

UNIT-IVSECTION-AEASY TYPE

1. Evaluate $\int_0^2 \int_0^x y dy dx$.

$$= \int_0^2 \left[\frac{y^2}{2} \right]_0^x dx$$

$$= \int_0^2 \left[\frac{x^2}{2} \right] dx$$

$$= \frac{1}{2} \int_0^2 x^2 dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2$$

$$= \frac{8/4}{6/3}$$

$$= \frac{4}{3}$$

2. Evaluate $\int_0^1 \int_0^1 \frac{dx dy}{\sqrt{(1-x^2)(1-y^2)}}$.

$$= \int_0^1 \int_0^1 \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1}(x) \right] dy$$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}} (\pi/2) dy$$

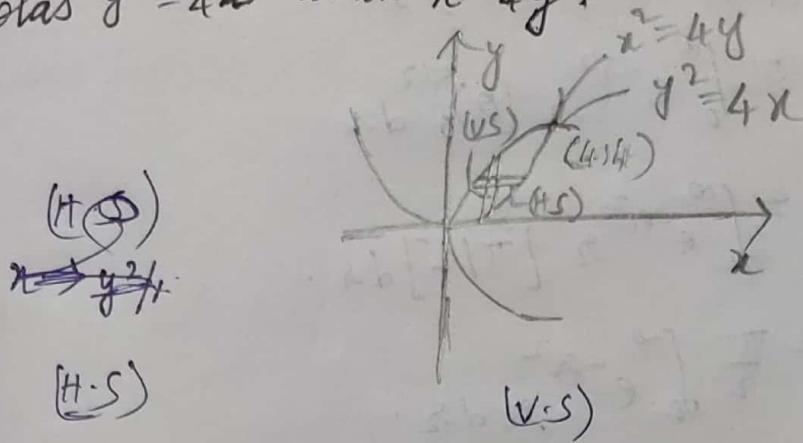
$$= \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-y^2}} dy$$

$$= \frac{\pi}{2} \left[\sin^{-1} y \right]_0^1$$

$$= \frac{\pi}{2} (\pi/2)$$

$$= \frac{\pi^2}{4}$$

3. Identify the limits of integration for $\iiint_R f(x,y) dxdy$
 where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.



$$x \rightarrow \frac{y^2}{4} \text{ to } 2\sqrt{y} \quad y \rightarrow \frac{x^2}{4} \text{ to } \sqrt{4x}$$

$$y \rightarrow 0 \text{ to } 4 \quad x \rightarrow 0 \text{ to } 4$$

4. Evaluate $\int_0^\infty \int_0^{\pi/2} e^{-r^2} r dr d\theta$

~~Let $r^2 = t$~~

~~$2r dr = dt$~~

~~$t \rightarrow 0 \text{ to } \pi^2/4$~~

~~= $\int_0^\infty \int_0^{\pi^2/4} e^{-t} \frac{dt}{2} d\theta$~~

~~= $\frac{1}{2} \int_0^\infty [-e^{-t}]_{0}^{\pi^2/4} d\theta$~~

~~= $\frac{1}{2} \int_0^\infty [e^{-\pi^2/4} + e^0] d\theta$~~

~~= $-\frac{1}{2} \int_0^\infty e^{-t} dt$~~

~~= $\frac{1}{2} \int_0^\infty [-e^{-\pi^2/4} + e^0] d\theta$~~

$$4. \text{ Evaluate } \int_0^\infty \int_0^{\pi/2} e^{-r^2} r dr d\theta.$$

$$= \int_0^\infty e^{-r^2} r \left(\int_0^{\pi/2} d\theta \right) dr.$$

$$= \int_0^\infty e^{-r^2} r [\theta]_0^{\pi/2} dr.$$

$$= \int_0^\infty e^{-r^2} r [\pi/2] dr.$$

$$= \frac{\pi}{2} \int_0^\infty e^{-r^2} r dr.$$

$$\text{Let } r^2 = t$$

$$2r dr = dt$$

$$t \rightarrow 0 \text{ to } \infty$$

$$= \frac{\pi}{2} \int_0^\infty e^{-t} \frac{dt}{2}$$

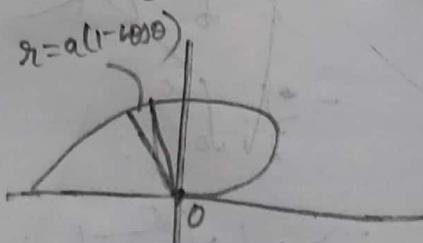
$$= \frac{\pi}{4} \int_0^\infty e^{-t} dt$$

$$= \frac{\pi}{4} [-e^{-t}]_0^\infty$$

$$= \frac{\pi}{4} [0+1]$$

$$= \frac{\pi}{4}$$

5. Identify the limits of integration for $\iint f(r, \theta) d\theta$ over the cardioid $r = a(1 - \cos \theta)$ above the initial line



Limits are:-

$$r \rightarrow 0 \text{ to } a(1 - \cos \theta)$$

$$\theta \rightarrow 0 \text{ to } \pi.$$

MEDIUM TYPE

1. find the new limits of integration after changing the order of integration for $\int_0^{\sqrt{a^2-y^2}} \int_a^{\sqrt{a^2-y^2}} f(x,y) dx dy$.
 Given limits are:-

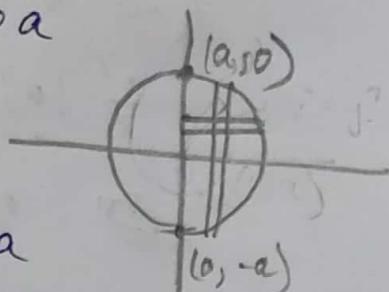
$$x \rightarrow 0 \text{ to } \sqrt{a^2-y^2}$$

$$y \rightarrow -a \text{ to } a$$

New limits are:-

$$y \rightarrow -\sqrt{a^2-x^2} \text{ to } \sqrt{a^2-x^2}, \quad x \rightarrow 0 \text{ to } a$$

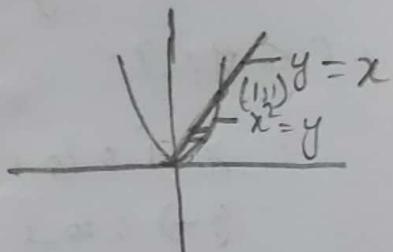
$$x \rightarrow 0 \text{ to } a, \quad y \rightarrow 0 \text{ to } 2a$$



2. find the new limits of integration after changing the order of integration for $\int_0^1 \int_{x^2}^x f(x,y) dx dy$.
 Given limits are :-

$$y \rightarrow x^2 \text{ to } x$$

$$x \rightarrow 0 \text{ to } 1$$



New limits are:-

$$x \rightarrow y \text{ to } \sqrt{y}$$

$$y \rightarrow 0 \text{ to } 1$$

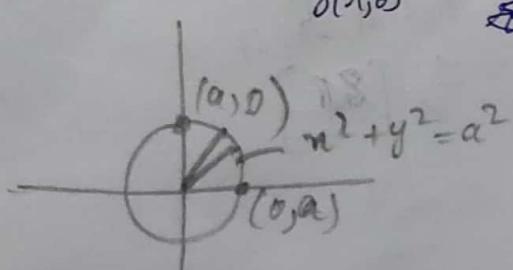
3. Convert into polar coordinates and then evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2) dy dx$.

Let, the polar coordinates be $x = r \cos \theta, y = r \sin \theta$
 $r^2 + y^2 = r^2$.

$$\frac{\partial(x,y)}{\partial(r,\theta)}$$

$$r \rightarrow 0 \text{ to } a$$

$$\theta \rightarrow 0 \text{ to } \pi/2$$



$$= \int_0^{\pi/2} \int_0^a r^2 dr d\theta.$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3} \right]_0^a d\theta.$$

$$= \int_0^{\pi/2} \left[\frac{a^3}{3} \right] d\theta.$$

$$= \frac{a^3}{3} \int_0^{\pi/2} d\theta$$

$$= \frac{a^3}{3} [\theta]_0^{\pi/2}$$

$$= \frac{\pi}{2} a^3 \cdot \frac{a^3}{3} = \frac{a^6 \pi}{6}.$$

$$= \frac{a^6 \pi}{6}$$

4. The value of $\iint_R x^2 y^3 dx dy$, where R is the region bounded by the rectangle $0 \leq x \leq 1$ and $0 \leq y \leq 3$.

$$x \rightarrow 0 \text{ to } 1$$

$$y \rightarrow 0 \text{ to } 3$$

$$= \int_0^3 \int_0^1 x^2 y^3 dx dy$$

$$= \int_0^3 y^3 \left[\frac{x^3}{3} \right]_0^1 dy$$

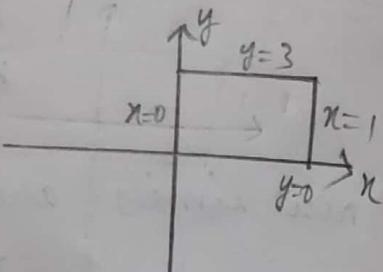
$$= \int_0^3 y^3 \left[\frac{1}{3} \right] dy$$

$$= \frac{1}{3} \int_0^3 y^3 dy$$

$$= \frac{1}{3} \left[\frac{y^4}{4} \right]_0^3$$

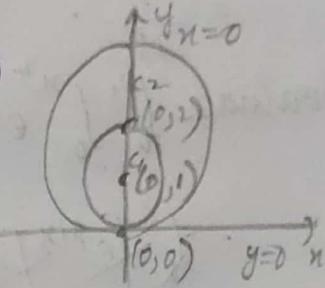
$$= \frac{1}{3} \left[\frac{81}{4} \right]$$

$$= \frac{27}{4}$$



5. Identify the limits of integration for $\iint_R f(r, \theta) dr d\theta$
 over the region bounded by the circles $r = 2 \sin \theta$
 and $r = 4 \sin \theta$.

$$\begin{aligned} r &= 2 \sin \theta \\ r &= 2 \left(\frac{y}{r}\right) \\ r^2 &= 2y \\ x^2 + y^2 &= 2y \\ x^2 + y^2 - 2y &= 0. \end{aligned}$$



$$\begin{aligned} C_1(0,1) \\ r_1 = 1 \end{aligned}$$

$$r = 4 \sin \theta$$

$$\begin{aligned} C_2(0,2) \\ r_2 = 2 \end{aligned}$$

The limits are:-

$$\begin{aligned} r &\rightarrow 2 \sin \theta \text{ to } 4 \sin \theta & x^2 + y^2 &= 4y \\ \theta &\rightarrow 0 \text{ to } \pi & x^2 + y^2 - 4y &= 0 \end{aligned}$$

SECTION-B

ESSAY QUESTIONS

EASY QUESTIONS

i. a. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dx dy$.

$$\begin{aligned} &= \int_0^1 \left(x^2 y + \frac{y^3}{3} \right) \Big|_x^{\sqrt{x}} dx \\ &= \int_0^1 \left(x^2 (\sqrt{x}) + \frac{(\sqrt{x})^3}{3} - x^3 - \frac{x^3}{3} \right) dx \\ &= \int_0^1 \left[x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3} \right] dx \\ &= \int_0^1 \left[\frac{3x^{5/2} + x^{3/2} - 3x^3 - x^3}{3} \right] dx \\ &= \int_0^1 \left[\frac{3x^{5/2} + x^{3/2} - 4x^3}{3} \right] dx \\ &= \frac{1}{3} \int_0^1 [3x^{5/2} + x^{3/2} - 4x^3] dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3} \left[3 \left(\frac{x^{7/2}}{7/2} \right) + \frac{x^{5/2}}{5/2} - 4 \left(\frac{x^4}{4} \right) \right] \Big|_0^1 \\ &= \frac{1}{3} \left[\frac{6}{7} x^{7/2} + \frac{2}{5} x^{5/2} - x^4 \right] \Big|_0^1 \end{aligned}$$

$$= \frac{1}{3} \left[\frac{6}{7} + \frac{2}{5} - 1 \right].$$

$$= \frac{1}{3} \left[\frac{9}{35} \right]$$

$$= \frac{3}{35},$$

b. Evaluate $\int_0^4 \int_0^{x^2} e^{y/x} dy dx$.

$$= \int_0^4 \left[e^{y/x} \left[\frac{x}{x^2} \right] \right] dy.$$

$$= \int_0^4 \left[\frac{e^{y/x}}{\left(\frac{x^2}{x^2} \right)} \right]_{x^2}^{x^2} dx.$$

$$= \int_0^4 \left[x e^{y/x} \right]_{x^2}^{x^2} dx$$

$$= \int_0^4 \left[x e^{x^2/x} - x e^{0/x} \right] dx.$$

$$= \int_0^4 [x e^x - x e^0] dx.$$

$$= \int_0^4 [x e^x - x] dx.$$

$$= \left[x e^x - e^x - \frac{x^2}{2} \right]_0^4.$$

$$= \left[4e^4 - e^4 - \frac{16}{2} \right].$$

$$= 3e^4 - \frac{24}{2}$$

$$= 4e^4 - 12$$

$$= 4(e^4 - 3)$$

$$\therefore \int x e^x = x e^x - e^x$$

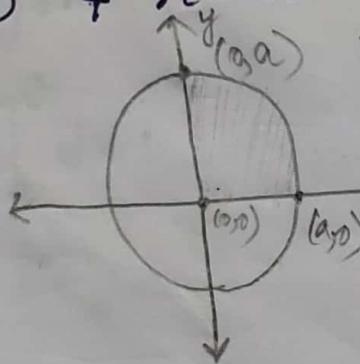
$$= 3e^4 - 8 + 1$$

$$= 3e^4 - 7.$$

$\int \int xy \, dx \, dy$ over the positive quadrant co-ordinate of the circle $x^2 + y^2 = a^2$.

\therefore The equation of the given circle is $x^2 + y^2 = a^2$

$$c = (0, 0) \text{ & } r = a.$$



$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

$$\int_0^a \left[\sqrt{a^2 - x^2} \right] dx$$

$$= \int_0^a \left[(a^2 - x^2)^{1/2} \right] dx$$

HORIZONTAL STRIP :-

In the a horizontal strip $x \rightarrow 0$ to $\sqrt{a^2 - y^2}$,
 $y \rightarrow 0$ to a .

(OR)

VERTICAL STRIP :-

In a vertical strip $y \rightarrow 0$ to $\sqrt{a^2 - x^2}$
 $x \rightarrow 0$ to a .

$$\begin{aligned}
 I &= \int_0^a \int_0^{\sqrt{a^2 - x^2}} ny dx dy \\
 &= \int_0^a \left[\int_0^{\sqrt{a^2 - y^2}} ny dx \right] dy \\
 &= \int_0^a y \left[\frac{x^2}{2} \right]_0^{\sqrt{a^2 - y^2}} dy \\
 &= \frac{1}{2} \int_0^a y (a^2 - y^2) dy \\
 &= \frac{1}{2} \int_0^a (a^2 y - y^3) dy \\
 &= \frac{1}{2} \left[a^2 \int_0^a y dy - \int_0^a y^3 dy \right] \\
 &= \frac{1}{2} \left[a^2 \left[\frac{y^2}{2} \right]_0^a - \left[\frac{y^4}{4} \right]_0^a \right] \\
 &= \frac{1}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right] \\
 \boxed{I = \frac{a^4}{8} \text{ sq units}}
 \end{aligned}$$

$T(0^{\circ} - 3)$

2. b. Evaluate $\iint (x+y)^2 dxdy$ over the area bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

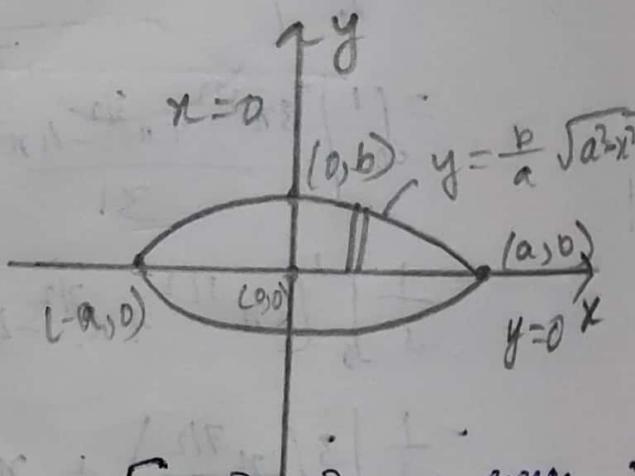
By using vertical strip

$$y \rightarrow -\frac{b}{a}\sqrt{a^2-x^2} \text{ to } \frac{b}{a}\sqrt{a^2-x^2}$$

$$x \rightarrow -a \text{ to } a$$

$$I = 4 \int_0^a \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + y^2 + 0) dy dx$$

$\because x^2 \text{ & } y^2$ are even functions & xy is odd function



$$= 4 \int_0^a \left[x^2 y + \frac{y^3}{3} \right]_0^a dx$$

$$= 4 \int_0^a \left[\frac{b}{a} x^2 \sqrt{a^2 - x^2} + \frac{b^3}{3a} (a^2 - x^2)^{3/2} \right] dx$$

let, $x = a \sin \theta$

$$dx = a \cos \theta d\theta$$

if $x=0 \Rightarrow \theta=0$

$$x=a \Rightarrow \theta=\pi/2$$

$$= 4 \int_0^{\pi/2} \left[\frac{b}{a} a^3 \sin^2 \theta d\theta + \frac{b^3}{3a^3} a^3 \cos^2 \theta \right] a \cos \theta d\theta$$

$$= 4 \int_0^{\pi/2} \left[b a^3 \sin^2 \theta \cos^2 \theta + \frac{b^3 a}{3} \cos^4 \theta \right] d\theta$$

~~sol~~

$$\therefore \int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{(m-1)(m-3) \cdots (n-1)(n-3) \cdots}{(m+n)(m+n-2) \cdots} \cdot \frac{\pi}{2}$$

$$\int_0^{\pi/2} \cos^n \theta d\theta = \frac{(n-1)(n-3) \cdots}{n(n-2) \cdots} \cdot \frac{\pi}{2}$$

$$= 4 \left[a^3 b \left[\frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right] + \frac{ab^3}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} \right]$$

$$= 4 \left[a^3 b \frac{\pi}{16} + ab^3 \frac{\pi}{16} \right]$$

$$= 4 \left[\frac{a^3 b \pi + ab^3 \pi}{16} \right]$$

$$= \frac{\pi}{4} [a^3 b + ab^3]$$

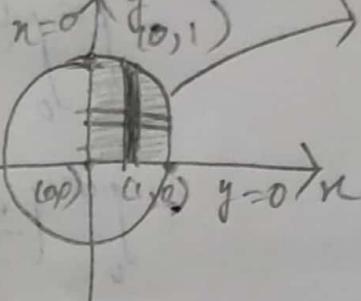
$$I = \frac{\pi}{4} ab (a^2 + b^2) \text{ sq. units}$$

Using change of order of integration evaluate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dy \, dx.$$

the given limit

$$x \rightarrow 0 \text{ to } \sqrt{1-y^2} \\ y \rightarrow 0 \text{ to } 1 \quad (\text{H.S})$$



$$x = \sqrt{1-y^2} \\ x^2 + y^2 = 1 \\ y^2 = 1-x^2 \\ y = \sqrt{1-x^2}$$

By using change of order of integration

$$y \rightarrow 0 \text{ to } \sqrt{1-x^2}$$

$$x \rightarrow 0 \text{ to } 1 \quad (\text{V.S})$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} x^3 y \, dy \, dx$$

$$= \int_0^1 x^3 \left[\frac{y^2}{2} \right]_0^{\sqrt{1-x^2}} \, dx$$

$$= \frac{1}{2} \int_0^1 x^3 (1-x^2) \, dx$$

$$= \frac{1}{2} \int_0^1 (x^3 - x^5) \, dx$$

$$= \frac{1}{2} \left[\frac{x^4}{4} - \frac{x^6}{6} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{4} - \frac{1}{6} \right]$$

$$= \frac{1}{2} \left[\frac{1}{24} \right]$$

$$= \frac{1}{24} \text{ sq. units}$$

3.b
By using ~~order~~ o change of order of integration evaluate $\int_0^3 \int_{\sqrt{4-y}}^y (x+y) dx dy$.

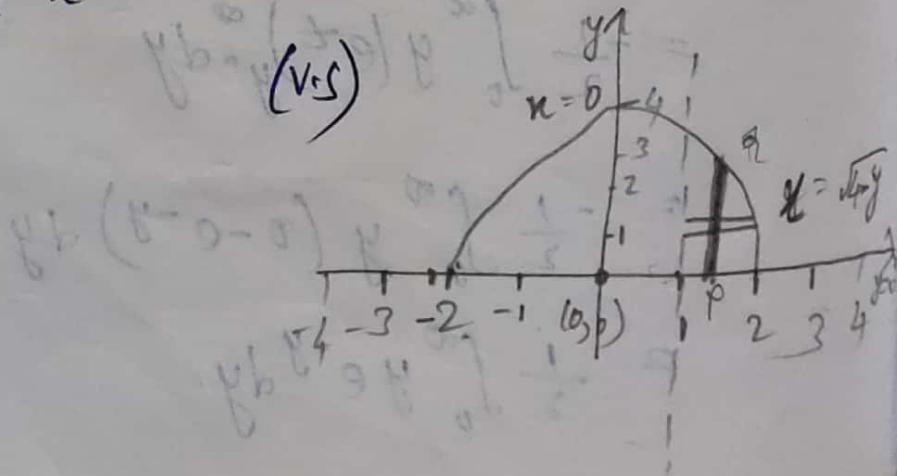
The given limits of horizontal strip are:-

$$x \rightarrow 1 \text{ to } \sqrt{4-y} \quad (\text{H.S})$$
$$y \rightarrow 0 \text{ to } 3$$

By ~~change~~ using change of order of integration

$$y \rightarrow 0 \text{ to } 4-x^2$$

$$x \rightarrow 1 \text{ to } 2 \quad (\text{V.S})$$



$$I = \int_0^2 \int_0^{4-x^2} (x+y) dy dx$$

$$= \int_0^2 \left[\int_0^{4-x^2} (x+y) dy \right] dx$$

$$= \int_0^2 \left(xy + \frac{y^2}{2} \right) \Big|_0^{4-x^2} dx$$

$$= \int_0^2 \left[x(4-x^2) + \frac{(4-x^2)^2}{2} \right] dx$$

$$= \frac{1}{2} \int_0^2 \left[(4x-x^3) + (16+x^4-8x^2) \right] dx$$

$$= \frac{1}{2} \left[\frac{8x^2}{2} - \frac{2x^4}{4} + 16x + \frac{x^5}{5} - \frac{8x^3}{3} \right]_0^2$$

$$= \frac{1}{2} \left[\frac{8}{2}(4) - \frac{2}{4}(2)^4 + 16(2) + \frac{(2)^5}{5} - \frac{8(2)^3}{3} - \left(\frac{8}{2} - \frac{2}{4} + 16 + \frac{1}{5} - \frac{8}{3} \right) \right]$$

$$= \frac{1}{2} \left[16 - 8 + 32 + \frac{32}{5} - \frac{64}{3} - \frac{8}{2} + \frac{1}{2} - 16 - \frac{1}{5} + \frac{8}{3} \right]$$

$$= \frac{1}{2} \left[24 + \frac{192 - 640 - 120 + 15 - 6 + 20}{30} \right]$$

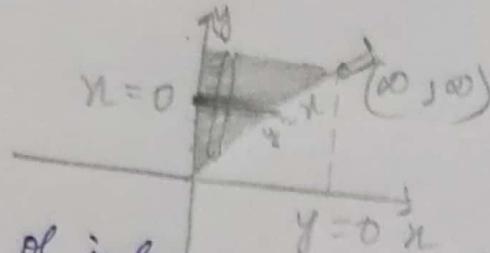
$$= \frac{1}{2} \left[\frac{24 \times 30 - 479}{30} \right]$$

$$= \frac{1}{2} \left[\frac{720 - 479}{30} \right]$$

$$= \frac{241}{60}$$

1.b. By changing the order of integration,
 evaluate the integral $\int_0^\infty \int_0^y \frac{e^{-y}}{y} dx dy$.
 The given limits are

$$\begin{aligned} y &\rightarrow n \text{ to } \infty \\ x &\rightarrow 0 \text{ to } y \quad (\text{V.S}) \end{aligned}$$



By using change of order of integration

$$n \rightarrow 0 \text{ to } \infty$$

$$y \rightarrow 0 \text{ to } \infty \quad (\text{H.S})$$

$$= \int_0^\infty \int_0^y \frac{e^{-y}}{y} dx dy$$

$$= \int_0^\infty \left(\frac{e^{-y}}{y} \right) (x) dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} (y) dy$$

$$= \left[-e^{-y} \right]_0^\infty$$

$$= - (e^{-\infty} - e^0)$$

$$= -(0 - 1)$$

$$= 1 \text{ sq. unit.}$$

10/9/19
By using change of order of integration evaluate
 $\int_0^{\infty} \int_y^{\infty} xe^{-x^2/y} dy dx$.

Q:- The given limits of vertical strip are:-

$$y \rightarrow 0 \text{ to } x \quad (\text{V.S})$$

$$x \rightarrow 0 \text{ to } \infty$$

By using change of order of integration

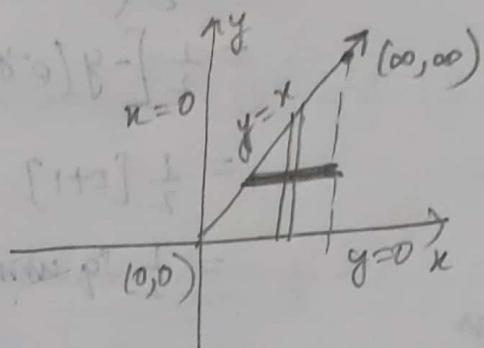
$$x \rightarrow y \text{ to } \infty$$

(H.S)

$$y \rightarrow 0 \text{ to } \infty$$

$$I = \int_0^{\infty} \int_y^{\infty} xe^{-x^2/y} dx dy.$$

$$= \int_0^{\infty} \left[\int_y^{\infty} xe^{-x^2/y} dx \right] dy.$$



$$\text{Let } \frac{x^2}{y} = t$$

$$2x dx = y dt$$

$$t \rightarrow y \text{ to } \infty$$

$$= \int_0^{\infty} \left(\int_y^{\infty} e^{-t} \frac{y dt}{2} \right) dy$$

$$= -\frac{1}{2} \int_0^{\infty} y (e^{-t})_{y}^{\infty} dy$$

$$= -\frac{1}{2} \int_0^{\infty} y (0 - e^{-y}) dy$$

$$= \frac{1}{2} \int_0^{\infty} y e^{-y} dy$$

$$= \frac{1}{2} \left[y(-e^{-y}) - \int (-e^{-y}) dy \right]_0^\infty$$

$$= \frac{1}{2} \left[\infty - (e^{-y})_0^\infty \right]$$

~~$$= \frac{1}{2} [\infty - \infty]$$~~

$$= \frac{1}{2} [\infty - (\cancel{e^{-\infty}} - 1)]$$

=

$$= \frac{1}{2} \left[y(-e^{-y}) - \int (-e^{-y}) dy \right]_0^\infty$$

~~$$= \frac{1}{2} \left(\int$$~~

$$= \frac{1}{2} \left[-y(e^{-y}) - e^{-y} \right]_0^\infty$$

$$= \frac{1}{2} [0 + 1]$$

$$= \frac{1}{2} \text{ Sq unit.}$$

POLAR
 Show that $\iint_R r^2 \sin \theta dr d\theta = \frac{2a^3}{3}$ where R is the semicircle $r = 2a \cos \theta$, above the initial line.

Let $x = r \cos \theta$ and $y = r \sin \theta$ be the polar coordinates of the circle.

Given:- $r = 2a \cos \theta$.

$$r = 2a \left(\frac{x}{r}\right).$$

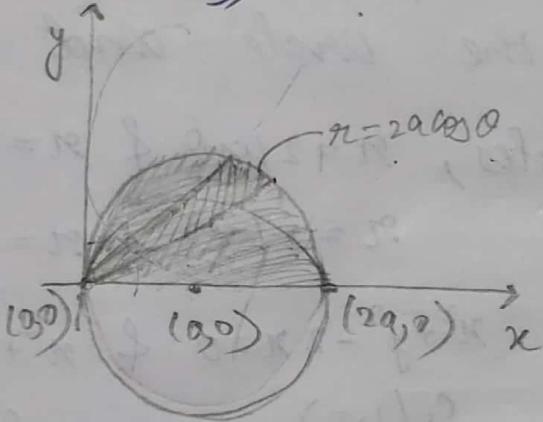
$$r^2 = 2ax$$

$$x^2 + y^2 - 2ax = 0.$$

$$c(a, 0), r = a.$$

$$\therefore \cos \theta = \frac{x}{r}.$$

$x = r \cos \theta$
$y = r \sin \theta$
$r^2 = x^2 + y^2$



$$r \rightarrow 0 \text{ to } 2a \cos \theta$$

$$\theta \rightarrow 0 \text{ to } \pi/2$$

$$I = \int_0^{\pi/2} \int_0^{2a \cos \theta} r^2 \sin \theta dr d\theta.$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3} \sin \theta \right]_0^{2a \cos \theta} d\theta.$$

$$= \int_0^{\pi/2} \sin \theta \left(\frac{8a^3 \cos^3 \theta}{3} \right) d\theta$$

$$= \frac{8a^3}{3} \int_0^{\pi/2} \sin \theta \cos^3 \theta d\theta.$$

$$\cos \theta = t \Rightarrow -\sin \theta d\theta = dt$$

$$t \rightarrow 1 \text{ to } 0$$

$$= \cancel{\frac{8a^3}{3} \int_1^0 t^3 (-dt)}$$

$$= \frac{8a^3}{3} \int_1^0 t^3 (-dt)$$

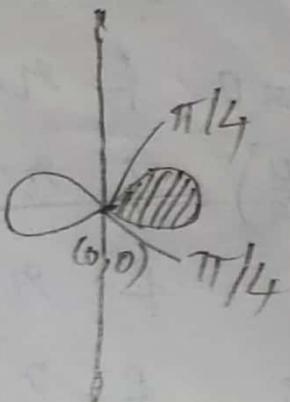
$$= -\frac{8a^3}{3} \left[\frac{t^4}{4} \right]_1^0$$

$$= -\frac{8a^3}{3} \left[0 - \frac{1}{4} \right]$$

$$\boxed{T = \frac{2a^3}{3}}$$

3a.
Evaluate $\iint_R \frac{r dr d\theta}{\sqrt{a^2 + r^2}}$ over one loop of the lemniscate
 $r^2 = a^2 \cos 2\theta$.

The equation of the lemniscate is $r^2 = a^2 \cos 2\theta$.



$$\theta \rightarrow -\pi/4 \text{ to } \pi/4$$

$$I = \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{a^2+r^2}} dr d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \frac{1}{2} \frac{2r}{\sqrt{a^2+r^2}} dr d\theta.$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(2\sqrt{a^2+r^2} \right)_0^{a\sqrt{\cos 2\theta}} dr$$

$$= \int_{-\pi/4}^{\pi/4} \left(\sqrt{a^2+a^2\cos 2\theta} - \sqrt{a^2} \right) dr$$

$$= \int_{-\pi/4}^{\pi/4} (\sqrt{a^2(2\cos 2\theta)} - a) dr$$

$$= \int_{-\pi/4}^{\pi/4} (\sqrt{2}a\cos\theta - a) dr$$

$$= a \left[\sqrt{2}\sin\theta - \theta \right]_{-\pi/4}^{\pi/4}$$

$$= a \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi}{4} - \left(-\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4} \right) \right]$$

$$= a \left[-\frac{\pi}{4} + 1 - \frac{\pi}{4} \right]$$

$$= a \left[2 - \frac{\pi}{2} \right].$$

3

Evaluate $\iint_R r^3 dr d\theta$ over the area between the two circles.

$$r = 2 \cos \theta \quad \text{and} \quad r = 4 \cos \theta.$$

Let $x = r \cos \theta$ & $y = r \sin \theta$ be the polar coordinates of the circle and $x^2 + y^2 = r^2$.

The given circles, $r = 2 \cos \theta$ & $r = 4 \cos \theta$.

$$r = 2 \left(\frac{x}{r} \right) \quad \text{and} \quad r = 4 \left(\frac{x}{r} \right).$$

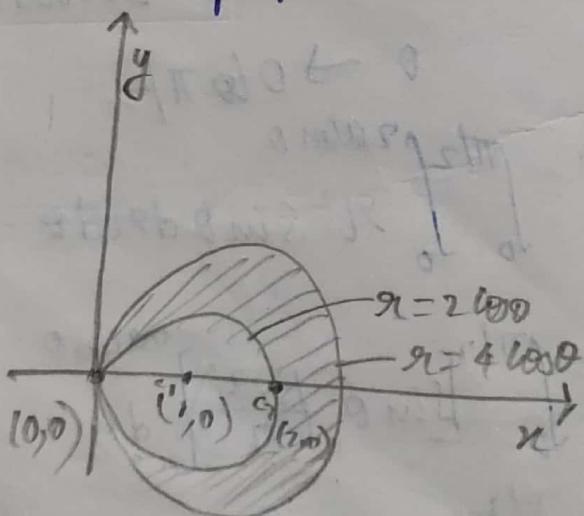
$$x^2 + y^2 - 2x = 0 \quad \text{and} \quad x^2 + y^2 - 4x = 0.$$

$$C_1(1, 0)$$

$$r_1 = 1$$

$$C_2(2, 0)$$

$$r_2 = 2$$



$$r \rightarrow 2 \cos \theta \text{ to } 4 \cos \theta$$

$$\theta \rightarrow -\pi/2 \text{ to } \pi/2$$

$$= \int_{-\pi/2}^{\pi/2} \int_{2\cos\theta}^{4\cos\theta} r^3 dr d\theta.$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{r^4}{4} \right) \Big|_{2\cos\theta}^{4\cos\theta} d\theta.$$

$$= \frac{1}{4} \int_{-\pi/2}^{\pi/2} (256 \cos^4\theta - 16 \cos^4\theta) d\theta.$$

$$= \frac{240}{4} \int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta.$$

$$= 60 \times 2 \int_0^{\pi/2} \cos^4\theta d\theta.$$

~~$$= 60 \times 2 \left[\frac{3x^2}{4x^2} \cdot \frac{\pi}{2} \right].$$~~

~~$$= +20 \times \frac{3}{4} \cdot \pi$$~~

~~$$= 15 \times \frac{3}{4} \cdot \pi$$~~

~~$$= 45 \pi.$$~~

~~$$= 60 \times 2 \left[\frac{3}{16} \cdot \pi \right]$$~~

~~$$= +20 \times \frac{3}{8} \cdot \frac{\pi}{2}$$~~

$$\boxed{I = \frac{45\pi}{2}}$$

$\therefore \int_{-\pi/2}^{\pi/2} f(x) dx = f(-x)$ is even

$$= 2 \int_0^{\pi/2} f(x) dx$$

TYPICAL QUESTIONS

1. a. Find the area lying inside the cardioid $r = a(1 + \cos \theta)$ and outside the circle $r = a$.

$$r \rightarrow a \text{ to } a(1 + \cos \theta)$$

$$\theta \rightarrow 0 \text{ to } \pi.$$

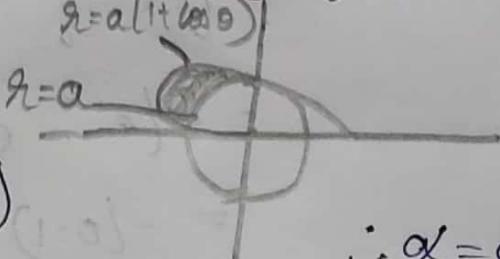
$$= \int_0^\pi \int_a^{a(1+\cos\theta)} r dr d\theta$$

$$= \iint_R r dr d\theta$$

$$= \int_0^\pi \int_a^{a(1+\cos\theta)} r dr d\theta$$

$$= \int_0^\pi \left[\frac{r^2}{2} \right]_a^{a(1+\cos\theta)} d\theta$$

$$= \int_0^\pi \left[\frac{[a(1+\cos\theta)]^2 - a^2}{2} \right] d\theta$$



$$\therefore \theta = \alpha(1 + \cos \theta)$$

$$1 + \cos \theta = 0.$$

$$\cos \theta = -1$$

$$\theta = \cos^{-1}(-1)$$

$$\theta = \pi.$$

10. 369

11. 231

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$$= \int_0^\pi \frac{a^2}{2} [(1+\cos\theta)^2 - 1] d\theta.$$

$$= 2 \int_0^{\pi/2} \frac{a^2}{2} [(1+\cos\theta)^2 - 1] d\theta.$$

$$= \frac{2a^2}{2} \int_0^{\pi/2} [(1+\cos\theta)^2 - 1] d\theta.$$

$$= a^2 \int_0^{\pi/2} [1 + \cos^2\theta + 2\cos\theta - 1] d\theta.$$

$$= a^2 \int_0^{\pi/2} (\cos^2\theta + 2\cos\theta) d\theta.$$

$$= a^2 \left[\frac{1}{2} \cdot \frac{\pi}{2} \right] + 2a^2 [\sin\theta]_0^{\pi/2}$$

$$= \frac{a^2\pi}{4} + 2a^2 [1].$$

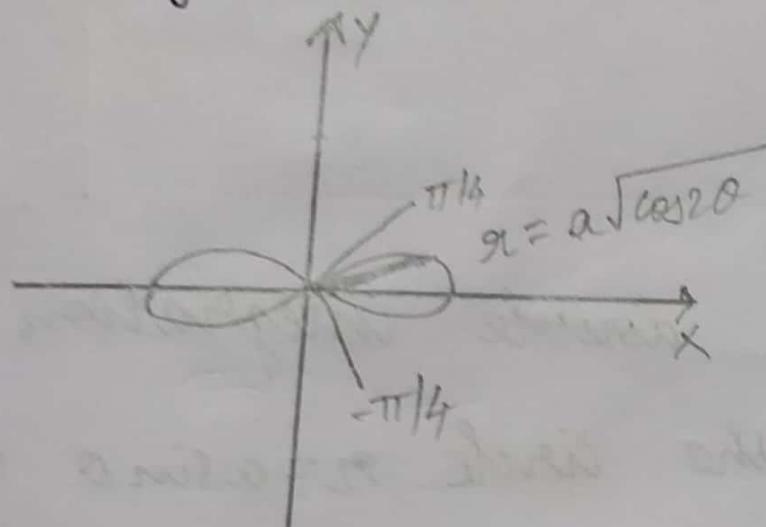
~~$\underline{\underline{=}}$~~

$$= \frac{a^2\pi}{4} + 2a^2 \text{ sq. units}$$

~~$\frac{a^2}{4} [1+8]$~~

1b
Find by double integration area of Lemniscate.

The equation of the Lemniscate scale $r^2 = a^2 \cos 2\theta$.



$$r \rightarrow 0 \text{ to } a\sqrt{\cos 2\theta}$$

$$\theta \rightarrow -\pi/4 \text{ to } \pi/4$$

$$\text{Area} = \iint_R r dr d\theta.$$

$$= 2 \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r dr d\theta.$$

$$= 2 \int_{-\pi/4}^{\pi/4} \left(\frac{r^2}{2} \right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} a^2 \cos 2\theta d\theta.$$

$$= \alpha^2 \left[\frac{\sin 20}{2} \right]^{n/4}$$

$$= \frac{\alpha^2}{2} [1+1]$$

$$\boxed{D = \alpha^2 \text{ sq units}}$$

2.a. Find the area common to the circles
 $r = a \cos \theta$ and $r = a \sin \theta$.

$$\text{Area} = \int_0^{\pi/4} \int_0^{a \sin \theta} r dr d\theta + \int_{\pi/4}^{\pi/2} \int_0^{a \cos \theta} r dr d\theta.$$

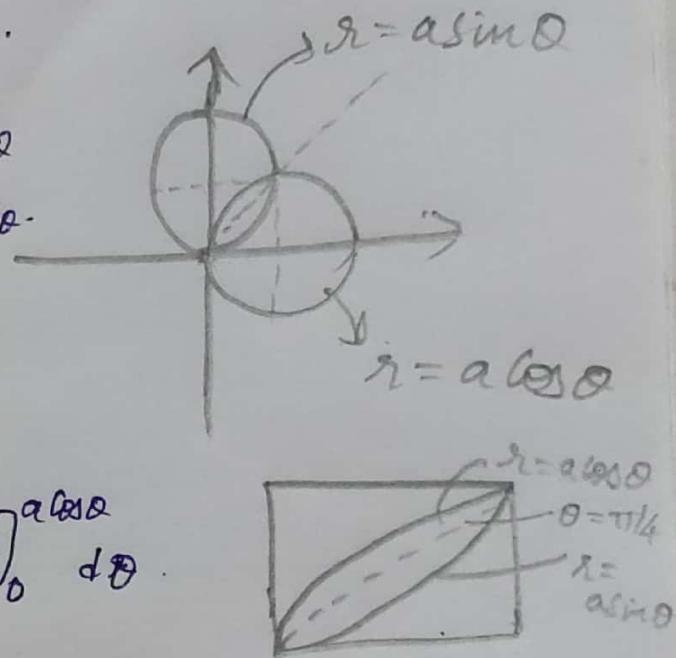
$$= \frac{a^2}{2} \int_0^{\pi/4} \sin^2 \theta d\theta$$

$$= \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_0^{a \sin \theta} d\theta + \int_{\pi/4}^{\pi/2} \left[\frac{r^2}{2} \right]_0^{a \cos \theta} d\theta.$$

$$= \int_0^{\pi/4} \left[\frac{a^2 \sin^2 \theta}{2} \right] d\theta + \int_{\pi/4}^{\pi/2} \left[\frac{a^2 \cos^2 \theta}{2} \right] d\theta.$$

$$= \frac{a^2}{2} \int_0^{\pi/4} \left[\frac{1 - \cos 2\theta}{2} \right] d\theta + \frac{a^2}{2} \int_{\pi/4}^{\pi/2} \left[\frac{1 + \cos 2\theta}{2} \right] d\theta.$$

$$= \frac{a^2}{4} \int_0^{\pi/4} (1 - \cos 2\theta) d\theta + \frac{a^2}{4} \int_{\pi/4}^{\pi/2} (1 + \cos 2\theta) d\theta.$$



$$= \frac{a^2}{4} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/4} + \frac{a^2}{4} \left[\theta + \frac{\sin 2\theta}{2} \right]_{\pi/4}^{\pi/2}$$

$$= \frac{a^2}{4} \left[\frac{\pi}{4} - \frac{1}{2} \right] + \frac{a^2}{4} \left[\frac{\pi}{2} - \frac{\pi}{4} \right]$$

$$= \frac{a^2}{4} \left[\frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{2} - \frac{\pi}{4} \right]$$

$$= \frac{a^2}{4} \left[\frac{\pi}{2} - \frac{1}{2} \right]$$

$$= \frac{a^2}{4} \left[\frac{\pi-1}{2} \right] \text{ sq. units}$$

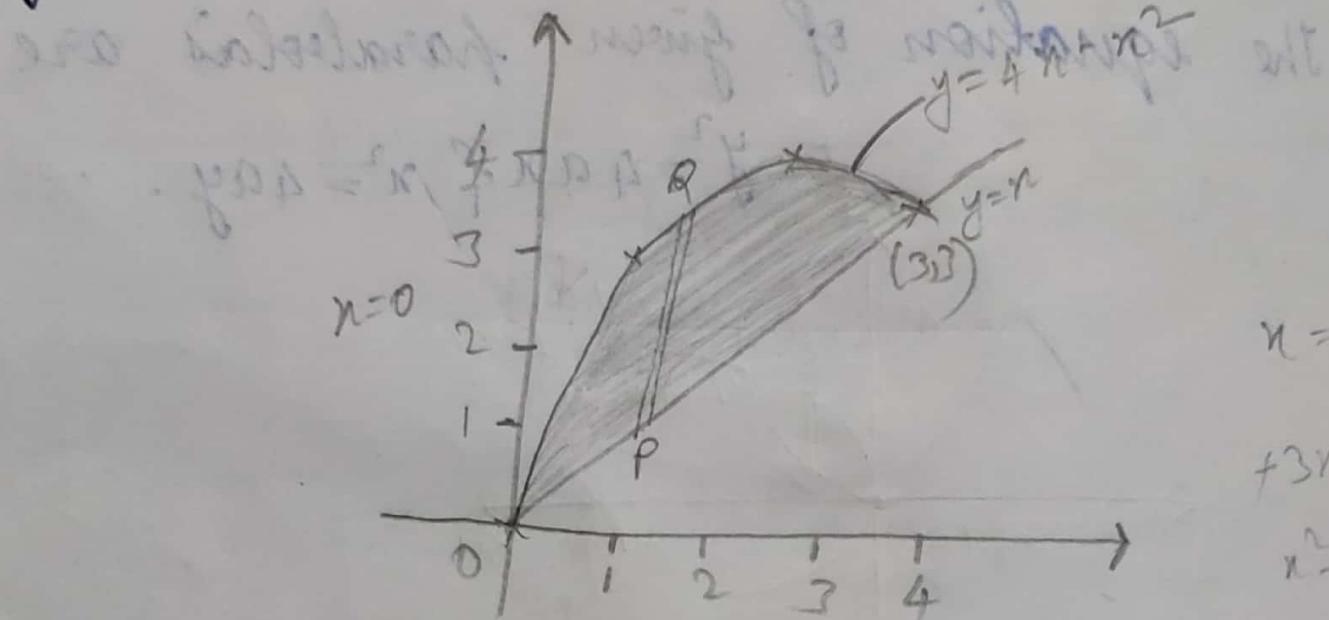
2.b. Find the area lying inside the cardioid

$r = 1 + \cos \theta$ and outside the parabola

$$r(1 + \cos \theta) = 1.$$

3. a) Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

:- The given curves are $y = 4x - x^2$ & $y = x$



x	0	1	2	3
y	0	3	4	3

$$y \rightarrow x \text{ to } 4x - x^2$$

$$x \rightarrow 0 \text{ to } 3$$

$$\text{Area} = \iint_R dx dy$$

$$= \int_0^3 \int_x^{4x-x^2} dy dx$$

$$= \int_0^3 \left[y \right]_{x^2}^{4x-x^2} dx$$

$$= \int_0^3 \left[4x - x^2 - \left(\sqrt{4x - x^2} \right)^2 \right] dx$$

$$= \int_0^3 [3x - x^2] dx$$

$$= \left[3\frac{x^2}{2} - \frac{x^3}{3} \right]_0^3$$

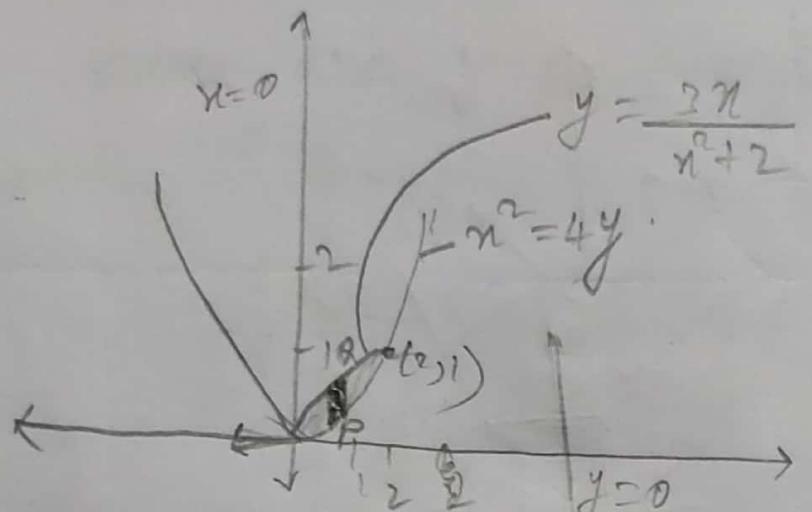
$$= \frac{27}{2} - 9$$

$$= \underline{\underline{27 - 18}}$$

$$\boxed{D = \frac{9}{2} \text{ units}}$$

Find the area enclosed by the curves $y = \frac{3x}{x^2+2}$
 $\& 4y = x^2$.

The given curves are $y = \frac{3x}{x^2+2}$ & $x^2 = 4y$.



$$\begin{aligned} x^2 &= 4y \\ y = 0 &\Rightarrow x = 0 \\ y = 1 &\Rightarrow x = \pm 2 \\ (0,0), (2,1), (-2,1) & \end{aligned}$$

$$\begin{aligned} y &= \frac{3x}{x^2+2} \\ y = 0 &\Rightarrow x = 0 \\ y = 1 &\Rightarrow x^2 + 2 - 3x = 0 \\ x &= 1, 2, \\ (0,0), (1,1), (2,1) & \end{aligned}$$

The intersected points are $(0,0)$, $(2,1)$ ~~are common points~~

$$\begin{aligned} y &\rightarrow \frac{x^2}{4} \text{ to } \frac{3x}{x^2+2} \\ x &\rightarrow 0 \text{ to } 2 \end{aligned}$$

$$\begin{aligned} \text{Area} &= \iint_R dxdy \\ &= \int_0^2 \int_{x^2/4}^{3x/(x^2+2)} dy dx. \end{aligned}$$

$$= \int_0^2 \left[y \right]_{x^2/4}^{3x/x^2+2} dx$$

$$= \int_0^2 \left[\frac{3x}{x^2+2} - \frac{x^2}{4} \right] dx.$$

$$= \left[\frac{3}{2} \log(x^2+2) - \frac{x^3}{12} \right]_0^2$$

$$= \frac{3}{2} \log 6 - \frac{8}{12} - \left[\frac{3}{2} \log 2 \right]$$

$$= \frac{3}{2} [\log 6 - \log 2] - \frac{2}{3}$$

$$= \frac{3}{2} [\log 3] - \frac{2}{3}$$

$$\boxed{I = \frac{3}{2} \log 3 - \frac{2}{3}}$$