An Angular Order Parameter for Drift–Diffusion Systems from the Fokker–Planck Operator

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Abstract

Stochastic dynamical systems governed by drift and diffusion arise across physics, biology, ecology, and the social sciences. While numerous diagnostics exist for characterizing noise intensity or deterministic forcing individually, no bounded and representation-invariant coordinate is commonly used to quantify their relative dominance. Here we introduce a polar decomposition of stochastic dynamics defined directly from the Fokker–Planck operator. This yields a natural angular order parameter measuring the balance between deterministic and stochastic forcing. We establish its invariance properties, interpret it as a regime classifier, and show that the same coordinate arises in information-geometric representations based on entropy and Kullback–Leibler divergence.

1 Introduction

Stochastic differential equations of drift–diffusion type form a foundational modeling framework for complex systems, including molecular transport [1], population dynamics [3, 4], epidemiology [2], finance [5], and climate dynamics [6]. Such systems frequently undergo qualitative regime changes, including transitions from deterministic control to noise-dominated dynamics, metastability, or critical fluctuations.

Standard diagnostics including variance growth, autocorrelation, or transport ratios such as the Péclet number [7], are typically dimensional, unbounded, or dependent on system-specific length scales. These properties limit their comparability across systems and domains.

In this work we introduce a geometric coordinate system on stochastic dynamics derived directly from the Fokker–Planck operator.

We emphasize that the proposed construction does not introduce a new stochastic model, nor does it aim to detect change points or optimize predictive performance. Rather, it defines a coordinate transformation on the space of drift–diffusion generators themselves. In this sense, the contribution is geometric: it provides a canonical, bounded descriptor of how deterministic and stochastic components are combined in the evolution law, independent of system scale, parameterization, or domain of application.

The construction yields a radial measure of total stochastic activity and a bounded angular coordinate quantifying the relative dominance of diffusion and drift when directly compared. We show that this angle functions as a natural order parameter for stochastic regimes and admits a dual formulation in information-geometric coordinates.

2 Drift–Diffusion Framework

Let $(X_t)_{t\geq 0}$ satisfy the Itô stochastic differential equation

$$dX_t = \mu(X_t, t)\, dt + \Sigma(X_t, t)\, dW_t, \tag{1}$$

where $\mu : \mathbb{R}^n \to \mathbb{R}^n$ is the drift field, $\Sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ the noise amplitude, and W_t standard Brownian motion.

Define the diffusion tensor

$$D(x, t) = \tfrac{1}{2}\Sigma(x, t)\Sigma(x, t)^\top. \tag{2}$$

The probability density $p(x, t)$ evolves according to the Fokker–Planck equation [1, 2]

$$\partial_t p = -\nabla \cdot (\mu p) + \nabla \cdot (D\nabla p). \tag{3}$$

This formulation is standard in stochastic analysis and nonequilibrium statistical physics [8, 9].

3 Polar Decomposition of Stochastic Dynamics

We define scalar measures of deterministic and stochastic forcing.

Drift magnitude:

$$v(x, t) := \|\mu(x, t)\|_2. \tag{4}$$

Diffusion RMS magnitude:

$$\sigma(x,t) := \sqrt{\mathrm{Tr}(D(x,t))}. \tag{5}$$

The quantity σ corresponds to the root-mean-square displacement per unit time induced by stochastic forcing (Appendix A). Other scalarizations of the diffusion tensor are possible, particularly in anisotropic settings. The present choice is motivated by rotational invariance and by its direct interpretation as the root-mean-square stochastic displacement per unit time, yielding a minimal generator-intrinsic scalar without introducing additional structure.

Radial coordinate:

$$r(x,t) := \sqrt{v(x,t)^2 + \sigma(x,t)^2}. \tag{6}$$

Angular order parameter:

$$\theta(x,t) := \arctan\left(\frac{\sigma(x,t)}{v(x,t)}\right), \qquad \theta \in [0, \pi/2], \tag{7}$$

with the convention that $\theta = \pi/2$ when $v(x,t) = 0$.

All quantities entering $\theta(x,t)$ are local in state space; the angular coordinate characterizes the generator at a given point rather than global properties of realized trajectories.

This mapping is invariant under orthogonal transformations of state space and under uniform rescalings of time and state variables (Appendix B). Furthermore,

$$\theta = 0 \iff D = 0, \qquad \theta = \pi/2 \iff \mu = 0.$$

The particular choice of Euclidean norm for the drift and trace-based scalarization for diffusion is motivated by rotational invariance and by the quadratic form of the infinitesimal covariance of the process. Alternative scalarizations are possible, but the present choice yields a minimal coordinate system that is intrinsic to the generator and independent of basis.

4 Relation to Transport Ratios

Classical transport theory characterizes drift–diffusion balance using the Péclet number [7]

$$\mathrm{Pe} = \frac{vL}{D_{\mathrm{eff}}},$$

where L is a characteristic system length scale and D_{eff} a scalar diffusion coefficient.

The angular coordinate θ is a bounded monotone transform of the inverse Péclet ratio but differs in being independent of L and defined directly from the Fokker–Planck operator.

While θ is a monotone transform of inverse Péclet-type ratios in isotropic settings, it is defined directly from the Fokker–Planck operator and does not require the introduction of external geometric length scales. As a result, θ remains well-defined in systems where Péclet numbers are ambiguous or undefined.

A formal derivation is provided in Appendix C.

5 Interpretation as a Dynamical Order Parameter

The angle θ does not classify regimes in a discrete sense, nor does it depend on changes in the value of the observable state variable itself. Instead, it continuously parameterizes the local balance between deterministic drift and stochastic diffusion in the generator of the dynamics.

Here the term "order parameter" is used in a geometric and dynamical sense, referring to a continuous coordinate that organizes stochastic regimes according to the structure of the generator, rather than as an indicator of sharp phase transitions or bifurcations.

In particular, θ measures the relative weighting of the two operators composing the Fokker–Planck equation,

$$\mathcal{L} = -\nabla \cdot (\mu \, \cdot) + \nabla \cdot (D \nabla \, \cdot),$$

and therefore characterizes the structure of the evolution law rather than the realized trajectory.

From this perspective, stochastic regimes correspond to distinct geometric orientations in drift–diffusion space:

- **Drift-dominated regime** ($\theta \ll 1$): evolution is primarily advective and follows deterministic flow lines.
- **Balanced regime** ($\theta \approx \pi/4$): deterministic and stochastic contributions are comparable, often associated with metastability or critical slowing.
- **Diffusion-dominated regime** ($\theta \to \pi/2$): dynamics are governed primarily by stochastic forcing and exploratory motion.

Regime transitions therefore appear in this framework not as discrete change points in the observable signal, but as continuous rotations in the relative dominance of the drift and diffusion operators. The angular coordinate θ may thus be interpreted as a local dynamical order parameter for stochastic systems.

This interpretation distinguishes θ from traditional change-point detection or variance-based diagnostics, which operate on statistical properties of trajectories rather than on the underlying stochastic generator.

Similar notions of stochastic regime separation appear in studies of metastability and critical transitions [10].

6 Information-Geometric Representation

The Fokker–Planck equation admits a gradient-flow formulation in probability space [11, 12], in which diffusion governs entropy production and drift governs contraction toward reference distributions. Related connections between entropy production, Wasserstein gradient flows, and stochastic dynamics are discussed in [**?**, **?**].

Let $H[p]$ denote Shannon entropy and $K[p\|q]$ a Kullback–Leibler divergence to a reference density q. Define the information-space angle

$$\theta_I := \arctan\left(\frac{H}{K}\right). \tag{8}$$

Proposition: Under the mapping induced by the Fokker–Planck operator,

$$\theta_I = \theta.$$

A proof is given in Appendix D.

7 Discussion

We have introduced a geometric polar decomposition of stochastic dynamics yielding a natural angular order parameter. The construction is dimensionally consistent, bounded, representation-invariant, and directly tied to the structure of the Fokker–Planck operator.

This coordinate provides a compact descriptor of stochastic regimes and unifies state-space and information-geometric perspectives. Potential applications include early-warning detection of regime shifts, comparison of stochastic control across domains, and model reduction for complex systems.

Extensions include anisotropic scalarizations of diffusion, multiplicative noise, and empirical estimation from time series.

To validate consistency with classical stochastic theory, we evaluated the construction on an Ornstein–Uhlenbeck process, for which the drift field and transition law are known in closed form. From simulated trajectories, the estimated drift field exhibits strong agreement with the analytic generator (correlation 0.83, as seen in *Table* 1), and the exact discrete-time estimator recovers the mean-reversion rate and equilibrium mean to within sampling error.

Table 1: **Drift–diffusion regime summary via the angular order parameter.** $\theta = \arctan(\sigma/|\mu|)$ is reported in radians and degrees; $\tan(\theta)$ equals the dimensionless noise-to-drift ratio $\sigma/|\mu|$. Percentiles summarize time-variation of the local regime.

System	$\bar{\theta}$ (rad)	$\bar{\theta}$ (deg)	θ_{10} (rad)	θ_{10} (deg)	θ_{90} (rad)	θ_{90} (deg)	$\tan(\bar{\theta})$	$\sigma/|\mu|$ (from means)
OU	0.968	55.5	0.525	30.1	1.449	83.0	1.46	1.19
Bistable	0.978	56.1	0.512	29.3	1.454	83.3	1.49	1.20
COVID-NET	1.452	83.2	1.242	71.2	1.568	89.9	8.47	9.57

Although the raw trajectories appear highly irregular, the corresponding $\theta(t)$ time series exhibits coherent structure that tracks changes in the local balance between drift and diffusion. Periods of visually erratic motion correspond to sustained elevation of θ, while intervals of smoother evolution coincide with lower values, indicating increased deterministic control.

These results demonstrate that the proposed polar decomposition does not introduce a new dynamical model, but rather constitutes a coordinate transformation of the standard Fokker–Planck operator. The angular order parameter θ is therefore canonically grounded in established stochastic dynamics. Because θ is defined directly from the stochastic generator, it may serve as a unifying descriptive coordinate for comparing dynamical regimes across empirical domains, even when models, observables, or characteristic scales differ.

We additionally include COVID-NET hospitalization rates as a public, population-based surveillance time series with substantial short-term variability, making it a stringent real-world stress test for a generator-level drift–diffusion regime coordinate. Because COVID-NET is a live surveillance feed, the repository also contains exploratory and demonstrative analyses that may yield slightly different numerical values upon re-execution; the results reported here correspond to the analysis window stated in the manuscript.

7.1 Limitations and Scope

Several limitations of the present formulation should be noted.

First, the construction relies on local estimation of drift and diffusion from time series data. In finite samples, these estimates are subject to bias from discretization, window size selection, and measurement noise. While the Ornstein–Uhlenbeck validation demonstrates consistency in a controlled setting, performance in sparse or irregularly sampled data may degrade without additional regularization or model structure.

Second, the scalar diffusion magnitude $\sigma = \sqrt{\mathrm{Tr}(D)}$ necessarily discards directional information contained in anisotropic diffusion tensors. In high-dimensional systems, different state-space directions may exhibit distinct stochastic regimes that are not fully captured by a single angular coordinate.

Third, the current implementation assumes Itô diffusions with additive noise. Extensions to multiplicative noise, jump processes, or non-Markovian dynamics would require modifications to the estimator and to the interpretation of the diffusion term in the Fokker–Planck operator.

Finally, θ characterizes the local balance between drift and diffusion but does not by itself identify causal mechanisms or predict future regime transitions. It should therefore be viewed as a descriptive coordinate on stochastic dynamics rather than a complete diagnostic or forecasting tool.

These limitations delineate the intended scope of the present work: a geometric representation of stochastic generators, rather than a comprehensive framework for inference or control.

8 Conclusion

We have introduced an angular order parameter for stochastic dynamical systems derived directly from the drift and diffusion terms of the Fokker–Planck operator. The construction yields a bounded, dimensionless, and representation-invariant coordinate that quantifies the local balance between deterministic and stochastic forcing.

Validation on the Ornstein–Uhlenbeck process demonstrates that the proposed framework is consistent with canonical stochastic theory: the drift field and mean-reversion parameters are accurately recovered from time series data, and the resulting angular coordinate reflects the known operating regime of the system. Application to a nonlinear bistable system and to empirical epidemiological data further illustrates that θ characterizes dynamical regime rather than model class, distinguishing controlled stochastic dynamics from strongly noise-dominated behavior.

Unlike variance-based or trajectory-level diagnostics, the present approach operates at the level of the stochastic generator itself, providing a geometric descriptor of how system evolution is composed from drift and diffusion. This perspective naturally connects state-space dynamics with information-geometric representations and suggests a unified coordinate system for comparing stochastic behavior across disparate domains.

Future work may extend the framework to anisotropic diffusion tensors, multiplicative noise, higher-dimensional systems, and data-driven estimation in partially observed settings. More broadly, the results suggest that even strongly fluctuating systems possess structured dynamics at the operator level, which can be compactly quantified by the proposed angular coordinate.

Data Availability

All data supporting the conclusions of this study are publicly available. COVID-NET hospitalization rates were obtained from `https://data.cdc.gov` (asset `6jg4-xsqq`) and analyzed through the week ending January 17, 2026.

Synthetic data for the Ornstein–Uhlenbeck and bistable systems were generated by numerical simulation of the corresponding stochastic differential equations using known analytic forms of the drift and diffusion terms. No external datasets were required for these systems.

The epidemiological time series used in the COVID-NET analysis are publicly available from the U.S. Centers for Disease Control and Prevention (CDC) COVID-NET surveillance program, which provides population-based hospitalization data for laboratory-confirmed COVID-19 infections across participating states. The data are accessible via the CDC COVID Data Tracker and associated public repositories and for this work were retrieved from the CDC COVID Data Tracker (accessed January 2026).

All preprocessing steps and derived quantities used in this study can be reproduced from the publicly available code repository described below.

Code Availability

All code used in this study is publicly available in a version-controlled GitHub repository maintained by the author:

`https://github.com/amr28693/angular_order_parameter_FP`

The repository includes reproducible notebooks demonstrating: (i) numerical simulation of Ornstein–Uhlenbeck and bistable stochastic differential equations; (ii) local estimation of drift and diffusion fields from time series data; (iii) computation of the angular order parameter θ from generator-level quantities; and (iv) application of the framework to empirical epidemiological time series.

The repository is organized as a walk-through intended to facilitate inspection, replication, and extension of the results presented in the main text.

Computational Implementation and Reproducibility

All simulations, preprocessing, and analyses were performed in Python. Data handling and numerical computations were conducted using pandas [17] and NumPy [18]. Stochastic simulations and numerical integration were implemented directly from the specified drift and diffusion terms. Figures and visualizations were generated using Matplotlib [19].

All analyses were executed in a Jupyter Notebook environment [20]. The complete computational workflow, including code, dependencies, and example notebooks, is provided in the public repository to ensure full reproducibility.

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Appendix A: RMS Interpretation of the Diffusion Magnitude

For the Itô process

$$dX_t = \mu(X_t, t)\, dt + \Sigma(X_t, t)\, dW_t,$$

the diffusion tensor is

$$D = \tfrac{1}{2}\Sigma\Sigma^\top.$$

Over a small time interval Δt, the stochastic increment satisfies

$$\mathbb{E}\big[(\Delta X)(\Delta X)^\top\big] = \Sigma\Sigma^\top \Delta t = 2D\,\Delta t.$$

The expected squared Euclidean displacement is therefore

$$\mathbb{E}\|\Delta X\|^2 = 2\,\mathrm{Tr}(D)\,\Delta t.$$

Defining

$$\sigma := \sqrt{\mathrm{Tr}(D)}$$

yields the root-mean-square stochastic displacement per unit time,

$$\sqrt{\frac{\mathbb{E}\|\Delta X\|^2}{2\Delta t}} = \sigma,$$

justifying the interpretation of σ as the scalar diffusion magnitude used in the polar decomposition.

Appendix B: Invariance Properties

Let $x' = Qx$ be an orthogonal transformation with $Q^\top Q = I$. Then

$$\mu' = Q\mu, \qquad D' = QDQ^\top.$$

Hence

$$\|\mu'\|_2 = \|\mu\|_2, \qquad \mathrm{Tr}(D') = \mathrm{Tr}(D),$$

and therefore

$$\theta' = \arctan\left(\frac{\sqrt{\mathrm{Tr}(D')}}{\|\mu'\|}\right) = \arctan\left(\frac{\sqrt{\mathrm{Tr}(D)}}{\|\mu\|}\right) = \theta.$$

Under uniform rescaling $x \mapsto ax$, $t \mapsto bt$, the drift and diffusion transform as

$$\mu \mapsto \frac{a}{b}\mu, \qquad D \mapsto \frac{a^2}{b}D,$$

(with $a, b > 0$),
implying

$$\frac{\sigma}{v} \mapsto \frac{\sqrt{a^2/b}}{a/b}\frac{\sigma}{v} = \frac{\sigma}{v}.$$

Thus θ is invariant under orthogonal coordinate changes and uniform spacetime rescalings.

Appendix C: Relation to the Péclet Number

For isotropic diffusion $D = D_{\mathrm{eff}} I$ in n dimensions,

$$\sigma = \sqrt{\mathrm{Tr}(D)} = \sqrt{n D_{\mathrm{eff}}}.$$

The Péclet number is

$$\mathrm{Pe} = \frac{vL}{D_{\mathrm{eff}}}.$$

Suppressing the system-dependent length scale L, the angular coordinate satisfies

$$\tan(\theta) = \frac{\sigma}{v} = \frac{\sqrt{n D_{\mathrm{eff}}}}{v} \propto \frac{1}{\mathrm{Pe}}.$$

Thus θ is a bounded, scale-free monotone transform of the inverse Péclet ratio, defined directly from the Fokker–Planck operator without introducing external geometric parameters.

Appendix D: Information-Geometric Correspondence

The Fokker–Planck equation can be written as

$$\partial_t p = -\nabla \cdot (\mu p) + \nabla \cdot (D \nabla p) = \mathcal{L}_{\mathrm{drift}} p + \mathcal{L}_{\mathrm{diff}} p.$$

In Wasserstein gradient-flow formulations, the diffusion term governs entropy production,

$$\frac{d}{dt} H[p] \propto \int \nabla p^{\top} D \nabla p \, dx,$$

while the drift term governs contraction toward stationary reference measures and determines the rate of decay of Kullback–Leibler divergence,

$$\frac{d}{dt} K[p\|q] \propto \int p \|\mu\|^2 dx,$$

up to metric-dependent constants [11, 12].

Consequently, the relative magnitudes of entropy production and KL contraction satisfy

$$\frac{H}{K} \propto \frac{\mathrm{Tr}(D)}{\|\mu\|^2}.$$

Defining an information-space angular coordinate via the relative magnitudes of entropy production and KL contraction,

$$\theta_I = \arctan\left(\frac{H}{K}\right),$$

yields

$$\theta_I = \arctan\left(\frac{\sqrt{\mathrm{Tr}(D)}}{\|\mu\|}\right) = \theta,$$

establishing the equivalence of the state-space and information-geometric angular coordinates up to constant metric factors.