UNIT-IV

SECTION-A

EASY TYPE

1. Evaluate $\int_0^2 \int_0^x y \, dy \, dx$.

$$= \int_0^2 \left[\frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^2 \left[\frac{x^2}{2}\right] dx$$

$$= \frac{1}{2}\int_0^2 x^2 dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2.$$

$$= \frac{\cancel{8}4}{\cancel{6}3}$$

$$= \frac{4}{3}.$$

2. Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$.

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\sin^{-1}(x)\right]_0^1 dy.$$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}}(\pi/2)\, dy.$$

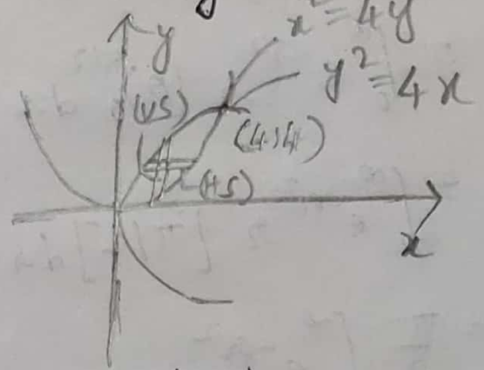
$$= \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1-y^2}} dy.$$

$$= \frac{\pi}{2}\left[\sin^{-1} y\right]_0^1$$

$$= \frac{\pi}{2}\left[\pi/2\right]$$

$$= \frac{\pi^2}{4}.$$

3. Identify the limits of integration for $\iint_R f(x,y)\,dx\,dy$ where R is the region bounded by the parabolas $y^2=4x$ and $x^2=4y$.



(H.S)

~~$x = y^2/4$~~

(H.S)

$x \to \frac{y^2}{4}$ to $2\sqrt{y}$.

$y \to 0$ to 4.

(V.S)

$y \to \frac{x^2}{4}$ to $\sqrt{2}x$

$x \to$ 0 to 4

4. Evaluate $\int_0^\infty \int_0^{\pi/2} e^{-r^2}\, r\, d\theta\, dr$

Let $r^2 = t$

$2r\,dr = dt$.

$t \to 0$ to $\pi^2/4$.

$= \int_0^\infty \int_0^{\pi^2/4} e^{-t} \frac{dt}{2}\, d\theta$.

$= \frac{1}{2}\int_0^\infty \left[-e^{-t}\right]_0^{\pi^2/4} d\theta$.

$= \frac{1}{2}\int_0^\infty \cancel{\left[-e^{-\pi^2/4} + e^0\right] d\theta}$

$= \frac{1}{2}\int_0^\infty$

$= \cancel{-\frac{1}{2}\int_0^\infty e^{-t}}$.

$= \frac{1}{2}\int_0^\infty \left[-e^{-\pi^2/4} + e^0\right] d\theta$

$= \frac{1}{2}\int_0^\infty \left[-e^{-\pi^2/4} + 1\right] d\theta$

$=$

Scanned by CamScanner

4. Evaluate $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}\, r\, d\theta\, dr$.

$$= \int_0^{\infty} e^{-r^2} r \left(\int_0^{\pi/2} d\theta\right) dr.$$

$$= \int_0^{\infty} e^{-r^2} r\, (\theta)_0^{\pi/2}\, dr.$$

$$= \int_0^{\infty} e^{-r^2} r\, [\pi/2]\, dr.$$

$$= \frac{\pi}{2}\int_0^{\infty} e^{-r^2} r\, dr.$$

Let $r^2 = t$

$2r\,dr = dt$

$t \to 0$ to ∞

$$= \frac{\pi}{2}\int_0^{\infty} e^{-t}\frac{dt}{2}$$

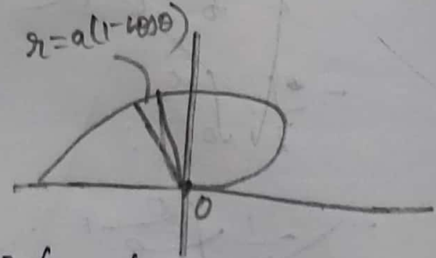
$$= \frac{\pi}{4}\int_0^{\infty} e^{-t}\, dt.$$

$$= \frac{\pi}{4}\left[-e^{-t}\right]_0^{\infty}.$$

$$= \frac{\pi}{4}[0+1]$$

$$= \frac{\pi}{4}.$$

5. Identify the limits of integration for $\iint_R f(r,\theta)\, dr\, d\theta$ over the cardiod $r = a(1-\cos\theta)$ above the initial line



Limits are:-

$r \to 0$ to $a(1-\cos\theta)$

$\theta \to 0$ to π.

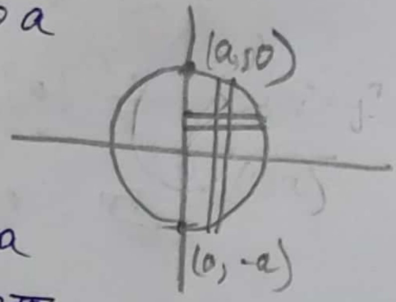
MODERATE TYPE

1. Find the new limits of integration after changing the order of integration for $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$.

Given Limits are:-

$x \to 0$ to $\sqrt{a^2-y^2}$

$y \to -a$ to a



New Limits are:-

$y \to -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$.

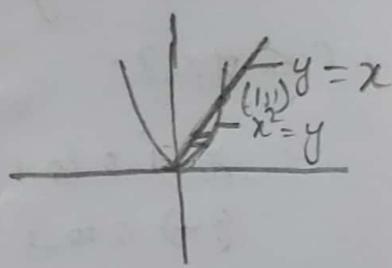
$x \to 0$ to a

2. Find the new limits of integration after changing the order of integration for $\int_0^1\int_{x^2}^{x} f(x,y)\,dx\,dy$.

Given Limits are :-

$y \to x^2$ to x

$x \to 0$ to 1



New Limits are :-

$x \to y$ to $\sqrt{y}$

$y \to 0$ to 1.

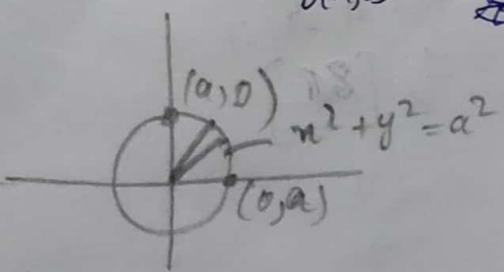
3. Convert into polar coordinates and then evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx$.

Let, the polar coordinates be $x = r\cos\theta$, $y = r\sin\theta$ & $x^2+y^2 = r^2$.

$\cancel{J = \frac{\partial(x,y)}{\partial(r,\theta)} = r}$

$r \to 0$ to a

$\theta \to 0$ to $\pi/2$



$$= \int_0^{\pi/2}\int_0^a r^2\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \int_0^{\pi/2} \left[\frac{a^3}{3}\right] d\theta$$

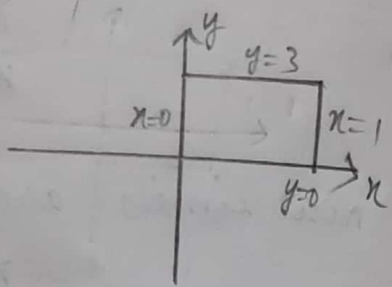
$$= \frac{a^3}{3}\int_0^{\pi/2} d\theta$$

$$= \frac{a^3}{3}[\theta]_0^{\pi/2}$$

$$= \frac{a^3}{3}\left[\frac{\pi}{2}\right]$$

$$= \frac{a^3\pi}{6}$$

4. The value of $\iint_R x^2y^3\,dx\,dy$, where R is the region bounded by the rectangle $0 \le x \le 1$ and $0 \le y \le 3$.



$x \to 0$ to 1

$y \to 0$ to 3

$$= \int_0^3\int_0^1 x^2y^3\,dx\,dy$$

$$= \int_0^3 y^3\left[\frac{x^3}{3}\right]_0^1 dy$$

$$= \int_0^3 y^3\left[\frac{1}{3}\right]dy$$

$$= \frac{1}{3}\int_0^3 y^3\,dy$$

$$= \frac{1}{3}\left[\frac{y^4}{4}\right]_0^3$$

$$= \frac{1}{\cancel{3}}\left[\frac{\overset{27}{\cancel{81}}}{4}\right]$$

$$= \frac{27}{4}$$

5. Identify the limits of integration for $\iint_R f(r,\theta)\,dr\,d\theta$ over the region bounded by the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

$r = 2\sin\theta$

$r = 2\left(\frac{y}{r}\right)$

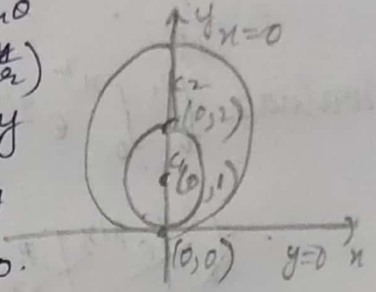
$r^2 = 2y$

$x^2+y^2 = 2y$

$x^2+y^2-2y = 0$.

$C_1(0,1)$

$r = 1$



$r = 4\sin\theta$

$r = 4\left(\frac{y}{r}\right)$

$r^2 = 4y$

$x^2+y^2 = 4y$

$x^2+y^2-4y = 0$.

$C_2(0,2)$

$r_2 = 2$

~~The limits are~~

The Limits are:-

$r \to 2\sin\theta$ to $4\sin\theta$

$\theta \to 0$ to π.

SECTION-B

ESSAY QUESTIONS

EASY QUESTIONS

1. a. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$.

$$= \int_0^1 \left(x^2y + \frac{y^3}{3}\right)_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left(x^2(\sqrt{x}) + \frac{(\sqrt{x})^3}{3} - x^3 - \frac{x^3}{3}\right) dx.$$

$$= \int_0^1 \left[x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3}\right] dx.$$

$$= \int_0^1 \left[\frac{3x^{5/2} + x^{3/2} - 3x^3 - x^3}{3}\right] dx.$$

$$= \int_0^1 \left[\frac{3x^{5/2} + x^{3/2} - 4x^3}{3}\right] dx.$$

$$= \frac{1}{3}\int_0^1 \left[3x^{5/2} + x^{3/2} - 4x^3\right] dx$$

$$= \frac{1}{3}\left[3\left(\frac{x^{7/2}}{7/2}\right) + \frac{x^{5/2}}{5/2} - 4\left(\frac{x^4}{4}\right)\right]_0^1$$

$$= \frac{1}{3}\left[\frac{6}{7}x^{7/2} + \frac{2}{5}x^{5/2} - x^4\right]_0^{1}.$$

$$= \frac{1}{3}\left[\frac{6}{7} + \frac{2}{5} - 1\right].$$

$$= \frac{1}{3}\left[\frac{9}{35}\right]$$

$$= \frac{3}{35}.$$

6. Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx$.

$$= \cancel{\int_0^4 \left[e^{y/x}\left[\frac{x}{x^2}\right]\right]}.$$

$$= \int_0^4 \left[\frac{e^{y/x}}{\cancel{\left(\frac{x}{x^2}\right)}}\right]_0^{x^2} dx.$$

$$= \int_0^4 \left[x e^{y/x}\right]_0^{x^2} dx$$

$$= \int_0^4 \left[x e^{x^2/x} - x e^{0/x}\right] dx.$$

$$= \int_0^4 \left[x e^x - x e^0\right] dx.$$

$$= \int_0^4 \left[x e^x - x\right] dx.$$

$$= \left[x e^x - e^x - \frac{x^2}{2}\right]_0^4.$$

$$\boxed{\because \int x e^x = x e^x - e^x}$$

$$= \left[4e^4 - e^4 - \frac{16}{2} + 1\right].$$

$$= \cancel{3e^4 - \frac{24}{2} 12}$$

$$= \cancel{4e^4 - 12}$$

$$= \cancel{4(e^4 - 3)}.$$

$$= 3e^4 - 8 + 1$$

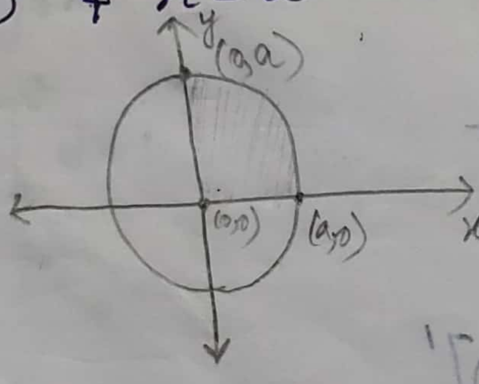
$$= 3e^4 - 7.$$

2a

$\iint xy\,dx\,dy$. over the positive ~~quar~~ co-ordinate of the circle $x^2+y^2=a^2$.

:- The equation of the given circle is $x^2+y^2=a^2$.

$c=(0,0)$ & $r=a$.



HORIZONTAL STRIP :-

In ~~hr~~ a horizontal ~~str~~ $x \to 0$ to $\sqrt{a^2-y^2}$.

$y \to 0$ to a.

(OR)

VERTICAL STRIP :-

In a vertical strip $y \to 0$ to $\sqrt{a^2-x^2}$

$x \to 0$ to a

$$I = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy \, dx \, dy.$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} xy \, dx \right] dy.$$

$$= \int_0^a y \left[\frac{x^2}{2} \right]_0^{\sqrt{a^2-y^2}} dy.$$

$$= \frac{1}{2} \int_0^a y (a^2-y^2) \, dy.$$

$$= \frac{1}{2} \int_0^a (a^2 y - y^3) dy.$$

$$= \frac{1}{2} \left[a^2 \int_0^a y \, dy - \int_0^a y^3 dy \right].$$

$$= \frac{1}{2} \left[a^2 \left[\frac{y^2}{2} \right]_0^a - \left[\frac{y^4}{4} \right]_0^a \right].$$

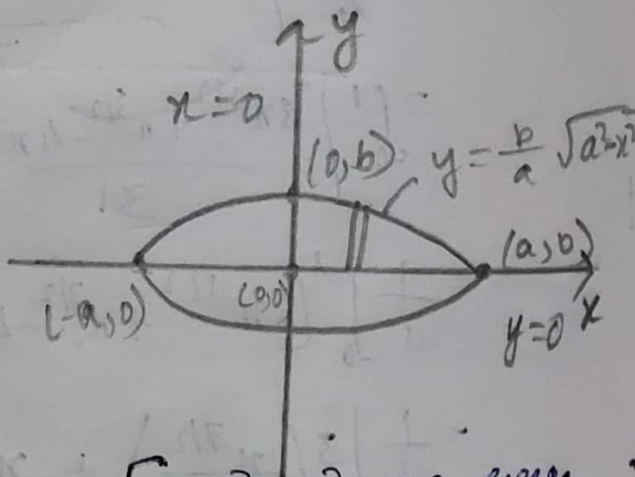
$$= \frac{1}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right].$$

$$\boxed{I = \frac{a^4}{8} \text{ sq units}}$$

***

Scanned by CamScanner

2. b. Evaluate $\iint (x+y)^2\, dx\,dy$ over the area bounded by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



By using vertical strip

$y \to -\frac{b}{a}\sqrt{a^2-x^2}$ to $\frac{b}{a}\sqrt{a^2-x^2}$

$x \to -a$ to a

$$I = 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+0)\,dy\cdot dx$$ [$\because x^2$ & y^2 are even functions & xy is odd func

$$= 4\int_0^a \left[x^2 y + \frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx.$$

$$= 4\int_0^a \left[\frac{b}{a} x^2 \sqrt{a^2-x^2} + \frac{b^3}{3a^3} (a^2-x^2)^{3/2}\right] dx$$

Let, $x = a\sin\theta$

$dx = a\cos\theta \, d\theta$

if $x = 0 \Rightarrow \theta = 0$

$x = a \Rightarrow \theta = \pi/2$

$$= 4\int_0^{\pi/2} \left[\frac{b}{a} a^3 \sin^2\theta \, d\theta + \frac{b^3}{3a^3} a^3 \cos^3\theta\right] a\cos\theta \, d\theta.$$

$$= 4\int_0^{\pi/2} ba^3 \sin^2\theta \cos^2\theta + \frac{b^3 a}{3} \cos^4\theta \,] d\theta.$$

$$\because \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \cdot \frac{\pi}{2}$$

$$\int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{(n-1)(n-3)\cdots}{n(n-2)\cdots} \cdot \frac{\pi}{2}..$$

$$= 4\left[a^3 b\left[\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] + \frac{ab^3}{3}\cdot\frac{3\cdot1}{4\cdot2}\cdot\frac{\pi}{2}\right].$$

$$= 4\left[\frac{a^3 b\pi}{16} + \frac{ab^3\pi}{16}\right]$$

$$= \cancel{4}\left[\frac{a^3 b\pi + ab^3\pi}{\cancel{16}\,4}\right]$$

$$= \frac{\pi}{4}\left[a^3 b + ab^3\right].$$

$$I = \frac{\pi}{4} ab(a^2+b^2) \text{ sq. units}.$$

Scanned by CamScanner

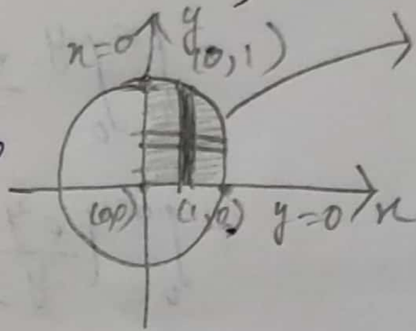
3.a. Using change of order of integration evaluate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy.$$

The given limits

$x \to 0$ to $\sqrt{1-y^2}$

$y \to 0$ to 1 (H.S)



$x = \sqrt{1-y^2}$

$x^2 + y^2 = 1$

$y^2 = 1 - x^2$

$y = \sqrt{1-x^2}$

By using change of order of integration

$y \to 0$ to $\sqrt{1-x^2}$

$x \to 0$ to 1 (V.S)

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} x^3 y \, dy \, dx$$

$$= \int_0^1 x^3 \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_0^1 x^3 (1-x^2) \, dx$$

$$= \frac{1}{2} \int_0^1 (x^3 - x^5) \, dx$$

$$= \frac{1}{2} \left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{4} - \frac{1}{6}\right]$$

$$= \frac{1}{\cancel{2}} \left[\frac{\cancel{2}}{24}\right]$$

$$= \frac{1}{24} \text{ Sq. units}$$

3.b

By using ~~order~~ change of order of integration evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$.

The given limits of horizontal strip are:-

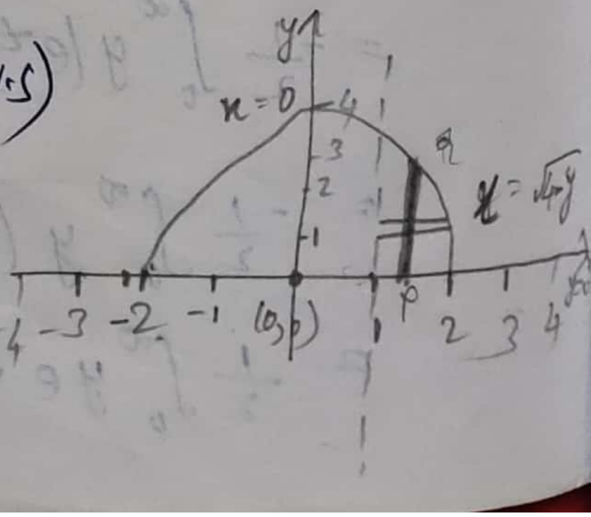
$x \rightarrow 1$ to $\sqrt{4-y}$

$y \rightarrow 0$ to 3 (H.S)

By ~~change~~ using change of order of integration

$y \rightarrow 0$ to $4-x^2$

$x \rightarrow 1$ to 2 (V.S)



$$I = \int_1^2 \int_0^{4-x^2} (x+y)\, dy\, dx$$

$$= \int_1^2 \left[\int_0^{4-x^2} (x+y)\, dy \right] dx.$$

$$= \int_1^2 \left(xy + \frac{y^2}{2} \right)_0^{4-x^2} dx$$

$$= \int_1^2 \left[x(4-x^2) + \frac{(4-x^2)^2}{2} \right] dx$$

$$= \frac{1}{2} \int_1^2 \left[2(4x - x^3) + (16 + x^4 - 8x^2) \right] dx$$

$$= \frac{1}{2} \left[\frac{8x^2}{2} - \frac{2x^4}{4} + 16x + \frac{x^5}{5} - \frac{8x^3}{3} \right]_1^2$$

$$= \frac{1}{2} \left[\frac{8}{2}(4) - \frac{2}{4}(2)^4 + 16(2) + \frac{(2)^5}{5} - \frac{8(2)^3}{3} - \left(\frac{8}{2} - \frac{2}{4} + 16 + \frac{1}{5} - \frac{8}{3} \right) \right]$$

$$= \frac{1}{2} \left[16 - 8 + 32 + \frac{32}{5} - \frac{64}{3} - \frac{8}{2} + \frac{1}{2} - 16 - \frac{1}{5} + \frac{8}{3} \right].$$

$$= \frac{1}{2} \left[24 + \frac{192 - 640 - 120 + 15 - 6 + 80}{30} \right]$$

$$= \frac{1}{2} \left[\frac{24 \times 30 - 479}{30} \right]$$

$$= \frac{1}{2} \left[\frac{720 - 479}{30} \right]$$

$$= \frac{241}{60}.$$

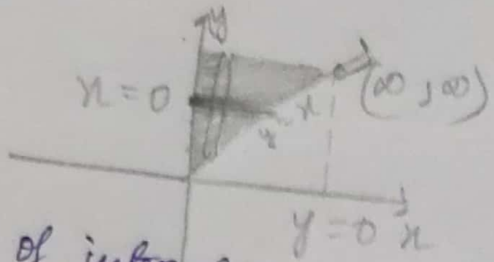
M

1b. By changing the order of integration, evaluate the integral $\int_0^\infty \int_0^\infty \frac{e^{-y}}{y} \, dx dy$.

The given limits :

$y \to x$ to ∞

$x \to 0$ to ∞ (V.S)



By using change of order of integration

$x \to 0$ to y

$y \to 0$ to ∞ (H.S)

$= \int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx dy$

$= \int_0^\infty \left(\frac{e^{-y}}{y}\right) (x)_0^y dy$

$= \int_0^\infty \frac{e^{-y}}{y} (y) dy$

$= \left(-e^{-y}\right)_0^\infty$

$= -\left(e^{-\infty} - e^{-0}\right)$

$= -(0-1)$

$= 1$ sq. unit.

Scanned by CamScanner

10/09/19

By using change of order of integration evaluate

$\int_0^{\infty}\int_0^{x} xe^{-x^2/y}\,dy\,dx$.

Sol:- The given limits of vertical strip are:

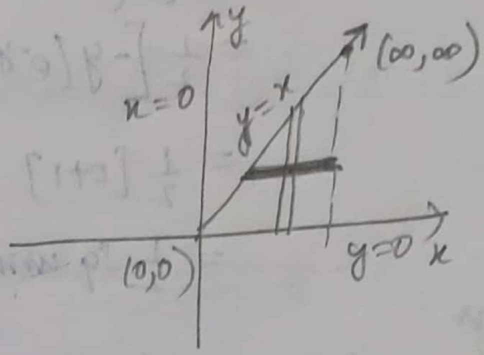
$y \to 0$ to x (V.S)

$x \to 0$ to ∞

By using change of order of integration

$x \to y$ to ∞ (H.S)

$y \to 0$ to ∞



$$I = \int_0^{\infty}\int_y^{\infty} x e^{-x^2/y}\,dx\,dy.$$

$$= \int_0^{\infty}\left[\int_y^{\infty} xe^{-x^2/y}\,dx\right]dy.$$

Let $\frac{x^2}{y} = t$

$2x\,dx = y\,dt$

$t \to y$ to ∞.

$$= \int_0^{\infty}\left(\int_y^{\infty} e^{-t}\,\frac{y\,dt}{2}\right)dy$$

$$= -\frac{1}{2}\int_0^{\infty} y\,(e^{-t})_y^{\infty}\,dy$$

$$= -\frac{1}{2}\int_0^{\infty} y\,(0-e^{-y})\,dy$$

$$= \frac{1}{2}\int_0^{\infty} ye^{-y}\,dy.$$

$$= \frac{1}{2}\left[\cancel{y(-e^{-y})} - \int (-e^{-y})\,dy\right]_0^\infty$$

$$= \frac{1}{2}\left[\infty - (e^{-y})_0^\infty\right]$$

$$= \cancel{\frac{1}{2}[\infty - \infty]}$$

$$= \frac{1}{2}\left[\infty - (e^{-\infty} - 1)\right]$$

$$=$$

$$= \frac{1}{2}\left[y(-e^{-y}) - \int(-e^{-y})\,dy\right]_0^\infty$$

$$\cancel{= \frac{1}{2}(-}$$

$$= \frac{1}{2}\left[-y(e^{-y}) - e^{-y}\right]_0^\infty$$

$$= \frac{1}{2}[0+1]$$

$= \frac{1}{2}$ sq unit.

OLAR

26. Show that $\iint_R r^2 \sin\theta \, dr\, d\theta = \frac{2a^3}{3}$ where R is the semicircle $r = 2a\cos\theta$, above the initial line.

Let $x = r\cos\theta$ and $y = r\sin\theta$ be the polar coordinate of the circle.

Given:- $r = 2a\cos\theta$.

$$r = 2a\left(\frac{x}{r}\right).$$

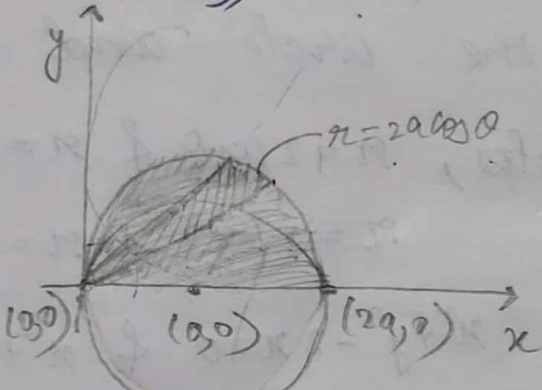
$$\boxed{\because \cos\theta = \frac{x}{r}}$$

$$r^2 = 2ax$$

$$\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ r^2 = x^2+y^2 \end{array}}$$

$$x^2+y^2-2ax = 0.$$

$$c(a,0),\ r = a.$$



$r \to 0$ to $2a\cos\theta$

$\theta \to 0$ to $\pi/2$

$$I = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2 \sin\theta \, dr\, d\theta.$$

$$= \int_0^{\pi/2} \left[\sin\theta \left(\frac{r^3}{3}\right)\right]_0^{2a\cos\theta} d\theta.$$

$$= \int_0^{\pi/2} \sin\theta \left(\frac{8a^3\cos^3\theta}{3}\right) d\theta$$

$$= \frac{8a^3}{3}\int_0^{\pi/2} \sin\theta \cos^3\theta \, d\theta.$$

$\cos\theta = t \Rightarrow -\sin\theta \, d\theta = dt$

$t \to 1$ to 0.

$= \cancel{\frac{8a^3}{3} \int_1^0 t \, dt}$

$= \frac{8a^3}{3} \int_1^0 t^3 (-dt)$

$= -\frac{8a^3}{3} \left[\frac{t^4}{4}\right]_1^0$

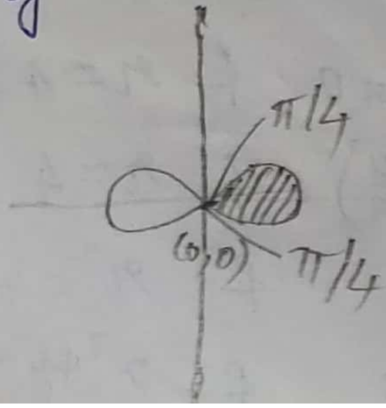
$= \frac{-8a^3}{3} \left[0 - \frac{1}{4}\right]$

$\boxed{I = \frac{2a^3}{3}}$

3a.

Evaluate $\iint_R \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$.

The equation of the lemniscate is $r^2 = a^2\cos 2\theta$.



$r \to 0$ to $a\sqrt{\cos 2\theta}$

$\theta \to -\pi/4$ to $\pi/4$.

$$I = \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{a^2+r^2}}\, dr\, d\theta .$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \frac{1}{2} \frac{2r}{\sqrt{a^2+r^2}}\, dr\, d\theta .$$

$$= \frac{1}{\cancel{2}} \int_{-\pi/4}^{\pi/4} \left(\cancel{2}\sqrt{a^2+r^2}\right)_0^{a\sqrt{\cos 2\theta}} d\theta .$$

$$= \int_{-\pi/4}^{\pi/4} \left(\sqrt{a^2+a^2\cos 2\theta} - \sqrt{a^2}\right) d\theta .$$

$$= \int_{-\pi/4}^{\pi/4} \left(\sqrt{a^2(2\cos\theta)} - a\right) d\theta .$$

$$= \int_{-\pi/4}^{\pi/4} \left(\sqrt{2}\, a\cos\theta - a\right) d\theta .$$

$$= a\left[\sqrt{2}\sin\theta - \theta\right]_{-\pi/4}^{\pi/4}$$

$$= a\left[\cancel{\sqrt{2}}\left(\frac{1}{\cancel{\sqrt{2}}}\right) - \frac{\pi}{4} - \left(-\cancel{\sqrt{2}}\left(\frac{1}{\cancel{\sqrt{2}}}\right) + \frac{\pi}{4}\right)\right]$$

$$= a\left[-\frac{\pi}{4} + 2 - \frac{\pi}{4}\right]$$

$$= a\left[2 - \frac{\pi}{2}\right].$$

Scanned by CamScanner

3

3b. Evaluate $\iint_R r^3\,dr\,d\theta$ over the area between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

Let $x = r\cos\theta$ & $y = r\sin\theta$ be the polar coordinates of the circle and $x^2+y^2=r^2$.

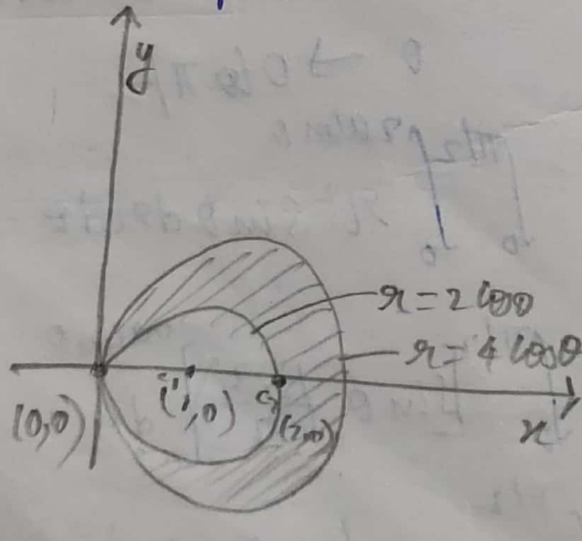
The given circles, $r = 2\cos\theta$ & $r = 4\cos\theta$.

$$r = 2\left(\frac{x}{r}\right) \;\&\; r = 4\left(\frac{x}{r}\right).$$

$$x^2+y^2-2x=0 \;\&\; x^2+y^2-4x=0.$$

$C_1(1,0)$ $\qquad$ $C_2(2,0)$

$r_1 = 1$ $\qquad$ $r_2 = 2$



$r \to 2\cos\theta$ to $4\cos\theta$

$\theta \to -\pi/2$ to $\pi/2$

Scanned by CamScanner

$$= \int_{-\pi/2}^{\pi/2} \int_{2\cos\theta}^{4\cos\theta} r^3 \, dr \, d\theta .$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{r^4}{4}\right)_{2\cos\theta}^{4\cos\theta} d\theta .$$

$$= \frac{1}{4} \int_{-\pi/2}^{\pi/2} \left(256\cos^4\theta - 16\cos^4\theta\right) d\theta .$$

$$= \frac{240}{4} \int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta .$$

$$\because \int_{-\pi/2}^{\pi/2} f(x)\,dx = f(-x) \text{ is even } = 2\int_0^{\pi/2} f(x)\,dx$$

$$= 60 \times 2 \int_0^{\pi/2} \cos^4\theta \, d\theta .$$

$$= 60 \times 2 \left[\frac{3 \times \cancel{2}}{4 \times \cancel{2}} \cdot \frac{\pi}{2}\right] .$$

$$= \cancel{120}^{\cancel{60}\,\cancel{30}\,15} \times \frac{3}{\cancel{8}_{\cancel{4}\,2}} \cdot \pi$$

$$= 45\pi .$$

$$= \cancel{60} \times 2 \left[\frac{3}{16} \cdot \pi\right]$$

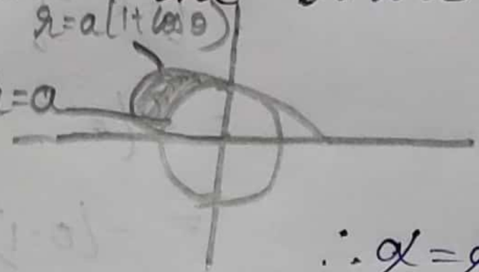
$$= \cancel{120}^{\cancel{60}\,\cancel{30}\,15} \times \frac{3}{\cancel{8}} \cdot \frac{\pi}{\cancel{2}_{\cancel{4}\,2}}$$

$$\boxed{I = \frac{45\pi}{2}} .$$

Scanned by CamScanner

TYPICAL QUESTIONS

1.a. Find the area lying inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$.



$r \rightarrow a$ to $a(1+\cos\theta)$

$\theta \rightarrow 0$ to π.

$\therefore a = a(1+\cos\theta)$

$1+\cos\theta = 0$.

$\cos\theta = -1$

$\theta = \cos^{-1}(-1)$

$\theta = \pi$.

$$= \int_0^{\pi} \int_a^{a(1+\cos\theta)}$$

$$= \iint_R r\,dr\,d\theta.$$

$$= \int_0^{\pi} \int_a^{a(1+\cos\theta)} r\,dr\,d\theta.$$

$$= \int_0^{\pi} \left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta.$$

$$= \int_0^{\pi} \left[\frac{[a(1+\cos\theta)]^2}{2} - \frac{a^2}{2}\right] d\theta.$$

$$= \int_0^{\pi} \frac{a^2}{2}\left[(1+\cos\theta)^2 - 1\right] d\theta.$$

$$= 2\int_0^{\pi/2} \frac{a^2}{2}\left[(1+\cos\theta)^2 - 1\right] d\theta.$$

$$= \frac{\cancel{2}a^2}{\cancel{2}}\int_0^{\pi/2} \left[(1+\cos\theta)^2 - 1\right] d\theta.$$

$$= a^2\int_0^{\pi/2} \left[\cancel{1}+\cos^2\theta + 2\cos\theta - \cancel{1}\right] d\theta.$$

$$= a^2\int_0^{\pi/2} (\cos^2\theta + 2\cos\theta)\, d\theta.$$

$$= a^2\left[\frac{1}{2}\cdot\frac{\pi}{2}\right] + 2a^2\left[\sin\theta\right]_0^{\pi/2}.$$

$$= \frac{a^2\pi}{4} + 2a^2[1].$$

$$= \frac{a^2\pi}{4} + 2a^2 \text{ sq. units.}$$

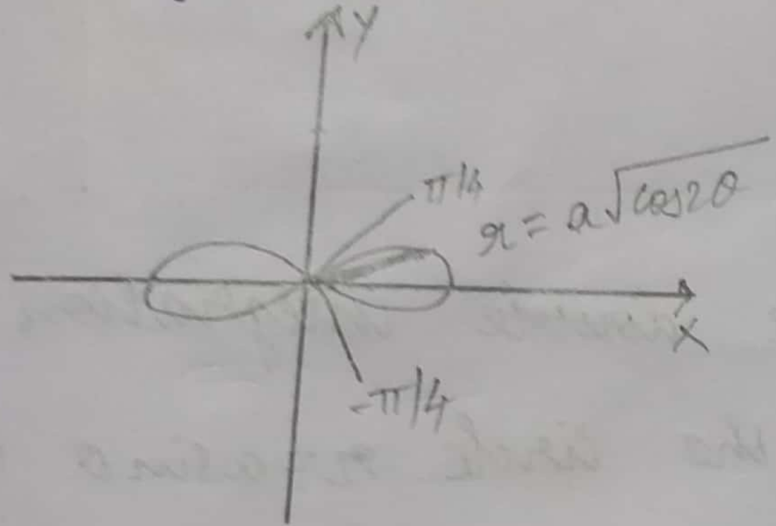
$$\cancel{= \frac{a^2}{4}[\pi + 8]}$$

3

16

Find by double integration area of Lemen scale.

The equation of the leminen scale $r^2 = a^2 \cos 2\theta$.



$r \to 0$ to $a\sqrt{\cos 2\theta}$

$\theta \to -\pi/4$ to $\pi/4$

$$\text{Area} = \iint_R r\,dr\,d\theta.$$

$$= 2\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta.$$

$$= 2\int_{-\pi/4}^{\pi/4}\left(\frac{r^2}{2}\right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

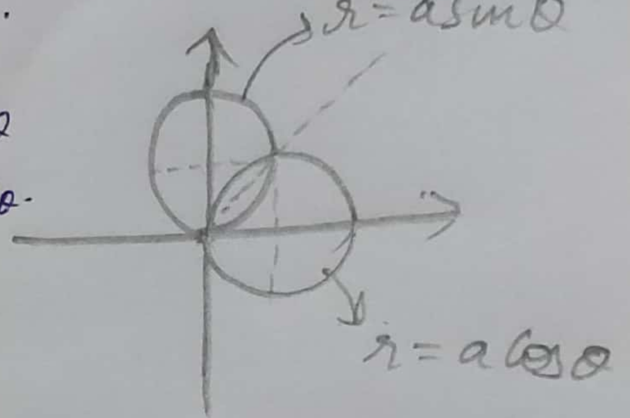
$$= \int_{-\pi/4}^{\pi/4} a^2 \cos 2\theta\, d\theta.$$

$$= a^2\left[\frac{\sin 2\theta}{2}\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{a^2}{2}[1+1]$$

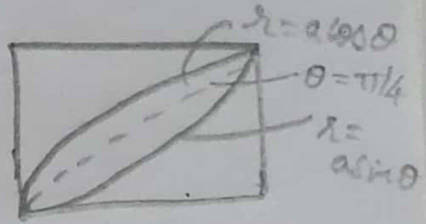
$$\boxed{I = a^2 \text{ sq units}}$$

2.a. Find the area common to the circles $r = a\cos\theta$ and $r = a\sin\theta$.



$$\text{Area} = \int_0^{\pi/4}\int_0^{a\sin\theta} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{a\cos\theta} r\,dr\,d\theta$$

$$\cancel{= \int_0^{\pi/4} \frac{a^2}{2}\sin^2\theta\,d\theta}$$



$$= \int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta + \int_{\pi/4}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta.$$

$$= \int_0^{\pi/4}\left[\frac{a^2\sin^2\theta}{2}\right] d\theta + \int_{\pi/4}^{\pi/2}\left[\frac{a^2\cos^2\theta}{2}\right] d\theta.$$

$$= \frac{a^2}{2}\int_0^{\pi/4}\left[\frac{1-\cos 2\theta}{2}\right] d\theta + \frac{a^2}{2}\int_{\pi/4}^{\pi/2}\left[\frac{1+\cos 2\theta}{2}\right] d\theta.$$

$$= \frac{a^2}{4}\int_0^{\pi/4}(1-\cos 2\theta)\,d\theta + \frac{a^2}{4}\int_{\pi/4}^{\pi/2}(1+\cos 2\theta)\,d\theta.$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4} + \frac{a^2}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{a^2}{4}\left[\frac{\pi}{4} - \frac{1}{2}\right] + \frac{a^2}{4}\left[\frac{\pi}{2} - \frac{1}{2} - \frac{\pi}{4}\right]$$

$$= \frac{a^2}{4}\left[\cancel{\frac{\pi}{4}} - \frac{1}{2} + \frac{\pi}{2} - \frac{1}{2} - \cancel{\frac{\pi}{4}}\right]$$

$$= \frac{a^2}{4}\left[\frac{\pi}{2} - \frac{2}{2}\right]$$

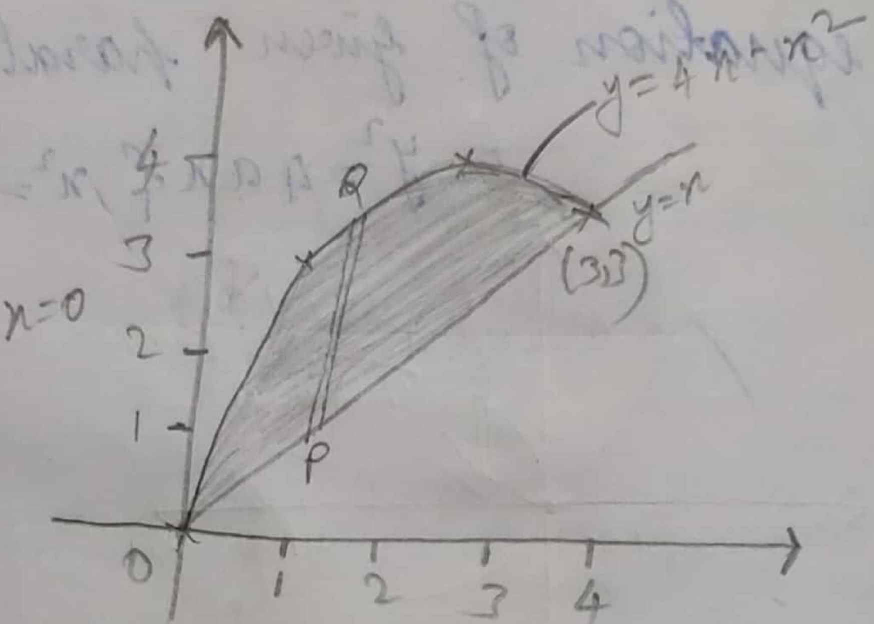
$$= \frac{a^2}{4}\left[\frac{\pi - 2}{2}\right] \text{ sq. units}$$

2.b. Find the area lying inside the cardioid $r = 1+\cos\theta$ and outside the parabola $r(1+\cos\theta) = 1$.

3.a

Find the area lying between the parabola $y=4x-x^2$ and the line $y=x$.

:- The given curves are $y=4x-x^2$ & $y=x$



$x = 4x-x^2$

$3x = x^2$

$x^2-3x=0$

$x=3$

x	0	1	2	3
y	0	3	4	3

$y \to x$ to $4x-x^2$

$x \to 0$ to 3

$$\text{Area} = \iint_R dx\,dy$$

$$= \int_0^3 \int_x^{4x-x^2} dy\,dx$$

$$= \int_0^3 \left[y \right]_x^{4x-x^2} dx$$

$$= \int_0^3 \left[4x - x^2 - x\right] dx$$

$$= \int_0^3 \left[3x - x^2\right] dx$$

$$= \left[3\frac{x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{27}{2} - 9$$

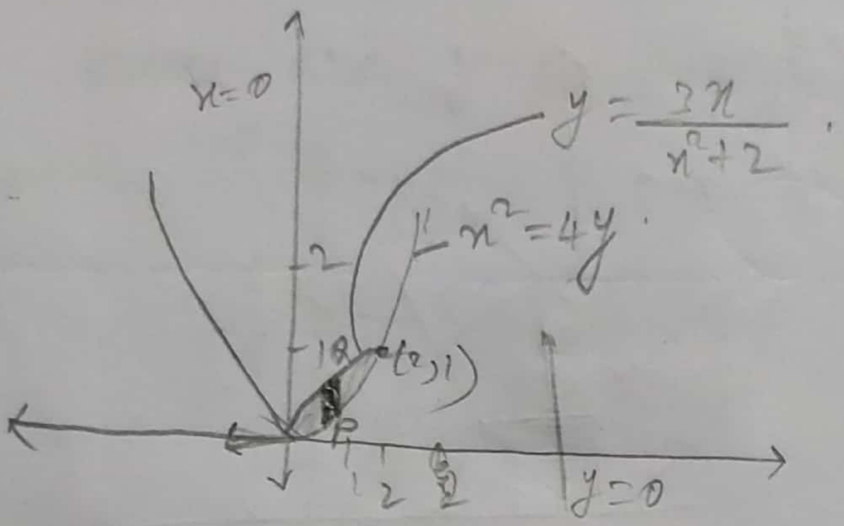
$$= \frac{27-18}{2}$$

$$\boxed{A = \frac{9}{2} \text{ sq unit.}}$$

Scanned by CamScanner

Find the area enclosed by the curves $y = 3x/x^2+2$ & $4y = x^2$.

The given curves are $y = \frac{3x}{x^2+2}$ & $x^2 = 4y$.



$x^2 = 4y$

$y=0 \Rightarrow x=0$

$y=1 \Rightarrow x=\pm 2$

$(0,0)$ $(2,1)$ $(-2,1)$

$y = \frac{3x}{x^2+2}$

$y=0 \Rightarrow x=0$

$y=1 \Rightarrow x^2+2-3x=0$

$x = 1, 2$

$(0,0)$ $(1,1)$ $(2,1)$

The intercepted points are $(0,0)$ $(2,1)$ ~~are common points~~.

$y \to \frac{x^2}{4}$ to $\frac{3x}{x^2+2}$

$x \to 0$ to 2

$$\text{Area} = \iint_R dx\,dy$$

$$= \int_0^2 \int_{x^2/4}^{3x/x^2+2} dy\,dx.$$

$$= \int_0^2 \left[y \right]_{x^2/4}^{3x/x^2+2} dx.$$

$$= \int_0^2 \left[\frac{3x}{x^2+2} - \frac{x^2}{4} \right] dx.$$

$$= \left[\frac{3}{2} \log |x^2+2| - \frac{x^3}{12} \right]_0^2$$

$$= \frac{3}{2} \log |6| - \frac{8}{12} - \left[\frac{3}{2} \log 2 \right]$$

$$= \frac{3}{2} \left[\log 6 - \log 2 \right] - \frac{2}{3}$$

$$= \frac{3}{2} \left[\log 3 \right] - \frac{2}{3}$$

$$\boxed{I = \frac{3}{2} \log 3 - \frac{2}{3}}$$