Perfect Tracing

Preface

This is a tool that calculates the expected cost (and variance) of an attempt to perfect-trace an equipment, taking into account costs of Spell Traces, Innocence Scrolls, Clean Slates, etc.

These calculations are done **analytically** rather than through simulations. The logic behind these will be explained here.

The rational player

Suppose you've set out to fully scroll a piece of equipment. You throw a bunch of spelltraces, but only 4 out of 8 slots pass. You have a decision here; do you throw an Innocence and try to get 5/8 and above, or do you just brute force the remaining slots with Clean Slates?

It's a decision that balances the costs of Clean Slates against those of Innocence Scrolls. Assuming the player is rational (very strong assumption, I know), the decision to Innocence or CSS will depend purely on the current state of the equipment; that is to say, the number of slots that passed or failed. This leads to the idea of a "target" number of slots; if you hit this number of success, you don't inno; if you miss it, you do inno.

Alternatively, we may formulate it in terms of the number of *fails* instead; if we fail a certain number of scrolls F, we inno; otherwise, we don't. Let p be the probability of success of whatever scroll/spelltrace we use, and n be the base number of slots on the equip (without considering hammers)

Since an Innocence Scroll sets us back at square one, we can try to investigate the probability behind a single *cycle*, basically just a round of throwing scrolls and seeing what sticks.

The probability table will look something like this:

Outcome	**Probability** q	**Scrolls**	**Hammers**	**Inno?**
F fails in a row	$(1-p)^F$	F	0	Y
1 pass before F fail	$\binom{F}{F-1}p(1-p)^F$	$F+1$	0	Y
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-F$ pass, F fail	$\binom{n-1}{F-1}p^{n-F}(1-p)^F$	n	0	Y
$F-1$ fail, rest pass, hammer fails	$\frac{1}{2}\binom{n}{F-1}p^{n-F+1}(1-p)^{F-1}$	n	1	Y
$F-1$ fail, rest pass, hammer + fail	$\frac{1}{2}\binom{n}{F-1}p^{n-F+1}(1-p)^{F}$	$n+1$	1	Y
$F-1$ fails, rest pass, hammer + pass	$\frac{1}{2}\binom{n}{F-1}p^{n-F+2}(1-p)^{F-1}$	$n+1$	1	N
$n-F+2$ passes, hammer fails	$\frac{1}{2}\binom{n}{F-2}p^{n-F+2}(1-p)^{F-2}$	n	1	N
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The hammer makes things complicated, but only slightly. It is possible to ignore the hammer entirely and pretend that the equipment has $n+1$ slots, since so few of them are needed on average, but we'll leave it in for the purpose of being thorough.

Sum up the probabilities in the rows that have "N" on inno, and call this some P. This is the chance, per cycle, that we decide not to use an Innocent Scroll.

The total cost T of perfect-tracing can hence be modelled as

$$T = I + c(A) + C$$

Where I is the resources (in units of scrolls, hammers, innos, css's) spent on outcomes that were rejected, A is the resources spent on the eventual-accepted outcome, and C is the amount spent on the final step of filling up failed slots with CSS and scrolls. Note that while I are independent of A and C, C is dependent on A.

I can be modelled as $\sum_{i=1}^{N} Q_i$, where N is a $Geom(P)$ random variable (here we use the convention where geometric r.v.s can be 0-valued). This is the number of times we'll end up inno-ing. For each of those N cycles before we Inno, the costs incurred are a random variable Q sampled according to the probability q in the table amongs the rows with Y under Inno, and we let Q_i be i.i.d. realisations of Q.

The following result will be used many times throughout this calculation:

Suppose that $(X_i)_{i\geq 1}$ are i.i.d., and Z is an independent $Geom_1(p)$ random variable. Then if $A = \sum_{i=1}^{Z} X_i$,

$$\begin{aligned}
\mathbb{E}(A|Z = n) &= n\mathbb{E}(X) \\
\mathrm{Var}(A|Z = n) &= n\mathrm{Var}(X) \\
\mathbb{E}(A) &= \sum_{n=1}^{\infty} \mathbb{E}(A|Z = n)\mathbb{P}(Z = n) \\
&= \mathbb{E}(Z)\mathbb{E}(X) \\
\mathrm{Var}(A) &= \mathbb{E}(Z)\mathrm{Var}(X)
\end{aligned}$$

Hence the mean and variance of I are $\frac{1-P}{P}\mathbb{E}(Q)$ and $\frac{1-P}{P}\mathrm{Var}(Q)$ respectively.

After we have inno'd for the Nth time, we finally hit on an "accepted" non-inno state; this state A is sampled from the probabilities under the rows with N under Inno, and has its own cost associated with it, $c(A)$.

C depends on the number of failed slots, s, which depends on A. C can be written as $\sum_{i=1}^{s} K_i$, where K is the number of Clean Slates and stat-scrolls spent on the equipment.

K has two components, the number of Clean Slates spent and the number of scrolls spent. The number of Clean Slates is itself a sum of independent $Geom(CSS)$ variables where CSS is the success chance of a Clean Slate (0.05 using traces outside of fever, or 0.1 during fever/using css10s). Hence $\mathbb{E}(K) = \frac{1}{p \times CSS}$, $\mathrm{Var}(K) = \frac{1-CSS}{p \times CSS^2}$, while the number of scrolls spent is a simple $Geom(p)$ random variable.

So, given an outcome A, and the number of failed slots $s(A)$ which depends on A, we have:

$$\begin{aligned}
\mathbb{E}(C|A = a) &= \frac{s(a)}{p \times CSS} \\
\mathrm{Var}(C|A = a) &= \frac{s(a)(1 - CSS)}{p \times CSS^2}
\end{aligned}$$

Note that since C is independent of the history A, I when conditioned on $A = a$ we can just add up the means and variances together to get the mean and variance of T.