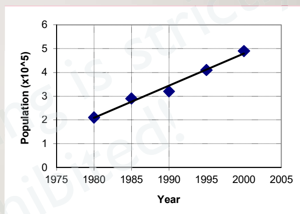
Linear Regression with Example

Year	Population
1980	2.1
1985	2.9
1990	3.2
1995	4.1
2000	4.9
2005	?

year	1980	1985	1990	1995	2000	2005
population	2.1	2.9	3.2	4.1	4.9	?

Linear Regression with Example

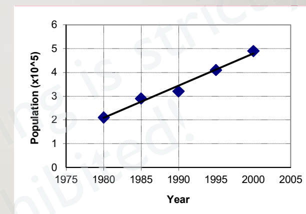
Year	Population
1980	2.1
1985	2.9
1990	3.2
1995	4.1
2000	4.9
2005	?



$y = slope * x + intercept$

Linear Regression with Example

Year	Population
1980	2.1
1985	2.9
1990	3.2
1995	4.1
2000	4.9
2005	$\hat{y} = slope * x + intercept$



$y = slope * x + intercept$

Manual Computation using Linear Regression Formula

$$\hat{y} = slope * x + intercept$$

$$slope = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2}$$

$$intercept = \bar{y} - slope \cdot \bar{x}$$

	x	y	xy	x^2
	Year	Population		
	1980	2.1		
	1985	2.9		
	1990	3.2		
	1995	4.1		
	2000	4.9		
Sum				
Average				
Count (n) =				

Slope	
Intercept	

Manual Computation using Linear Regression Formula

$$\hat{y} = slope * x + intercept$$

$$slope = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2}$$

$$intercept = \bar{y} - slope \cdot \bar{x}$$

	x	y	xy	x^2
	Year	Population		
	1980	2.1	4158	3920400
	1985	2.9	5756.5	3940225
	1990	3.2	6368	3960100
	1995	4.1	8179.5	3980025
	2000	4.9	9800	4000000
Sum	9950	17.2	34262	19800750
Average	1990	3.44		
Count (n) =	5			

Slope	0.136
Intercept	-267.2

Linear Regression with Example

Year	Population
1980	2.1
1985	2.9
1990	3.2
1995	4.1
2000	4.9
2005	?

Verify Using Excel Function			
	Slope		
	Intercept		
Predict	For Year		Prediction
	2005		

Linear Regression with Example

Year	Population
1980	2.1
1985	2.9
1990	3.2
1995	4.1
2000	4.9
2005	?

Verify Using Excel Function			
	Slope	0.136	
	Intercept	-267.2	
Predict	For Year		Prediction
	2005		5.48

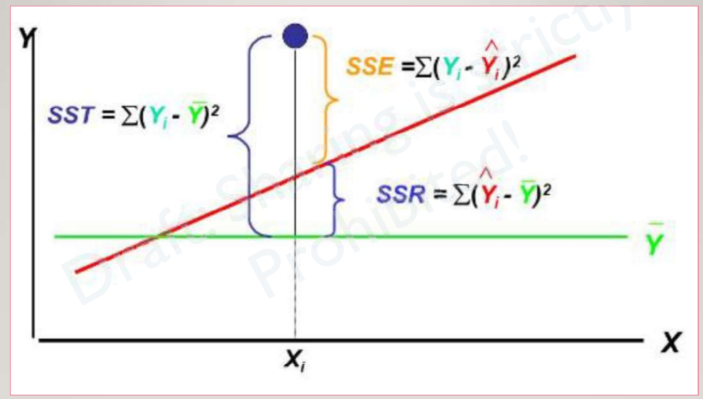
Regression Goodness of Fit

Several indices are used to determine the goodness of fit of the model.

- R-squared, or coefficient of determination
- Adjusted R-squared
- Standard Error
- F statistics
- t statistics

R-squared (Measures of Variation)

R^2 = SSR/SST = (SST-SSE)/SST
= 1 - SSE/SST



R-squared (Measures of Variation)

- Total variation is made up of two parts:

$$SST = SSR + SSE$$

Total Sum of Squares	Regression Sum of Squares	Error Sum of Squares
$SST = \sum(y_i - \bar{y})^2$	$SSR = \sum(\hat{y}_i - \bar{y})^2$	$SSE = \sum(y_i - \hat{y}_i)^2$

where:

$\bar{y}$ = Average value of the dependent variable

y_i = Observed values of the dependent variable

$\hat{y}_i$ = Predicted value of y for the given x_i value

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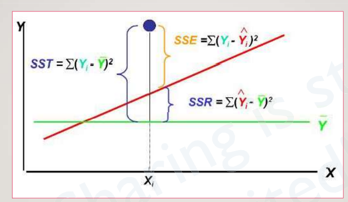
R-squared (Measures of Variation)



R^2 = SSR / SST
= (SST-SSE) / SST
= 1 - SSE/SST

R-squared (Measures of Variation)

$$R^2 = 1 - \frac{SSE}{SST}$$

Zero Regression Error

$$R^2 = 1 - \frac{0}{SS_{Total}} \rightarrow 1.0$$

R-squared: Calculate SST, SSE & SSR

	x	y				
	Year	Population	Prediction	Error	Square Error	Sq. Mean Difference
	1980	2.1				
	1985	2.9				
	1990	3.2				
	1995	4.1				
	2000	4.9				
Sum					SSE	SST
Mean	avg(x)	avg(y)			MSE	MST
Count (n)					df	df

Slope	0.136
Intercept	-267.2

Number of Coefficients	
R Square	
MSE	
MST	
Adjusted R Square	
Standard Error	

R-squared: Calculate SST, SSE & SSR

	x	y				
	Year	Population	Prediction	Error	Square Error	Sq. Mean Difference
	1980	2.1	2.08	-0.02	0.0004	1.80
	1985	2.9	2.76	-0.14	0.0196	0.29
	1990	3.2	3.44	0.24	0.0576	0.06
	1995	4.1	4.12	0.02	0.0004	0.44
	2000	4.9	4.8	-0.1	0.01	2.13
Sum					0.088	4.71
Mean	avg(x)	avg(y)			MSE	MST
Count (n)					df	df

Slope	0.136
Intercept	-267.2

Number of Coefficients	
R Square	
MSE	
MST	
Adjusted R Square	
Standard Error	

R-squared: Calculate SST, SSE & SSR

$$R^2 = 1 - \frac{SSE}{SST}$$

$$R^2_{adj} = 1 - \frac{MSE}{MST}$$

$$MSE = SSE/(n-q)$$

$$Std\ Error = \sqrt{MSE} = \sqrt{\frac{SSE}{n-q}}$$

$$MST = SST/(n-1)$$

$$F = \frac{MSR}{MSE}$$

$$MSR = \frac{\sum_i (\hat{y}_i - \bar{y})}{q-1} = \frac{SST - SSE}{q-1}$$

	x	y				
	Year	Population	Prediction	Error	Square Error	Sq. Mean Difference
	1980	2.1	2.08	-0.02	0.0004	1.80
	1985	2.9	2.76	-0.14	0.0196	0.29
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	1995	4.1	4.12	0.02	0.0004	0.44
	2000	4.9	4.8	-0.1	0.01	2.13
Sum					0.088	4.71
Mean	avg(x)	avg(y)			MSE	MST
Count (n)					df	df

Number of Coefficients	
R Square	
MSE	
MST	
Adjusted R Square	
Standard Error	

R-squared: Calculate SST, SSE & SSR

	Degree of freedom	Sum of square	Mean square	F
Regression	$q-1$	$SST - SSE = \sum_i (\hat{y}_i - \bar{y})$	$MSR = \frac{SST - SSE}{q-1}$	$F = \frac{MSR}{MSE}$
Residual (Error)	$n-q$	$SSE = \sum_i (y_i - \hat{y}_i)^2$	$MSE = \frac{SSE}{n-q}$	
Total	$n-1$	$SST = \sum_i (y_i - \bar{y})^2$		

Linear Regression Equation: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \ldots + \beta_n X_n$

Simple Linear Regression: $Y = \beta_0 + \beta_1 X_1$

= Intercept + X1* Slope

No. of Coefficients q = 2 (β_0 = Intercept, β_1 = Slope)

R-squared: Calculate SST, SSE & SSR

$$R^2 = 1 - \frac{SSE}{SST}$$

$$R^2_{adj} = 1 - \frac{MSE}{MST}$$

$$MSE = SSE/(n-q)$$

$$Std\ Error = \sqrt{MSE} = \sqrt{\frac{SSE}{n-q}}$$

$$MST = SST/(n-1)$$

$$F = \frac{MSR}{MSE}$$

$$MSR = \frac{\sum_i (\hat{y}_i - \bar{y})}{q-1} = \frac{SST - SSE}{q-1}$$

	x	y				
	Year	Population	Prediction	Error	Square Error	Sq. Mean Difference
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	1995	4.1	4.12	0.02	0.0004	0.44
	2000	4.9	4.8	-0.1	0.01	2.13
Sum					0.088	4.71
Mean	avg(x)	avg(y)			MSE	MST
Count (n)					df	df

Number of Coefficients	
R Square	
MSE	
MST	
Adjusted R Square	
Standard Error	

R-squared: Calculate SST, SSE & SSR

$$R^2 = 1 - \frac{SSE}{SST}$$

$$R^2_{adj} = 1 - \frac{MSE}{MST}$$

$$MSE = SSE/(n-q)$$

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	x	y				
	Year	Population	Prediction	Error	Square Error	Sq. Mean Difference
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	1985	2.9	2.76	-0.14	0.0196	0.29
	1990	3.2	3.44	0.24	0.0576	0.06
	1995	4.1	4.12	0.02	0.0004	0.44
	2000	4.9	4.8	-0.1	0.01	2.13
Sum					0.088	4.71
Mean	avg(x)	avg(y)			0.029	1.178
Count (n) = 5					df = 3	df = 4

Number of Coefficients	2
R Square	0.981
MSE	0.029
MST	1.178
Adjusted R Square	0.975
Standard Error	0.1712698

Calculate Statistics (Using Tool)

Regression Statistics	
Multiple R	0.991
R Square	0.981
Adjusted R Square	0.975
Standard Error	0.171
Observations	5

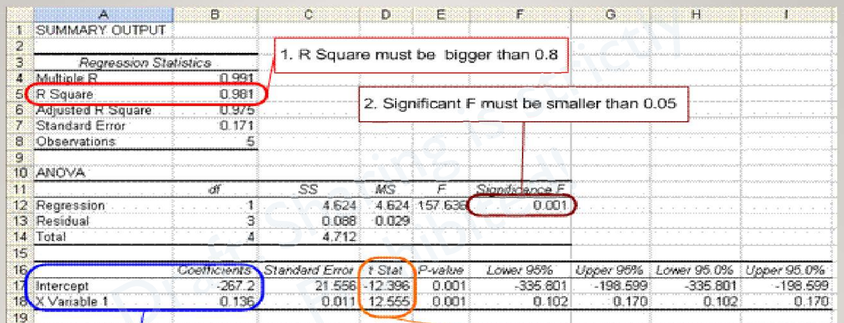
ANOVA

	df	SS	MS	F	Significance F
Regression	1	4.624	4.624	157.636	0.001
Residual	3	0.088	0.029		
Total	4	4.712			

t & p Statistics

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%
Intercept	-267.2	21.556	-12.396	0.001	-335.801	-198.599	-335.801	-198.599
X Variable 1	0.136	0.011	12.555	0.001	0.102	0.170	0.102	0.170

Calculate Statistics (Using Tool)



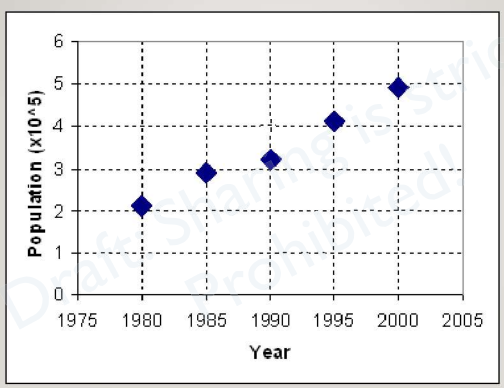
	A	B	C	D	E	F	G	H	I
1	SUMMARY OUTPUT								
2									
3	Regression Statistics								
4	Multiple R	0.991							
5	R Square	0.981							
6	Adjusted R Square	0.975							
7	Standard Error	0.171							
8	Observations	5							
9									
10	ANOVA								
11		df	SS	MS	F	Significance F			
12	Regression	1	4.624	4.624	157.636	0.001			
13	Residual	3	0.088	0.029					
14	Total	4	4.712						
15									
16		Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%
17	Intercept	-267.2	21.556	-12.396	0.001	-335.801	-198.599	-335.801	-198.599
18	X Variable 1	0.136	0.011	12.555	0.001	0.102	0.170	0.102	0.170
19									

Regression Goodness of Fit

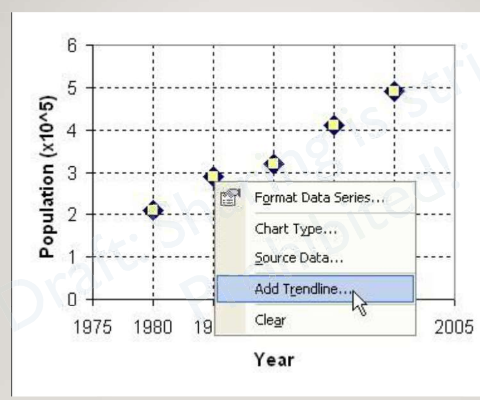
Regression model needs to pass all the criteria below:

1. The R square must be bigger than 0.80
2. The significant F (from ANOVA) must be smaller than 0.05
3. The absolute value of t-statistics must be larger than 1.96 for α=0.05 and must larger than 1.645 for α=0.10

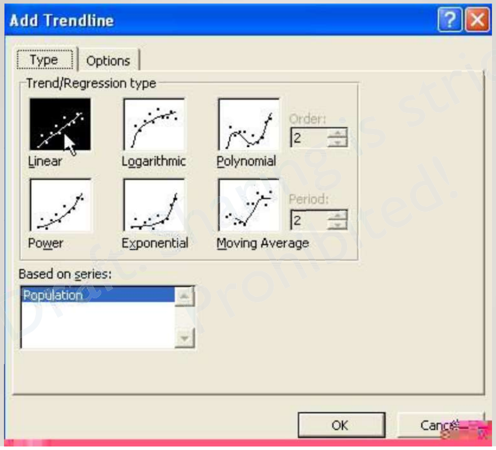
Regression analysis using chart (Scatter Plot)



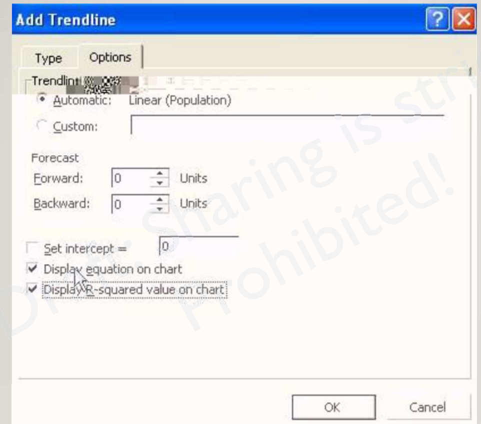
Regression analysis using chart



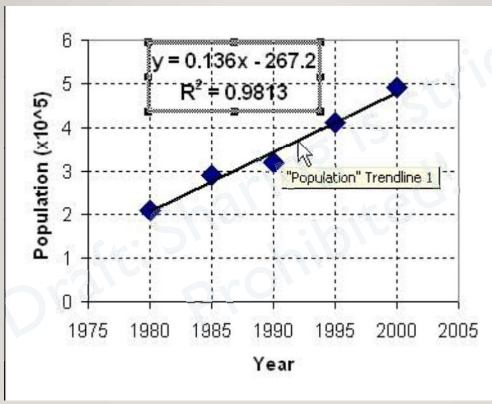
Regression analysis using chart



Regression analysis using chart



Regression analysis using chart



Manual Computation we did

$$\hat{y} = slope * x + intercept$$

$$slope = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2}$$

$$intercept = \bar{y} - slope \cdot \bar{x}$$

	x	y	xy	x^2
	Year	Population		
	1980	2.1		
	1985	2.9		
	1990	3.2		
	1995	4.1		
	2000	4.9		
Sum				
Average				
Count (n) =				

Slope	
Intercept	

Manual Computation we did

$$\hat{y} = slope * x + intercept$$

$$slope = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2}$$

$$intercept = \bar{y} - slope \cdot \bar{x}$$

	x	y	xy	x^2
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Sum	9950	17.2	34262	19800750
Average	1990	3.44		
Count (n) =	5			

Slope	0.136
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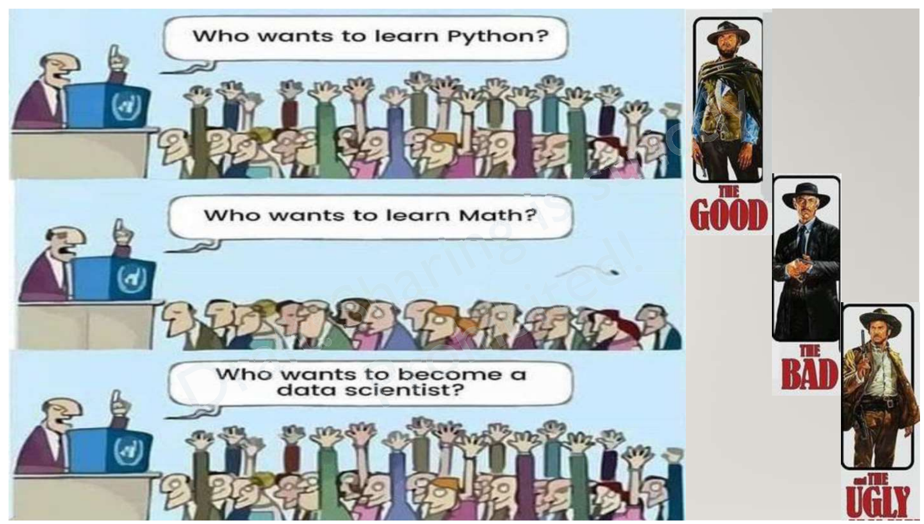

Linear Regression with Linear Algebra

$$y = m \cdot x + c$$

$$m \cdot x_1 + c = y_1$$
$$m \cdot x_2 + c = y_2$$
$$\vdots$$
$$m \cdot x_n + c = y_n$$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \cdot \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\mathbf{A} = [\mathbf{x} \;\; 1] \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} m \\ c \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Linear Regression with Linear Algebra

$$y = m \cdot x + c$$

$$m \cdot x_1 + c = y_1$$
$$m \cdot x_2 + c = y_2$$
$$\vdots$$
$$m \cdot x_n + c = y_n$$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \cdot \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\mathbf{A} \cdot \mathbf{b} = \mathbf{y}$$

$$(\mathbf{A}^t \cdot \mathbf{A}) \cdot \mathbf{b} = \mathbf{A}^t \cdot \mathbf{y}$$

$$(\mathbf{A}^t \cdot \mathbf{A})^{-1} \cdot (\mathbf{A}^t \cdot \mathbf{A}) \cdot \mathbf{b} = (\mathbf{A}^t \cdot \mathbf{A})^{-1} \cdot \mathbf{A}^t \cdot \mathbf{y}$$

$$\boxed{\mathbf{b} = (\mathbf{A}^t \cdot \mathbf{A})^{-1} \cdot \mathbf{A}^t \cdot \mathbf{y}}$$

$$\mathbf{A} = [\mathbf{x} \;\; 1] \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} m \\ c \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\mathbf{A}^t \cdot \mathbf{A} =$$

$$\mathbf{A}^t \cdot \mathbf{y} =$$

Linear Regression with Linear Algebra

$$y = m \cdot x + c$$

$$m \cdot x_1 + c = y_1$$
$$m \cdot x_2 + c = y_2$$
$$\vdots$$
$$m \cdot x_n + c = y_n$$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \cdot \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\mathbf{A} \cdot \mathbf{b} = \mathbf{y}$$

$$(\mathbf{A}^t \cdot \mathbf{A}) \cdot \mathbf{b} = \mathbf{A}^t \cdot \mathbf{y}$$

$$(\mathbf{A}^t \cdot \mathbf{A})^{-1} \cdot (\mathbf{A}^t \cdot \mathbf{A}) \cdot \mathbf{b} = (\mathbf{A}^t \cdot \mathbf{A})^{-1} \cdot \mathbf{A}^t \cdot \mathbf{y}$$

$$\boxed{\mathbf{b} = (\mathbf{A}^t \cdot \mathbf{A})^{-1} \cdot \mathbf{A}^t \cdot \mathbf{y}}$$

$$\mathbf{A} = [\mathbf{x} \;\; 1] \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} m \\ c \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\mathbf{A}^t \cdot \mathbf{A} = \begin{bmatrix} \sum x^2 & \sum x \\ \sum x & n \end{bmatrix}$$

$$\mathbf{A}^t \cdot \mathbf{y} = \begin{bmatrix} \sum xy \\ \sum y \end{bmatrix}$$

Linear Regression with Linear Algebra

A = [x 1]		y
1980	1	2.1
1985	1	2.9
1990	1	3.2
1995	1	4.1
2000	1	4.9

A'				
1980	1985	1990	1995	2000
1	1	1	1	1
A'A				
19800750	9950			
9950	5			
Inverse of A'A				
0.004	-7.96			
-7.96	15840.6			
L = Inv(A'A).A'				
-0.04	-0.02	8.88E-16	0.02	0.04
79.8	40	0.2	-39.6	-79.4
b = Inv(A'A).A' * y = L*y				
0.136				
-267.2				

THANK YOU!