Topological Phases of Matter - A Whirlwinding Number Review

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Abstract

Topological phases of matter are an active area of research in modern physics. To the unindoctrinated, several immediate questions come to mind: is there a deeper meaning behind the use of the word topological? What is all the buzz about? What is the relevant mathematical structure? This review comes from a quantum information theorist, and attempts to provide a minimal scaffolding to address these questions.

I. INTRODUCTION

The discovery of the integer quantum Hall effect [1] began the study of topological phases of matter. It was found that applying a magnetic field to a material can induce conducting edge states, leaving the bulk an insulator [2], and that this phenomenon is robust to perturbations and sample imperfections. Topological phases of matter have several characteristic properties, among which are a bulk-boundary correspondence and insensitivity to continuous deformations of the Hamiltonians that leave the gap open. In this paper we review the essential physics of the integer quantum Hall effect, which leads us naturally to a discussion of geometric phase. After developing Berry's theory, we present a minimal physical model for a system with both topological invariants and a Berry phase. After this, we consider a slightly more detailed model - the Su-Schrieffer-Heeger model - that allows us to discover more features of topological systems, such as a bulk boundary correspondence between parity of the number of edge states and a winding number of the Hamiltonian. Finally, we conclude with a formal discussion of topological phases, and a high-level derivation of the so-called periodic table of topological insulators, developed by Altland and Zirnbauer [3], Ryu et. al [4], and Kitaev [5].

Topological phases of matter are an exciting topic in modern physics, with interesting theoretical behavior that also gives way to novel experimental phenomena. Many of these properties suggest that these materials will have practical utility. Even more interestingly, as a researcher in quantum information, they are relevant to understanding modern advances in the field, such as topological entanglement entropy. This review thus aims to serve as a minimal entry point that captures the *je ne sais quoi* of both topological phases, and their algebraic topological nature. Thus, rather than being comprehensive, this review aims to be self-contained enough to suggest that topological phases are rich and interesting, but makes no promises of self-contained enough to be pedagogical or not-requiring standard references. An analogy the author like to consider is digging a deep hole, with only the minimal supporting buttresses to avoid collapse.

In this paper we review the integer quantum Hall effect, and Berry phase - two early and fundamentally topological and geometric physical phenomena. After understanding these two examples, we move on to exploring a simple topological model that gives the Chern number as a topological invariant and the Su-Schrieffer-Heeger (SSH) model which hosts a bulk-boundary correspondence. We conclude with a complete mathematical classification of topological phases of non-interacting matter.

II. THE INTEGER QUANTUM HALL EFFECT

The integer quantum Hall effect (IQHE) is the quantum generalization of the classical Hall effect. To this end, we start by considering an electron confined to two dimensions, with a magnetic field pointing perpendicular to the plane. Without loss of generality, we take the magnetic field to point in the Z-direction. Then, we can choose the Landau gauge to give the vector potential as $A = (0, xB, 0)$. In the presence of a vector potential, the momentum transforms as the gauge covariant derivative, namely $\partial \to \partial + iA$, so that the Schrödinger equation is separable and is given in one dimension as

$$-\frac{\hbar^2}{2m_e}\frac{\partial^2\Phi(x)}{\partial x^2} + \frac{\hbar^2}{2m_e}(k_y + \frac{eB}{\hbar}x)^2\Phi(x) = E\Phi(x). \tag{1}$$

This has solutions that are those of a simple harmonic oscillator, and so if we impose periodic boundary conditions in the other two dimensions we arrive at a model with Landau levels - the harmonic oscillator has levels separated by $\hbar\omega$ that are independent of k_y. Thus, calling L_y the length of the sample in the y direction and L_x the length in the x direction, we can approximate that $|k_y| \leq eBL_y/(2\hbar)$, and is quantized in multiples of $2\pi/L_x$.

To study the microscopic physics of the IQHE we now apply a transverse electric field. We note that the magnetic field can be recast as an applied electric field via the Lorentz force. Suppose there is a current density $j_y = \sigma_0 E_y$. Then we have that

$$j = \sigma_0(E - \frac{j \times B}{n_e e}), \tag{2}$$

where n_e is the electron density. From our previous argument, we know that the number of possible k_y associated with each Landau level is

$$N_{k_y} = \frac{eB \times L_x L_y}{2\pi\hbar} \tag{3}$$

and so the electron density is of the form

$$n_e = f\frac{eB}{\hbar} \tag{4}$$

and we have that [6]

$$\sigma_{yy} = \frac{j_y}{E_y} = \frac{n_e e^2 t_0/m_e^*}{1 + (et_0 B/m_e^*)^2}, \tag{5}$$

where we have used the Hall conductance

$$\sigma_0 = \frac{n_e e^2 t_0}{m_e^*}, \tag{6}$$

with the effective electron mass in the material.

First, we consider σ_{yy}, the longitudinal conductance - experimentally we see that this is zero [7]. We can see this should be the case - resistance is caused by scattering events in the material, and during a scattering event electrons scatter to different energy levels. If the temperature is kept low enough, scattering to high levels is energetically forbidden, and other states at and below the Fermi level are filled, so that the time between collisions, t_0 should go to infinity. Comparing with our expression for σ_{yy} we see then that $\sigma_{yy} \to 0$. Thus the material is an inductor in the longitudinal direction.

The transverse component of the conductivity is given as [6]

$$\sigma_{xy} = \frac{j_x}{E_y} = f\frac{e^2}{h}(1 - \frac{1}{1 + (et_0 B/m^* e)^2}), \tag{7}$$

and to argue that the proportionality constant f should be quantized requires a more involved argument, attributed to Laughlin [8]. We will present this argument after a short review of the Berry phase.

III. THE ADIABATIC APPROXIMATION

Berry's guess at the form of the phase acquired in a closed loop was motivated by physical reasoning, as we will see now. The following discussion of the adiabatic approximation follows closely from [9]. Consider a Hamiltonian, parameterized by a time dependent control, $s(t)$, $H(s(t))$. The adiabatic approximation follows intuition from the adiabatic theorem - if the control is varied "slowly enough", the system should stay in the ground state, and should thus only acquire a phase. "Slow enough" in this context means relative to any energy differences in the system - thus this result will not hold for systems with degeneracies.

$H(s(t))$ is Hermitian and therefore admits an instantaneous orthonormal basis. Consequently, we can relate the instantaneous eigenbasis to the initial eigenbasis $\Phi(s(0))_n$ via a unitary transformation $U(t)$. Intuitively, moving into this basis will let us "track" the lowest energy eigenstate. A wavefunction in a rotating frame satisfies Schrödinger's equation, as can be seen from the following argument

$$i\hbar\dot{\tilde{\psi}} = i\hbar(\dot{U}(t)\psi + U(t)\dot{\psi}) = i\hbar\dot{U}(t)U^\dagger(t)\tilde{\psi} + U(t)H(s(t))U^\dagger\tilde{\psi}. \tag{8}$$

Thus we see in the rotating frame, there is an effective Hamiltonian

$$H_{\text{eff}} = i\hbar\dot{U}(t)U^\dagger(t) + U(t)H(s(t))U^\dagger. \tag{9}$$

While the first term is rapidly oscillating, it cannot be ignored. This is because the rapidity of the oscillations is mollified by the length of the interval of integration. We would therefore like to again change into a frame that tracks the slow dynamics, so that the only relevant time-scale is the slow dynamics.

Naming

$$\Delta(t) = i\hbar\dot{U}(t)U^\dagger(t)\tilde{\psi}, \tag{10}$$

the Hamiltonian in the rotating frame is given by

$$H_{\text{rot}}(t) = V(t)\Delta(t)V^\dagger(t), \tag{11}$$

where $V(t)$ is the propagator of the Schrödinger equation with Hamiltonian

$$\tilde{H}(t) = U(t)H(s(t))U^\dagger(t). \tag{12}$$

In fact, inspection of this expression reveals that the time evolution under $V(t)$ gives nothing other than the "dynamical phase", accumulating proportionately to the instantaneous eigenenergy. Thus, it's easily seen that the evolution in the rotating frame is proportional to the difference of these dynamical phases - the diagonal elements accumulate no dynamical phase in this basis, as the basis tracks their evolution.

In particular, we find that $\Delta(t)$ in this frame evolves as

$$\Delta_{nm}(t) = \exp\left(\frac{i}{\hbar}\int_0^t (E_n(s(t)) - E_m(s(t)))dt\right). \tag{13}$$

If the gap is much larger than any fluctuations of the gap, then this will integrate to zero - this is the adiabatic approximation. Under this approximation, we arrive at a functional form for the evolution of a quantum state evolved along a circuit as

$$\psi(t) = \exp(i\gamma(t))\exp(i\phi(t))\psi(0), \tag{14}$$

which we see gives a phase, specifically the geometric phase, in addition to the expected dynamical phase. We can further derive the form of the geometric phase, via the Schrödinger equation. We find that

$$\dot{\gamma}(t) = i\left\langle n(R(t))|\nabla_R n(R(t))\cdot\dot{R}(t)\right\rangle. \tag{15}$$

This of course can be integrated, and together with vector identities and Stokes' theorem, we arrive at

$$\gamma(C) = i \int_C \langle n(R)|\nabla_R n(R)\rangle \cdot dR = -Im \int_C dS \cdot \sum_{m\neq n} \langle \nabla n|m\rangle \times \langle m|\nabla n\rangle \,. \tag{16}$$

Now we return to Laughlin's thought experiment.

IV. LAUGHLIN'S THOUGHT EXPERIMENT

We return to the setting of the IQHE, and consider a sheet of material - to study the current, we want to look at the expectation of the current operator. In electromagnetic contexts, the curvature corresponds to the field strength. We consider then the effect of Hamiltonian with an external flux, parameterized by its angle on the XY-axis. From the Hamilton equations for electromagnetism, we know that current and flux are conjugate variables, namely

$$J_x = c\frac{\partial H}{\partial \Phi_x}. \tag{17}$$

Furthermore, the curvature is given as

$$\partial_{\Phi_x} \langle \psi|\partial_{\Phi_y}\psi\rangle - \partial_{\Phi_y} \langle \psi|\partial_{\Phi_x}\psi\rangle = 2iIm \langle \partial_{\Phi_x}\psi|\partial_{\Phi_y}\psi\rangle = \mathcal{K}(\Phi_x, \Phi_y). \tag{18}$$

We would like to compute this quantity. To do this, we imagine deforming the plane to enclose a flux above the plane in the x-direction, and increasing it by a flux quantum, and then opening and deforming it to enclose the a flux above the plane in the y-direction and doing the same. If the system is large enough, the change in boundary conditions can be neglected [10], and if the system is deformed adiabatically, we expect there to only be a gauge-invariant Berry phase at the end. Varying Φ_x the Schrödinger equation gives

$$i\hbar \langle \partial_{\Phi_y}\psi|\partial_t\psi|\partial_{\Phi_x}\psi\rangle = \langle \partial_{\Phi_y}\psi|H\psi\rangle \tag{19}$$

Now, by adding this equation to its own complex conjugate, using the the Feynman-Hellman theorem, and using the fact that the ground state energy is independent of the flux, we arrive at

$$\langle \psi|J_x|\psi\rangle = c\hbar\partial_t\phi_x\mathcal{K}(\Phi_x, \Phi_y) \tag{20}$$

Finally, we have that the transverse Hall conductivity is given as

$$\sigma_{xy}(\Phi_x, \Phi_y) = \hbar c^2\mathcal{K}(\Phi_x, \Phi_y) \tag{21}$$

Intuitively we are done, since the Gauss-Bonet-Chern formula [11] tells use the integral of a curvature K over a closed surface is

$$\frac{1}{2\pi}\int\int Kds = 2 - 2g \tag{22}$$

where g is the genus of the surface. In our example, the two fluxes Φ_x, Φ_y parameterize a Hilbert space curvature, so while the quantization holds, it's not given directly by the genus of the phase space torus. So we see that we should expect the Hall conductance to be quantized. As we will see later, this quantization is no coincidence - the IQHE can be described with a topological invariant called a Chern number.

V. A SIMPLE TOPOLOGICAL MODEL

The key essence of the geometric phase derived in the previous two sections can be understood via simple model, presented in [2]. Consider a quantum systems with three states, and a process that moves the system between three states, acquiring a phase at each step.

Quantum mechanics is a $U(1)$ gauge theory, with the manifold of $\mathbb{RP}(1)$ as the base space, and $U(1)$ forming the fibers, together giving a Hermitian line bundle, with the standard inner product between quantum states on fibers. The phase associated with any state is gauge dependent, and therefore nonphysical, but we should expect that the phase acquired around a loop is gauge invariant. Indeed, each choice of local gauge will appear twice, with opposite sign. In our original problem, taking the continuum limit, we see that we should also expect the evolution to contribute a gauge invariant phase, since the Schrödinger equation together with the adiabatic theorem suggests that that is the only evolution the state can have along a closed circuit.

Taking our simple model a step further, we can see that this Berry phase really is a topological quantity. Consider, instead of a loop of states, a grid of states as in Fig. 1. By standard Stokes-like arguments, any closed circuit is equivalent to a closed circuit around the entire grid, and the so called Berry flux can be broken into a product of the Berry phases for each plaquette - however we only have equality mod 2π. One place this difference can be seen to come from is two alternative ways to compute the phase around a loop - we can add the argument of the phases, or take the log of the product of the phases - the difference of these two quantities, divided by 2π, is called the number of vortices.

A final interesting twist, that can be physically motivated by Bloch theory, is to consider periodic boundary conditions. If we associated the two pairs of sides, we end up with the topology of a torus. This identification is shown in Fig. 1, where the green sides are associated and the yellow sides are associated. Moreover, the phase along the edges cancel. As previously mentioned, this is only equal to the sum of the Berry phases of the plaquettes mod 2π - in fact this counts the number of vortices. This is the Chern number of this model.

grid.png

FIG. 1. A grid of quantum states that accumulate phase $e^{i\phi_{ij}}$ between nodes i and j. The coloring identifies pairs of edges as equivalent, thereby giving the grid the topology of a torus. Under this equivalence $0 \equiv 6 \equiv 8$, $3 \equiv 5$, and $1 \equiv 7$.

VI. THE IQHE REVISISTED

We will now see that our original example in Sec. II is nothing more than a Chern number. To achieve this, we will consider a tight binding model. We can use a technique called Peierls substitution [12, 13] to replace the applied vector potential with modified hopping potentials. We review this procedure now.

Consider a two-dimensional tight-binding model, given as

$$H = \sum -t\exp\left(-i\theta^y_{m,n}\right)|m+a,n\rangle\langle m,n| + -t\exp\left(-i\theta^x_{m,n}\right)|m,n+a\rangle\langle m,n| + \epsilon|m,n\rangle\langle m,n| + h.c. \tag{23}$$

To lowest order, we have that the matrix elements are proportional to the momentum operator

$$|m+a\rangle\langle m| \sim \left(1 - \frac{ip_x}{\hbar}a - \frac{p_x^2}{2\hbar^2}\right)|m\rangle\langle m| \tag{24}$$

while the phases are expanded with the usual Taylor expansion

$$e^{i\theta} \sim 1 + \frac{iaqA_x}{\hbar} - \frac{a^2q^2A_x^2}{2\hbar^2} \tag{25}$$

Then, to lowest order we find,

$$e^{i\theta} |m+a\rangle \langle m| + h.c. = (-a^2/\hbar^2(p_x - qA_x)^2 + 2) |m\rangle \langle m| , \tag{26}$$

in each dimension, which leads to

$$H = \frac{1}{2m}(p - qA)^2 + \epsilon_0, \tag{27}$$

where $\epsilon_0 = \epsilon - 4$, and $m = \frac{\hbar^2}{2ta^2}$ is the effective mass. Thus we might expect the investigation of a two dimensional tightbinding model to reveal the topological nature of the IQHE. In this review we consider the SSH model, the simplicity of which will ease analysis. We can motivate a reduction to this model, a 1D chain with staggered tunneling amplitudes, as follows. By considering one of the k-coordinates in the Bloch Hamiltonian of a 2D tight-binding model as a drive for a 1D tight-binding model, we recover an instance of the Rice-Mele model. This reduction is for a particular choice of drive, but to simply understand that this model should admit topological behavior we will consider the simplified situation in the next section.

VII. THE SSH MODEL

Consider a 1D chain of sites, given by the Hamiltonian

$$H = \sum u |m, A\rangle \langle m, B| + v |m, B\rangle \langle m+1, A| + h.c, \tag{28}$$

where u, v are hopping potentials. An illustration is provided in Fig. 2, showing the cells m highlighted in purple, with sites A given as red dots, and sites B given as blue dots.

From this model we will derive several characteristic properties of topological materials:

1. A gap.
2. A winding number (in this case, arising from chiral symmetry) - the winding number in fact labels the topological phases, and depends on the relative sizes of u and v.
3. A bulk-boundary correspondence - we can use the winding number of the bulk to make predictions about the low energy physics at the edge (the number of edge eigenstates).

To begin our analysis, we consider the thermodynamics limit of the model with periodic boundary conditions, so that we can ignore the behavior on the edges [10]. This gives a system that admits solutions via Bloch's theorem, as we have the eigenequation

$$\hat{H}(k) = \begin{pmatrix} 0 & u + ve^{-ik} \\ u + ve^{ik} & 0 \end{pmatrix} = d_x\sigma_x + d_y\sigma_y + d_z\sigma_z \tag{29}$$

ssh.png

FIG. 2. The SSH model. Comparing with Eq. 28, sites m are highlighted in purple, with intrasite hopping potentials u. Sites A are colored in red, while sites B are colored in blue, and intersite hopping potentials are labeled with v.

$$\hat{H}(k)\begin{pmatrix} a(k) \\ b(k) \end{pmatrix} = E(k)\begin{pmatrix} a(k) \\ b(k) \end{pmatrix}$$

where this is now a Schrödinger equation in reciprocal space, and $d_x = u + v\cos(k)$, $d_y = v\sin(k)$, $d_z = 0$. This gives a dispersion relation of

$$E(k) = \pm\sqrt{u^2 + v^2 + 2uv\cos(k)}. \tag{30}$$

There are two interesting properties to note - the first, is that for $u = v$, there are Dirac points - the gap closes at $k = -\pi, \pi$ Second, if we plot d_x and d_y as a function of k, the path can be seen to enclose the origin, or not enclose the origin. This value, 0 or 1, is called the winding number, and is an example of a Chern number. The Hamiltonian is seen in Eq. 29 to not have any d_z component, and thus there is a well-defined winding number that labels the topological phase. That the Hamiltonian does not have any d_z component is a consequence of its chiral symmetry, which we discuss now.

Chiral symmetry, also known as sublattice symmetry, is a symmetry of the Hamiltonian that is unitary and transforms the Hamiltonian as

$$\hat{\Gamma}\hat{H}\hat{\Gamma}^\dagger = -\hat{H}. \tag{31}$$

For this model, we see that

$$\hat{\Gamma} = \hat{P}_A - \hat{P}_B \tag{32}$$

where $\hat{P}_A, \hat{P}_B$ are projectors onto the different sublattices. This implies that each nonzero eigenvalue has chiral symmetric negative energy partner, and has equal support on both sublattices. Relatedly, zero energy eigenvalues can be written to have support on only a single sublattice. In the case of the SSH model, chiral symmetry can thus be defined via the operators $\hat{P}_A = \sum |m, A\rangle\langle m, A|$, $\hat{P}_B = \sum |m, B\rangle\langle m, B|$.

At the same time, we can consider finite-size effects. Note that if $u = 0$, the ends of the chain are disconnected, and the model hosts zero-energy eigenstates. This can be seen since in that case

$$\hat{H}|1, A\rangle = \hat{H}|N, B\rangle = 0. \tag{33}$$

These are a trivial example of what are known as edge modes, and a more detailed derivation shows they exist in general, with energies tending to zero in the thermodynamic limit. This follows intuitively, if the bulk Hamiltonian is gapped, the support of zero energy eigenstates has to be exponentially suppressed as you move into the bulk. Moreover we can show they are also a topological invariant - if the gap doesn't close, the difference of the number of edge states on sites A and B must remain constant.

In fact, the presence of these two topological invariants simultaneously is not a coincidence - they are dual. While a general proof of this result is beyond the scope of this paper [14, 15], in this simple case we can show that the two are related - the difference of the number of edge states on sites A and B is equal to winding number of the vector $\vec{d}(k) = (d_x, d_y, d_z)$, and is an element of $\mathbb{Z}_2$. When $u = v$ the gap is closed, when $v > u$ both are one, and otherwise both are zero. That the difference of the number of edge states is a topological invariant is clear - any adiabatic deformation can only create or destroy pairs of edge states since each has a chiral partner. We also see that the edge states can't be pulled into the bulk - the gap prevents any non-negligible support. That there should be a correspondence between topological invariants and edge states is perhaps intuitive - if there is no adiabatic connection between two gapped, inequivalent Hamiltonians that keeps the gap open, then the gap must close somewhere. It's precisely at these points where there are edge states.

This concludes our discussion of the SSH model and the IQHE. We have seen that both of these examples host properties we call *topological phases*. While we have not explicitly motivated this name, some thought reveals that the phases described are robust to imperfections and deformations in the Hamiltonians considered. However, one still may wonder if there is a general theory behind these materials. In the following section we describe one such theory that completely classifies non-interacting topological superconductors and insulators. Moreover, it motivates the name - it relies heavily on the tools of algebraic geometry and K-theory, fundamentally topological fields of mathematics.

VIII. THE PERIODIC TABLE OF TOPOLOGICAL INSULATORS

In contrast to conventional Ginzberg-Landau theory, recent work has studied topological phases of matter which do not have an order parameter in the conventional sense. Thus, the most sensible question to ask seems to be, "in what sense are they phases at all"? The literature suggests that topological phases have a number of signature properties such as degenerate ground states, large scale entanglement, robustness to imperfections and bulk-boundary correspondences. Moreover, as long as the gap doesn't close, the phase remains the same.

While producing an exhaustive of these properties seems challenging, another approach to such a discussion is to give an axiomatic mathematical definition. We would expect from the name, that such a classification would involve topology - we will see that it will in fact involve algebraic topology. We start with some examples of what physicists mean when they say topological phase.

1. The Chern number. The phases are labeled by $\mathbb{Z}$.

2. The Kitaev chain. The phases are labeled by $\mathbb{Z}_2$.

3. A free electron - there is only a topologically trivial phase. The simple picture is given by considering where the topological properties come from. They comes from the nontrivial topology in reciprocal space - for a free electron, the first Brillouin zone is all of $\mathbb{R}^3$, and the Chern numbers take value in the integral over the boundary of the Berry curvature. The curvature is zero - the Hamiltonian has no dependence on space, so that there is only one phase - the topologically trivial phase. The phases are thus labeled by $\{0\}$.

4. Pwave superconductors. with no symmetries. These can be wound around threaded fluxes to give $\mathbb{Z}$ [16].

In the case of non-interacting systems (there are counterexamples with non-interacting systems [17]), there is a classification that captures all known topological phases in terms of their symmetries and dimension, and moreover demonstrates a periodic structure [5]. What this means, specifically, is presented in Fig. **??** taken from [5]. d labels the dimension of the system, so for a Majorana chain $d = 1$, and for the IQHE (in the insulators table) $d = 2$. The diagonals of the table consist of systems with the same symmetries, for example time reversal symmetry or charge conjugation. For example, from BCS theory we have the Hamiltonian for a $p_x + ip_y$ spinless superconductor [16]

$$H = \int d^2r(\psi^\dagger(-\frac{\nabla^2}{2m} - \mu)\psi + \frac{\Delta}{2}(e^{i\phi}\psi(\partial_x + i\partial_y)\psi + h.c)). \quad (34)$$

This Hamiltonian by itself violates time-reversal symmetry, since it can be shown to accumulate a signed relative phase depending on the direction of time. Additionally, attempts to engineer the system will introduce a magnetic field to spin polarize the electrons - this too will break time reversal symmetry. Because Cooper pairs have a net charge and are free to condense, this too violates charge conservation, so that the Hamiltonian is without symmetry.

However, what is shown is that there is a *periodicity* in the classifying space, as shown in the column label $\pi_0(R_q)$. This notation is suggestive - it is the fundamental group of a space arising in K-theory. Later we will suggest that these groups R_q can be derived from properties of the Hamiltonian, but for now we note simply that this periodicity is in fact the same as a periodicity seen in algebraic topology. In particular, Bott showed that

$$\pi_n(O(\infty)) \sim \pi_{n+8}(O(\infty)), \quad (35)$$

with the first eight homotopy groups being given as $\mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$, and $O(\infty)$ being the direct limit of the orthogonal groups - the limiting space by considering injective homomorphisms

$$O(1) \to ... \to O(n) \to ... \quad (36)$$

To better understand the setting for Bott periodicity, we now will discuss elementary ideas in K-theory.

We start, as is the case with Berry phase, by studying vector bundles. A vector bundle is a line bundle, where the fibers are in general vector spaces. We can define an addition on vector bundles to form a semigroup via the Whitney sum - for vector bundles ξ and η we can define a vector bundle $\xi \oplus \eta$ as the vector bundle pull back along the diagonal map - equivalently we can define it as the vector bundle whose fibers above x are $\xi_x \oplus \eta_x$, the direct sum of the fibers above x of the summands. A standard technique for extending semigroups to groups is by establishing equivalence classes, such as $(a, b) \sim (c, d)$ if $a \oplus c \oplus e = b \oplus c \oplus e$, which implies that $(\eta, \xi) = (\xi, \eta)^{-1}$, since $(\eta, \xi) + (\xi, \eta)$ is then equivalent to $(\varepsilon, \varepsilon)$ - two copies of the zero dimensional trivial bundle on the base space, since $\varepsilon \oplus \xi \oplus \eta = \eta \oplus \xi \oplus \varepsilon$. This is equivalent to taking the collection of vector bundles and imposing the equivalence relation that $\eta \sim \xi$ if $\xi \oplus \mathbb{C}^n \sim \eta \oplus \mathbb{C}^n$. This definition allows us to generalize to reduced K-groups, $\tilde{K}$, defined as $\eta \sim \xi$ if $\xi \oplus \mathbb{C}^n \sim \eta \oplus \mathbb{C}^m$, for n not necessarily equal to m. Furthermore we define $\tilde{K}^{-n}(X) = \tilde{K}(S^n X)$ where S is the suspension.

Now, let us study the K-theory of the sphere. We know that for a contractible space, the space of vector bundles is $\mathbb{Z}_{\geq 0}$, which just counts the dimension of the fiber. One way then to study the K-theory on a sphere is to study how the two contractible hemispheres glue together, via a so-called "clutching function".

$$f : S^{n-1} \to GL_k(\mathbb{C}). \quad (37)$$

From this, we find that there is an isomorphism from isomorphism classes of rank k vector bundles on S^n to homotopy classes of maps $S^{n-1} \to GL_k(\mathbb{C})$. Since GL($\mathbb{C}$) is path connected, every bundle over S^1 is trivial, and so $V(S^1)$, the space of vector bundles over the circle, is $\mathbb{Z}_{\geq 0}$. Therefore the reduced K-group is the trivial group. Following a more involved argument, we arrive at the conclusion that $\tilde{K}(S^2) \equiv \mathbb{Z}$, and further there exists an isomorphism from $\tilde{K}(X) \to \tilde{K}(S^2 X)$. But then, since $S^n \equiv S(S^{n-1})$ for n$\geq$ 1, we have the result of Bott periodicity in complex k-theory, that there is a periodicity in the reduced K-groups of the sphere.

But, another fact about cohomology theories is that they have classifying spaces - this follows from the Brown representability theorem. Maps into these classifying spaces are in bijection with the objects themselves, and so for the K-theory of the sphere, BU is the classifying space and so $S^n \to BU$ are the maps, but these are precisely $\pi_n(BU)$. A final fact to put the story together is that $\pi_n(BU) \equiv \pi_{n+1}(U_\infty)$, and so we find the final result for complex K-theory where the objects are homotopy groups of U_∞ rather than O_∞.

Now, the important question to answer is how does this show up in physical systems? Kitaev outlines a construction that gives an answer to this question in [5], in the case of noninteracting systems. We start with a system given by

$$\hat{H} = \sum X_{jk} a_j^\dagger a_k, \tag{38}$$

with X Hermitian. We consider gapped systems, so that we consider a single band where the eigenvalues ε_j can be constrained to be $\Delta \leq |\varepsilon_j| \leq E_{\max}$. We consider a "spectral flattening" procedure that reduces all Hamiltonians to a canonical form, where all positive eigenvalues are sent to +1, and all negative eigenvalues are sent to -1. Now, this family of matrices form the set $\bigcup_{0\leq k\leq n} U(n)/(U(k) \times U(n-k))$. This is seen to be similar to the first classifying space in Fig. **??**. It remains to understand why we would consider the fundamental group of this space - this we will argue now.

We consider two Hamiltonians to be equivalent in this class if they are homotopic. For the collection defined, two Hamiltonians are homotopic if they have the same size and have the same number of negative eigenvalues. We define equivalence between these "admissable" Hamiltonians in a way motivated by K-theory

$$X' \sim X'' \text{ if } X' \oplus Y \equiv X'' \oplus Y \tag{39}$$

for some Y. Finally, we consider difference classes of matrices, given by

$$(A', B') \sim (A'', B'')$$
$$\text{if } A' \oplus B'' \sim A'' \oplus B'.$$

These difference classes are characterized by a single integer, however to characterize difference classes between families of matrices parameterized by Λ, we instead consider functions from Λ to the classifying space given in Fig. **??**. This gives a complete picture of difference classes of parameterized families of Hamiltonians to the classifying spaces considered in complex K-theory. It's interesting to ask, physically, what considering these Hamiltonians up to homotopy means. By direct summing with ancillary systems, we are considering adding trivial flat bands, such as disjoint atomic orbitals. However, by adding on these spaces, we introduce more space to continuously deform one into another.

The conversation is nearly complete in spirit. We have argued how to find one entry in the table - we still don't know how to move between cells in the table. For this, we need to consider symmetries and the dimension of the Hamiltonian. Morally, the introduction of symmetries will change the classifying space, and we can show that adding on dimensions acts like adding on symmetries via the introduction of γ matrices coming from Dirac equations. We will now introduce, briefly, the mathematical formalism for this.

The mathematical object we consider is a Clifford algebra. A Clifford algebra is given by two collections of generators - one set whose elements square to one and one set whose elements square to minus one. Additionally, all non-equal generators must anticommute. Symmetries, such as charge conservation and time reversal, when expressed as matrices acting on fermionic mode operators, can have these properties, with the Hamiltonian represented as a generator as well that squares to minus one (via the introduction of a global phase.)

Finally, we can also write a Dirac operator

$$D = \sum_{\alpha} \gamma_{\alpha\alpha} + M. \tag{40}$$

To see that these appear naturally, consider the Hamiltonian given by

$$H = \frac{i}{2}(u \sum_{l=1}^{n} \hat{c}_{2l-1}\hat{c}_{2l} + v \sum_{l=1}^{n-1} \hat{c}_{2l}\hat{c}_{2l+1}) \tag{41}$$

We can write an effective Hamiltonian near the phase transition as

$$H = \frac{i}{2} \int dx \hat{\eta}^T \begin{pmatrix} \partial & m \\ -m & -\partial \end{pmatrix} \hat{\eta} \tag{42}$$

With $M = mi\sigma^y$ and $\gamma = \sigma^z$. Looking back at Eq. 40, we take these γ to anticommute, and square to the identity, while M squares to the identity, by replacing it with $\tilde{M}$. As described in [5], these act as generators that "cancel out" the existing generators. Physically this means that a continuous spectrum is equivalent (in the sense described above) to a Dirac spectrum with these additional symmetries. Via techniques not discussed here, these Clifford algebras are elements of the classifying spaces in Fig. **??**. While this later step is important, we have included all of the relevant details of the physical theory in the Clifford algebras, and so one might view the remainder of the task as an exercise in mathematics. Thus we have presented a way to organize non-interacting topological phases of matter, and outlined an explanation of their organization.

IX. CONCLUSION

We have started with an exposition of the integer quantum Hall effect - a quantum mechanical experiment whose observable edge currents are quantized and robust to perturbations. To understand this robustness, we started by introducing the Berry phase, and motivating it with the adiabatic approximation. By generalizing this slightly, we arrived at a simple model that exhibits a Chern number. This Chern number is the topological invariant that characterizes the IQHE, and so by reducing the IQHE to a tightbinding model, we applied a simple analysis of a 1D chain to show that we should have expected topological phenomena all along. Moreover, we gave an example of the well-known bulk-boundary correspondence. Finally, we concluded by showing that there is a mathematically rigorous classification of non-interacting topological phases that uses the symmetries and dimension of the system.

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