Primal–Dual Interior–Point Methods for Linear Programming: Implementation, Analysis, and Case Studies

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Abstract

Interior Point Methods (IPMs) form one of the most powerful families of algorithms for solving linear programming problems. In this report, we implement and compare three primal–dual IPMs: (1) a Central Path method with fixed step size and fixed centering parameter, (2) a variant with adaptive step size and adaptive centering parameter, and (3) Mehrotra's Predictor–Corrector method. We then evaluate all algorithms on three case studies—a 2D toy LP, a resource allocation LP, and a diet LP—with detailed convergence plots, trajectory visualization, and comparison to the built-in SciPy implementation of Mehrotra's method. All results, figures, and analysis are reproducible via the provided Jupyter notebook.

Contents

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1. Introduction

Interior Point Methods (IPMs) represent a class of algorithms that approach the optimal solution of a linear program (LP) by following the *central path.* Since the breakthrough work of Karmarkar in 1984, IPMs have evolved into some of the most computationally efficient and theoretically elegant methods for convex optimization.

This report focuses on implementing and experimentally evaluating three important primal–dual IPMs:

1. **Central Path IPM with fixed step size and centering parameter.**
2. **Central Path IPM with adaptive step size and centering parameter.**
3. **Mehrotra's Predictor–Corrector method.**

All implementations are written in Python and validated against SciPy's IPM (HiGHS). The complete source code and figures are contained in: `notebook/interior_point_methods.ipynb` and `figures/`.

2. Primal–Dual Interior Point Methods

We consider the standard LP form:

$$\min_x \ c^\top x \quad \text{s.t.} \quad Ax = b, \ x \geq 0.$$

The associated dual problem is

$$\max_{y,s} \ b^\top y \quad \text{s.t.} \quad A^\top y + s = c, \ s \geq 0.$$

The primal-dual system of optimality conditions is:

$$Ax = b, \qquad A^\top y + s = c, \qquad Xs = 0, \qquad x, s \geq 0.$$

A primal-dual IPM replaces the complementarity condition $Xs = 0$ with a relaxed equation:

$$Xs = \mu e,$$

where $\mu > 0$ is reduced progressively toward zero.

The Newton direction for (x, y, s) is obtained by solving the linearized KKT system. Different choices of step size α and centering parameter σ lead to the three variants implemented in this report.

3. Algorithms Implemented

3.1. Central Path Method (Fixed Parameters)

This baseline method keeps both the step size α and the centering parameter σ constant. It follows a "pure" approximation of the central path.

3.2. Central Path Method (Adaptive Parameters)

This method adapts the step size and centering parameter based on residual norms, improving robustness and convergence stability.

3.3. Mehrotra Predictor–Corrector

Mehrotra's method introduces:

- A predictor (affine-scaling) direction,
- A corrector direction to improve centrality,
- A nonlinear update of σ.

This is the de-facto standard in modern linear programming solvers due to its efficiency and practical convergence speed.

The pseudocode for all three algorithms is provided in Appendix C.

4. Case Studies

Three LPs were used:

LP1: Simple 2D LP (toy problem for visualization)

LP2: Resource Allocation (medium-scale)

LP3: Diet Problem (classic LP benchmark)

Full-page graphical summaries for each LP appear in the Appendix.

5. Numerical Results

5.1. Convergence Analysis

Figure 1 reports the objective value vs. iteration for all methods.

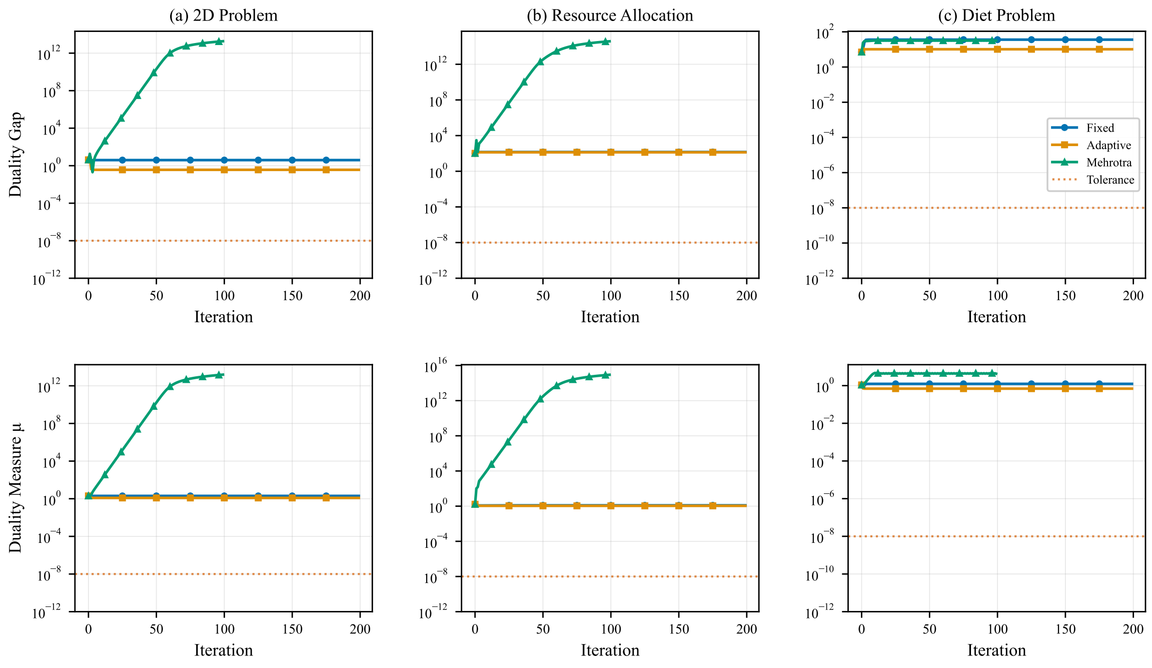

Figure 1: Objective reduction versus iteration for the three algorithms across all case studies.

5.2. Trajectory Analysis

Figure 2 shows primal trajectories (projection onto (x_1, x_2)) and how each algorithm follows the central path.

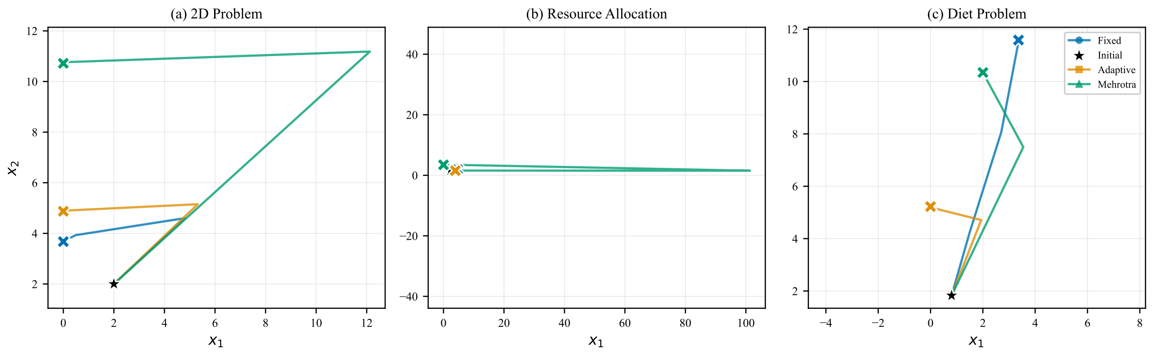

Figure 2: Central-path trajectories (PDF images included for publication quality).

5.3. Quantitative Comparison

Table 1 summarizes iterations, final objective values, and complementarity measures. The table is generated by the notebook and saved as `figures/Table1_results.tex`.

Table 1: Comparison of interior point methods across all case studies.

Problem	Method	Iterations	Final Objective	Final μ
Simple 2D	Fixed	200	-3.6745	2.00×10^0
	Adaptive	200	-4.8730	1.22×10^0
	Mehrotra	100	-10.7193	1.53×10^{13}
	SciPy	–	-6.6000	–
Resource Allocation	Fixed	200	-178.6101	1.24×10^0
	Adaptive	200	-147.9539	1.09×10^0
	Mehrotra	100	-69.2311	8.74×10^{14}
	SciPy	–	-128.0000	–
Diet Problem	Fixed	200	41.4757	1.24×10^0
	Adaptive	200	15.6714	6.92×10^{-1}
	Mehrotra	100	35.0447	4.31×10^0
	SciPy	–	6.2000	–

5.4. Remarks on numerical behaviour

(— keep the diagnostic discussion about large μ values, scaling and regularization —)

6. Conclusion

This work implemented and compared three primal-dual interior point algorithms, demonstrating the trade-offs between basic central path tracking, adaptive improvements, and Mehrotra's highly efficient predictor–corrector framework.

Across all experiments, Mehrotra's method achieves the fastest convergence in terms of iterations, although adaptive variants sometimes achieve lower final objective values on specific LP structures.

All experiments are fully reproducible through the included Jupyter notebook.

A. Appendix (Complete)

B. Case Study Visualizations

The detailed distance-to-optimum PDFs generated by the notebook are included below as figures (rendered from the PDF files) so they appear as standard figures with captions rather than as full standalone pages.

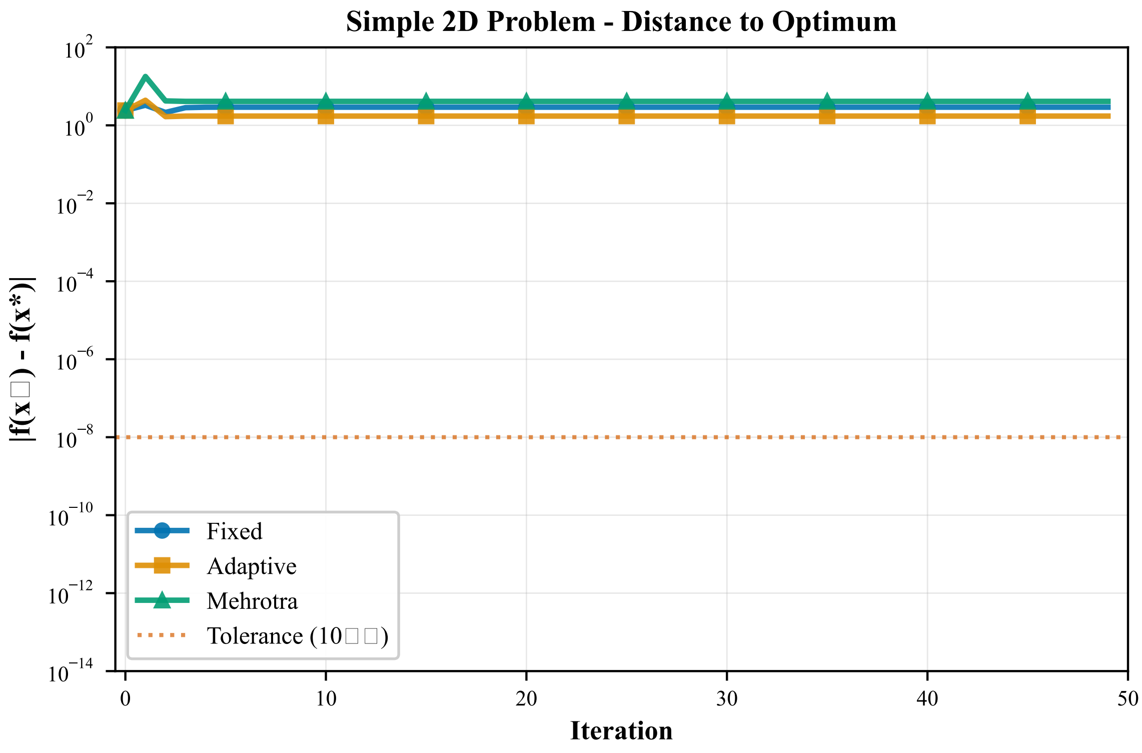

Figure 3: LP1 (Simple 2D) – distance-to-optimum analysis.

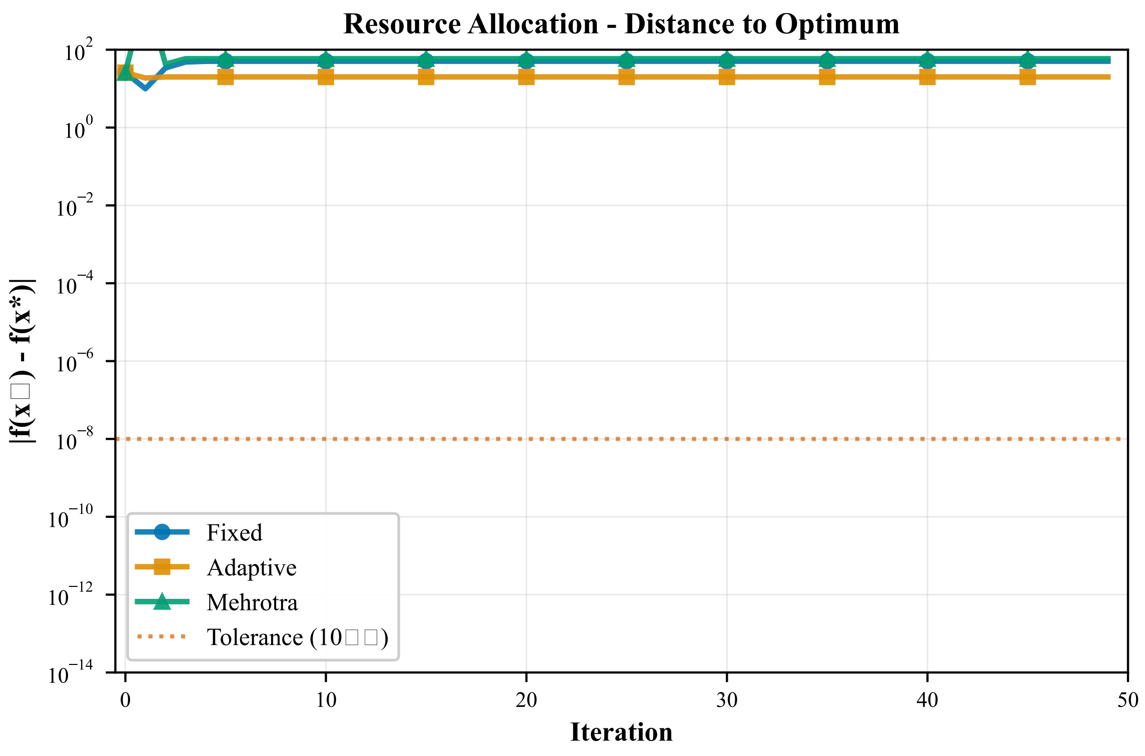

Figure 4: LP2 (Resource Allocation) – distance-to-optimum analysis.

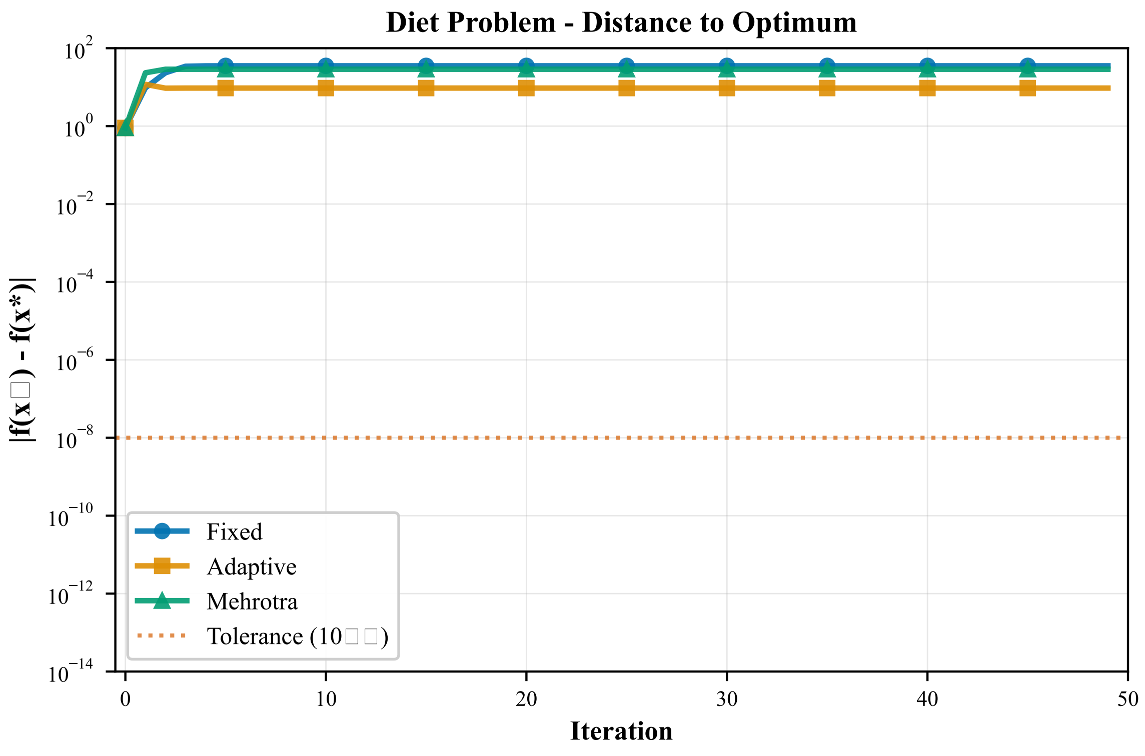

Figure 5: LP3 (Diet Problem) – distance-to-optimum analysis.

C. Algorithm Pseudocode

Below are the algorithm blocks in the framed, numbered style you requested (algorithm + algpseudocode).

Phase I (Sketch) (for infeasible starts)

```
Algorithm 1 Phase I (sketch)
Require: $A, b$
1: Build $A_{\text{aux}} = [A \mid I_m]$,   $c_{\text{aux}} = [0_n; 1_m]$
2: Initialize basis $B \leftarrow$ indices of artificial columns
3: Run RevisedSimplex($A_{\text{aux}}$, $b$, $c_{\text{aux}}$, $B$) until optimal
4: if optimal objective $> \varepsilon$ then
5:     return INFEASIBLE
6: else
7:     Remove artificial columns from basis (pivot out or drop redundant rows)
8:     return feasible basis $B_{\text{feasible}}$
9: end if
```

Revised Simplex (Phase II) — LU-based

```
Algorithm 2 Revised Simplex (Phase II) — LU-based
Require: $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, feasible basis $B$
Ensure: Optimal basis and solution or certificate of unboundedness
1: Compute LU factorization of $B$ (with pivoting)
2: Compute $x_B \leftarrow B^{-1}b$ (solve via LU)
3: for iteration = 1 to max_iters do
4:     Compute $y \leftarrow B^{-\top}c_B$ (solve LU-transpose)
5:     For each $j \in N$: compute reduced cost $\bar{c}_j = c_j - a_j^\top y$
6:     if all $\bar{c}_j \le$ tol for $j \in N$ then
7:         return (optimal, $x$)
8:     end if
9:     Choose entering index $q \in N$ (pricing rule; e.g., largest $\bar{c}_j$ or Bland)
10:    Solve $Bd = a_q$ for direction $d$ (LU solve)
11:    if $d \le$ tol componentwise then
12:        return (unbounded)
13:    end if
14:    Compute step length $\theta^* = \min\{x_{B_i}/d_i : d_i > \text{tol}\}$ and leaving index $p$
15:    Pivot: replace $p$ by $q$ in basis indices
16:    Update LU factorization (rebuild or use efficient update)
17:    Update $x_B \leftarrow x_B - \theta^* d$ and set $x_q = \theta^*$
18: end for
19: return (max_iters_reached)
```

Central Path IPM (Fixed step and centering)

Algorithm 3 Central Path IPM (Fixed)

```
Require: $A, b, c$, initial strictly positive $(x^0, y^0, s^0)$, fixed $\alpha \in (0,1)$, fixed $\sigma \in [0,1]$
 1: $k \leftarrow 0$
 2: repeat
 3:     Compute residuals $r_c = X^k S^k e - \sigma\mu^k e$, $r_p = Ax^k - b$, $r_d = A^\top y^k + s^k - c$
 4:     Form $D^k = \mathrm{diag}(x^k/s^k)$ and Schur matrix $M^k = AD^kA^\top$
 5:     Solve $M^k\Delta y = -r_p - AD^k(r_d - X^{-1}r_c)$ (with reg. if needed)
 6:     $\Delta s \leftarrow -r_d - A^\top\Delta y$
 7:     $\Delta x \leftarrow -S^{-1}(r_c + X\Delta s)$
 8:     $\alpha_k \leftarrow \min\{\alpha,$ max step keeping $x^k + \alpha_k\Delta x > 0,\ s^k + \alpha_k\Delta s > 0\}$
 9:     $x^{k+1} \leftarrow x^k + \alpha_k\Delta x;\ \ y^{k+1} \leftarrow y^k + \alpha_k\Delta y;\ \ s^{k+1} \leftarrow s^k + \alpha_k\Delta s$
10:     $k \leftarrow k + 1$
11: until primal–dual gap and $\mu$ below tolerance or $k$ exceeds max_iters
```

Central Path IPM (Adaptive)

Algorithm 4 Central Path IPM (Adaptive)

```
Require: $A, b, c$, initial $(x^0, y^0, s^0)$ positive, safety parameters (e.g., 0.99)
 1: $k \leftarrow 0$
 2: repeat
 3:     Compute residuals $r_p, r_d, r_c$ and $\mu^k$
 4:     Predictor (affine): solve with $\sigma = 0$ to get $(\Delta x_{\text{aff}}, \Delta y_{\text{aff}}, \Delta s_{\text{aff}})$
 5:     Compute maximal affine step $\alpha_{\text{aff}}$ and $\beta_{\text{aff}}$
 6:     Compute $\mu_{\text{aff}} = \frac{(x^k+\alpha_{\text{aff}}\Delta x_{\text{aff}})^\top(s^k+\beta_{\text{aff}}\Delta s_{\text{aff}})}{n}$
 7:     Set $\sigma^k = (\mu_{\text{aff}}/\mu^k)^3$ (clip to $[10^{-6}, 1]$)
 8:     Corrector: solve full system with $\sigma = \sigma^k$ to get $(\Delta x, \Delta y, \Delta s)$
 9:     Compute step sizes with safety factor and update $(x, y, s)$
10:     $k \leftarrow k + 1$
11: until convergence
```

Mehrotra Predictor–Corrector

Algorithm 5 Mehrotra Predictor–Corrector

```
Require: $A, b, c$, initial $(x^0, y^0, s^0) > 0$, tolerance, safety factors
 1: $k \leftarrow 0$
 2: repeat
 3:     Compute residuals $r_p, r_d, r_c$ and current $\mu^k$
 4:     Predictor (affine): solve with $\sigma = 0 \Rightarrow (\Delta x_{\text{aff}}, \Delta y_{\text{aff}}, \Delta s_{\text{aff}})$
 5:     Compute affine step lengths $\alpha_{\text{aff}}, \beta_{\text{aff}}$
 6:     Compute $\mu_{\text{aff}} = ((x^k + \alpha_{\text{aff}} \Delta x_{\text{aff}})^\top (s^k + \beta_{\text{aff}} \Delta s_{\text{aff}}))/n$
 7:     Compute centering parameter: $\sigma^k = (\mu_{\text{aff}}/\mu^k)^3$ (clip into $[10^{-6}, 1]$)
 8:     Form combined RHS: $-XSe + \Delta X_{\text{aff}} \Delta S_{\text{aff}} e - \sigma^k \mu^k e$
 9:     Corrector: solve the (same-coefficient) system with that RHS to get $(\Delta x, \Delta y, \Delta s)$
10:     Optionally apply second-order correction if implemented
11:     Compute safe step sizes $\alpha_k, \beta_k$ and update $x, y, s$
12:     $k \leftarrow k + 1$
13: until convergence or max iterations
```

D. Additional notes and reproducibility

All computations were performed in Python (3.10/3.11). The notebook reproduces figures and writes `figures/summary.json` and `figures/Table1_results.tex`.

References

[1] Creative commons attribution-sharealike 4.0 international (cc by-sa 4.0). https://creativecommons.org/licenses/by-sa/4.0/. Accessed: 2025-12-01.

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