Logistic Regression

Introduction to Binary Outcomes

Continuous vs. Categorical Variables

- General linear regression model:
  $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$
- Independent variables (x's):
  - Continuous: age, income, height → use numerical value.
  - Categorical: gender, city, ethnicity → use dummies.
- Dependent variable (y):
  - Continuous: consumption, time spent → use numerical value.
  - Categorical: yes/no → use dummies.

Examples of Binary Outcomes

- Should a bank give a person a loan or not?
- Is an individual transaction fraudulent or not?
- What determines admittance into a school?
- Which people are more likely to vote against a new law?
- Which customers are more likely to buy a new product?

Representing the Binary Outcomes

- There are two outcomes: Yes and No
- We will create a dummy variable to indicate if an observation is a Yes or a No:
  - $y = 1$ if Yes
  - $y = 0$ if No
- If we code the variable the other way around, our coefficients will have the same magnitudes but opposite signs.

A linear model?

- Aside from being binary, there's really nothing special about our dependent variable (y).
- Its value is higher (from a 0 to a 1) if a customer subscribes, so whatever makes it higher increases the likelihood of subscription.
- We can then run:

$$subscribe = \beta_0 + \beta_1\, age + \varepsilon$$

Result of Linear Model

gretl: model 1

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```
Model 1: OLS, using observations 1-1000
Dependent variable: subscribe

             coefficient   std. error   t-ratio    p-value
  ---------------------------------------------------------
  const      -1.70073      0.0638035    -26.66    1.20e-118 ***
  age         0.0645433    0.00178736    36.11    2.52e-183 ***

Mean dependent var   0.573000   S.D. dependent var   0.494890
Sum squared resid    106.0736   S.E. of regression   0.326016
R-squared            0.566464   Adjusted R-squared   0.566030
F(1, 998)            1304.002   P-value(F)           2.5e-183
Log-likelihood      -297.1275   Akaike criterion     598.2550
Schwarz criterion    608.0705   Hannan-Quinn         601.9855
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$$subscribe = -1.700 + 0.064\;\, age$$

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- The result can be read as:
  $$\mathrm{P}(subscribe = 1) = p = -1.700 + 0.064\ age$$
- Every additional year of age increases the probability of subscription by 6.4%.

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- What about people with 25 and 45 years of age?
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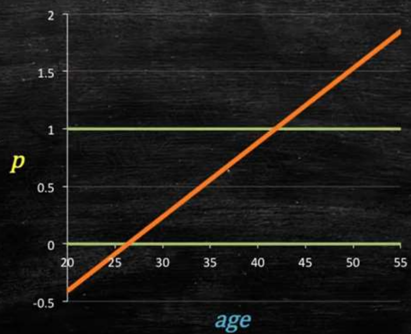
  $p = -1.700 + 0.064 \times 35 = 0.54$
- What about people with 25 and 45 years of age?
  $p = -1.700 + 0.064 \times 25 = -0.09$
  $p = -1.700 + 0.064 \times 45 = 1.20$

Linear Model Plot



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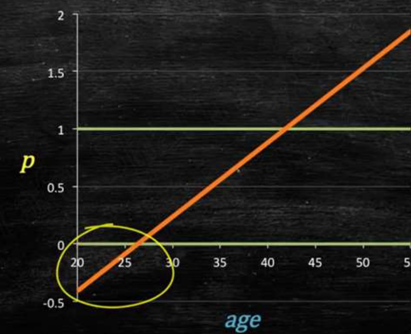

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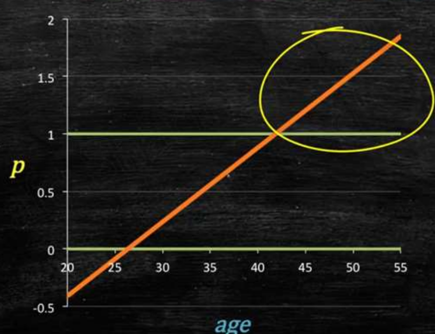

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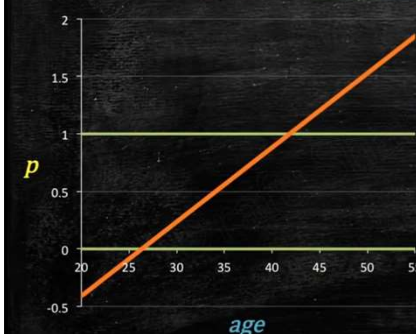

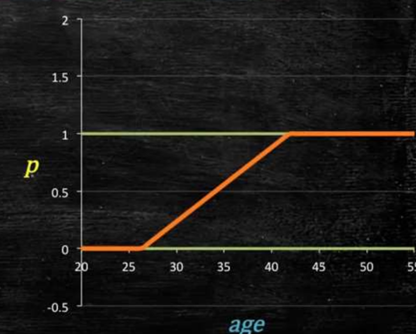

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- What must $f(\bullet)$ satisfy to always produce reasonable forecasts?
- $f(\bullet)$ must satisfy two things:
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f(x)=x^2

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$$p = \frac{\exp(\beta_0 + \beta_1\, age)}{\exp(\beta_0 + \beta_1\, age) + 1} = \frac{e^{\beta_0 + \beta_1\, age}}{e^{\beta_0 + \beta_1\, age} + 1}$$

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- Even though the probability of a customer subscribing (p) is not a linear function of age, the simple transformation is a linear function of age.
- The above equation is the one used in **logistic regressions**.

Result of Logistic Regression

gretl: model 2

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Model 2: Logit, using observations 1-1000
Dependent variable: subscribe
Standard errors based on Hessian

             coefficient   std. error     z        slope
  ------------------------------------------------------------
  const      -26.5240      1.82819      -14.51
  age          0.781053    0.0535623     14.58   0.154207

Mean dependent var   0.573000   S.D. dependent var   0.494890
McFadden R-squared   0.636613   Adjusted R-squared   0.633683
Log-likelihood      -247.9937   Akaike criterion     499.9873
Schwarz criterion    509.8028   Hannan-Quinn         503.7179

Number of cases 'correctly predicted' = 884 (88.4%)
f(beta'x) at mean of independent vars = 0.197
Likelihood ratio test: Chi-square(1) = 868.915 [0.0000]

           Predicted
              0     1
  Actual 0  350    77
         1   39   534
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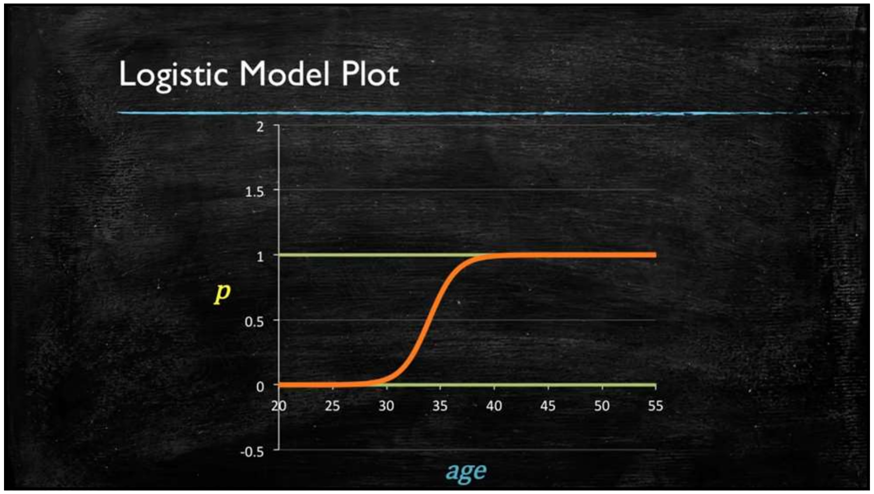
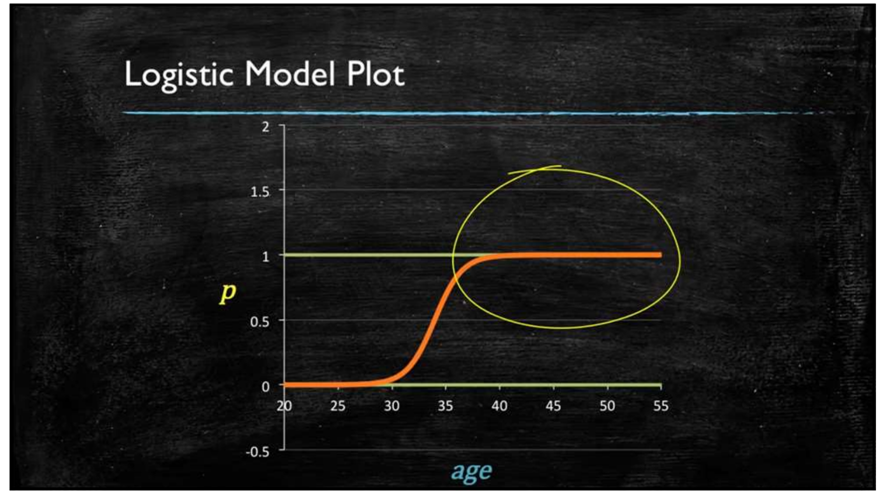
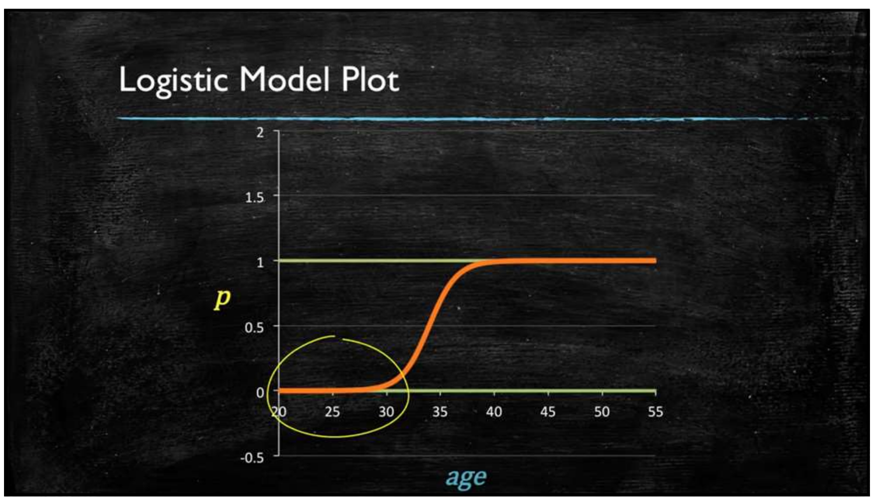
- Or written in terms of the probability p we have:

$$p = \frac{\exp(-26.52 + 0.78\ age)}{\exp(-26.52 + 0.78\ age) + 1} = \frac{e^{-26.52 + 0.78\ age}}{e^{-26.52 + 0.78\ age} + 1}$$







Logistic Regression

- Supervised learning method for classification.
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$$odds = \frac{P(event)}{1 - P(event)}$$

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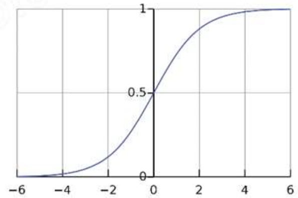

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What is the problem with this, if any? For purely predictive purposes, this actually is not a crazy idea—it tends to give decent predictions. But there are two drawbacks:

1. We cannot use any of the well-established routines for statistical inference with least squares (e.g., confidence intervals, etc.), because these are based on a model in which the outcome is continuously distributed. At an even more basic level, it is hard to precisely interpret $\hat{\beta}$
2. We cannot use this method when the number of classes exceeds 2.

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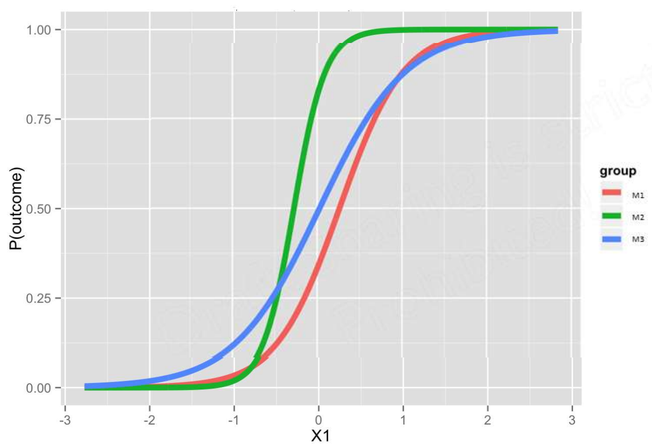
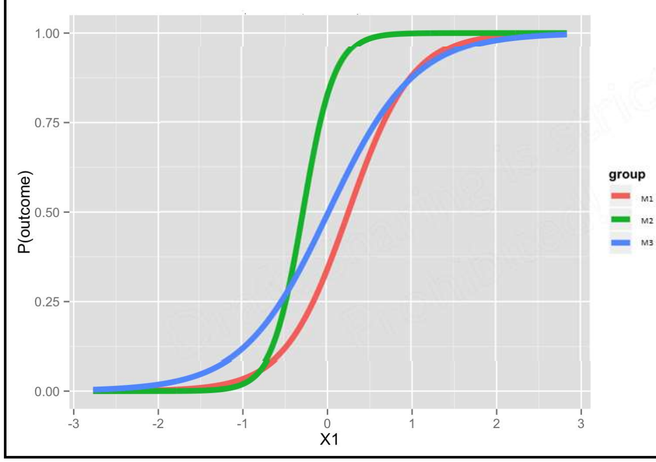
Intuition?





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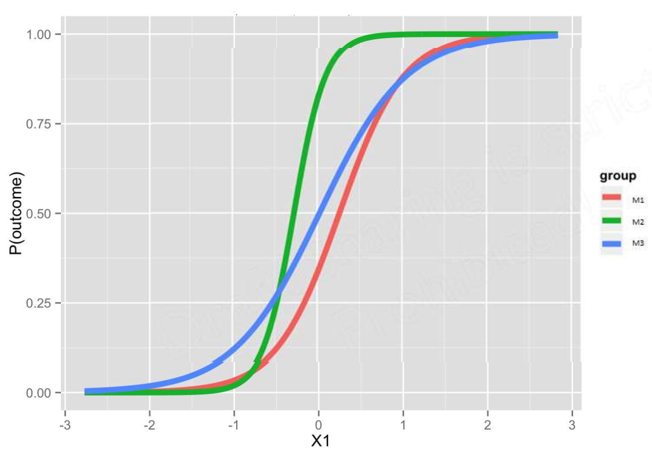

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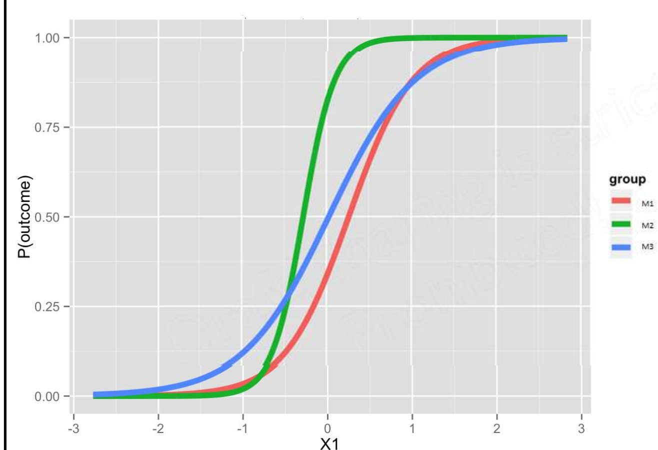

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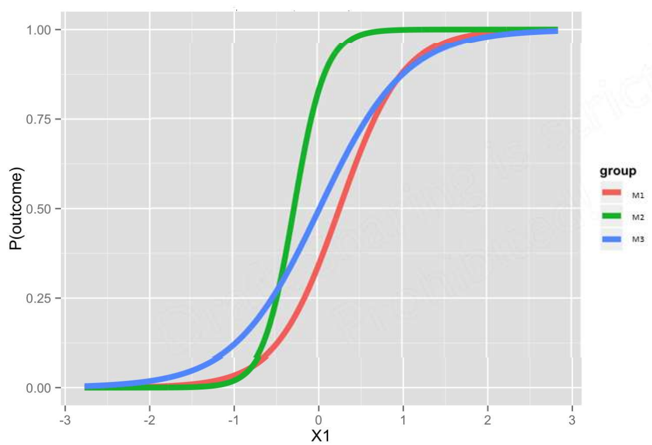

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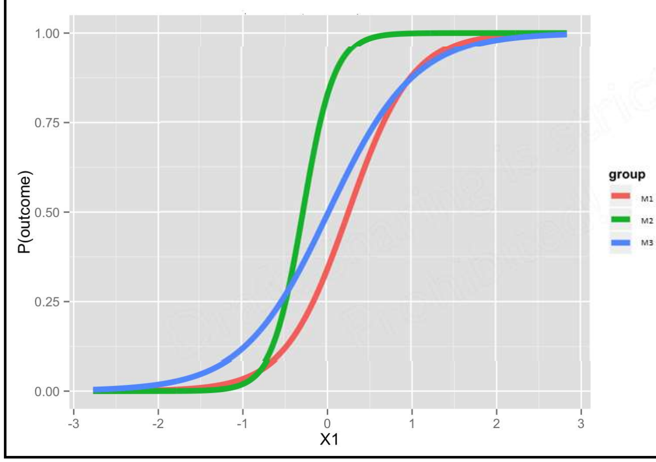

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$$\prod_{s \text{ in } yi=1} p(x_i)$$

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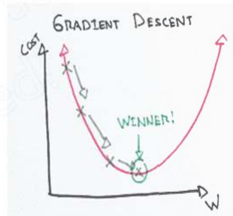
$$l(\beta) = \sum_{i=1}^{n} y_i \log(p(x_i)) + (1 - y_i) \log(1 - p(x_i))$$

The method most commonly used for logistic regression to estimate parameters is gradient descent.

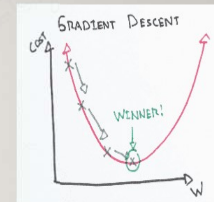


But, for gradient descent, we try to minimize the cost function. So, we can use a simple trick. maximizing $l(\beta)$ is equivalent to minimizing $-l(\beta)$. Therefore, our job turns into:

Minimize $$\sum_{i=1}^{n} -y_i \log(p(x_i)) - (1 - y_i) \log(1 - p(x_i))$$

Sum up: Convexity, Gradient Descent, and Log-Likelihood

- The error function is the function through which we optimize the parameters of a machine learning model
- This optimization takes place through algorithmic methods for learning
- The method most commonly used for logistic regression is gradient descent
- Gradient descent requires convex cost functions
- Mean Squared Error, commonly used for linear regression models, isn't convex for logistic regression. This is because the logistic function isn't always convex (another reason for not using OLS)
- The logarithm of the likelihood function is however always convex



We, therefore, elect to use the log-likelihood function as a cost function for logistic regression.

Logistic Regression

Interpretation of Coefficients and Forecasting

Leveraging the Similarities with Linear Models

gretl: model 2

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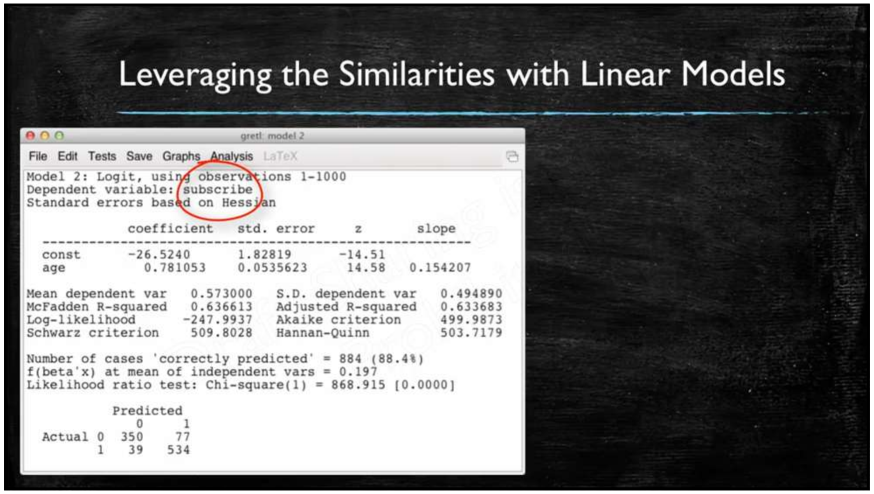
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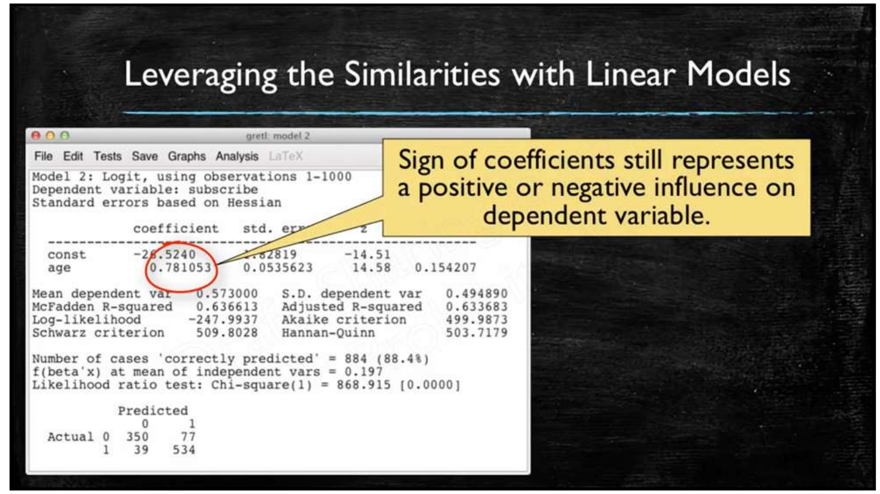
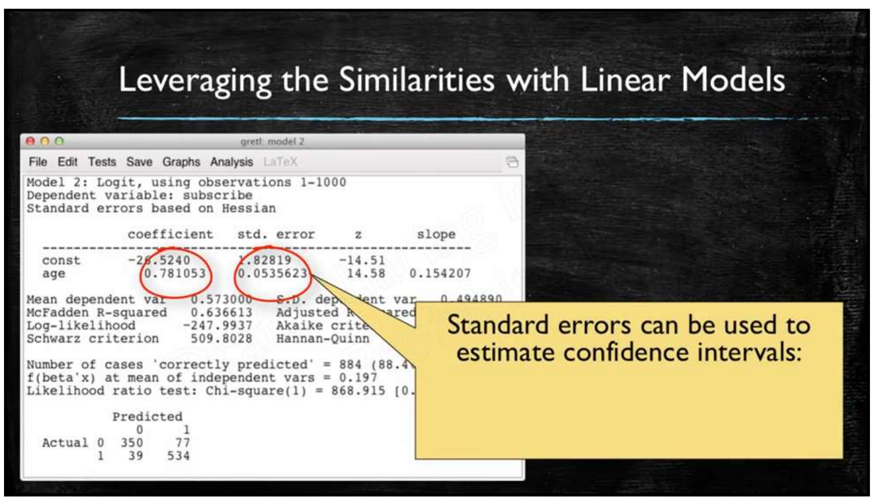
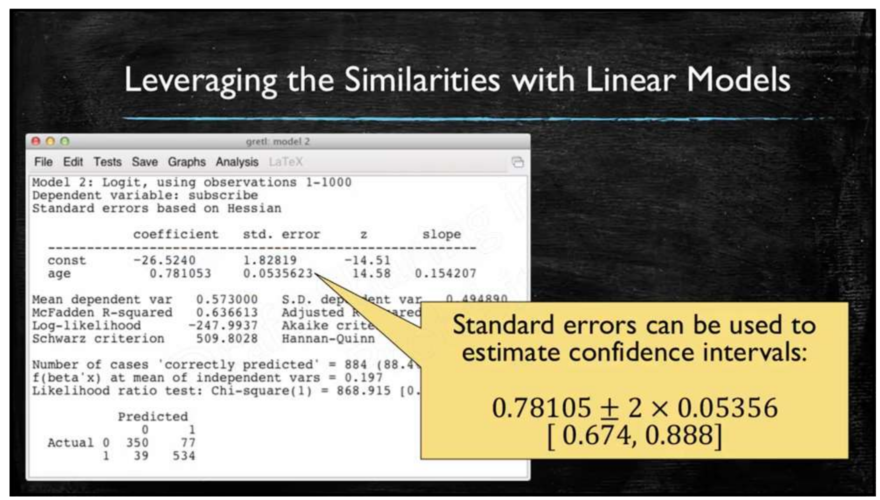
Standard errors based on Hessian
coefficient std. error z slope
const -26.5240 1.82819 -14.51
age 0.781053 0.0535623 14.58 0.154207
Mean dependent var 0.573000 S.D. dependent var 0.494890
McFadden R-squared 0.636613 Adjusted R-squared 0.633683
Log-likelihood -247.9937 Akaike criterion 499.9873
Schwarz criterion 509.8028 Hannan-Quinn 503.7179
Number of cases 'correctly predicted' = 884 (88.4%)
f(beta'x) at mean of independent vars = 0.197
Likelihood ratio test: Chi-square(1) = 868.915 [0.0000]
Predicted
0 1
Actual 0 350 77
1 39 534


Leveraging the Similarities with Linear Models
Sign of coefficients still represents a positive or negative influence on dependent variable.
Model 2: Logit, using observations 1-1000
Dependent variable: subscribe
Standard errors based on Hessian
const -26.5240 1.82819 -14.51
age 0.781053 0.0535623 14.58 0.154207


Leveraging the Similarities with Linear Models
Standard errors can be used to estimate confidence intervals:
Model 2: Logit, using observations 1-1000
Dependent variable: subscribe
const -26.5240 1.82819 -14.51
age 0.781053 0.0535623 14.58 0.154207


Leveraging the Similarities with Linear Models
Standard errors can be used to estimate confidence intervals:
0.78105 ± 2 × 0.05356
[0.674, 0.888]
Model 2: Logit, using observations 1-1000
Dependent variable: subscribe
const -26.5240 1.82819 -14.51
age 0.781053 0.0535623 14.58 0.154207

What changed?

What changed?

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- What is the meaning of 0.78 in our estimated model?

$$\ln\left(\frac{p}{1-p}\right) = -26.524 + 0.781\ age$$

- For every unit increase of age, $\ln\left(\frac{p}{1-p}\right)$ increases 0.78 units.

- But what is $\ln\left(\frac{p}{1-p}\right)$? Let us call it y^*

From y^* to p

- If we have: $y^* = \ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1\ age$

From y^* to p

- If we have: $y^* = \ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1\ age$

- Then: $p = \dfrac{\exp(\beta_0 + \beta_1\ age)}{\exp(\beta_0 + \beta_1\ age) + 1} = \dfrac{e^{\beta_0 + \beta_1\ age}}{e^{\beta_0 + \beta_1\ age} + 1}$

From y^* to p

- If we have: $y^* = \ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1\, age$
- Then: $p = \dfrac{\exp(\beta_0 + \beta_1\, age)}{\exp(\beta_0 + \beta_1\, age) + 1} = \dfrac{e^{\beta_0 + \beta_1\, age}}{e^{\beta_0 + \beta_1\, age} + 1}$
- Or simply put: $p = \dfrac{\exp(y^*)}{\exp(y^*) + 1} = \dfrac{e^{y^*}}{e^{y^*} + 1}$

From y^* to p in Excel

Variable	Coefficients	35y
Constant	-26.524000	1
Age	0.781053	35
y* = ln(p/(1-p))		0.813
p = exp(y*)/(exp(y*)+1)		0.693

From y^* to p in Excel

Variable	Coefficients	35y	36y
Constant	-26.524000	1	1
Age	0.781053	35	36
y* = ln(p/(1-p))		0.813	1.594
p = exp(y*)/(exp(y*)+1)		0.693	0.831

From y^* to p in Excel

Variable	Coefficients	35y	36y
Constant	-26.524000	1	1
Age	0.781053	35	36
y* = ln(p/(1-p))		0.813	1.594
p = exp(y*)/(exp(y*)+1)		0.693	0.831
Change			0.138

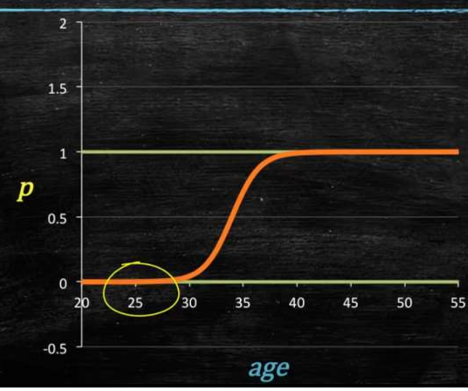
From y^* to p in Excel

Variable	Coefficients	35y	36y	25y	26y
Constant	-26.524000	1	1	1	1
Age	0.781053	35	36	25	26
y* = ln(p/(1-p))		0.813	1.594	-7	-6.22
p = exp(y*)/(exp(y*)+1)		0.693	0.831	9E-04	0.002
Change			0.138		0.001

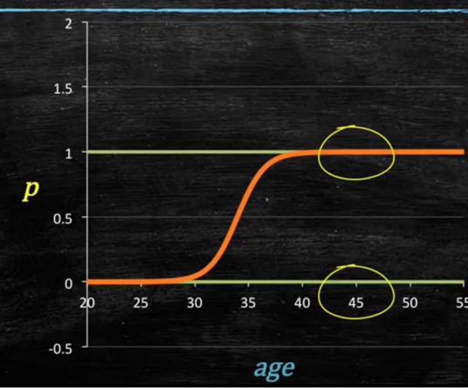
From y^* to p in Excel

Variable	Coefficients	35y	36y	25y	26y	45y	46y
Constant	-26.524000	1	1	1	1	1	1
Age	0.781053	35	36	25	26	45	46
y* = ln(p/(1-p))		0.813	1.594	-7	-6.22	8.623	9.404
p = exp(y*)/(exp(y*)+1)		0.693	0.831	9E-04	0.002	1	1
Change			0.138		0.001		1E-04

Logistic Model Plot



Logistic Model Plot



Multiple Logistic Regression

- Moving from a single regression with a single independent variable (age) to a multiple regression model with more than one (age and gender) is very simple.

Multiple Logistic Regression

- Moving from a single regression with a single independent variable (age) to a multiple regression model with more than one (age and gender) is very simple.

- Let's add a dummy variable for gender and run

$$y^* = \beta_0 + \beta_1\, age + \beta_2\, woman$$

Result of Multiple Logistic Regression

gretl: model 3

File Edit Tests Save Graphs Analysis LaTeX

```
Model 3: Logit, using observations 1-1000
Dependent variable: subscribe
Standard errors based on Hessian

             coefficient   std. error      z        slope
  ---------------------------------------------------------
  const      -26.4653      1.84246      -14.36
  age          0.787213    0.0542546     14.51     0.155652
  woman       -0.557795    0.231171     -2.413    -0.110022

Mean dependent var   0.573000   S.D. dependent var   0.494890
McFadden R-squared   0.640941   Adjusted R-squared   0.636545
Log-likelihood      -245.0401   Akaike criterion     496.0803
Schwarz criterion    510.8035   Hannan-Quinn         501.6762

Number of cases 'correctly predicted' = 886 (88.6%)
f(beta'x) at mean of independent vars = 0.198
Likelihood ratio test: Chi-square(2) = 874.822 [0.0000]

           Predicted
              0     1
  Actual 0  367    60
         1   54   519
```

- Estimated model is:

$$y^* = -26.47 + 0.79 \times age - 0.56 \times woman$$

Result of Multiple Logistic Regression

gretl: model 3

File Edit Tests Save Graphs Analysis LaTeX

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File  Edit  Tests  Save  Graphs  Analysis  LaTeX
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```

- Estimated model is:
  $y^* = -26.47 + 0.79 \times age - 0.56 \times woman$
- We can interpret signs and compute confidence intervals.
- We want p.

From y^* to p in Excel

Variable	Coefficients
Constant	-26.465300
Age	0.787213
Woman	-0.557795

y* = ln(p/(1-p))

p = exp(y*)/(exp(y*)+1)

Change

From y^* to p in Excel

Variable	Coefficients	35y W	35y M
Constant	-26.465300	1	1
Age	0.787213	35	35
Woman	-0.557795	1	0
y* = ln(p/(1-p))		0.529	1.087
p = exp(y*)/(exp(y*)+1)		0.629	0.748
Change			0.119

From y^* to p in Excel

Variable	Coefficients	35y W	35y M	36y W	36 M
Constant	-26.465300	1	1	1	1
Age	0.787213	35	35	36	36
Woman	-0.557795	1	0	1	0
y* = ln(p/(1-p))		0.529	1.087	1.317	1.874
p = exp(y*)/(exp(y*)+1)		0.629	0.748	0.789	0.867
Change			0.119		

From y^* to p in Excel

Variable	Coefficients	35y W	35y M	36y W	36 M
Constant	-26.465300	1	1	1	1
Age	0.787213	35	35	36	36
Woman	-0.557795	1	0	1	0
y* = ln(p/(1-p))		0.529	1.087	1.317	1.874
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p = exp(y*)/(exp(y*)+1)		0.629	0.748	0.789	0.867
Change			0.119		0.078

Example : continuation of previously shown linear regression example

Categorical dependent variable

What if Y is categorical?

$$\underbrace{Ln\left(\frac{P_i}{1-P_i}\right) = \beta_0 + \beta_1(Price_i) + \beta_2(Pink\ Slip_i) + \varepsilon_i}$$

BINOMIAL LOGISTIC REGRESSION

Categorical dependent variable

What if Y is categorical?

REMEMBER!
"Chance" = "Probability"
≠ "Odds"

If the probability of rain tomorrow is 20%,
what are the odds of rain tomorrow?

Categorical dependent variable

What if Y is categorical?

If the probability of rain tomorrow is 20%,
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Categorical dependent variable

What if Y is categorical?

If the probability of rain tomorrow is 20%,
what are the odds of rain tomorrow?

$$Odds_i = \frac{P}{1-P} = \frac{0.2}{0.8} = 0.25$$

Categorical dependent variable

What if Y is categorical?

If the probability of rain tomorrow is 20%,
what are the odds of rain tomorrow?

$$Odds_i = \frac{P}{1-P} = \frac{0.2}{0.8} = 0.25 \quad OR \quad 1:4$$

Categorical dependent variable

What if Y is categorical?

$$Ln\left(\frac{P_i}{1-P_i}\right) = \beta_0 + \beta_1(Price_i) + \beta_2(Pink\ Slip_i) + \varepsilon_i$$

DV: Log Odds (Sold)	Coef	SE	Z	P-value	OR	95% LCL	95% UCL
Intercept	0.396	0.480	0.82	0.4097			
Price ($000s)	-0.173	0.057	-3.04	0.0023	0.84	0.75	0.94
Pink slip	1.555	0.531	2.93	0.0034	4.73	1.67	13.41

Categorical dependent variable

What if Y is categorical?

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$$Ln\widehat{\left(\frac{P_i}{1-P_i}\right)} = 0.396 - 0.173(Price_i) + 1.555(Pink\ Slip_i)$$

Categorical dependent variable

What if Y is categorical?

$$Ln\widehat{\left(\frac{P_i}{1-P_i}\right)} = 0.396 - 0.173(Price_i) + 1.555(Pink\ Slip_i)$$

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Pink slip	1.555	0.531	2.93	0.0034	4.73	1.67	13.41

For a $1000 increase in price, the log-odds of selling the car decreases by 0.173, on average, holding all else constant

Categorical dependent variable

What if Y is categorical?

$$Ln\widehat{\left(\frac{P_i}{1-P_i}\right)} = 0.396 - 0.173(Price_i) + 1.555(Pink\ Slip_i)$$

DV: Log Odds (Sold)	Coef	SE	Z	P-value	OR	95% LCL	95% UCL
Intercept	0.396	0.480	0.82	0.4097			
Price ($000s)	-0.173	0.057	-3.04	0.0023	0.84	0.75	0.94
Pink slip	1.555	0.531	2.93	0.0034	4.73	1.67	13.41

For a $1000 increase in price, the odds of selling the car decreases by 16%, on average, holding all else constant.

Categorical dependent variable

What if Y is categorical?

$$Ln\left(\widehat{\frac{P_i}{1-P_i}}\right) = 0.396 - 0.173(Price_i) + 1.555(Pink\ Slip_i)$$

DV: Log Odds (Sold)	Coef	SE	Z	P-value	OR	95% LCL	95% UCL
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Pink slip	1.555	0.531	2.93	0.0034	4.73	1.67	13.41

If the car has a pink slip, the log odds of sale increases by 1.555, on average, holding price constant.

Categorical dependent variable

What if Y is categorical?

$$Ln\left(\widehat{\frac{P_i}{1-P_i}}\right) = 0.396 - 0.173(Price_i) + 1.555(Pink\ Slip_i)$$

DV: Log Odds (Sold)	Coef	SE	Z	P-value	OR	95% LCL	95% UCL
Intercept	0.396	0.480	0.82	0.4097			
Price ($000s)	-0.173	0.057	-3.04	0.0023	0.84	0.75	0.94
Pink slip	1.555	0.531	2.93	0.0034	4.73	1.67	13.41

Cars with a pink slip have 4.73 times the odds of being sold compared to cars without a pink slip, on average, holding all else constant.

Categorical dependent variable

What if Y is categorical?

DV: Log Odds (Sold)	Coef	SE	Z	P-value	OR	95% LCL	95% UCL
Intercept	0.396	0.480	0.82	0.4097			
Price ($000s)	-0.173	0.057	-3.04	0.0023	0.84	0.75	0.94
Pink slip	1.555	0.531	2.93	0.0034	4.73	1.67	13.41

(a) Find the probability that a $4,500 car with a pink slip will sell.

(b) Find the probability that a $4,500 car <u>without</u> a pink slip will sell.

(c) calculate the odds in parts (a) and (b) and find their ratio.

Categorical dependent variable

What if Y is categorical?

DV: Log Odds (Sold)	Coef	SE	Z	P-value	OR	95% LCL	95% UCL
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Pink slip	1.555	0.531	2.93	0.0034	4.73	1.67	13.41

(a) Find the probability that a $4,500 car with a pink slip will sell

$$Ln\left(\frac{P}{1-P}\right) = 0.396 - 0.173(Price_i) + 1.555(Pink\ Slip_i)$$

Categorical dependent variable

What if Y is categorical?

DV: Log Odds (Sold)	Coef	SE	Z	P-value	OR	95% LCL	95% UCL
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Pink slip	1.555	0.531	2.93	0.0034	4.73	1.67	13.41

(a) Find the probability that a $4,500 car with a pink slip will sell

$$Ln\left(\frac{P}{1-P}\right) = 0.396 - 0.173(4.5) + 1.555(1)$$

Categorical dependent variable

What if Y is categorical?

DV: Log Odds (Sold)	Coef	SE	Z	P-value	OR	95% LCL	95% UCL
Intercept	0.396	0.480	0.82	0.4097			
Price ($000s)	-0.173	0.057	-3.04	0.0023	0.84	0.75	0.94
Pink slip	1.555	0.531	2.93	0.0034	4.73	1.67	13.41

(a) Find the probability that a $4,500 car with a pink slip will sell

$$Ln\left(\frac{P}{1-P}\right) = 0.396 - 0.173(4.5) + 1.555(1)$$

$$Ln\left(\frac{P}{1-P}\right) = 1.1725$$

Categorical dependent variable

What if Y is categorical?

(a) Find the probability that a $4,500 car with a pink slip will sell.

$$Ln\left(\frac{P}{1-P}\right) = 1.1725$$

Categorical dependent variable

What if Y is categorical?

(a) Find the probability that a $4,500 car with a pink slip will sell.

$$Ln\left(\frac{P}{1-P}\right) = 1.1725$$

$$\frac{P}{1-P} = e^{1.1725} = 3.23$$

Categorical dependent variable

What if Y is categorical?

(a) Find the probability that a $4,500 car with a pink slip will sell.

$$Ln\left(\frac{P}{1-P}\right) = 1.1725$$

$$\frac{P}{1-P} = e^{1.1725} = 3.23$$

$$P = 3.23(1 - P)$$

$$P + 3.23P = 3.23$$

$$P = \frac{3.23}{4.23} = 0.764$$

Categorical dependent variable

What if Y is categorical?

(a) Find the probability that a $4,500 car with a pink slip will sell.

ANSWER:

A car with a sale price of $4,500 and a pink slip has a probability of sale of 76.4%

Categorical dependent variable

What if Y is categorical?

(a) Find the probability that a $4,500 car with a pink slip will sell.

ANSWER:

A car with a sale price of $4,500 and a pink slip has a probability of sale of 76.4%

(b) Find the probability that a $4,500 car without a pink slip will sell.

Categorical dependent variable

What if Y is categorical?

(a) Find the probability that a $4,500 car with a pink slip will sell.

ANSWER:

A car with a sale price of $4,500 and a pink slip has a probability of sale of 76.4%

(b) Find the probability that a $4,500 car without a pink slip will sell.

$$Ln\left(\frac{P}{1-P}\right) = 0.396 - 0.173(4.5) + 1.555(0)$$
$$= 40.6$$

Categorical dependent variable

What if Y is categorical?

(a) Find the probability that a $4,500 car with a pink slip will sell.

ANSWER:

A car with a sale price of $4,500 and a pink slip has a probability of sale of 76.4%

(b) Find the probability that a $4,500 car without a pink slip will sell.

A car with a sale price of $4,500 without a pink slip has probability of sale of 40.6%

Categorical dependent variable

What if Y is categorical?

(c) calculate the odds in parts (a) and (b) and find the odds ratio

Categorical dependent variable

What if Y is categorical?

(c) calculate the odds in parts (a) and (b) and find the odds ratio

	(a)	(b)
Car	$4500, pink slip	$4500, no pink slip

Categorical dependent variable

What if Y is categorical?

(c) calculate the odds in parts (a) and (b) and find the odds ratio

	(a)	(b)
Car	$4500, pink slip	$4500, no pink slip
Prob	76.4%	40.6%

Categorical dependent variable

What if Y is categorical?

(c) calculate the odds in parts (a) and (b) and find the odds ratio

	(a)	(b)
Car	$4500, pink slip	$4500, no pink slip
Prob	76.4%	40.6%
Odds	76.4%/23.6% = 3.23	40.6%/59.4% = 0.68

Categorical dependent variable

What if Y is categorical?

(c) calculate the odds in parts (a) and (b) and find the odds ratio

	(a)	(b)
Car	$4500, pink slip	$4500, no pink slip
Prob	76.4%	40.6%
Odds	76.4%/23.6% = 3.23	40.6%/59.4% = 0.68

Odds Ratio: 3.23/0.68 = 4.73

Categorical dependent variable

What if Y is categorical?

(c) calculate the odds in parts (a) and (b) and find the odds ratio

DV: Log Odds (Sold)	Coef	SE	Z	P-value	OR	95% LCL	95% UCL
Intercept	0.396	0.480	0.82	0.4097			
Price ($000s)	-0.173	0.057	-3.04	0.0023	0.84	0.75	0.94
Pink slip	1.555	0.531	2.93	0.0034	4.73	1.67	13.41

Odds Ratio: 3.23/0.68 = 4.73

THANK YOU!