Faculty of Science
Dept. of Mathematics

SYSTEM SIMULATION

An Introduction to Simulation and Modelling

For 4th level students

Prepared

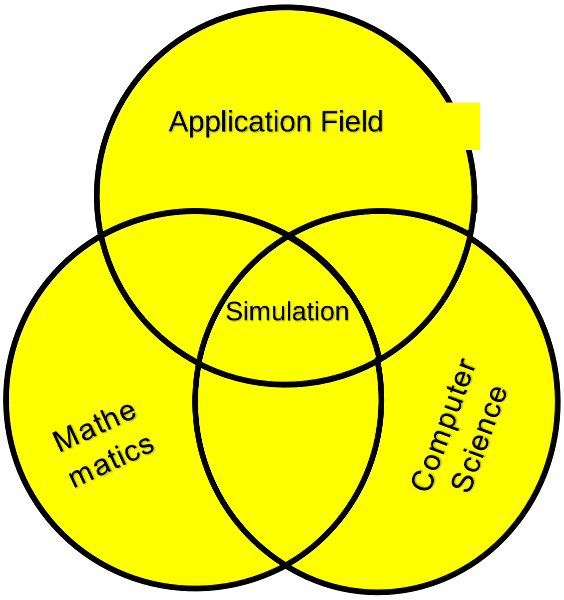
by

Department of Mathematics

2022

SYSTEM SIMULATION

An Introduction to Simulation and Modelling



Course outlines

- Introduction to Basic simulation models
- Modelling and Simulation
- Probability in Simulation
- Discrete-Event Simulation
- Continuous System Simulation
- Random number
- Other simulation models
- Simulation for Aircraft Model
- Queuing System Simulation

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Chapter 1

Introduction to Simulation

Definition

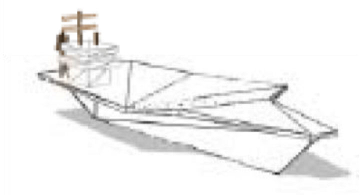
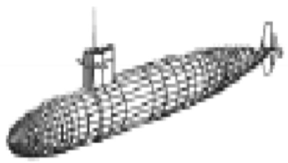
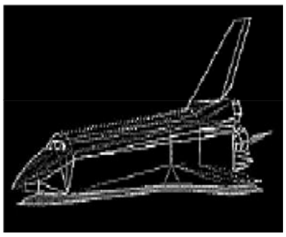
"Simulation is the process of designing a model of a real system and conducting experiments with this model for the purpose of either understanding the behavior of the system and/or evaluating various strategies for the operation of the system."

Introduction to Simulation

- **Simulation**
  - the imitation of the operation of a real-world process or system over time
  - to develop a set of assumptions of mathematical, logical, and symbolic relationship between the entities of interest, of the system.
  - to estimate the measures of performance of the system with the simulation-generated data
  - The behavior of a system as it evolves over time is studied by developing a simulation model.

Simulating

To simulate any system we need at first to build a model for that system then simulate this model

Types of simulation models

- Physical simulation models

- Mathematical simulation models –Staticvs. dynamic
  –Deterministicvs. stochastic
  –Continuousvs. discrete

(Most operational models are dynamic, stochastic, and discrete – will be called discrete-event simulation models)

Why we need simulation ?

Because the real system is:

- too cumbersome (شديد التعقيد)
- too costly
- too dangerous

- too slow

Simulation allows us to:

- Model complex systems in a detailed way
- Describe the behavior of systems
- Construct theories or hypotheses that account for the observed behavior
- Use the model to predict future behavior, that is, the effects that will be produced by changes in the system
- Analyze proposed systems

Goal of modeling and simulation

A model : can be used to investigate a wide verity of "what if" questions about real-world system.

❑ Potential changes to the system can be simulated and predicate their impact on the system.

❑ Find adequate parameters before implementation

So simulation can be used as

➢ Analysis tool for predicating the effect of changes

➢ Design tool to predicate the performance of new system (It is better to do simulation before Implementation).

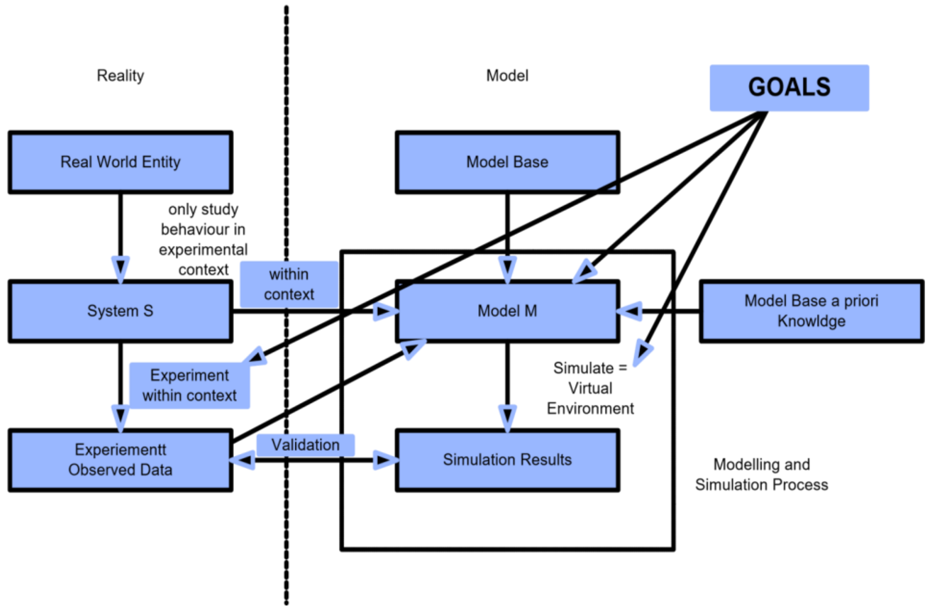


Figure 5: Modeling and Simulation

When Simulation is the Appropriate Tool (1)

- Simulation enables the study of, and experimentation with, the internal interactions of a complex system, or of a subsystem within a complex system.
- Informational, organizational, and environmental changes can be simulated, and the effect of these alterations on the model's behavior can be observed.
- The knowledge gained in designing a simulation model may be of great value toward suggesting improvement in the system under investigation.
- By changing simulation inputs and observing the resulting outputs, valuable insight may be obtained into which variables are most important and how variables interact.

- Simulation can be used to experiment with new designs or policies prior to implementation, so as to prepare for what may happen.
- Simulation can be used to verify analytic solutions.
- By simulating different capabilities for a machine, requirements can be determined.
- Animation shows a system in simulated operation so that the plan can be visualized.
- Simulation can be used as a pedagogical device to reinforce analytic solution methodologies

When Simulation is not Appropriate

- When the problem can be solved using common sense.
- When the problem can be solved analytically.
- When it is easier to perform direct experiments.
- When the simulation costs exceed the savings.
- When the resources or time are not available.

- When system behavior is too complex or can’t be defined.
- When isn’t the ability to verify and validate the mode

Difficulties of Simulation

- Provides only individual, not general solutions
- Manpower and time-consuming
- Computing memory and time-intensive
- Difficult so experts are required
- Hard to interpret results
- Expensive

Advantages of Simulation

1. Can handle large and complex systems
2. Can answer “what-if” questions
3. Does not interfere with the real system
4. Allows study of interaction among variables
5. “Time compression” is possible
6. Handles complications that other methods can t’
7. Easy to compare alternatives
8. Control experimental conditions

Disadvantages of Simulation

1. Can be expensive and time consuming
2. Managers must choose solutions they want to try
   ("what-if" scenarios)
3. Each model is unique
4. Model building is an art as well as a science. The quality of the analysis depends on the quality of the model and the skill of the
5. Simulation results are sometimes hard to interpret.
6. Should not be used when an analytical method would provide for quicker results.

System

definition:

System is a collection of entities (people, parts, messages, machines, servers, …) that act and interact together toward some end (Schmidt and Taylor, 1970)

- In practice, depends on objectives of study
- Might limit the boundaries (physical and logical) of the system

Systems and System Environment

- **System**
- is defined as a group of objects that are joined together in some e regular interaction or interdependence toward the accomplishment of some purpose.

- **An Example:** is a production system manufacturing automobiles.

- **System Environment:**

  - The decision on the boundary between the system and its environment may depend on the purpose of the study.

  - A system is often affected by changes occurring outside the system.

Components of a System

- **Entity** : an object of interest in the system.
- **Attribute** : a property of an entity.
- **Activity** : a time period of specified length.
- **State :** the collection of variables necessary to describe the system at any time, relative to the objectives of the study.
- **Event :** an instantaneous occurrence that may change the state of the system.
- **Endogenous :** to describe activities and events occurring within a system.
- **Exogenous :** to describe activities and events in an environment that affect the system.

DISCRETE AND CONTINUOUS SYSTEMS

- **DISCRETE SYSTEMS:**

is one in which the state variable(s) change only at a discrete set of points in time.

- **An example:** Is the bank

- **CONTINUOUS SYSTEMS:**

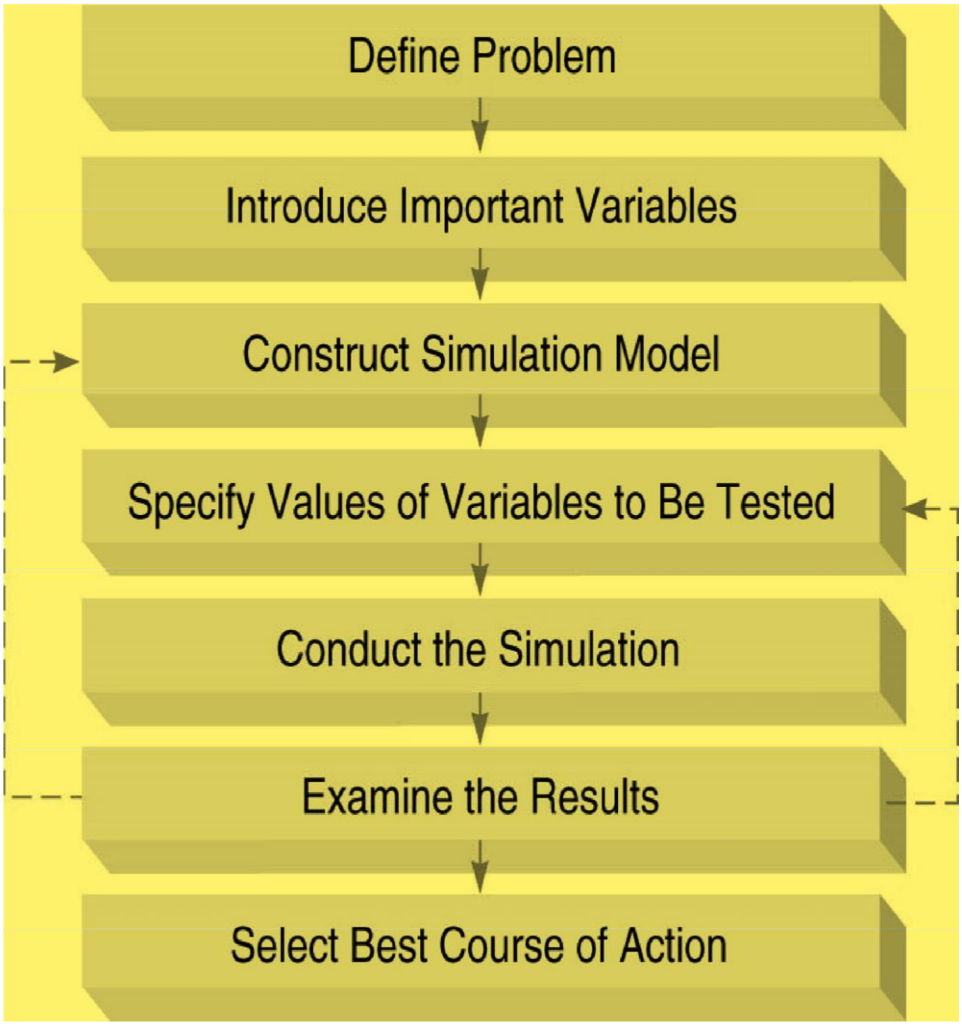
- is one in which the state variable(s) change continuously.
- **An example: water dam.**

EXAMPLES OF SYSTEMS AND COMPONENTS:

System	Entities	Attributes	Activities	Events	State Variables
Banking	Customers	Checking account balance	Making deposits	Arrival; Departure	number of busy tellers; number of customers waiting

SYSTEM	Entities	Attributes	Activities	Events	State variables
Banking	Customers	Checking account balance	Making deposits	Arrival; departure	Number of busy tellers; number of Customers waiting
Railways	Riders	Origination destination	Traveling	Arrival at Station; Arrival at destination	Number of riders waiting at each station; number of riders in transit
Production	Machines	Speed; capacity breakdown rate	Welding stamping	Breakdown	Status of machines (busy , idle, or down)
Communications	Messages	Length; destination	Transmitting	Arrival at destination	Number waiting to be transmitted
Inventory	Warehouse	Capacity	Withdrawing	Demand	Levels of inventory ; backlogged demands

The Process of Simulation



Models

Chapter 2

Introduction to Modelling

Modelling is a large discipline in itself and creating a system requires a lot of mathematical ability and understanding of the system.

Models are usually composed of variables and relationships between them. Exactly what these variables represent and what the relationships between them are can vary.

Modelling Concepts

There are several concepts underlying simulation. These include system and model, events, system state variables, entities and attributes, list processing, activities and delays, and finally the definition of discrete-event simulation.

The process of making and testing hypotheses about models and then revising designs or theories has its foundation in the experimental sciences. Similarly, computational scientists use modeling to analyze complex, real-world problems in order to predict what might happen with some course of action.

For example, Dr. Jerrold Marsden, a computational physicist at CalTech, models space mission trajectory design (Marsden). Dr. Julianne Collins, a genetic epidemiologist (statistical genetics) at the Greenwood Genetics Center, runs genetic analysis programs and analyzes epidemiological studies using the Statistical Analysis Software (SAS) (Greenwood Genetics Center). Some of the projects on which she has worked involve analyzing data from a genome scan of Alzheimer's disease, performing linkage analyses of X-linked mental retardation families, determining the recurrence risk in nonsyndromic mental retardation, analyzing folic acid levels from a nutritional survey of Honduran women, and researching new methods to detect genes or risk factors involved in autism. Scientists in areas such as cognitive psychology and social psychology at the Human-Technology Interaction Center of The University of Oklahoma perform research on the interaction of people with modern technologies (Human-Technology Interaction Center). Some of the studies involve “strategic planning in air traffic control” and “designing interfaces for effective information retrieval from collections of multimedia.” Buried land mines are a serious danger in many areas of the world (Weldon et al. 2001). Scientists are using a combination of mathematics, signal processing, and scientific visualization to model, image, and discover land mines. Lourdes Esteva, Cristobal Vargas, and Jorge Velasco- Hernandez have modeled the oscillating patterns of the disease dengue fever, for which an estimated 50 to 100 million cases occur globally each year (Esteva and Vargas 1999).

The model

- A *model* construct a conceptual framework that describes a system
- The model takes a set of expressed assumptions:
  - Mathematical, logical
    - Symbolic relationship between the *entities*

The variables of the model must represent the *state* of the system. The state is split into different components to represent the different parts of the system. These are sometimes called *model components*.

For example :-

A car in a traffic simulator may have a *position*, a *size* and a *velocity*.

The relationships between these model components define the behavior of the system. Going back to our previous example some rules may be:

The position of the car changes based on the velocity. If the distance to the car in front is less than X, decelerate.

MODELING AND SIMULATION

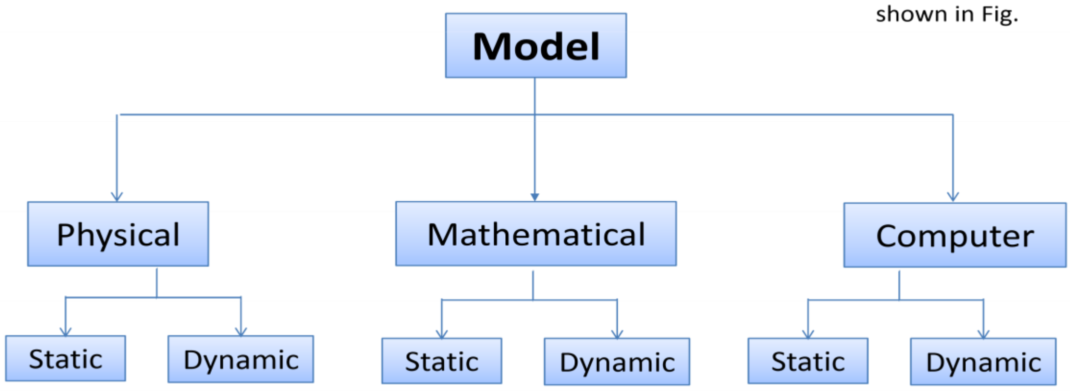
While building a model certain basic principles are to be followed. While making a model one should keep in mind five basic steps.

- Block building
- Relevance
- Accuracy
- Aggregation
- Validation

Types of Models

In the following sections, we will discuss in details the various types of models as

shown in Fig.



Types of computer model

Dynamic System: If a system changes with time, it is called a dynamic system.

Static System: If a system does not change with time, it is called a static system.

PHYSICAL MODELS

Physical models are of two types, static and dynamic.

Static physical model: is a scaled down model of • a system which does not change with time. An architect before constructing a building, makes a scaled down model of the building, which reflects all it rooms, outer design and other important features. This is an example of static physical model

Dynamic physical models are ones which change • with time or which are function of time.

In wind tunnel, small aircraft models (static models) are kept and air is blown over them with different

velocities and pressure profiles are measured with the help of transducers embedded in the model.

Here wind velocity changes with time and is an example of dynamic physical model. A model of a hanging wheel of vehicle is another case of dynamic physical model

What is Mathematical Model?

A set of mathematical equations (e.g., differential eqs.) that describes the input-output behavior of a system.

What is a model used for?

- Simulation
- Prediction/Forecasting
- Prognostics/Diagnostics
- Design/Performance Evaluation
- Control System Design

Mathematical Modelling Basics

Mathematical model of a real world system is derived using a combination of *physical* laws and/or *experimental* means

• Physical laws are used to determine the model structure (linear or nonlinear) and order.
• The parameters of the model are often estimated and/or validated experimentally.
• Mathematical model of a dynamic system can often be expressed as a system of differential (difference in the case of discrete-time systems) equations

Classification of Mathematical Models

- Linear vs. Non-linear

- Deterministic vs. Probabilistic (Stochastic)
- Static vs. Dynamic
- Discrete vs. Continuous
- White box, black box and gray box

How a model can be developed?

Mathematical Methods

- Probability theory, algebraic method ,…
- Their results are accurate
- They have a few Number of parameters
- It is impossible for complex systems

Numerical computer-based simulation

It is simple

It is useful for complex system

Six Step Approach to Dynamic System Problems

- Define the system and its components
- Formulate the mathematical model and list the necessary assumptions
- Write the differential equations describing the model
- Solve the equations for the desired output variables
- Examine the solutions and the assumptions
- If necessary, reanalyze or redesign the system

Differential Equation of Physical Systems

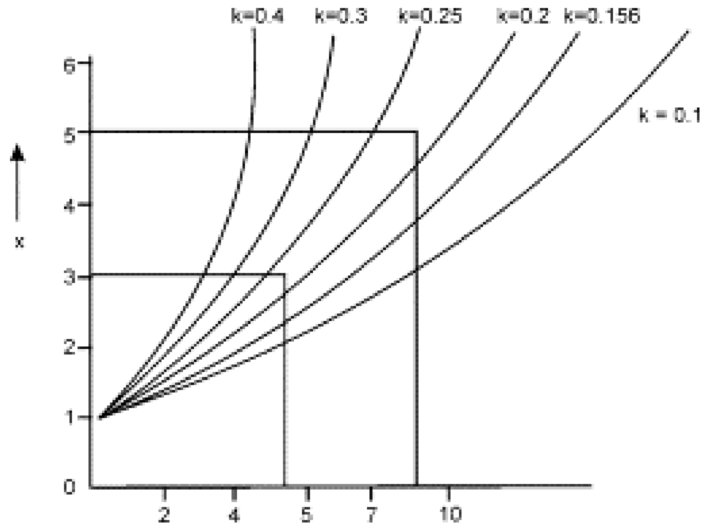
There are various activities in nature, in which rate of change of an entity is proportional to itself. Such entities grow exponentially and study of such models are known as exponential growth models. To understand this concept, let us consider the birth rate of monkeys. Unlike other animals, birth rate of monkeys grows very fast. If not controlled, their population grows exponentially. Let us try to model this simple natural problem mathematically. If in a region, say *x* is the number of monkeys at time *t*, then their rate of growth at time *t* is proportional to their number *x* at that time. Let proportionality constant be *k*. This type of functions can be expressed in the form of differential equations as:

$$\frac{dx}{dt} = kx$$

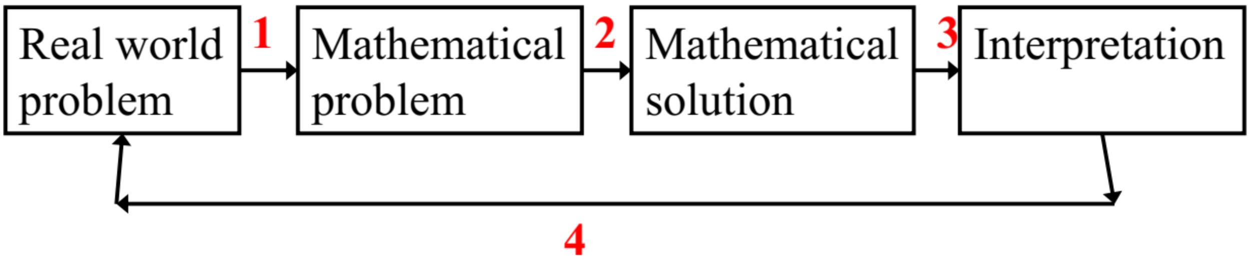
with the condition $x = x_0$ at $t = 0$.

This is first order differential equation and its solution is

$$x = x_0 e^{kt}$$



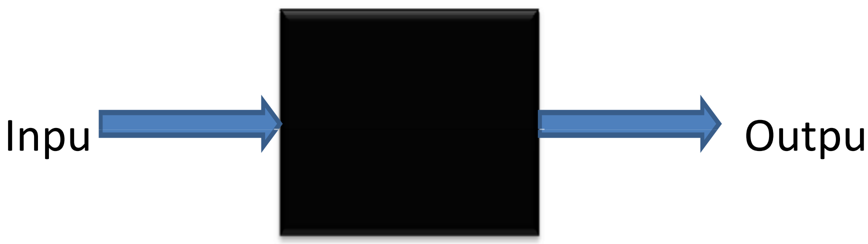
The idea of a (mathematical) model



1- Reality to mathematics
2- Mathematical solution
3- Interpreting the model outputs
4- Using the results in the real world

Black Box Model

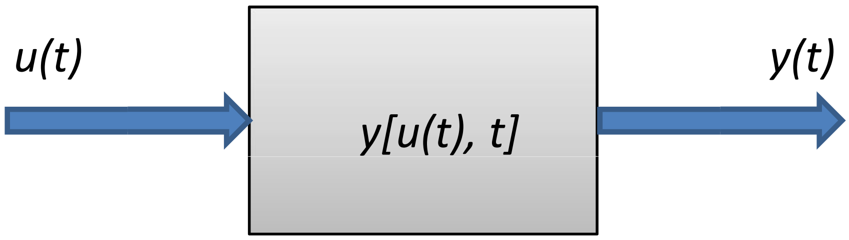
- When only input and output are known.
- Internal dynamics are either too complex or unknown.



Easy to Model •

Grey Box Model

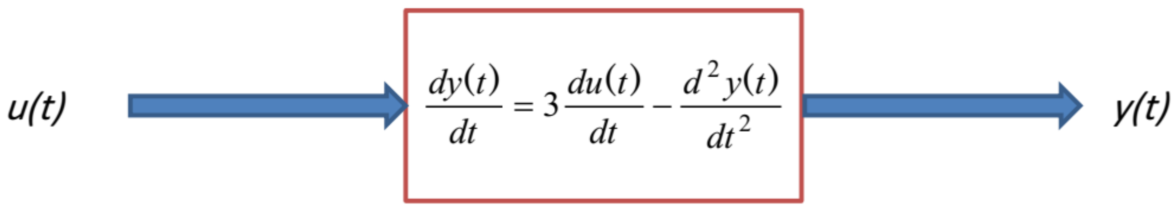
When input and output and some information about the internal dynamics of the system is known.



Easier than white box Modelling.

White Box Model

When input and output and internal dynamics • of the system is known.



One should know and have complete knowledge of the system to derive a white box model.

COMPUTER MODELS

Distributed Lag Models—Dynamic Models •

When a model involves number of parameters and hefty data, one has to opt for computer. Models that have the properties of changing only at fixed intervals of time, and of basing current values of the variables on other current values and values that occurred in previous intervals, are called ***distributed lag models.***

Characterizing a Simulation Model

- **Static or Dynamic Simulation Models**
  - Static simulation model (called Monte Carlo simulation) represents a system at a particular point in time.
  - Dynamic simulation model represents systems as they change over time
- **Deterministic or Stochastic Simulation Models**
  - Deterministic simulation models contain no random variables and have a known set of inputs which will result in a unique set of outputs
  - Stochastic simulation model has one or more random variables as inputs. Random inputs lead to random outputs.
- The model of interest in this class is discrete, dynamic, and stochastic.

Discrete-Event System Simulation

- The simulation models are analyzed by numerical rather than by analytical methods
  - Analytical methods employ the deductive reasoning of mathematics to solve the model.
  - Numerical methods employ computational procedures to solve mathematical models.

Stochastic: some state variables are random Dynamic: time evolution is important

Discrete-Event: significant changes occur at discrete time instances

Model Development

How to develop a model:

1) Determine the goals and objectives
2) Build a ***conceptual*** model
3) Convert into a ***specification*** model
4) Convert into a ***computational*** model
5) Verify
6) Validate

Typically an iterative process

Three Model Levels

1- Conceptual

- Very high level
- How comprehensive should the model be?
- What are the *state variables,* which are dynamic, and which are important?

2- Specification

- On paper
- May involve equations, pseudocode, etc.
- How will the model receive input?

3- Computational

- A computer program
- General-purpose PL or simulation language?

Verification vs. Validation

❑ ***Verification***

- Computational model should be consistent with specification model
- Did we build the model right?

- ***Validation***
  - Computational model should be consistent with the system being analyzed
  - Did we build the right model?
  - Can an expert distinguish simulation output from system output?
- Interactive graphics can prove valuable

Steps in a Simulation Study

- Problem formulation
  - Policy maker/Analyst understand and agree with the formulation.
- Setting of objectives and overall project plan
- Model conceptualization
  - The art of modeling is enhanced by an ability to abstract the essential features of a problem, to select and modify basic assumptions that characterize the system, and then to enrich and elaborate the model until a useful approximation results.
- Data collection
  - As the complexity of the model changes, the required data elements may also change.
- Model translation

  – GPSS/HTM or special-purpose simulation software
- Verified?
  - Is the computer program performing properly?
  - Debugging for correct input parameters and logical structure
- Validated?

- The determination that a model is an accurate representation of the real system.

- Validation is achieved through the calibration of the model

- Experimental design

  - The decision on the length of the initialization period, the length of simulation runs, and the number of replications to be made of each run.

- Production runs and analysis

  - To estimate measures of performances

- More runs?

- Documentation and reporting

  - Program documentation : for the relationships between input parameters and output measures of performance, and for a modification

  - Progress documentation : the history of a simulation, a chronology of work done and decision made.

- Implementation

- Four phases

- First phase : a period of discovery or orientation

  (step 1, step2)

- Second phase : a model building and data collection

  (step 3, step 4, step 5, step 6, step 7) –

  Third phase : running the model

(step 8, step 9, step 10)

- Fourth phase : an implementation (step 11, step 12)

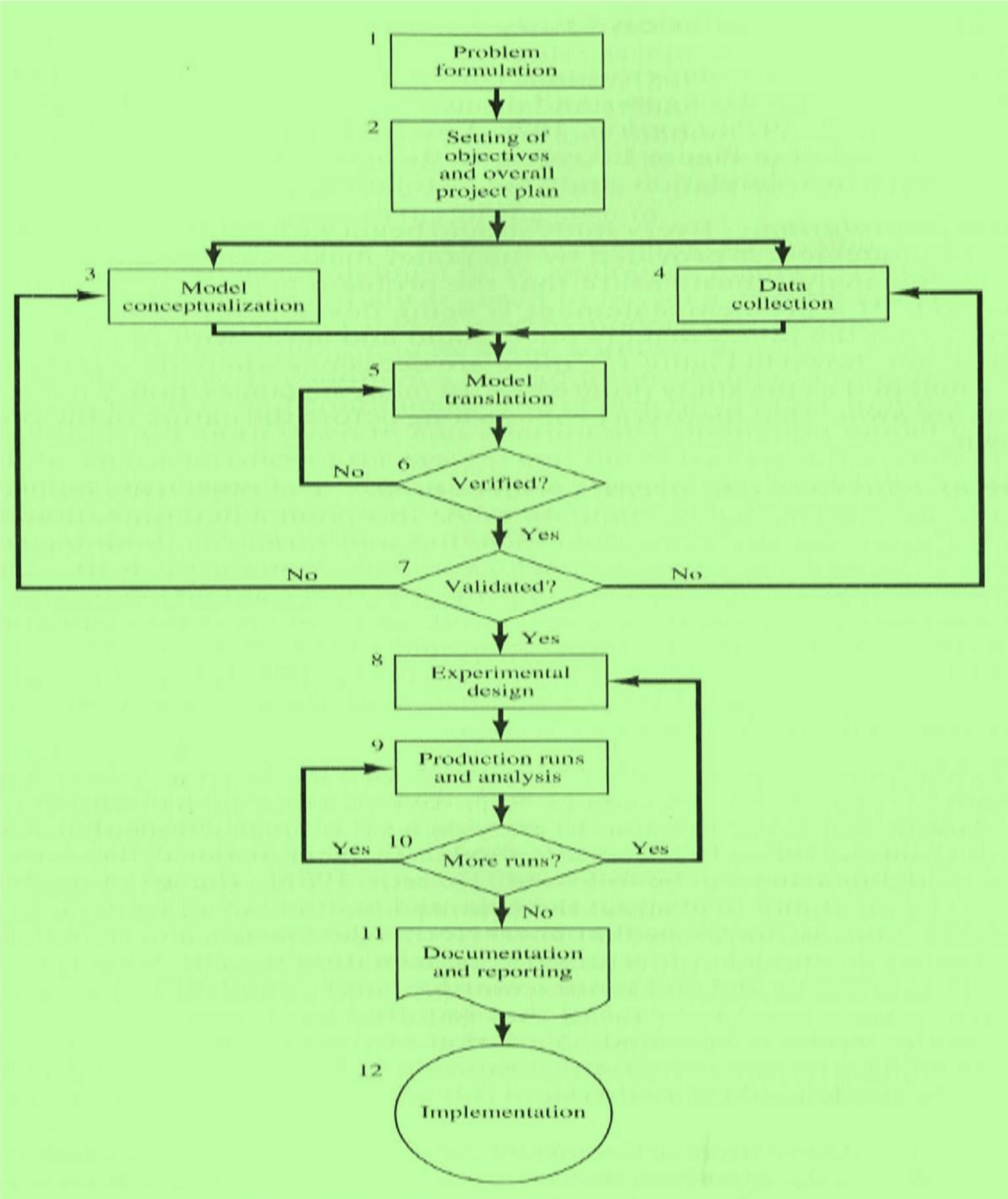
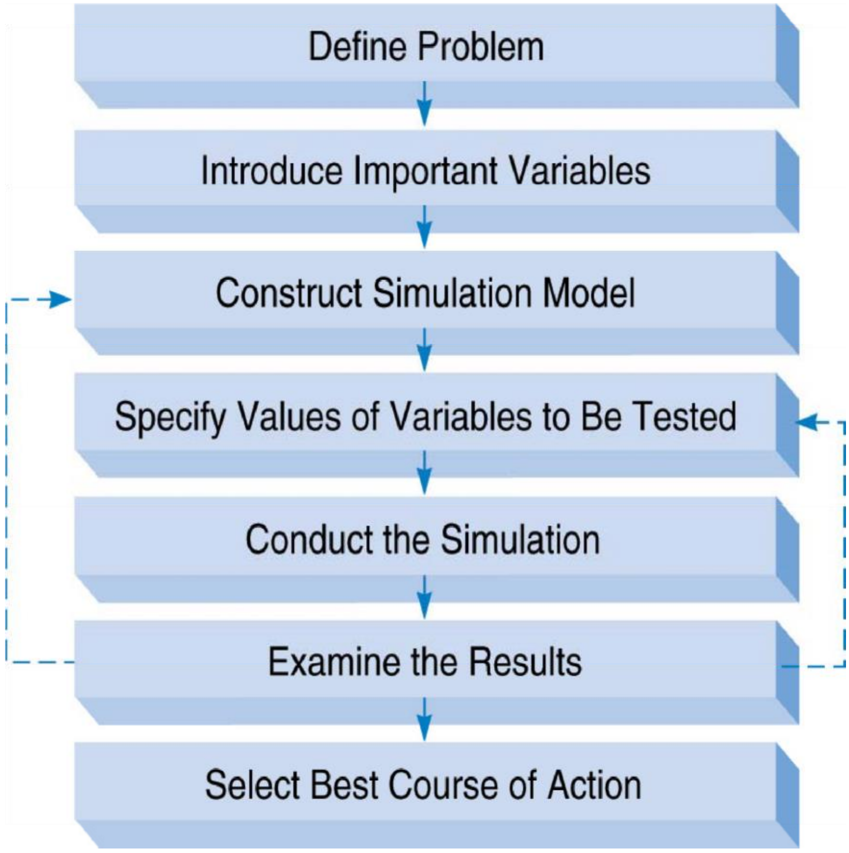


Figure 1.3. Steps in a simulation study.

The Process of Simulation



EXERCISE

1- What is the difference between static and dynamic models?

2- Give an example of a dynamic mathematical model

3- What are the basic steps to be followed while making a model?

4- What are distributed lagged models?

5- Name several entities, attributes, activities for the following systems.

Chapter 3:

Computational modeling

Introduction

Simulation and modeling

Probably the single most important scientific use – of computing today Having an impact on quantitative fields, such as – chemistry, biology, medicine, meteorology, ecology, geography, economics, and so on.

Computational Modeling: Introduction to Systems and Models

- The scientific method
  - Observe the behavior of a system
  - Formulate a hypothesis about system behavior
  - Design and carry out experiments to prove or disprove the validity of the hypothesis
- Often a model of the system is used

A model

An abstraction of the system being studied that we claim behaves much like the original

Computer simulation

- A physical system is modeled as a set of mathematical equations and/or algorithmic procedures
- Model is translated into a high-level language and executed on the Von Neumann computer
- Computational models
- Also called simulation models
- Used to
- Design new systems
- Study and improve the behavior of existing systems
- Allow the use of an interactive design methodology (sometimes called computational steering)
- Used in most branches of science and engineering

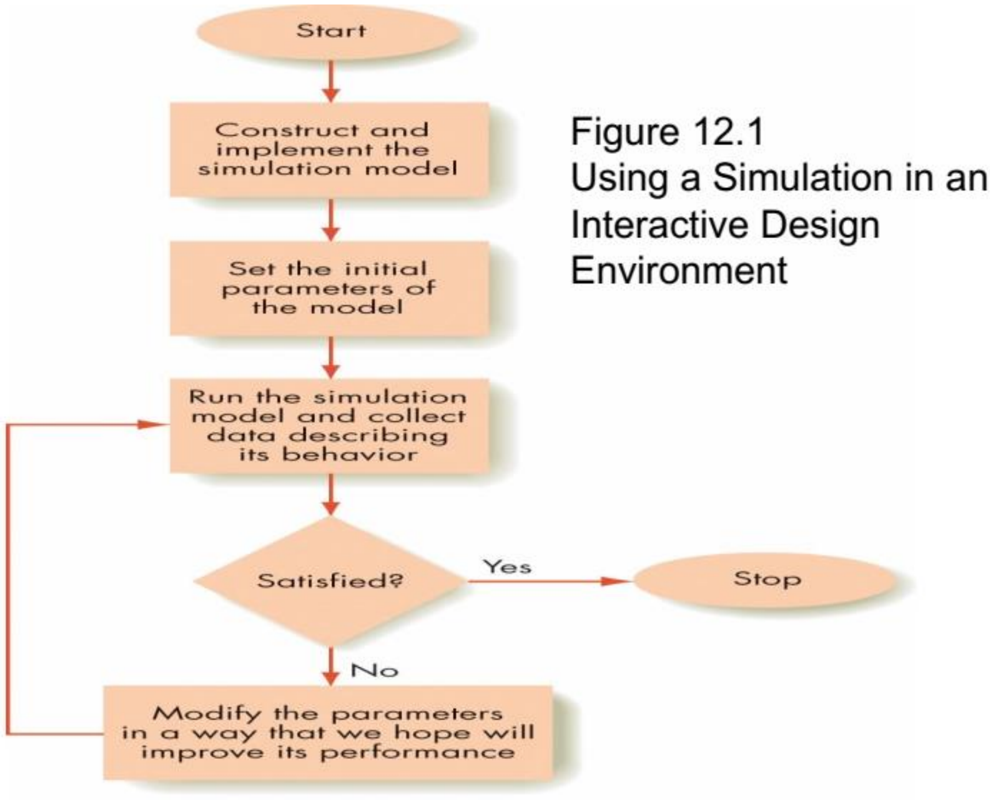

Start
Construct and implement the simulation model
Set the initial parameters of the model
Run the simulation model and collect data describing its behavior
Satisfied?
Yes
Stop
No
Modify the parameters in a way that we hope will improve its performance
Figure 12.1
Using a Simulation in an Interactive Design Environment

Figure 3-1

Computational Models, Accuracy, and Errors

Proper balance between accuracy and complexity must be achieved
A model must be both –

- An accurate representation of the physical system
- Simple enough to implement as a program and solve on a computer in a reasonable amount of time

To build a model

- Include important factors that act on the system
- Omit unimportant factors that only make the model harder to build, understand, and solve

Continuous model

- A set of equations that describe the behavior of a system as a continuous function of time *t*

Models that use statistical approximations

- Needed for systems that cannot be modeled using precise mathematical equations

An Example of Model Building

Discrete event simulation

1- One of the most p popular and widely used techniques for building computer models
2- The behavior of a system is modeled only at an explicit and finite set of times
   a- Only the times when an event takes place are modeled
3- Event: An activity that changes the state of the system

4- To process an event Change the state of the simulated system in the same way that the actual system would change if the event had occurred in real life
5- Once finished, move to the next event
6- When simulation is complete, the program displays results that characterize the system's behavior

Practical Problem

You are the owner of a new take-out restaurant, McBurgers, currently under construction You want to determine the proper number of checkout stations needed You decide to build a model of McBurgers to determine the optimal number of servers

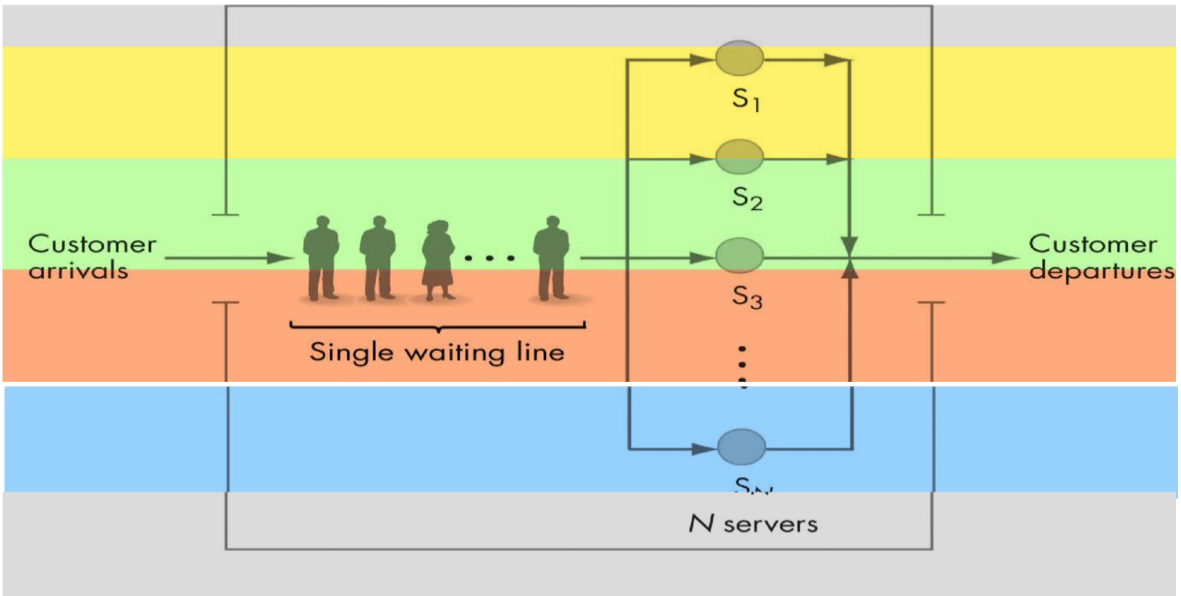


Figure 3-2

System to Be Modeled

First: Identify the events that can change the system

1- A new customer arriving –
2- An existing customer departing after receiving food and paying

Next: Develop an algorithm for each event

1- Should describe exactly what happens to the system when this event occurs

New Customer Arrival
  Record the time that this customer entered the restaurant
  Check if any one of the N servers $S_1, S_2, \ldots, S_N$ is currently idle
  If everyone is busy then
    Put this customer at the end of the waiting line
    Increase the length of the waiting line by 1
  Else (server S_i is idle)
    Mark that server S_i is now busy
    Determine how long it will take to serve this customer, call that value T_{serve}
    Schedule a customer departure event for (*current time* + T_{serve})
    Increase the total time that server S_i has worked by T_{serve}
End of New Customer Arrival

Figure 3.3 Algorithm for New Customer Arrival

The algorithm for the new customer arrival event uses a statistical distribution (Figure 3.3) to determine the time required to service the customer Can model the statistical distribution of customer service time using the algorithm in Figure 3.5

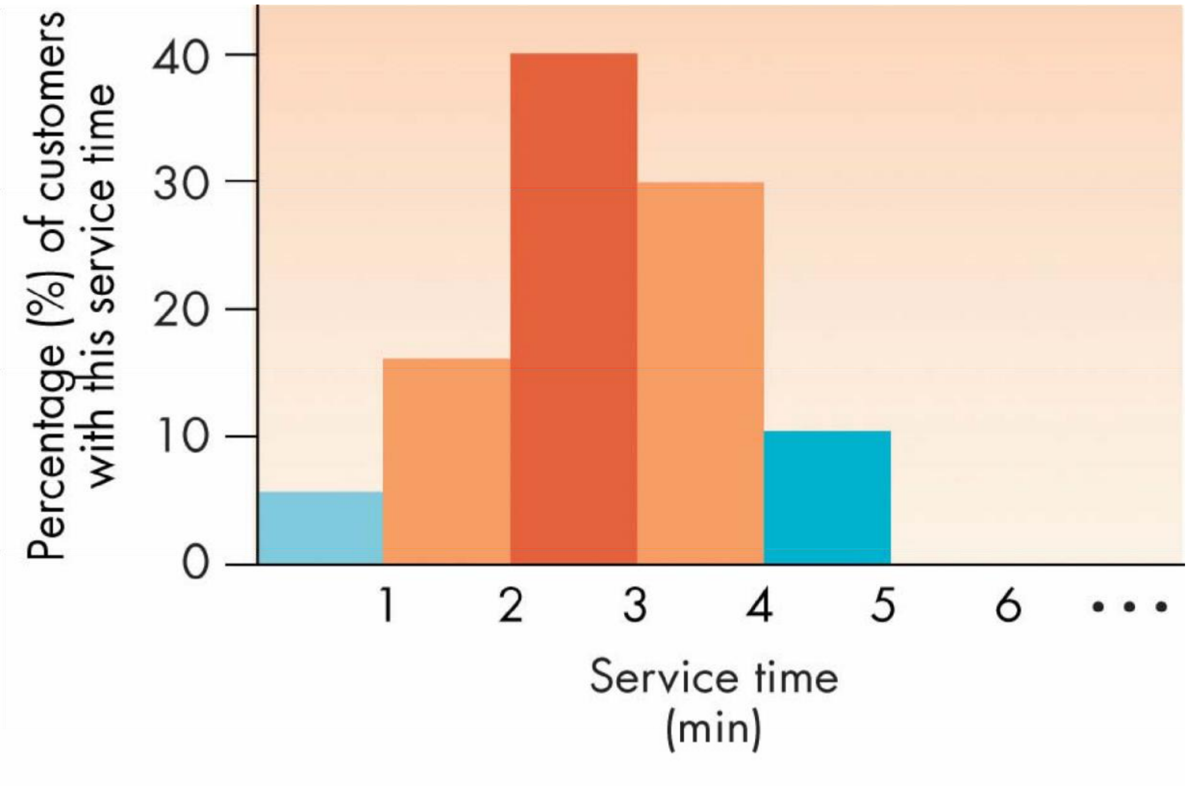


Figure 3.4 Statistical Distribution of Customer Service Time

Generate a uniform random integer value v between 1 and 100
If v is in the range 1–5, then
    Set T_{serve} to a uniform random number between 0.0 and 1.0
Else if v is in the range 6–20, then
    Set T_{serve} to a uniform random number between 1.0 and 2.0
Else if v is in the range 21–60, then
    Set T_{serve} to a uniform random number between 2.0 and 3.0
Else if v is in the range 61–90, then
    Set T_{serve} to a uniform random number between 3.0 and 4.0
Else
    Set T_{serve} to a uniform random number between 4.0 and 5.0

Figure 3-5 Algorithm for Generating Random Numbers That Follow the Distribution Given in Figure 3.4

Customer Departure from Server S_i
    Determine the total time that this customer spent in the restaurant
    If there is someone in line then
        Take the next customer out of line and decrease the waiting line size by one
        Determine how long this new customer will take to be served,
        call that value T_{serve}
        Schedule a customer departure event for (current time + T_{serve})
        Increase the total time that server S_i has worked by T_{serve}
    Else
        Mark this server as idle
End Customer Departure

Figure 3-6 Algorithm for Customer Departure Event

Must initialize parameters to the model
Model must collect data that accurately measures performance of the McBurgers restaurant

When simulation is ready, the computer will

1- Run the simulation
2- Process all *M* customers
3- Print out the results –

```
Main Part of the Simulation Model
    Set current time to 0
    Set the waiting line size to 0
    Get an input value for N, the number of servers
    Set all N servers, S1, S2, ... , SN to idle
    Get an input value for M, the total number of customers
    Schedule M customer arrivals and put them on the list of events
        Each arrival occurs Tinterval time units after the previous one
    While there is still a scheduled event on the list do
        Get the next event on the list
        Move current time to the time of this event
        If this is a customer arrival event
            Execute the arrival algorithm of Figure 12.4
        Else
            Execute the departure algorithm of Figure 12.7
        Remove this event from the list of all scheduled events
    End of the loop
    Print out a set of data that describes the behavior of our system
Stop
```

Figure 3-7 The Main Algorithm of Our Simulation Model

Running the Model and Visualizing Results

Scientific visualization

a- Visualizing data in a way that highlights its important characteristics and simplifies its interpretation
b- An important part of computational modeling
c- Different from computer graphics

Scientific visualization is concerned with
1- Data extraction: Determine which data values are important to display and which are not
2- Data manipulation: Convert the data to other – forms or to different units to enhance display

Output of a computer model can be represented visually using:
A two-dimensional graph – A three-dimensional image – Visual representation of data helps identify important features of the model's output

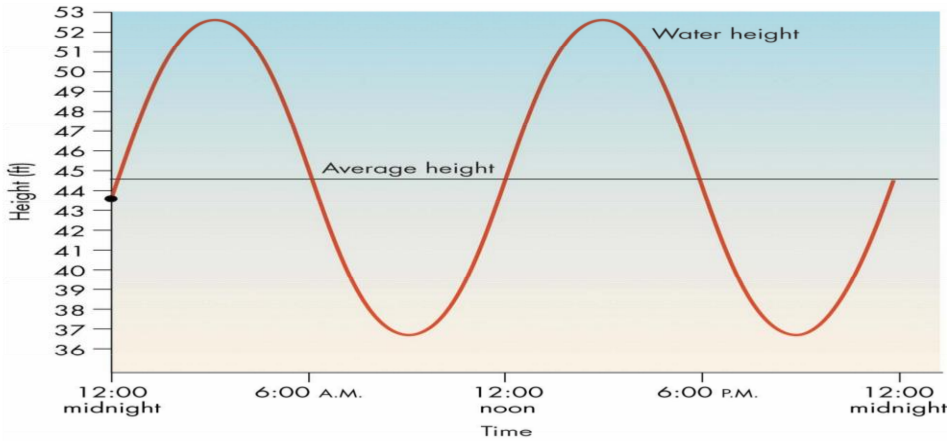


Figure 3-8 Using a Two-Dimensional Graph to Display Output

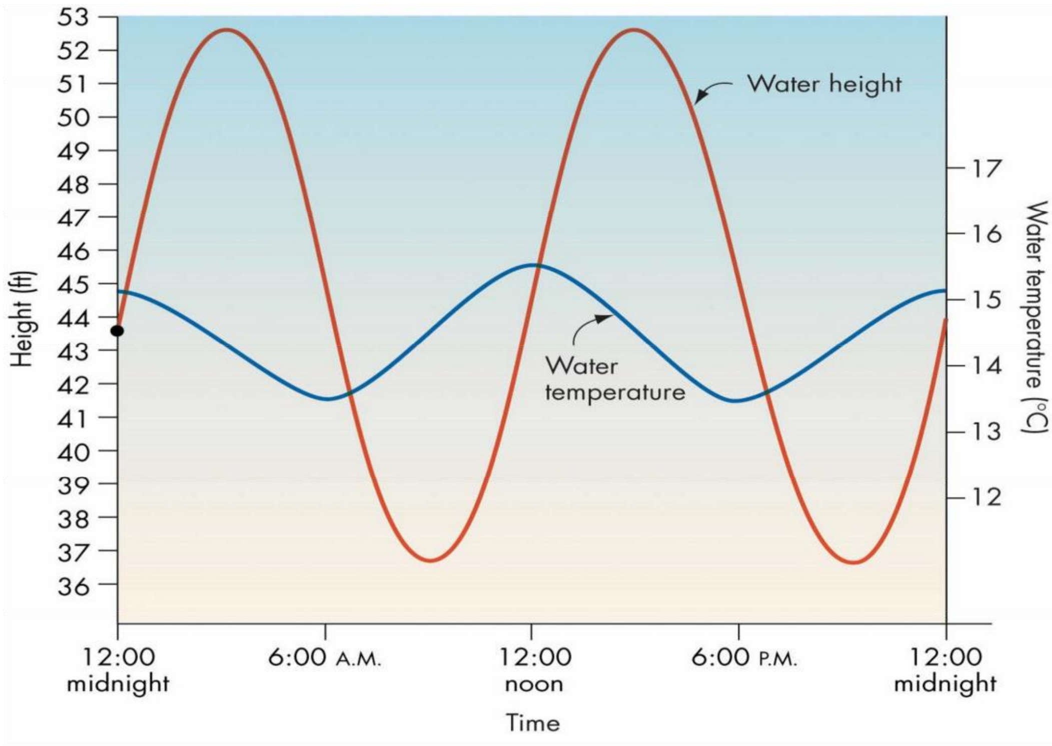


Figure 3-9 Using a Two-Dimensional Graph represents a Comparison of Two Data Values

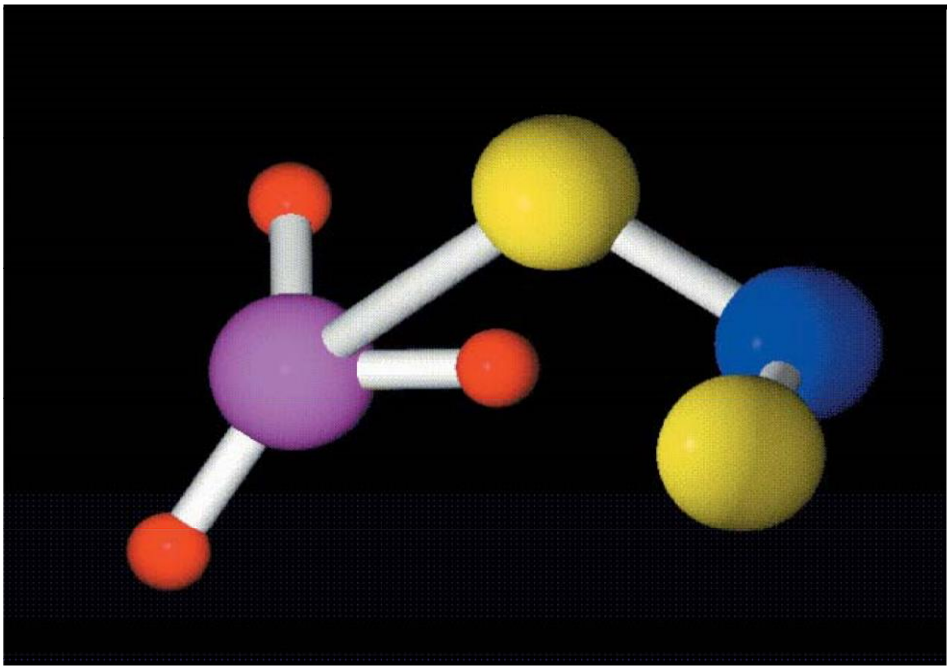


Figure 3-10 Three-Dimensional Model of a Methyl Nitrite Molecule

Image animation

One of the most powerful and useful forms of – visualization Shows how model's output changes over time Created using many images, each showing system state at a slightly later point in time

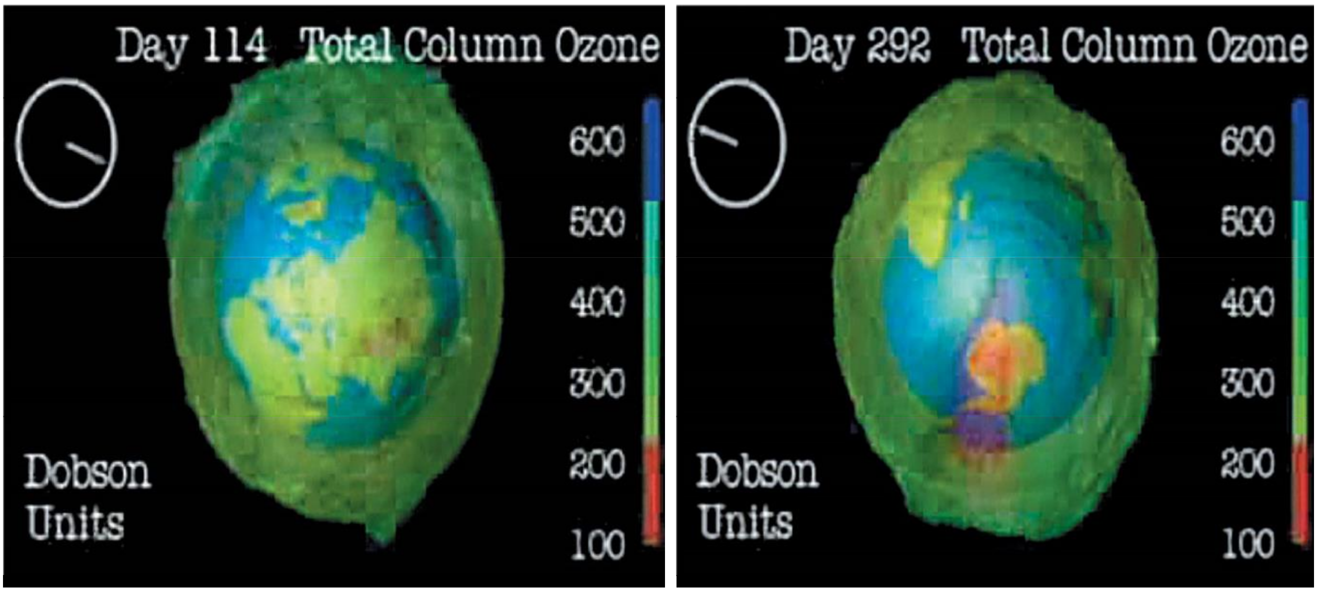


Figure 3-11 Use of Animation to Model Ozone Layers

Chapter 4:

Simulation Software and Monte Carlo Simulation

Simulation

Simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical models over extended period of real time.

We thus define *system simulation as the • technique of solving problems by the observation of the performance, over time, of a dynamic model of the system*. In other words, we can define simulation as an experiment of physical scenario on the computer

Simulation can also be defined as a technique of • performing sampling experiments on the model of the system. This is called stochastic simulation and is a part of simulation techniques. Because sampling from a particular probability distribution involves the use of random numbers, stochastic simulation is sometimes called *Monte Carlo Simulation*

Historically, Monte Carlo method is considered to be a technique, using random or pseudo random numbers. It is important to know what random numbers are.

Let us take a simple example of tossing a coin. If coin is unbiased, probability of coming head is 0.5. If we generate two numbers say, 0 and 1, so that occurrence of both is equally likely. Let us assume that number 1 depicts head and 0, tail. These numbers are called uniform random numbers.

We will discuss stochastic simulation in chapter four •

Monte Carlo Simulation

We give below some differences between the Monte Carlo method and simulation:

1. In the Monte Carlo method, time does not play as substantial role, a role as it does in stochastic simulation.
2. The observations in the Monte Carlo method, as a rule, are independent. In simulation, however, we experiment with the model over time so, as a rule, the observations are serially correlated.
3. In the Monte Carlo method, it is possible to express the response as a rather simple function of the stochastic input variates. In simulation the response is usually a very complicated one and can be expressed explicitly only by the computer program

Monte Carlo Simulation

Use the fundamental theory and logic of the Monte Carlo Simulation technique to solve the following optimization problem:

Maximize $Z = (e^{X_1} + X_2)^2 + 3 (1 - X_3)^2$

Subject to:

$0 \le X_1 \le 1$

$0 \le X_2 \le 2$

$2 \le X_3 \le 3$

Step 1:

Set j = 1 and choose a large value for N where:

j = trial number

N = total number of trials

Step 2:

Generate a proper random number, RN1 , over 0 and 1.

Step 3:

Generate another proper random number, RN2 , over 0 and 2.

Step 4:

Generate another proper random number, RN3 , over 2 and 3.

Step 5:

Substitute RN

1 for X1 , RN2 for X2 , and RN3 for X3 in the objective function. Store its value in Z(j) and record the corresponding values for X1 , X2 , and X3.

Step 6:

Add 1 to j. If j > N, go to Step 7; otherwise, go to Step 2.

Step 7:

The approximate solution of the problem is determined by the

values of X

1 (= RN1), X2 (= RN2), and X3 (= RN3), which

correspond to the maximum value of { Z(j), j = 1, 2, 3, ..., N }.

• NOTE: As $N \to \infty$, $X1 \to 1$, $X2 \to 2$, and $X3 \to 3$.

Simulation Software

1- GENERAL PURPOSE LANGUAGES USED FOR SIMULATION

2- GENERAL PURPOSE SIMULATION LANGUAGES

3- SPECIAL PURPOSE PACKAGES USED FOR SIMUL (Simulation Environment)

GEN. PURPOSE LANGUAGES USED FOR SIMULATION
FORTRAN
Probably more models than any other language. –
PASCAL
Not as universal as FORTRAN –
MODULA
Many improvements over PASCAL –
ADA
Department of Defense attempt at standardization
C, C++
Object-oriented programming language –
MODELING W/ GENERAL SIMULATION LANGUAGES

- Advantages:

Standardized features often needed in modeling –

Shorter development cycle for each model –

Much assistance in model verification –

Very readable code –

- Disadvantages:

Higher software cost (up-front) –

Additional training required – Limited portability –

GENERAL PURPOSE SIMULATION LANGUAGES

- GPSS

Block-structured Language –

Interpretive Execution –

FORTRAN-based (Help blocks) –

World-view: Transactions/Facilities –

- SIMSCRIPT II.5

English-like Problem Description Language –

Compiled Programs –

Complete language (no other underlying language) – *World-view:* Processes/ Resources/ Continuous –

MODSIM III ●

Modern Object-Oriented Language –

Modularity Compiled Programs –

Based on Modula2 (but compiles into C) –

World-view: Processes –

SIMULA ●

ALGOL-based Problem Description Language –

Compiled Programs – *World-view:* Processes –

SLAM ●

Block-structured Language –

Interpretive Execution –

FORTRAN-based (and extended) –

World-view: Network / event / continuous –

CSIM ●

process-oriented language –

C-based (C++ based) –

World-view: Processes –

MODELING W/ SPECIAL-PURPOSE SIMUL. PACKAGES

Advantages ●

Very quick development of complex models –

Short learning cycle –

No programming--minimal errors in usage –

Disadvantages ●

High cost of software –

Limited scope of applicability –

Limited flexibility (may not fit your specific application) –

SPECIAL PURPOSE PACKAGES USED FOR SIMUL.

NETWORK II.5 ●

Simulator for computer systems –

OPNET ●

Simulator for communication networks, including – wireless networks

COMNET III ●

Simulator for communications networks –

SIMFACTORY ●

Simulator for manufacturing operations –

selection of simulation software

You choose simulation software depending on:

- **Model building feature:**

. 1. input data analysis capability
2. graphical model building
3. conditional routing
4. simulation programming
5. Syntax
6. input flexibility
7. modelling conciseness
8. specialized components and templates
9. user-built objects
10. interface with general programming

- **Runtime environment:**
  1. execution speed
  2. model size
  3. interactive debugger
  4. model status and statistic

Chapter 5:

Introduction to Probability

Objective:

-BASIC PROBABILITY CONCEPTS

-Students will be able to find the probability of a simple event.

-Students will be able to understand the distinction between simple events and compound events.

Essential Question:

(1) How do I find the probability of a simple event?

(2) How can I distinguish between a simple and compound event?

probability theory :

The theory of probability plays an important role in the applications and theories of statistics and the theory of probability means the study of random experiments

What threw a stone in the air within the certainty that it falls on the ground, but not sure that the number (4) example "will appear and this and most examples of probability is shown on The following experiments:

(1) Dice, (2) Spinners

Definition

Experiment ==> any planned process of data collection. It consists of a number of trials (replications) under the same condition.

In probability, an experiment is any process that can be repeated in which the results are uncertain.

Definition

-Probability deals with **experiments** that yield random short-term results or outcomes, yet reveal long-term predictability.

-The long-term proportion with which a certain outcome is observed is the probability of that outcome**.**

Random – outcomes that occur at random if each outcome is equally likely to occur.

Definition

A **simple event:** is any single outcome from a probability experiment. Each simple event is denoted using small letters such as (ei).

An event: is any collection of outcomes from a probability experiment. An event may consist of one or more simple events. Events are denoted using capital letters such as (E).

The **probability of an event,** denoted P(E), is the likelihood of that event occurring.

Definition

Sample Point : Each possible outcome of an experiment is called a **sample point**.

The sample space, S, of a probability experiment is the collection of all possible simple events. In other words, the sample space is a list of all possible outcomes of a probability experiment. **Examples that use sample space** Roll a dice, S= {1,2,3,4,5,6}

Definition

Complementary Events – the events of one ▪ outcome happening and that outcomes not happening are complimentary; the sum of the probabilities of complementary events is 1.

Outcome : one possible result of a probability

An **unusual event** is an event that has a low probability of occurring.

Probability of Simple Events

What is a PROBABILITY?

- Probability is the chance that some event will happen

- It is the ratio of the number of ways a certain event can occur to the number of possible outcomes

Probability of Simple Events

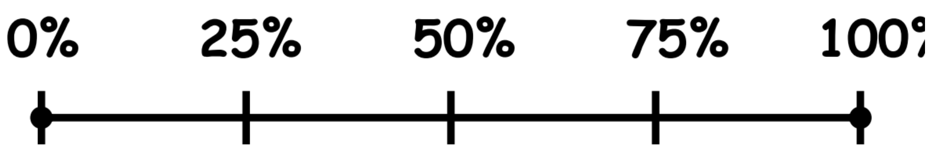
What is the probability

$$P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$$

Examples that use Probability:

(1) Dice, (2) Spinners, (3) Coins, (4) Playing cards, (5) Evens/Odds, (6) Alphabet, etc.

Probability of Simple Events



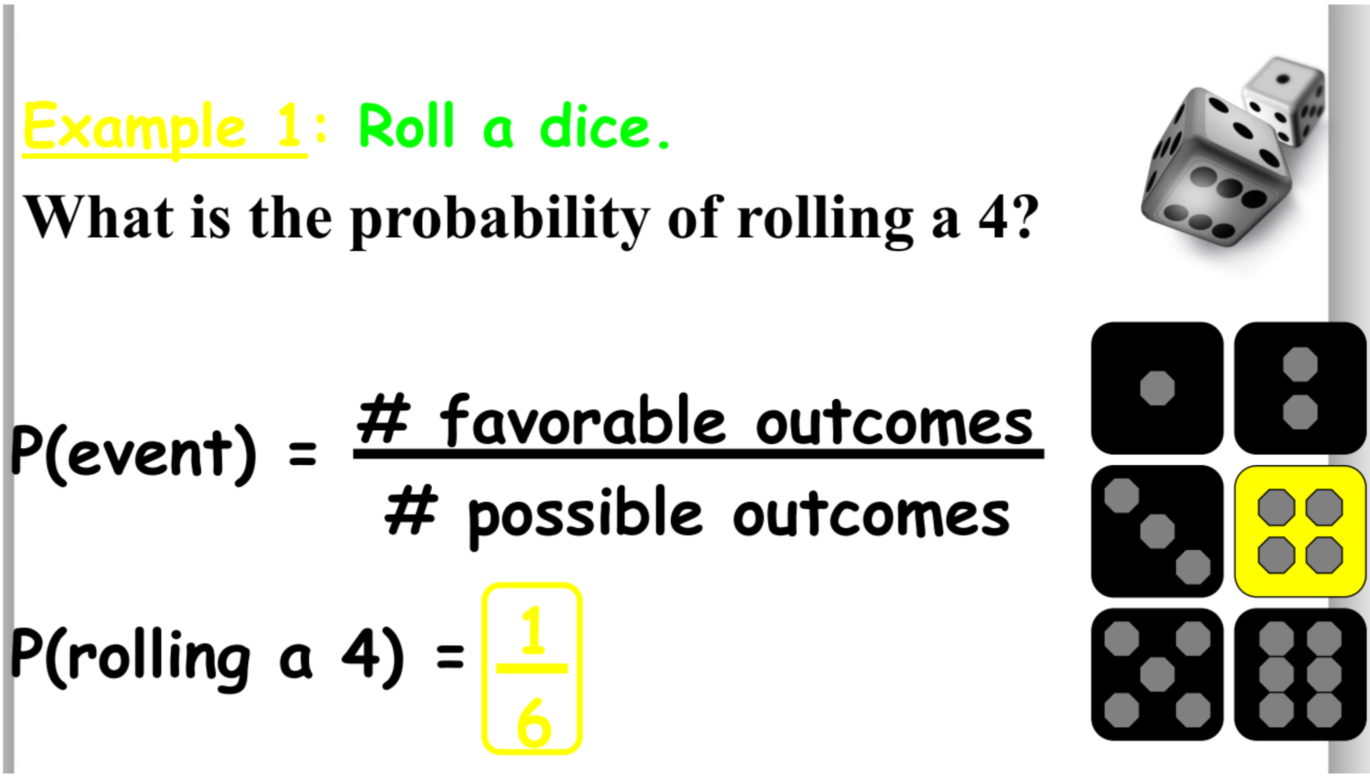
0	$\frac{1}{4}$ or .25	$\frac{1}{2}$ Or .5	$\frac{3}{4}$ or .75	1
Impossible	Not Very Likely	Equally Likely	Somewhat Likely	Certain

Identify gin Events and the Sample EXAMPLE Space of a Probability Experiment
Consider the probability experiment of having two children.

(a) Identify the simple events of the probability experiment. (b) Determine the sample space. (c) Define the event E= “have one boy”.

Answer:

a) S={ (boy, girl), (girl, girl), (boy, boy)}
b)S={ (boy, girl), (girl, girl), (boy, boy)}
c)S= { (boy, girl), (boy, boy)}



Example 2: Roll a dice.

What is the probability of rolling an even number?

The probability of rolling an even number is 3 out of 6 or .5 or 50%

Example 3: Spinners. What is the probability of spinning green?

$$P(\text{event}) = \frac{\#\text{ favorable outcomes}}{\#\text{ possible outcomes}}$$

$$P(\text{green}) = \frac{1}{4} = \boxed{\frac{1}{4}}$$

The probability of spinning green is 1 out of 4 or .25 or 25%

Probability of Simple Events

Example 4: Flip a coin. What is the probability of flipping a tail?

$$P(\text{event}) = \frac{\#\text{ favorable outcomes}}{\#\text{ possible outcomes}}$$

$$P(\text{tail}) = \frac{1}{2} = \boxed{\frac{1}{2}}$$

The probability of spinning green is 1 out of 2 or .5 or 50%

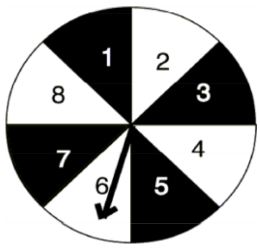
- **Probability is the chance that some event will happen**

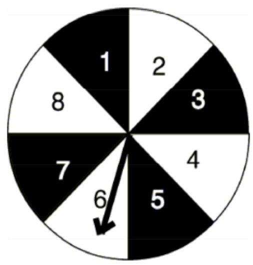
- **It is the ratioof the <u>number of ways a certain even can occur</u> to the <u>total number of possible outcomes</u>**

<u>Guided Practice:</u> Calculate the probability of each independent event.

1) P(black) = 2) P(1) = 3) P(odd) = 4) P(prime) =

<u>Guided Practice:</u> Answers.





1) P(black) = $^4/_8$

2) P(1) = $^1/_8$

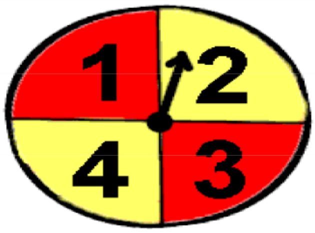
3) P(odd) = $^1/_2$

4) P(prime) = $^1/_2$

<u>Independent Practice:</u> Calculate the probability of each independent event.

1) P(red) = 2) P(2) = 3) P(not red) = 4) P(even) =

Independent Practice: Answers.



1) P(red) = $^1/_2$ 2) P(2) = $^1/_4$ 3) P(not red) = $^1/_2$ 4) P(even) = $^1/_2$

Probability of Simple Events

Real World Example:

A computer company manufactures [1],500 computers each day. An average of 100 of these computers are returned with defects. What is the probability that the computer you purchased is not defective?

= 2,400 = 24 P(not defective) = # not defective

Experimental vs. Theoretical

Experimental probability:

P(event) = number of times event occurs total number of trials Theoretical probability: P(E) = number of favorable outcomes total number of possible outcomes

[1] **2,500 25total # manufactured**

Experimental probability

Experimental probability is found by repeating an **experiment** and observing the **outcomes**.

P(head)= 3/10

A head shows up 3 times out of 10 trials,

P(tail) = 7/10 A tail shows up 7 times out of 10 trials

Theoretical probability

P(head) = 1/2

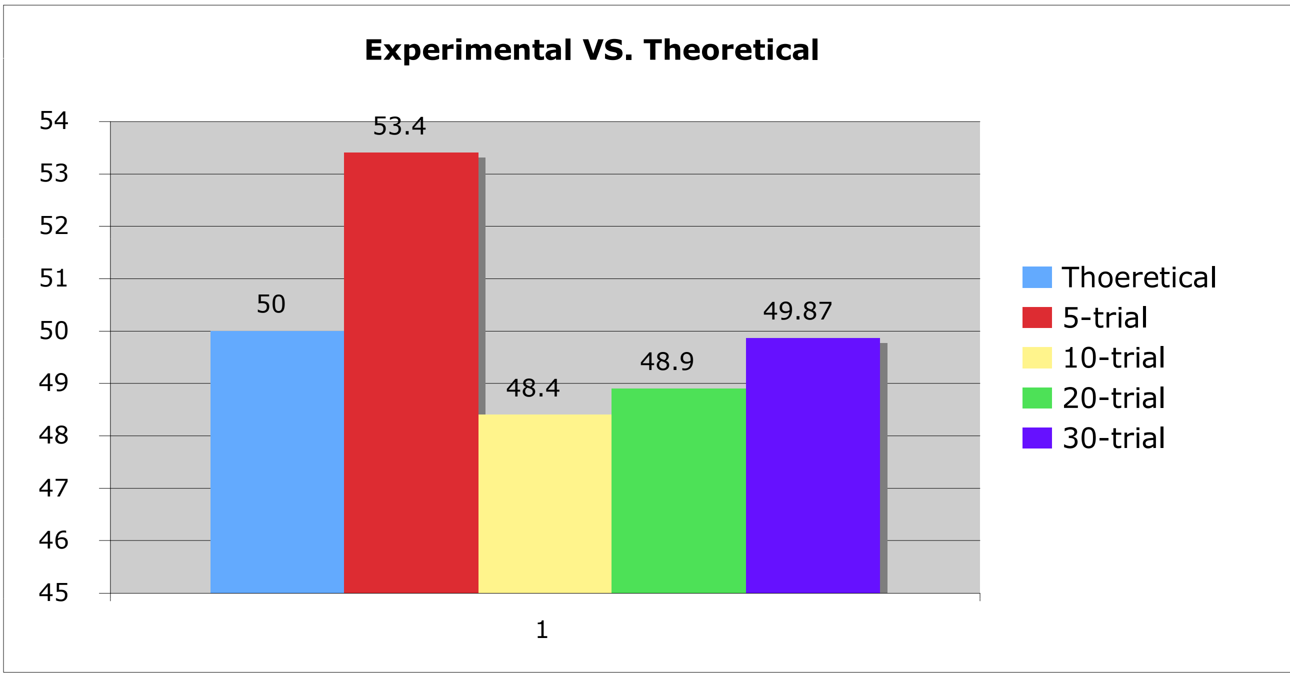
P(tail) = 1/2

Since there are only two outcomes, you have 50/50 chance to get a head or a tail.

How come I never get a theoretical value in both experiments? Tom asked.

If you repeat the experiment many times, the results will • getting closer to the theoretical value.

Law of the Large Numbers •



Law of the Large Numbers

The Law of Large Numbers was first published in • 1713 by Jocob Bernoulli.

It is a fundamental concept for probability and • statistic.

This Law states that as the number of trials • increase, the experimental probability will get closer and closer to the theoretical probability.

Contrast experimental and theoretical probability

Experimental probability is the result of an experiment.

Theoretical probability is, what is expected to happen.

*Three students tossed a coin 50 times individu*ally.

- Lisa had a head 20 times. (20/50 = 0.4)
- Tom had a head 26 times. (26/50 = 0.52) Experimental
- **Al had a head 28 times. (28/50 = 0.56)**
- Please compare their results with the theoretical probability.
- It should be 25 heads. (25/50 = 0.5) Theoretical

Properties of Probabilities

The probability of any event E, P(E), must be .١ between 0 and 1 inclusive.

That is, $0 \leq P(E) \leq 1$.

2. If an event is **impossible**, the probability of the event is 0.
3. If an event is a **certainty,** the probability of the event is 1.
4. If S= $\{e_1, e_2, \ldots, e_n\}$, then

$$P(e_1) + P(e_2) + \ldots + P(e_n) = 1.$$

Three methods for determining the probability of an event:

(1) the classical method

(2) the empirical method

(3) the subjective method

The classical method

The classical method of computing probabilities requires equally likely outcomes.

An experiment is said to have **equally likely outcomes** when each simple event has the same probability of occurring.

If an experiment has unequally likely simple events and if the number of ways that an event
Ecan occur is m, then the probability of E, P(E), is

$$P(E) = \frac{\text{Number of way that E can occur}}{\text{Number of Possible Outcomes}} = \frac{m}{n}$$

So, if S is the sample space of this experiment,

$$P(E) = \frac{N(E)}{N(S)}$$

Computing Probabilities Using the EXAMPLE Classical Method

Suppose a “fun size” bag of M&Ms contains 9 brown candies, 6 yellow candies, 7 red candies, 4 orange candies, 2 blue candies, and 2 green candies. Suppose that a candy is randomly selected. (a) What is the probability that it is brown? (b) What is the probability that it is blue?

Empirical Method

The probability of an event E is approximately the number of times event E is observed divided by the number of repetitions of the experiment.

$$P(E) \approx \text{relative frequency of E}$$

$$= \frac{\text{frequency of E}}{\text{number of trials of experiment}}$$

Using Relative Frequencies to EXAMPLE Approximate Probabilities

The following data represent the number of homes with various types of home heating fuels based on a survey of 1,000 homes.

(a) Approximate the probability that a randomly selected home uses electricity as its home heating fuel.

(b) Would it be unusual to select a home that uses coal or coke as its home heating fuel?

HOUSE HEATING FUEL	Frequency
Utility gas	504
Bottled, tank, or LP gas	64
Electricity	307
Fuel oil, kerosene, etc.	94
Coal or coke	2
Wood	17
Solar energy	1
Other fuel	4
No fuel used	7

Subjective method

Subjective probabilities are probabilities obtained based upon an educated guess.
It is an estimate that reflects a person’s opinion, or best guess about whether an outcome will occur.

For example, there is a 40% chance of rain tomorrow.

Chapter 6:

Probability of Simple Events

1. Events, Sample Spaces, and Probability
2. Unions and Intersections
3. Complementary Events
4. The Additive Rule and Mutually Exclusive Events
5. Conditional Probability
6. The Multiplicative Rule and Independent Events

Experiments

example: Observe Gender

1. **Mutually Exclusive** **two** outcomes can not occur at the same time

— Male & Female in same person

2. **Collectively Exhaustive**

One outcome in sample space must occur.

— Male or Female

Events

1. Specific collection of sample points
2. Simple Event Contains only one sample point
1. Compound Event Contains two or more sample points

Probability Rules for Sample Points

Let p_i represent the probability of sample point *i*.

1. All sample point probabilities *must* lie between 0 and 1 (i.e., $0 \leq p_i \leq 1$).
2. The probabilities of all sample points within a sample space *must* sum to 1 (i.e., $\Sigma p_i = 1$).

Steps for Calculating Probability

1. Define the experiment; describe the process used to make an observation and the type of observation that will be recorded
2. List the sample points
3. Assign probabilities to the sample points
4. Determine the collection of sample points contained in the event of interest
5. Sum the sample points probabilities to get the event probability

Combinations Rule

A sample of *n* elements is to be drawn from a set of *N* elements. The, the number of different samples possible

is denoted by $\binom{N}{n}$ and is equal to

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}$$

where the factorial symbol (!) means that

$$n! = n(n-1)(n-2)\ldots(3)(2)(1)$$

For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ 0! is defined to be 1.

Statistical dependent

1. Event occurrence does ***not*** affect probability of another event
   - Toss 1 coin twice
2. Causality not implied
3. Tests for independence
   - $P(A \mid B) = P(A)$
   - $P(B \mid A) = P(B)$
   - $P(A \cap B) = P(A) \times P(B)$

Addition Rules of Probability

If events A and B are mutually exclusive, then

(ME)→ $P(A \text{ or } B) = P(A) + P(B)$

If events A and B are NOT mutually exclusive, then

(NOT)→ $P(A \text{ or } B) = P(A) + P(B) - P(\text{both})$

Coming up with the probabilities in our probability model:

Example: A couple wants three children. What are the numbers of girls they could end up with? What are the probabilities for each outcome? Create a probability table. To calculate the probabilities for each possible value, we can use the **addition rule**.

Sample space: *Let X = possible number of girls*: {0, 1, 2, 3}

➔ *P*(X = 0) = *P*(BBB) = 1/8

➔ *P*(X = 1) = *P*(BBG or BGB or GBB) = *P*(BBG) + *P*(BGB) + *P*(GBB) = 3/8

➔ *P*(X = 2) = *P*(BGG or GBG or GGB) = *P*(BBG) + *P*(BGB) + *P*(GBB) = 3/8 ➔ *P*(X = 3) = *P*(GGG) = 1/8

Value of X	0	1	2	3
Probability	1/8	3/8	3/8	1/8

Important: Note that the sample space in this question is NOT {BBB, BBG, BGB, GBB, GGB, GBG, BGG, GGG}

Remember that it is vitally important to <u>properly</u> identify your sample space!!!

Examples using a Probability Table

X = # of girls in a household with 3 children

Value of X	0	1	2	3
Probability	1/8	3/8	3/8	1/8

Example: What percentage of households has fewer than 2 girls? Answer:

$P(X<2) = 3/8 + 1/8 = 0.5$

Example: What is the probability that a randomly selected household has at least one girl?

Answer: $P(X>=1) = 3/8 + 3/8 + 1/8 = 7/8$

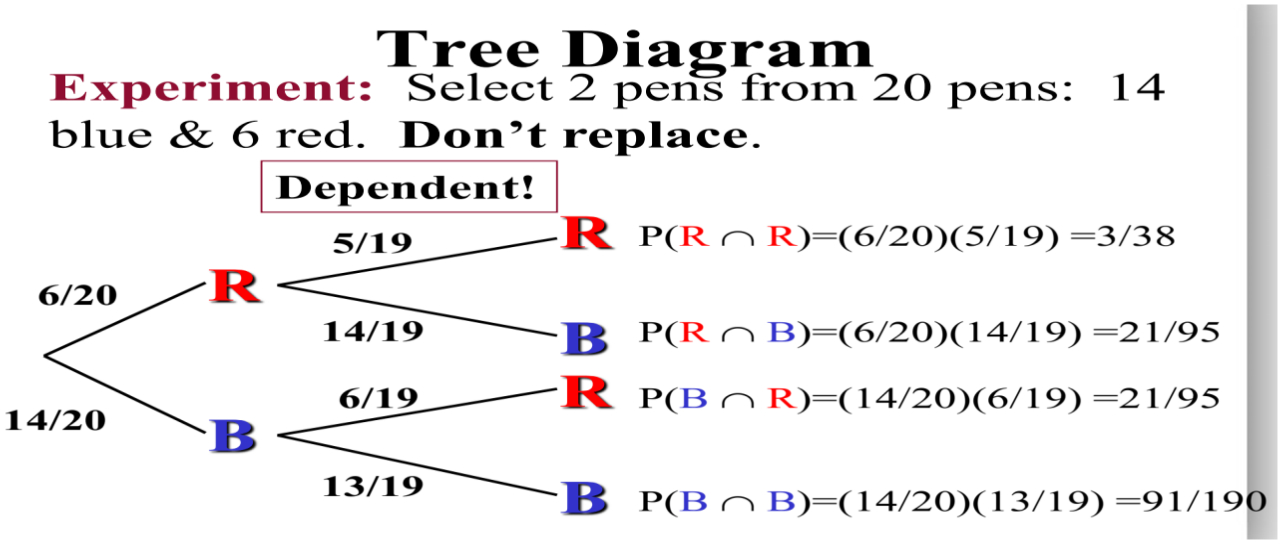
3.5

Conditional Probability

1. **Event probability *given* that another event occurred**

2. Revise original sample space to account for *new* information Eliminates certain outcomes•

3. $$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(A \cap B)}{P(B)}$$



Exercises 1

Experiment 1: In a math class of 30 students, 17 are boys and 13 are girls. On a unit test, 4 boys and 5 girls made an A grade. If a student is chosen at random from the class, what is the probability of choosing a girl or an A student?

Exercises : 2

A day of the week is chosen at random. What is the probability of choosing a Monday or Tuesday?

Exercises :3

A number from 1 to 10 is chosen at random. What is the probability of choosing a 5 or an even number?

Exercises :4

A single 6-sided die is rolled. What is the probability of rolling a number greater than 3 or an even number?

Chapter 7:

Discrete Random Variables

Prepare by

Random Variable

A **random variable** is a variable that assumes numerical values associated with the random outcomes of an experiment, where one (and only one) numerical value is assigned to each sample point.

Random variable is a variable related to a random event

Discrete Random Variable

Random variables that can assume a countable number (finite or infinite) of values are Called **discrete**.

Continuous Random Variable

Random variables that can assume values corresponding to any of the points contained in one or more intervals (i.e., values that are infinite and uncountable) are called **continuous**.

Two Types of Random Variables Examples

D**iscrete random variables**

- Number of sales
- Number of calls
- Shares of stock
- People in line
- Mistakes per page

Continuous random variables

Length –
Depth –
Volume –
Time –
Weight –

Discrete Probability Distribution

The **probability distribution** of a **discrete random variable** is a graph, table, or formula that specifies the probability associated with each possible value the random variable can assume.

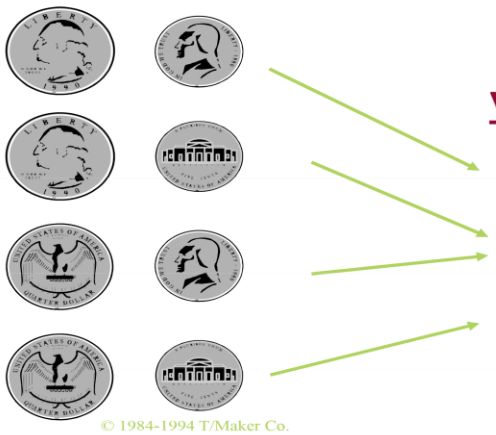
Requirements for the Probability Distribution of a Discrete Random Variable x.

The **probability distribution** of a discrete random variable is defined as a function that specifies the probability associated with each possible outcome
the random variable can assume

1. $p(x) \geq 0$ for all values of x
2. $\sum p(x) = 1$ where the summation of $p(x)$ is over all possible values of x.
3.

Discrete Probability Distribution Example

Experiment: Toss 2 coins. Count number of tails.



Probability Distribution

Values, x	Probabilities, $p(x)$
1/4 = .25	0
2/4 = .50	1
1/4 = .25	2

Visualizing Discrete Probability Distributions

Visualizing Discrete Probability Distributions

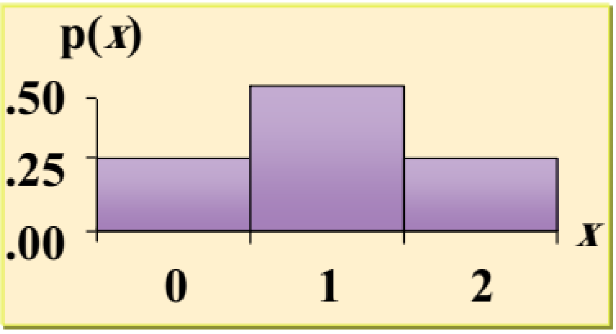
Listing

{ (0, .25), (1, .50), (2, .25) }

Table

# Tails	f(x) Count	p(x)
0	1	.25
1	2	.50
2	1	.25

Graph



Formula

$$p(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Summary Measures

1. Expected Value (Mean of probability distribution)
   - Weighted average of all possible values
   - $\mu = E(x) = \Sigma x\, p(x)$
2. Variance
   - Weighted average of squared deviation about mean
   - $\sigma^2 = E[(x-\mu)^2] = \Sigma\,(x-\mu)^2\, p(x)$
3. Standard Deviation
   - $\sigma = \sqrt{\sigma^2}$

Example:

You toss 2 coins. You're interested in the number of tails. What are the expected value, variance, and standard deviation of this random variable, number of tails**?**

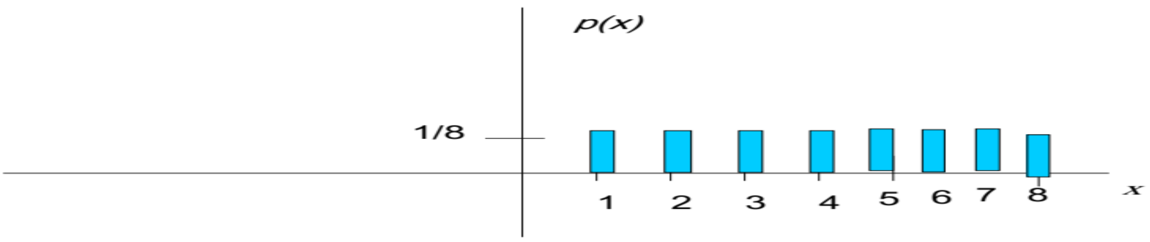
Expected Value & Variance Solution*

x	$p(x)$	$x\,p(x)$	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2 p(x)$
0	**.25**	**0**	**–1.00**	**1.00**	**.25**
1	**.50**	**.50**	**0**	**0**	**0**
2	**.25**	**.50**	**1.00**	**1.00**	**.25**
		$\mu = 1.0$			$\sigma^2 = .50$
					$\sigma = .71$

Probability functions

- A probability function maps the possible values of *x* against their respective probabilities of occurrence, *p(x)*
- *p(x)* is a number from 0 to 1.0.
- The area under a probability function is always 1.

Discrete example: roll of a die

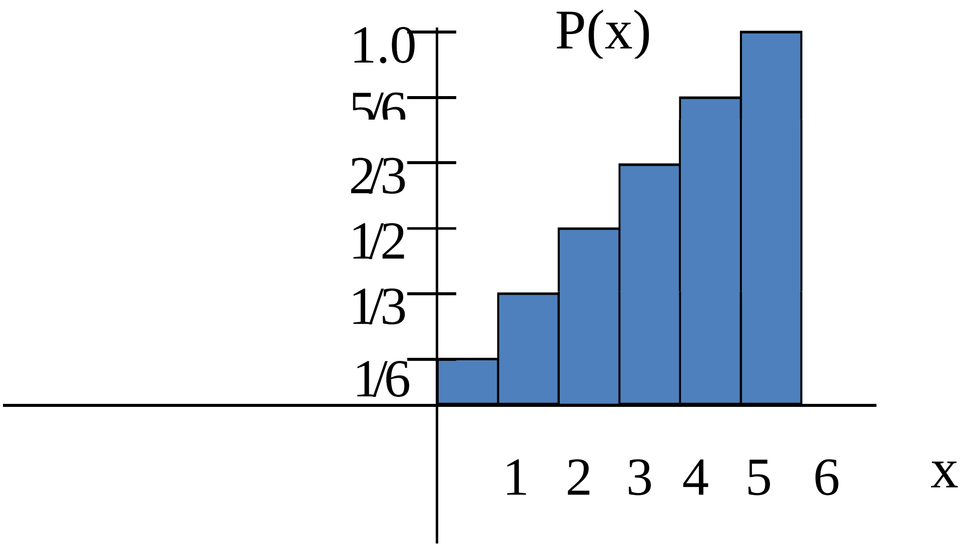


$$\sum_{\text{all } x} P(x) = 1$$

Probability mass function (pmf)

x	p(x)
1	p(x=1)=1/6
2	p(x=2)=1/6
3	p(x=3)=1/6
4	p(x=4)=1/6
5	p(x=5)=1/6
6	p(x=6)=1/6
	1.0

Cumulative distribution function (CDF)



Cumulative distribution function

x	$P(x \le A)$
1	$P(x \le 1) = 1/6$
2	$P(x \le 2) = 2/6$
3	$P(x \le 3) = 3/6$
4	$P(x \le 4) = 4/6$
5	$P(x \le 5) = 5/6$
6	$P(x \le 6) = 6/6$

Example:

- The number of patients seen in the ER in any given hour is a random variable represented by *x*. The probability distribution for *x* is:

x	10	11	12	13	14
P(x)	.4	.2	.2	.1	.1

Find the probability that in a given hour:

a. exactly 14 patients arrive $p(x=14)= .1$

b. At least 12 patients arrive $p(x\geq 12)= (.2 + .1 +.1) = .4$

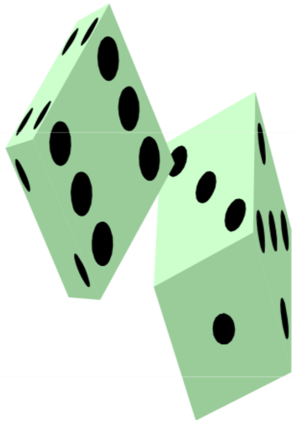
c. At most 11 patients arrive $p(x\leq 11)= (.4 +.2) = .6$

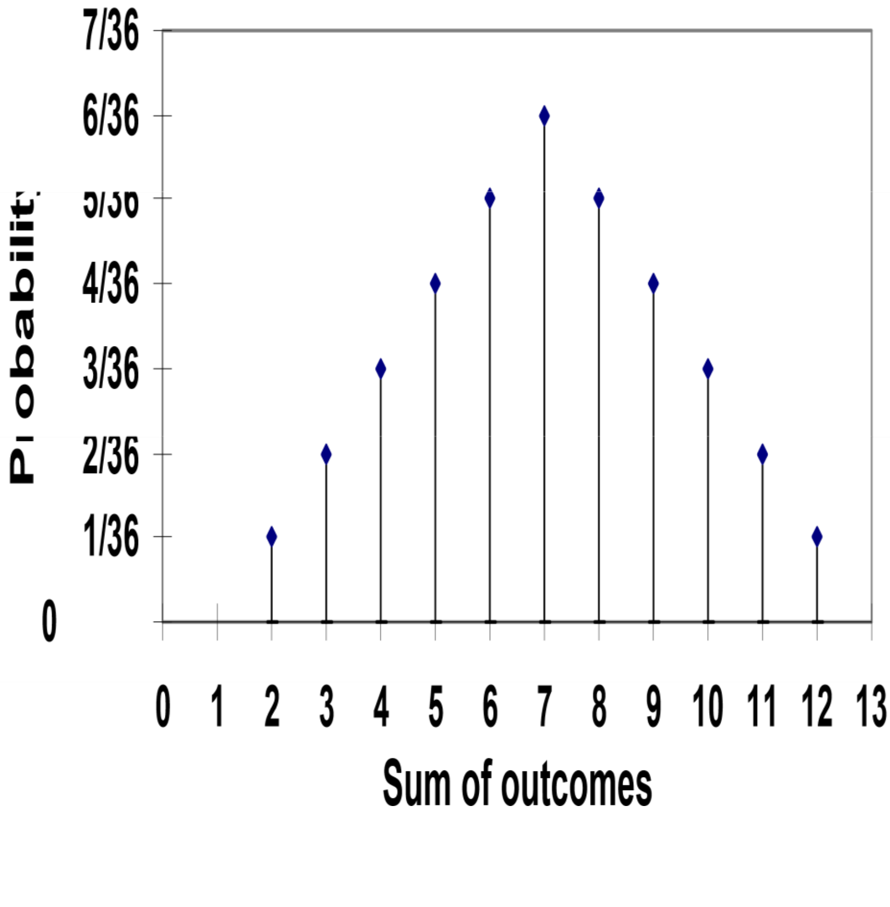
If you toss a die, what's the probability that you roll a 3 or less?

In the case that we toss two dice

The Cumulative Distribution:

Cumulative distribution function F(x) equals the probability to get at most x.




7/36
6/36
5/36
4/36
3/36
2/36
1/36
0
0 1 2 3 4 5 6 7 8 9 10 11 12 13
Sum of outcomes

When playing two dice the sum of outcomes lies between 2-12. Using cumulative distribution we can easily find probabilities for different events:

$P(X<7) = 15/36 \approx 0{,}42$

$P(X>9) = 1 – 30/36 = 6/36 \approx 0{,}17$

$P(4<X<9) = 26/36 – 6/36 = 20/36 \approx 0{,}56$

x	F(x)
2	1/36
3	3/36
4	6/36
5	10/36
6	15/36
7	21/36
8	26/36
9	30/36
10	33/36
11	35/36
12	36/36

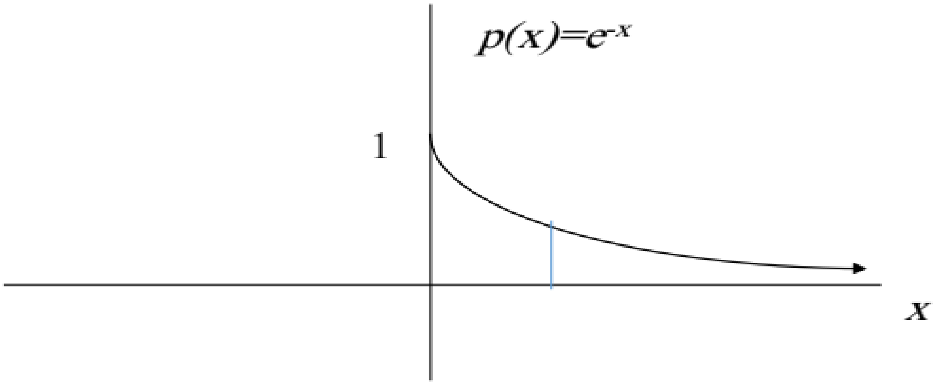
The probability function that accompanies a continuous random variable is a continuous mathematical function that integrates to 1.

For example, recall the negative exponential function (in probability, this is called an “exponential distribution”):

$$f(x) = e^{-x}$$

This function integrates to 1:

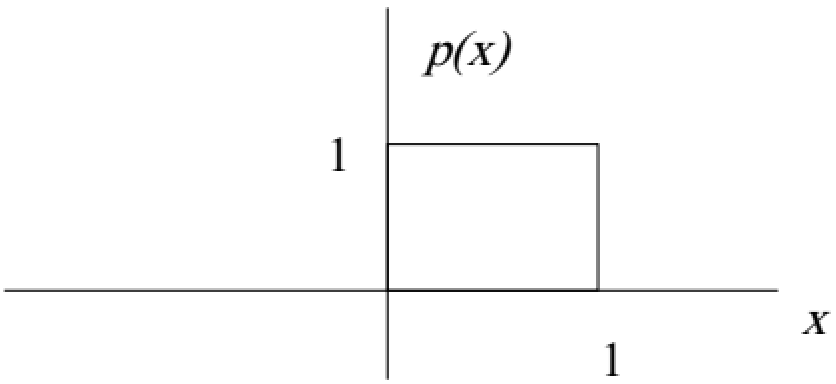
$$\int_0^{+\infty} e^{-x} = -e^{-x} \Big|_0^{+\infty} = 0 + 1 = 1$$



Continuous case: “probability density function” (pdf)

Example : Uniform distribution

The uniform distribution: all values are equally likely. f(x)= 1 , for 1≥ x ≥0



We can see it’s a probability distribution because it integrates to 1 (the area under the curve is 1):

$$\int_0^1 1 = x \Big|_0^1 = 1 - 0 = 1$$

Expected Value and Variance

- All probability distributions are characterized by an expected value (mean) and a variance (standard deviation squared).

Expected value, formally Discrete case:

$$\boxed{E(X) = \sum_{\text{all x}} x_i \, p(x_i)}$$

Continuous case:

$$\boxed{E(X) = \int_{\text{all x}} x_i \, p(x_i) dx}$$

Expected Value

- Expected value is just like the mean in empirical distributions

Examples:

- When playing a dice the expected value equals 3,5
- Insurance company is interested in the expected value of indemnities
- Investor is interested in the expected value of portfolio's revenue

Expected value calculation

- The expected value for a discrete random variable is obtained by multiplying each possible outcome by its probability and then sum these products

Example: expected value

Recall the following probability distribution of ER arrivals:

x	10	11	12	13	14
$P(x)$	.4	.2	.2	.1	.1

^

$$\boxed{\sum_{i=1}^{5} x_i\, p(x) = 10(.4) + 11(.2) + 12(.2) + 13(.1) + 14(.1) = 11.3}$$

$$\mu = E(X) = \sum xp(x).$$

E(X) is read as “expected value of X” or “mean of X”

Expected Values of Discrete Random Variables

The variance of a discrete random variable *X* is

$$\sigma^2 = E\left[(X-\mu)^2\right] = \sum\left[(x-\mu)^2\, p(x)\right].$$

The standard deviation of a discrete random variable *X* is

$$\sigma = \sqrt{\sigma^2} = \sqrt{E\left[(X-\mu)^2\right]} = \sqrt{\sum\left[(x-\mu)^2 p(x)\right]}.$$

Expected Values of Discrete Random Variables

• In a roulette wheel in a U.S. casino, a $1 bet on "even" wins $1 if the ball falls on an

even number (same for "odd," or "red," or "black").

• The odds of winning this bet are 47.37%

$$P(win\ \$1) = 0.4737$$
$$P(lose\ \$1) = 0.5263$$
$$\mu = 1 \times 0.4737 + (-1) \times 0.5263 = -0.0526$$
$$\sigma^2 = E\left[(X-\mu)^2\right] = (1+.0526)^2(0.4737) + (-1+.0526)^2(0.5263)$$
$$= 0.5248 + 0.4724 = 0.9972$$
$$\sigma = \sqrt{0.9972} = 0.9986$$

On average, bettors lose about five cents for each dollar they put down on a bet like this. (These are the best bets for patrons.)

Example

The expected value and variance of a coin toss (H=1, T=0) are?

The expected value and variance of a coin toss are?

Binomial Probability Distribution

- A fixed number of observations (trials), n
- e.g., 15 tosses of a coin; 20 patients; 1000 people surveyed
- A binary outcome
- e.g., head or tail in each toss of a coin; disease or no disease
- Generally called "success" and "failure"
- Probability of success is p, probability of failure is 1 – p
- Constant probability for each observation
- e.g., Probability of getting a tail is the same each time we toss the coin

Binomial Probability Distribution

$$p(x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$p(x)$ = Probability of x 'Successes'

p = Probability of a 'Success' on a single trial

q = $1 - p$

n = Number of trials

x = Number of 'Successes' in n trials ($x = 0, 1, 2, ..., n$)

$n - x$ = Number of failures in n trials

Example

Experiment: Toss 1 coin 5 times in a row. Note number of tails. What's the probability of 3 tails?

$$p(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$p(3) = \frac{5!}{3!(5-3)!} .5^3 (1-.5)^{5-3}$$

$$= .3125$$

Solution:

One way to get exactly 3 heads: HHHTT What's the probability of this exact arrangement?

P(heads)xP(heads) xP(heads)xP(tails)xP(tails) $=(1/2)^3 \text{ x } (1/2)^2$

Another way to get exactly 3 heads: THHHT

Probability of this exact outcome = $(1/2)^1 \times (1/2)^3 \times (1/2)^1 = (1/2)^3 \times (1/2)^2$

In fact, $(1/2)^3 \times (1/2)^2$ is the probability of each unique outcome that has exactly 3 heads and 2 tails. So, the overall probability of 3 heads and 2 tails is: $(1/2)^3 \times (1/2)^2 + (1/2)^3 \times (1/2)^2 + (1/2)^3 \times (1/2)^2 + \ldots$.. for as many unique arrangements as there are— but how many are there??

The Binomial Distribution

A Binomial Random Variable with:

- **n identical trials**
- **Two outcomes per trial: Success or Failure**
  - **P(S) = p; P(F) = q = 1 – p**
  - **Trials are independent**
  - **x is the number of Successes in n trial**

Example

Say 40% of the light bulbs in a production line are defective.
What is the probability that 6 of the 10 randomly selected bulbs are defective?

$$P(x) = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{10}{6} \left(0.4^6\right)\left(0.6^{10-6}\right)$$

$$= 210(0.004096)(0.1296)$$

$$= 0.1115$$

A Binomial Random Variable has

Mean: $\mu = E(X) = np$

Variance: $\sigma^2 = Var(X) = npq$

Standard Deviation: $\sigma = SD(X) = \sqrt{npq}$

Example 2

For 1,000 coin flips,

$$\mu = np = 1000 \times 0.5 = 500$$
$$\sigma^2 = npq = \ 1000 \times 0.5 \times 0.5 = 250$$
$$\sigma = \sqrt{npq} = \sqrt{250} \cong 16$$

Q1- A coin toss can be thought of as an example of a binomial distribution with N=1 and p=.5. What are the expected value and variance of a coin toss?

Q2- If I toss a coin 10 times, what is the expected value and variance of the number of heads?

Poisson Distribution

Number of events that occur in an interval

• events **per unit**

— Time, Length, Area, Space

Examples

• Number of customers arriving in 20 minutes

• Number of strikes per year in the U.S.

• Number of defects per lot (group) of DVD’s

Characteristics of a Poisson Random Variable

1. Consists of counting number of times an event occurs during a given unit of time or in a given area or volume (any unit of measurement).

2. The probability that an event occurs in a given unit of time, area, or volume is the same for all units.

3. The number of events that occur in one unit of time, area, or volume is independent of the number that occur in any other mutually exclusive
unit.

4. The mean number of events in each unit is denoted by λ.

Poisson Probability Distribution Function

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad (x = 0, 1, 2, 3, \ldots)$$

$$\mu = \lambda$$

$$\sigma^2 = \lambda$$

p(x) = Probability of x given λ
λ = Mean (expected) number of events in unit
e = 2.71828 . . . (base of natural logarithm)
x = Number of events **per unit**

The mean is $\mu = E(x) = \lambda$

The variance is σ^2, $\sigma = \sqrt{\lambda}$

Example

Customers arrive at a rate of 72 per hour. What is the probability of 4 customers arriving in 3 minutes?

Solution:

72 Per Hr. = 1.2 Per Min. = 3.6 Per 3 Min. Interval

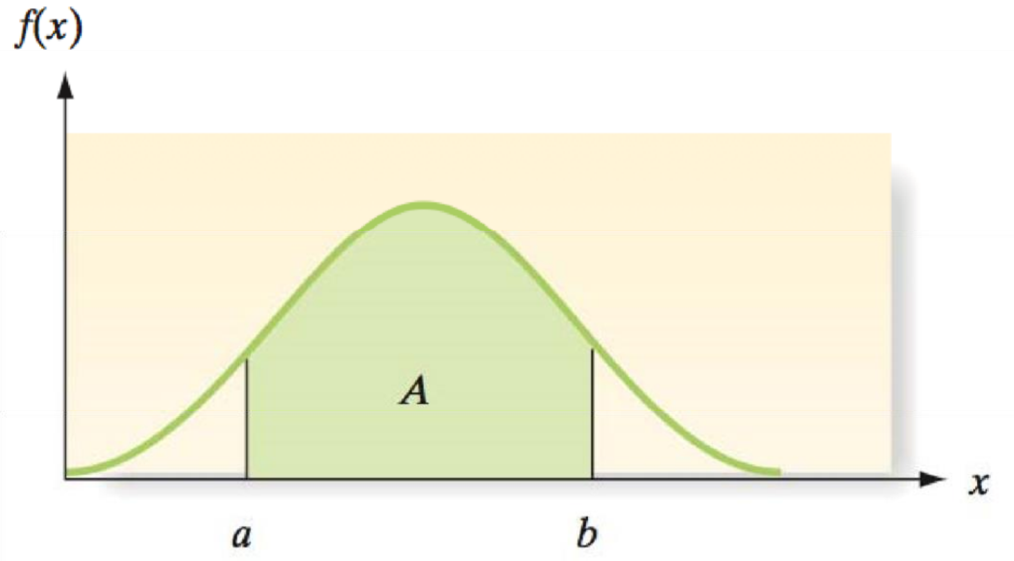
$$p(x) = \frac{\lambda^{x} e^{-\lambda}}{x!}$$

$$p(4) = \frac{(3.6)^{4} e^{-3.6}}{4!} = .1912$$

Probability Distributions for Continuous Random Variables

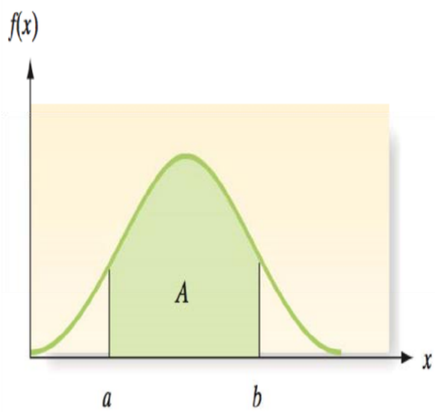
Continuous Probability Density Function

The graphical form of the probability distribution for a continuous random variable x is a smooth curve



This curve, a function of *x*, is denoted by the symbol *f*(*x*) and is variously called a probability density function (pdf), a frequency function, or a probability distribution.

The areas under a probability distribution correspond to probabilities for x. The area A beneath the curve between two points a and b is the probability that x assumes a value between a and b.



Chapter 8

Random Number

Random numbers are numbers that occur in a sequence such that two conditions are met:
• The values are uniformly distributed over a defined interval or set, and
• It is impossible to predict future values based on past or present ones.

Uses of Random Number

Randomness has many uses in science, art, statistics, cryptography, gaming, gambling, and other fields. For example, random assignment in randomized controlled trials helps scientists to test hypotheses, and random numbers or pseudorandom numbers help video games such as video poker.

A **pseudo-random number** generator (PRNG) is a program written for, and used in, probability and statistics applications when large quantities of random digits are needed. Most of these programs produce endless strings of single-digit numbers, usually in base 10, known as the decimal system.

How do you generate random number in C++

- One way to generate these numbers in C++ is to use the function rand(). Rand is defined as: #include <cstdlib> int rand(); The rand function takes no arguments and returns an integer that is a pseudo-random number between 0 and RAND_MAX.

- RAND_MAX. Maximum value returned by rand. This macro expands to an integral constant expression whose value is the maximum value returned by the rand function. This value is library-dependent, but is guaranteed to be at least 32767 on any standard library implementation.

- These created values are not truly "random" because a mathematical formula is used to generate the values.

  srand(x) used to set the starting value (seed) for generating a sequence of pseudo-random integer values. The srand(x) function sets the seed of the random number generator algorithm used by the function **rand**( ).

- Pseudo Random Number Generator (PRNG) refers to an algorithm that uses mathematical formulas to produce sequences of random numbers. PRNGs generate a sequence of numbers approximating the properties of random numbers.

- On a completely deterministic machine you can't **generate** anything you could really call a random sequence of numbers," . "because the machine is following the same algorithm to generate them. ... Not all randomness is pseudo, however, There are ways that machines can generate truly random numbers

Random Number Generators algorithm

- Random number generation is a method of producing a sequence of numbers that lack any discernible pattern.
- Random Number Generators (RNGs) have applications in a wide range of different fields and are used heavily in computational simulations. Pseudo Random Number Generator (PRNG)Algorithms
- Linear Congruential Generator.

  $X_{n+1} = (a X_n + c) \bmod m$
- Mid-square random number generator.
- Lagged Fibonacci Generator

$X_n = X_{n-1} + X_{n-2}$

- Add-with-carry & Subtract-with-borrow

AWC: $X_n=(X_{n-l}+X_{n-k}+carry) \bmod m$

SWB: $X_n=(X_{n-l}-X_{n-k}-carry) \bmod m$

Linear Congruential Generator

- **Many built-in RNGs use the Linear Congruential Generator or LCG. This generator is very fast, has low memory requirements but for most serious applications where randomness actually matters, it is useless.**
- **A sequence of random values Xn where:**
- **$X_{n+1} = (a X_n + c) \bmod m$**
- **with well-chosen values of a, c, m**

Congruential or Residual Generators

One of the common methods, used for generating the pseudo uniform random numbers is the congruence relationship given by,

$$X_{i+1} \equiv (aX_i + c)(\bmod m), \quad i = 1,2,\ldots,n \tag{1}$$

where multiplier a, the increment c and modulus m are non-negative integers.

Equation (1) means, if (aXi + c) is divided by m, then the remainder is X_{i+1} . In this equation m is a large number such that $m \leq 2^w - 1$, where w is the word length of the computer in use for generating the (m – 1) numbers and (i = 0) is seed value.

By seed value, we mean any initial value used for generating a set of random numbers. Seed value should be different for different set of random numbers. In order, the numbers falling between 0 and 1, we must divide all Xis' by (m – 1).

@In order to have non-repeated period m, following conditions are to be satisfied,

(i) c is relatively prime to m, i.e., c and m have no common divisor.

(ii) a ≡ 1 (mod g) for every prime factor g of m.

(iii) a ≡ 1 (mod 4) if m is a multiple of 4.

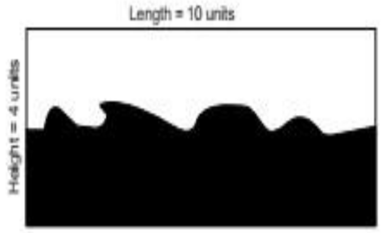
Computation of Irregular Area using Monte Carlo Simulation

To further understand Monte Carlo simulation, let us examine a simple problem. Below is a rectangle for which we know the length [10 units] and height [4 units]. It is split into two sections which are identified using different colours. What is the area covered by the black colour?

Due to the irregular way in which the rectangle is split, this problem cannot be easily solved using analytical methods. However, we can use Monte Carlo simulation to easily find an approximate answer.

The procedure is as follows:

1. Randomly select a location (point) within the rectangle.
2. If it is within the black area, record this instance a hit.
3. Generate a new location and follow 2.
4. Repeat this process 10,000 times



What is the area covered by Black?

$$\text{Black area} = \frac{\text{number of black hits}}{10{,}000 \text{ hits}} \times 40 \text{ square units}$$

Multiplicative Generator Method

Another widely used method is multiplicative generator method and is given as,

$$X_{i+1} = (aX_i)(\text{mod } m),\ i = 1,2,\ldots,n$$

This equation is obtained from (1) by putting c = 0. One important condition in this is that X0 is prime to m and a satisfies certain congruence conditions. In this case too, in order to generate random numbers between 0 and 1, we divide Xi+1 by m.

Linear Congruential Generator

Example Sequences:

m=10, a=2, c=1	m=10, a=1, c=7
1	1
3 (2*1 + 1 % 10)	8 (1*1 + 7 % 10)
7 (2*3 + 1 % 10)	5 (1*8 + 7 % 10)
5 (2*7 + 1 % 10)	2 (1*5 + 7 % 10)
1 (2*5 + 1 % 10)	9 (1*2 + 7 % 10)
3 (2*1 + 1 % 10)	6 (1*9 + 7 % 10)
7 (2*3 + 1 % 10)	3 (1*6 + 7 % 10)
5 (2*7 + 1 % 10)	0 (1*3 + 7 % 10)
…	…

Linear Congruential Generator

Various sources use different parameters for the LCG:

Numerical Recipes

$m = 2^{32}$ a = 1664525 c = 1013904223

GCC $m = 2^{32}$ a = 1103515245 c = 12345

MMIX

$m = 2^{64}$ a = 6364136223846793005 c = 1442695040888963407

Mid Square Random Number Generator

This is one of the earliest method for generating the random numbers. This was used in 1950s, when the principle use of simulation was in designing thermonuclear weapons. Method is as follows:

1. Take some n digit number.
2. Square the number and select n digit number from the middle of the square number.
3. Square again this number and repeat the process.

Example 4.1: Generate random numbers using Mid Square Random Number Generator.

Solution: Let us assume a three digit seed value as 123.

Step 1: Square of 123 is 15129. We select mid three numbers which is 512.

Step 2: Square of 512 is 262144. We select mid two numbers which is 21.

Repeat the process. Thus random numbers are 512, 21, ...

Lagged Fibonacci Generator

The Lagged Fibonacci Generator is based on a generalisation of the Fibonacci sequence

A sequence of 25 random numbers with seed value 3459 is given below.

9645	0260	0675	4555	7479	9353	4785	8962	3173
678	4596	1231	5153	5534	0250	0625	3905	2489
1950	8025	4005	0400	1599	5567	9913	2675	

1, 1, 2, 3, 5, 8, 13, 21 ....

Obtained from the sequence

$X_n = X_{n-1} + X_{n-2}$

Lagged Fibonacci Generator

The LFG generator is defined by:

$X_n = (X_{n-j} \text{ OP } X_{n-k}) \bmod m$

Where:

OP is some binary operator +, -, *, /,
XOR. $0 < j < k$

This algorithm produces a higher quality of random numbers but requires the storage of **k** past states.

Random Walk Problem

One of the important applications of random numbers is a drunkard walk. A drunkard is trying to go in a direction (say *y*-axis in *xy*-plane). But sometimes he moves in forward direction and some times left, right or ackward direction. Random walk has many applications in the field of Physics. Brownian motion of molecules is like random walk. Probabilities of drunkard's steps are given as follows.

Probability of moving forward	=	0.5
Probability of moving backward	=	0.1
Probability of moving right	=	0.2
Probability of moving left	=	0.2

Truly Random Numbers

- While most computers are incapable of producing truly random numbers there are some modern devices capable of generating a truly random number.
- A simulation that uses a truly random number is not repeatable.
- These quantum random number generators are not seeded and each simulation will be different.

TESTING OF RANDOM NUMBERS

To find out whether a given series of random numbers are truly random, there are several tests available. Random numbers are considered random if

- The numbers are uniformly distributed i.e., every number has equal chance to occur.
- The numbers are not serially autocorrelated.

Meaning of second point is that once a random number is generated, next can not be generated by some correlation with one. Based on the above specifications, to test whether given random numbers are random or not, below we are give some tests.

The Kolmogrov-Smirnov Test

chi-square (χ^2) Test

Poker's Method

Chapter 9

Queuing Models

The Purpose

■Simulation is often used in the analysis of queueing models.

■Queueing models provide the analyst with a powerful tool for designing and evaluating the performance of queueing systems.

Examples of Real World Queuing Systems?

- Commercial Queuing Systems

- Ex. bank, ATM, gas stations, …

- Transportation service systems
  - Ex. Vehicles waiting at toll stations and traffic lights, trucks or ships waiting to be loaded …

- Business-internal service systems
  - Ex. Inspection stations, computer support …

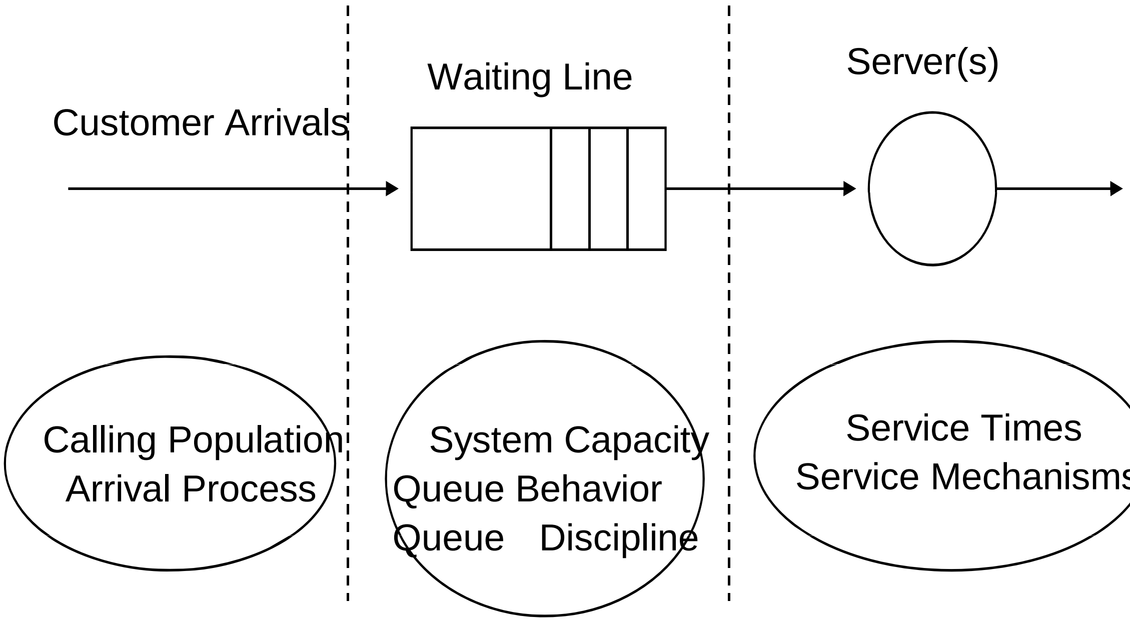
- Social service systems
  - Ex. the ER at a hospital, student dorm rooms

Characteristics of Queueing Systems

(1): Key Elements

- Key elements of queueing systems:
  - Customer: refers to anything that arrives at a facility and requires service, e.g., people, machines, trucks, emails.
  - Server: refers to any resource that provides the requested service,

e.g., repair persons, retrieval machines, runways at airport.



(2): Calling Population

- Calling population: the population of potential customers, may be assumed to be finite or infinite.
  - Finite population model: if arrival rate depends on the number of customers being served and waiting, e.g., model of one corporate jet, if it is being repaired, the repair arrival rate becomes zero.
  - Infinite population model: if arrival rate is not affected by the number of customers being served and waiting, e.g., systems with large population of potential customers.

(3): System Capacity

- System Capacity: a limit on the number of customers that may be in the waiting line or system.

- Limited capacity, e.g., an automatic car wash only has room for 10cars to wait in line to enter the mechanism.
- Unlimited capacity, e.g., concert ticket sales with no limit on the number of people allowed to wait to purchase tickets.

(4): Arrival Process (1)

- For infinite-population models:
  - Usually characterized in terms of inter-arrival times of successive customers. Arrivals may occur at random or scheduled times
  - Random arrivals:inter-arrival times usually characterized by a probability distribution.
    - Most important model: Poisson arrival process (with rate l), where $A_n$represents the inter-arrival time between customer n-1and customer n, and is exponentially distributed (with mean 1/l).
  - Scheduled arrivals: inter-arrival times can be constant or constant plus or minus a small random amount (jitter) to represent early or late arrivals.
- e.g.scheduled airline flight arrivals to an airport.
  - At least one customer is assumed to always be present, so the server is never idle, e.g., sufficient raw material for a machine.

(4): Arrival Process (2)

- For finite-population models:
  - Define customer as “pending” when the customer is outside the queueing system, e.g., machine-repair problem: a machine is “pending” when it is operating, it becomes “not pending” the instant it demands service form the repairman.
  - Define “runtime” of a customer as the length of time from departure from the queueing system until that customer’s next

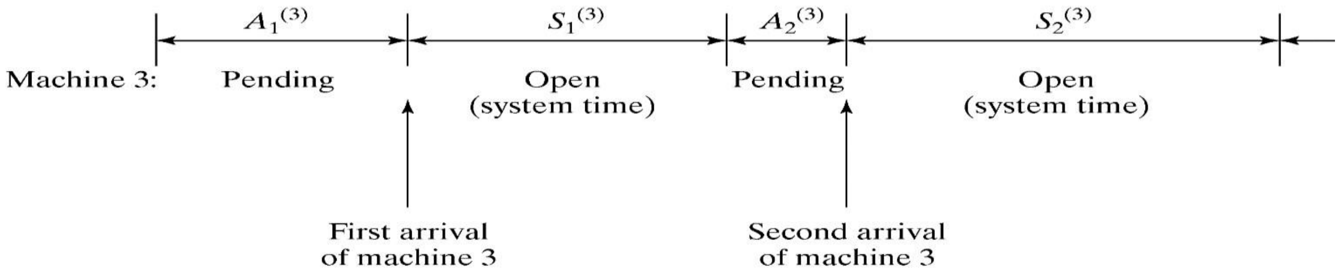
arrival to the queue, e.g., machine-repair problem, machines are customers and a runtime is time to failure.

Let $A_1^{(i)}$, $A_2^{(i)}$, … be the successive runtimes of customer i, and $S_1^{(i)}$, $S_2^{(i)}$be the corresponding successive system times: that is

– $S_n^{(i)}$is the total time spent in the system by customer iduring the n^{th}visit.

(4): Arrival Process (3)

- Finite Population Models



– The total arrival process is the superposition of the arrival times of all customers.
One important application of finite models is the machine-repair problem. Machines are the customers and runtime is time to failure. When a machine fails, it “arrives” at the queueing system

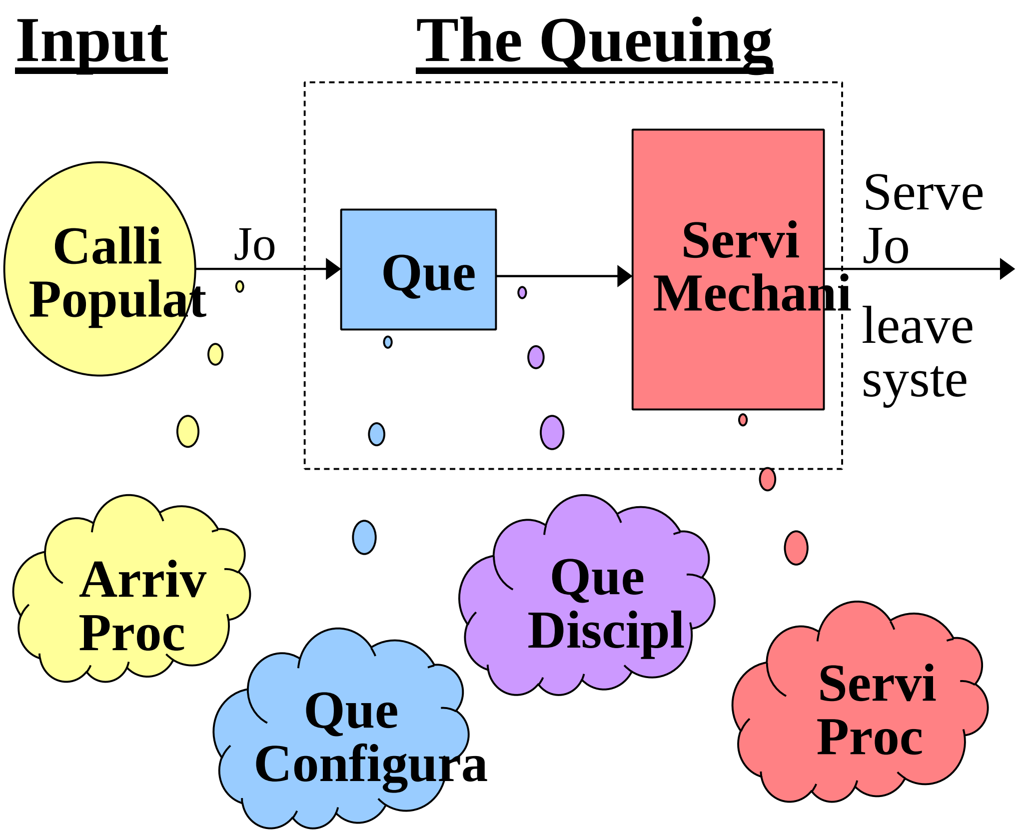
and remains there until it is served. Time to failure is chracterized by exponential, Weibull and Gamma distributions.

(5): Queue Behavior and Queue Discipline

- Queue behavior:refers to the actions of customers while in a queue waiting for service to begin, for example:
  - Balk: leave when they see that the line is too long,
  - Renege: leave after being in the line when its moving too slowly, – Jockey: move from one line to a shorter line.

- Queue discipline: refers to the logical ordering of customers in a queue that determines which customer is chosen for service when a server becomes free, for example:
  - First-in-first-out (FIFO)
  - Last-in-first-out (LIFO)
  - Service in random order (SIRO)
  - Shortest processing time first (SPT)
  - Service according to priority (PR). (e.g., type, class, priority)

Components of a Basic Queuing Process

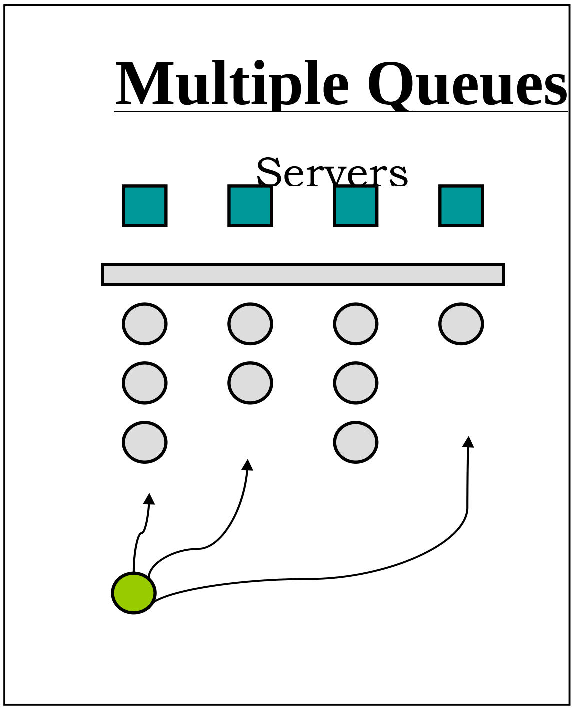


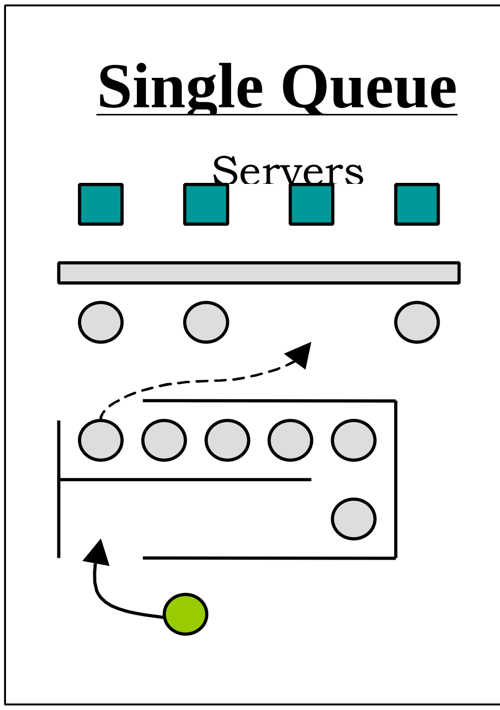
Example – Two Queue Configurations

1. **The service provided can be differentiated**
   – Ex. Supermarket express lanes
2. **Labor specialization possible**
3. **Customer has more flexibility**
4. **Balking behavior may be deterred**

1. **Guarantees fairness**
   – FIFO applied to all arrivals
2. **No customer anxiety regarding choice of queue**
3. **Avoids “cutting in” problems**
4. **The most efficient set up for minimizing time in the queue**
5. **Jockeying (line switching)**

Multiple v.s. Single Customer Queue Configuration





Queuing or Waiting Line Analysis

(waiting lines) affect people everyday

ry goal is finding the best level of service

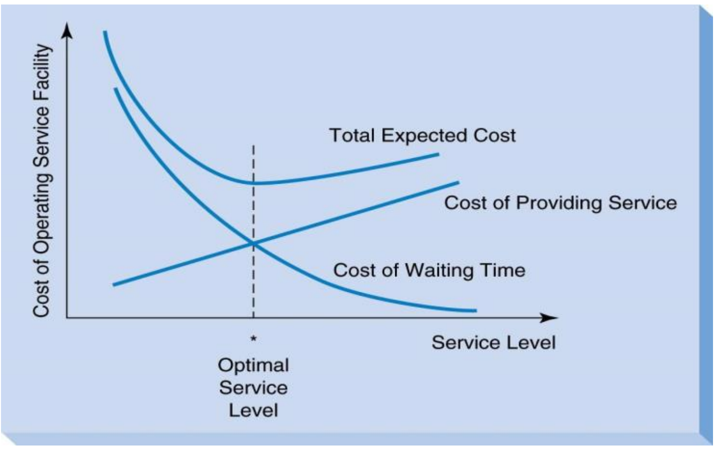
al modeling (using formulas) can be used for many queues

omplex situations, computer simulation is needed

Queuing System Costs

st of providing service

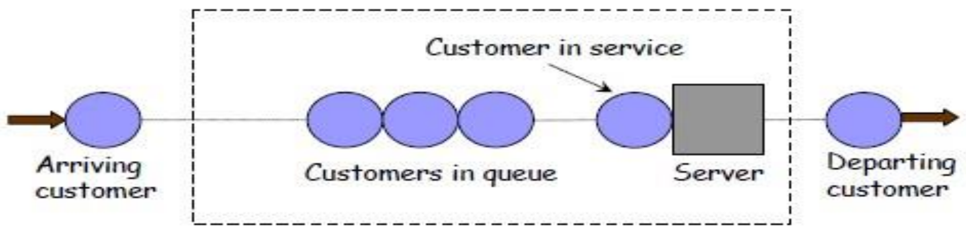
st of <u>not</u> providing service (waiting time)

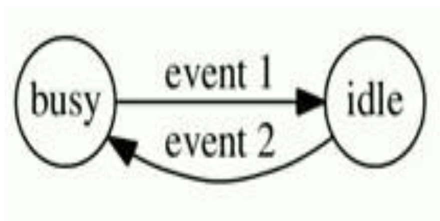


Queueing system components

- Entities: server, queue
- State:
  - number of units (customers for the bank example) in the system
  - Server status: busy/idle, S = {B, I}.
- Events:
  - Arrivals
  - Departures
- Simulation clock: tracks simulated time
- Actions
  - Different actions, depending on the type of the event (arrival or departure) and the current system state
  - Flowcharts for actions following both arrival and departure even

Single server queuing system simulation

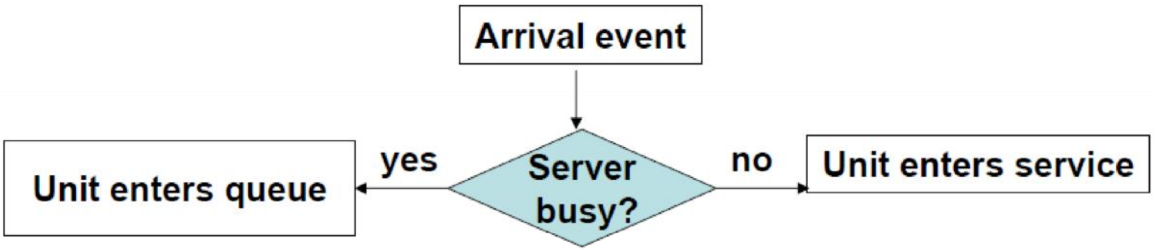




.- In a single server, what are the ”state variables”? The status of the server idle & busy
The number of customers waiting in queue.
The time of arrival of each customers waiting in queue.

2- What are the ”events” in a single server model?
Events
The arrival time of customer.

Flowchart for arrival



How does system state change?

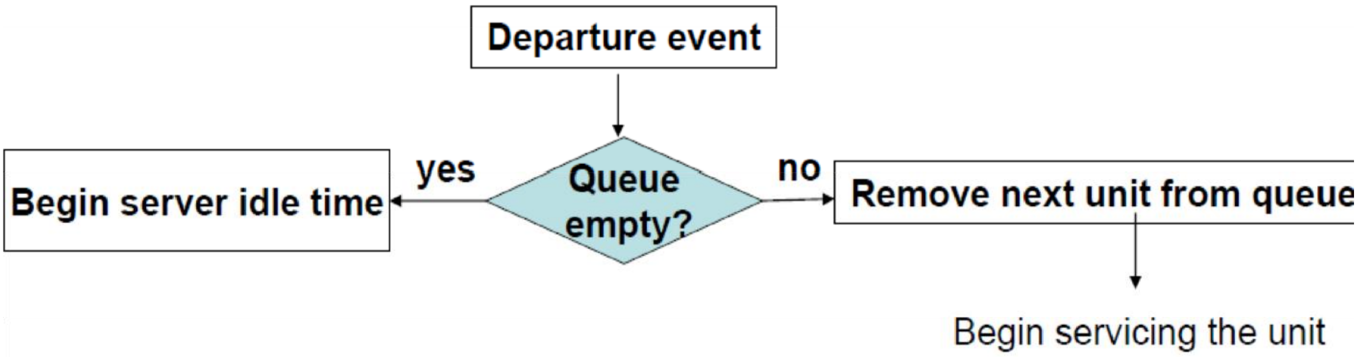
If $S \neq B$, $Q = Q$; $S = B$;

If $S = B$, $Q = Q+1$; $S = B$.

Is this valid: If $S = B$, $Q = Q+1$, $S = I$?

The departure time of customer after being served

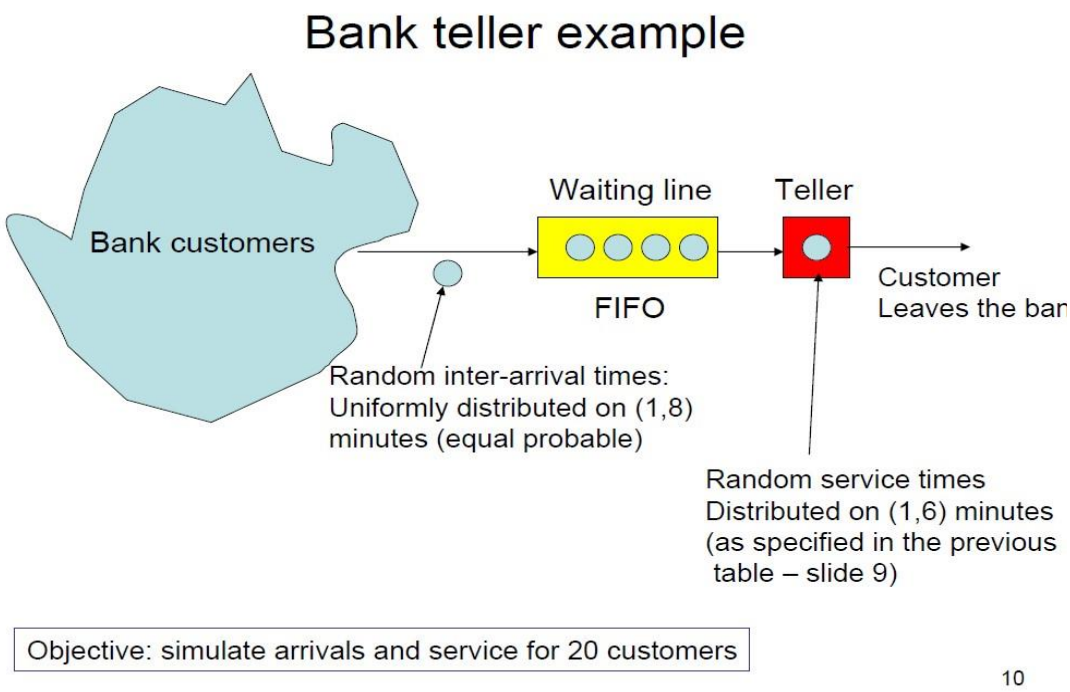
Flowchart for departure



How does system state change?
If $Q \neq 0$, $Q = Q-1$; $S = B$;
If $Q = 0$, $Q = 0$; $S = I$.

Are these valid: If $Q=0$, $Q=0$, $S=B$?
If $Q \neq 0$, $Q = Q$; $S = I$?

Examples



Simulate Bank system:

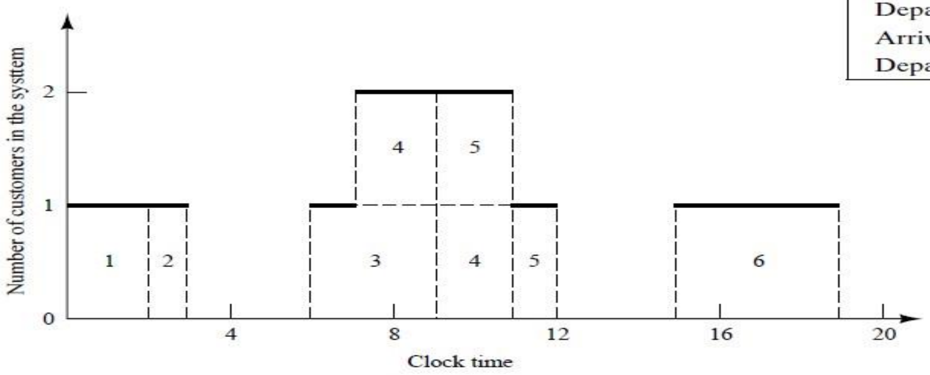
with 6 expected customers In a single-channel queuing system interarrival times and service times are generated from the distributions of these random variables. For simplicity, assume that the times between arrivals were generated by rolling a die five times and recording the up face. These five interarrival times are used to compute the arrival times of six customers at the queuing system.

Customer	Interarrival Time	Arrival Time on Clock
1	—	0
2	2	2
3	4	6
4	1	7
5	2	9
6	6	15

Customer	Service Time
1	2
2	1
3	3
4	2
5	1
6	4

A	B	C	D	E
Customer Number	Arrival Time (Clock)	Time Service Begins (Clock)	Service Time (Duration)	Time Service Ends (Clock)
1	0	0	2	2
2	2	2	1	3
3	6	6	3	9
4	7	9	2	11
5	9	11	1	12
6	15	15	4	19

Event Type	Customer Number	Clock Time
Arrival	1	0
Departure	1	2
Arrival	2	2
Departure	2	3
Arrival	3	6
Arrival	4	7
Departure	3	9
Arrival	5	9
Departure	4	11
Departure	5	12
Arrival	6	15
Departure	6	19



Simulation table : Bank teller example

Simulation table : Bank teller example customer	Inter arrival	arrival	Time Ser. Begins	Service time Duration	Time ser. ends	Time in queue	Time cust. Spends in sys.	Idle time service
1	0	0	0	2	2	0	2	0
2	2	2	2	1	3	0	1	0
3	4	6	6	3	9	0	3	3
4	1	7	9	2	11	2	4	0
5	2	9	11	1	12	2	3	0
6	6	15	15	4	19	0	4	3
Total	**15**	**39**	**43**	**13**	**56**	**4**	**17**	**6**
Average	**3**	**6.5**		**2.1**		**0.67**	**2.8**	**1**

The average time between arrival = (sum time between arrival) /total number of arrivals (customers) -1

Compare between the expected and calculated average time between arrival

E(A)= the mean of the discrete uniform distribution for rolling a die whose endpoints are a = 1 and b = 6. is equal 3.5 !!! ((1+6)/2 =3.5)

The average time Customer spends on the system , will be compare with the total of: Average service time+ time in queue

Characteristics of a Queuing System

The queuing system is determined by:

- **Arrival characteristics**
- **Queue characteristics**
- **Service facility characteristics**

Arrival Characteristics

- **Size of the arrival population – either infinite or limited**
- **Arrival distribution:**

–Either fixed or random

–Either measured by time between consecutive arrivals, or arrival rate

–The Poisson distribution is often used for random arrivals

The Poisson Process

The standard assumption in many queuing models is that the arrival process is Poisson

Two equivalent definitions of the Poisson Process

1. The times between arrivals are independent, identically distributed and exponential

2. X(t) is a Poisson process with arrival rate l iff.

a) X(t) have independent increments

b) For a small time interval h it holds that

- P(exactly 1 event occurs in the interval [t, t+h]) = lh + o(h)
- P(more than 1 event occurs in the interval [t, t+h]) = o(h)

Properties of the Poisson Process

- Poisson processes can be aggregated or disaggregated and the resulting processes are also Poisson processes

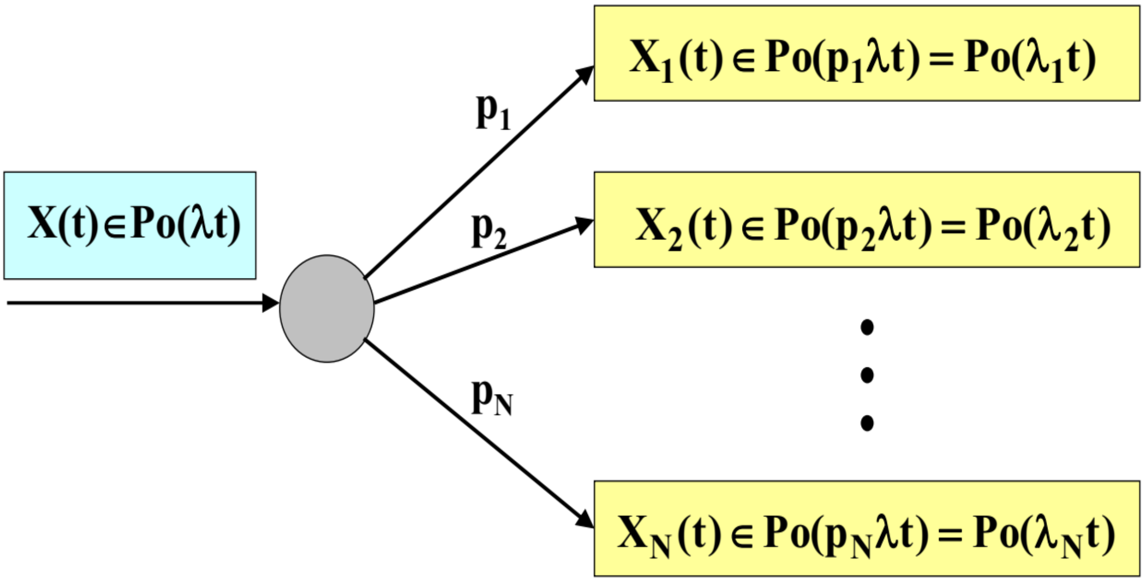
   a) Aggregation of N Poisson processes with intensities $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ renders a new Poisson process with intensity $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

   b) Disaggregating a Poisson process $X(t) \in Po(\lambda t)$ into N sub-processes $\{X_1(t), X_2(t), , \ldots, X_3(t)\}$ (for example N customer types) where $X_i(t) \in Po(\lambda_i t)$ can be done if

      – For every arrival the probability of belonging to sub-process $i = p_i$

      – $p_1 + p_2 + \ldots + p_N = 1$, and $\lambda_i = p_i \lambda$

Disaggregating a Poisson Process



$X(t) \in Po(\lambda t)$
p_1
p_2
p_N
$X_1(t) \in Po(p_1\lambda t) = Po(\lambda_1 t)$
$X_2(t) \in Po(p_2\lambda t) = Po(\lambda_2 t)$
$X_N(t) \in Po(p_N\lambda t) = Po(\lambda_N t)$

Terminology and Notation

- The state of the system = the number of customers in the system
- Queue length = (The state of the system) – (number of customers being

served)

N(t) = Number of customers/jobs in the system at time t

P_n(t) = The probability that at time t, there are n customers/jobs in the system.

λ_n = Average arrival intensity (= # arrivals per time unit) at n customers/jobs in the system

μ_n = Average service intensity for the system when there are n customers/jobs in it. (Note, the total service intensity for all ***occupied*** servers)

ρ = The utilization factor for the service facility. (= The expected fraction of the time that the service facility is being used)

Example – Service Utilization Factor

Consider an M/M/1 queue with arrival rate = λ and service intensity = μ

- λ = Expected capacity demand per time unit
- μ = Expected capacity per time unit

$$\boxed{\rho = \frac{\text{Capacity Demand}}{\text{Available Capacity}} = \frac{\lambda}{\mu}}$$

Similarly if there are c servers in parallel, i.e., an M/M/c system but the expected capacity per time unit is then c*μ

$$\rho = \frac{\text{Capacity Demand}}{\text{Available Capacity}} = \frac{\lambda}{c * \mu}$$

Poisson Distribution

- Average arrival rate is known
- Average arrival rate is constant for some number of time periods
- Number of arrivals in each time period is independent
- As the time interval approaches 0, the average number of arrivals approaches 0
- λ = the average arrival rate per time unit P(x) = the probability of exactly x

  arrivals occurring during one time period

$$P(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

Exponential Distribution

μ = average service time

t = the length of service time ($t \geq 0$)

P(t) = probability that service time will be greater than

$$P(t) = e^{-\mu t}$$

Measuring Queue Performance

- ρ = utilization factor (probability of all servers being busy)
- L_q = average number in the queue
- L = average number in the system
- W_q = average waiting time
- W = average time in the system
- P_0 = probability of 0 customers in system
- P_n = probability of exactly n customers in system

Kendall's Notation

A / B / s

A = Arrival distribution (M for Poisson, D for deterministic, and G for general)

B = Service time distribution (M for exponential, D for deterministic, and G for general)

S = number of servers

The Queuing Models Covered Here All Assume

1. Arrivals follow the Poisson distribution

2. FIFO service

3. Single phase

4. Unlimited queue length

5. Steady state conditions

We will look at 5 of the most commonly used queuing systems.

Name (Kendall Notation)	Models Covered Example
Simple system (M / M / 1)	Customer service desk in a store
Multiple server (M / M / s)	Airline ticket counter
Constant service (M / D / 1)	Automated car wash
General service (M / G / 1)	Auto repair shop

Limited population (M / M / s / ∞ / N)	An operation with only 12 machines that might break

Single Server Queuing System (M/M/1)

- Poisson arrivals
- Arrival population is unlimited
- Exponential service times

- All arrivals wait to be served
- λ is constant
- $\mu > \lambda$ (average service rate > average arrival rate)

Operating Characteristics for M/M/1 Queue

1. Average server utilization $\rho = \lambda / \mu$
2. Average number of customers waiting $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)}$
3. Average number in system $L = L_q + \lambda / \mu$
4. Average waiting time

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu - \lambda)}$$

5. Average time in the system $W = W_q + 1/ \mu$
6. Probability of 0 customers in system $P_0 = 1 - \lambda/\mu$
7. Probability of exactly n customers in system

$$P_n = (\lambda/\mu)^n P_0$$

Total Cost of Queuing System

Total Cost = C_w x L + C_s x s

C_w = cost of customer waiting time per time period

L = average number customers in system

C_s = cost of servers per time period s = number of servers

Multiple Server System (M / M / s)

- Poisson arrivals
- Exponential service times
- s servers
- Total service rate must exceed arrival rate

  ($s\mu > \lambda$)

- Many of the operating characteristic formulas are more complicated

Single Server System With Constant Service Time (M/D/1)

- Poisson arrivals
- Constant service times (not random)
- Has shorter queues than M/M/1 system - Lq and Wq are one-half as large

Single Server System With General Service Time (M/G/1)

- Poisson arrivals
- General service time distribution with known mean (μ) and standard deviation (σ)
- $\mu > \lambda$

Example – Simulation of a M/M/1 Queue

- Assume a small branch office of a local bank with only one teller.

- Empirical data gathering indicates that inter-arrival and service times are exponentially distributed.
  - The average arrival rate = λ = 5 customers per hour
  - The average service rate = μ = 6 customers per hour
- Using our knowledge of queuing theory we obtain
  - σ = the server utilization = 5/6 $\approx$ 0.83
  - L_q = the average number of people waiting in line
  - W_q = the average time spent waiting in line

$L_q = 0.83^2/(1-0.83) \approx 4.2$ $\quad W_q = L_q/\lambda \approx 4.2/5 \approx 0.83$

Muti-Server System With Finite Population (M/M/s/∞/N)

- Poisson arrivals
- Exponential service times
- s servers with identical service time distributions

- Limited population of size N
- Arrival rate decreases as queue lengthens

Department of Commerce Example

- Uses 5 printers (N=5)
- Printers breakdown on average every 20 hours

  $\lambda = \frac{1 \text{ printer}}{20 \text{ hours}} = 0.05$ printers per hour

- Average service time is 2 hours $\mu = \frac{1 \text{ printer}}{2 \text{ hours}} = 0.5$ printers per hour

More Complex Queuing Systems

- When a queuing system is more complex, formulas may not be available
- The only option may be to use computer simulation

Reference

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3. Banks, Carson, Nelson & Nichol, *Discrete Event System Simulation*, Prentice Hall.
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