(UNIT - 1) (Section B)

Typical Questions.

(1) Given $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$$(A - \lambda I) = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$(6-\lambda)((3-\lambda)(3-\lambda)-(-1)(-1)) - (-2)((-2)(3-\lambda)-(-1)(2))$
$+ 2((-2)\times(-1) - (3-\lambda)(2)) = 0$

$\Rightarrow (6-\lambda)((9-6\lambda+\lambda^2)-1) + 2((-6+2\lambda)-(-2)) + 2(2-(6-2\lambda)) = 0$

$\Rightarrow (6-\lambda)(8-6\lambda+\lambda^2) + 2(-4+2\lambda) + 2(-4+2\lambda) = 0$

$\therefore (48-44\lambda+12\lambda^2-\lambda^3) + (-8+4\lambda) + (-8+4\lambda) = 0$

$\Rightarrow (-\lambda^3+12\lambda^2-36\lambda+32) = 0$

$-(\lambda-2)(\lambda-2)(\lambda-8) = 0$

$\therefore (\lambda-2) = 0$ (or) $(\lambda-2) = 0$ (or) $(\lambda-8) = 0$

$\therefore$ The Eigen values of the matrix A are by $\lambda = 2, 8$

For $\lambda = 8$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_3 = t$, $v_1 = 2t$, $v_2 = -t$

$$v = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} t$$

For $\lambda = 2$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_2 = t$, $v_3 = s$ then $v_1 = -\frac{s}{2} + \frac{t}{2}$, $v_2 = t$, $v_3 = s$

$$\begin{bmatrix} \frac{s}{2} + \frac{t}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} t = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} s$$

The Egen vectors compose the columns of matrix P

$$\therefore P = \begin{bmatrix} 1/2 & -1/2 & 2 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The Diagonal matrix of Eigen values

$$\therefore D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Now P^{-1}

$$|P| = \begin{bmatrix} 1/2 & -1/2 & 2 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$1/2 \times \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} + \frac{1}{2} \times \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times (0 \times 1 - (-1) \times 1) + \frac{1}{2} \times (1 \times 1 - (-1) \times 0) + 2 \times (1 \times 1 - 0 \times 0)$$

$$= \frac{1}{2} \times (0 - 1) + \frac{1}{2} \times (1 + 0) + 2 \times (1 - 0)$$

$$= \frac{1}{2} \times (1) + \frac{1}{2} \times (1) + 2 \times (1)$$

$$= \frac{1}{2} + \frac{1}{2} + 2$$

$$= 3$$

$$V = \begin{bmatrix} -\frac{s}{2} + \frac{t}{2} \\ t \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} t = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} s$$

The eigen vectors compose the columns of matrix P

$$\therefore P = \begin{bmatrix} 1/2 & -1/2 & 2 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The Diagonal matrix of eigen values

$$\therefore D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Now P^{-1}

$$|P| = \begin{bmatrix} 1/2 & -1/2 & 2 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$1/2 \times \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} + \frac{1}{2} \times \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times (0 \times 1 - (-1) \times 1) + \frac{1}{2} \times (1 \times 1 - (-1) \times 0) + 2 \times (1 \times 1 - 0 \times 0)$$

$$= \frac{1}{2} \times (0 - 1) + \frac{1}{2} \times (1 + 0) + 2 \times (1 - 0)$$

$$= \frac{1}{2} \times (1) + \frac{1}{2} \times (1) + 2 \times (1)$$

$$= \frac{1}{2} + \frac{1}{2} + 2$$

$$= 3$$

$$\operatorname{Adj} P = \operatorname{Adj}\begin{bmatrix} 1/2 & -1/2 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} +\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} -\frac{1}{2} & 2 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} \frac{1}{2} & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{vmatrix} \\ +\begin{vmatrix} -\frac{1}{2} & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & 2 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(0\times 1-(-1)\times 1) & -(1\times 1-(-1)\times 0) & +(1+1-0\times 0)\times \\ -(-1/2\times 1-2\times 1) & +(\frac{1}{2}\times 1-2\times 0 & -(\frac{1}{2}\times 1-(-\frac{1}{2})\times 0) \\ +(-1/2\times(-1)-2\times 0 & -(\frac{1}{2}\times(-1)-2\times 1) & +(\frac{1}{2}\times 0-(-\frac{1}{2})\times 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} +(0+1) & -(1+0) & +(1+0) \\ -(-1/2-2) & +(1/2+0) & -(\frac{1}{2}+0) \\ +(1/2+0) & -(-1/2-2) & +(0+\frac{1}{2}) \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} & \frac{1}{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 5/2 & 1/2 \\ -1 & 1/2 & 5/2 \\ 1 & -1/2 & 1/2 \end{bmatrix}$$

Now, $P^{-1} = \frac{1}{|P|} \times Adj(P)$

$$= \frac{1}{3} \times \begin{bmatrix} 1 & \frac{5}{2} & 1/2 \\ -1 & 1/2 & 5/2 \\ 1 & -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 5/6 & 1/6 \\ -1/3 & 1/6 & 5/6 \\ 1/3 & -1/6 & 1/6 \end{bmatrix}$$

Now verify that $A = PDP^{-1}$

$$P \times D = \begin{bmatrix} \frac{1}{2} & -1/2 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \times 2 - \frac{1}{2} \times 0 + 2 \times 0 & \frac{1}{2} \times 0 - \frac{1}{2} \times 2 \times 0 & 1/2 \times 0 - \frac{1}{2} \times 0 + 2 \times 8 \\ 1 \times 2 + 0 \times 0 - 1 \times 0 & 1 \times 0 + 0 \times 2 - 1 \times 0 & 1 \times 0 + 0 \times 0 - 1 \times 8 \\ 0 \times 2 + 1 \times 0 + 1 \times 0 & 0 \times 0 + 1 \times 2 + 1 \times 0 & 0 \times 0 + 1 \times 0 + 1 \times 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0-1+0 & 0+0+16 \\ 2+0+0 & 0+0+0 & 0+0-8 \\ 0+0+0 & 0+2+0 & 0+0+8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 16 \\ 2 & 0 & -8 \\ 0 & 2 & 8 \end{bmatrix}$$

$$\therefore (P \times D) \times (P^{-1}) = \begin{bmatrix} 1 & -1 & 16 \\ 2 & 0 & -8 \\ 0 & 2 & 8 \end{bmatrix} \times \begin{bmatrix} \frac{1}{3} & \frac{5}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times \frac{1}{3} - 1 \times -\frac{1}{3} + 16 \times \frac{1}{3} & 1 \times \frac{5}{6} - 1 \times \frac{1}{6} + 16 \times -\frac{1}{6} & 1 \times \frac{1}{6} - 1 \times \frac{5}{6} + 16 \times \frac{1}{6} \\ 2 \times \frac{1}{3} + 0 \times -\frac{1}{3} - 8 \times \frac{1}{3} & 2 \times \frac{5}{6} + 0 \times \frac{1}{6} - 8 \times -\frac{1}{6} & 2 \times \frac{1}{6} + 0 \times \frac{5}{6} - 8 \times \frac{1}{6} \\ 0 \times \frac{1}{3} + 2 \times -\frac{1}{3} + 8 \times \frac{1}{3} & 0 \times \frac{5}{6} + 2 \times \frac{1}{6} + 8 \times -\frac{1}{6} & 0 \times \frac{1}{6} + 2 \times \frac{5}{6} + 8 \times \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} + \frac{1}{3} + \frac{16}{3} & \frac{5}{6} - \frac{1}{6} - \frac{8}{3} & \frac{1}{6} - \frac{5}{6} + \frac{8}{3} \\ \frac{2}{3} + 0 - \frac{8}{3} & \frac{5}{3} + 0 + \frac{4}{3} & \frac{1}{3} + 0 - \frac{4}{3} \\ 0 - \frac{2}{3} + \frac{8}{3} & 0 + \frac{1}{3} - \frac{4}{3} & 0 + \frac{5}{3} + \frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\therefore PDP^{-1} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} //$$