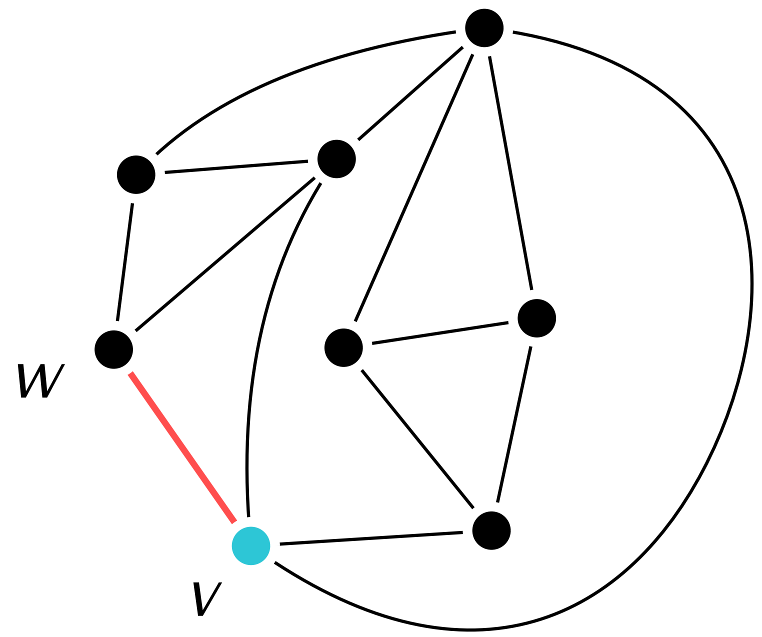
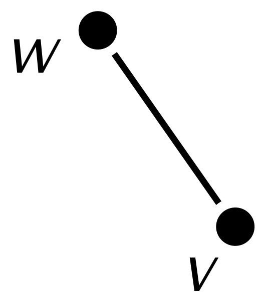
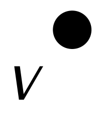


Alice

4. Answers $(a, b) \in [k]^2$ with $a \neq b$ following distribution p_e



Bob



4. Answers $c \in [k]$ following distribution p_v

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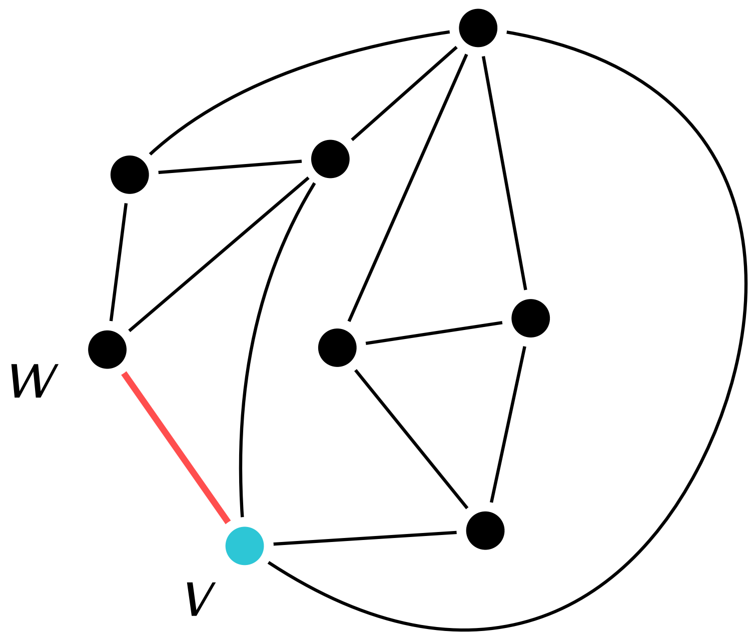

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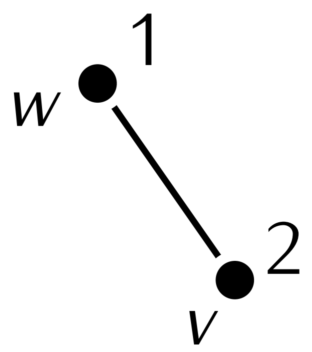

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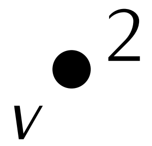

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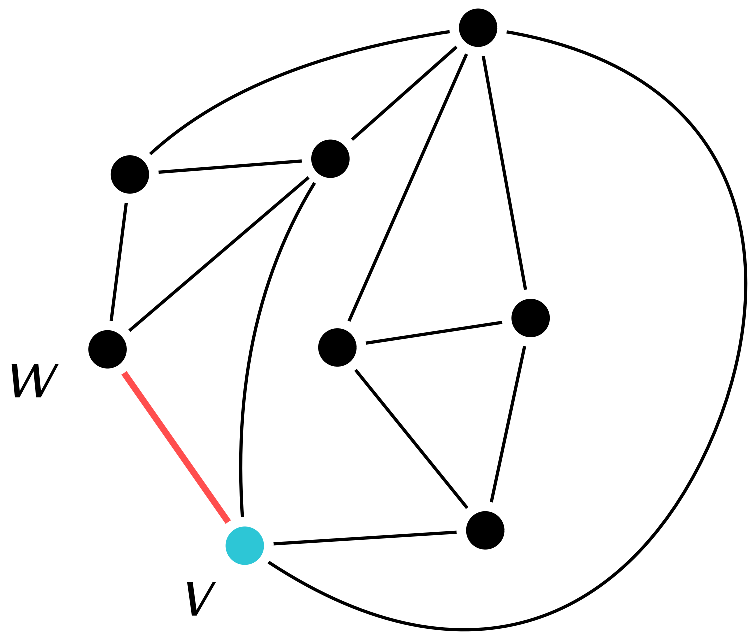

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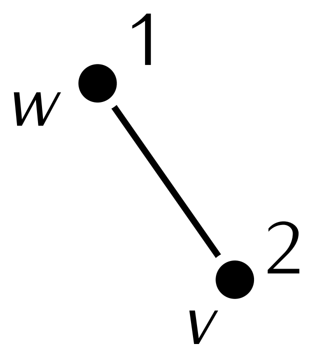

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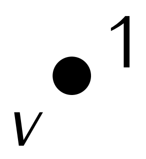

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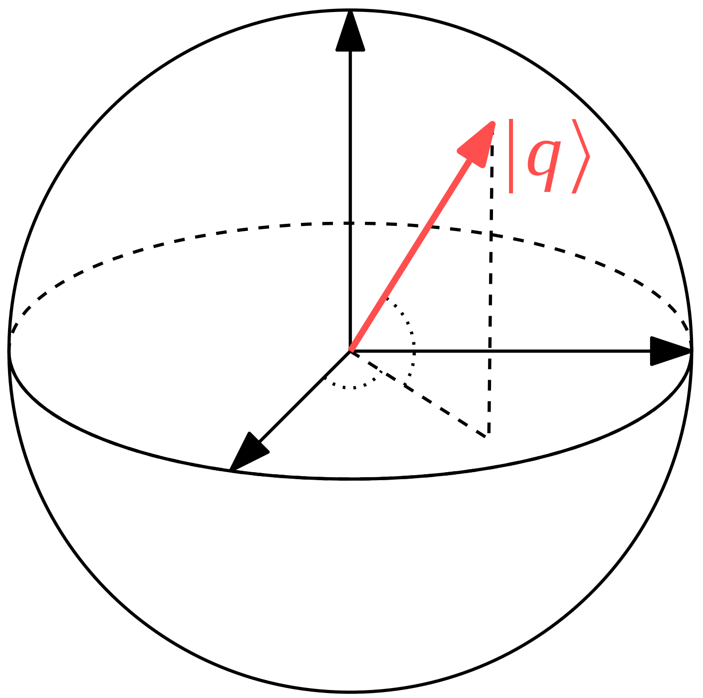

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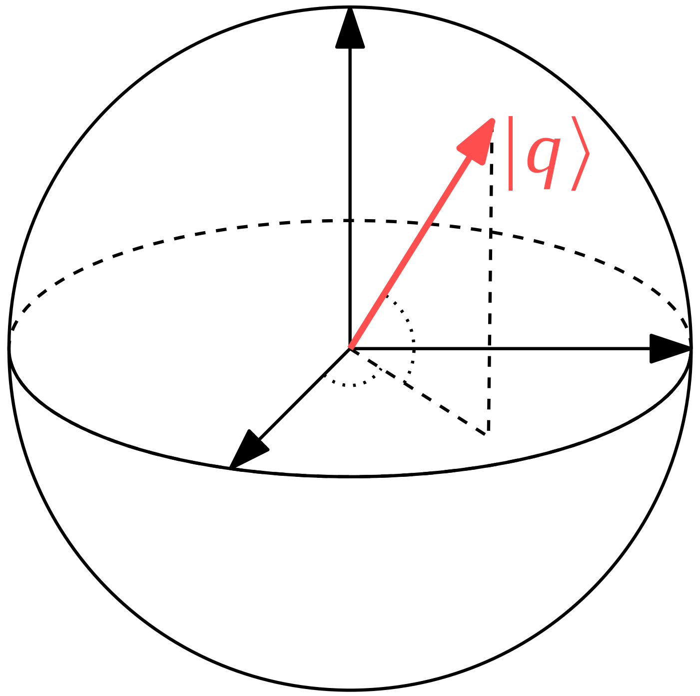

How Alice performs a measurement:

- Choose projectors $Q_{a,b}$ summing to id
- Output of measurement is (a,b) with probability $\langle q|Q_{a,b}|q\rangle$

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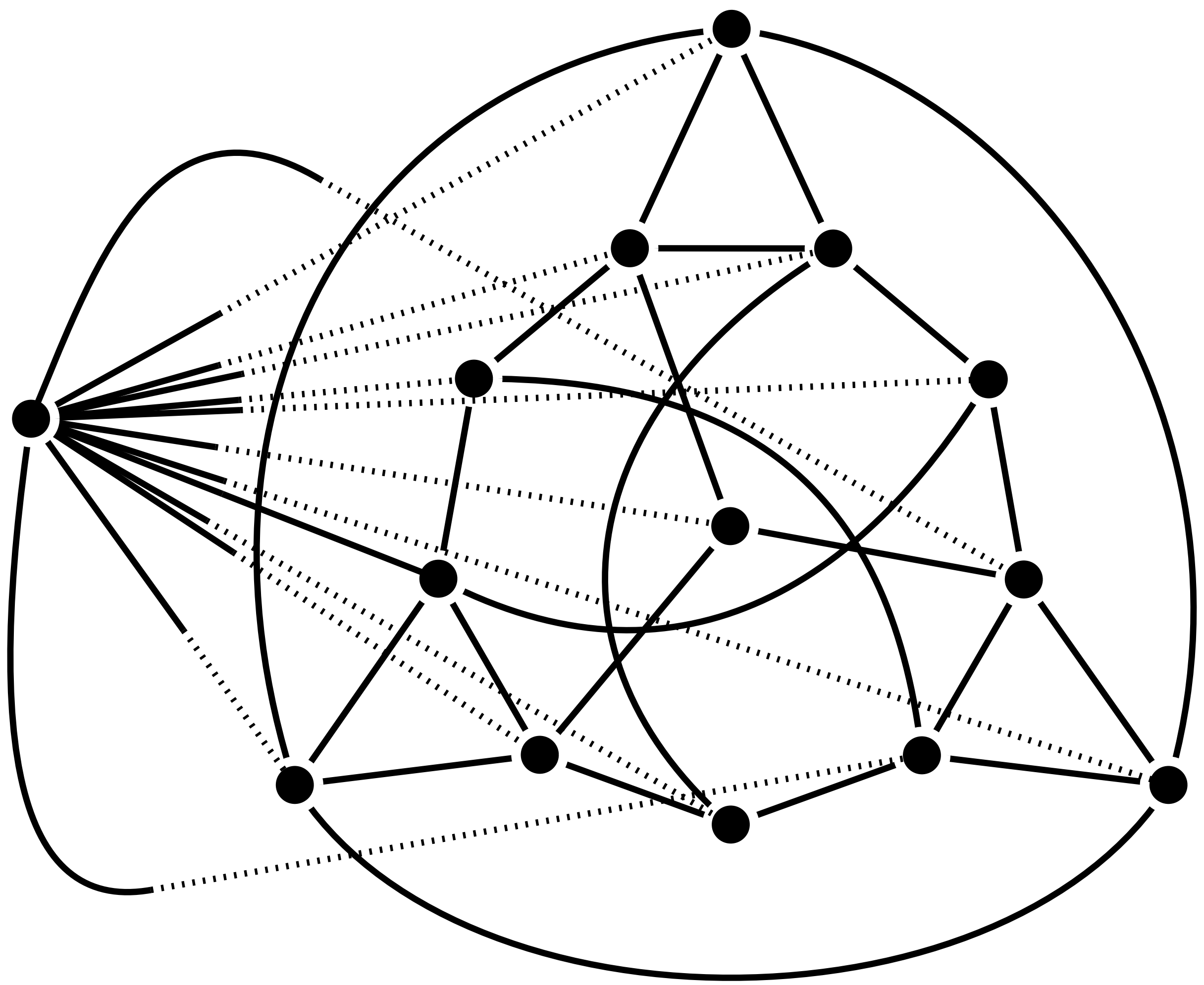

Quantum 4-colorable graph with chromatic number 5
[Mančinska-Roberson '18]

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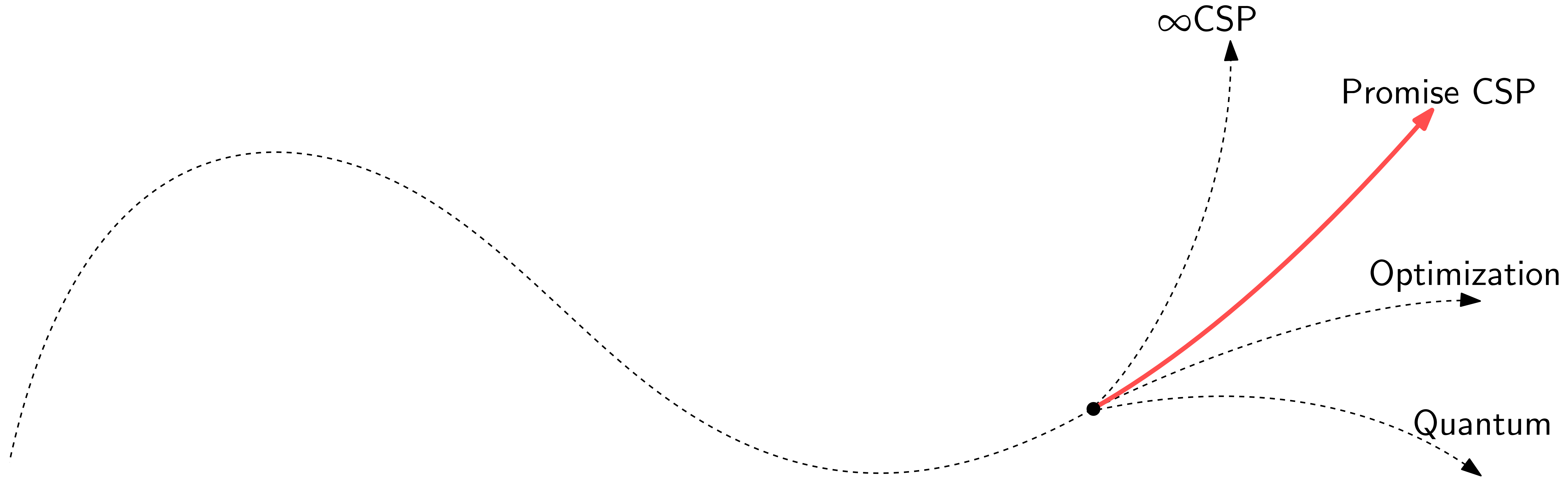
Project (with L. Ciardo and G. Joubert)**.** Understand the complexity landscape for entangled CSPs.

Research in Promise Constraint Satisfaction



Solving CSPs under structural assumptions on the instances:

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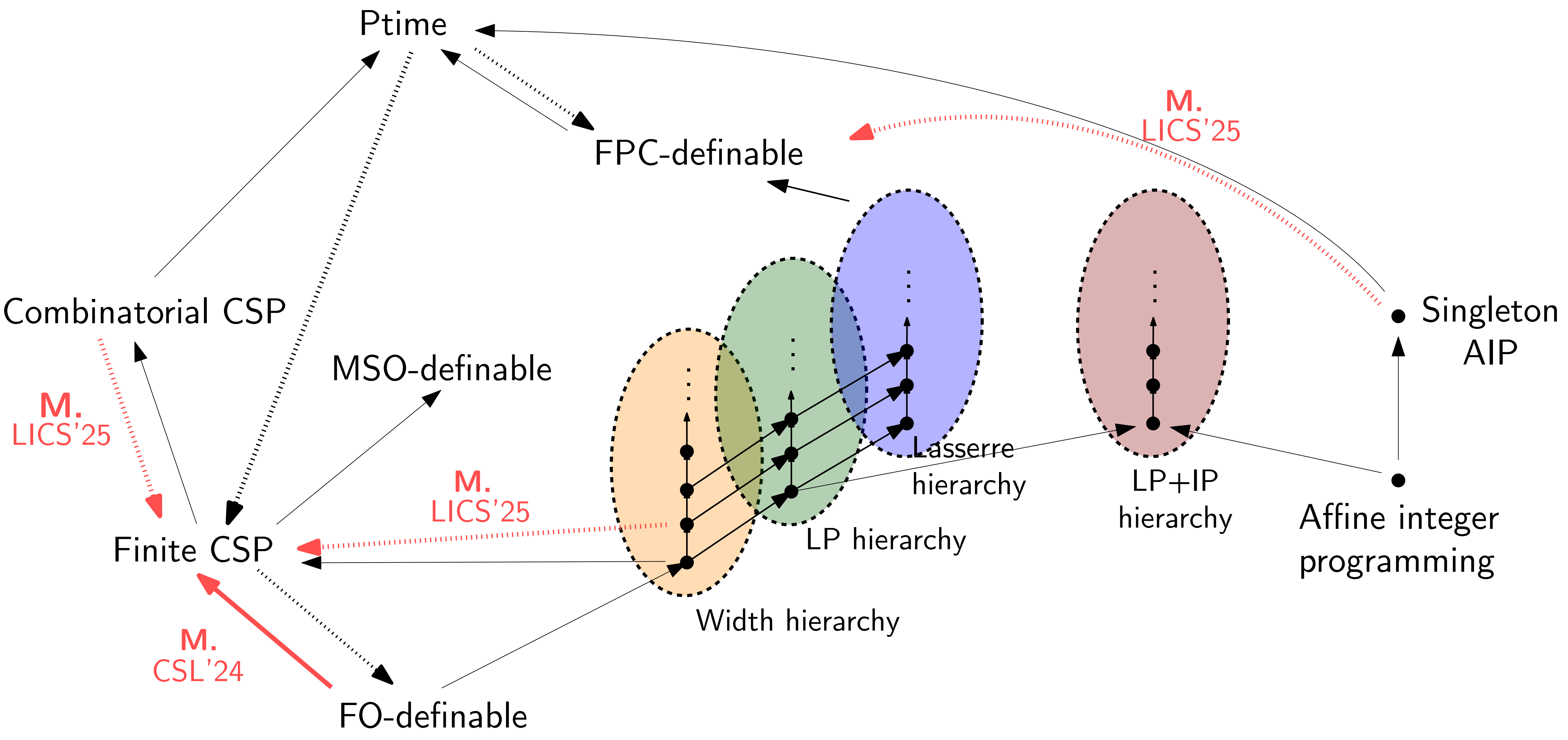
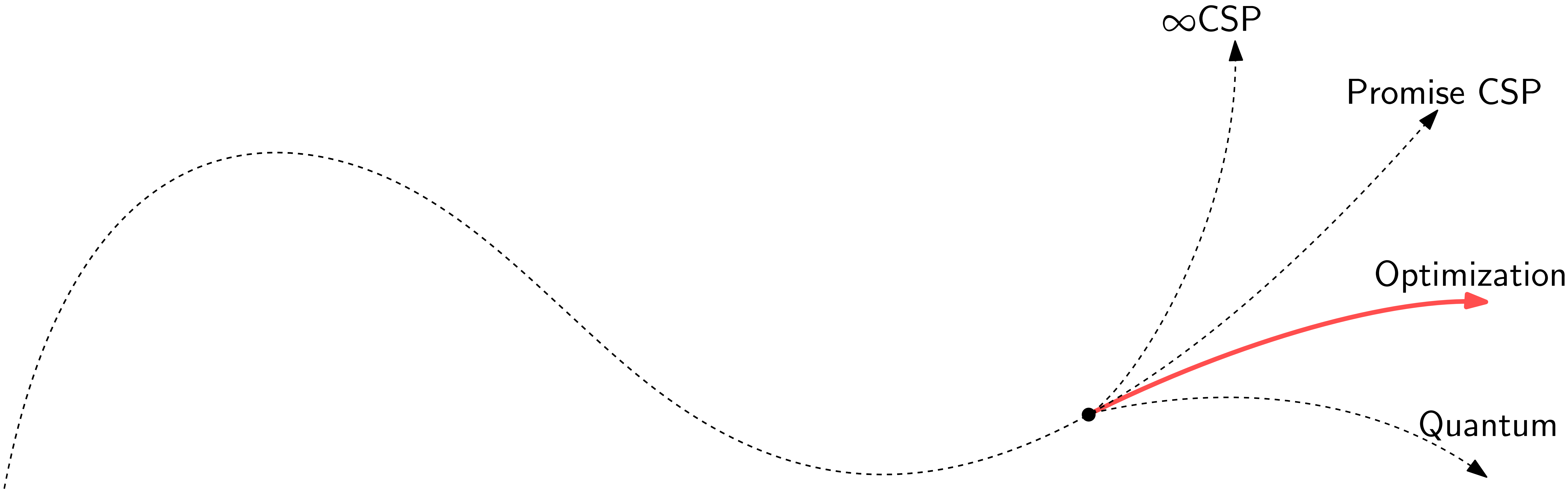
- Massive framework for the study of approximation problems
- Existence of a suitable generalization to the algebraic approach
- Surprising connection betweeen finite-domain PCSPs and combinatorial CSPs:

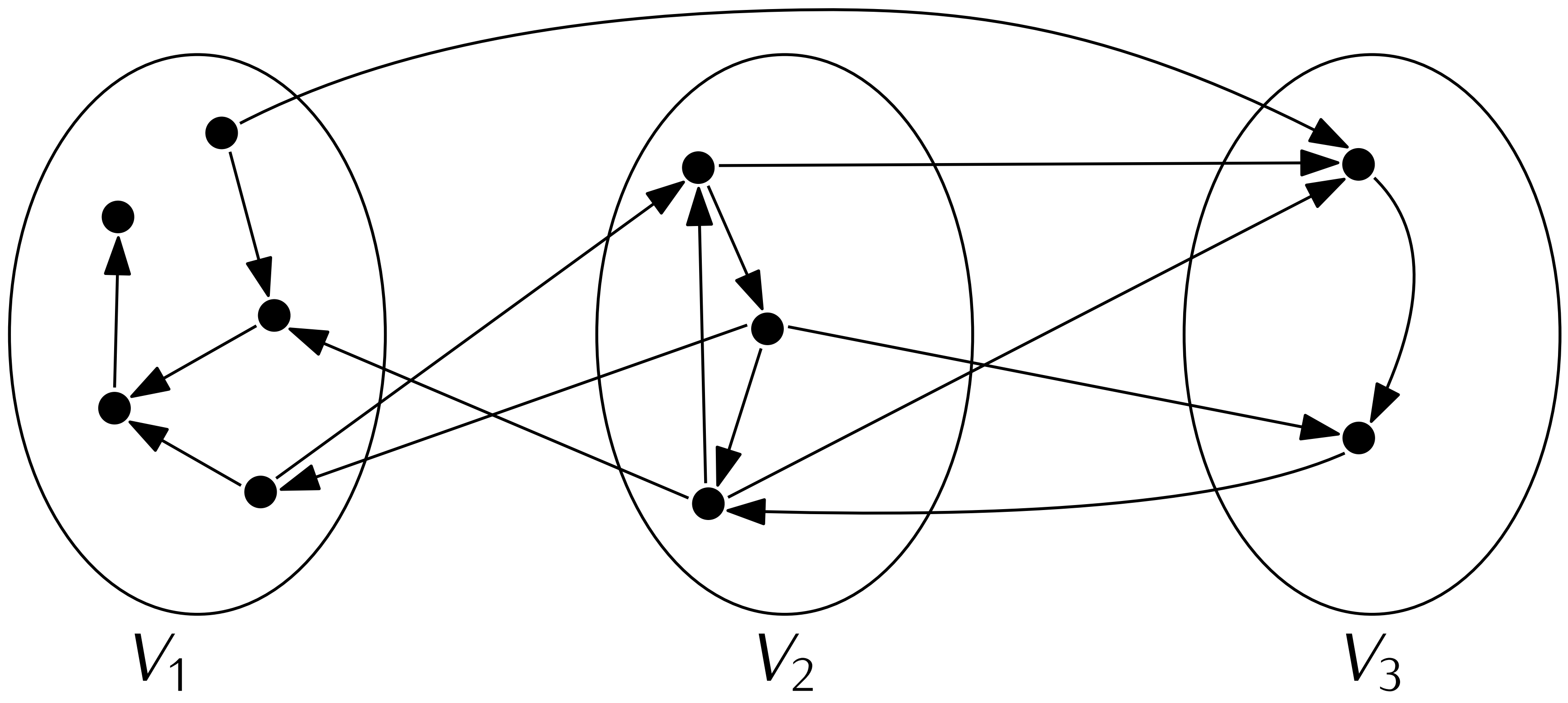
Theorem (**M.** LICS'25)**.** The following hold:
- Combinatorial CSP $\simeq$ $\mathrm{PCSP}(\text{combinatorial}, \text{finite})$
- There are finite PCSPs whose tractability can be shown by combinatorial CSPs and not by finite CSPs.
- There exists a uniform algorithm for temporal CSPs obtained by a reduction to finite PCSPs.


Ptime
FPC-definable
M.
LICS'25
Combinatorial CSP
MSO-definable
M.
LICS'25
M.
LICS'25
Singleton
AIP
Lasserre
hierarchy
LP+IP
hierarchy
Finite CSP
LP hierarchy
Affine integer
programming
Width hierarchy
M.
CSL'24
FO-definable

Research in Optimization



k-layering of a digraph: map $\ell\colon V \to \{1, \ldots, k\}$



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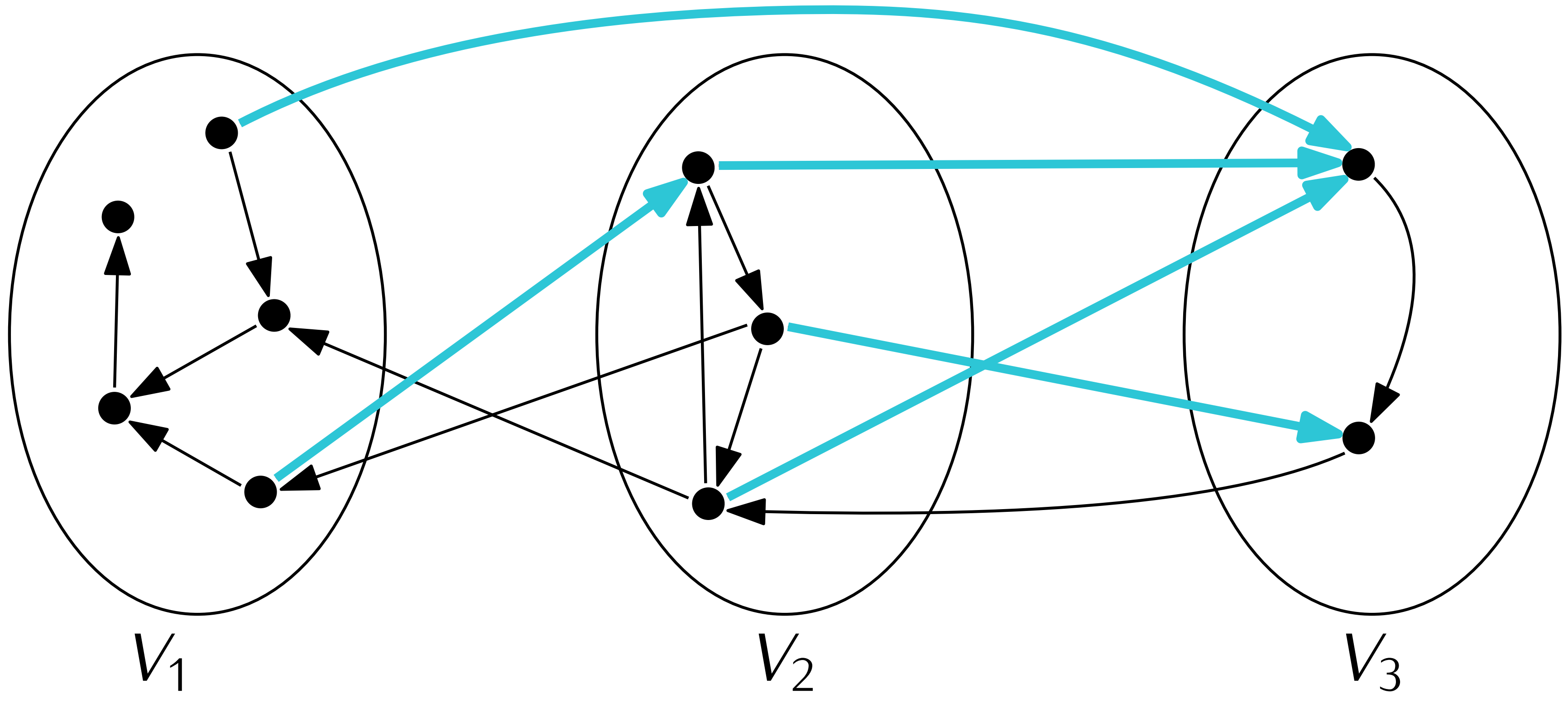

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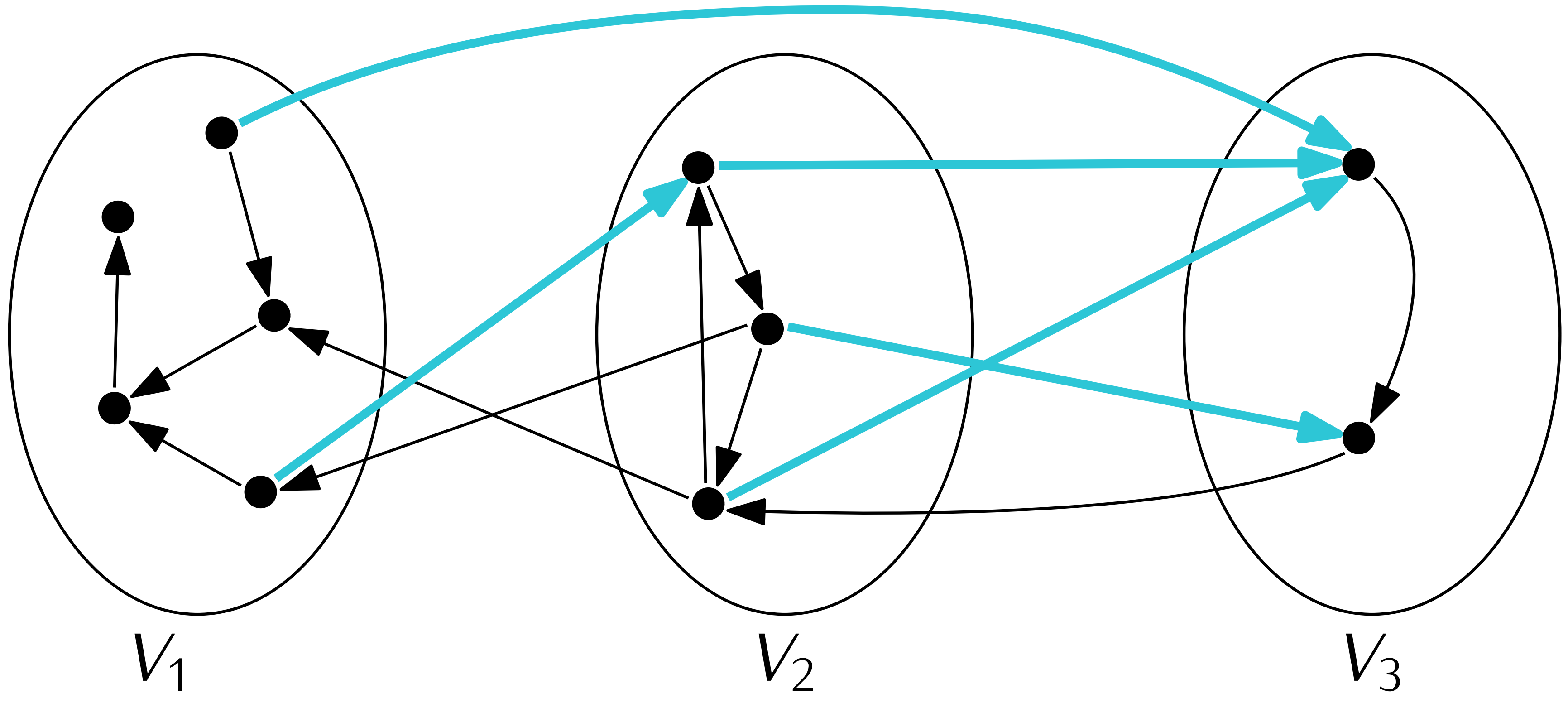
Note: every layering with value ρ is an acyclic subgraph with ρ edges



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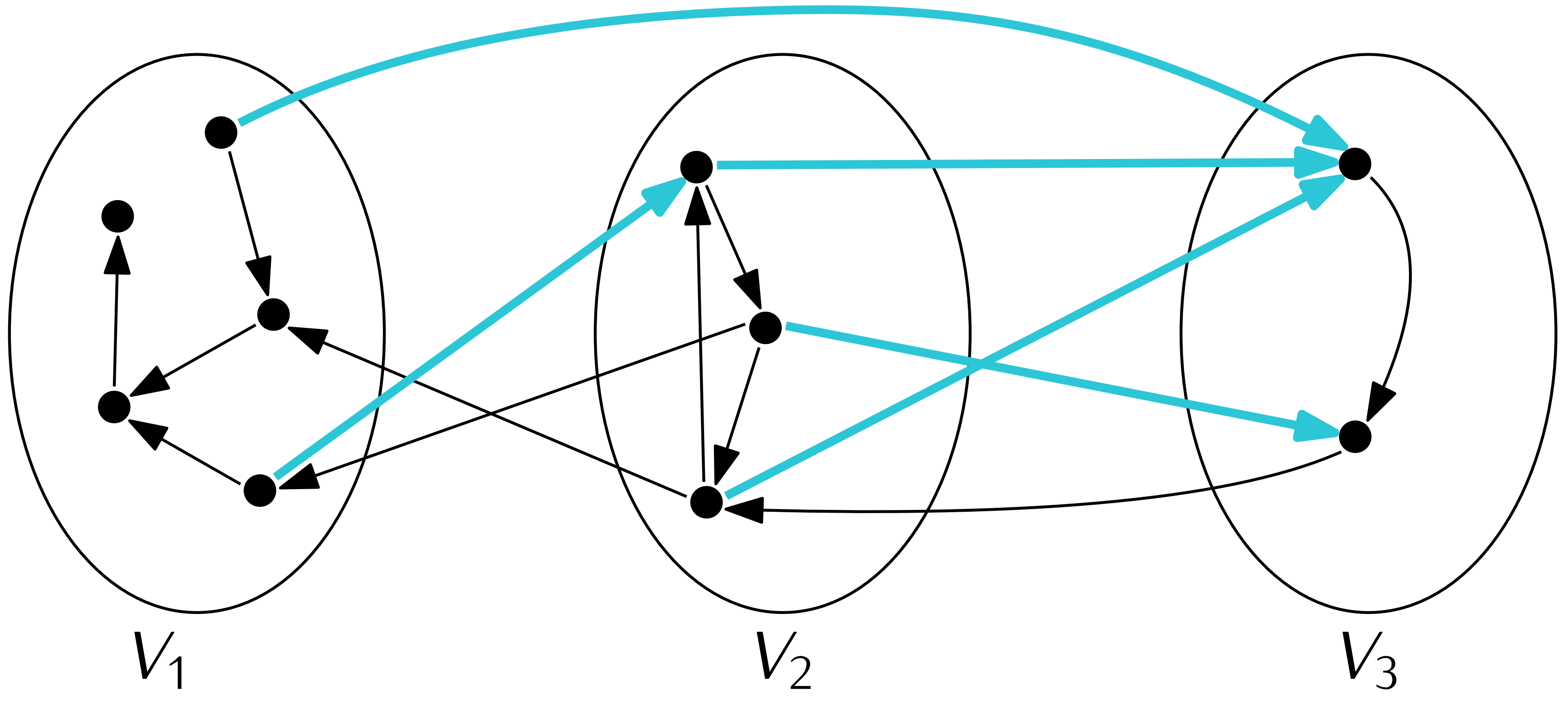
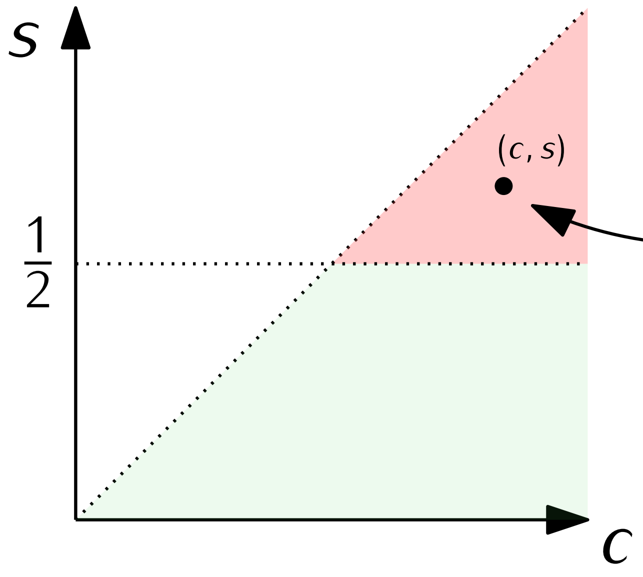


Approximation diagram for Maximum Acyclic Subgraph



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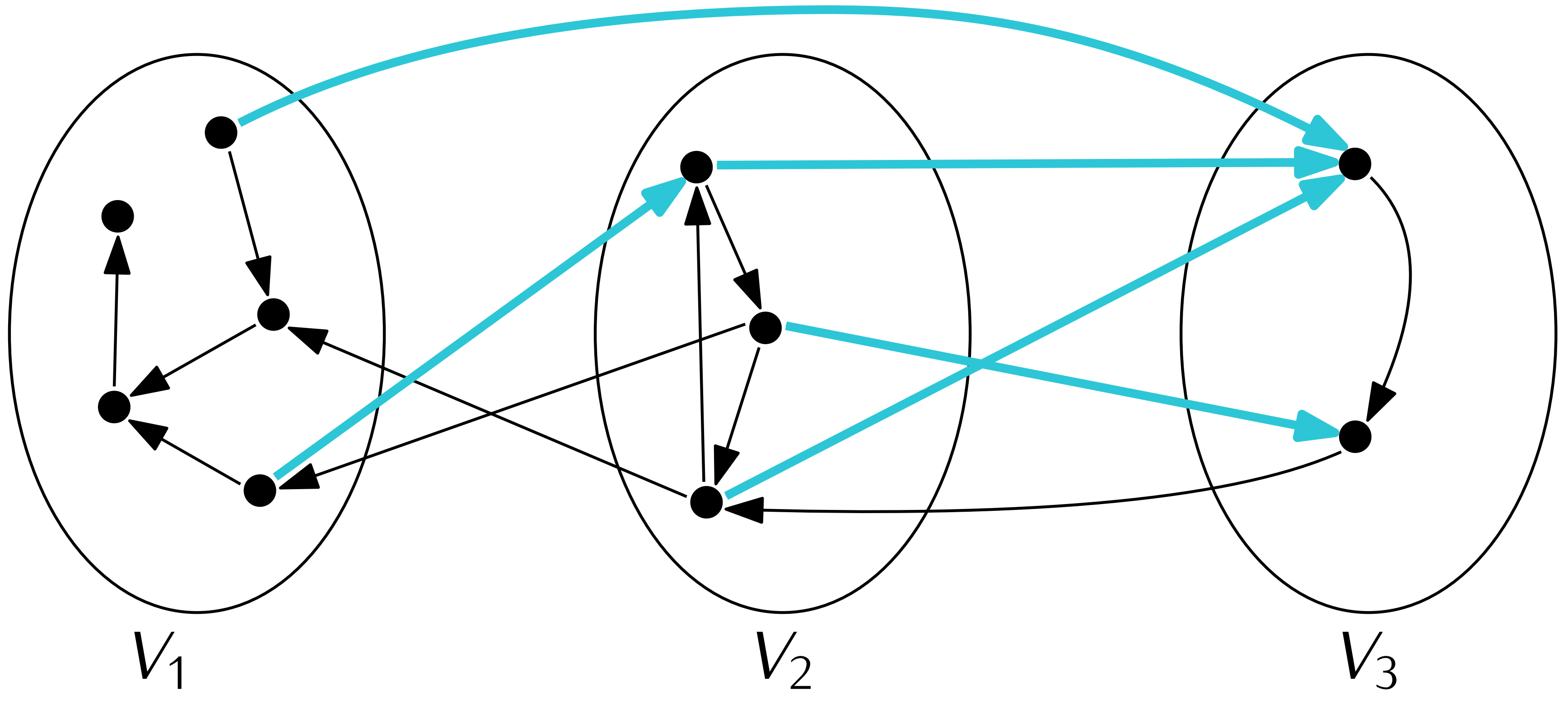

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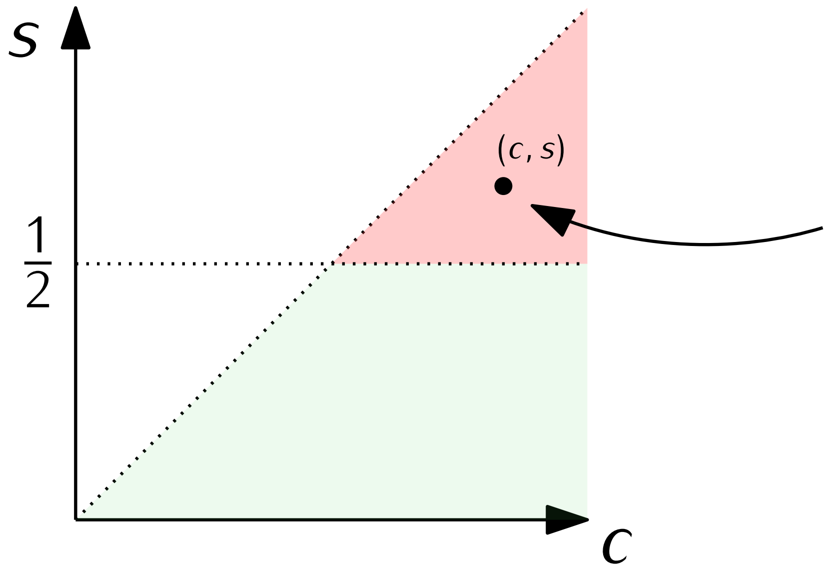

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$\mathrm{PCSP}(\mathrm{LO}_k, \mathrm{LO}_\infty)$ (maximization version): given a directed graph G admitting a k-layering of value ρ, find an acyclic subgraph of G containing at least ρ edges.

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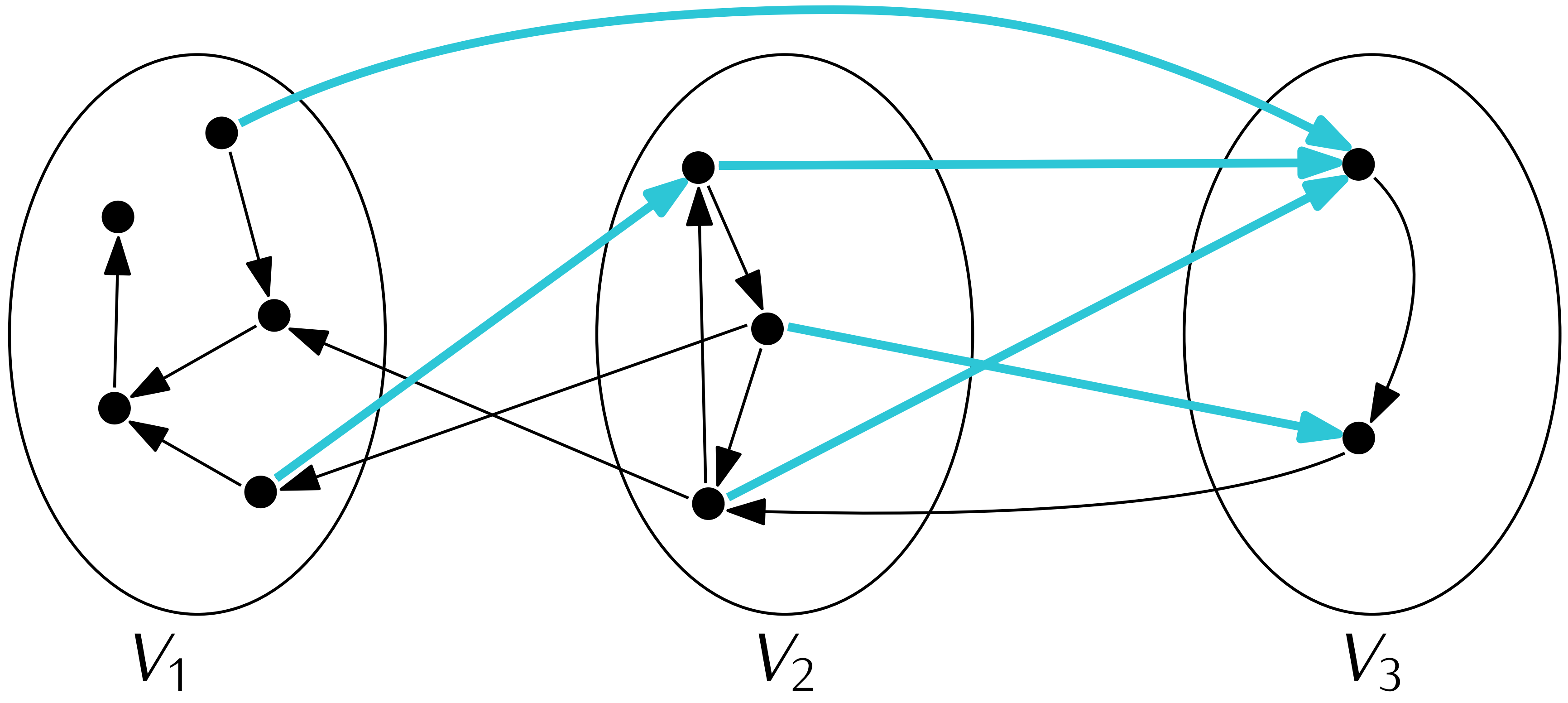

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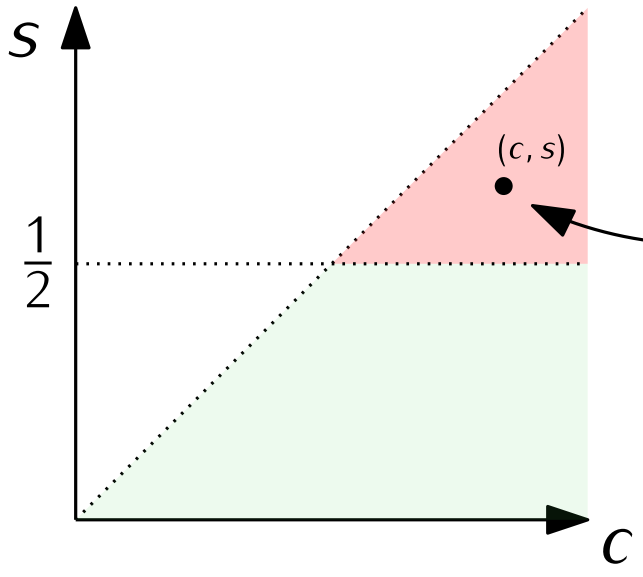

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- Solvable in polynomial time for $k = 2$ (Nakajima-Živný, **M.** '25)
- NP-hard for $k \geq 4$, but unknown tractability boundary in approximation diagram
- Open complexity for $k = 3$