UNIT-V

SECTION-A

EASY TYPE

1. Evaluate $\int_0^1\int_1^2\int_2^3 x^2y^3z^2\,dx\,dy\,dz$.

$= \int_0^1\int_1^2 y^3z^2\left[\frac{x^3}{3}\right]_2^3 dy\,dz$.

$= \int_0^1\int_1^2 y^3z^2\left[\frac{27}{3} - \frac{8}{3}\right]dy\,dz$.

$= \int_0^1\int_1^2 y^3z^2\left[\frac{19}{3}\right]dy\,dz$.

$= \frac{19}{3}\int_0^1\int_1^2 y^3z^2\,dy\,dz$.

$= \frac{19}{3}\int_0^1 z^2\left[\frac{y^4}{4}\right]_1^2 dz$

$= \frac{19}{3}\int_0^1 z^2\left[\frac{16}{4} - \frac{1}{4}\right]dz$.

$= \frac{19}{3}\int_0^1 z^2\left[\frac{15}{4}\right]dz$

$= \frac{19}{\cancel{3}} \times \frac{\cancel{15}^{\,5}}{4}\int_0^1 z^2\,dz$

$= \frac{95}{4}\left[\frac{z^3}{3}\right]_0^1$.

$= \frac{95}{4}\left[\frac{1}{3}\right]$

$= \frac{95}{12}$.

2. Evaluate $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx$.

$= \int_0^1\int_0^1\left[(x^2+y^2)z + \frac{z^3}{3}\right]_0^1 dy\,dx$.

$= \int_0^1\int_0^1\left[x^2+y^2+\frac{1}{3}\right]dy\,dx$.

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} + \frac{1}{3}y\right]_0^1 dx.$$

$$= \int_0^1 \left[x^2 + \frac{1}{3} + \frac{1}{3}\right] dx.$$

$$= \int_0^1 \left[x^2 + \frac{2}{3}\right] dx.$$

$$= \left[\frac{x^3}{3} + \frac{2}{3}x\right]_0^1$$

$$= \frac{1}{3} + \frac{2}{3}$$

$$= \cancel{\frac{3}{3}}\,1$$

$$= 1.$$

3\. If $\int_0^K \int_1^2 \int_2^3 xyz\,dx\,dy\,dz = \frac{15}{8}$.

$$\frac{15}{8} = \int_0^K \int_1^2 \left[\frac{x^2}{2} yz\right]_2^3 dy\,dz$$

$$\frac{15}{8} = \int_0^K \int_1^2 \left[\frac{9}{2} yz - \frac{4}{2} yz\right] dy\,dz.$$

$$\frac{15}{8} = \int_0^K \int_1^2 \left[\frac{5}{2} yz\right] dy\,dz.$$

$$\frac{15}{8} = \frac{5}{2} \int_0^K \left[\frac{y^2}{2} z\right]_1^2 dz.$$

$$\frac{15}{8} = \frac{5}{2} \int_0^K \left[\frac{4}{2} z - \frac{1}{2} z\right] dz.$$

$$\frac{15}{8} = \frac{5}{2} \int_0^K \left[\frac{3}{2} z\right] dz$$

$$\frac{15}{8} = \frac{5}{2} \times \frac{3}{2} \int_0^K z\,dz$$

$$\frac{15}{8} = \frac{15}{4} \left[\frac{z^2}{2}\right]_0^K.$$

$$\frac{15}{8} = \frac{15}{4} \left[\frac{K^2}{2}\right]$$

$$\cancel{\frac{15}{8}} = \frac{\cancel{15} K^2}{\cancel{8}}$$

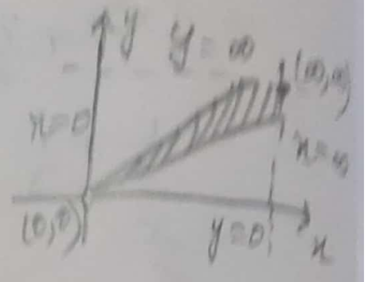
$$K^2 = 1$$

$$\boxed{K = \pm 1}$$

4. Transform the double integral $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ into the polar form.

The polar coordinates are $x = r\cos\theta,\ y = r\sin\theta$, $x^2+y^2 = r^2$

The given limits are $y \to 0$ to ∞
$x \to 0$ to ∞



The limits are
$r \to 0$ to ∞
$\theta \to 0$ to $\pi/2$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^1} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta.$$

5. Express the triple integral $\iiint_R f(x,y,z)\,dx\,dy\,dz$ in spherical polar coordinates.

The spherical co-ordinates are $x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$
$|J| = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta.$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R f[r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta]\, r^2\sin\theta\, dr\,d\theta\,d\phi$$

MODERATE TYPE

1. Evaluate $\int_0^2\int_1^z\int_0^{yz} xyz\,dx\,dy\,dz.$

$$= \int_0^2\int_1^z \left[\frac{x^2}{2}yz\right]_0^{yz} dy\,dz.$$

$$= \int_0^2\int_1^z \left[\frac{y^2z^2}{2}yz\right] dy\,dz$$

$$= \int_0^2\int_1^z \left[\frac{y^3z^3}{2}\right] dy\,dz.$$

$$= \int_0^2 \left[\frac{y^4}{8} z^3\right]_1^z dz.$$

$$= \frac{1}{8}\int_0^2 \left[z^4 \cdot z^3 - 1 \cdot z^3\right] dz.$$

$$= \frac{1}{8}\int_0^2 \left[z^7 - z^3\right] dz$$

$$= \frac{1}{8}\left[\frac{z^8}{8} - \frac{z^4}{4}\right]_0^2.$$

$$= \frac{1}{8}\left[\frac{128}{8} - \frac{16}{4}\right].$$

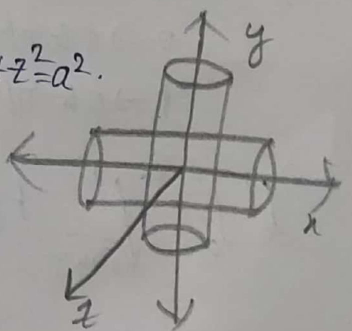
$$= \frac{1}{8}\left[\frac{128-32}{8}\right]$$

$$= \frac{\cancel{86}\,43}{\cancel{8}\,4}$$

$$= \frac{43}{4}.$$

2. The Volume Common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$ is given by the double integral $2\int_{K_1}^{K_2}\int_{f(x)}^{g(x)} z\,dx\,dy$, find $K_1, K_2, f(x)$ and $g(x)$. 1.

The given cylinders are $x^2+y^2=a^2$ & $x^2+z^2=a^2$.



Given,

$$2\int_{K1}^{K2}\int_{f(x)}^{g(x)} z\,dx\,dy$$

$y \to -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

$x \to -a$ to a.

$$= 2\int_{-a}^{a}\int_{-\sqrt{a^2-x}}^{\sqrt{a^2-x^2}} z\,dx\,d\,dy.$$

$\therefore$ $K_1 = -a$

$K_2 = a$

$f(x) = -\sqrt{a^2-x^2}$.

$g(x) = \sqrt{a^2-x^2}$

3. Transform the triple integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dx\,dy\,dz$ into spherical polar coordinates.

The spherical polar coordinates are $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$|J| = \dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R'} f[r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta]\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

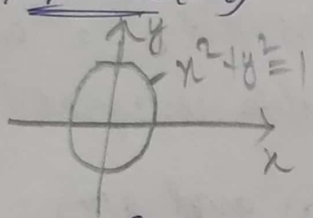
The given limits are

$z \to \sqrt{x^2+y^2}$ to 1

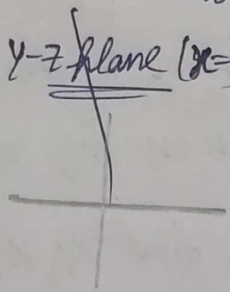
$y \to 0$ to $\sqrt{1-x^2}$

$x \to 0$ to 1

x-y plane ($z=0$)



y-z plane ($x=0$)



$r \to 0$ to 1

$\theta \to 0$ to $\pi/2$

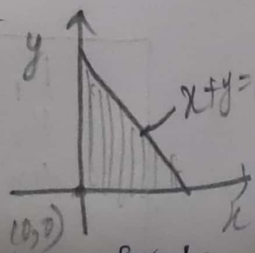
$\phi \to 0$ to $\pi/2$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dx\,dy\,dz = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 f(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

4. Using the transformation $x+y=u$, $y=uv$ transform the double integral $\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dy\,dx$ into uv-coordinate system.

The given limits are:- $y \to 0$ to $1-x$

$x \to 0$ to 1



If $y=0 \Rightarrow uv=0$, $u=0$ & $v=0$.

$y=1-x \Rightarrow x+y=1 \Rightarrow u=1$

$x=0 \Rightarrow x+uv=u \Rightarrow x=u-uv \Rightarrow u(1-v)=0 \Rightarrow u=0$ & $v=1$.

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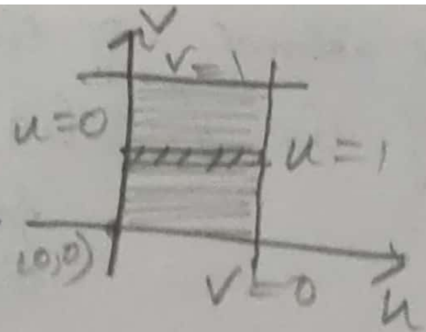
$u \to 0$ to 1

$v \to 0$ to 1



$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(\phi(u,v), \psi(u,v))\,|J|\,du\,dv.$$

$$\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}}\,dx\,dy = \int_0^1 \int_0^1 e^{\frac{uv}{u}}\,u\,du\,dv.$$

5. The volume bounded by the x-y plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$ is given by $\iint_R (3-x-y)\,dx\,dy$. Find x, y limits.

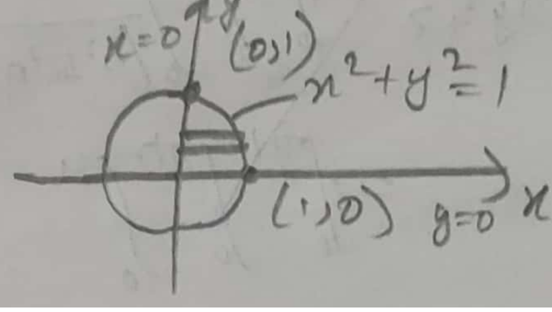
The given curves $x^2+y^2=1$ (circle)

$x+y+z=3$ (plane)

The limits are

$x \to 0$ to $\sqrt{1-y^2}$

$y \to 0$ to 1.



3) $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx.$

$= \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx.$

$x= \quad y= \quad z=$

$\cancel{= \int_0^a \int_0^x \int_0^{x+y}}$

$= \int_0^a \int_0^x \left(e^{x+y+z} \right)_0^{x+y} dy \, dx.$

$= \int_0^a \int_0^x \left(e^{2x+2y} - e^{x+y} \right) dy \, dx.$

$= \int_0^a \left(\frac{e^{2x+2y}}{2} - \frac{e^{x+y}}{1} \right)_0^x dx.$

$$= \int_0^a \left[\frac{e^{4x}}{2} - e^{2x} + \left(\frac{e^{2x}}{2} - e^x\right)\right] dx.$$

$$= \int_0^a \left(\frac{e^{4x}}{2} - \frac{3}{2}e^{2x} + e^x\right) dx.$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^x\right]_0^a.$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \left[\frac{1}{8} - \frac{3}{4} + 1\right].$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \left[\frac{3}{8}\right].$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}.$$

4. $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$

③ $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \cdot dx\,dy\,dz.$

$$= \int_1^e \int_1^{\log y} (z \log z - z)_1^{e^x} \, dx\,dy.$$

$$= \int_1^e \int_1^{\log y} (e^x \log e^x - e^x + 1) \, dx\,dy.$$

$$= \int_1^e \int_1^{\log y} (xe^x - e^x + 1) \, dx\,dy.$$

$$= \int_1^e (xe^x - e^x - e^x + x)_1^{\log y} \, dy.$$

$$= \int_1^e [y \log y - 2y + \log y - \cancel{e^1} + \cancel{e^1} + e^1 - 1] \, dy.$$

$$= \int_1^e [y \log y - 2y + \log y + e - 1] \, dy.$$

$$\boxed{\int y \log y = \log y\left(\frac{y^2}{2}\right) - \int \ldots \quad = \frac{y^2}{2}\log y - \frac{y^2}{4}}$$

$$= \left[\frac{y^2}{2}\log y - \frac{y^2}{4} - \frac{2y^2}{2} + y\log y - y + (e-1)y\right]_1^e.$$

$$= \left[\frac{e^2}{2}(1) - \frac{e^2}{4} - \cancel{e^2} + e\cancel{(1)} - \cancel{e} + \cancel{e^2} - e + \frac{1}{4} + 1 + 1 - e + 1\right].$$

$$= \frac{e^2}{2} - \frac{e^2}{4} - 2e + \frac{13}{4}.$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}.$$

1. Evaluate $\iiint_V \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ by changing into spherical co-ordinates, where V is the volume of the sphere $x^2+y^2+z^2=a^2$.

The given sphere is $x^2+y^2+z^2=a^2$.

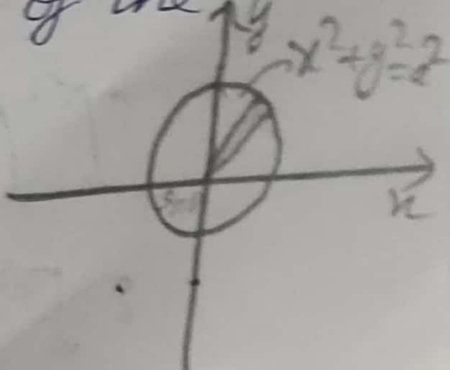
The ~~sphe~~ spherical co-ordinates of the sphere is

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$J = r^2\sin\theta$.



The volume of the sphere is 8 times the volume of the portion in the positive octant.

$r \to 0$ to a

$\theta \to 0$ to $\frac{\pi}{2}$.

$\phi \to 0$ to $\frac{\pi}{2}$.

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R'} f(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\, r^2\sin\theta \ldots$$

$$\iiint_V \frac{dx\,dy\,dz}{x^2+y^2+z^2} = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \frac{1}{(r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta)}\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \frac{1}{\left(r^2\sin^2\theta(\sin^2\phi+\cos^2\phi)\right)+r^2\cos^2\theta}\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \frac{1}{(r^2\sin^2\theta + r^2\cos^2\theta)}\, r^2\sin\theta\,dr\,d\theta\,d\phi.$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \frac{1}{\cancel{r^2}(\sin^2\theta+\cos^2\theta)}\,\cancel{r^2}\sin\theta\,dr\,d\theta\,d\phi.$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \sin\theta\,dr\,d\theta\,d\phi.$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left[\frac{r^2}{2}\right]_0^a d\theta\,d\phi.$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \left[\frac{a^2}{2}\right] d\theta\, d\phi$$

$$= \cancel{8}^{4}\left(\frac{a^2}{\cancel{2}}\right)\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\, d\theta\, d\phi$$

$$= 4a^2\int_0^{\pi/2} (-\cos\theta)_0^{\pi/2}\, d\phi$$

$$= 4a^2\int_0^{\pi/2} (-0+1)\, d\phi$$

$$= 4a^2\int_0^{\pi/2} d\phi$$

$$= 4a^2\,[\phi]_0^{\pi/2}$$

$$= \cancel{4}^{2}a^2\left[\frac{\pi}{\cancel{2}}\right]$$

$$= 2a^2\pi \text{ cubic-unit}$$

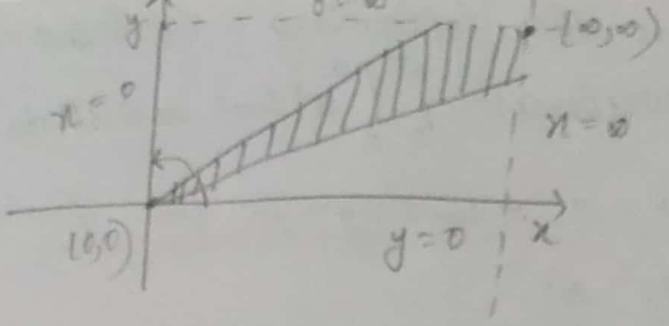
5. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$ by changing to polar co-ordinates and hence S.T $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

Let, the polar co-ordinates be

$$x = r\cos\theta,\ y = r\sin\theta \ \& \ x^2+y^2=r$$

The given limits

$$y \to 0 \text{ to } \infty$$
$$x \to 0 \text{ to } \infty$$



$r \to 0$ to ∞

$\theta \to 0$ to $\pi/2$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta .$$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta .$$

$r^2 = t \Rightarrow 2r\,dr = dt$

$t \to 0$ to ∞

$$= \int_0^{\pi/2}\int_0^\infty e^{-t}\frac{dt}{2}\,d\theta .$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(-e^{-t}\right)_0^\infty d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} (0-1)\,d\theta .$$

$$= \frac{1}{2}(\theta)_0^{\pi/2}$$

$$= \frac{\pi}{4} .$$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4} .$$

$y = x \Rightarrow dy = dx$

$$\left[\int_0^\infty e^{-x^2}\right]^2 = \frac{\pi}{4} .$$

$$\boxed{\int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2}.}$$

6. Evaluate $\iint_R xy\sqrt{1-x-y}\,dx\,dy$ region bounded by $x=0$, $y=0$ & $x+y=1$. using the transformation $x+y=u$, $y=uv$.

7. $\int_0^a$

MODERATE QUESTIONS

2. Find the Volume of the Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

The limits are $z \to 0$ to $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$y \to 0$ to $b\sqrt{1-\frac{x^2}{a^2}}$.

$x \to 0$ to a

$$= 8\iiint dx\,dy\,dz$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right] dy\,dx.$$

$$= \frac{8c}{b}\int_0^a \int_0^p \sqrt{p^2-y^2}\,dy\,dx$$

let $b\sqrt{1-\frac{x^2}{a^2}} = p$

$= \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$.

$= \sqrt{p^2-y^2}$.

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{p^2-y^2}+\frac{p^2}{2}\sin^{-1}\left(\frac{y}{p}\right)\right]_0^p dx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{p}{2}(0)+\frac{p^2}{2}\left[\frac{\pi}{2}\right]\right] dx$$

$$= \frac{8c\pi}{b4}\int_0^a [p^2]\,dx$$

$$= \frac{8c}{b}\cdot\frac{\pi}{4}\int_0^a \left[b^2\left(1-\frac{x^2}{a^2}\right)\right]dx.$$

$$= \frac{8c\pi}{b4}\int_0^a \left[b^2 - \frac{b^2x^2}{a^2}\right]dx$$

$$= \frac{8c\pi}{b4}\left[b^2x - \frac{b^2x^3}{a^2 3}\right]_0^a.$$

$$= \frac{8c\pi}{b4}\left[b^2a - \frac{b^2a^3}{a^2 3}\right].$$

$$= \frac{8c\pi}{b4}\left[b^2a - \frac{b^2 a}{3}\right].$$

$$= \frac{8c\pi}{b4}\left[\frac{3ba^2 - b^2a}{3}\right]$$

d.

$$= \frac{\overset{2}{\cancel{4}\cancel{8}}c\pi}{\cancel{b}\cancel{4}_2}\left[\frac{2b^2a}{3}\right]$$

$$= \frac{4abc\pi}{3}.$$

TYPICAL <u>QUESTIONS</u>

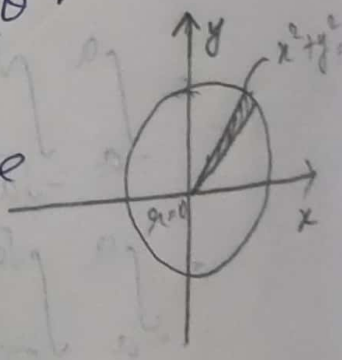
5. Find by the triple integration the volume of the sphere $x^2+y^2+z^2=a^2$.

The equation of the given sphere is $x^2+y^2+z^2=a^2$.

The spherical co-ordinates of the sphere is $x=r\sin$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$
$$J = r^2\sin\theta.$$

The volume of the sphere is 8 times the volume of the portion in ~~pos~~. the positive ~~octant~~ ~~octane~~ octant.



$$r \to 0 \text{ to } a$$
$$\theta \to 0 \text{ to } \pi/2$$
$$\phi \to 0 \text{ to } \pi/2$$

$$V = 8\iiint_R r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^2 \sin\theta \, dr\, d\theta\, d\phi.$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \left[\frac{r^3}{3}\right]_0^a d\theta\, d\phi.$$

$$= \frac{8a^3}{3}\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \, d\theta\, d\phi.$$

$$= \frac{8a^3}{3}\int_0^{\pi/2} \left[-\cos\theta\right]_0^{\pi/2} d\phi.$$

$$\cancel{= \frac{8a^3}{3}\int_0^{\pi/2}}$$

$$= -\frac{8a^3}{3}\int_0^{\pi/2} [-1]\, d\phi$$

$$= \frac{8a^3}{3}\int_0^{\pi/2} d\phi$$

$$= \frac{\overset{4}{\cancel{8}}a^3}{3}\left[\frac{\pi}{\cancel{2}}\right]$$

$$= \frac{4a^3\pi}{3} \text{ cubic units.}$$

... the cylinder

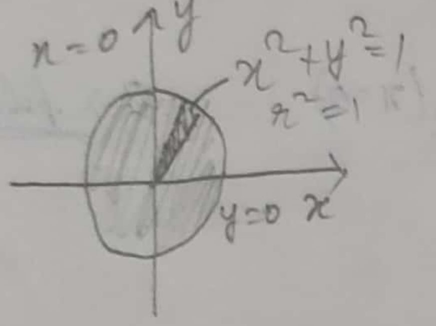
Q: Find the volume bounded by the x-y plane, to the cylinder $x^2+y^2=1$ & the plane $x+y+z=3$,

$z=0$.

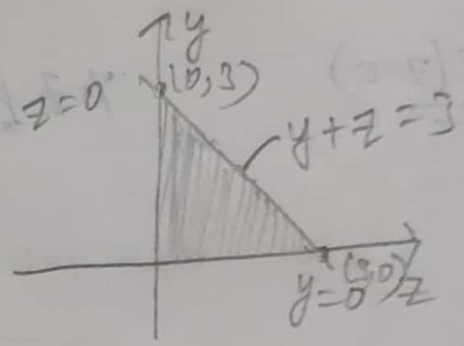
The given curves $x^2+y^2=1$ (circle)

$x+y+z=3$ (plane).

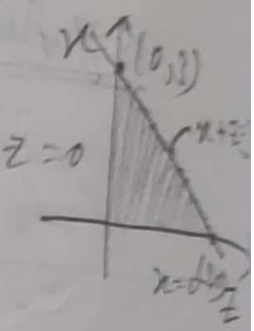
X-Y plane ($z=0$)



Y-Z plane ($x=0$)



Z-X plane ($y=0$)



$r \to 0$ to 1

$\theta \to 0$ to 2π

$z \to 0$ to $3-x-y$

$$V = \iiint dx\,dy\,dz$$

$$= \iint_{x^2+y^2=1} \int_0^{3-x-y} dz\,dx\,dy$$

$$= \iint_{x^2+y^2=1} \int_0^{3-x-y} dz\,dx\,dy.$$

$$= \iint_{x^2+y^2=1} (3-x-y)\,dx\,dy.$$

Let,

$x = r\cos\theta,\ y = r\sin\theta$ & $x^2+y^2=r^2$, $J=r$

$r \to 0$ to 1

$\theta \to 0$ to 2π

$$= \int_0^{2\pi}\int_0^1 (3 - r\cos\theta - r\sin\theta)\, r\,dr\,d\theta.$$

$$= \int_0^{2\pi}\int_0^1 \left(3r - r^2(\cos\theta + \sin\theta)\right) dr\,d\theta.$$

$$= \int_0^{2\pi} \left[3\frac{r^2}{2} - \frac{r^3}{3}(\cos\theta + \sin\theta)\right] d\theta.$$

$$= \int_0^{2\pi} \left[\frac{3}{2} - \frac{1}{3}(\cos\theta + \sin\theta)\right] d\theta.$$

$$= \left[\frac{3}{2}\theta - \frac{1}{3}(\sin\theta - \cos\theta)\right]_0^{2\pi}.$$

$$= \left[\frac{3}{2}(2\pi) - \frac{1}{3}\left[(0-1) - (0-1)\right]\right]$$

$$= 3\pi - \frac{1}{3}(-1+1)$$

$= 3\pi$ cubic units.

Find the Volume of the common to the cylinders $x^2+y^2=a^2$ & $x^2+z^2=a^2$.

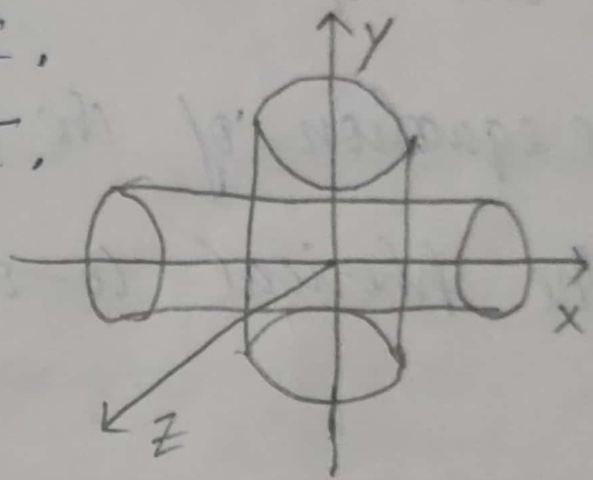
The given cylindricals are $x^2+y^2=a^2$, $x^2+z^2=a^2$

$$V = \iiint_R dx\,dy\,dz.$$

$z \rightarrow -\sqrt{a^2-x^2}$ to $+\sqrt{a^2-x^2}$.

$y \rightarrow -\sqrt{a^2-x^2}$ to $+\sqrt{a^2-x^2}$.

$x \rightarrow -a$ to a



$$\cancel{\iiint_{-a}^{a}}$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx.$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\right) dy\,dx.$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(2\sqrt{a^2-x^2}\right) dy\,dx.$$

$$= 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, dy\,dx.$$

$$= 2\int_{-a}^{a} \sqrt{a^2-x^2}\left(\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\right)dx.$$

$$= 4\int_{-a}^{a} (a^2-x^2)\,dx.$$

$$= 4\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}.$$

$$= 4\left[a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3}\right]$$

$$\cancel{= -\frac{8a^3}{3}}.$$

$$= 4\left[2a^3 - \frac{2a^3}{3}\right]$$

$$= 8\left[\frac{2a^3}{3}\right]$$

$$= \frac{16a^3}{3} \text{ cubic units.}$$

3. Find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ay$.

The cylindrical co-ordinates are:-

$$x = r\cos\phi,\ y = r\sin\phi,\ z = z \ \& \ J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

Given, $x^2+y^2+z^2=a^2$ (SPHERE)

$$r^2+z^2=a^2$$

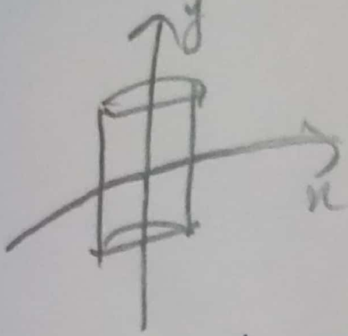
$$x^2+y^2=ay \text{ (CYLINDER)}$$

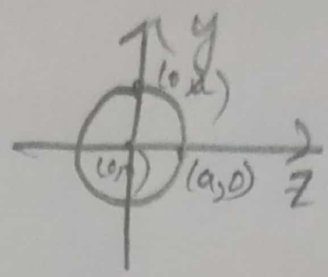
$$r^2=a(r\sin\phi)$$

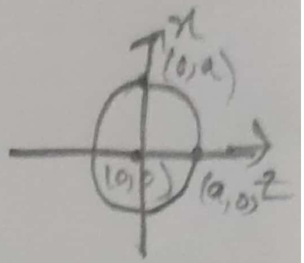
$$r=a\sin\phi$$

$\int_0 (1-\cos\phi)\ldots$

x-y plane (z=0) | y-z plane (x=0) | z-x plane (y=0)







$r \to 0$ to $a\sin\phi$ $\qquad$ $z \to 0$ to $\sqrt{a^2-r^2}$.

$\theta \to 0$ to π

$$= 2\int_0^{\pi}\int_0^{a\sin\phi}\int_0^{\sqrt{a^2-r^2}} r\cdot dz\cdot dr\, d\phi.$$

$$= 2\int_0^{\pi}\int_0^{a\sin\phi} r\,[z]_0^{\sqrt{a^2-r^2}}\, dr\cdot d\phi$$

$$= 2\int_0^{\pi}\int_0^{a\sin\phi} r\left[\sqrt{a^2-r^2}\right] dr\cdot d\phi$$

$$= 2\int_0^{\pi}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^{a\sin\phi} d\phi.$$

$$= \frac{2a^3}{3}\int_0^{\pi}(1-\cos^3\phi)\,d\phi$$

$$= \frac{2a^3}{3}\int_0^{\pi} 1\,d\phi - \frac{2a^3}{3}\,2\int_0^{\pi/2}\cos^3\phi\,d\phi.$$

$$= \frac{2a^3}{3}[\pi] - \frac{4a^3}{3}\left[\frac{2}{3}\cdot 1\right]$$

$$= \frac{2a^3}{3}[\pi] - \frac{8a^3}{9}.$$

$$= \frac{2a^3}{9}[3\pi-4].$$