# SECTION-B

## EASY

1) a) If $z(x+y) = x^2+y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

A) * $\boxed{\dfrac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2} = \frac{\partial}{\partial x}\left(\frac{u}{v}\right)}$ $\rightarrow \frac{\partial z}{\partial x} = \frac{(x+y)(2x) - x^2+y^2(1)}{(x+y)^2}$

$z = \frac{x^2+y^2}{x+y}$ $\rightarrow \frac{\partial z}{\partial x} = \frac{2x^2+2xy-x^2+y^2}{(x+y)^2} \rightarrow \frac{\partial z}{\partial x} = \frac{x^2-y^2+2xy}{(x+y)^2}$

$\boxed{L.H.S}$

$\rightarrow \frac{\partial z}{\partial y} = \frac{y^2-x^2+2xy}{(x+y)^2}$ $\rightarrow \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{x^2-y^2+2xy-y^2+x^2-2xy}{(x+y)^2}$

$\rightarrow \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2}$

$\boxed{R.H.S}$

$\rightarrow 4\left(1 - \left(\frac{x^2-y^2+2xy}{(x+y)^2}\right) - \left(\frac{y^2-x^2+2xy}{(x+y)^2}\right)\right) = 4\left(\frac{x^2+y^2+2xy - x^2+y^2-2xy - y^2+x^2-2xy}{(x+y)^2}\right)$

$= 4\left(\frac{x^2+y^2-2xy}{(x+y)^2}\right) = \frac{4(x-y)^2}{(x+y)^2}$

Hence, $\underline{L.H.S = R.H.S}$

b) let $r^2 = x^2+y^2+z^2$ and $v = r^m$, P.T $v_{xx}+v_{yy}+v_{zz} = m(m+1)r^{m-2}$

A) $r = \sqrt{x^2+y^2+z^2}$ $\Rightarrow \frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{x^2+y^2+z^2}}(2x)$

$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$

$v = r^m \rightarrow v_x = mr^{m-1}\frac{\partial r}{\partial x}$

$\Rightarrow v_x = mr^{m-1}\left(\frac{x}{r}\right)$

$\rightarrow v_x = \underline{mr^{m-2}x}$

$\Rightarrow v_{xx} = xm(m-2)r^{m-3}\left(\frac{\partial r}{\partial x}\right) + r^{m-2}(1)m$

$= m\left[x(m-2)r^{m-3}\left(\frac{x}{r}\right) + r^{m-2}\right]$

$v_{xx} = m(m-2)x^2r^{m-4} + mr^{m-2}$

$\rightarrow v_{yy} = m(m-2)y^2r^{m-4} + mr^{m-2}$

$\Rightarrow v_{zz} = m(m-2)z^2r^{m-4} + mr^{m-2}$

$\Rightarrow m(m-2)r^{m-4}(x^2+y^2+z^2) + 3mr^{m-2}$

$\Rightarrow m(m-2)r^{m-2} + 3mr^{m-2}$

$\Rightarrow mr^{m-2}(m+1)$

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a) If $u = \log(x^3+y^3+z^3-3xyz)$ S.T $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

let $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$ similarly, $\Rightarrow \frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$

$\Rightarrow \frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$\Rightarrow \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\left(\frac{3x^2-3yz+3z^2+3y^2-3xz-3xy}{x^3+y^3+z^3-3xyz}\right)$

$\boxed{\text{If } a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)}$

$\Rightarrow 3\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\left(\frac{1}{x+y+z}\right) = 3\left(\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right)$

$= 3\left[-1(x+y+z)^{-2}(1+0+0) + \ldots\right]$

$= 3\left[\frac{-1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2}\right] = \frac{-9}{(x+y+z)^2}$ //

b) If $v = x^y y^x$, prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v(x+y+\log v)$

$\log v = \log(x^y y^x)$

$\log v = \log x^y + \log y^x$

$\Rightarrow \log v = y\log x + x\log y$ — ①

$\Rightarrow$ Diff ① w.r.t 'x' partially, we get

$\frac{1}{v}\frac{\partial v}{\partial x} = y\left(\frac{1}{x}\right) + \log y(1)$

$\frac{\partial v}{\partial x} = v\cdot\frac{y}{x} + v\log y$

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Partially diff ① w.r.t y, we get

$$\frac{1}{v}\cdot\frac{\partial v}{\partial y} = 1(\log x) + x\frac{1}{y}$$

$$\frac{\partial v}{\partial y} = v\log x + v\cdot\frac{x}{y}$$

$$\Rightarrow x\cdot\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = x\left(v\cdot\frac{y}{x} + v\log y\right) + y\left(v\log x + v\cdot\frac{x}{y}\right)$$

$$= v\cdot y + vx\log y + vy\log x + vx$$

$$= v\left[x + y + (x\log y + y\log x)\right]$$

$$= v\left[x + y + \log v\right]$$

Hence L.H.S = R.H.S

3) a) If $u = \sin^{-1}(x-y)$, $x = 3t$ and $y = 4t^3$, S.T $\frac{\partial u}{\partial t} = \frac{3}{\sqrt{(1-t)^2}}$

A) $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{1}{\sqrt{1-(x-y)^2}}\cdot(3) - \frac{1}{\sqrt{1-(x-y)^2}}(12t^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}}$$

$$\frac{\partial u}{\partial t} = \frac{3\left[1-4t^2\right]}{\sqrt{1-16t^6+24t^4-9t^2}}$$

$$\frac{\partial x}{\partial t} = 3$$

$$\frac{\partial u}{\partial t} = \frac{3(1-4t^2)}{\sqrt{(1-t)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}$$

$$\frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-(x-y)^2}}$$

$$\frac{\partial y}{\partial t} = 12t^2$$

If $u = x^2+y^2+z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$
Find $\frac{du}{dt}$ as a total derivative

A) $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$

$\frac{\partial u}{\partial x} = 2x$ ; $\frac{\partial x}{\partial t} = 2e^{2t}$ ; $\frac{\partial u}{\partial y} = 2y$ ; $\frac{\partial y}{\partial t} = e^{2t}\cdot(-3\sin 3t) + \cos 3t(2e^{2t})$

$\frac{\partial y}{\partial t} = e^{2t}(-3\sin 3t + 2\cos 3t)$ ; $\frac{\partial z}{\partial t} = e^{2t}\,3\cos 3t + 2e^{2t}\sin 3t$

$\frac{\partial u}{\partial z} = 2z$

$\frac{\partial u}{\partial t} = 2x(2e^{2t}) + 2y\left(e^{2t}(2\cos 3t - 3\sin 3t)\right) + 2z\left(e^{2t}(3\cos 3t + 2\sin 3t)\right)$

$= 2(e^{2t})(2e^{2t}) + 2(e^{2t}\cos 3t)(2\cos 3t - 3\sin 3t)$
$\quad + 2(e^{2t}\sin 3t)(e^{2t}(3\cos 3t + 2\sin 3t))$

$= 4e^{4t} + 4e^{4t}\cos^2 3t - 6e^{4t}\sin 3t\cos 3t + 4e^{4t}\sin^2 3t$
$\quad + 6e^{4t}\cos 3t\sin 3t$

$$\boxed{\frac{\partial u}{\partial t} = 8e^{4t}}$$

⇒ MODERATE

1) By the substitution, $u = x^2 - y^2$, $v = 2xy$, $f(x,y) = \theta(u,v)$.
S.T $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2}\right)$

A) Now, $\frac{\partial f}{\partial x} = \frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial\theta}{\partial v}\cdot\frac{\partial v}{\partial x}$

$\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial\theta}{\partial u}(2x) + \frac{\partial\theta}{\partial v}(2y) \Rightarrow \frac{\partial}{\partial x} = \left[2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}\right]$

$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \left[2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}\right]\left[2x\frac{\partial\theta}{\partial u} + 2y\frac{\partial\theta}{\partial v}\right]$

$$\frac{\partial^2 f}{\partial x^2} = 4x^2\frac{\partial^2 \theta}{\partial u^2} + 4xy\frac{\partial^2 \theta}{\partial u \partial v} + 4xy\frac{\partial^2 \theta}{\partial u \partial v} + 4y^2\frac{\partial^2 \theta}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial x^2} = 4x^2\frac{\partial^2 \theta}{\partial u^2} + 8xy\frac{\partial^2 \theta}{\partial u \partial v} + 4y^2\frac{\partial^2 \theta}{\partial v^2} \quad — (1)$$

Similarly, $\frac{\partial f}{\partial y} = \frac{\partial \theta}{\partial y}$

$$\frac{\partial f}{\partial y} = \frac{\partial \theta}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \theta}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial \theta}{\partial u}\cdot(-2y) + \frac{\partial \theta}{\partial v}(2x)$$

$$\Rightarrow \frac{\partial}{\partial y} = -2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v} \quad — (2) \qquad \rightarrow \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]$$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = \left[-2y\cdot\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}\right]\left[-2y\frac{\partial \theta}{\partial u} + 2x\frac{\partial \theta}{\partial v}\right]$$

$$\rightarrow \frac{\partial^2 f}{\partial y^2} = 4y^2\frac{\partial^2 \theta}{\partial u^2} - 4xy\frac{\partial^2 \theta}{\partial u \partial v} - 4xy\frac{\partial^2 \theta}{\partial u \partial v} + 4x^2\frac{\partial^2 \theta}{\partial v^2}$$

$$\rightarrow \frac{\partial^2 f}{\partial y^2} = 4x^2\frac{\partial^2 \theta}{\partial v^2} - 8xy\frac{\partial^2 \theta}{\partial u \partial v} + 4y^2\frac{\partial^2 \theta}{\partial u^2} \quad — (2)$$

$\because$ L.H.S

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4x^2\frac{\partial^2 \theta}{\partial u^2} + \cancel{\frac{8xy\,\partial^2 \theta}{\partial u \partial v}} + 4y^2\frac{\partial^2 \theta}{\partial v^2} + 4x^2\frac{\partial^2 \theta}{\partial v^2} - \cancel{8xy\frac{\partial^2 \theta}{\partial u \partial v}} + 4y^2\frac{\partial^2 \theta}{\partial v^2}$$

$$= 4(x^2+y^2)\left[\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right]$$

L.H.S = R.H.S

Hence Proved

Q) 1) If $x = r\cos\theta,\ y = r\sin\theta$, evaluate $\frac{\partial(r,\theta)}{\partial(x,y)}, \frac{\partial(x,y)}{\partial(r,\theta)}$ S.T $\frac{\partial(r,\theta)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = 1$

A) $\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$

Given

$\Rightarrow x = r\cos\theta \Rightarrow y = r\sin\theta$

$\Rightarrow \cos\theta = \frac{x}{r} \Rightarrow \sin\theta = \frac{y}{r}$

$\Rightarrow r = \sqrt{x^2+y^2}$

$\rightarrow \frac{\partial r}{\partial x} = (x^2+y^2)^{1/2} = \frac{1}{2\sqrt{x^2+y^2}}(2x) \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

$\rightarrow$ Given, $x = r\cos\theta,\ y = r\sin\theta$

$\frac{y}{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\frac{y}{x}$ //

* $\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)$ * $\frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$

* $\frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2}$ * $\frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2}$

$\rightarrow \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}} + \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$

$\Rightarrow \frac{x^2+y^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{1}{r}$

$\rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r$ //

$\rightarrow \frac{1}{r} \cdot r = 1$ // Hence Proved.

b) a) If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ where $x = r\cos\theta$, $y = r\sin\theta$,

S.T $\frac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$

A) $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$u,v \to x, y \to r,\theta$ $\quad = \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} \cdot \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= (-4xy + 16xy)(r)$

$= 12xyr = 12(r\cos\theta)(r\sin\theta)r$

$\rightarrow 12r^3 \sin\theta\cos\theta = 6r^3 \sin 2\theta$

3) a) If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

A) $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$

$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$

$yz(2y-2z) - xz(2x-2z) + xy(2x-2y)$

$2y^2z - 2yz^2 - 2x^2z + 2xz^2 + 2x^2y - 2xy^2$

$= 2(y^2z - yz^2 - x^2z + xz^2 + x^2y - xy^2)$

$\Rightarrow \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{1}{2(y^2z - yz^2 - x^2z + xz^2 + x^2y - xy^2)}$

Q) If $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$. Are u and functionally related? If so, find the relation

A) If $J\left(\dfrac{u,v}{x,y}\right) = 0$, then functionally dependant

$\dfrac{\partial u}{\partial x} = \dfrac{1-xy(1) - x+y(-y)}{(1-xy)^2} = \dfrac{1-xy+xy+y^2}{(1-xy)^2} = \dfrac{1+y^2}{1-xy}$

$\dfrac{\partial u}{\partial y} = \dfrac{1+x^2}{1-x^2}$ ; $\dfrac{\partial v}{\partial x} = \dfrac{1}{1+x^2}$ ; $\dfrac{\partial v}{\partial y} = \dfrac{1}{1+y^2}$

$\Rightarrow \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0$ * So functionally dependant

$\rightarrow$ Relation $\rightarrow u = \dfrac{x+y}{1-xy}$ ; $v = \tan^{-1}x + \tan^{-1}y$

$\rightarrow \boxed{v = \tan^{-1}u}$ $\qquad = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) - \tan^{-1}x$

$6xy^2 - 12x^2y^2 - 3xy^3$ TYPICAL $(x^3y - 8x^3y - 9x^3y^2)$

① Discuss maxima & minima of $f(x,y) = x^3y^2(1-x-y)$

A) $f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

$\Rightarrow \frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$\Rightarrow x^2y^2(3 - 4x - 3y)$

$\boxed{x=0, y=0}$ ; $\boxed{4x+3y=3}$

$\rightarrow \frac{\partial f}{\partial y} = x^3 2y - 2yx^4 - 3y^2x^3 = 0$

$\Rightarrow x^3y(2 - 2x - 3y) = 0$ — ③

$\boxed{x=0, y=0} \Rightarrow \boxed{2-2x-3y}$

Solve, $4x + 3y = 3$
$2x + 3y = 2$

$2x = 1$

$x = \frac{1}{2}$ , $y = \frac{1}{3}$

∴ $\boxed{(x,y) = \left(\frac{1}{2}, \frac{1}{3}\right)}$

Now, partially diff ② & ③ w.r.t $x$ & $y$

$\frac{\partial^2 f}{\partial x^2} = r = 6xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(1 - 2x - y)$

$\frac{\partial^2 f}{\partial y^2} = t = 2x^3(1 - x - 3y)$ ; $\frac{\partial^2 f}{\partial x \partial y} = s = x^2y(6 - 8x - 9y)$

Now, at point $\left(\frac{1}{2}, \frac{1}{3}\right) \Rightarrow r = \frac{-1}{9}$, $t = \frac{-1}{8}$, $s = \frac{-1}{12}$ $\Rightarrow \boxed{rt - s^2 = \frac{1}{144}}$

as $rt - s^2 > 0$, check $r = \frac{-1}{9} < 0$

so, $\left(\frac{1}{2}, \frac{1}{3}\right)$ is point of maxima //

b) Discuss the maxima & minima of $f(x,y) = x^3 + y^3 - 3axy$.

A) $\frac{\partial f}{\partial x} = 3x^2 - 3ay = 0$ ; $\frac{\partial f}{\partial y} = 3y^2 - 3ax = 0$

$x^2 = ay$     $y^2 = ax$     $\Rightarrow (x,y) = (0,0)$ & $(a,a)$

$y = \frac{x^2}{a}$     $\Rightarrow \left(\frac{x^2}{a}\right)^2 = ax$

Now, partially diff eq ② & ③ further

$\frac{\partial^2 f}{\partial x^2} = \boxed{6x = r} \Rightarrow \frac{\partial^2 f}{\partial y^2} = \boxed{t = 6y}$

$\rightarrow \frac{\partial^2 f}{\partial x \partial y} = \boxed{s = -3a}$ Now, $rt - s^2 = 36xy - 9a^2$

$\frac{x^4}{a^2} = ax$

$x^4 = a^3x$

$x^3 = a^3$

$x = a$

∴ At Point $(0,0)$, $rt - s^2 = -9a^2 < 0$

as $rt - s^2 < 0$, saddle point $= (0,0)$

$\Rightarrow$ At point $(a,a)$

$rt - s^2 = 36(a)(a) - 9a^2 = 27a^2 > 0$

as $rt - s^2 > 0$, check $\boxed{r}$

$r = 6x = 6a > 0 \Rightarrow r > 0$ Point of minima

∴ $(a,a)$ is point of minima

9) Find the volume of the greatest rectangular parallelopiped that can be inscribed in ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Given, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$

Here, $f(x,y,z)=8xyz$ [objective function]

$\phi(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$ [constrained function]

$\Rightarrow \boxed{F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)}$

$\Rightarrow F(x,y,z)=8xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$

$\frac{\partial F}{\partial x}=0 \Rightarrow 8yz+\lambda\left(\frac{2x}{a^2}\right)=0 \Rightarrow \boxed{\frac{4yza^2}{x}=-\lambda}$

Similarly $\Rightarrow \frac{\partial F}{\partial y}=0 \Rightarrow \boxed{\frac{4zxb^2}{y}=-\lambda}$ $\Rightarrow \frac{\partial F}{\partial z}=0 \Rightarrow \boxed{\frac{4xyc^2}{z}=-\lambda}$

$\Rightarrow \frac{4yz}{x}\cdot a^2=\frac{4zx}{y}\cdot b^2=\frac{4xy}{z}c^2=-\lambda$

$\Rightarrow \frac{y}{x}\cdot a^2=\frac{x}{y}\cdot b^2$ $\Rightarrow \frac{z}{y}b^2=\frac{y}{z}c^2$

$\Rightarrow y^2a^2=x^2b^2$ $\Rightarrow \frac{z^2}{c^2}=\frac{y^2}{b^2}$

$\Rightarrow \frac{y^2}{b^2}=\frac{x^2}{a^2}$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$\Rightarrow \frac{3x^2}{a^2}=1$

$\Rightarrow x=\frac{a}{\sqrt{3}}$

$\Rightarrow y=\frac{b}{\sqrt{3}}$

$\Rightarrow z=\frac{c}{\sqrt{3}}$

$\boxed{\text{Required volume}=8xyz=\frac{8abc}{3\sqrt{3}}}$

3) a) Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq.cm.

A) Let x, y, z be the dimensions of rectangular box.

Volume = $V = xyz$; Total surface area $= 2xy + 2yz + 2zx$

but box is open at top

$\Rightarrow xy + 2yz + 2zx = 432$

$F = xyz + \lambda(xy + 2yz + 2zx - 432)$

$F_x = 0 \Rightarrow yz + \lambda(y + 2z) = 0 \Rightarrow yz = -\lambda(y + 2z)$

$F_y = 0 \Rightarrow xz + \lambda(x + 2z) = 0 \Rightarrow xz = -\lambda(x + 2z)$

$F_z = 0 \Rightarrow xy + \lambda(2y + 2x) = 0 \Rightarrow xy = -\lambda(2x + 2y)$

$\Rightarrow \frac{y}{-\lambda} = \frac{y+2z}{x+2z} \Rightarrow x = y \Rightarrow \frac{z}{y} = \frac{x+2z}{2x+2y} \Rightarrow y = 2z$

So, $xy + 2xz + 2zy = 432 \Rightarrow 4z^2 + 4z^2 + 4z^2 = 432$

$\Rightarrow z = 6 \Rightarrow x = 12, y = 12$ The dimensions are (12, 12, 6)

b) The sum of 3 no's is constant. P.T their product is max when they are equal.

A) Given $x + y + z = b \Rightarrow x + y + z - b = 0$ & product is $\boxed{P = xyz}$ should be max

Here, objective function :- $f(x,y,z) = xyz$

constrained function :- $\phi(x,y,z) = x + y + z - b$

$\boxed{F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)} \Rightarrow F(x,y,z) = xyz + \lambda(x + y + z - b)$

$\frac{\partial F}{\partial x} = yz + \lambda = 0$ $\quad \frac{\partial F}{\partial y} = xz + \lambda = 0$ $\quad \frac{\partial F}{\partial z} = xy + \lambda = 0$

$\lambda = -yz \qquad \lambda = -xz \qquad \lambda = -xy$

$\Rightarrow x = y = z \qquad x + x + x = b$ so $\left(\frac{b}{3}, \frac{b}{3}, \frac{b}{3}\right) \Rightarrow x = \frac{b}{3} = y = z$

max value $= b^3/27$