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CMSE

HW7

1

1.1 Adams Bashforth

1.1.1 1 step

$$\sigma(\zeta) = 1$$
$$\rho(\zeta) = \zeta - 1$$

1.1.2 2 step

$$\sigma(\zeta) = \frac{-1}{2} + \frac{3}{2}\zeta$$
$$\rho(\zeta) = \zeta^2 - \zeta$$

1.1.3 3 step

$$\sigma(\zeta) = \frac{5}{12} - \frac{16}{12}\zeta + \frac{23}{12}\zeta^2$$
$$\rho(\zeta) = \zeta^3 - \zeta^2$$

1.2 Adams Moulton

1.2.1 1 step

$$\sigma(\zeta) = \frac{1}{2} + \frac{1}{2}\zeta$$
$$\rho(\zeta) = \zeta - 1$$

1.2.2 2 step

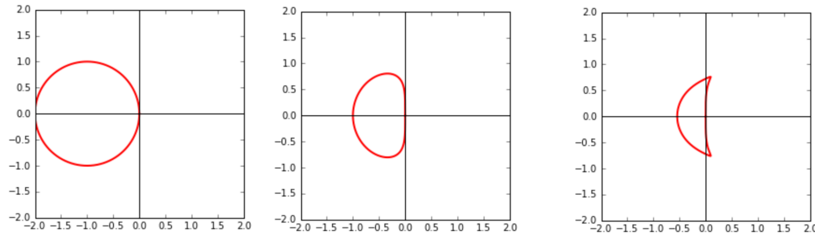
$$\sigma(\zeta) = \frac{1}{12}(-1 + 8\zeta + 5\zeta^2)$$
$$\rho(\zeta) = \zeta^2 - \zeta$$

1.2.3 3 step

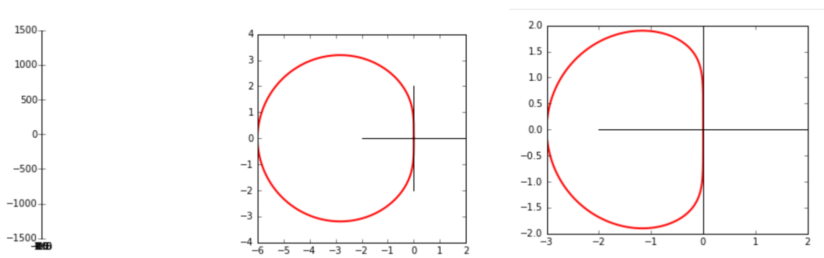
$$\sigma(\zeta) = \frac{1}{24}(1 - 5\zeta + 19\zeta^2 + 9\zeta^3)$$
$$\rho(\zeta) = \zeta^3 - \zeta^2$$

1.3

Adam-Bashforth figures



They are ordered, 1-, 2-, and 3- step.
Adam-Moulton figures



They are ordered, 1-, 2-, and 3-step.
Region of stability decreases as accuracy increases. Region of stability by the way is within the red shapes. For the Adam-Moulton 1 step I believe it is the entire left side of the axis.

2 Crank Nicolson LTE

$$\frac{U_i^{n+1} - U_i^n}{k} = \frac{1}{2} D^2 U_i^n + \frac{1}{2} D^2 U_i^{n+1}$$
$$D^2 U = \frac{1}{h^2}(U(\bar{x} - h) - 2U(\bar{x}) + U(\bar{x} + h))$$
$$\tau = \frac{U(x, t + k) - U(x, t)}{k} - \frac{1}{2h^2}(U(x - h, t) - 2U(x, t) + U(x + h, t))$$
$$-\frac{1}{2h^2}(U(x - h, t + k) - 2U(x, t + k) + U(x + h, t + k))$$

Expansions:

$$U(x + h, t) = U(x, t) + hU_x(x, t) + \frac{h^2}{2}U_{xx}(x, t) + \frac{h^3}{3!}U_{xxx}(x, t) + \frac{h^4}{4!}U_{xxxx}(x, t) + \mathcal{O}(h^5)$$
$$U(x, t) = U(x, t)$$
$$U(x - h, t) = U(x, t) - hU_x(x, t) + \frac{h^2}{2}U_{xx}(x, t) - \frac{h^3}{3!}U_{xxx}(x, t) + \frac{h^4}{4!}U_{xxxx}(x, t) + \mathcal{O}(h^5)$$

Expansions:

$$U(x+h,t+k) = U(x,t) + hU_x(x,t) + kU_t(x,t) + \frac{h^2}{2}U_{xx}(x,t) + \frac{k^2}{2}U_{tt}(x,t) + hkU_{tx}(x,t) + \frac{h^3}{6}U_{xxx}(x,t)$$
$$+\frac{h^2k}{2}U_{xxt}(x,t) + \frac{hk^2}{2}U_{xtt}(x,t) + \frac{k^3}{6}U_{ttt}(x,t) + \mathcal{O}$$
$$U(x-h,t+k) = U(x,t) - hU_x(x,t) + kU_t(x,t) + \frac{h^2}{2}U_{xx}(x,t) + \frac{k^2}{2}U_{tt}(x,t) - hkU_{tx}(x,t)$$
$$-\frac{h^3}{6}U_{xxx}(x,t) + \frac{h^2k}{2}U_{xxt}(x,t) - \frac{hk^2}{2}U_{xtt}(x,t) + \frac{k^3}{6}U_{ttt}(x,t) + \mathcal{O}$$

Expansions:

$$U(x,t+k) = U(x,t) + kU_t(x,t) + \frac{k^2}{2}U_{tt}(x,t) + \frac{k^3}{3!}U_{ttt}(x,t) + \frac{k^4}{4!}U_{tttt}(x,t) + \mathcal{O}(k^5)$$

$$\tau = \frac{U(x,t) + kU_t(x,t) + \frac{k^2}{2}U_{tt}(x,t) + \frac{k^3}{3!}U_{ttt}(x,t) + \frac{k^4}{4!}U_{tttt}(x,t) + \mathcal{O}(h^5) - U(x,t)}{k}$$
$$-\frac{1}{2h^2}(U(x-h,t) - U(x,t) + U(x+h,t)) - \frac{1}{2h^2}(U(x-h,t+k) - U(x,t+k) + U(x+h,t+k))$$
$$= \frac{kU_t(x,t) + \frac{k^2}{2}U_{tt}(x,t) + \frac{k^3}{3!}U_{ttt}(x,t) + \frac{k^4}{4!}U_{tttt}(x,t) + \mathcal{O}(k^5)}{k}$$
$$-\frac{1}{2h^2}(U(x-h,t) - U(x,t) + U(x+h,t)) - \frac{1}{2h^2}(U(x-h,t+k) - U(x,t+k) + U(x+h,t+k))$$

In the interest of saving space just focus on this part:

$$U(x-h,t) - 2U(x,t) + U(x+h,t)$$
$$= U(x,t) - hU_x(x,t) + \frac{h^2}{2}U_{xx}(x,t) - \frac{h^3}{3!}U_{xxx}(x,t) + \frac{h^4}{4!}U_{xxxx}(x,t) - 2U(x,t)$$
$$+U(x,t) + hU_x(x,t) + \frac{h^2}{2}U_{xx}(x,t) + \frac{h^3}{3!}U_{xxx}(x,t) + \frac{h^4}{4!}U_{xxxx}(x,t) + \mathcal{O}(h^5)$$
$$= h^2U_{xx}(x,t) + \frac{h^4}{12}U_{xxxx}(x,t) + \mathcal{O}(h^5)$$

In the interest of saving space just focus on this part:

$$U(x-h,t+k) - 2U(x,t+k) + U(x+h,t+k)$$
$$= U(x,t) - hU_x(x,t) + kU_t(x,t) + \frac{h^2}{2}U_{xx}(x,t) + \frac{k^2}{2}U_{tt}(x,t) - hkU_{tx}(x,t)$$
$$-\frac{h^3}{6}U_{xxx}(x,t) + \frac{h^2k}{2}U_{xxt}(x,t) - \frac{hk^2}{2}U_{xtt}(x,t) + \frac{k^3}{6}U_{ttt}(x,t) - 2U(x,t) - 2kU_t(x,t) - \frac{2k^2}{2}U_{tt}(x,t) -$$
$$\frac{2k^3}{3!}U_{ttt}(x,t) - \frac{2k^4}{4!}U_{tttt}(x,t) + U(x,t) + hU_x(x,t) + kU_t(x,t) + \frac{h^2}{2}U_{xx}(x,t) + \frac{k^2}{2}U_{tt}(x,t) + hkU_{tx}(x,t)$$
$$+\frac{h^3}{6}U_{xxx}(x,t) + \frac{h^2k}{2}U_{xxt}(x,t) + \frac{hk^2}{2}U_{xtt}(x,t) + \frac{k^3}{6}U_{ttt}(x,t) + \mathcal{O}$$

$$= h^2 U_{xx}(x,t) + h^2 k U_{xxt}(x,t) + \mathcal{O}$$

I added in higher order terms just incase and already crossed them out.

$$= h^2 U_{xx}(x,t) + h^2 k U_{xxt}(x,t)$$
$$+ \frac{h^4}{12} U_{xxxx}(x,t) + \frac{h^2 k^2}{2} U_{xxtt}(x,t) + \mathcal{O}$$

Back to the main equation:

$$= U_t(x,t) + \frac{k}{2} U_{tt}(x,t) + \frac{k^2}{3!} U_{ttt}(x,t) + \frac{k^3}{4!} U_{tttt}(x,t)$$
$$- \frac{1}{2h^2} (h^2 U_{xx}(x,t) + \frac{h^4}{12} U_{xxxx}(x,t))$$
$$- \frac{1}{2h^2} (h^2 U_{xx}(x,t) + h^2 k U_{xxt}(x,t) + \frac{h^4}{12} U_{xxxx}(x,t) + \frac{h^2 k^2}{2} U_{xxtt}(x,t) + \mathcal{O})$$

I took out all the terms that don't have a h or k in front of it because it won't contribute to LTE.

$$= U_t(x,t) + \frac{k}{2} U_{tt}(x,t) + \frac{k^2}{3!} U_{ttt}(x,t) + \frac{k^3}{4!} U_{tttt}(x,t)$$
$$- \frac{1}{2} (U_{xx}(x,t) + \frac{h^2}{12} U_{xxxx}(x,t))$$
$$- \frac{1}{2} (U_{xx}(x,t) + k U_{xxt}(x,t) + \frac{h^2}{12} U_{xxxx}(x,t) + \frac{k^2}{2} U_{xxtt}(x,t) + \mathcal{O})$$

Then I used the fact that $U_t = U_{xx}$ so $U_{tt} = U_{xxt}$

$$= \frac{k^2}{3!} U_{ttt}(x,t) + \frac{k^3}{4!} U_{tttt}(x,t)$$
$$- \frac{1}{2} (\frac{h^2}{12} U_{xxxx}(x,t))$$
$$- \frac{1}{2} (\frac{h^2}{12} U_{xxxx}(x,t) + \frac{k^2}{2} U_{xxtt}(x,t) + \mathcal{O})$$

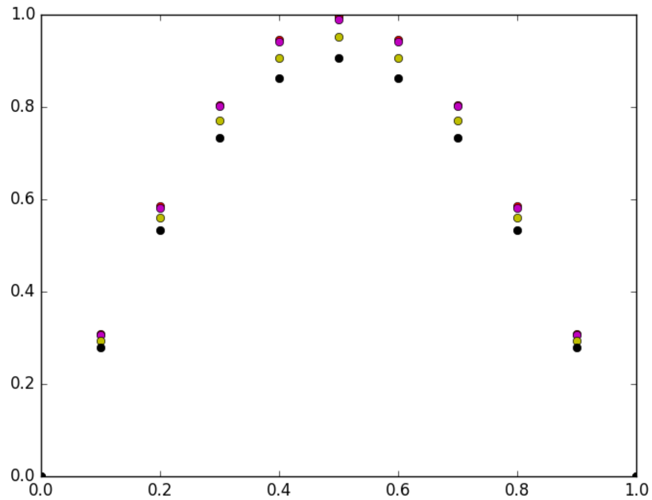
Therefore error is of order $\mathcal{O}(h^2 + k^2)$.

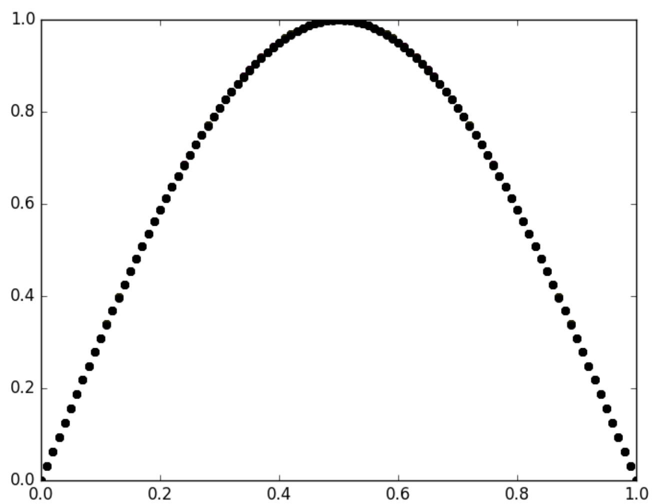
3 Solve equations using MOL

Stability condition is below

$$\frac{k}{h^2} \leq \frac{1}{2}$$

For given h=0.1, $k_{max} = 0.005$. For given h=0.01, $k_{max} = 0.00005$





Points are extremely close together in the 2nd graph. Pictured here is the graph of U(x,t) (Temperature equation?) for different t. The same t values were used for the 1st and second graph.