MMSNP: Proof of the Universal-Algebraic Dichotomy Conjecture

Manuel Bodirsky, Florent Madelaine, **Antoine Mottet**

September 25, 2018

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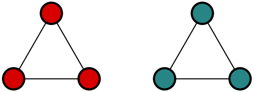

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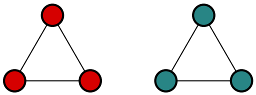

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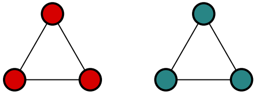

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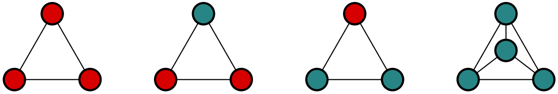

- Complexity: in P.

In general, for some fixed set $\mathcal{F}$ of vertex-coloured graphs, the problem FPP($\mathcal{F}$) is:

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MMSNP and FPP are computationally equivalent.

New proofs of known results:

- For every finite set $\mathcal{F}$ of forbidden patterns, the corresponding problem is in P or NP-complete,
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- Solve question posed by Lutz and Wolter (ICDT'15),
- Confirmation of conjecture by Bodirsky-M (LICS'16).

Outline

Definition

$\mathfrak{B} = (B; E)$ a graph.
CSP($\mathfrak{B}$) is the problem:

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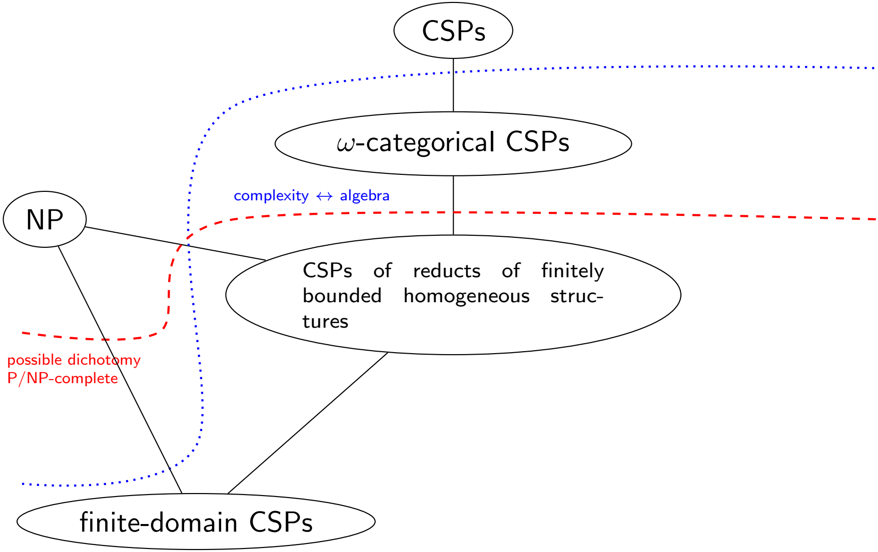
In general, the forbidden patterns problem (FPP) for $\mathcal{F}$ is not a CSP, but a finite union of CSPs.

Proposition

Every FPP reduces in polynomial-time to a finite number of FPP of connected structures.

Context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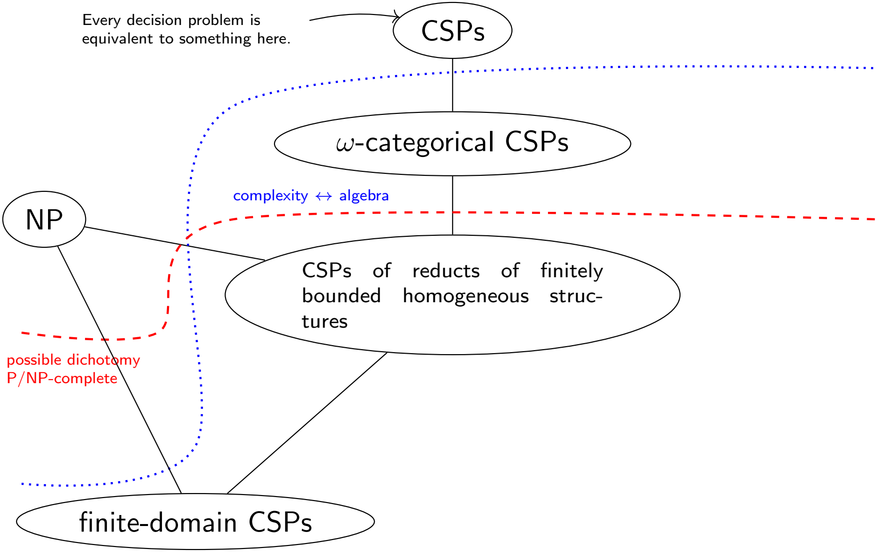

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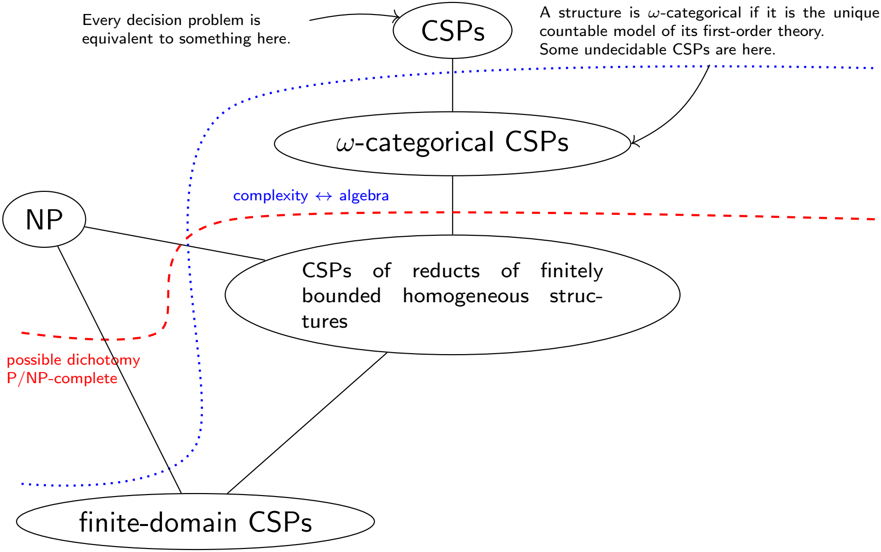
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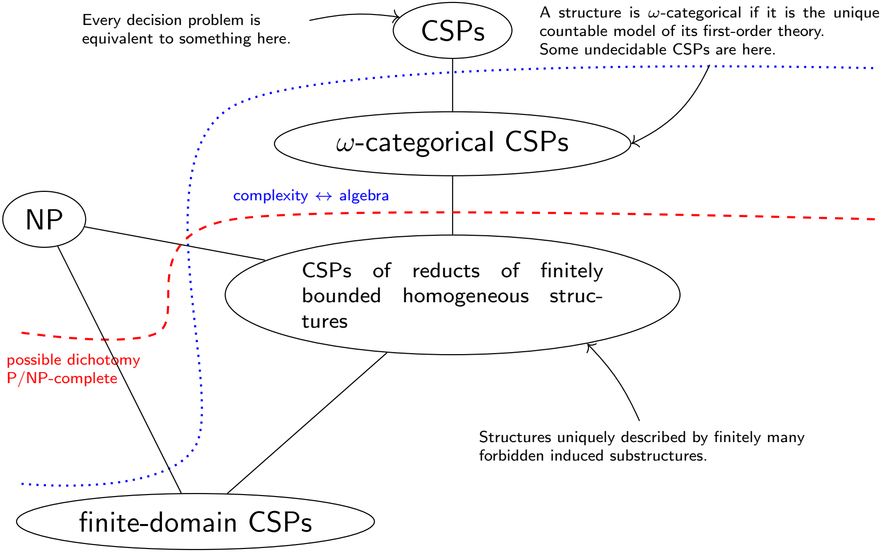
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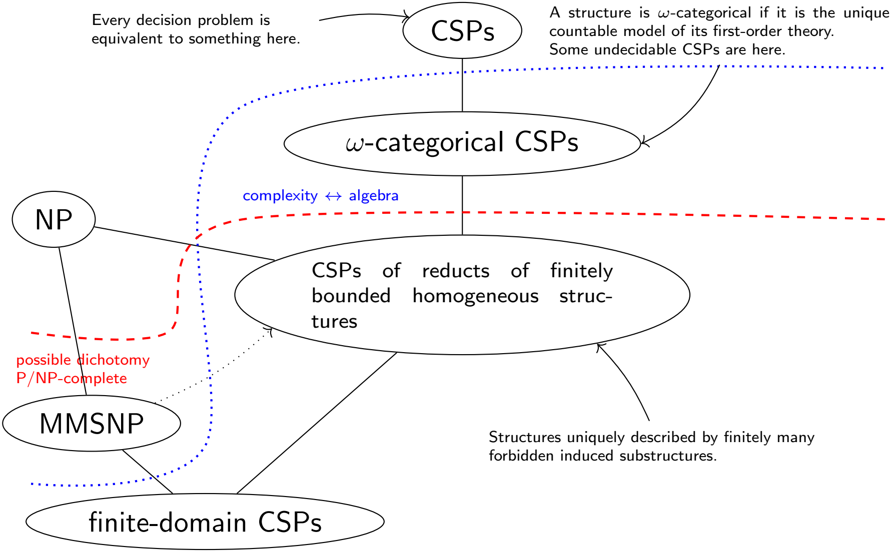
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MMSNP is essentially a subset of CSP

Theorem (Cherlin-Shelah-Shi, Adv. Appl. Math. 1999)

For every finite set $\mathcal{F}$ of finite connected coloured graphs, there exists an ω-categorical partially coloured graph $\mathfrak{B}^*$ such that $\mathfrak{A}^* \to \mathfrak{B}^*$ iff $\mathfrak{A}^*$ avoids $\mathcal{F}$.

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For this talk: we call $\mathfrak{B}$ an MMSNP structure.

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$$\Downarrow$$

CSPs in the BP class

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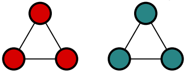

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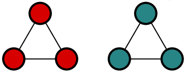

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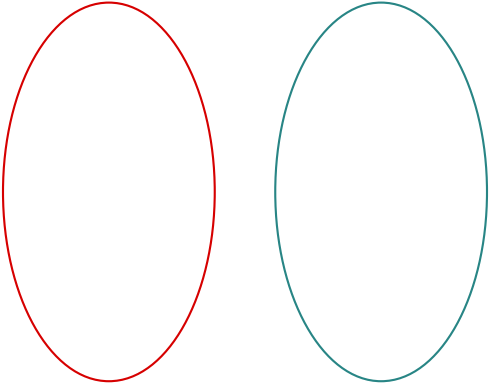

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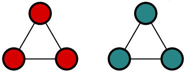

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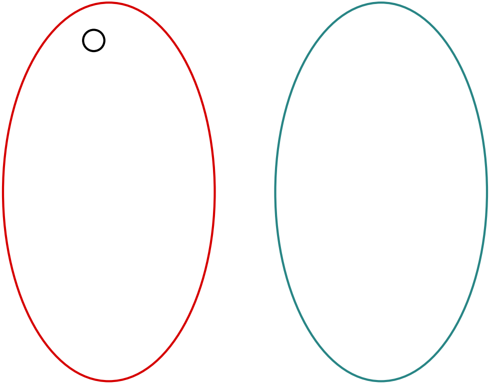

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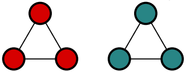

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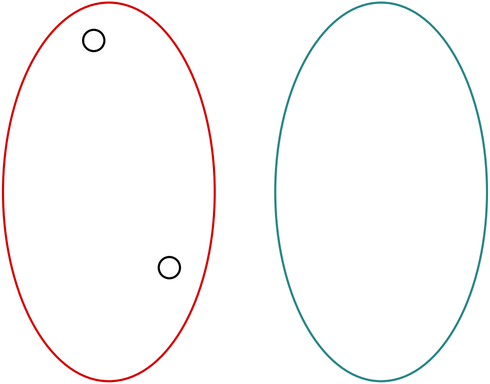

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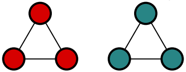

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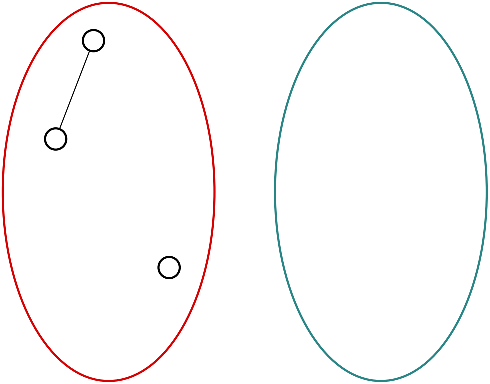

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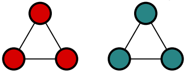

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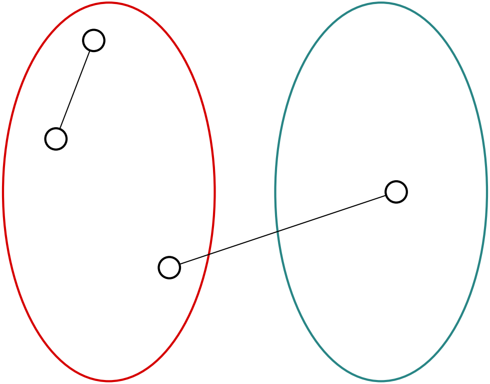

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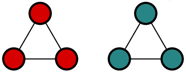

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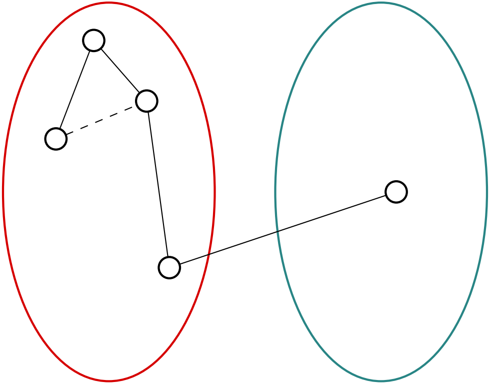

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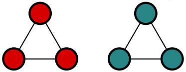

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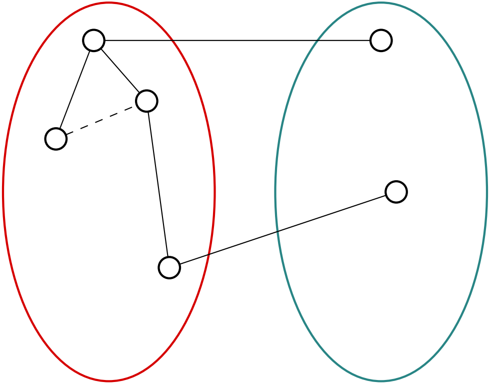

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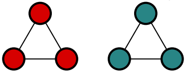

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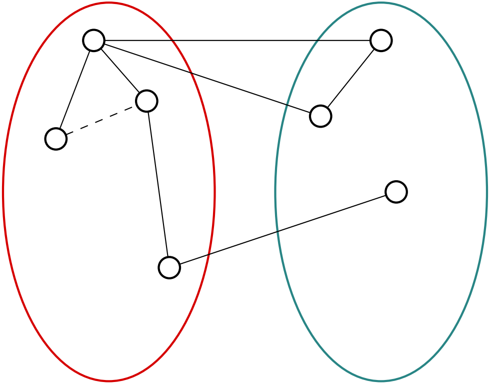

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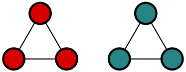

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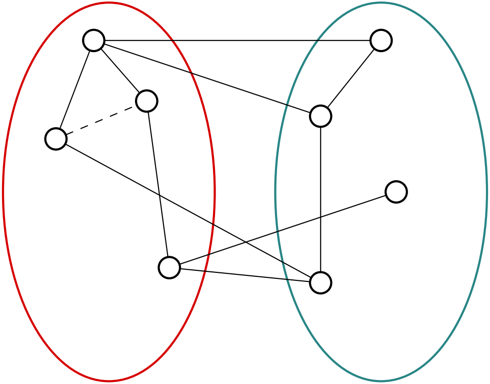

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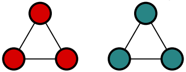

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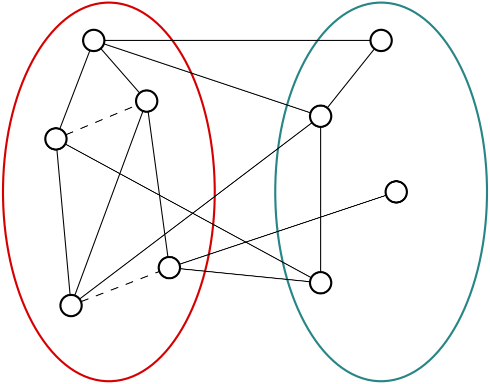

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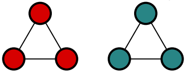

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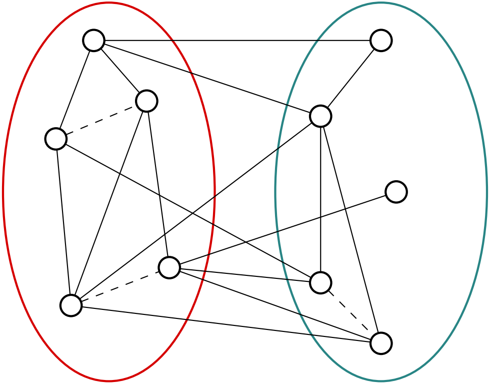

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Conjecture (Bodirsky-Pinsker, 2011)

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Theorem (Bodirsky-Madelaine-M, 2018)

Let $\mathfrak{B}$ be ω-categorical such that $\mathrm{CSP}(\mathfrak{B})$ *is in MMSNP. If there is no uniformly continuous h1 homomorphism* $\mathrm{Pol}(\mathfrak{B}) \to \mathscr{P}$, *then* $\mathrm{CSP}(\mathfrak{B})$ *is in P.*

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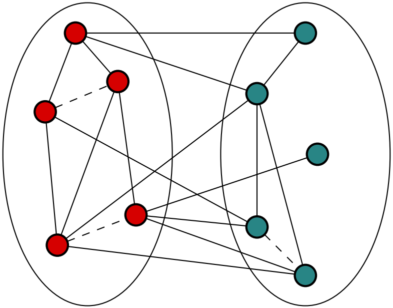
Rephrased: do CSP($\mathfrak{B}$, ●, ●) and CSP($\mathfrak{B}$) have same complexity?



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Good news: we can choose the MMSNP structure $\mathfrak{B}$ so that $(\mathfrak{B}, \neq)$ is an ω-categorical model-complete core.

Precolouring gadget

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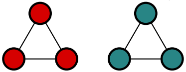

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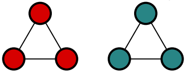

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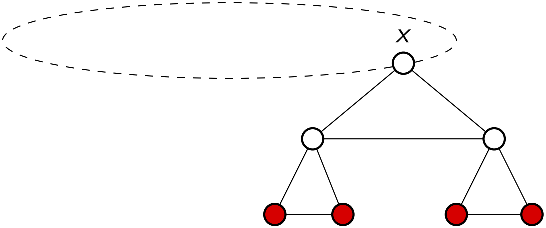

Precolouring gadget

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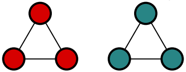

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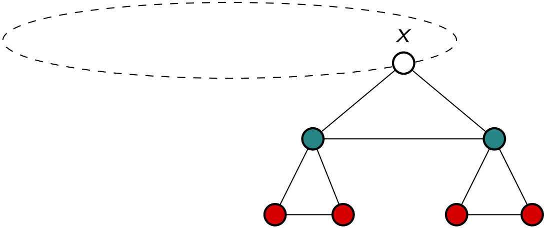

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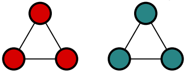

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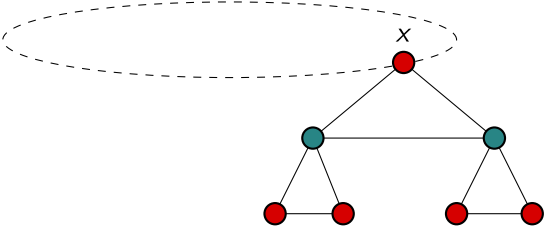

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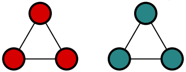

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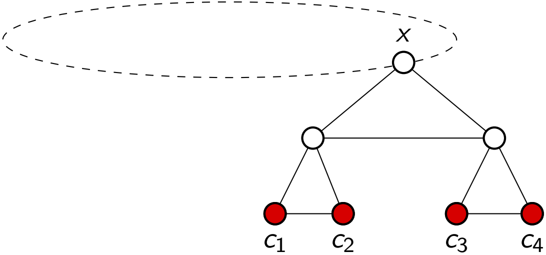

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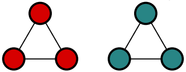

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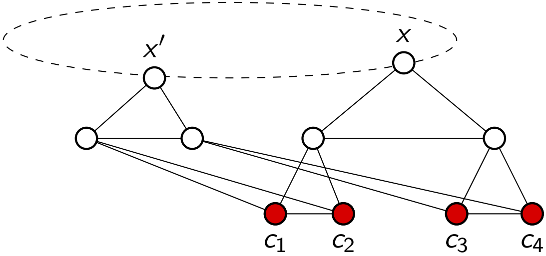

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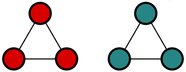

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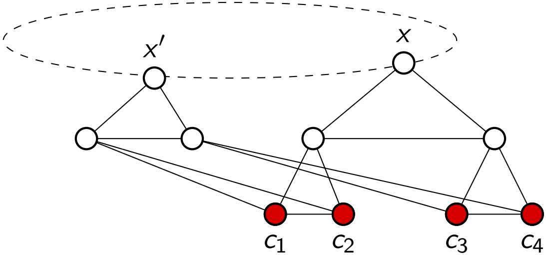

Proposition

The input precoloured graph is colourable iff the graph obtained by adding the gadgets is colourable.

Outline

Definition

$f\colon B^k \to B$, a group $\mathcal{G}$ acting on B. f is canonical (wrt $\mathcal{G}$) if for every finite subset $S \subseteq B$ of B and $\alpha_1, \dots, \alpha_k \in \mathcal{G}$, there exists $\beta \in \mathcal{G}$ such that $\beta \circ f|_S = f \circ (\alpha_1, \dots, \alpha_k)|_S$.

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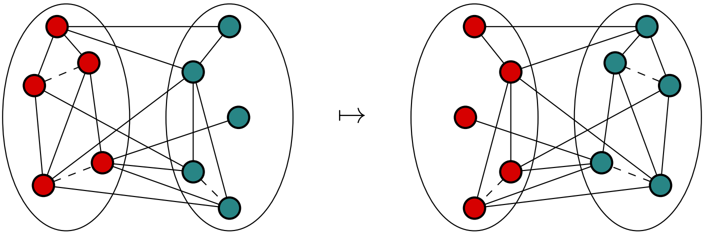

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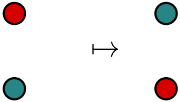

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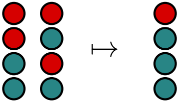

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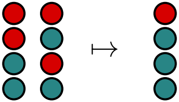

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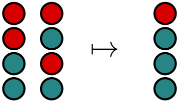

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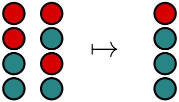

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$\Rightarrow$ let $\mathscr{C}$ be the clone of canonical polymorphisms of the MMSNP structure $\mathfrak{B}$. If there is no h1 homomorphism $\mathscr{C} \to \mathscr{P}$, then $\mathrm{CSP}(\mathfrak{B})$ is in P.

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4. Define an h1 homomorphism $\mathrm{Pol}(\mathfrak{B}) \to \mathscr{P}$ by canonizing and composing with ξ.

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σ: set of colour symbols. A trivial subfactor of $\mathscr{C}$ is a partition $S \uplus T \subseteq \sigma$ such that the equivalence relation with blocks S and T is a congruence of $\mathscr{C}$ with the property that the clone induced by $\mathscr{C}$ on $\{S, T\}$ is isomorphic to $\mathscr{P}$.

Proposition

S, T trivial subfactor of $\mathscr{C}$. $\exists(B, E)$ undirected graph s.t.:

- *(B, E) contains an edge from S to T but does not contain pseudo-loops;*
- *the connected components of N are included in S, included in T, or bipartite with the bipartition induced by S and T;*
- *E is preserved by* $\mathrm{Pol}(\mathfrak{B})$.

We say that $\{S, T\}$ is a Cthulhu partition of (B, E).

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Getting canonical functions

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Given $f \in \mathrm{Pol}(\mathfrak{B})$, and $\xi\colon \mathscr{C} \to \mathscr{P}$ given by subfactor $\{S, T\}$, define $\phi(f) := \xi(g)$ where g is an arbitrary function in $\mathscr{C} \cap \overline{\mathrm{Aut}(\mathfrak{B}, <) f \, \mathrm{Aut}(\mathfrak{B}, <)}$.

Proposition

$\phi\colon \mathrm{Pol}(\mathfrak{B}) \to \mathscr{P}$ is a well-defined uniformly continuous height 1 homomorphism.

Summing up:

Theorem

Let $\mathfrak{B}$ be a MMSNP structure.
Then either the following equivalent statements hold:

1. *there is no uniformly continuous height 1 homomorphism* $\mathrm{Pol}(\mathfrak{B}) \to \mathscr{P}$,

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Items 3. and 4. can be checked effectively.

- When is CSP($\mathfrak{B}$) in Datalog? Is it decidable?
  Answered for monochromatic obstructions.
- MMSNP_2: instead of colouring vertices, we colour edges. It is more expressive than MMSNP, but it is open whether it has a complexity dichotomy (Lutz et al.).
  Example: is it possible to colour the edges of an input graph and avoid: