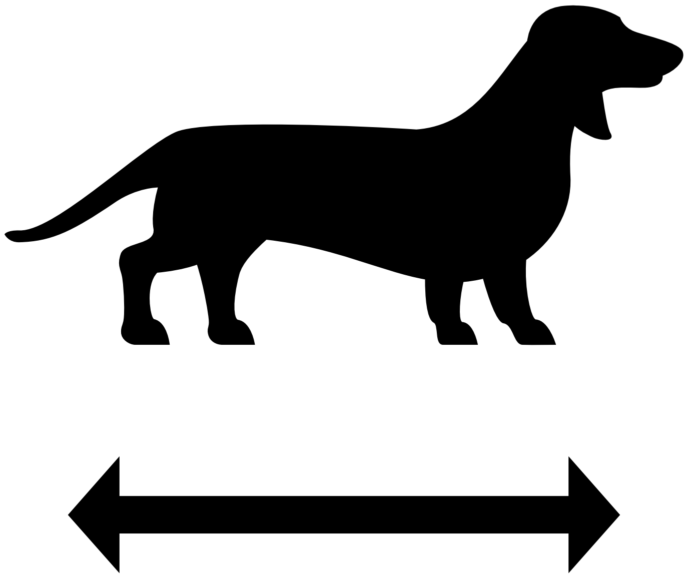
Singular Value Decomposition

Luis Serrano

Transformations

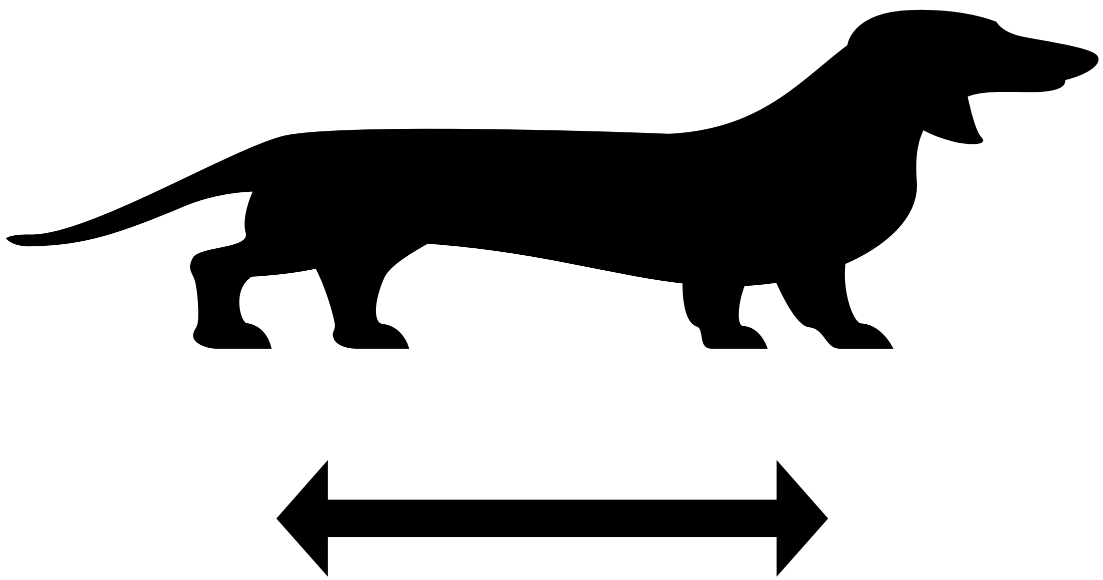
Transformations

Stretch (or compress) horizontally



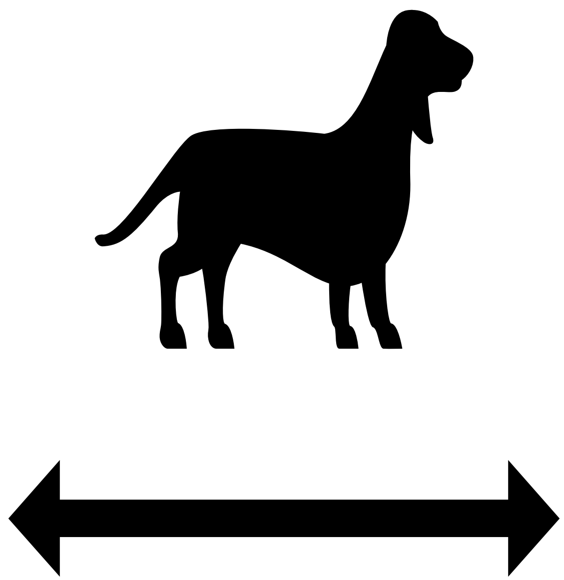
Transformations

Stretch (or compress) horizontally



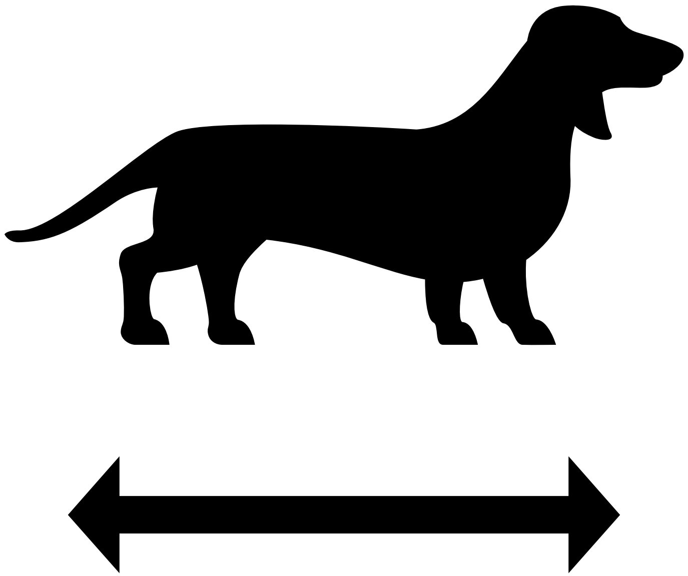
Transformations

Stretch (or compress) horizontally



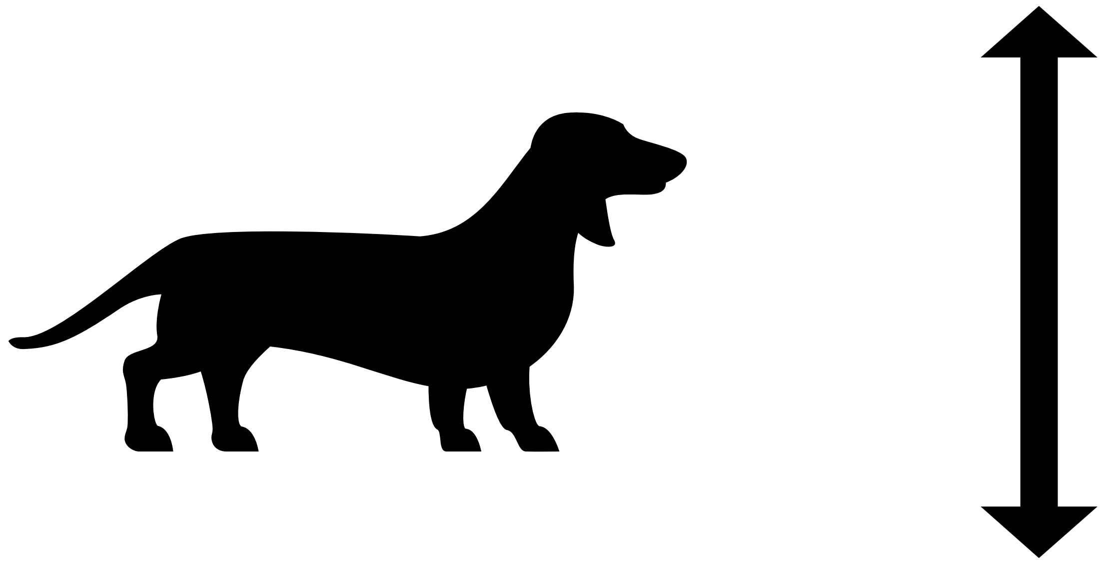
Transformations

Stretch (or compress) horizontally



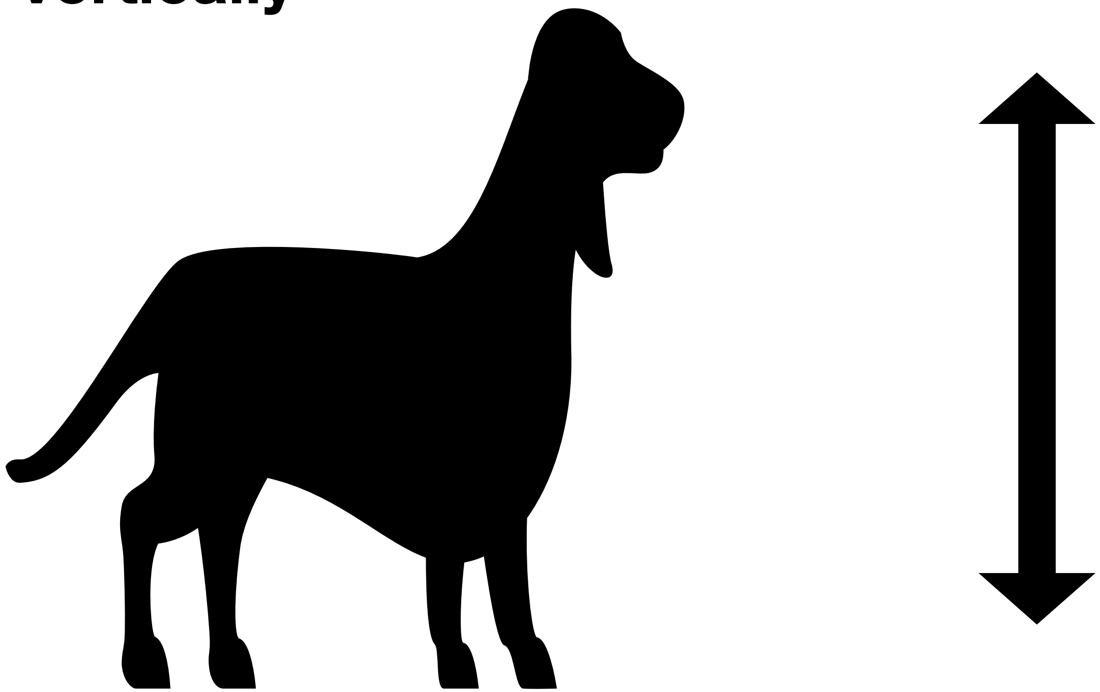
Transformations

Stretch (or compress) vertically



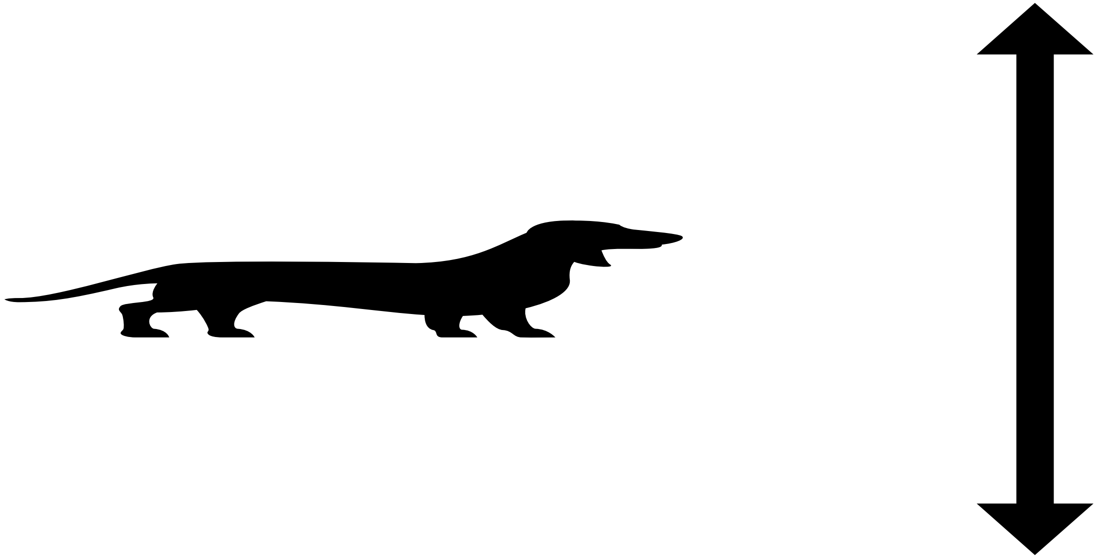
Transformations

Stretch (or compress) vertically



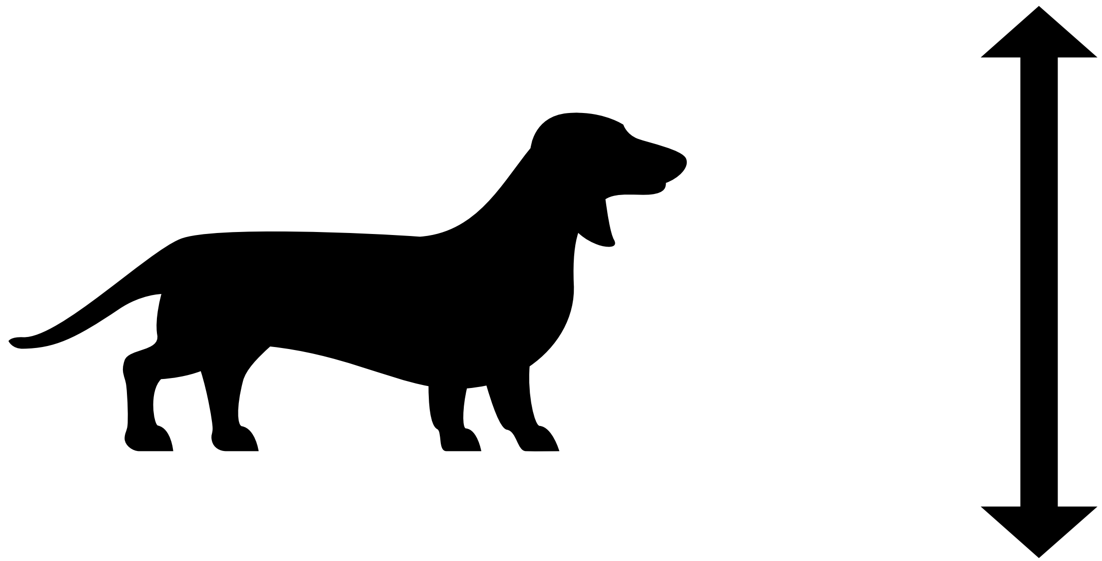
Transformations

Stretch (or compress) vertically



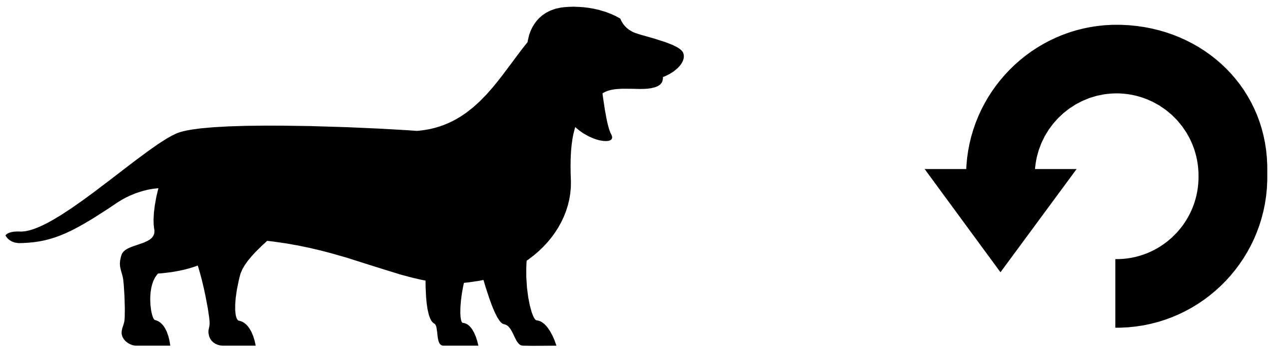
Transformations

Stretch (or compress) vertically



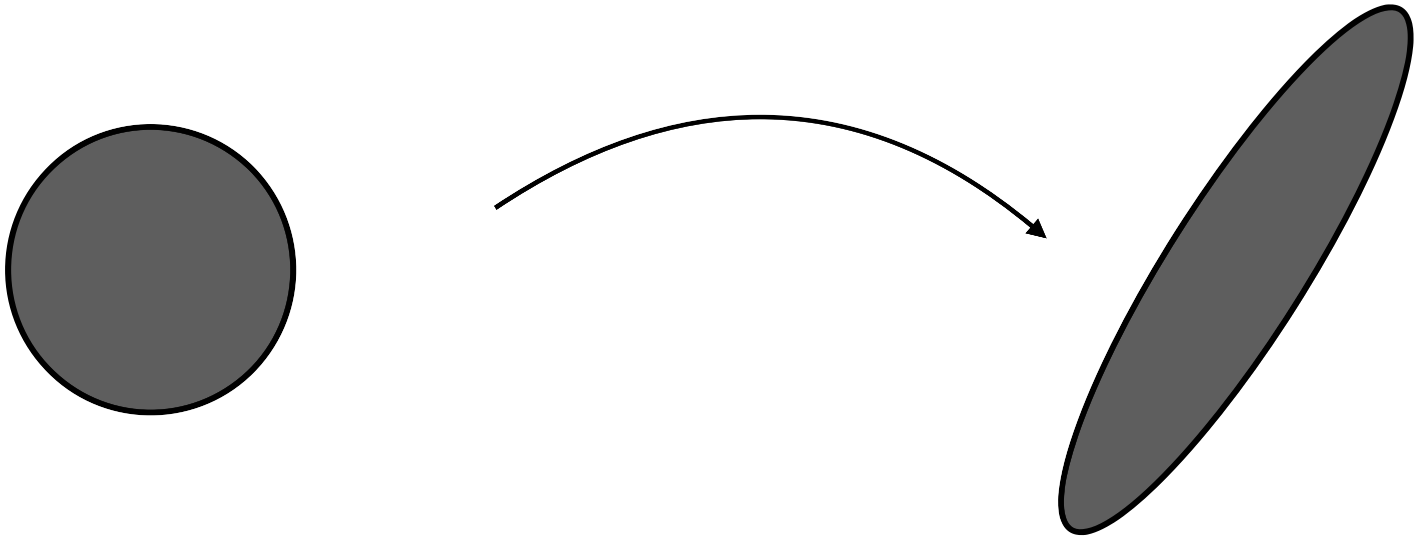
Transformations

Rotate

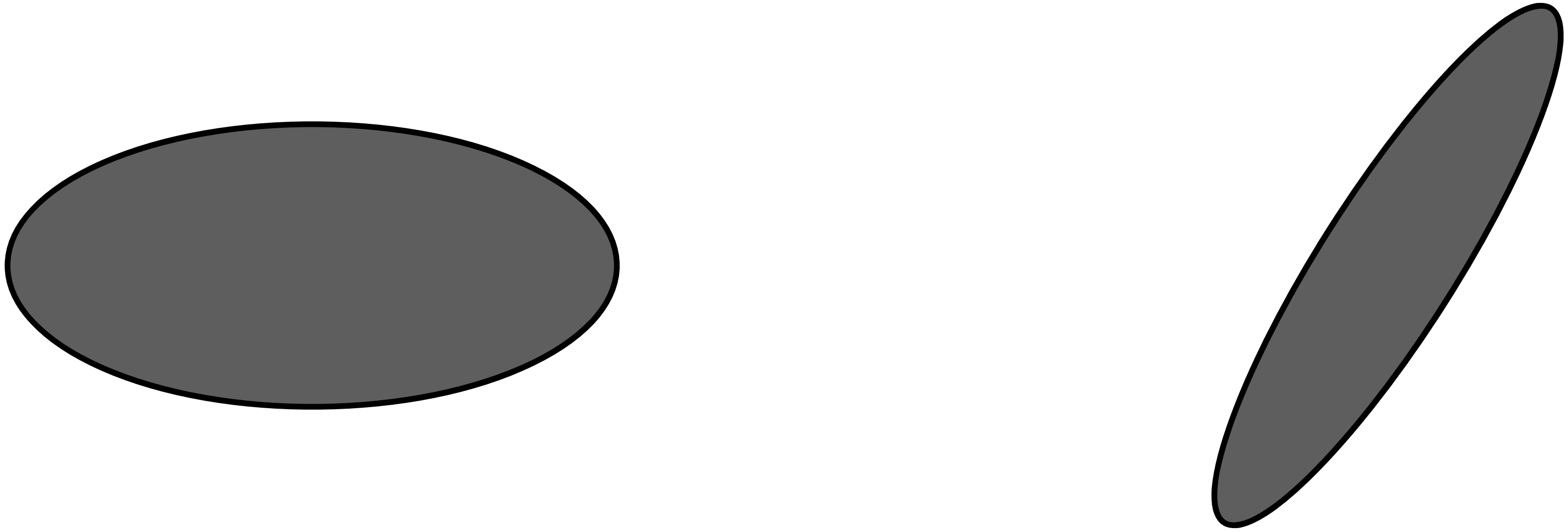


Puzzle (easy)

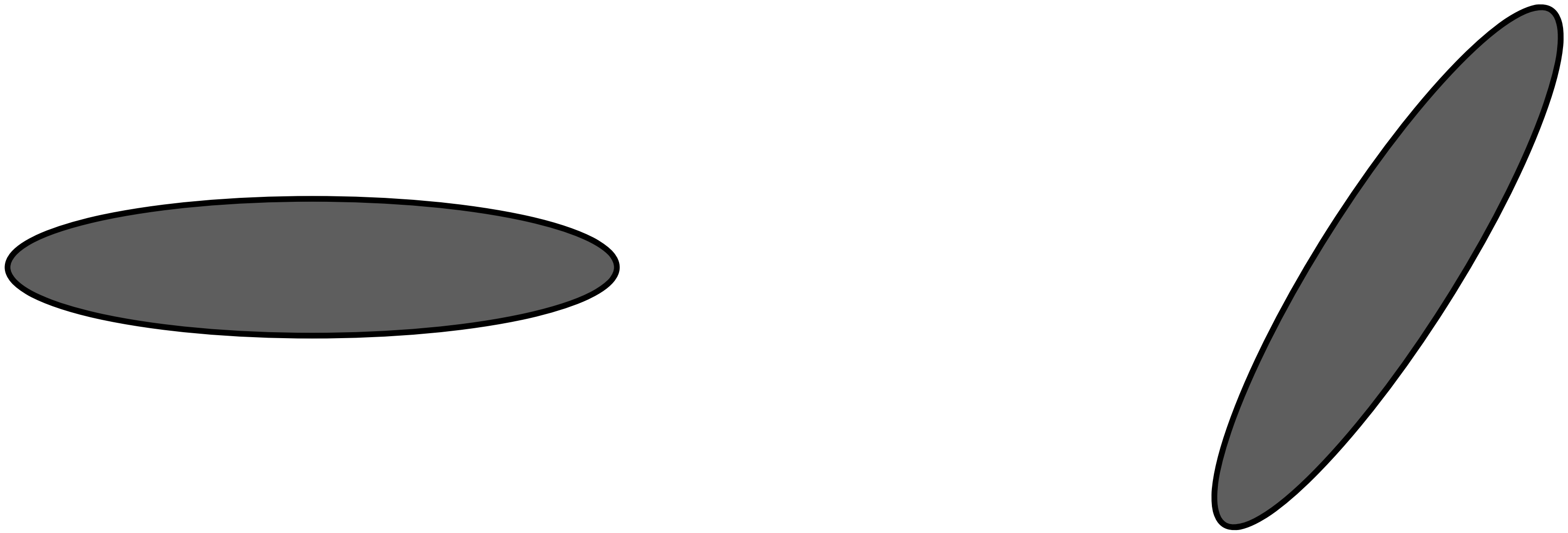
Puzzle (easy)



Puzzle (easy)



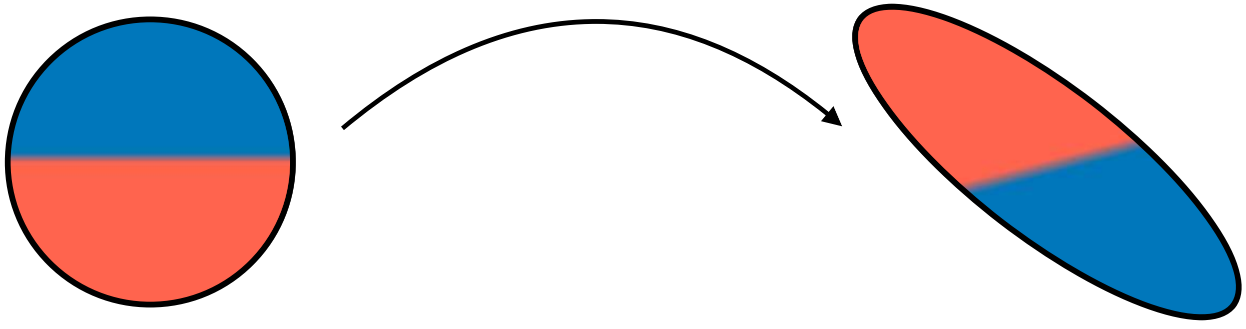
Puzzle (easy)



Puzzle (easy)

Puzzle (hard)

Puzzle (hard)



Puzzle (hard)

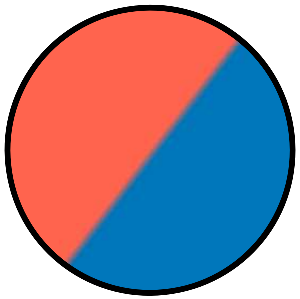
Puzzle (hard)

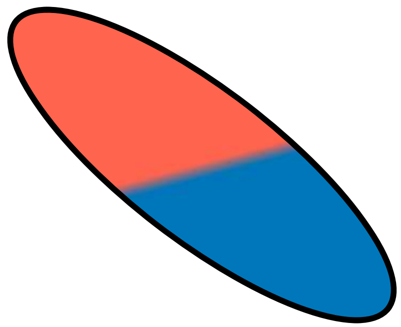
Puzzle (hard)

Solution

Solution

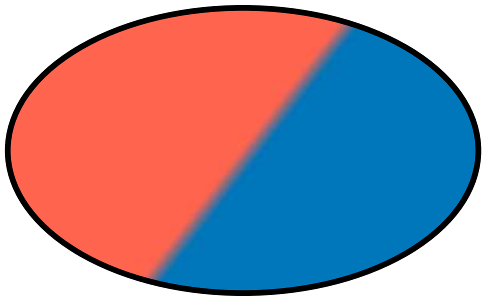
Solution

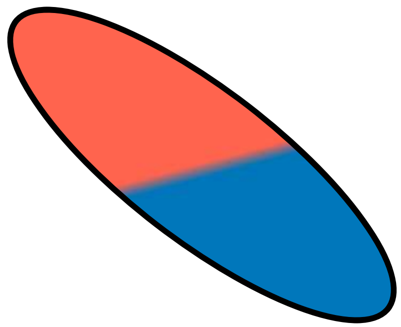




Solution

Solution





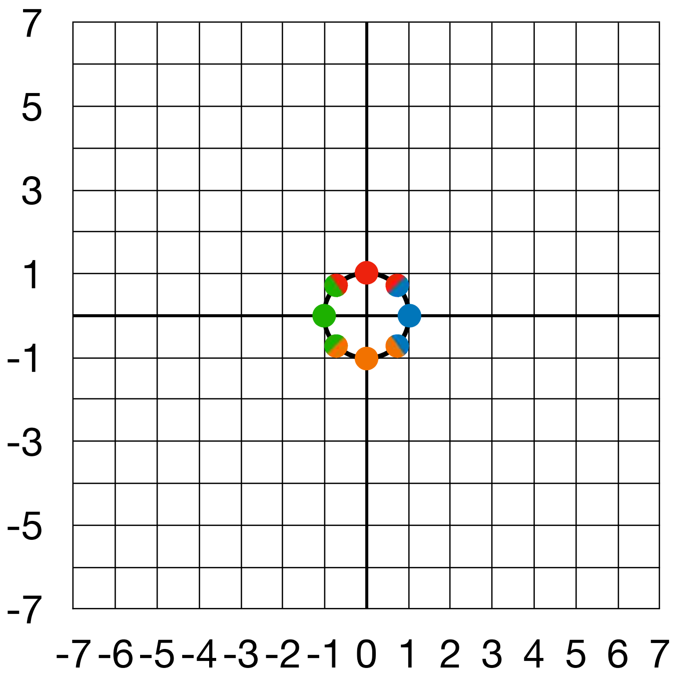
Solution

Solution

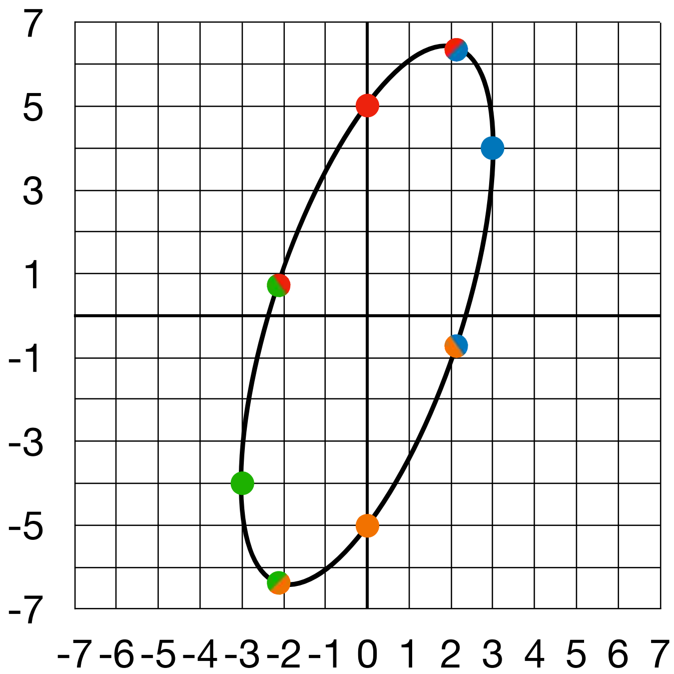
Linear transformations

What does this have to do with matrices?

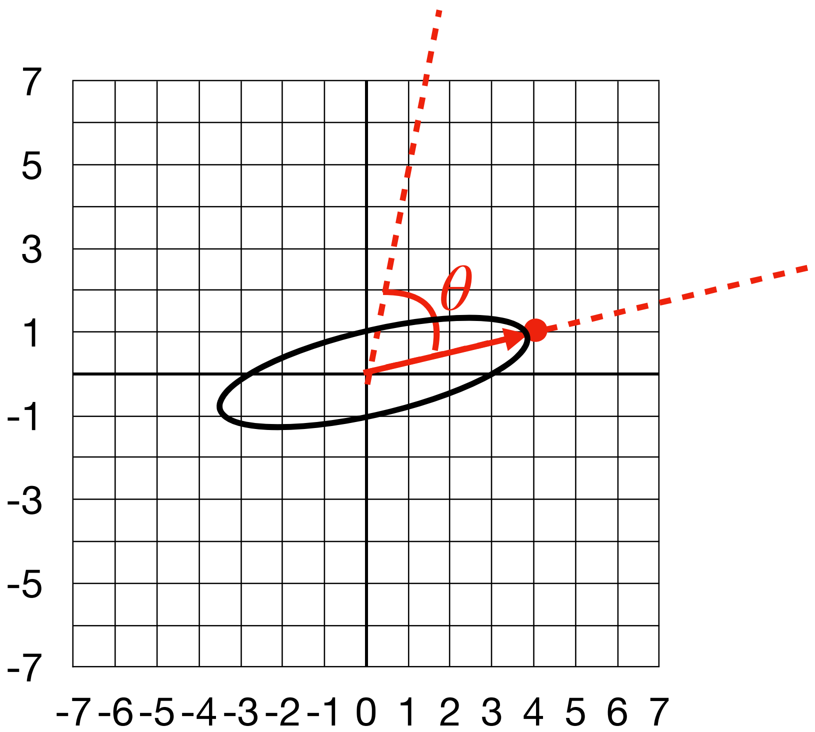
$$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$$



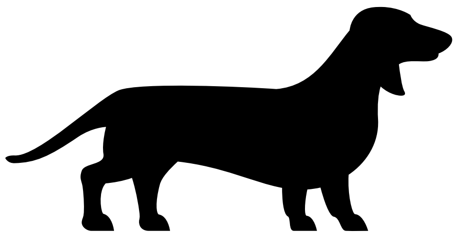
(p,q)	(3p+0q, 4p+5q)
(1,0)	(3, 4)
(0,1)	(0, 5)
(-1,0)	(-3, -4)
(0,-1)	(0, -5)



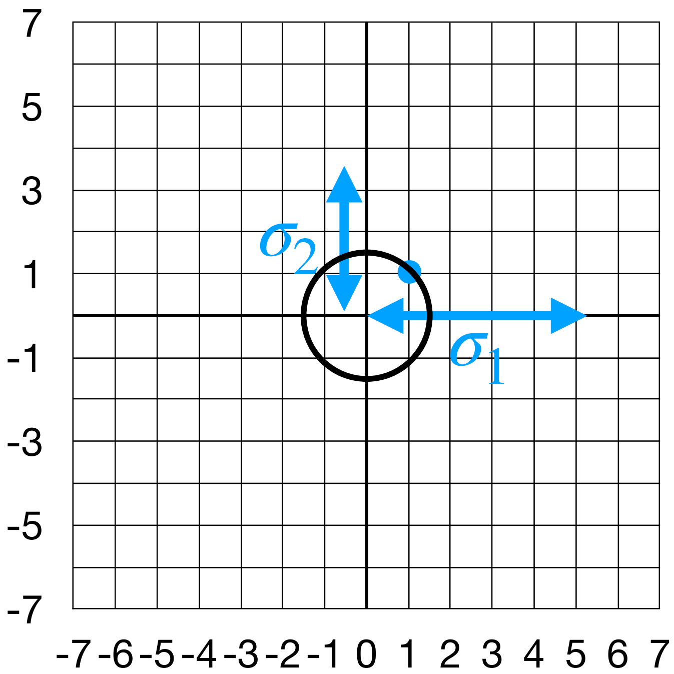
Rotation matrices



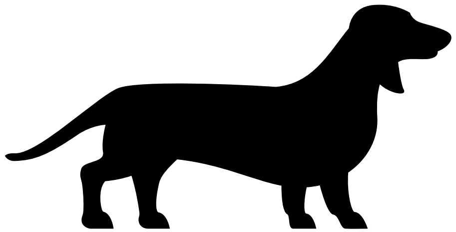
$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$



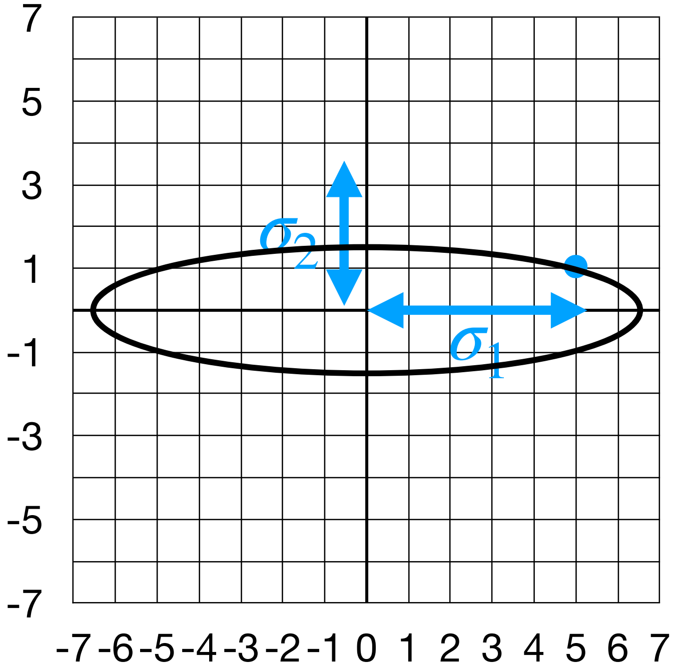
Stretching matrices



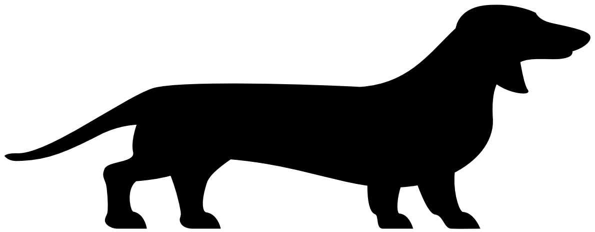
$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$



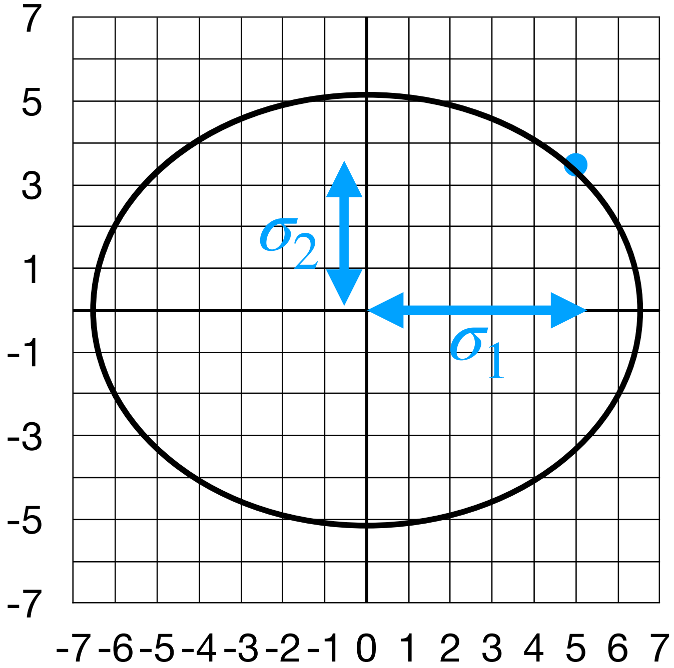
Stretching matrices



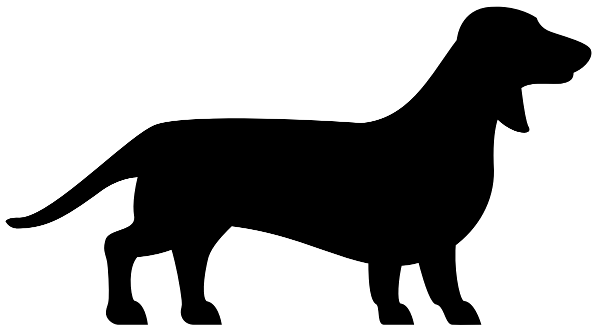
$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$



Stretching matrices

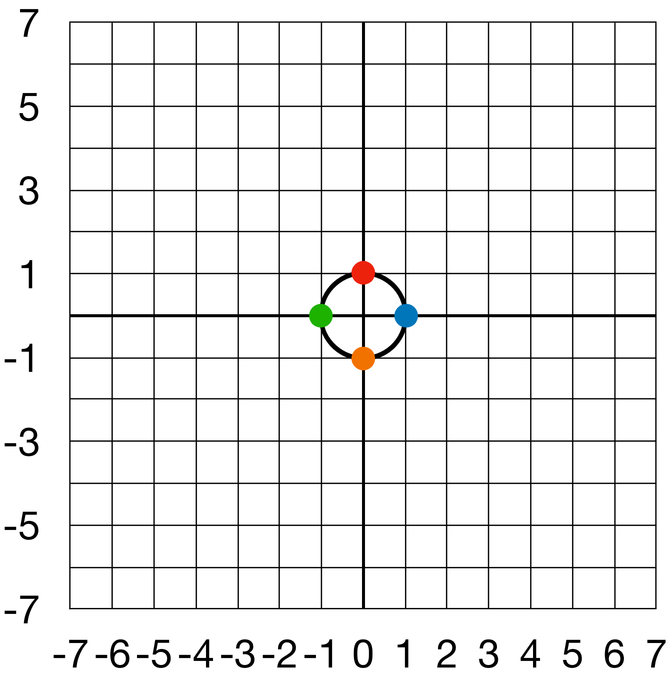


$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$



What does this have to do with matrices?

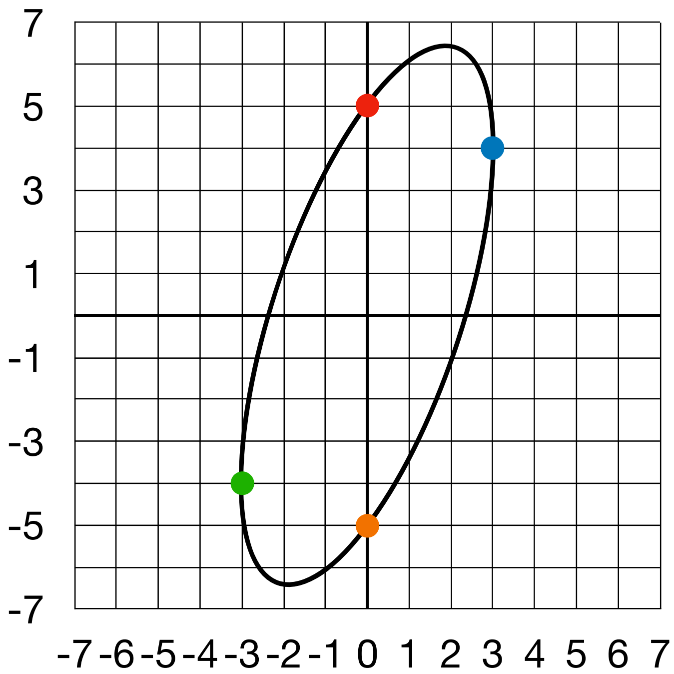
$$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$$



$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

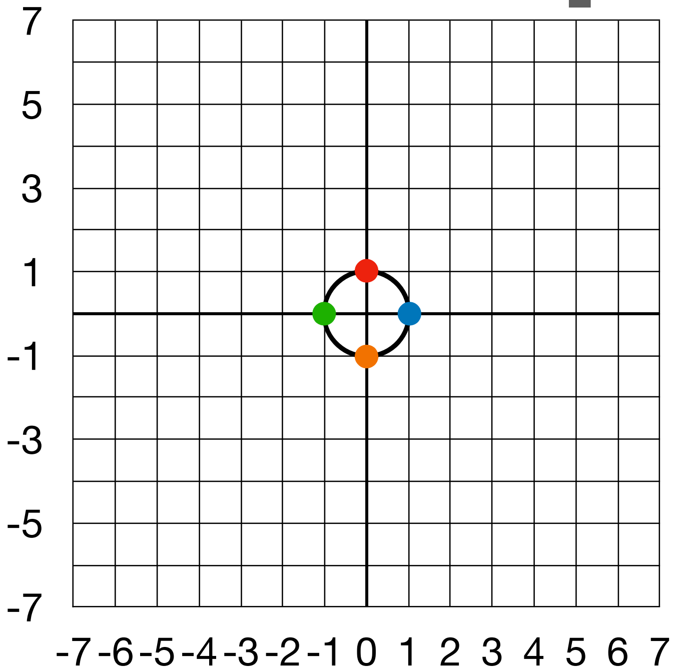
$$\begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$$



SVD

$$A = U\Sigma V^{\dagger}$$

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0.316 & -0.949 \\ 0.949 & 0.316 \end{bmatrix}\begin{bmatrix} 6.708 & 0 \\ 0 & 2.236 \end{bmatrix}\begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$



$$\begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}\begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}\boxed{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

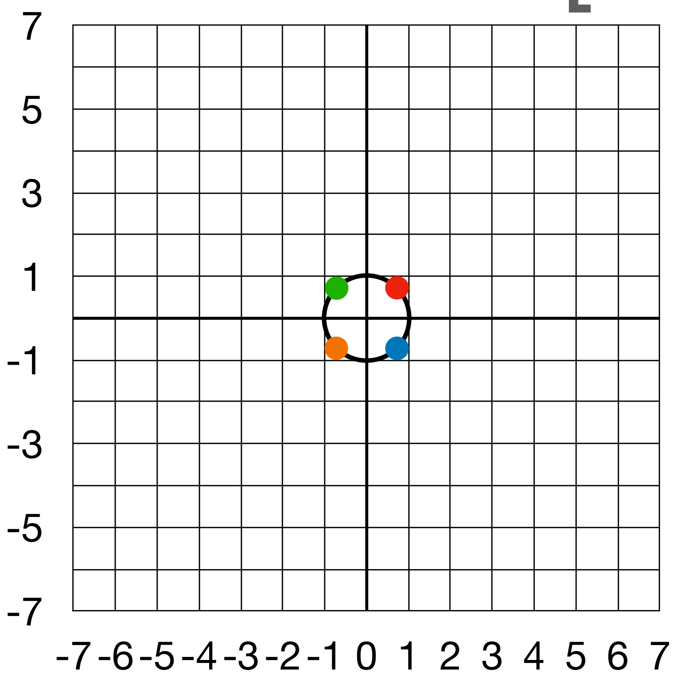
Rotation of $\theta = -\dfrac{\pi}{4} = -45^o$

Rotation

$$A = U\Sigma V^{\dagger}$$

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0.316 & -0.949 \\ 0.949 & 0.316 \end{bmatrix} \begin{bmatrix} 6.708 & 0 \\ 0 & 2.236 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \boxed{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

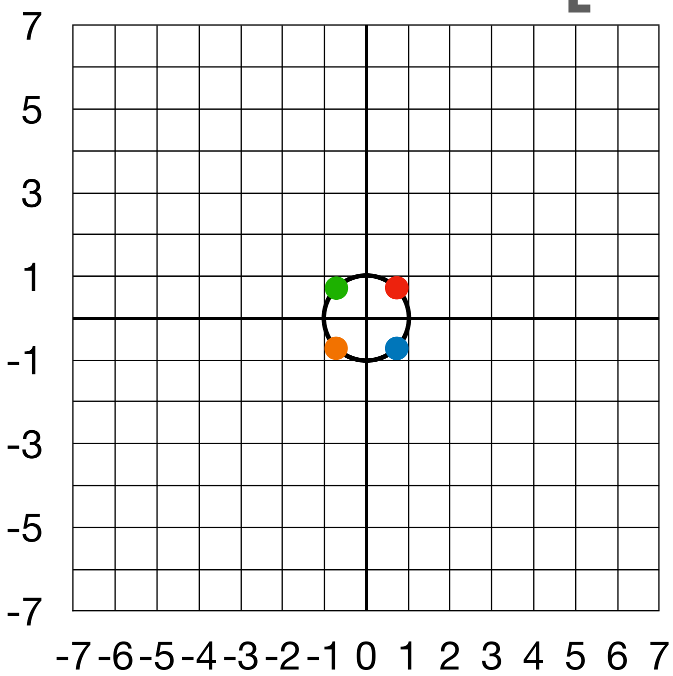


Rotation of $\theta = -\dfrac{\pi}{4} = -45^{o}$

Rotation

$$A = U\Sigma V^{\dagger}$$

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0.316 & -0.949 \\ 0.949 & 0.316 \end{bmatrix}\begin{bmatrix} 6.708 & 0 \\ 0 & 2.236 \end{bmatrix}\begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$



$$\begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}\begin{bmatrix} \boxed{3\sqrt{5}} & 0 \\ 0 & \sqrt{5} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

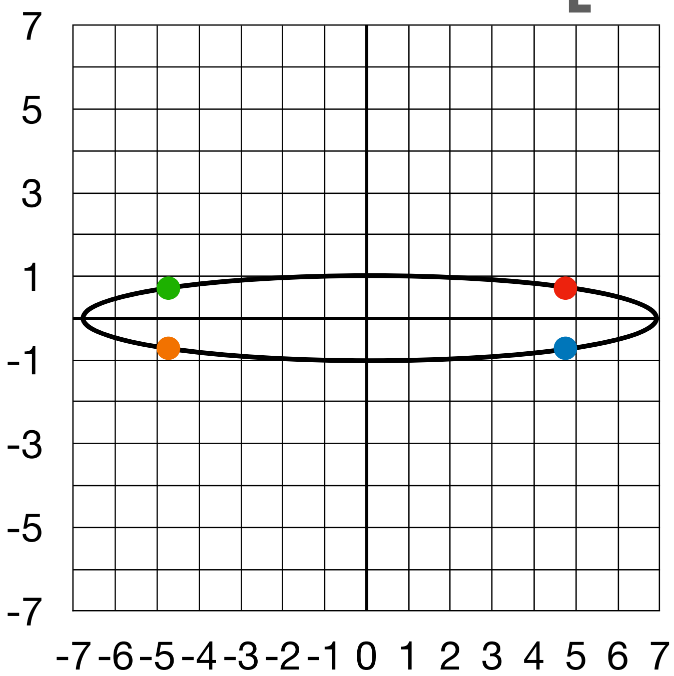
Horizontal scaling by $3\sqrt{5}$

Scaling

$$A = U\Sigma V^\dagger$$

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0.316 & -0.949 \\ 0.949 & 0.316 \end{bmatrix}\begin{bmatrix} 6.708 & 0 \\ 0 & 2.236 \end{bmatrix}\begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}\begin{bmatrix} \boxed{3\sqrt{5}} & 0 \\ 0 & \boxed{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$



Horizontal scaling by $3\sqrt{5}$

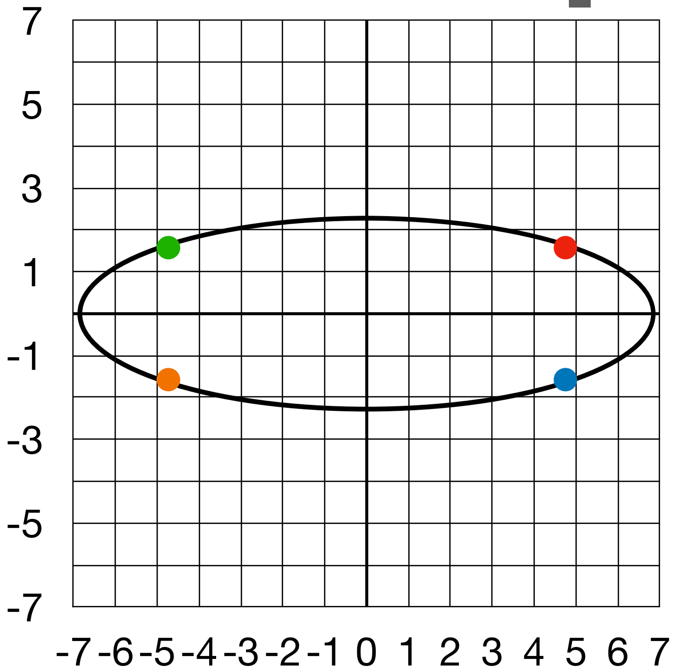
Vertical scaling by $\sqrt{5}$

Rotation

$$A = U\Sigma V^{\dagger}$$

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0.316 & -0.949 \\ 0.949 & 0.316 \end{bmatrix} \begin{bmatrix} 6.708 & 0 \\ 0 & 2.236 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \boxed{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

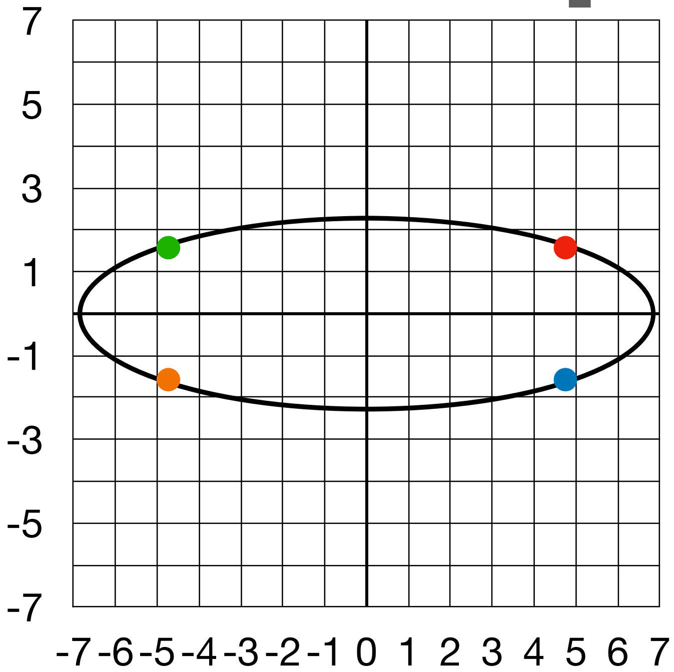


Vertical scaling by $\sqrt{5}$

Rotation

$$A = U\Sigma V^\dagger$$

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0.316 & -0.949 \\ 0.949 & 0.316 \end{bmatrix} \begin{bmatrix} 6.708 & 0 \\ 0 & 2.236 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$



$$\boxed{\begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}} \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

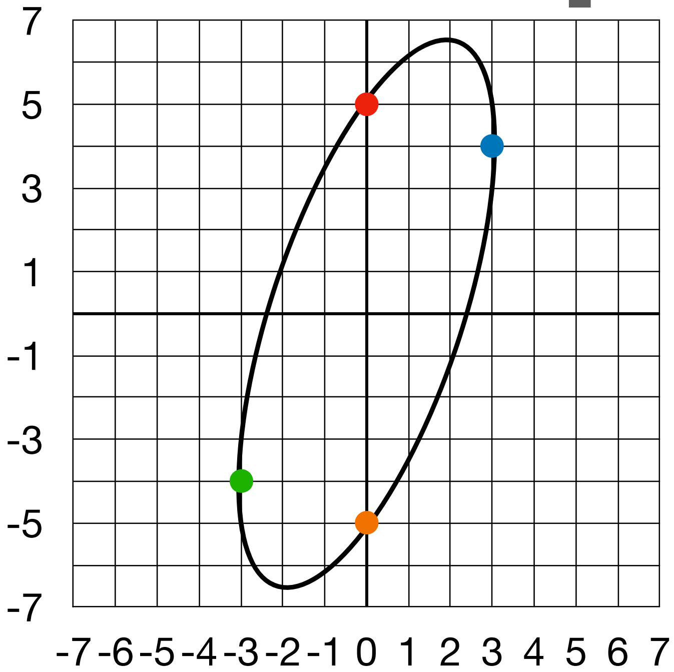
Rotation of $\theta = \arctan(3) = 71.72^o$

Rotation

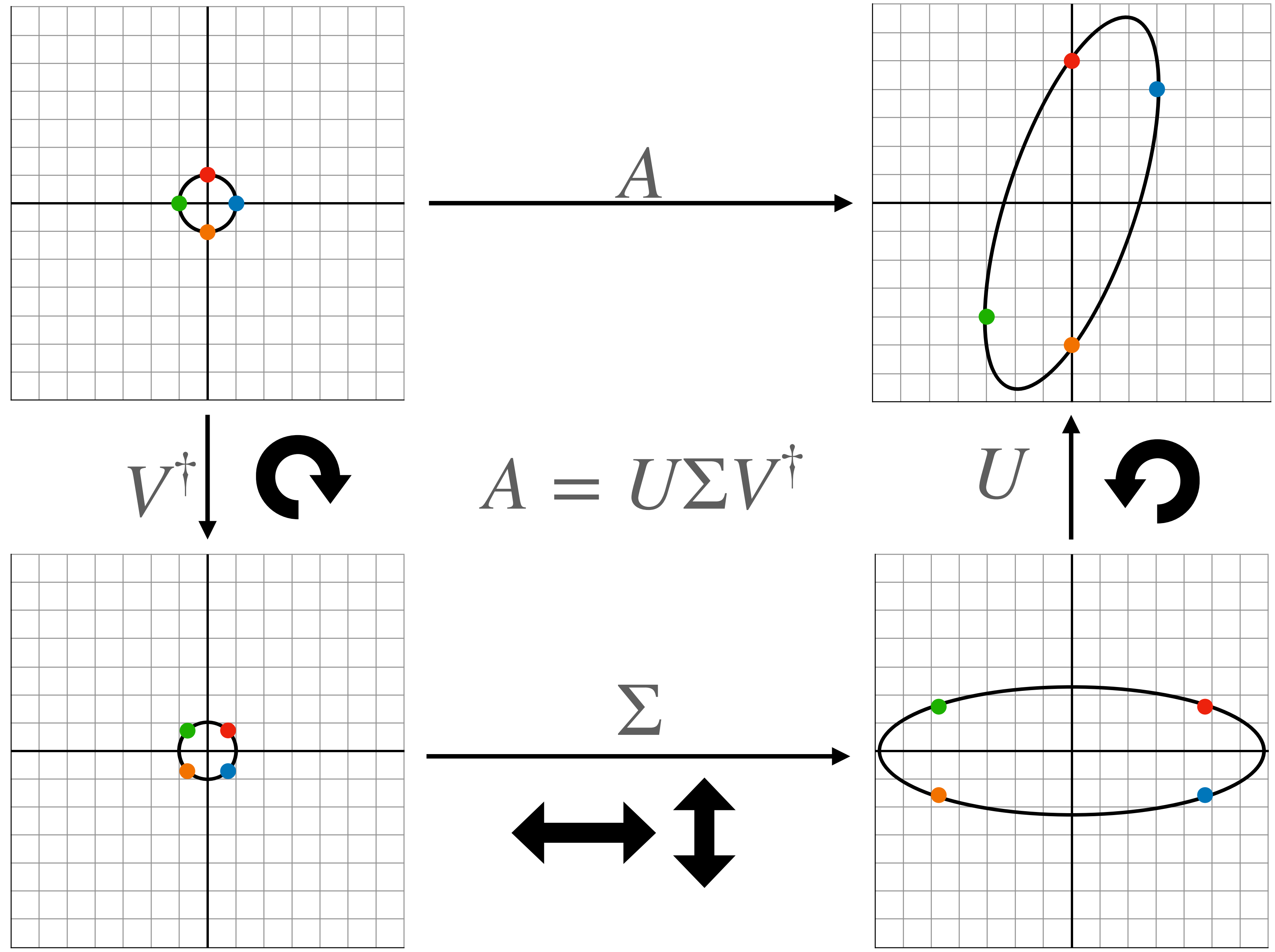
$$A = U\Sigma V^{\dagger}$$

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0.316 & -0.949 \\ 0.949 & 0.316 \end{bmatrix} \begin{bmatrix} 6.708 & 0 \\ 0 & 2.236 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}} \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$



Rotation of $\theta = \arctan(3) = 71.72^{o}$


A
$V^\dagger$
$A = U\Sigma V^\dagger$
U
Σ

Dimensionality reduction

Difference between these two matrices?

$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 2 & 3 & 4 \\
-1 & -1 & -2 & -3 & -4 \\
2 & 2 & 4 & 6 & 8 \\
10 & 10 & 20 & 30 & 40
\end{array}
$$

Difference between these two matrices?

	1	2	3	4
1	1	2	3	4
-1	-1	-2	-3	-4
2	2	4	6	8
10	10	20	30	40

	?	?	?	?
?	3	1	4	1
?	5	9	2	6
?	5	3	5	8
?	9	7	9	3

Rank 1 matrices

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \\ 2 & 4 & 6 & 8 \\ 10 & 20 & 30 & 40 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 10 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

16 numbers

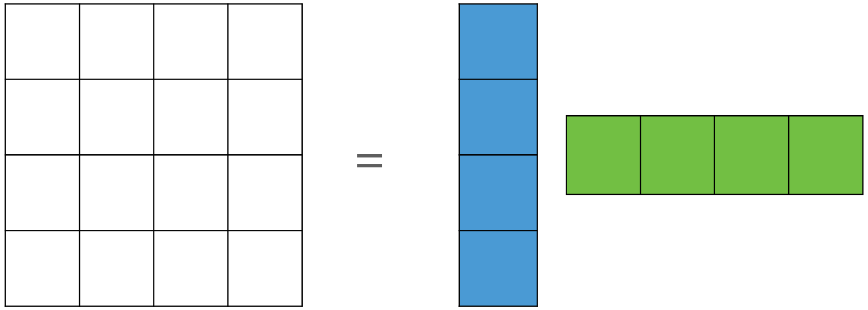
8 numbers

Higher rank matrices

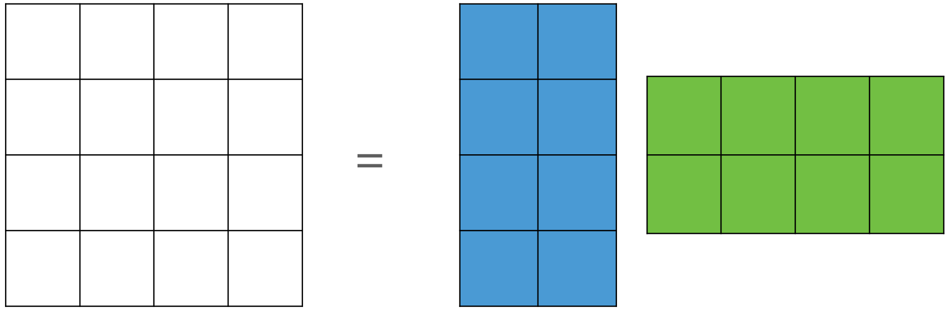
$$\begin{bmatrix} 3 & 1 & 4 & 1 \\ 5 & 9 & 2 & 6 \\ 5 & 3 & 5 & 8 \\ 9 & 7 & 9 & 3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \begin{bmatrix} ? & ? & ? & ? \end{bmatrix}$$

16 numbers

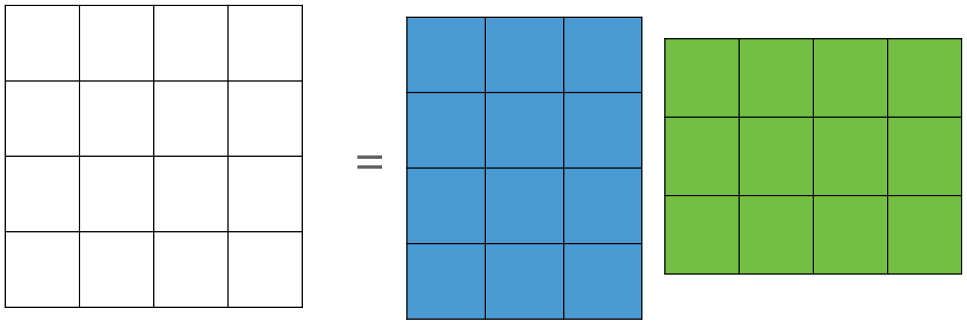
Rank of a matrix



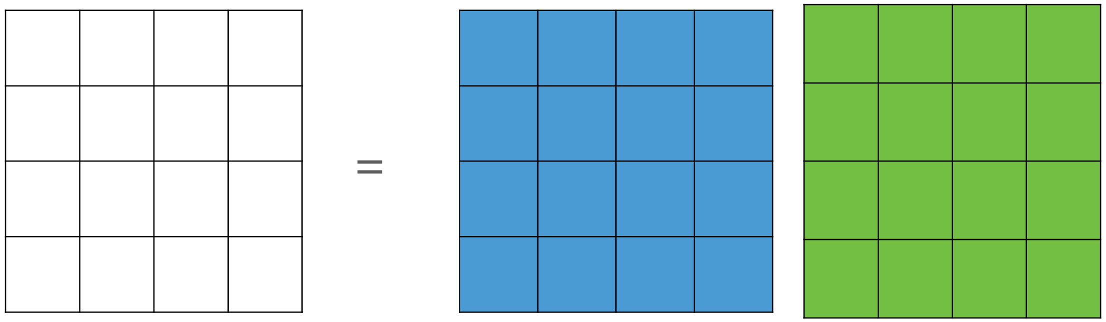
Rank 1



Rank 2



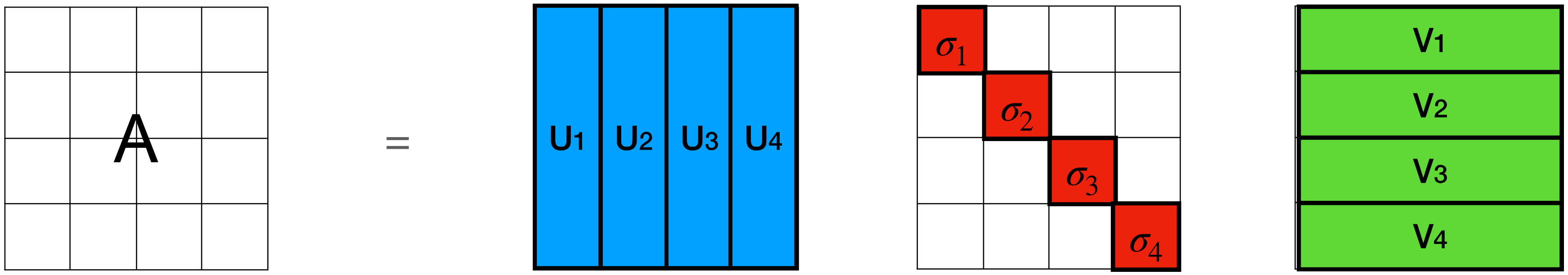
Rank 3



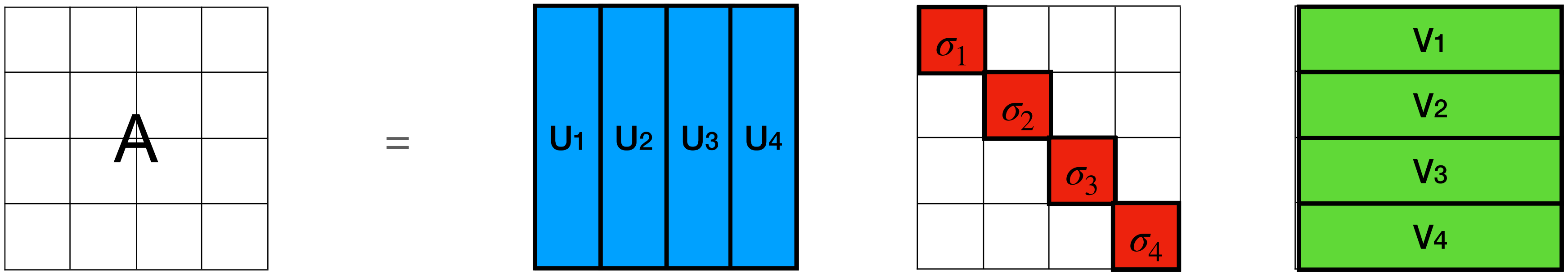
Rank 4

Approximation by a rank one matrix

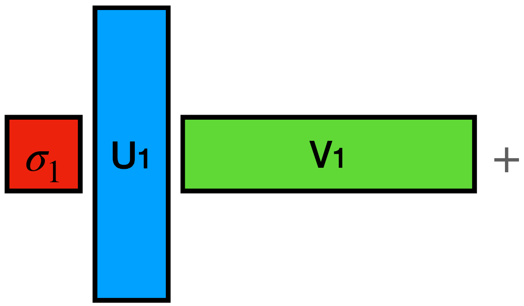
$$\begin{bmatrix} 3 & 1 & 4 & 1 \\ 5 & 9 & 2 & 6 \\ 5 & 3 & 5 & 8 \\ 9 & 7 & 9 & 3 \end{bmatrix} \sim \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \begin{bmatrix} ? & ? & ? & ? \end{bmatrix}$$



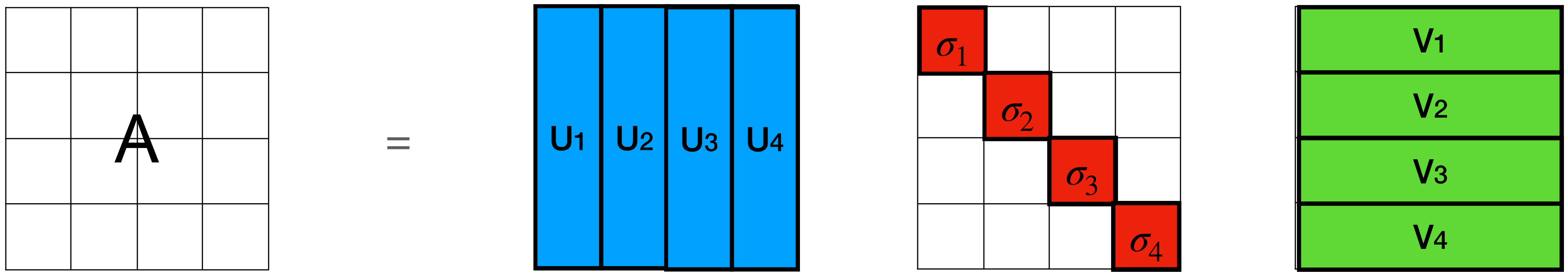
A
=
U1
U2
U3
U4
σ1
σ2
σ3
σ4
V1
V2
V3
V4

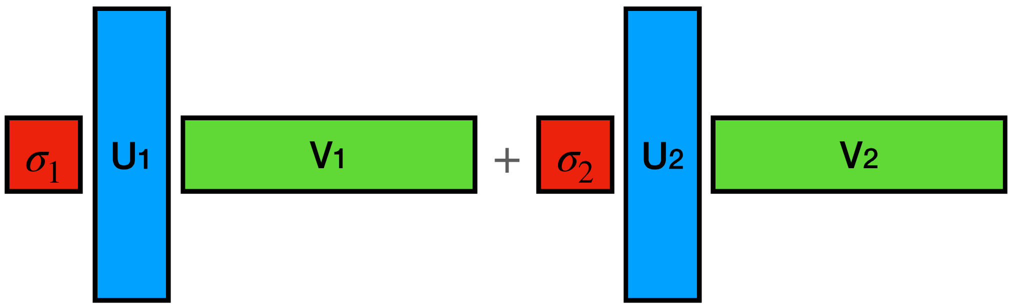


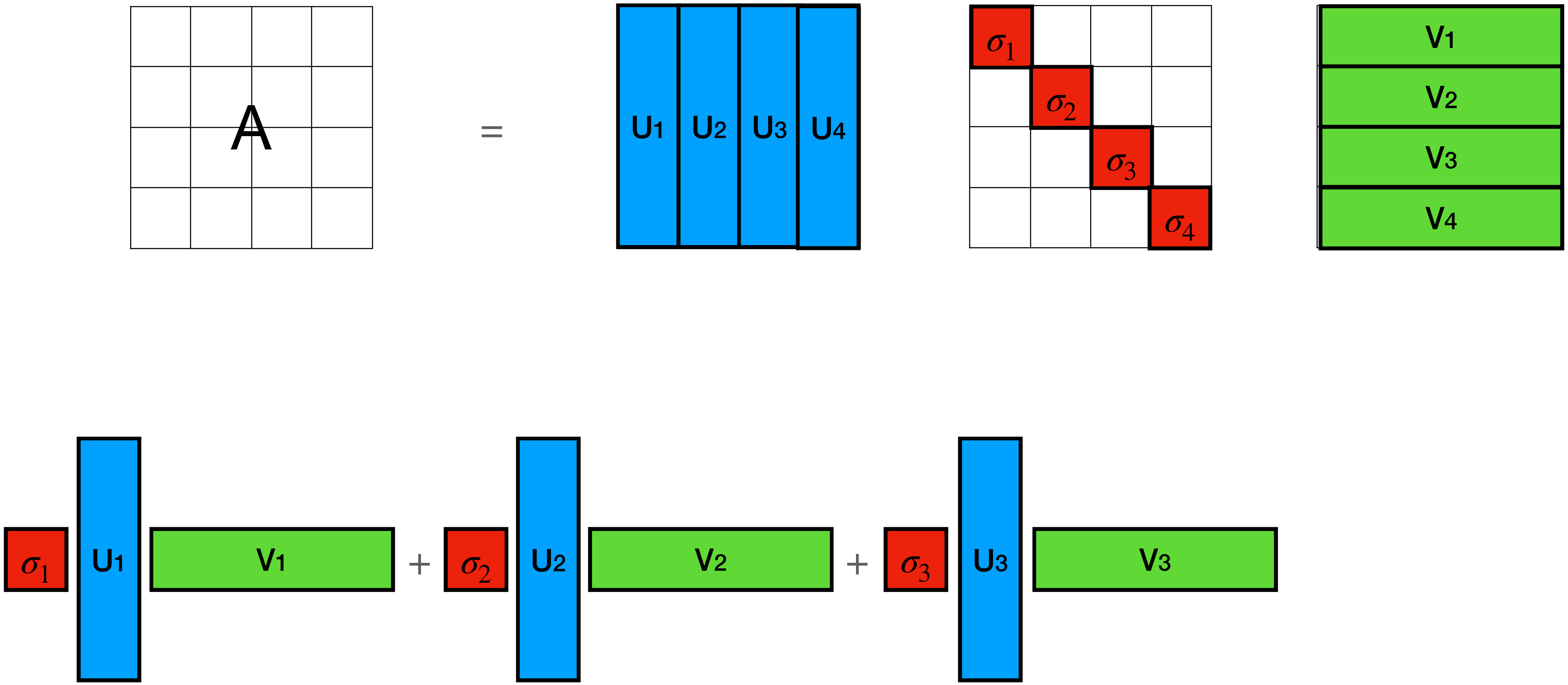
A
=
U1
U2
U3
U4
σ_1
σ_2
σ_3
σ_4
V1
V2
V3
V4

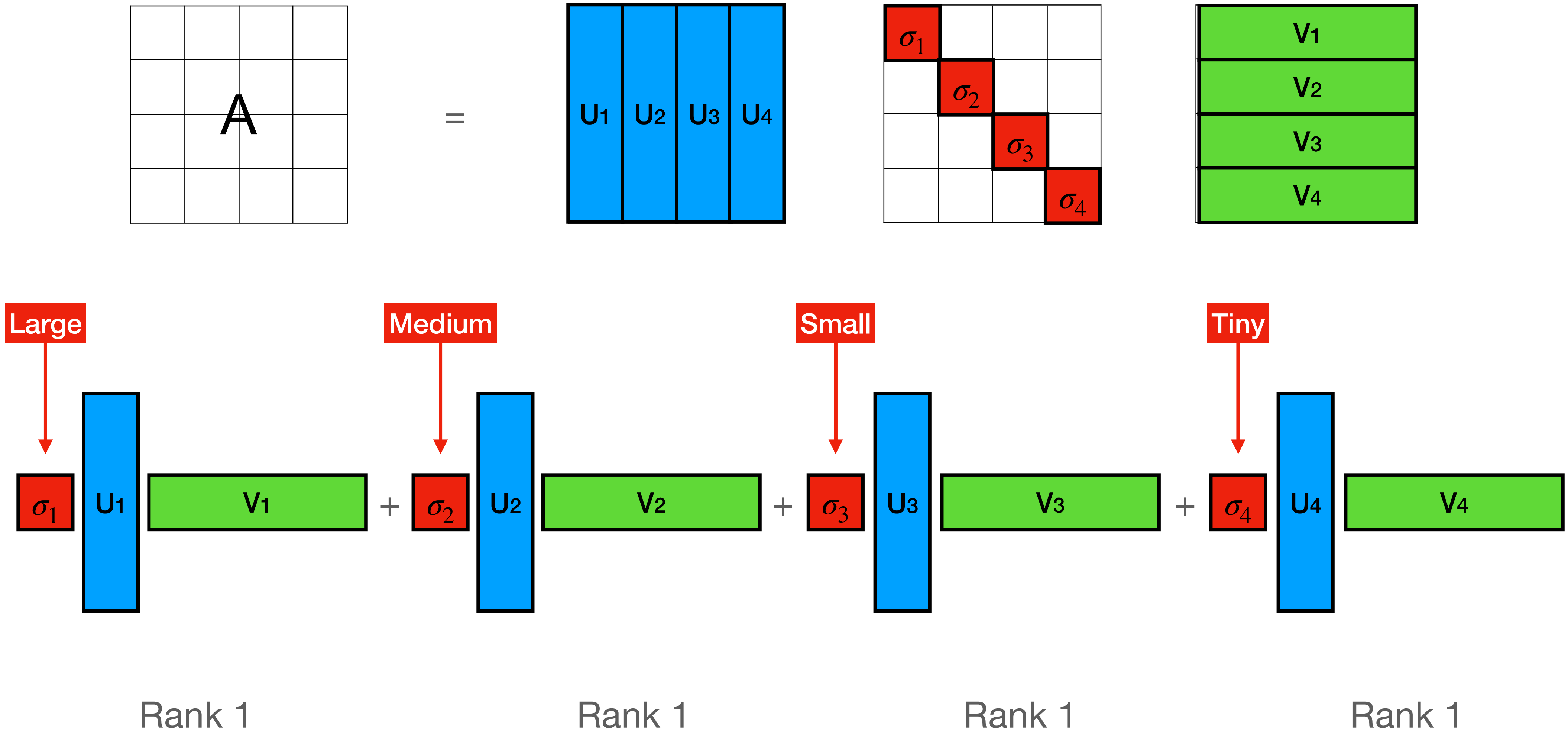


σ_1
U1
V1
+

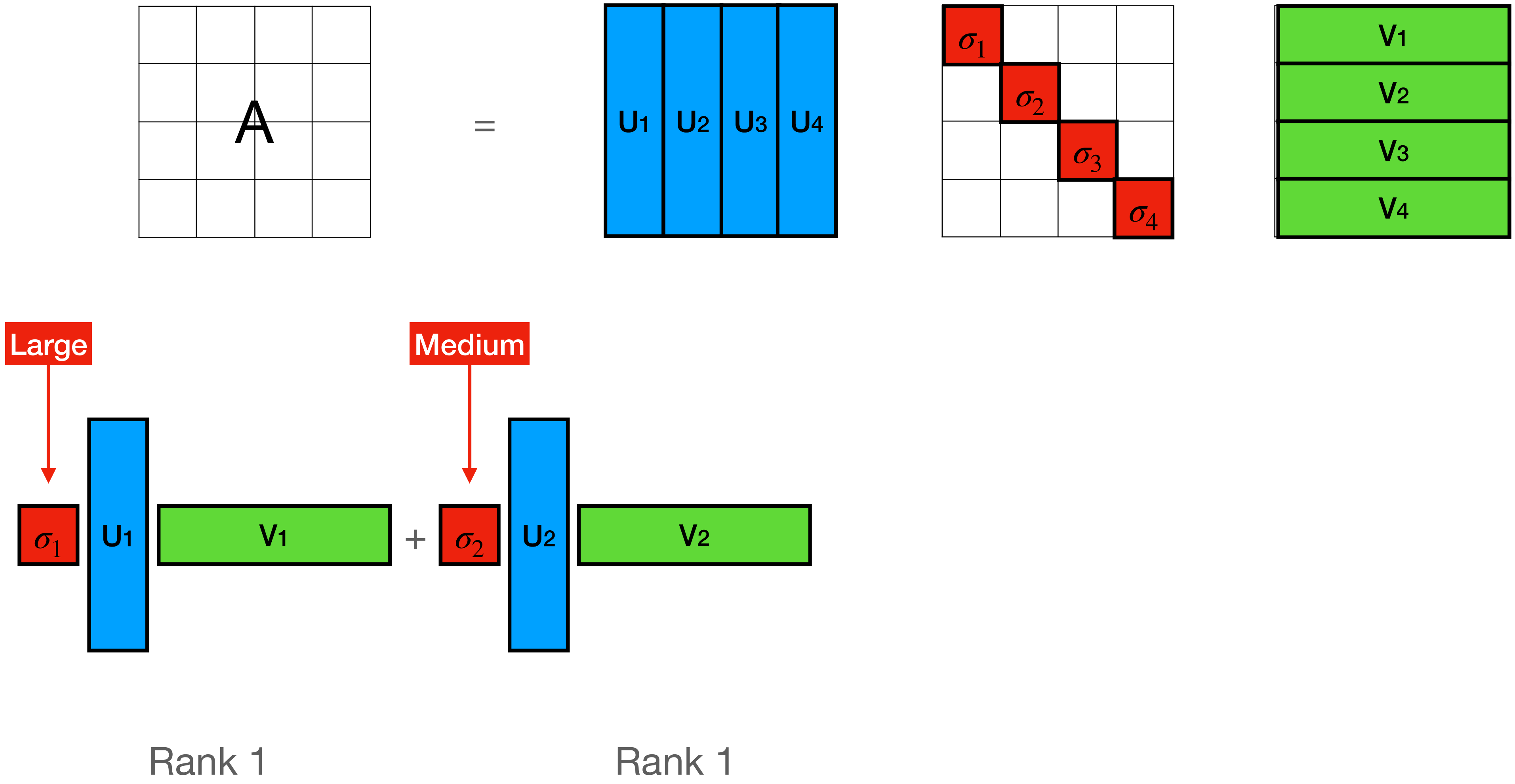

A
=
U1
U2
U3
U4
σ1
σ2
σ3
σ4
V1
V2
V3
V4


σ1
U1
V1
+
σ2
U2
V2


A
=
U1
U2
U3
U4
σ1
σ2
σ3
σ4
V1
V2
V3
V4
σ1
U1
V1
+
σ2
U2
V2
+
σ3
U3
V3



A
=
U1
U2
U3
U4
σ_1
σ_2
σ_3
σ_4
V1
V2
V3
V4
Large
Medium
Small
Tiny
σ_1
U1
V1
+
σ_2
U2
V2
+
σ_3
U3
V3
+
σ_4
U4
V4
Rank 1
Rank 1
Rank 1
Rank 1



A
=
U1
U2
U3
U4
σ1
σ2
σ3
σ4
V1
V2
V3
V4
Large
Medium
σ1
U1
V1
+
σ2
U2
V2
Rank 1
Rank 1

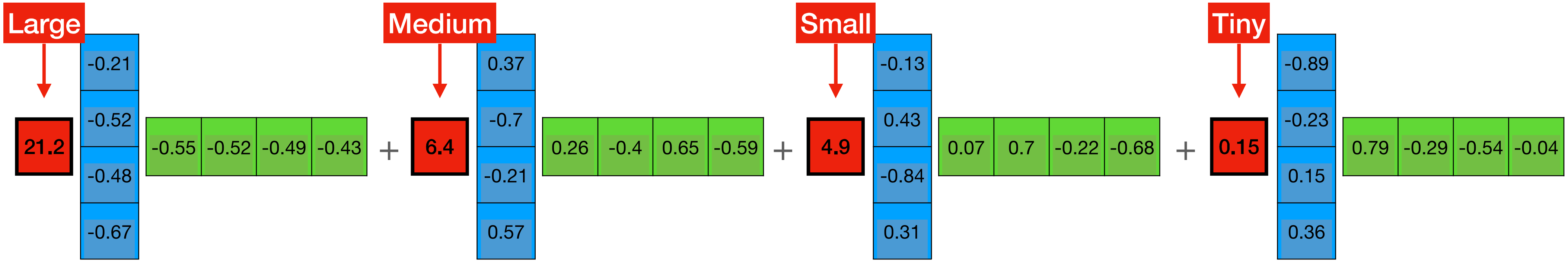
3	1	4	1
5	9	2	6
5	3	5	8
9	7	9	3

=

-0.21	0.37	-0.13	-0.89
-0.52	-0.7	0.43	-0.23
-0.48	-0.21	-0.84	0.15
-0.67	0.57	0.31	0.36

21.2			
	6.4		
		4.9	
			0.15

-0.55	-0.52	-0.49	-0.43
0.26	-0.4	0.65	-0.59
0.07	0.7	-0.22	-0.68
0.79	-0.29	-0.54	-0.04



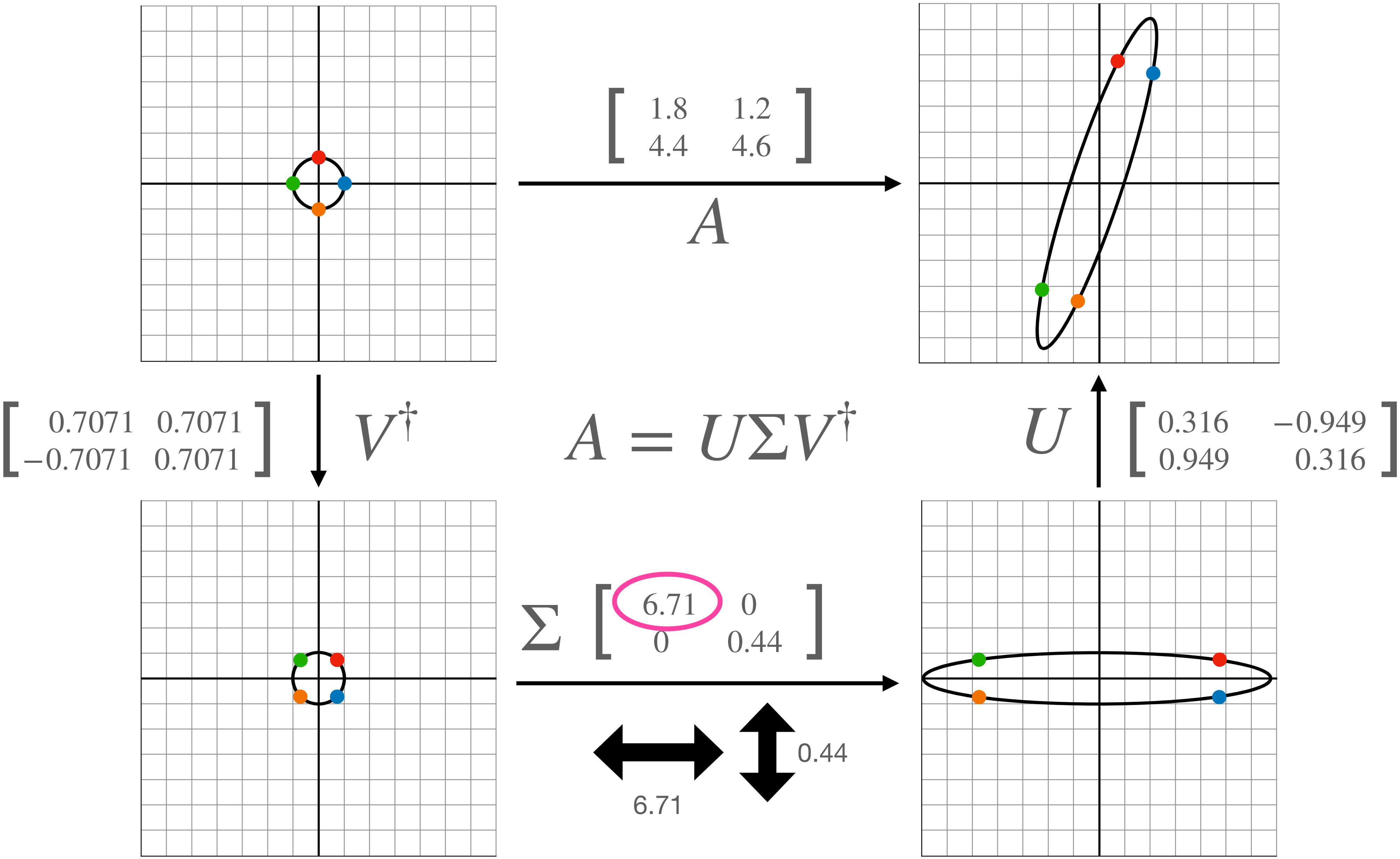
2.51	2.37	2.22	1.97
6.07	5,72	5.37	4.77
5.63	5,31	4.99	4.43
7.88	7.43	6.98	6.19

3.15	1.41	3.79	0.56
4.87	7.53	2.44	7.43
5.28	5.85	4.12	5.22
8.85	5.96	9.36	4.03

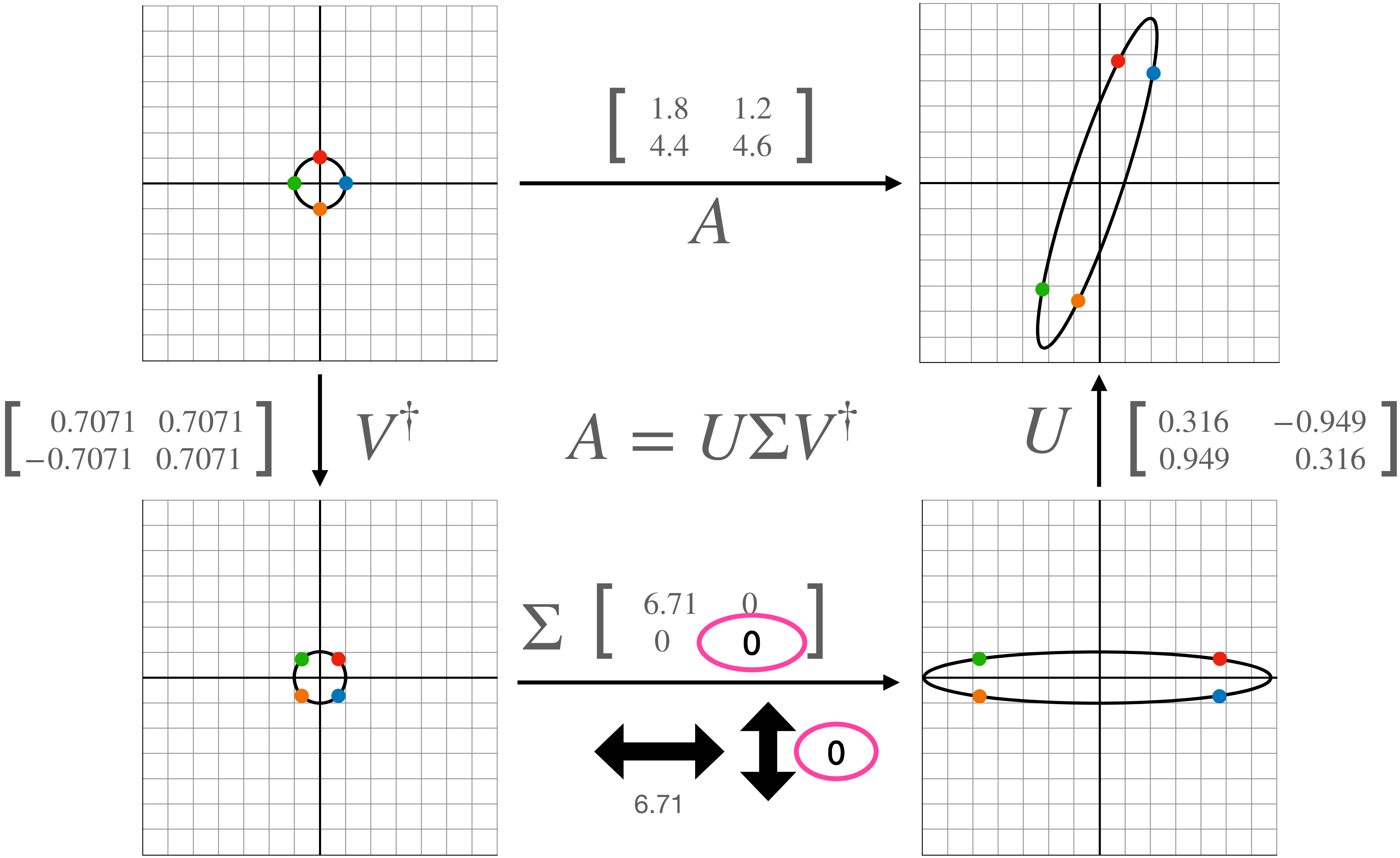
3.1	0.96	3.93	0.99
5.03	8.99	1.98	6
4.98	3.01	5.01	8
8.96	7.02	9.03	3

3	1	4	1
5	9	2	6
5	3	5	8
9	7	9	3

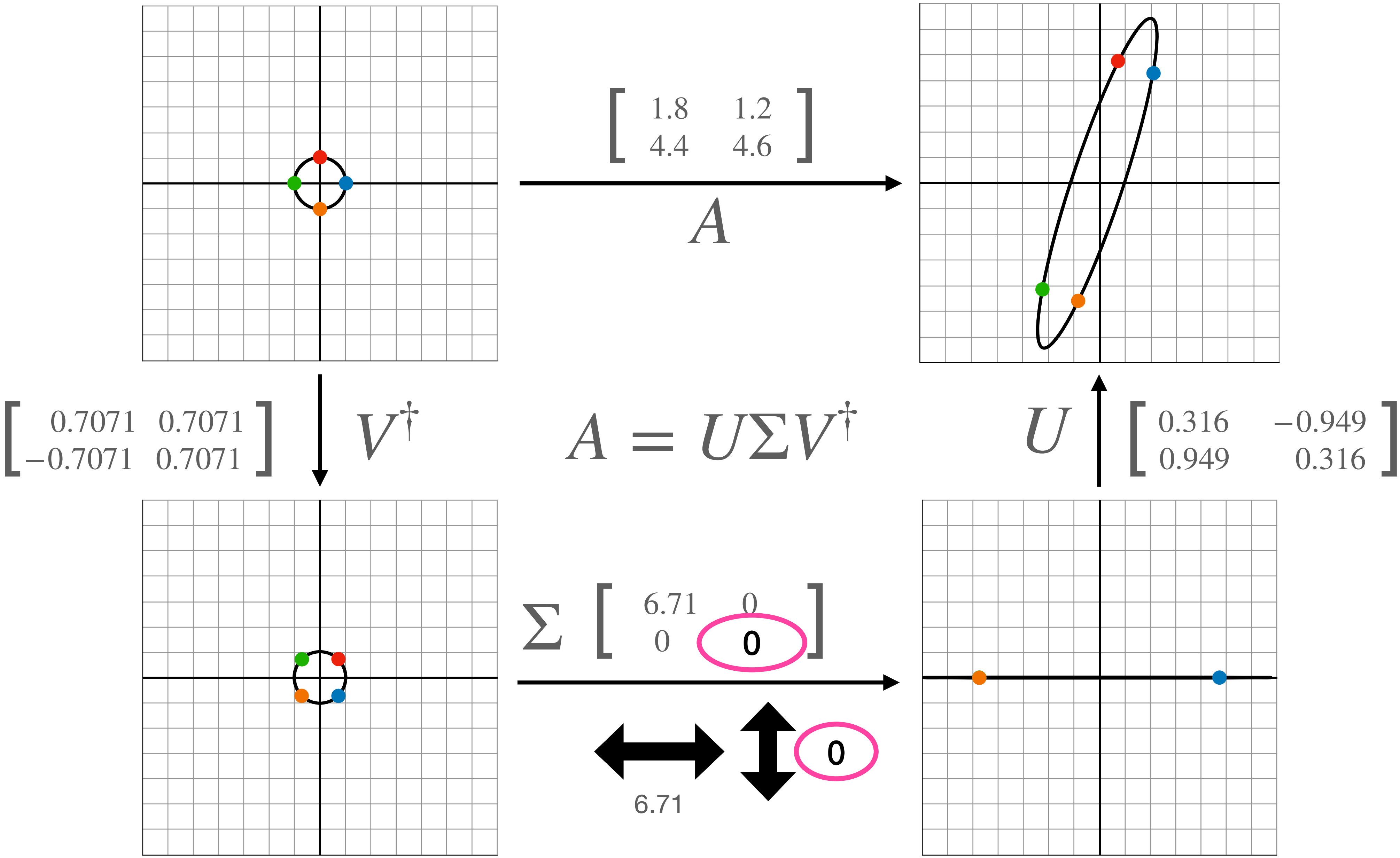
Dimensionality reduction



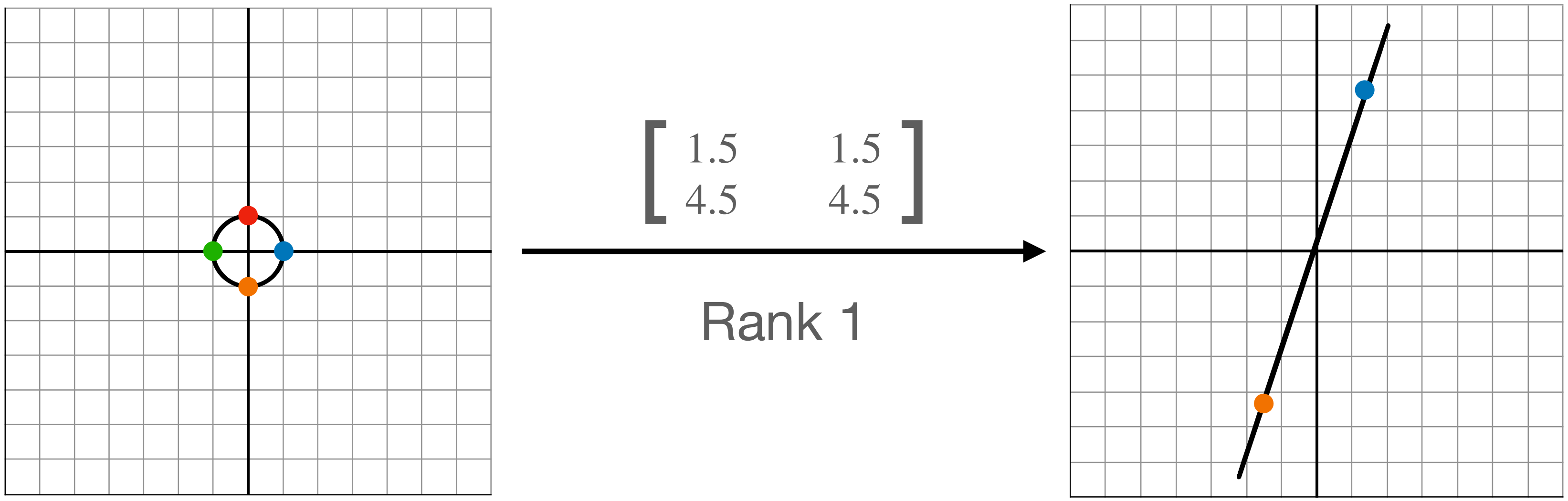
$\begin{bmatrix} 1.8 & 1.2 \\ 4.4 & 4.6 \end{bmatrix}$
A
$\begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$
$V^\dagger$
$A = U\Sigma V^\dagger$
U
$\begin{bmatrix} 0.316 & -0.949 \\ 0.949 & 0.316 \end{bmatrix}$
Σ
$\begin{bmatrix} 6.71 & 0 \\ 0 & 0.44 \end{bmatrix}$
6.71
0.44



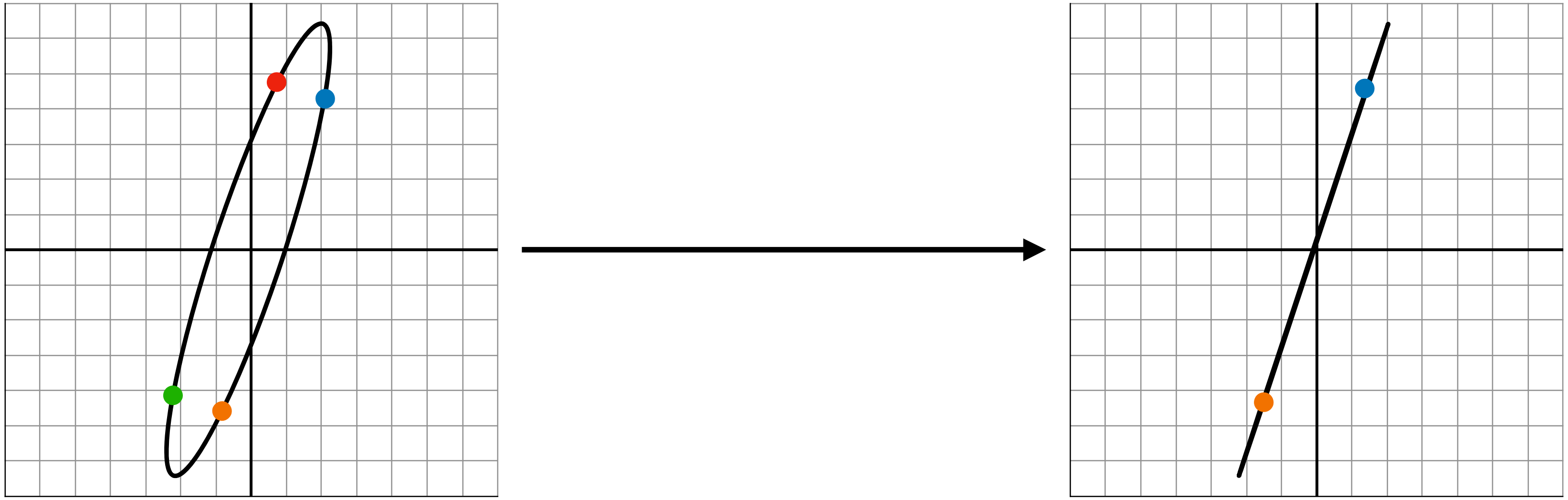
$\begin{bmatrix} 1.8 & 1.2 \\ 4.4 & 4.6 \end{bmatrix}$
A
$\begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$
$V^\dagger$
$A = U\Sigma V^\dagger$
U
$\begin{bmatrix} 0.316 & -0.949 \\ 0.949 & 0.316 \end{bmatrix}$
Σ
$\begin{bmatrix} 6.71 & 0 \\ 0 & 0 \end{bmatrix}$
0
6.71
0



1.8 1.2
4.4 4.6
A
0.7071 0.7071
−0.7071 0.7071
V†
A = UΣV†
U
0.316 −0.949
0.949 0.316
Σ
6.71 0
0 0
6.71
0



$\begin{bmatrix} 1.5 & 1.5 \\ 4.5 & 4.5 \end{bmatrix}$
Rank 1



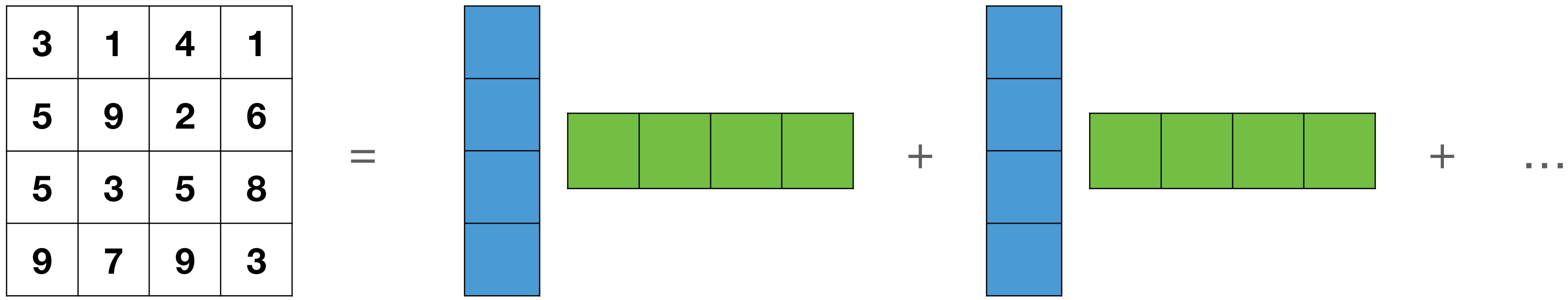
$$\begin{bmatrix} 1.8 & 1.2 \\ 4.4 & 4.6 \end{bmatrix}$$

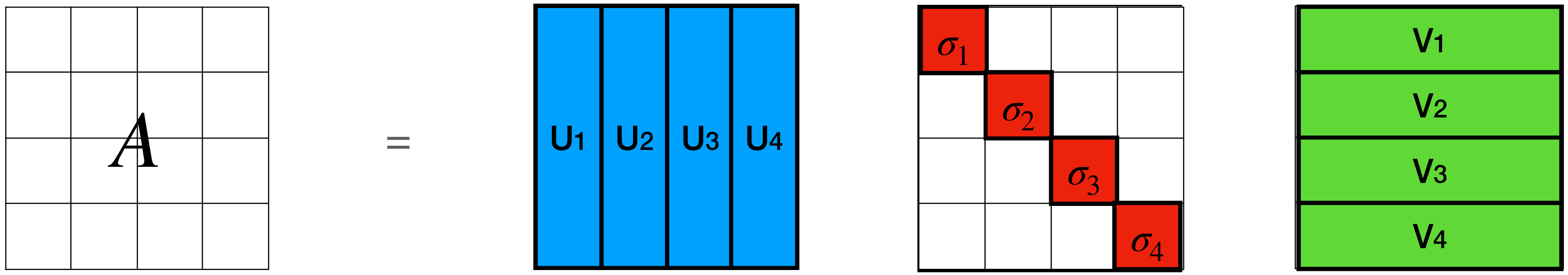
Rank 2

$$\begin{bmatrix} 1.5 & 1.5 \\ 4.5 & 4.5 \end{bmatrix}$$

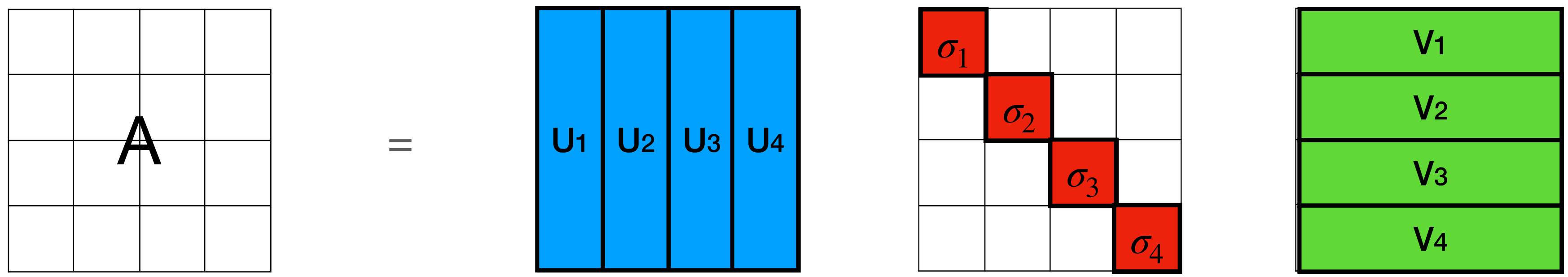
Rank 1

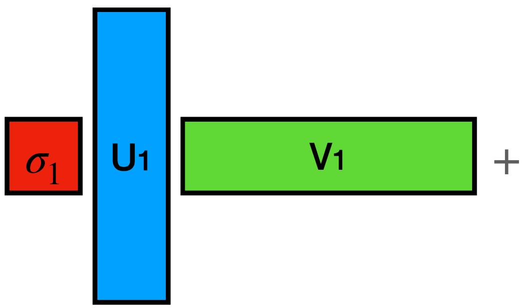
Approximation by rank one matrices

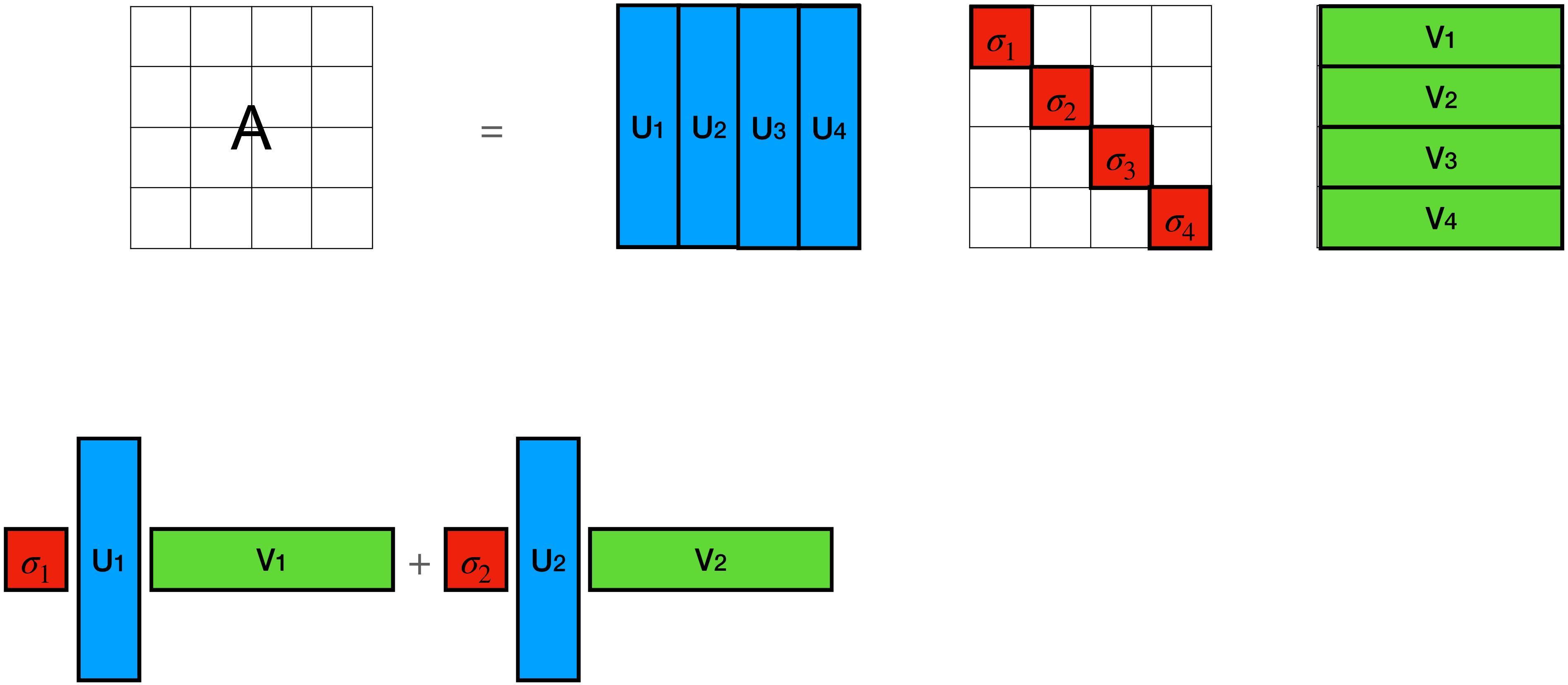


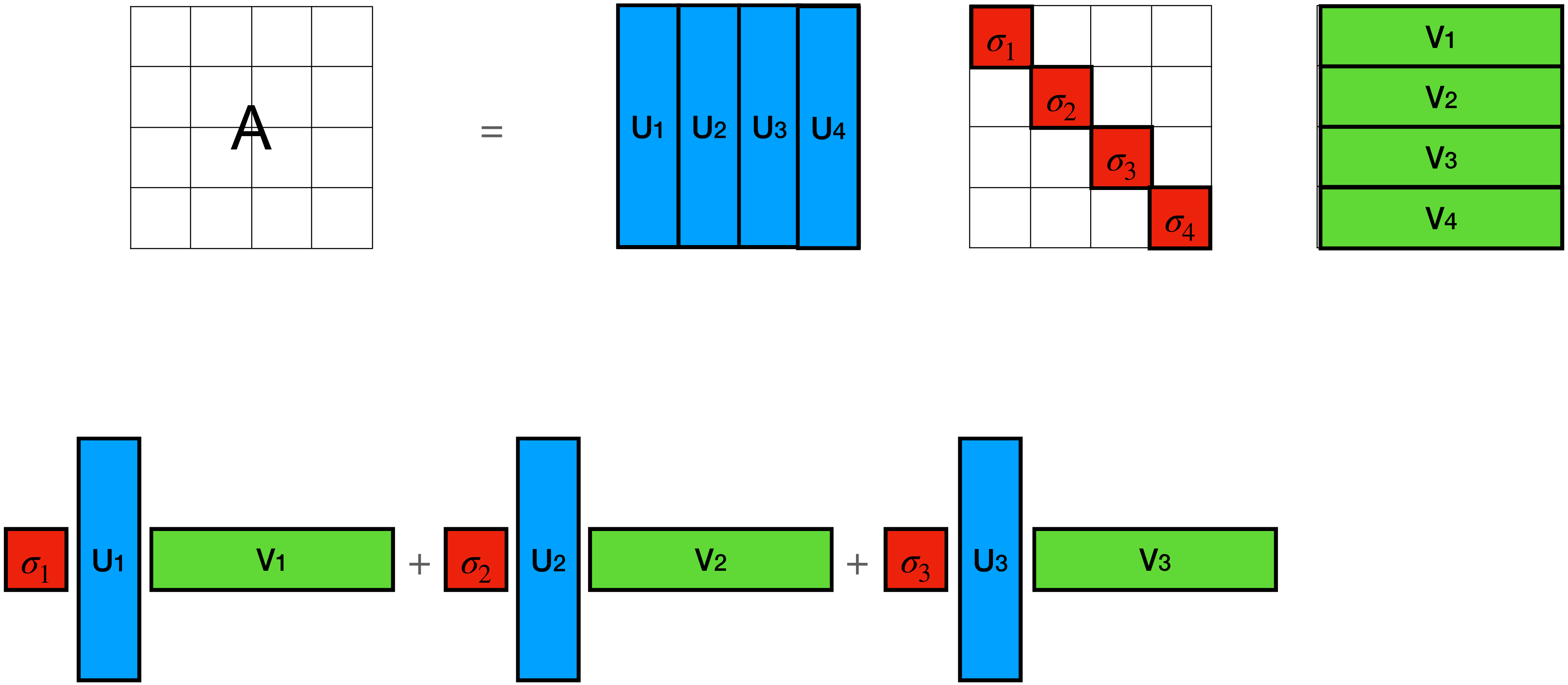


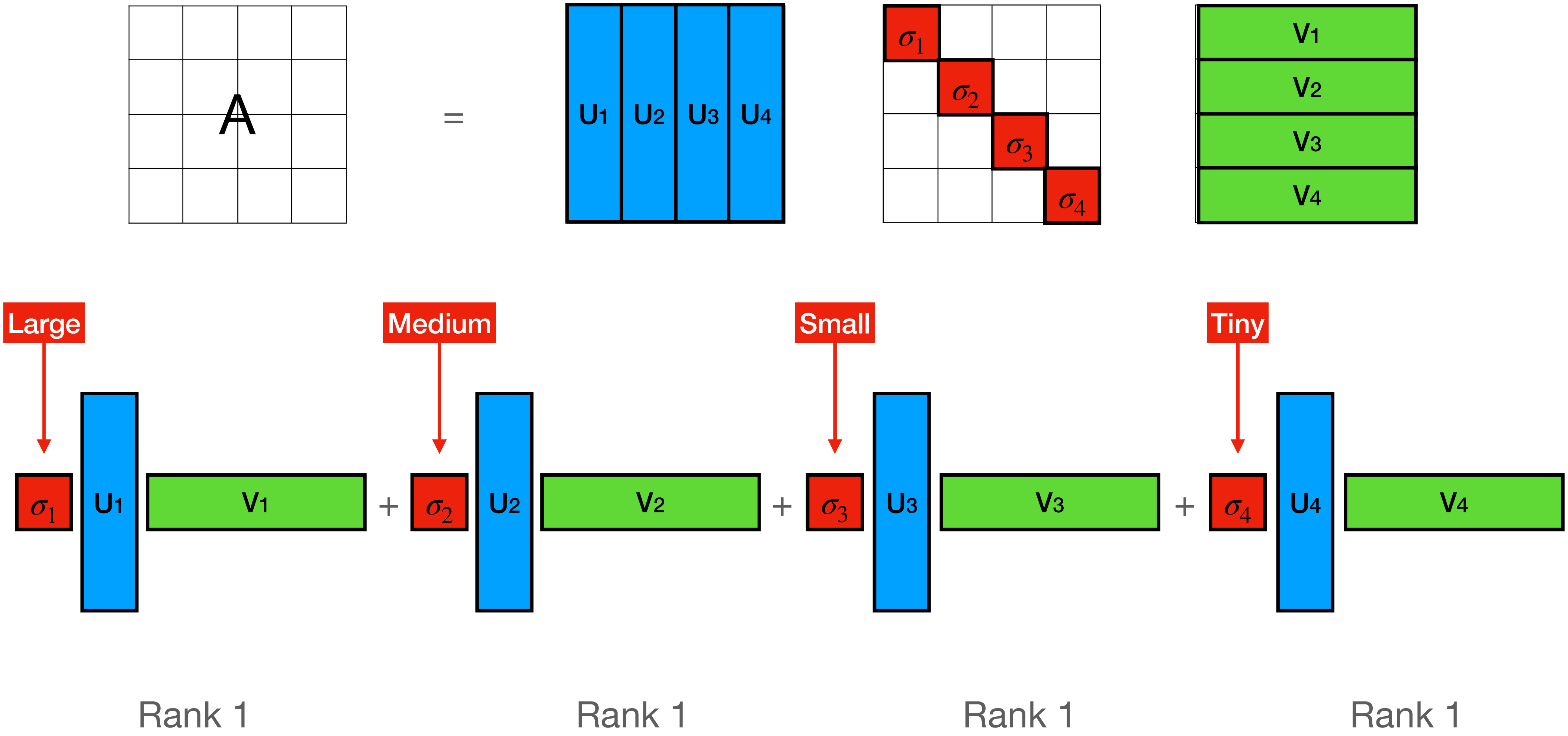
A
=
U1
U2
U3
U4
σ1
σ2
σ3
σ4
V1
V2
V3
V4


A
=
U1 U2 U3 U4
σ_1 σ_2 σ_3 σ_4
V1
V2
V3
V4

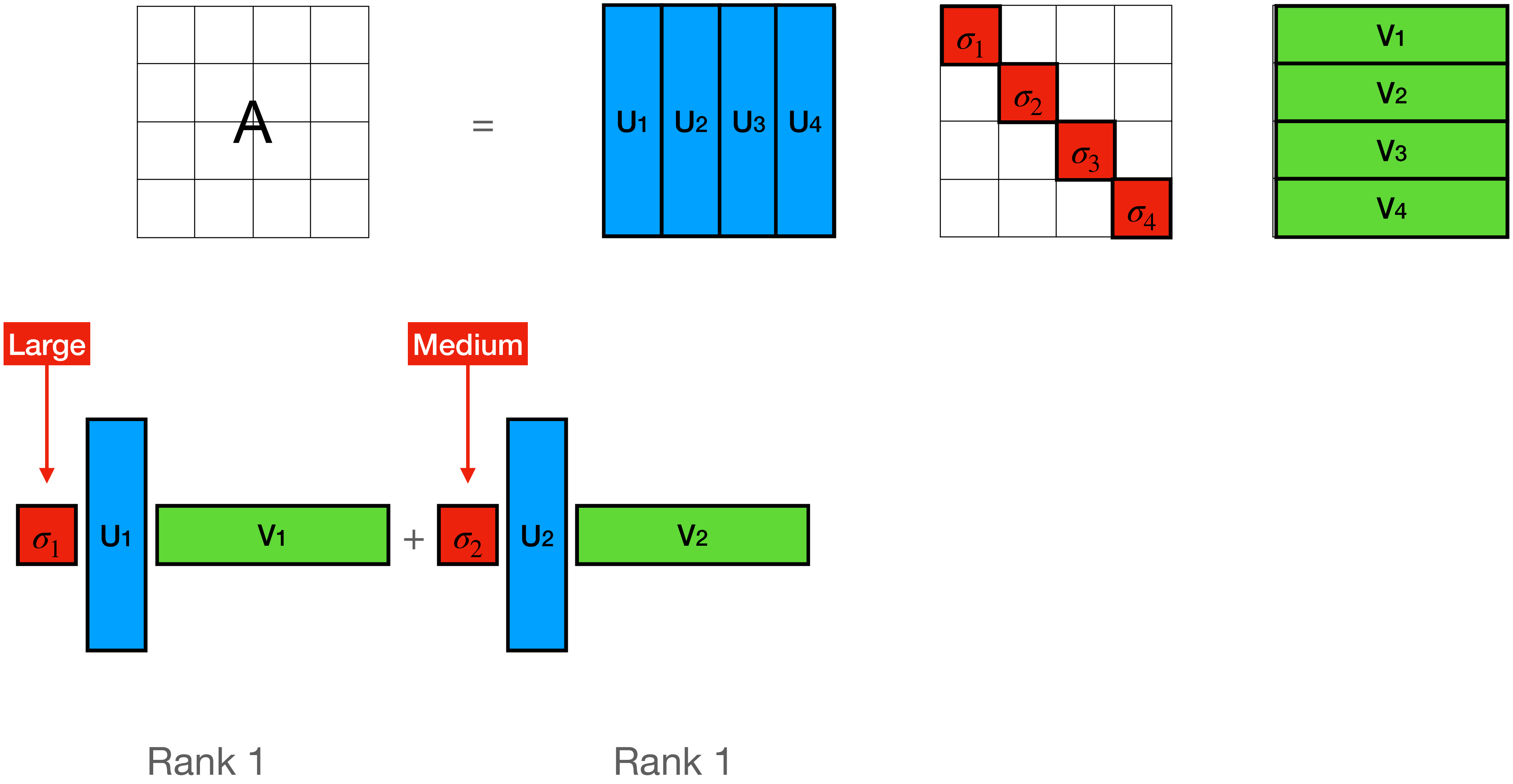

σ_1
U1
V1
+


A
=
U1
U2
U3
U4
σ1
σ2
σ3
σ4
V1
V2
V3
V4
σ1
U1
V1
+
σ2
U2
V2


A
=
U1
U2
U3
U4
σ_1
σ_2
σ_3
σ_4
V1
V2
V3
V4
σ_1
U1
V1
+
σ_2
U2
V2
+
σ_3
U3
V3



A
=
U1 U2 U3 U4
σ_1 σ_2 σ_3 σ_4
V1
V2
V3
V4
Large
Medium
Small
Tiny
σ_1 U1 V1 + σ_2 U2 V2 + σ_3 U3 V3 + σ_4 U4 V4
Rank 1
Rank 1
Rank 1
Rank 1



A
=
U1 U2 U3 U4
σ1 σ2 σ3 σ4
V1 V2 V3 V4
Large
Medium
σ1 U1 V1 + σ2 U2 V2
Rank 1
Rank 1

Rectangular matrices

No square matrix? No problem!

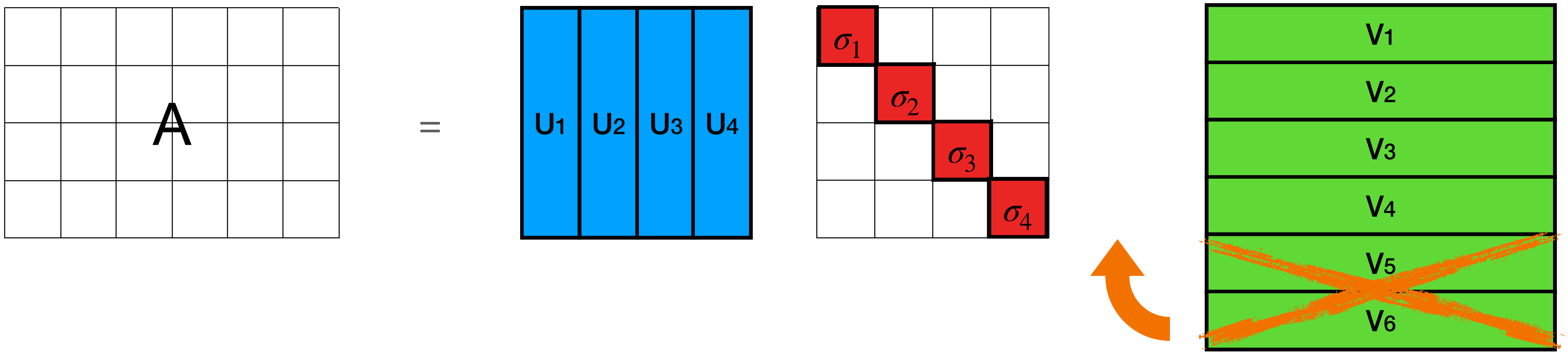


Image compression

GitHub

www.github.com/luisguiserrano/singular_value_decomposition

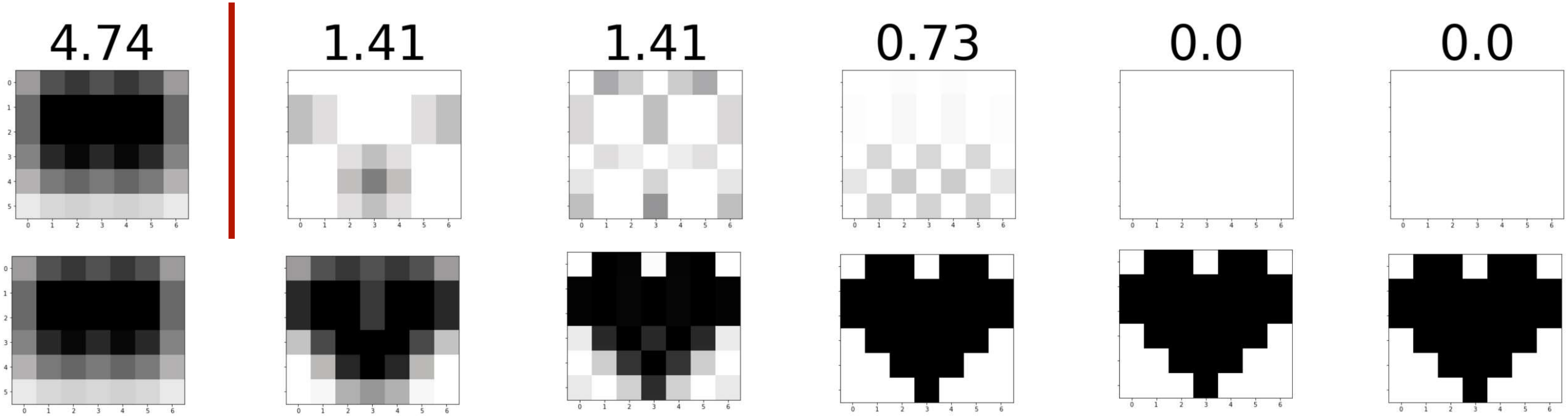
0	1	1	0	1	1	0
1	1	1	1	1	1	1
1	1	1	1	1	1	1
0	1	1	1	1	1	0
0	0	1	1	1	0	0
0	0	0	1	0	0	0

Rank 4

```
[[-0.36 -0.   -0.73 -0.05  0.56  0.13]
 [-0.54  0.35  0.27 -0.08 -0.16  0.69]
 [-0.54  0.35  0.27 -0.08  0.16 -0.69]
 [-0.45 -0.35 -0.27  0.52 -0.56 -0.13]
 [-0.28 -0.71  0.18 -0.62 -0.   -0.  ]
 [-0.08 -0.35  0.46  0.57  0.56  0.13]]
```

```
[[4.74 0.   0.   0.   0.   0.   0.  ]
 [0.   1.41 0.   0.   0.   0.   0.  ]
 [0.   0.   1.41 0.   0.   0.   0.  ]
 [0.   0.   0.   0.73 0.   0.   0.  ]
 [0.   0.   0.   0.   0.   0.   0.  ]
 [0.   0.   0.   0.   0.   0.   0.  ]]
```

```
[[-0.23 -0.4  -0.46 -0.4  -0.46 -0.4  -0.23]
 [ 0.5   0.25 -0.25 -0.5  -0.25  0.25  0.5 ]
 [ 0.39 -0.32 -0.19  0.65 -0.19 -0.32  0.39]
 [-0.22  0.42 -0.44  0.42 -0.44  0.42 -0.22]
 [ 0.56 -0.43  0.03  0.   -0.03  0.43 -0.56]
 [-0.42 -0.55 -0.16  0.    0.16  0.55  0.42]
 [-0.12 -0.11  0.69 -0.   -0.69  0.11  0.12]]
```



Thank you!