Reasoning with discrete time

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QuantLA Workshop 2016

Outline

Temporal reasoning: preparing the workshop

A scheduling problem:

- Find results,
- Write abstract,
- Write slides,
- Rehearse,

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- Find results, before writing the slides
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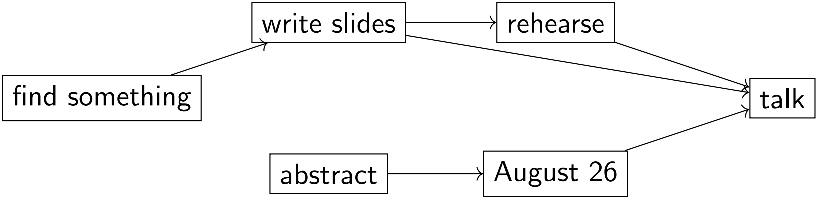

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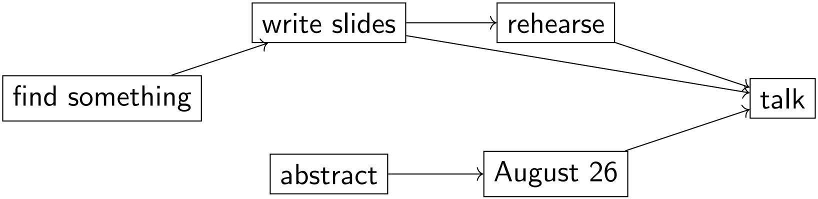

Reducts

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Definition (CSP(Γ))

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- Γ reduct of $(\mathbb{Z}, <) \Rightarrow$ CSP(Γ) is in NP.

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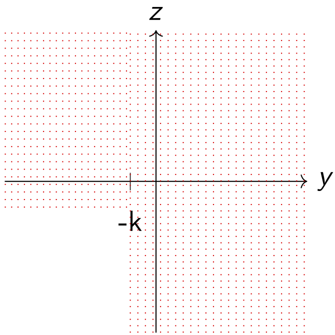

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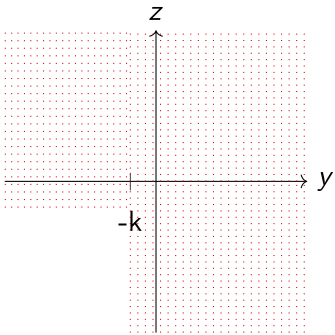

- Equivalent to deciding winner in deterministic mean-payoff games.
- In P, if k given in unary.

Difference logic with modular constraints

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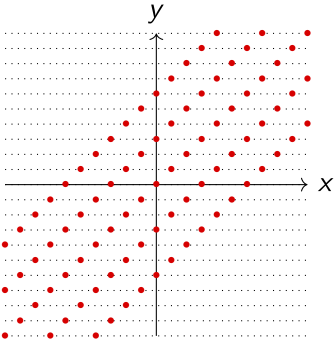

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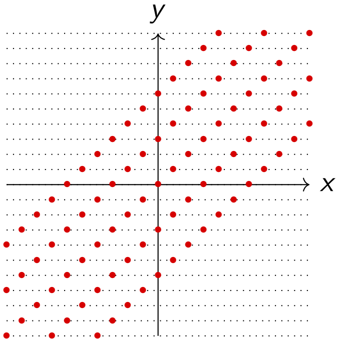

- If $d = 1$, difference logic
- For all $d \geq 1$, in P.

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Example

- $(\mathbb{Q}, <)$ itself: in P, digraph acyclicity
- $(\mathbb{Q}, x = y \Rightarrow u = v, \leq, \neq)$: in P, Ord-Horn (Nebel, Bürckert)
- $(\mathbb{Q}, x < y < z \vee z < y < x)$: NP-complete, Betweenness

The discrete case

Theorem (Bodirsky, Martin, M)

Let Γ be a reduct of $(\mathbb{Z}; <)$. Then $\mathrm{CSP}(\Gamma)$ is in P, NP-complete, or is $\mathrm{CSP}(\Delta)$ for a finite structure Δ.

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2. *Δ is a reduct of $(\mathbb{Q}; <)$.*
3. *The endomorphisms of Δ are isometries.*

Let's prove it!

Definition (Endomorphism)

$\Gamma = (D, R_1, \ldots, R_s)$ a structure, $f \colon D \to D$. f is an endomorphism of Γ if

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Example

$\Gamma = (\mathbb{Z}, |x - y| = 1)$. Then $f\colon x \mapsto x \bmod 2$ is an endomorphism.

Bounded endomorphisms

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- Then $\forall x, y, k \in \mathbb{Z}$, $\exists e \in \mathrm{End}(\Gamma)$ such that $|e(x) - e(y)| > k$.
- So what?

Definition

$(\mathbb{Q}.\mathbb{Z}, <)$ is the structure on $\mathbb{Q} \times \mathbb{Z}$ with the lexicographic ordering.
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Going to the saturated extension

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- $\exists e \in \mathrm{End}(\mathbb{Q}.\Gamma)$ such that

$$\forall x, y \in \mathbb{Q}.\mathbb{Z}, e(x) \neq e(y) \Rightarrow e(x) - e(y) = \infty.$$

Theorem

Let Γ *be a reduct of* $(\mathbb{Z}, <)$ *with finite signature and without finite-range endomorphisms. Exactly one of the following applies:*

- *There exists a reduct* Δ *of* $(\mathbb{Q}, <)$ *with* $\mathsf{CSP}(\Gamma) = \mathsf{CSP}(\Delta)$*.*
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Note: if $t = 1$, every endomorphism of $\mathbb{Q}.\Gamma$ is an isometry.

Going from t to 1

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Going from t to 1

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- $\mathrm{CSP}(\Gamma) = \mathrm{CSP}(\Delta)$.

Final step

We proved:

Theorem

Γ *a reduct of* $(\mathbb{Z}; <)$ *with finite signature.* $\exists\Delta$ *with* $\mathrm{CSP}(\Delta) = \mathrm{CSP}(\Gamma)$ *and at least one of the following cases applies:*

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Preservation theorem: we can assume that Δ contains the relation $y = x + 1$ or $|y - x| = 1$.

Outline

Horn over *succ*

Constraints: conjunctions of clauses of the form

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- If no new implication is found, accept

Constraints: conjunctions of clauses of the form $(\bigwedge x_i = y_i + p_i) \Rightarrow u = v + q$

- A relation R is preserved by a binary function f iff

$$\begin{array}{ccccc} \forall & (x_1 & \dots & x_n) & \in R \\ \forall & (y_1 & \dots & y_n) & \in R \\ \Rightarrow & (f(x_1, y_1) & \dots & f(x_n, y_n)) & \in R. \end{array}$$

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- We use f to produce a solution of the instance, such that all the premises are violated.

Definition (Modular maximum)

$d \geq 1$. $\max_d : \mathbb{Z}^2 \to \mathbb{Z}$ defined by

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Algorithm: essentially local consistency.

Outline

Non-positive case

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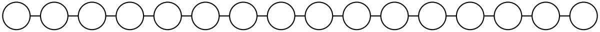

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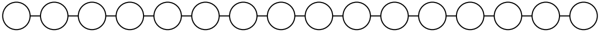

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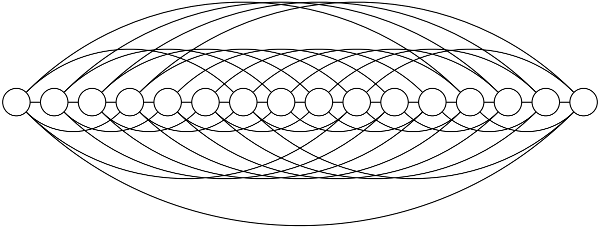

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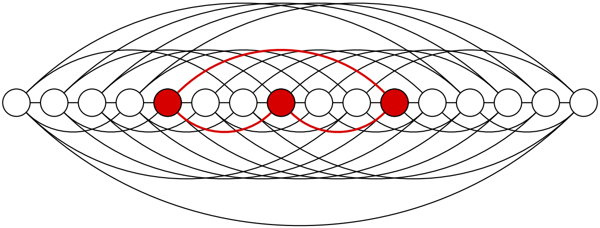

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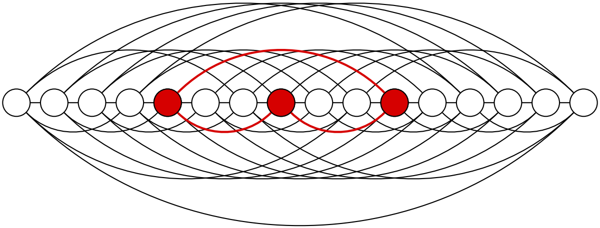

Theorem (Hell, Nešetril, Journ. Comb. Th. Series B 1990)

G undirected graph. CSP(G) *in P if G bipartite, NP-complete otherwise.*

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Theorem (BDalmauMMPinsker, Inf. Comp. 2016)

Let Γ be a reduct of $(\mathbb{Z}, <)$ which is locally finite. If Γ is not preserved by $\max_d$ for any $d \geq 1$, then $\mathrm{CSP}(\Gamma)$ is NP-complete.

Outline

Theorem (Bodirsky, Martin, M)

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1. *Δ has a finite domain.*
2. *Δ is a reduct of $(\mathbb{Q}; <)$.*
3. *The relations of Δ are Horn, and $\mathrm{CSP}(\Delta)$ is in P.*
4. *The relations of Δ are preserved by $\max_d$, for some $d \geq 1$, and $\mathrm{CSP}(\Delta)$ is in P.*
5. *$\mathrm{CSP}(\Delta)$ is NP-complete.*

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4. *The relations of* Δ *are preserved by* $\max_d$*, for some* $d \geq 1$*, and* $\mathrm{CSP}(\Delta)$ *is in P.*
5. $\mathrm{CSP}(\Delta)$ *is NP-complete.*

Complexity dichotomy modulo the Feder-Vardi conjecture.

What's next?

- Reductions of $(\mathbb{Z}, <, 0)$:
  - all finite-domain CSPs (Zhuk'16?!),
  - reducts of equality with constants (Bodirsky, M LICS'16)
- Long-term goal: reducts of $(\mathbb{Z}, +, <)$, i.e., complexity of CSPs within Presburger arithmetic.