Cyclic terms, CSP, MMSNP

Antoine Mottet
(j.w. Manuel Bodirsky, Florent Madelaine)

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  - shown in a beautiful picture (never).

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Theorem (Barto-Kozik + Barto-Opršal-Pinsker)

Let $\mathcal{A}$ be a finite relational structure. Then exactly one of the following holds:

(+) $\mathrm{Pol}(\mathcal{A})$ *contains a cyclic operation,*

(-) *there exists a clonoid homomorphism* $\mathrm{Pol}(\mathcal{A}) \to \mathscr{P}$.

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- Clonoid homomorphisms: required to be uniformly continuous.

- **Goal:** find other structures $\mathcal{A}$ for which the cyclic term theorem holds.

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Definition (CSP)

$\mathrm{CSP}(\mathcal{B}) := \{\mathcal{A} \mid \mathcal{A} \text{ finite}, \mathcal{A} \to \mathcal{B}\}$.

- $\mathrm{CSP}(K_3) =$ all finite 3-colourable graphs,
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Building structures with given CSPs:

Theorem (Cherlin, Shelah, Shi)

Let $\mathfrak{F}$ be a finite set of finite connected graphs. There exists an ω-categorical $\mathcal{B}$ such that $\mathcal{A} \to \mathcal{B}$ iff $\forall \mathcal{F} \in \mathfrak{F}$, $\mathcal{F} \not\to \mathcal{A}$.

Example: there exists an ω-categorical graph $\mathcal{B}$ such that

$$\mathrm{CSP}(\mathcal{B}) = \{\mathcal{A} \mid \mathcal{A} \text{ is finite and triangle-free}\}.$$

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- Cherlin-Shelah-Shi: There exists an ω-categorical graph $\mathcal{B}$ such that $\mathrm{CSP}(\mathcal{B}) = \{\mathcal{A} \mid \mathcal{A} \models \Phi_{\text{No-mono-tri}}\}$.

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- More generally, we consider formulas with:
  - existential unary second-order quantifiers,
  - universal first-order quantifiers,
  - a conjunction of forbidden patterns.

Theorem

Let $\mathcal{B}$ be ω-categorical and such that $\mathrm{CSP}(\mathcal{B}) \in$ *MMSNP. Then:*

- **(-)** *There is a uniformly continuous clonoid homomorphism from* $\mathrm{Pol}(\mathcal{B})$ *to* $\mathscr{P}$*, or*
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- Only focus on particular structures $\mathcal{C}_\Phi$, for particular MMSNP sentences Φ.
- Make a bet: if $\mathrm{Pol}(\mathcal{C}_\Phi)$ has a cyclic operation, it has a very regular one.

Definition

$f\colon B^k \to B$, a group $\mathscr{G}$ acting on B. f is canonical (wrt $\mathscr{G}$) if for every finite subset $S \subseteq B$ of B and $\alpha_1, \ldots, \alpha_k \in \mathscr{G}$, there exists $\beta \in \mathscr{G}$ such that $\beta \circ f|_S = f \circ (\alpha_1, \ldots, \alpha_k)|_S$.

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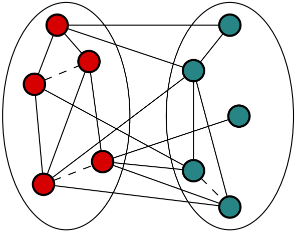

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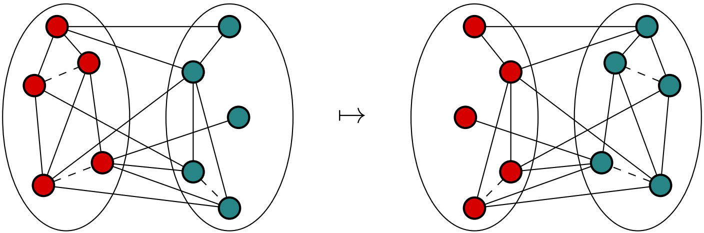
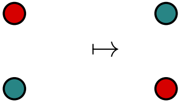


Every canonical function determines a behaviour on colours.



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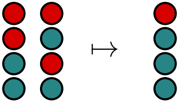

Behaviours

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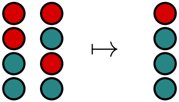

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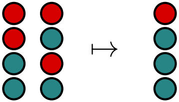

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Theorem (Bodirsky-Pinsker-Pongrácz)

Let $\mathscr{C}$ be a clone consisting of canonical functions with respect to a homogeneous structure. Then:

(+) *$\mathscr{C}$ contains a cyclic operation, or*

(-) *there exists a clone homomorphism $\mathscr{C} \to \mathscr{P}$.*

Theorem (Hubička-Nešetřil, 2016)

There is a linear order $<$ *on* $\mathcal{C}_\Phi$ *such that* $\mathrm{Aut}(\mathcal{C}_\Phi, <)$ *is oligomorphic and extremely amenable.*

Obtaining canonical functions

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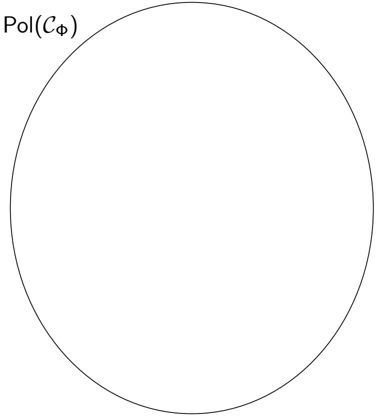
Theorem (Bodirsky-Pinsker-Tsankov, 2010)

Suppose that $\mathscr{G} := \mathrm{Aut}(\mathcal{B})$ *is oligomorphic and extremely amenable. For every* $f \colon B^k \to B$*, there exists a function* $g \in \overline{\mathscr{G} f \mathscr{G}}$ *that is canonical with respect to* $\mathscr{G}$*.*

- $\mathscr{C} \subseteq \mathrm{Pol}(\mathcal{C}_\Phi)$, so simply extend $\mathscr{C} \to \mathscr{P}$ to $\mathrm{Pol}(\mathcal{C}_\Phi) \to \mathscr{P}$!



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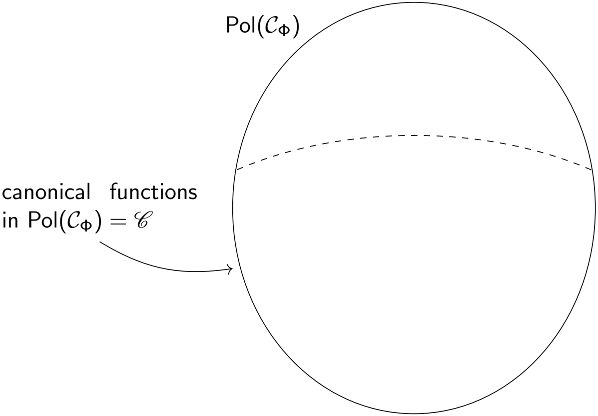

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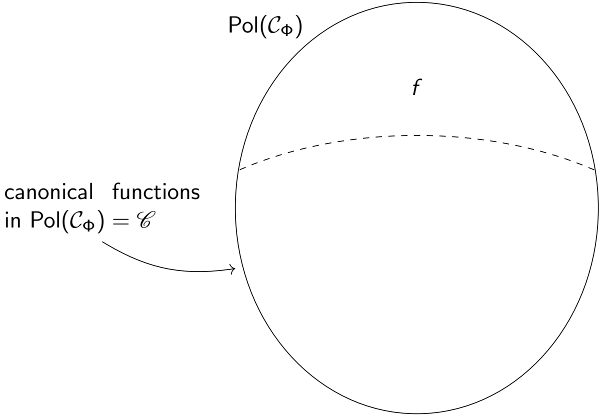

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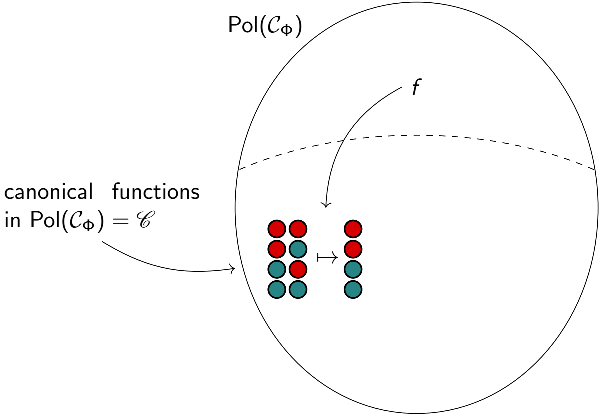

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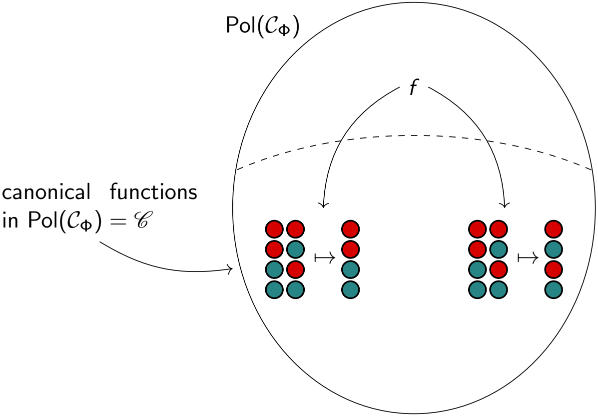

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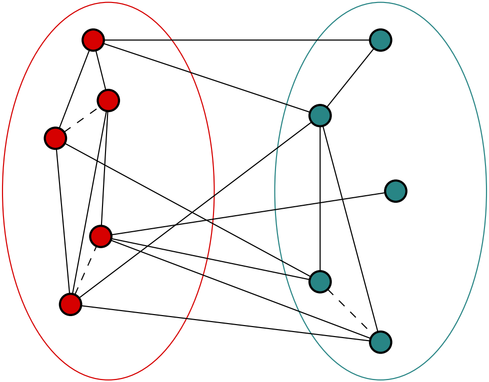

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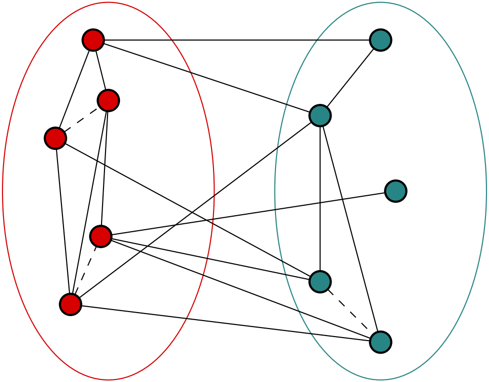

Example, the 2-colour case

- $\Phi = \Phi_{\text{No-mono-tri}}$
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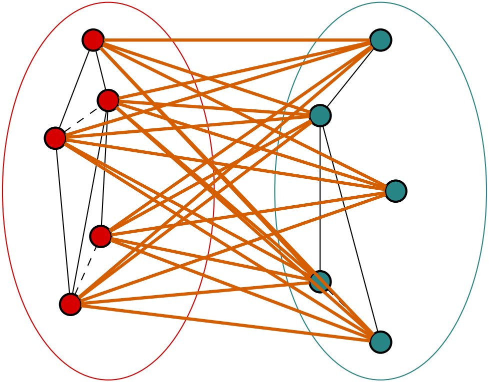
- The relation $N \subseteq (\mathcal{C}_\Phi)^2$ is invariant under $\mathrm{Pol}(\mathcal{C}_\Phi)$.



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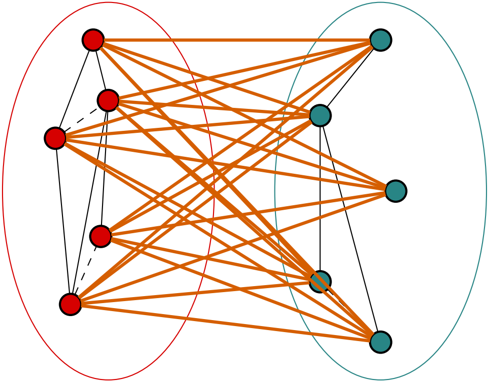
- So $\mathrm{Pol}(\mathcal{C}_\Phi) = \mathscr{C}$!



More colours?

- In general, $\mathscr{C} \subsetneq \mathrm{Pol}(\mathcal{C}_\Phi)$ and N is not invariant under $\mathrm{Pol}(\mathcal{C}_\Phi)$.
- But some almost-bipartite simple graph is.
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Theorem

Φ MMSNP formula. One of the following holds:

(-) *There is a uniformly continuous clonoid homomorphism from* $\mathrm{Pol}(\mathcal{C}_\Phi)$ *to $\mathscr{P}$, or*

(+) $\mathrm{Pol}(\mathcal{C}_\Phi)$ *contains a pseudo-cyclic operation.*