Promise Constraints and Cheese

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$\mathbb{A}, \mathbb{B}$ ... relational structures with $\mathbb{A} \to \mathbb{B}$

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$(\mathbb{A}, \mathbb{B})$ sandwiches $\mathbb{C}$ if $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ $\quad$ ($\Rightarrow \mathrm{PCSP}(\mathbb{A}, \mathbb{B}) \leq \mathrm{CSP}(\mathbb{C})$, search version ...)

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Observation:

If $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is in P, there is $\mathbb{C}$ s.t. $\mathrm{CSP}(\mathbb{C})$ is in P and $\mathrm{PCSP}(\mathbb{A}, \mathbb{B}) \leq \mathrm{CSP}(\mathbb{C})$.

Proof: M solving $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ $\quad L(M) \equiv \mathrm{CSP}(\mathbb{C})$ $\quad$ (Bodirsky-Grohe)

$\square$

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Not quite a sandwich...

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Observation 2:

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Proof: $\mathbb{C}$ = disjoint union of structures accepted by algorithm.

Corollary: $\emptyset$

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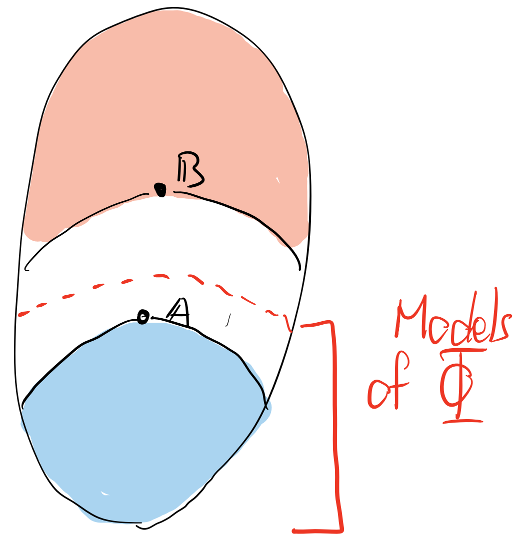
Theme: Find nicer cheese under assumption that $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is solved by specific algorithm.

- search $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$
- $\mathrm{Pol}(\mathbb{C}) \to \mathrm{Pol}(\mathbb{A}, \mathbb{B})$

① AC^0 / FIRST ORDER LOGIC AND TIME TRAVEL

$PCSP(\mathbb{A},\mathbb{B})$ is solved by Φ if:

- $\mathbb{X} \to \mathbb{A} \Rightarrow \mathbb{X} \models \Phi$
- $\mathbb{X} \not\to \mathbb{B} \Rightarrow \mathbb{X} \not\models \Phi$ / $\mathbb{X} \models \Phi \Rightarrow \mathbb{X} \to \mathbb{B}$.



Question: (Opršal '20)

Can one characterize $(\mathbb{A},\mathbb{B})$ s.t. $PCSP(\mathbb{A},\mathbb{B})$ is in FO?

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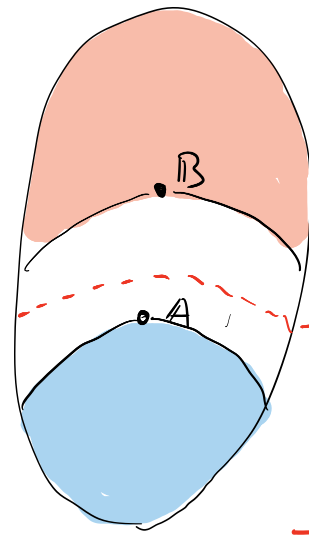

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Can one characterize $(\mathbb{A},\mathbb{B})$ s.t. $PCSP(\mathbb{A},\mathbb{B})$ is in FO?

Theorem: (Atserias)

$\mathbb{A}$ finite. $CSP(\mathbb{A})$ is in FO iff $\exists \mathcal{F}$ finite s.t. $\mathcal{X} \to \mathbb{A} \Leftrightarrow \nexists \mathbb{F} \in \mathcal{F}: \mathbb{F} \to \mathcal{X}$.

Finite duality

Question: (Opršal '21)

$PCSP(\mathbb{A},\mathbb{B})$ in FO iff $\exists \mathcal{F}$ finite s.t.
$\mathcal{X} \to \mathbb{A} \Rightarrow (\nexists \mathbb{F} \in \mathcal{F}: \mathbb{F} \to \mathcal{X})$ and $(\nexists \mathbb{F} \in \mathcal{F}: \mathbb{F} \to \mathcal{X}) \Rightarrow \mathcal{X} \to \mathbb{B}$?

① AC^0 / FIRST ORDER LOGIC AND TIME TRAVEL

PCSP$(\mathbb{A},\mathbb{B})$ is solved by Φ if:

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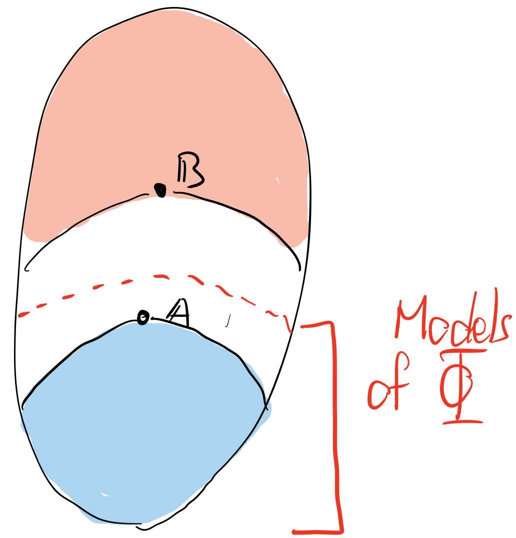

Theorem: (Atserias)

A finite. CSP(A) is in FO iff $\exists \mathcal{F}$ finite s.t. $X \to A \Leftrightarrow \nexists F \in \mathcal{F}: F \to X$.

Question: Opršal '22

CLASSIFY SOME ALGORITHMS.

- local consistency
- Sherali-Adams
- CLAP [Ciardo, Živný]
- $\boxed{\text{finite duality}}$
- definability in various logics.

Theorem: (Rossman '08)

Yes.

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For finite CSPs:

- finite duality $\Rightarrow$ tree duality
- characterization by "1-tolerant" polymorphisms
- "CSP(A) $\in$ FO?" in NP

Larose, Loten, Tardif

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Larose, Loten, Tardif

Theorem: (M.)

For finite $\mathbb{A}, \mathbb{B}$, TFAE:

- $PCSP(\mathbb{A}, \mathbb{B}) \in FO$
- $(\mathbb{A}, \mathbb{B})$ has finite duality

Rossman

- $(\mathbb{A}, \mathbb{B})$ has finite tree duality
- $\exists\, \mathbb{C}$ finite s.t. $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ and $CSP(\mathbb{C}) \in FO$.

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- $\exists\, \mathbb{C}$ finite s.t. $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ and $\mathrm{CSP}(\mathbb{C}) \in \mathrm{FO}$.

But:

- no characterization by 1-tolerant polymorphisms (necessary not sufficient)
- "$\mathrm{PCSP}(\mathbb{A}, \mathbb{B}) \in \mathrm{FO}$?" decidable?

Proof: $\mathcal{F}$ finite duality for $(\mathbb{A},\mathbb{B})$: $\quad \mathbb{X}\to\mathbb{A} \Rightarrow \nexists\, \mathbb{F}\in\mathcal{F}: \mathbb{F}\to\mathbb{X}$

$\quad \mathbb{X}\not\to\mathbb{B} \Rightarrow \exists\, \mathbb{F}\in\mathcal{F}: \mathbb{F}\to\mathbb{X}$

Wlog every $\mathbb{F}\in\mathcal{F}$ is connected.

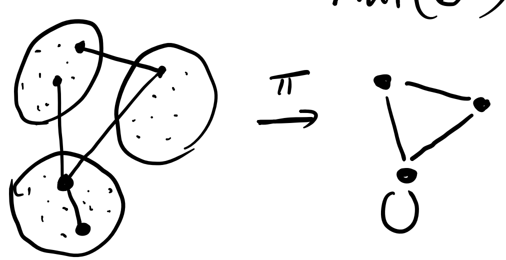
$\exists\, \mathbb{U}$ ω-categorical s.t. $\mathbb{X}\to\mathbb{U}$ iff $\nexists\, \mathbb{F}\in\mathcal{F}: \mathbb{F}\to\mathbb{X}$.
(Cherlin, Shelah, Shi)

$$\mathbb{A}\to\mathbb{A} \Rightarrow \nexists\, \mathbb{F}\in\mathcal{F}: \mathbb{F}\to\mathbb{A} \Rightarrow \mathbb{A}\to\mathbb{U}$$

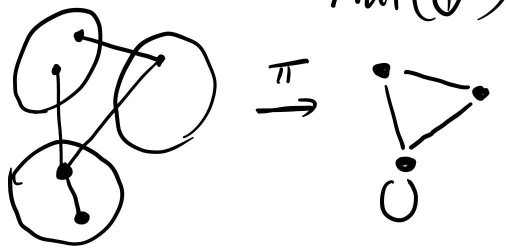
$$\mathbb{U}\to\mathbb{U} \Rightarrow \nexists\, \mathbb{F}\in\mathcal{F}: \mathbb{F}\to\mathbb{U} \Rightarrow \mathbb{U}\to\mathbb{B}.$$

$\exists\, \mathbb{U}^{+}$ ω-categorical Ramsey expansion of $\mathbb{U}$
(Hubička, Nešetřil)

$$\mathbb{A}\to\mathbb{U}\xrightarrow{\pi}\mathbb{U}/_{\mathrm{Aut}(\mathbb{U}^{+})}\xrightarrow{\tilde{h}}\mathbb{B}$$



Proof (cont'd): $\mathbb{A} \to \mathbb{U} \xrightarrow{\pi} \mathbb{U}/_{\mathrm{Aut}(\mathbb{U}^+)} \xrightarrow{\tilde{h}} \mathbb{B}$ and $\mathbb{U}$ has finite duality.



$\mathbb{U}$ has 1-tolerant polymorphism: $f: U^k \to U$ s.t. $f(R^{\mathbb{U}}, \dots, R^{\mathbb{U}}, U^n, R^{\mathbb{U}}, \dots, R^{\mathbb{U}}) \subseteq R^{\mathbb{U}}$

$\forall$ relation $R^{\mathbb{U}}$ of $\mathbb{U}$

[BPT]

$\Rightarrow$ $\mathbb{U}$ has 1-tolerant polymorphism that is canonical w.r.t. $\mathbb{U}^+$

$\Rightarrow$ $\mathbb{U}/_{\mathrm{Aut}(\mathbb{U}^+)}$ has 1-tolerant polymorphism

$\Rightarrow$ $\underbrace{\mathbb{U}/_{\mathrm{Aut}(\mathbb{U}^+)}}_{\mathbb{C}}$ has finite duality. $\square$

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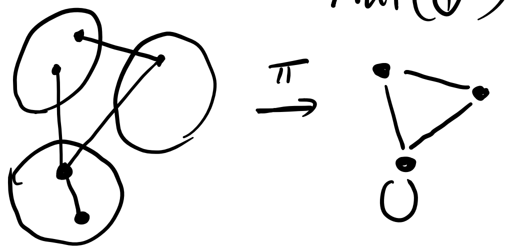

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$\Rightarrow$ $\underbrace{\mathbb{U}/\mathrm{Aut}(\mathbb{U}^+)}_{\mathbb{C}}$ has finite duality. $\square$

$(\mathbb{K}_3, \mathbb{K}_9)$ has binary 1-tolerant polymorphism but $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_9)$ not in FO.

Open: bound on $|\mathbb{C}|$?

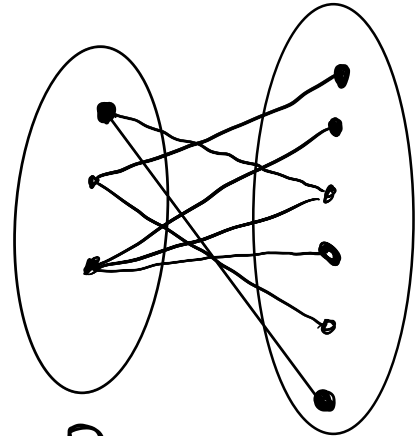
2 PCSPs WITH BOUNDED WIDTH

(k,ℓ)-consistency solves $\mathrm{PCSP}(\mathbb{A},\mathbb{B})$ if

$$\mathbb{X} \ (k,\ell)\text{-consistent w.r.t. } \mathbb{A} \Rightarrow \mathbb{X} \to \mathbb{B}.$$

$$\Updownarrow$$

$\exists$ family of sets $P_K \subseteq A^K$ ($K \subseteq X$, $|K| \leq \ell$)
s.t. $P_K \subseteq$ partial homomorphisms $\mathbb{X}[K] \to \mathbb{A}$
and $\forall\, K \subseteq L$



P_K P_L

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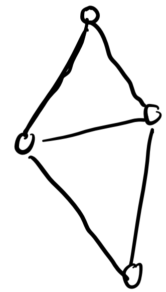
$\boxed{\text{Fix } \mathbb{A}}$

$\mathbb{X}$ (k,l)-consistent



$\mathbb{X}$

$\mathbb{A}$

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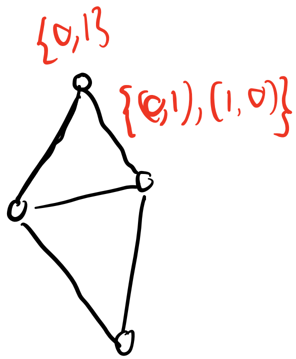
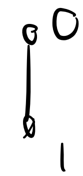
$\mathbb{X}$ (k,ℓ)-consistent

$(\mathbb{X}, P)$ decoration by a (k,ℓ)-system



$(\mathbb{X}, P)$



$\mathbb{A}$

$$K^+ := \left\{ \mathbb{X} + \begin{array}{c}\text{non-trivial}\\ (k,\ell)\text{-system}\end{array} \;\middle|\; \mathbb{X}\ (k,\ell)\text{-consistent w.r.t. } \mathbb{A} \right\}$$

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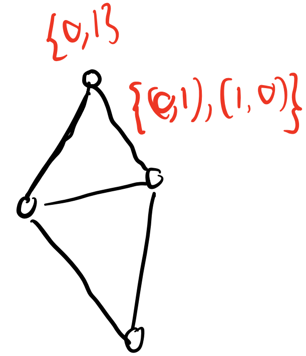

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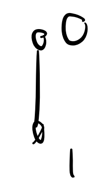

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$(\mathbb{X},P)$ decoration by a (k,ℓ)-system

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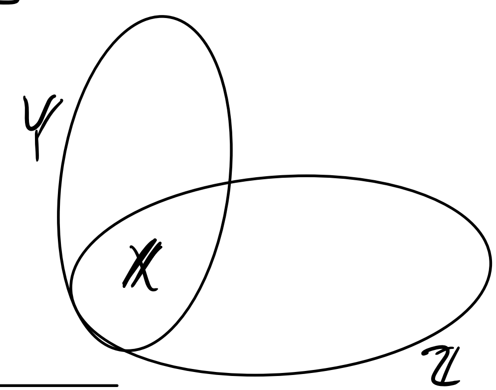
Lemma:

$\mathcal{K}^+$ has the (strong) amalgamation property:

$(\mathbb{Y},P),(\mathbb{Z},Q)$ (k,ℓ)-consistent and

$$P|_{Y\cap Z} = Q|_{Y\cap Z}$$

$\Rightarrow$ $\mathbb{Y}\cup\mathbb{Z}$ (k,ℓ)-consistent.



$\rightsquigarrow$ $\exists(\mathbb{U},P)$ universal limit object, ω-categorical (and more)

$$K^+ := \left\{ \mathbb{X} + \text{non-trivial } (k,\ell)\text{-system} \;\middle|\; \mathbb{X}\ (k,\ell)\text{-consistent w.r.t. } \mathbb{A} \right\}$$

$$\boxed{\mathbb{X} \to \mathbb{U} \iff \mathbb{X}\ (k,\ell)\text{-consistent w.r.t. } \mathbb{A}}$$

- $\mathbb{A}$ (k,ℓ)-consistent w.r.t. $\mathbb{A}$ $\Rightarrow$ $\mathbb{A} \to \mathbb{U}$

$$\mathcal{K}^+ := \left\{ \mathbb{X} + \begin{array}{c}\text{non-trivial}\\ (k,l)\text{-system}\end{array} \;\middle|\; \mathbb{X}\ (k,l)\text{-consistent w.r.t. } \mathbb{A} \right\}$$

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- $\mathbb{U} \to \mathbb{U} \Rightarrow \mathbb{U}$ (k,l)-consistent w.r.t. $\mathbb{A}$

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- $\mathbb{U} \to \mathbb{U}$ $\Rightarrow$ $\mathbb{U}$ (k,ℓ)-consistent w.r.t. $\mathbb{A}$
- $\mathbb{U}$ has width (k,ℓ):

$$\begin{aligned} &\mathbb{X}\ (k,\ell)\text{-consistent w.r.t. } \mathbb{U} \\ \Rightarrow\ &\mathbb{X}\ (k,\ell)\text{-consistent w.r.t. } \mathbb{A} \\ \Rightarrow\ &\mathbb{X} \to \mathbb{U}. \end{aligned}$$

Theorem: (M.)

$\mathbb{A}$, $1 \leq k \leq \ell$. There exists $\mathbb{U}$ ω-categorical^{++} s.t. $\mathbb{A} \to \mathbb{U}$, $\mathbb{U}$ has width (k,ℓ), and for all $\mathbb{B}$, TFAE:

- $(\mathbb{A}, \mathbb{B})$ has width (k,ℓ)
- $\mathbb{U} \to \mathbb{B}$.

$\mathbb{U}$ is universal with these properties.

Polymorphisms?

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Polymorphisms?

Theorem: (Atserias, Dalmau)

If $(\mathbb{A}, \mathbb{B})$ has bounded width then $(\mathbb{A}, \mathbb{B})$ has $WNU(n)$ $\forall n \geq 3$.

"Indicator" structure:

$$\Sigma = w(yxx) \approx w(xyx) \approx w(xxy)$$

$$\Sigma(\mathbb{A}) = \mathbb{A}^3 / \sim_\Sigma$$

$A = \{0, 1\}$

$001 \sim_\Sigma 010 \sim_\Sigma 100$

Observation: $\Sigma(\mathbb{A}) \to \mathbb{A}$ iff $\mathrm{Pol}(\mathbb{A})$ satisfies Σ

$$\boxed{\mathbb{X} \to \mathbb{U} \iff X \ (k,l)\text{-consistent w.r.t. } \mathbb{A}}$$

Proposition:

For every set Σ of height 1 identities, TFAE:

- $\mathrm{Pol}(\mathbb{U})$ satisfies Σ
- $\mathrm{Pol}(\mathbb{A}, \mathbb{U})$ satisfies Σ
- $\Sigma(\mathbb{A})$ (k,l)-consistent w.r.t. $\mathbb{A}$

$\forall n \quad WNU(n)$ ✗ (not even $\forall n$ large)

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$\forall n \; dWNU(n)$ ✓

(Barto terms)

$WNU(n) \Rightarrow dWNU(n)$

$n=3$:

$$f_1(yxx) \approx f_3(xyx) \approx f_4(xyx)$$
$$f_1(xyx) \approx f_2(xyx) \approx f_3(xxy)$$
$$f_1(xxy) \approx f_2(xxy) \approx f_3(xxy)$$
$$f_3(yxx) \approx f_2(yxx) \approx f_4(yxx)$$

$\forall n$ large $pWNU(n)$ ✓

(not height 1)

$$\alpha_1 f(yx\cdots x) \approx \alpha_2 f(xyx\cdots x) \approx \cdots$$

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Corollary:

$\mathbb{A}$ ω-categorical with bounded width.

$\exists\ \mathbb{U}$ ω-categorical^{++} s.t. $\mathbb{A} \rightleftarrows \mathbb{U}$ and $\mathbb{U}$ has dWNU polymorphisms of all arities.

Existence of dWNU(3):

$$f_1(yxx) \approx f_3(xyx) \approx f_4(xyx)$$
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$$\vdots$$

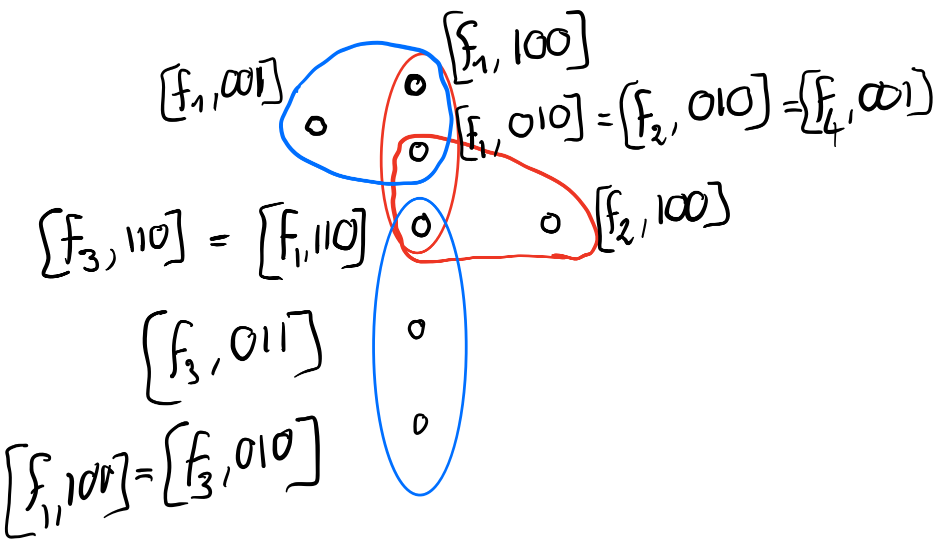
$$\mathbb{A} = \left(\{0,1\},\ \overbrace{x+y+z=0},\ \overbrace{x+y+z=1}\right)$$

$\Sigma(\mathbb{A})$ consistent?

Domain $= [f_1, 100], [f_1, 010], \ldots$

$[f_1, 100] = [f_3, 010]$

Let Π be a „tree" with $\Pi \to \Sigma(\mathbb{A})$.



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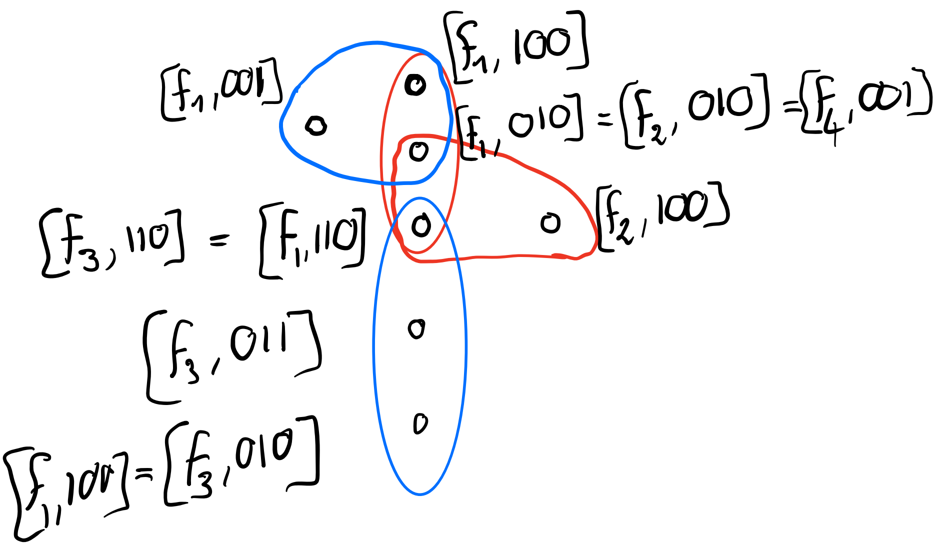
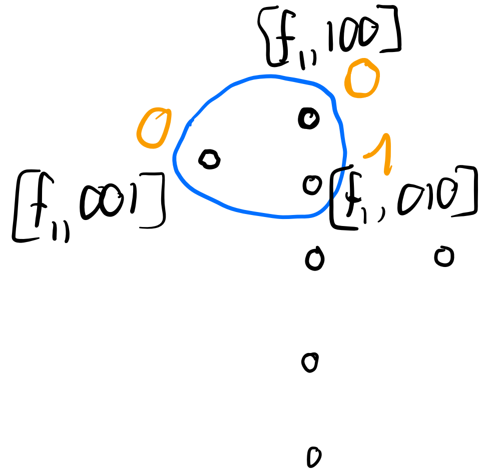
$\Sigma(\mathbb{A})$ consistent?

Domain = $[f_1, 100], [f_1, 010], \dots$

$[f_1, 100] = [f_3, 010]$

Let $\mathbb{T}$ be a „tree" with $\mathbb{T} \to \Sigma(\mathbb{A})$.





Existence of dWNU(3):

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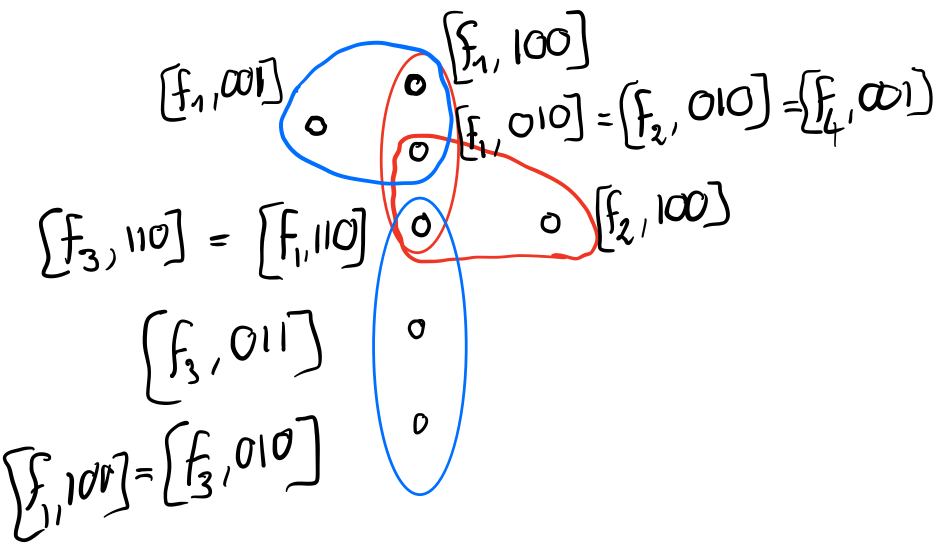

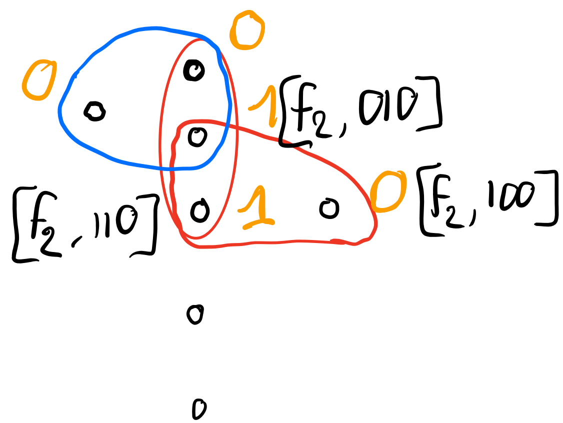

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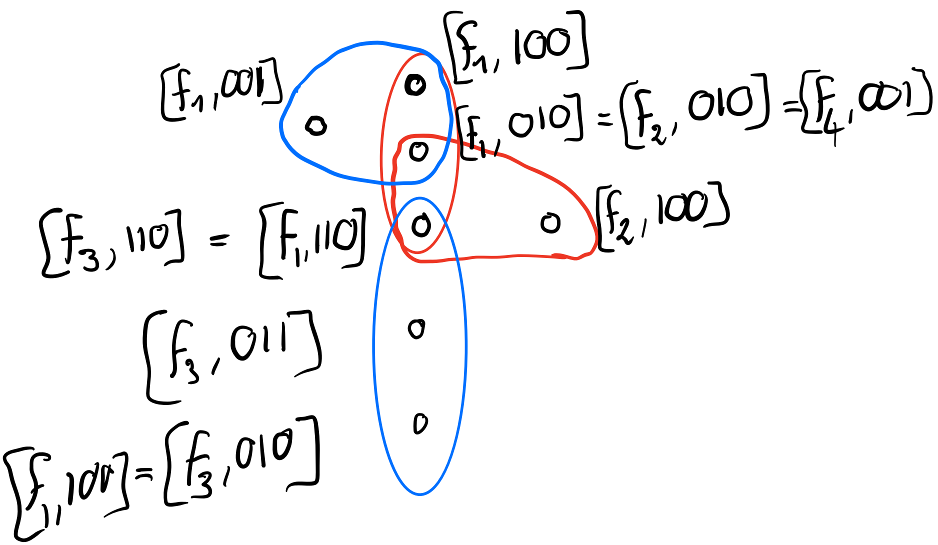
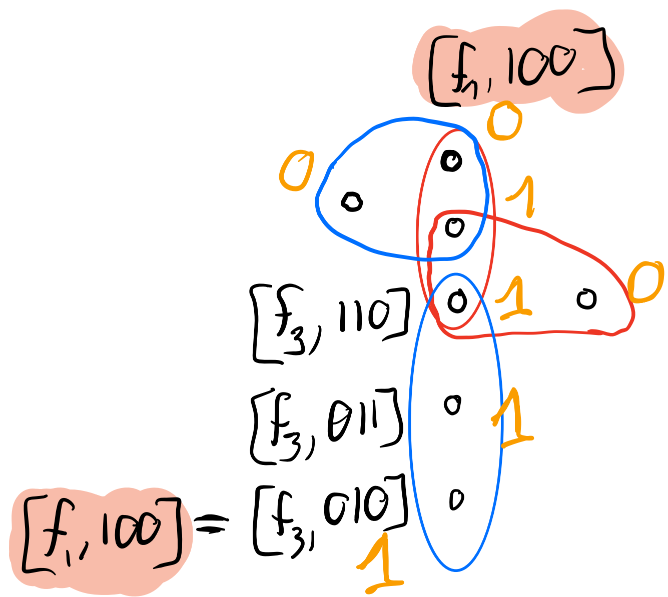
Let Π be a "tree" with $\Pi \to \Sigma(\mathbb{A})$.





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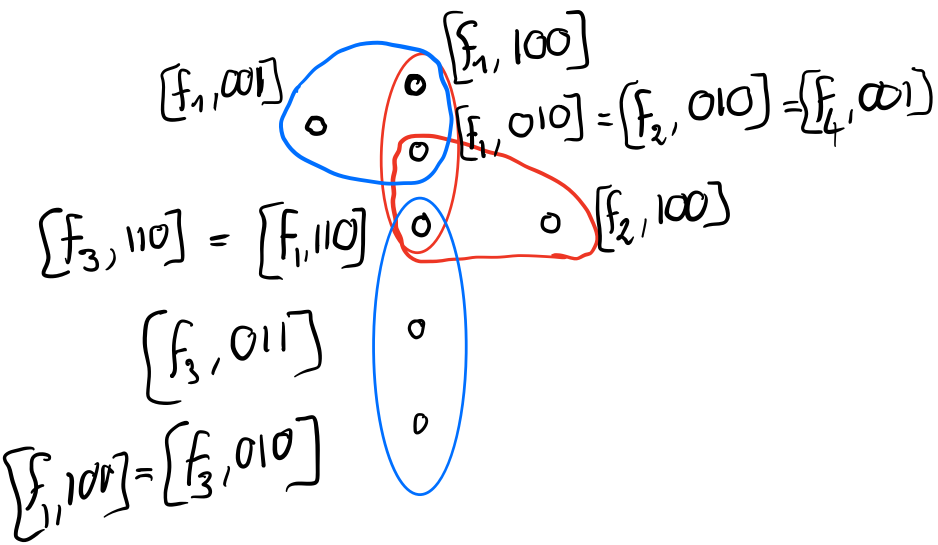

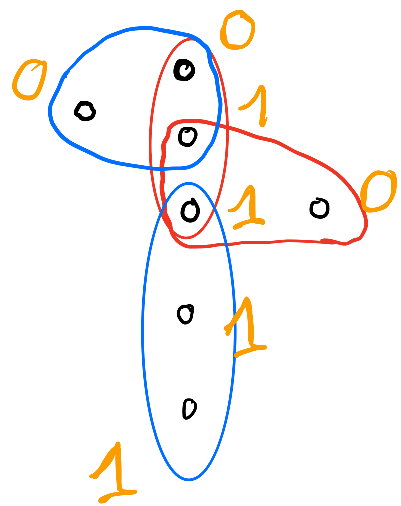

So $\boxed{\mathbb{T} \to \mathbb{A}}$

Open: can $\mathbb{U}$ be taken finite? decidable property?

Proof strategy:

1. construct Ramsey expansion (exists for width 1)
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CLASSIFY SOME ALGORITHMS.

- local consistency ?
- Sherali–Adams
- CLAP [Ciardo, Živný]
- finite duality ✓
- definability in various logics.

General argument:

Suppose one has a pair $\mathcal{A}$ ... algorithm s.t.

$\mathcal{M}$ ... minion

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- $\mathbf{F}_{\mathcal{M}}(\mathbb{A}) \to \mathbb{U}$

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- $\mathbf{F}_{\mathcal{M}}(\mathbb{A}) \to \mathbb{U}$ ✗