Reducts of finitely bounded structures, and lifting tractability from finite-domain constraint satisfaction

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Outline

- Relational structure: $\mathcal{A} = (A, R_1^{\mathcal{A}}, \ldots, R_k^{\mathcal{A}})$ with $R_i^{\mathcal{A}} \subseteq A^{r_i}$

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Let $\mathcal{A}$ be a relational structure, in a fixed finite signature τ. We consider the following problem:

Definition (Hom($\mathcal{A}$))

Input: a finite τ-structure $\mathcal{B}$
Question: decide the existence of a homomorphism $\mathcal{B} \to \mathcal{A}$.

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Definition (CSP($\mathcal{A}$))

Input: a first-order sentence ϕ of the form

$$\exists x_1 \ldots \exists x_k \bigwedge_i R_i(x_{i1}, \ldots, x_{is_i}) \qquad (R_i \in \tau)$$

Question: $\mathcal{A} \models \phi$?

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CSP($\mathcal{A}$) and Hom($\mathcal{A}$) are equivalent.

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Main tool: universal-algebraic approach.

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Suppose that $\mathcal{A}$ has $f(x_1, \dots, x_k) = a$ as a polymorphism for some $a \in \mathcal{A}$. If $h \colon \mathcal{B} \to \mathcal{A}$ is a homomorphism, then $f(h, \dots, h) \colon x \mapsto a$ is also a homomorphism.

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Concrete example: a group G, $f \colon x \mapsto e_G$.

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Let $\mathcal{A}$ be a finite structure. Then:

- *$\mathcal{A}$ has a cyclic polymorphism – something satisfying*

  $$f(x_1, \dots, x_k) = f(x_2, \dots, x_k, x_1),$$

  and $\mathrm{CSP}(\mathcal{A})$ *is in P,*
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  - structures for which the CSP is "manageable": finitely bounded structures,
  - structures for which finite substructures can move around: homogeneity

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All we need for the induction is the following:

Definition

A class $\mathcal{K}$ of finite structures is said to be an amalgamation class if for every $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in \mathcal{K}$

$$\begin{array}{ccc} \mathcal{A} & & \mathcal{B}_1 \\ & & \\ \mathcal{B}_2 & & \end{array}$$

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$$
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Finitely bounded structures

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Lemma

Let $\mathcal{A}$ be a finitely bounded homogeneous structure, and let $\mathcal{B}$ be first-order definable in $\mathcal{A}$. Then CSP($\mathcal{B}$) *is in NP.*

Definition

A function $f\colon A^k \to A$ is canonical with respect to $\mathcal{A}$ if for every finite $S \subset A$ and $\alpha_1, \ldots, \alpha_k \in \mathrm{Aut}(\mathcal{A})$, there exists $\beta \in \mathrm{Aut}(\mathcal{A})$ such that

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Canonical functions

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Example

Let $\mathcal{G}$ be the countable random graph. There is an infinite clique in $\mathcal{G}$, say K_∞. Then an injection $f\colon \mathcal{G} \to K_\infty$ is canonical with respect to $\mathcal{G}$.

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- *there is a polynomial-time reduction from $\mathrm{CSP}(\mathcal{A})$ to $\mathrm{CSP}(T_{\mathcal{B}}(\mathcal{A}))$.*

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Proposition

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Proposition

No (in general).

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Question: find the biggest class of structures for which CSP($\mathcal{A}$) and CSP($T_{\mathcal{B}}(\mathcal{A})$) are equivalent. This class contains (a posteriori):

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Theorem (Bodirsky-M, LICS '16)

Let $\mathcal{A}$ *be a finite-signature structure that is first-order definable in* $\mathcal{B} = (\mathbb{N}; 0, 1, \dots)$*. Then* $\mathrm{CSP}(\mathcal{A})$ *and* $\mathrm{CSP}(T_{\mathcal{B}}(\mathcal{A}))$ *are polynomial-time equivalent.*

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Corollary

The algebraic conjecture for finite-domain CSPs is equivalent to the statement: if $\mathcal{A}$ is definable in $(\mathbb{N}, 0, 1, \dots)$, then:

- *if $\mathcal{A}$ has a cyclic polymorphism modulo endomorphisms, then* $\mathrm{CSP}(\mathcal{A})$ *is in P,*
- *or* $\mathrm{CSP}(\mathcal{A})$ *is NP-hard.*

Proof based on clone homomorphisms.

Conclusion

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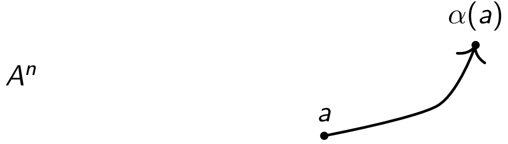
Conjecture

Let $\mathcal{A}$ be a finite-signature structure such that $\mathrm{CSP}(\mathcal{A})$ is definable in MMSNP. Then there is a polynomial-time reduction from $\mathrm{CSP}(T(\mathcal{A}))$ to $\mathrm{CSP}(\mathcal{A})$.

Let $\mathcal{A}$ be a structure. There is a natural partition of A^n for all n:



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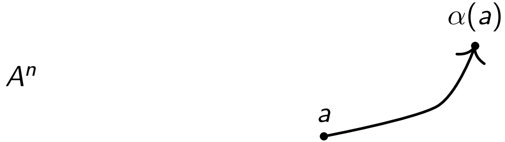

Example

Let $\mathcal{A} = (\mathbb{Q}; <)$. $\mathrm{Aut}(\mathcal{A})$ is the group of increasing bijections on $\mathbb{Q}$.

- $n = 1$: only one orbit
- $n = 2$: three orbits given by $x < y$, $x = y$, $x > y$.

The orbit structure

The domain of $T_{\mathcal{B}}(\mathcal{A})$ is the set of orbits of A^m under $\mathrm{Aut}(\mathcal{B})$.
The relations of $T_{\mathcal{B}}(\mathcal{A})$ are:

- "the tuples of the orbit x is in the relation R of $\mathcal{A}$", for each relation of $\mathcal{A}$,
- "the tuples of the orbits x and y are in the same orbit, if we restrict them to I and J", for each $I, J \subseteq \{1, \ldots, m\}$.