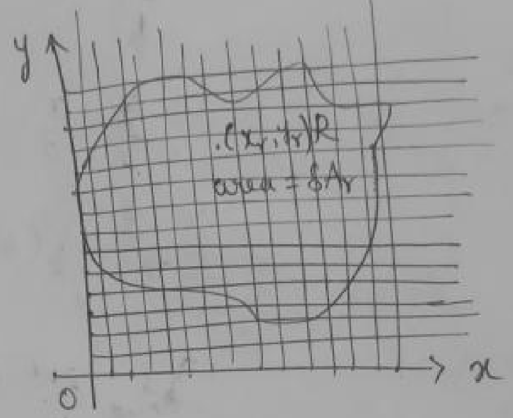
Double Integrals:

Consider a function $f(x,y)$ of the independent Variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, ---, \delta A_n$. Let (x_r, y_r) be any point within the r^{th} elementary areas δA_r.



Consider the Sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + --- + f(x_n, y_n)\,\delta A_n$$

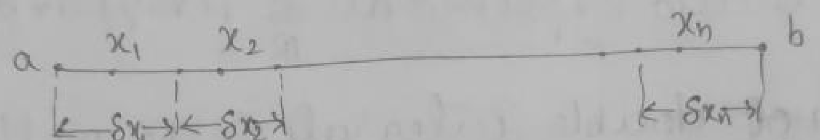
$$\text{i.e., } \sum_{r=1}^{n} f(x_r, y_r) \cdot \delta A_r$$

The limit of this Sum; if it exists, as the number of Subdivisions increases indefinitely and area of each Sub-division decreases to zero and is defined as the double integral of $f(x,y)$ over the region R and is written as $\iint_R f(x,y)\, dA$.

$$\text{Thus } \iint_R f(x,y)\, dA = \lim_{\substack{n \to \infty \\ \delta A \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r .$$

Multiple Integrals & Beta, Gamma Functions

Double Integrals: The definite Integral $\int_a^b f(x)\,dx$ is defined as the limits of the Sum $f(x_1)\delta x_1 + f(x_2)\delta x_2 + \cdots + f(x_n)\delta x_n$ where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots, \delta x_n$ tends to Zero. Here $\delta x_1, \delta x_2, \ldots, \delta x_n$ are n-Sub intervals into which the range $b-a$ has been divided and $x_1, x_2, \ldots, x_n$ are values of x lying respectively in the first, Second, ..., n^{th} Sub interval.



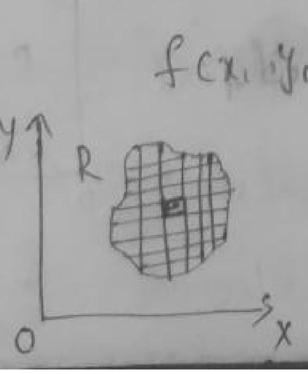
A double integral is its Counter part in two dimensions. Let a Single valued and bounded function $f(x,y)$ of two independent variables x, y be defined in a Closed Region R of the xy-plane. Divide the Region R into Sub Regions by drawing lines $\parallel^{lel}$ to co-ordinates axes. Number of rectangles which lie entirely inside the region R, from 1 to n.

Let (x_r, y_r) be the any pt inside the r^{th} rectangle whose area is δA_r.

Consider the Sum

$$f(x_1,y_1)\,\delta A_1 + f(x_2,y_2)\,\delta A_2 + \cdots + f(x_n,y_n)\,\delta A_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \longrightarrow (1)$$



Let the no. of Sub-regions increase indefinitely, Such that the Largest linear dimension (i.e; diagonal) of δA_r approaches Zero. The limit of the Sum (1) if it exists, is called the double integral of $f(x,y)$ over the region R and is denoted by

$$\iint_R f(x,y)\, dA.$$

(or) $$\lim_{\substack{n\to\infty \\ \delta A_r\to 0}} \sum_{r=1}^{n} f(x_r, y_r)\, \delta A_r = \iint_R f(x,y)\, dA.$$

which is also expressed as $\iint_R f(x,y)\, dx\, dy$ (or) $\iint_R f(x,y)\, dy\, dx$

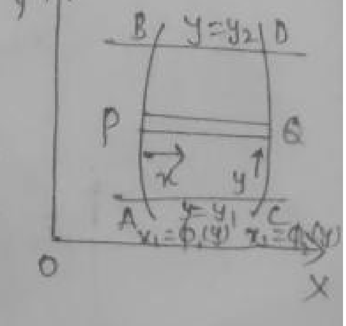
<u>Evaluation of double Integrals</u>: The method of Evaluating the double integrals depend upon the nature of the curves bounding the Region R.

Let the region R be bounded by the curves $x=x_1, x=x_2$ and $y=y_1, y=y_2$.

(i) <u>When x_1, x_2 are fun's of y & y_1, y_2 are Constants</u>:

Let AB and CD be curves $x_1=\phi_1(y)$ & $x_2=\phi_2(y)$.

Take a horizontal strip PQ of width δy.



Here the double integral is evaluated first w.r.to x. The resulting expression which is a fun of y is integrated w.r.to y b/w the

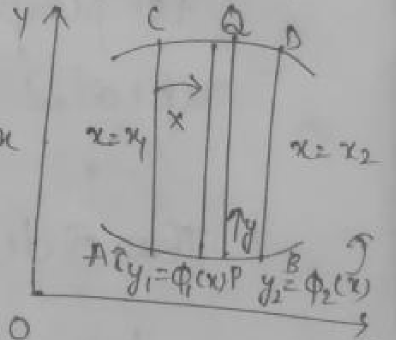
the limits $y = y_1$ & $y = y_2$. $\iint_R f(x,y)\,dx\,dy = \left[\int_{y_1}^{y_2}\left[\int_{x_1=\phi_1(y)}^{x_2=\phi_2(y)} f(x,y)\,dx\right]dy\right]$

Thus the integration being carried from the inner to the outer rectangle. Geometrically, the integral in the inner rectangle indicates that the integration is performed along the horizontal strip PQ (y const) while the outer rectangle corresponds to the sliding of the strip PQ from AC to BD thus covering the entire region ABDC of integration.

(ii) <u>when y_1, y_2 are ~~const~~ functions of x and x_1, x_2 are consts:</u>

Let AB & CD be the curves $y_1 = \phi_1(x)$ & $y_2 = \phi_2(x)$. Take a vertical strip PQ of width δx. Here the double integral is evaluated first w.r.to y then the resulting expression is integrated w.r.to x b/w the limits $x = x_1$ & $x = x_2$

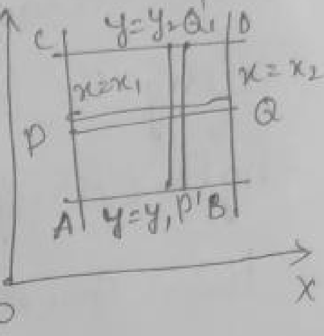
$$\iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\left[\int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} f(x,y)\,dy\right]dx$$



(iii) <u>when x_1, x_2, y_1, y_2 are constants:</u>

Here the Region of Integration R is the rectangle ABDC. It is immaterial whether we integrate first along the horizontal strip PQ and then slide it from AB to CD. (or) we integrate first along the vertical strip P'Q' and then slide it from AC to BD. Thus the order of integration is immaterial, provided the limits of integration are changed accordingly.

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]dy = \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right]dx$$



<u>Triple Integrals:</u> Consider the function $f(x,y,z)$ which is constant at every point of a finite region V of the three dimensional space. Divide the region V into n sub-regions of respective volumes $\delta v_1, \delta v_2, ---, \delta v_n$.

Let (x_r, y_r, z_r) be an arbitrary point in the r^{th} sub-region

Consider the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta v_r$.

The limit of this sum as $n \to \infty$ and $\delta v_r \to 0$, if it exists, is called the triple integral of $f(x,y,z)$ over the region V and is denoted by $\iiint_V f(x,y,z)\,dv$.

for purpose of evaluation, it can be expressed as the repeated integral $\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz \to (1)$

The order of integration is depending upon the limits.

Let $x_1 = f_1(y,z)$, $x_2 = f_2(y,z)$, $y_1 = \phi_1(z)$, $y_2 = \phi_2(z)$ and $z_1 = a$, $z_2 = b$.

Then the integral (1) is evaluated as follows.

$$\int_{z_1=a}^{z_2=b}\left[\int_{y_1=\phi_1(z)}^{y_2=\phi_2(z)}\left[\int_{x_1=f_1(y,z)}^{x_2=f_2(y,z)} f(x,y,z)\,dx\right]dy\right]dz.$$

Exercise 7.1

Evaluate the following Integrals:

(i) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$

Sol: $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy = \int_0^1 \left(x^2 \cdot y \Big|_x^{\sqrt{x}} + \frac{y^3}{3}\Big|_x^{\sqrt{x}} \right) dx$

$$= \int_0^1 \left(x^2(\sqrt{x}-x) + \frac{1}{3}(x^{3/2}-x^3) \right) dx$$

$$= \int_0^1 \left(x^{5/2} - x^3 + \frac{x^{3/2}}{3} - \frac{x^3}{3} \right) dx$$

$$= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3} \right) dx$$

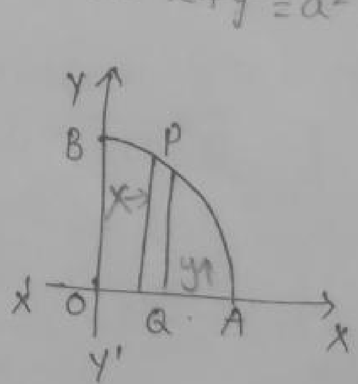
$$= \left[\frac{x^{5/2+1}}{5/2+1} + \frac{x^{3/2+1}}{3(3/2+1)} - \frac{4}{3}\frac{x^4}{4} \right]_0^1$$

$$= \left[\frac{2}{7} + \frac{2}{15} - \frac{1}{3} \right] = \frac{30+14-35}{105} = \frac{9}{105}$$

$$= 3/35$$

(ii) $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$

Sol: positive quadrant means $x \geq 0, y \geq 0$.



Eq $x^2+y^2=a^2$

$\Rightarrow y = \sqrt{a^2-x^2}$

$$\therefore \iint_R xy\,dx\,dy = \int_{x=0}^{x=a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\,dx\,dy$$

$$= \int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} y\,dy$$

$$= \int_0^a x\,dx \cdot \left(y^2/2 \right)_0^{\sqrt{a^2-x^2}}$$

$$= \int_0^a x \cdot \left(\frac{a^2 - x^2}{2}\right) dx$$

$$= \frac{1}{2} \cdot \int_0^a (a^2 \cdot x - x^3) dx$$

$$= \frac{1}{2}\left[a^2 \left(\frac{x^2}{2}\right)_0^a - \left(\frac{x^4}{4}\right)_0^a \right]$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4} \right] = \frac{a^4}{2}\left[\frac{2-1}{4}\right] = \frac{a^4}{8}.$$

Here the region of integration is the shaded quadrant of the circle OAB.

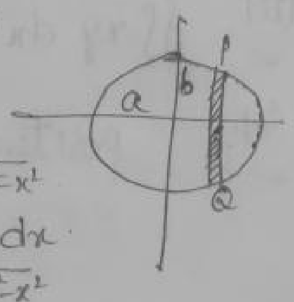
y varies from 0 to PQ i.e; 0 to $\sqrt{a^2 - x^2}$.

x varies from 0 to OA i.e; 0 to a

(iii) $\iint_R (x+y)^2 \, dx\, dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

<u>Sol:</u>

$$\iint_R (x+y)^2 \, dx\, dy = \int_{x=-a}^{x=a} \int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{y=\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+2xy)\, dx\, dy$$



$$= \int_{-a}^{a} \left[x^2 y + \frac{y^3}{3} + 2x \frac{y^2}{2} \right]_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} \left[x^2 \left(\frac{2b}{a}\sqrt{a^2-x^2} \right) + \frac{1}{3}\left[\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^3 - \left(-b/a\sqrt{a^2-x^2}\right)^3 \right] + x\left[\frac{b^2}{a^2}\cancel{(a^2-x^2)} - \frac{b^2}{a^2}\cancel{(a^2-x^2)} \right] \right] dx$$

$\left(\because y^2/b^2 = 1 - \frac{x^2}{a^2} \right.$

$\Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

$\Rightarrow y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$

$\therefore \frac{x^2}{a^2} = 1$

$\left.\Rightarrow x = \pm a\right)$

$$= \int_{-a}^{a} \left[x^2 \left(\frac{2b}{a}\sqrt{a^2-x^2}\right) + \frac{1}{3}\left[\frac{2b^3}{a^3}(a^2 - x^2)^{3/2} \right]\right] dx$$

$$= 2\int_{-a}^{a} \left[x^2\, b/a \sqrt{a^2 - x^2} + b^3/3a^3 \,(a^2 - x^2)^{3/2} \right] dx$$

$$= 4\int_0^a \left[x^2 \frac{b}{a}\sqrt{a^2-x^2} + \left(\frac{b^3}{3a^3}\right)(a^2-x^2)^{3/2}\right] dx$$

$\because$ function is even.

put $x = a\sin\theta$, $dx = a\cos\theta \, d\theta$

$$= 4\int_0^{\pi/2} \left[a^2\sin^2\theta \frac{b}{a}\sqrt{a^2\cos^2\theta} + \frac{b^3}{3a^3}(a^2 - a^2\sin^2\theta)^{3/2}\right] a\cos\theta \, d\theta$$

$$= 4\int_0^{\pi/2} \left[\left(\frac{a^3 b}{a}\sin^2\theta \cdot \cos\theta\right) + \frac{b^3}{3a^3} \cdot a^3\cos^3\theta\right] a\cos\theta \, d\theta$$

$$= 4\int_0^{\pi/2} \left(a^3 b\sin^2\theta\cos^2\theta + \frac{b^3 a}{3}\cos^4\theta\right) d\theta$$

$$= 4a^3 b\int_0^{\pi/2} \sin^2\theta\cos^2\theta \, d\theta + \frac{4ab^3}{3}\int_0^{\pi/2} \cos^4\theta \, d\theta$$

we know that $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{(m-1)(m-3)\cdots \times (n-1)(n-3)\cdots}{(m+n)(m+n-2)(m+n-4)\cdots} \times \frac{\pi}{2}$

$$\therefore \int_0^{\pi/2} \sin^2\theta\cos^2\theta \, d\theta = \frac{(2-1)(2-1)}{4(4-2)} \times \frac{\pi}{2} = \frac{\pi}{16} \quad (n \text{ is even})$$

$$\& \int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{(n-1)(n-3)\cdots}{(n)(n-2)(n-4)\cdots} \times \frac{\pi}{2} \quad (\text{if } n \text{ is even})$$

$$\Rightarrow \int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{(4-1)(4-3)}{4\cdot(4-2)} \times \frac{\pi}{2} = \frac{3\pi}{16}$$

$$\therefore \iint_R (x+y)^2 \, dx \, dy = 4a^3 b\left(\frac{\pi}{16}\right) + 4\frac{b^3 a}{3}\left(\frac{3\pi}{16}\right)$$

$$= \frac{4\pi}{16}(a^3 b + ab^3) = \frac{\pi}{4}ab(a^2+b^2).$$

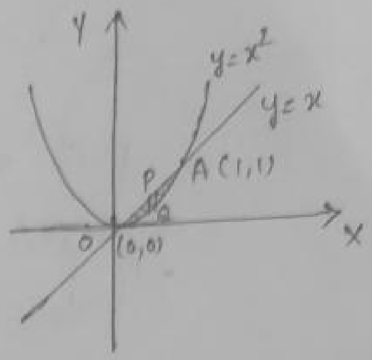
⑦ $\iint xy(x+y)\,dx\,dy$ over the area b/w $y=x^2$ and $y=x$

Sol: Here y varies from x^2 to x

when $x=0,\ y=0$

$x=1,\ y=1$

$\therefore$ x varies from 0 to 1



$$\therefore \iint_R xy(x+y)\,dx\,dy = \int_0^1\int_{x^2}^{x}(x^2y+xy^2)\,dx\,dy$$

x	0	1	-1	-2	2
y	0	1	1	4	4

$$= \int_0^1 \left[x^2\frac{y^2}{2}+x\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\frac{x^2}{2}(x^2-x^4)+\frac{x}{3}(x^3-x^6)\right]dx$$

$$= \int_0^1 \left(\frac{x^4}{2}-\frac{x^6}{2}+\frac{x^4}{3}-\frac{x^7}{3}\right)dx$$

$$= \int_0^1 \left(\frac{5}{6}x^4-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$= \left[\frac{5}{6}\cdot\frac{x^5}{5}-\frac{1}{2}\frac{x^7}{7}-\frac{1}{3}\frac{x^8}{8}\right]\Big|_0^1$$

$$= \frac{1}{6}-\frac{1}{14}-\frac{1}{24} = 3/56$$

④ $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Sol: Let $I = \int_0^1\int_0^{\sqrt{1+x^2}}\frac{dx\,dy}{1+x^2+y^2}$

$$= \int_0^1\left[\int_0^{\sqrt{1+x^2}}\frac{dy}{(\sqrt{1+x^2})^2+(y)^2}\right]dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \left[\frac{y}{\sqrt{1+x^2}} \right] \right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \tan^{-1}(1)\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \pi/4 \, dx$$

$$= \pi/4 \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$$

$$= \pi/4 \left[\log(x + \sqrt{1+x^2}) \right]_0^1$$

$$= \pi/4 \left[\log(1+\sqrt{2}) - \log 1 \right]$$

$$= \pi/4 \log(1+\sqrt{2})$$

① $\left\{ \int_1^2 \left[\int_1^3 xy^2\, dy \right] dx \right\}$

Sol: ✓

$$\int_1^2 \left[\int_1^3 xy^2\, dy \right] dx = \int_1^2 \left[x \frac{y^3}{3} \Big|_1^3 \right] dx$$

$$= \int_1^2 \left[\frac{x}{3} (27-1) \right] dx$$

$$= \frac{26}{3} \left[\int_1^2 x\, dx \right]$$

$$= \frac{26}{3} \left[\frac{x^2}{2} \Big|_1^2 \right]$$

$$= \frac{26}{\cancel{3}} \cdot \frac{1}{2} (4-1) = \underline{\underline{13}}.$$

③ ✓ $\int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx$

Sol:

$$\int_0^4 dx \int_0^{x^2} e^{y/x}\, dy = \int_0^4 \frac{e^{y/x}}{1/x} \Big|_0^{x^2} dx$$

$$= \int_0^4 x\, e^{y/x} \Big|_0^{x^2} dx = \int_0^4 (x \cdot (e^x - 1))\, dx$$

$$= x e^x \Big|_0^4 - \int_0^4 1 \cdot e^x dx - x^2/2 \Big|_0^4$$

Change of order of Integration:

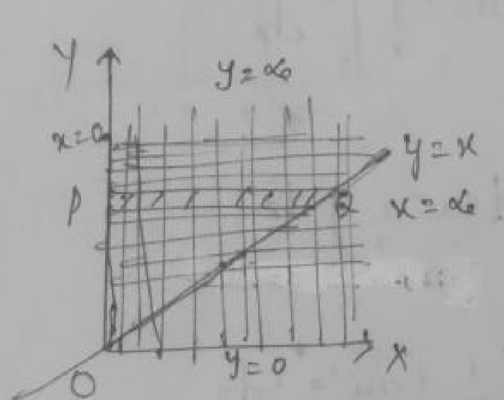
(1) In case we want to change the order of Integration in a given Interval for which the limits are given then from the given limits we first of all try to find out the region of integration and obtain the limits of integration as desired.

(1) Change the order of Integration and Evaluate $\int_0^\infty \int_x^\infty \left(\frac{e^{-y}}{y}\right) dy\, dx$
(14) ✓

Sol: The given limits shows that the integration region is defined by $y=x$ and $x=0$. In the given case y varies from x to ∞ and x varies from 0 to ∞.

When we want to change the order of integration then first we shall Integrate by taking strips $\parallel^{el}$ to x-axis and the given integral reduces to

$$\int_{y=0}^{y=\infty} \left[\int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy\right] = \int_{y=0}^{y=\infty} \left\{\int_{x=0}^{x=y} 1.\, dx\right\} \frac{e^{-y}}{y} dy$$



$$= \int_0^\infty x\Big|_0^y \frac{e^{-y}}{y}\, dy$$

$$= \int_0^\infty \cancel{y} \cdot \frac{e^{-y}}{\cancel{y}}\, dy$$

$$= \int_0^\infty e^{-y}\, dy = -e^{-y}\Big|_0^\infty$$

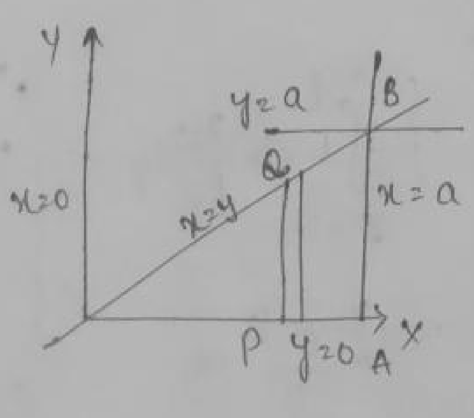
$$= -e^{-\infty} + e^{0} = 1$$

(2) Change the order of integration in $\int_0^a \int_y^a \frac{x\,dx}{x^2+y^2}\,dy$ and
(8) hence Evaluate the Same.

Sol: From the limits of integration, it is clear that the region of integration is bounded by $x=y$ and $x=a$, $y=0$ and $y=a$. Thus the region of integration is the ΔOAB. and is divided into horizontal strips.

For changing the order of integration, we divide the region of integration into vertical strips. The new limits of integration becomes 'y' varies from 0 to x and x varies from 0 to a.

$$\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \int_0^a \int_0^x \frac{x\,dy\,dx}{x^2+y^2}$$



$$= \int_0^a x\left[\int_0^x \frac{1}{x^2+y^2}\,dy\right] dx$$

$$= \int_0^a x\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$= \int_0^a x\left(\frac{1}{x}\tan^{-1}\frac{x}{x}\right) dx$$

$$= \int_0^a \cancel{x}\left(\frac{1}{\cancel{x}}\tan^{-1}1\right) dx$$

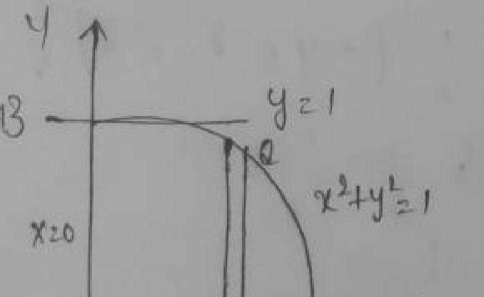
$$= \int_0^a \frac{\pi}{4}\,dx = \frac{\pi}{4}\,x\Big|_0^a = \frac{\pi a}{4}$$

(3) $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy$

Sol: From the limits of integration, it is clear that region of integration is bounded by $x=0$, $x=\sqrt{1-y^2}$ i.e., $x^2+y^2=1$ & $y=0$ & $y=1$. Thus the region of integration is $\Delta^{le}OAB$ and is divided into horizontal strips.

For changing the order of integration, we divide the region of integration into vertical strips. Thus the new limits of integration becomes 'y' varies from 0 to $\sqrt{1-x^2}$ and x from 0 to 1.

$$\therefore \int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy = \int_{x=0}^{x=1} x^3 \left[\int_{y=0}^{y=\sqrt{1-x^2}} y \, dy \right] dx$$



$$= \frac{1}{2} \int_0^1 x^3 \left[(1-x^2)^{1/2} \right]^2 dx$$

$$= \frac{1}{2} \int_0^1 x^3 \left[(1-x^2) \right] dx$$

$$= \frac{1}{2} \int_0^1 (x^3 - x^5) \, dx$$

$$= \frac{1}{2} \left[\frac{x^4}{4} - \frac{x^6}{6} \right]_0^1 = \frac{1}{2} \left[\frac{1}{4} - \frac{1}{6} \right]$$

$$= \frac{1}{2} \left[\frac{6-4}{24} \right] = \frac{2}{48} = \frac{1}{24}$$

(14)
(15) $\int_0^{\infty}\int_0^{x} x e^{-x^2/y}\, dy\, dx$

Sol: The limits of y are 0 to x. i.e; the x-axis and the line $y=x$. The limits for x are constants.
So the integral should be evaluated w.r.to y first and then w.r.to x. The order is to be changed, so that the integration is to be done w.r.to x first and then w.r.to y. So the limits for x are y to ∞ and the limits for y are 0 to ∞.

$$\therefore \int_0^{\infty}\int_0^{x} x e^{-x^2/y}\, dy\, dx = \int_0^{\infty}\left[\int_y^{\infty} x e^{-x^2/y}\, dx\right] dy$$

put $x^2 = t$
$2x\,dx = dt$

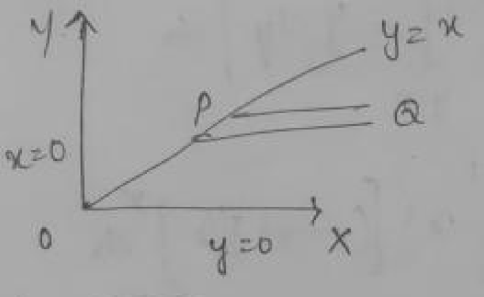
$$= \int_0^{\infty}\left[-\frac{y}{2} e^{-x^2/y}\right]_{y^2}^{\infty} dy$$

$\therefore$ when $x=y$, $t=y^2$
$x=\infty$, $t=\infty$

$$= \int_0^{\infty} \frac{y}{2} e^{-y}\, dy$$

$$= \frac{1}{2}\left[-y e^{-y} - e^{-y}\right]_0^{\infty}$$

$$= \frac{1}{2}$$



(16) $\int_0^{1}\int_x^{\sqrt{2-x^2}} \frac{x\, dy\, dx}{\sqrt{x^2+y^2}}$

Sol: The limits for y are $y=x$ & $y=\sqrt{2-x^2}$

i.e; $y=x$ & $x^2+y^2=2$

$\therefore$ The given integral is to be evaluated w.r.to y first and then w.r.to x.

The order is to be changed so that the integral is to be evaluated w.r. to x first and then w.r. to y. Therefore the limits for x should be functions of y and the limits for y are constants.

Region is ΔABO.

Dividing the region ABO into two regions OACO & CABC.

$$\therefore \iint_R \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \iint_{R_1} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} + \iint_{R_2} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$$

To find the pt of intersection

$$y = x, \quad x^2+y^2 = 2$$

$$\Rightarrow 2x^2 = 2 \Rightarrow x = \pm 1.$$

(But in first quadrant $x = 1$)

$\therefore y = 1$

$\therefore A = (1,1)$.

for R_1, The limits for y are 0 to 1.

The limits for x are 0 to y.

for R_2, The limits for y are 1 to $\sqrt{2}$

" " " x are 0 to $\sqrt{2-y^2}$.

$$\therefore \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \int_0^1 \int_0^y \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$$

$$= \int_0^1 \left[\int_0^y \frac{x\,dx}{\sqrt{x^2+y^2}} \right] dy + \int_1^{\sqrt{2}} \left[\int_0^{\sqrt{2-y^2}} \frac{x\,dx}{\sqrt{x^2+y^2}} \right] dy$$

Let $\sqrt{x^2+y^2} = t$

$$\frac{2x}{2\sqrt{x^2+y^2}}\,dx = dt$$

ΔAM, when $x=0,\ t=y$

$x=y,\ \sqrt{2y^2}=t$

$\sqrt{2}\,y = t$

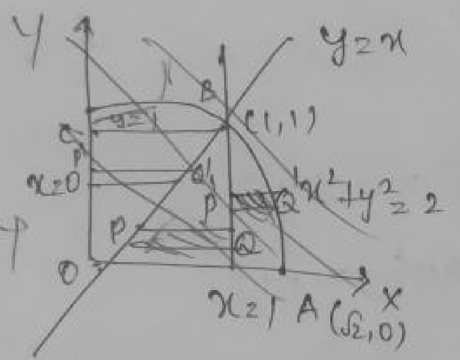
$$= \int_0^1 \left[\int_y^{\sqrt{2}y} dt\right] dy + \int_1^{\sqrt{2}} \left[\int_y^{\sqrt{2}} dt\right] dy$$

ΔMB, $x=0,\ t=y$

$x=\sqrt{2-y^2},\ t=\sqrt{2}$

$$= \int_0^1 \left(\sqrt{2}y - (y)\right) dy + \int_1^{\sqrt{2}} (\sqrt{2}-y)\,dy$$

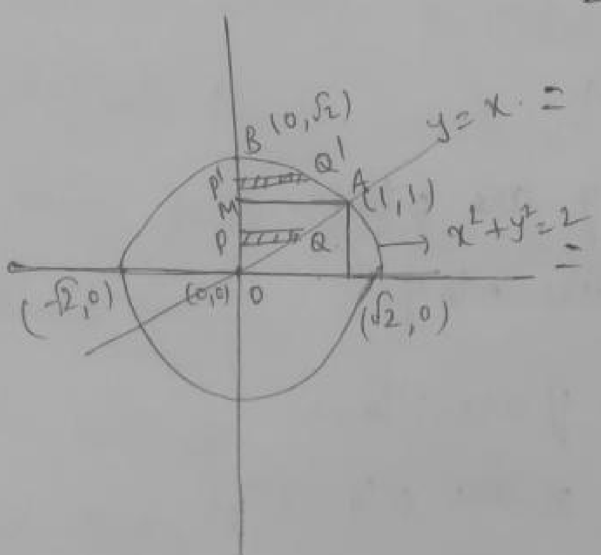
$$= (\sqrt{2}-1)\frac{y^2}{2}\Big|_0^1 + \sqrt{2}y - \frac{y^2}{2}\Big|_1^{\sqrt{2}}$$



$$= (\sqrt{2}-1)\,{}^1\!/_2 + \sqrt{2}(\sqrt{2}-1) - \frac{1}{2}(2-1)$$

$$= (\sqrt{2}-1)\,{}^1\!/_2 + 2-\sqrt{2}-1+{}^1\!/_2$$

$$= \frac{\sqrt{2}}{2} - \frac{1}{2} + 2 - \sqrt{2} - 1 + {}^1\!/_2$$



$$= 1 - \sqrt{2} + {}^1\!/_{\sqrt{2}}$$

$$= \frac{\sqrt{2}-2+1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}.$$

Change the order of integration in the double integral

(16) $\int_{0}^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V \, dx \, dy$.

Sol: The limits for y are $y = \sqrt{2ax - x^2}$ i.e; $(x-a)^2 + y^2 = a^2$ and $y^2 = 2ax$. Eq $(x-a)^2 + y^2 = a^2$ represents a circle c having centre at $(a, 0)$ and radius a.

Also the eqn $y^2 = 2ax$ represents the parabola. This parabola touches the circle at $(0,0)$. Also the line $x = 2a$ meets the parabola $y^2 = 2ax$ at $L(2a, 2a)$ and touches the circle at $M, (2a, 0)$.

When we change the order of integration, we have to integrate first w.r.to 'x' by drawing strips $\parallel^{lel}$ to x-axis. Such strips start from the arc OKL of the parabola $y^2 = 2ax$. But some of these end on the line ML while rest of these end on the arc ON of the semicircle and restart from the arc NM of the semi circle and end on the line ML.

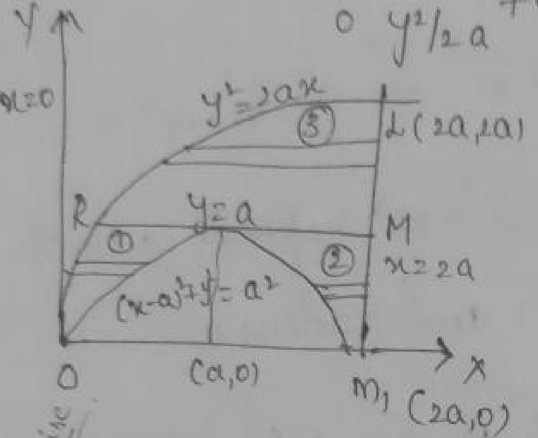
Thus the integral is converted into [integral for region (1) + Integral for region (2) + Integral for region (3)].

$$= \iint_1 f(x,y)\,dx\,dy + \iint_2 f(x,y)\,dx\,dy + \iint_3 f(x,y)\,dx\,dy \rightarrow (4)$$

from $x^2+y^2 = 2ax$, we get $x = a \pm \sqrt{a^2-y^2}$

$$\therefore \int_0^a \int_{y^2/2a}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx\,dy + \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx\,dy$$

$$+ \int_{y=a}^{2a} \int_{y^2/2a}^{2a} f(x,y)\,dx\,dy.$$



Exercise

Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$ and hence Evaluate the same.

Sol: from the limits of integration, it is clear that we have to integrate first w.r.to y which varies from $y=x^2$ to $y=2-x$ and then w.r.to x which varies from $x=0$ to $x=1$. The region of integration is divided into vertical strips. for changing the order of integration, we divided the region of integration into horizontal strips.

Solving $y=x^2$ & $y=2-x$, the co-ordinates of A are (1,1)

Draw $AM \perp OY$. The region of integration is divided into two parts. OAM and MAB.

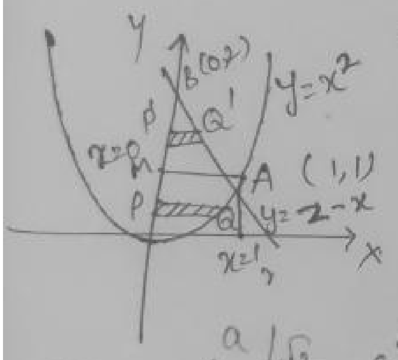
for the region OAM, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1. for the region MAB, x varies from 0 to $2-y$ and y varies from 1 to 2.

$$\therefore \int_0^1 \int_{x^2}^{2-x^2} xy\,dx\,dy = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\left(x^2/2\right)_0^{2-y} dy$$

$$= 1/2 \int_0^1 y^2\,dy + 1/2 \int_1^2 y(2-y)^2\,dy$$

$$= 1/2\,(y^3/3)_0^1 + 1/2 \int_1^2 [4y - 4y^2 + y^3]\,dy$$



$$= \frac{1}{3} + \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_1^2$$

$$= 3/8\ (7/6)$$

(15) (ii) $$\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\,dx\,dy$$

Sol: The given integral is bounded by the limits $x = y$ & $x^2+y^2 = a^2$ and $y=0$ & $y = a/\sqrt{2}$.

Then the region of integration is OAB.

It can be divided into two regions OCBD & CABC

for the region OCBO, y varies from 0 to x & x

varies from 0 to $a/\sqrt{2}$.

for the region CABC, y varies from 0 to $\sqrt{a^2-x^2}$ & x varies from $a/\sqrt{2}$ to a.

$$\therefore \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\, dx\, dy = \int_0^{a/\sqrt{2}} \int_0^{x} \log(x^2+y^2)\, dy\, dx$$

$$+ \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-x^2}} \log(x^2+y^2)\, dy\, dx$$

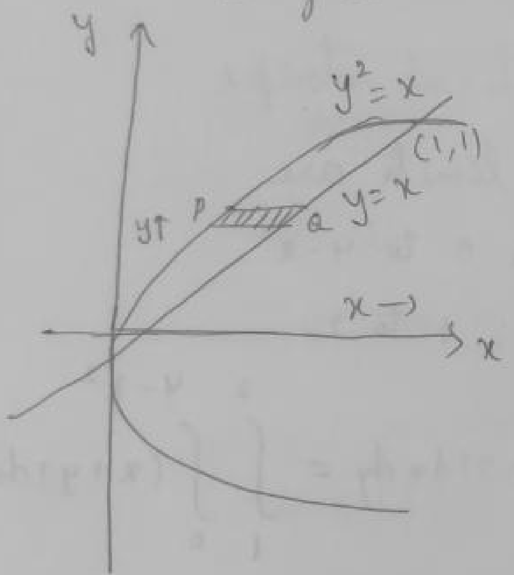
$$\int_0^{a/\sqrt{2}} \int_0^{x} \log(x^2+y^2)\, dy\, dx = \int_0^{a/\sqrt{2}} \Big[\log(x^2+y^2) \cdot y \Big|_0^x$$

$$- \int_0^x \frac{2y}{x^2+y^2} \cdot y\, dy \Big] dx$$

(12) Change the order of integration and evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \, dx\, dy$

Sol: From the limits of integration, it is clear that the function is integrated w.r.to y and then w.r.to 'x'.

The region of integration is bounded by $y=x$, $y=\sqrt{x}$, $x=0$ and $x=1$. To change the order of integration i.e, to integrate first w.r.to 'x' and the region is divided into horizontal strips. The new limits of integration becomes y varies from 0 to 1 and x varies from y^2 to y



Hence the given integral after change of order can be written as

$$\int_0^1 \int_x^{\sqrt{x}} xy \, dy\, dx = \int_0^1 \int_{y^2}^{y} xy \, dx\, dy$$

$$= \int_0^1 y \left. \frac{x^2}{2} \right|_{y^2}^{y} dy$$

$$= \int_0^1 \frac{y}{2}\left(y^2 - y^4 \right) dy$$

$$= \frac{1}{2} \int_0^1 \left(y^3 - y^5 \right) dy$$

$$= \frac{1}{2} \left[\frac{y^4}{4} - \frac{y^6}{6} \right]_0^1$$

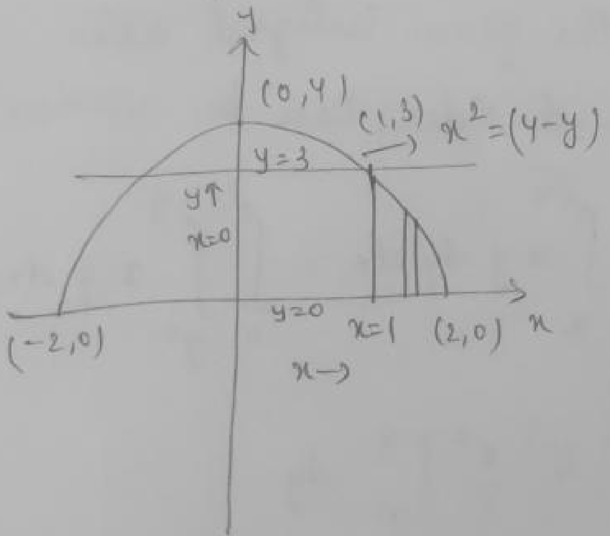
$$= \frac{1}{2} \left[\frac{1}{4} - \frac{1}{6} \right] = \frac{1}{2} \left[\frac{2}{24} \right] = 1/24$$

(9) Change the order of integration and Evaluate

$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$$

Sol: From the limits of integration, it is clear that the region of integration is bounded by $y=0$, $y=3$, $x=1$ and $x=\sqrt{4-y}$. Thus the region of integration is divided into horizontal strips.

For changing the order of integration, we divide the region of integration into vertical strips.



The New limits are

y : 0 to $4-x^2$

x : 1 to 2

$$\therefore \int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy = \int_1^2 \int_0^{4-x^2} (x+y)\,dx\,dy$$

$$= \int_1^2 \left(xy + \frac{y^2}{2}\right)_0^{(4-x^2)} dx$$

$$= \int_1^2 \left[x(4-x^2) + \frac{(4-x^2)^2}{2}\right] dx$$

$$= \int_1^2 \left(4x - x^3 + \frac{(4-x^2)^2}{2}\right) dx$$

$$= \cancel{\frac{4x^2}{2}\Big|_1^2 - \frac{x^4}{4}\Big|_1^2 + 2x\Big|_1^2}$$

$$= \frac{4}{2}(4-1) - \frac{1}{4}(16-1) + 2\cdot 1 - \frac{1}{6}(8-1) - \frac{x^3}{6}\Big|_1^2$$

=

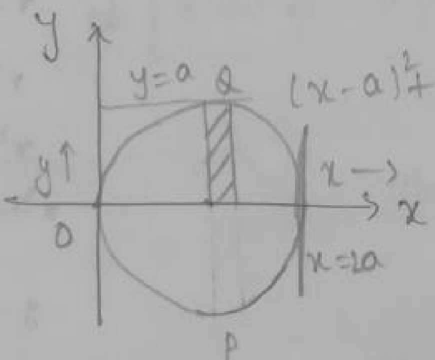
$= \frac{241}{60}$

x	± 2	0	± 1	
y	0	4	3	

(13) Change the order of integration and Evaluate $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\,dx\,dy$

Sol: From the limits of integration, it is clear that the function is integrated w.r.to x and then w.r.to 'y'. The region of integration is bounded by $x = a \pm \sqrt{a^2-y^2}$, $y=0$ and $y=a$.

To change the order of integration i.e., first w.r.to 'y' and the region is divided into vertical strips. The new limits of integration becomes 'y' varies from 0 to $\sqrt{2ax-x^2}$ and x varies from 0 to $2a$.



$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\,dx\,dy = \int_0^{2a}\int_0^{\sqrt{2ax-x^2}} xy\,dy\,dx$$

$$= \int_0^{2a} x \left.\frac{y^2}{2}\right|_0^{\sqrt{2ax-x^2}} dx$$

$$= \frac{1}{2}\int_0^{2a} x(2ax-x^2)\,dx$$

$$= \frac{1}{2}\left[2a\frac{x^3}{3} - \frac{x^4}{4}\right]_0^{2a}$$

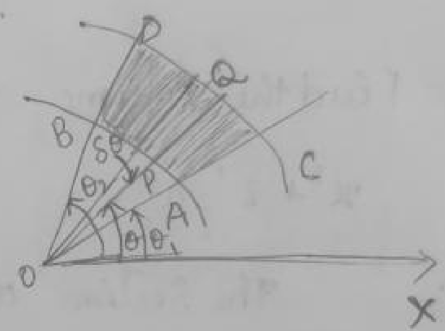
$$= \frac{1}{2}\left[\frac{16}{3}a^4 - \frac{16a^4}{4}\right]$$

$$= \frac{8a^4}{1}\left[\frac{4-3}{12}\right] = \frac{8a^4}{12} = \frac{2}{3}a^4$$

Double Integrals in polar co-ordinates:

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$, we first integrate w.r. to r between limits $r=r_1$ and $r=r_2$ keeping θ fixed and the resulting expression is integrated w.r. to θ from θ_1 to θ_2. In this integral r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$.



Then $\int_{r_1}^{r_2} f(r,\theta)\,dr$ indicates that the integration is along PQ from P to Q while the integration w.r. to θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area ACDB.

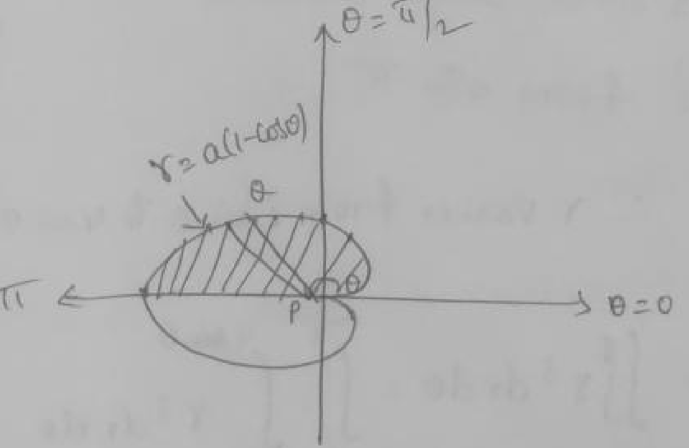
The order of integration may be changed with appropriate changes in the limits.

Eg: Evaluate $\iint r\sin\theta \, dr \, d\theta$ over the cardioid $r = a(1-\cos\theta)$ above the initial line.

Sol: The Region of integration R is the radial strips whose ends are at $r=0$ and $r = a(1-\cos\theta)$.

The angle θ varies from 0 to π

$$\therefore \iint r\sin\theta \, dr\, d\theta = \int_0^{\pi}\int_0^{a(1-\cos\theta)} r\sin\theta \, dr \, d\theta$$



$$= \int_0^{\pi} \sin\theta \left. \frac{r^2}{2} \right|_0^{a(1-\cos\theta)} d\theta$$

$$= \int_0^{\pi} \frac{\sin\theta \, a^2 (1-\cos\theta)^2}{2} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \sin\theta (1-\cos\theta)^2 d\theta$$

$$= \frac{a^2}{2}\left[\int_0^{\pi} \sin\theta (1+\cos^2\theta - 2\cos\theta) \, d\theta\right]$$

$r = a(1-\cos\theta)$

when $\theta = 0$, $r = 0$

$\theta = \pi/2$, $r = a$

$\theta = \pi$, $r = 2a$.

$\theta = 30°$; $r = (0.133)a$

$\theta = 45°$; $r = (0.2928)a$

$\theta = 60°$, $r = (0.5)a$

$\theta = 120°$, $r = (1.5)a$

$\theta = 135°$, $r = (1.707)a$

$\times$ $\left(= \frac{a^2}{2}\left[\int_0^{\pi} \sin\theta + \sin\theta\cos^2\theta - 2\cos\theta\sin\theta \, d\theta\right]\right.$

$$= \frac{a^2}{2}\left[-\cos\theta \Big|_0^{\pi} + \int \right) \times$$

$$= \frac{a^2}{2}\int_0^{\pi} (f(\theta))^2 f'(\theta) \, d\theta$$

$$= \frac{a^2}{2} \left. \frac{(1-\cos\theta)^3}{3} \right|_0^{\pi}$$

$$= \frac{a^2}{2} \frac{(1-\cos\theta)^3}{3} \Bigg|_0^{\pi}$$

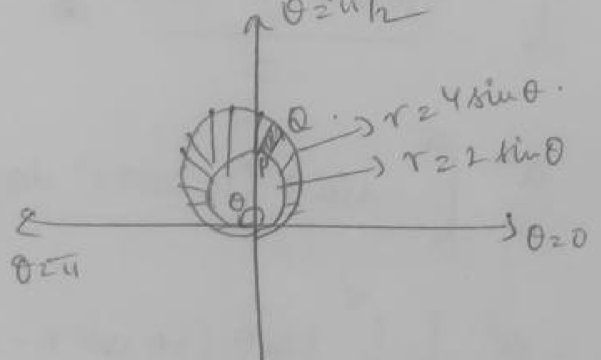
$$= \frac{a^2}{6}(2-0)^3 = \frac{2^3 a^2}{6} = \frac{8a^2}{6} = \frac{4a^2}{3}$$

Pb: Calculate $\iint r^3 \, dr \, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Sol: The Region of integration R is the radial strips whose ends are at $r = 2\sin\theta$ and $r = 4\sin\theta$.

The angle θ varies from 0 to π

$\therefore$ r varies from $2\sin\theta$ to $4\sin\theta$



$$\therefore \iint r^3 \, dr \, d\theta = \int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r^3 \, dr \, d\theta$$

$$= \int_0^{\pi} \frac{r^4}{4} \Bigg|_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4} \int_0^{\pi} (4^4 - 2^4) \sin^4\theta \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi} (256 - 16) \sin^4\theta \, d\theta$$

$$= 60 \times 2 \int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$= 120 \cdot \frac{(n-1)(n-3)}{n \cdot (n-2)} \cdot \pi/2 = \cancel{120}^{\,\cancel{30}\,15} \frac{3 \cdot 1}{\cancel{4} \cdot \cancel{2}} \pi/2$$

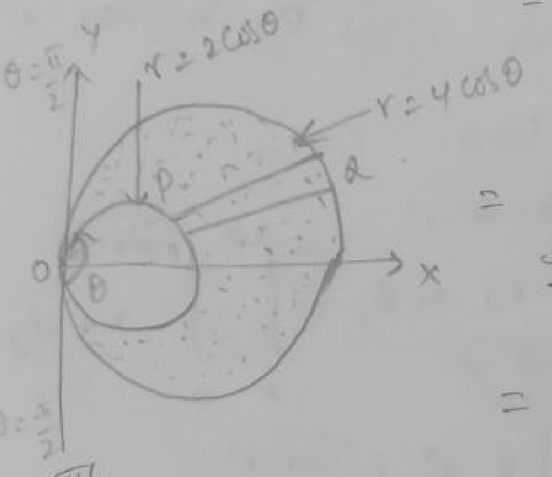
$$= 45\pi/2$$

(19) Evaluate $\iint r^3\,dr\,d\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$

<u>Sol:</u> The Region of integration R is shown shaded. Here r varies from $2\cos\theta$ to $4\cos\theta$ while θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$\therefore \iint_R r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{r^4}{4}\Big|_{2\cos\theta}^{4\cos\theta} d\theta$$



$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}\left[4^4\cos^4\theta - 16\cos^4\theta\right] d\theta$$

$$= 60\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta$$

$\int_0^{\pi/2} \cos^n\theta\,d\theta = \frac{\cancel{\phi}(n-1)(n-3)\cdots}{n(n-2)(n-4)\cdots}\cdot\frac{\pi}{2}$

$$= 120\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= 120\cdot\frac{3\times1}{4\times2}\cdot\frac{\pi}{2} = \frac{45}{2}\pi$$

$r = 2\cdot\frac{x}{r}$

$r^2 = 2x$

$x^2+y^2-2x=0$

$x^2+y^2+2gx+2fy+c=0$

$g=-1$

$(-g,-f)=(1,0)$

$r=1$

$r = 4\frac{x}{r}$

$r^2 = 4x$

$x^2+y^2-4x=0$

$x^2+y^2+2gx+2fy+c=0$

$g=-2,\ f=0$

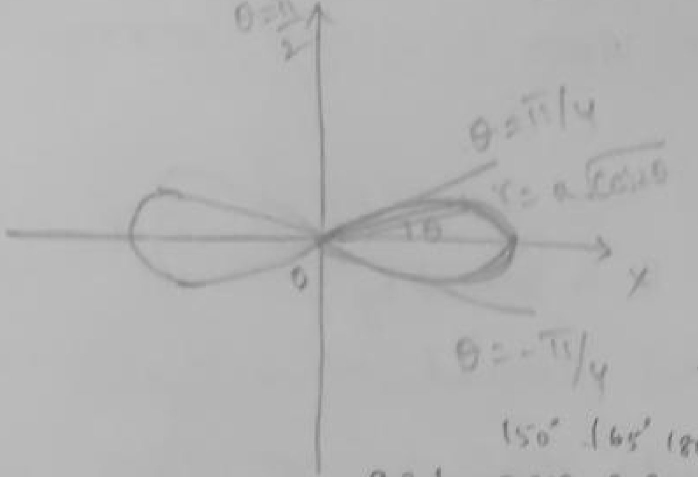
$\therefore (-g,-f)$

$(2,0)$

$r=2$

(18) Evaluate $\iint \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$

<u>Sol:</u> The region of integration R is covered by radial strips whose ends are $r=0$ and $r = a\sqrt{\cos 2\theta}$, the strips starting from $\theta = -\frac{\pi}{4}$ ~~to~~ and ending at $\theta = \pi/4$



150° 165° 180°
a=1 0.707 0.9 1

θ: 0 30° 45° 60° 90° 135°
r: 1 0.930 0.840 0.70 0 0

Lemniscate is a plane curve with a characteristic shape and consisting of two loops which meet at a central point.

When $\theta = 0$ or π then $r = \pm a$

when $r = 0$, then $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

1) when $-\pi/4 < \theta < \pi/4$, $\cos 2\theta$ is positive and r is real.

r increases as θ increases from $-\pi/4$ to 0 and r decreases as θ increases from 0 to $\pi/4$

2) when $\pi/4 < \theta < 3\pi/4$, r is negative.

$\therefore$ r is imaginary when θ lies b/w $\pi/4$ to $3\pi/4$ $(\pi/4 < \theta < 3\pi/4)$

$$\iint_R \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}} = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{2r\,dr\,d\theta}{2\sqrt{r^2+a^2}}$$

$$= \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{1}{2}(r^2+a^2)^{-1/2}\cdot 2r\,dr\,d\theta$$

$$\int_{-\pi/4}^{\pi/4} \cancel{\frac{1}{2}}\left[\frac{(r^2+a^2)^{1/2}}{\cancel{1/2}}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[(a^2\cos 2\theta + a^2)^{1/2} - a\right] d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}\left[(1+\cos 2\theta)^{1/2} - 1\right] d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}\left[(2\cos^2\theta)^{1/2} - 1\right] d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}\left[\sqrt{2}\cos\theta - 1\right] d\theta$$

$$= 2a\int_{\cancel{-\pi/4}\;0}^{\pi/4}(\sqrt{2}\cos\theta - 1)\,d\theta$$

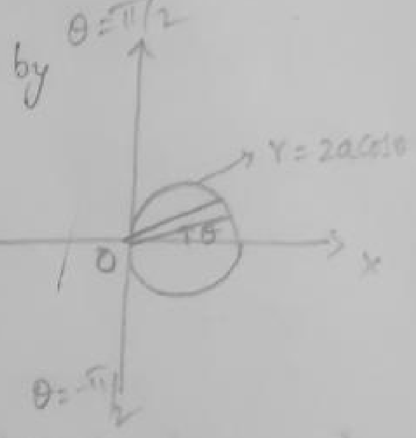
$$= 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4}$$

$$= 2a\left[\sqrt{2}\sin\pi/4 - \pi/4\right]$$

$$= 2a\left[1 - \pi/4\right]$$

(17) Show that $\iint_R r^2 \sin\theta \, dr \, d\theta = 2a^3/3$, R is the semicircle $r = 2a\cos\theta$ above the initial line.

Sol: The region of integration R is covered by radial strips whose ends are $r=0$ and $r = 2a\cos\theta$, the strips starting from $\theta = 0$ and ending at $\theta = \pi/2$



$$\therefore \iint_R r^2 \sin\theta \, dr \, d\theta = \int_0^{\pi/2} \int_0^{2a\cos\theta} r^2 \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left. \frac{r^3}{3} \right|_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} 8a^3 \cos^3\theta \sin\theta \, d\theta$$

$$= -\frac{8a^3}{3} \int_0^{\pi/2} (\cos\theta)^3 (-\sin\theta) \, d\theta$$

$$= -\frac{\cancel{8}^{2} a^3}{3} \left[\frac{(\cos\theta)^4}{\cancel{4}} \right]_0^{\pi/2}$$

$$= -\frac{2}{3} a^3 [-1] = \frac{2a^3}{3}.$$

$r = 2a \cdot \frac{x}{r}$

$r^2 = 2ax$

$x^2 + y^2 - 2ax = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

$(-g, -f)$

$(a, 0)$

$r = \sqrt{g^2 + f^2 - c}$

$r = \sqrt{a^2} = a.$

$\frac{}{2}$

Area by Double Integration:

(i) Cartesian Co-ordinates: The Area of the region enclosed by the curves $y_1 = f_1(x)$ & $y_2 = f_2(x)$ and the ordinates $x=a, x=b$ is given by $A = \int_a^b \int_{f_1(x)}^{f_2(x)} dx\, dy$.

The Area of the region enclosed by the curves $x_1 = \phi_1(y)$, $x_2 = \phi_2(y)$ and the ordinates $y=c, y=d$ is given by

$$A = \int_c^d \int_{\phi_1(y)}^{\phi_2(y)} dx\, dy$$

(ii) polar co-ordinates: The area of the region bounded by the curves $r_1 = f_1(\theta)$, $r_2 = f_2(\theta)$ and the lines $\theta = \alpha, \theta = \beta$ is given by

$$A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\, dr\, d\theta .$$

Eg: S.T the area b/w the parabolas $y^2 = 4ax$, $x^2 = 4ay$ is $\frac{16}{3}a^2$

Sol: The pt of intersection of $y^2 = 4ax$ & $x^2 = 4ay$

is $\left(\frac{x^2}{4a}\right)^2 = 4ax$

$$\Rightarrow \frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow x^4 = 64a^3 x$$

$$\Rightarrow x(x^3 - 64a^3) = 0$$

$$x = 0 \text{ or } 4a.$$

when $x = 0$, $y = 0$

$x = 4a$, $y = \frac{16a^2}{4a} = 4a$.

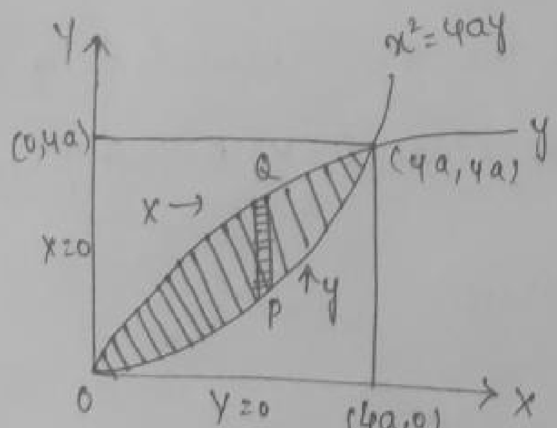
As such for the shaded area b/w these parabolas

x varies from 0 to $4a$.

y " " strip P to Q.

i.e; $x^2/4a$ to $2\sqrt{ax}$

Hence the required Area is



$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$$

$$= \int_0^{4a} y\Big|_{x^2/4a}^{2\sqrt{ax}} \cdot dx$$

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= 2\sqrt{a}\cdot\frac{2}{3}\, x^{3/2}\Big|_0^{4a} - \frac{1}{4a}\,\frac{x^3}{3}\Big|_0^{4a}$$

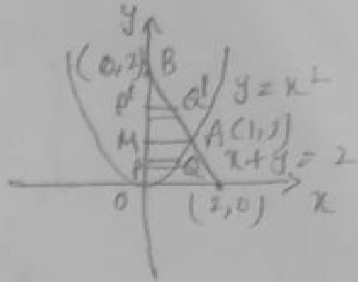
$$= \frac{4}{3}\sqrt{a}\,(4a)^{3/2} - \frac{(4a)^3}{(4a)3}$$

$$= \frac{a^2}{3}(32 - 16) = 16/3\, a^2$$

7.2.11

2) Find the area lying b/w the parabola $y = x^2$ and the line $x+y = 2$

sol: The point of intersection of these two curves is



x	2	0	1
y	0	2	1

$$2 - x = x^2$$

$$x^2 + x - 2 = 0$$

$$x = \frac{-1 \pm \sqrt{1+8}}{2}$$

$$x = \frac{-1 \pm 3}{2} = 1, -2$$

The points are $(1,1)$ & $(-2,4)$

Draw $AM \perp OB$, Divide the region OAB into two regions OAM and AMB.

$$\text{Required Area} = \iint_R dx\,dy = \iint_{OAB} dx\,dy$$

$$= \iint_{OAM} dx\,dy + \iint_{AMB} dx\,dy$$

In the region OAM, $x \to 0$ to $\sqrt{y}$
$y \to 0$ to 1

In the region AMB, $x \to 0$ to $2-y$
$y \to 1$ to 2

$$= \int_0^1 \int_0^{\sqrt{y}} dx\,dy + \int_1^2 \int_0^{2-y} dx\,dy$$

$$= \int_0^1 x\Big|_0^{\sqrt{y}} dy + \int_1^2 x\Big|_0^{2-y} dy$$

$$= \int_0^1 \sqrt{y}\,dy + \int_1^2 (2-y)\,dy$$

$$= \frac{2}{3} y^{3/2}\Big|_0^1 + \left(2y - \frac{y^2}{2}\right)\Big|_1^2$$

$$= \frac{2}{3} + 2 - \frac{3}{2} = \frac{4+12-9}{6} = \frac{7}{6}$$

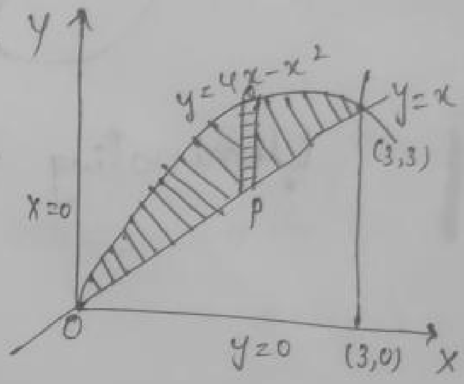
$$= 2\sqrt{a} \cdot \frac{2}{3} \cdot 4^{3/2} \cdot a^{3/2} - \frac{1}{4a} \cdot 64 \cdot \frac{a^3}{3}$$

$$= \frac{32}{3} a^2 - \frac{16}{3} a^2 = \frac{16}{3} a^2$$

Exercise - 7.2

① Find by double integration; the area lying b/w the parabola $y = 4x - x^2$ and the line $y = x$

Sol: The two curves intersect at pts whose abscissae are given by



$$4x - x^2 = x$$

$$\Rightarrow \cancel{x(4-x^2)} \; x(3-x) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 3.$$

$\therefore$ x varies from 0 to 3.

y " " strip P to Q.

i.e., x to $4x - x^2$.

$$\therefore \text{Required Area} = \int_0^3 \int_x^{4x-x^2} dy\, dx$$

$$= \int_0^3 y \Big|_x^{4x-x^2} \cdot dx$$

$$= \int_0^3 (4x - x^2 - x)\, dx$$

$$= \int_0^3 (3x - x^2)\, dx$$

$$= 3\frac{x^2}{2} - \frac{x^3}{3} \Big|_0^3$$

$$= \frac{3}{2}(9) - \frac{1}{3}(27)$$

$$= 27/2 - 27/3 = 27/6 = 9/2$$

Problems – 7.2

(1) Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

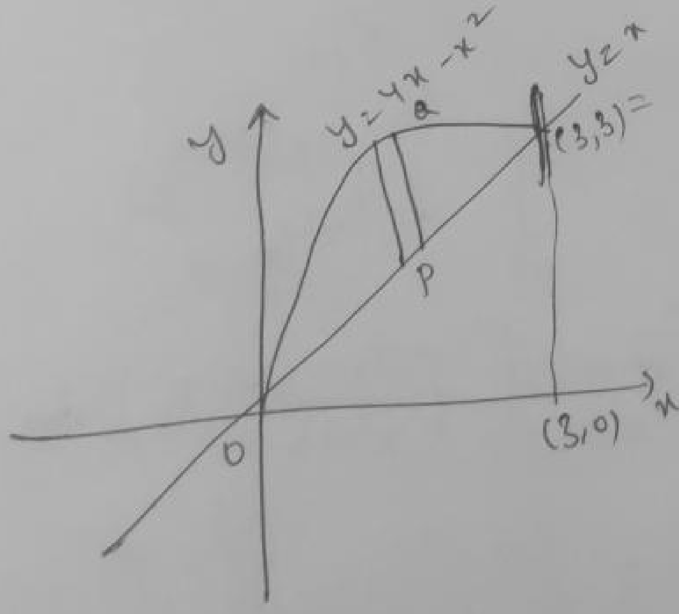
Sol: The two curves intersect at points whose abscissae are given by $4x - x^2 = x$

$$x^2 - 3x = 0 \quad i.e., \quad x = 0, 3.$$

Using vertical strips, the required area lies between $x = 0, x = 3$ and $y = x$, $y = 4x - x^2$

$$\therefore \text{Required area} = \int_0^3 \int_x^{4x-x^2} dy\, dx$$

$$= \int_0^3 y \Big|_x^{4x-x^2} dx$$



$$= \int_0^3 (4x - x^2 - x)\, dx$$

$$= \int_0^3 (3x - x^2)\, dx = 3\frac{x^2}{2}\Big|_0^3 - \frac{x^3}{3}\Big|_0^3$$

$$= \frac{3}{2}(9) - \frac{27}{3} = 27\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{27}{6} = \frac{9}{2}$$

(4) Find, by double integration, the area enclosed by the curves $y = \frac{3x}{x^2+2}$ and $x^2 = 4y$

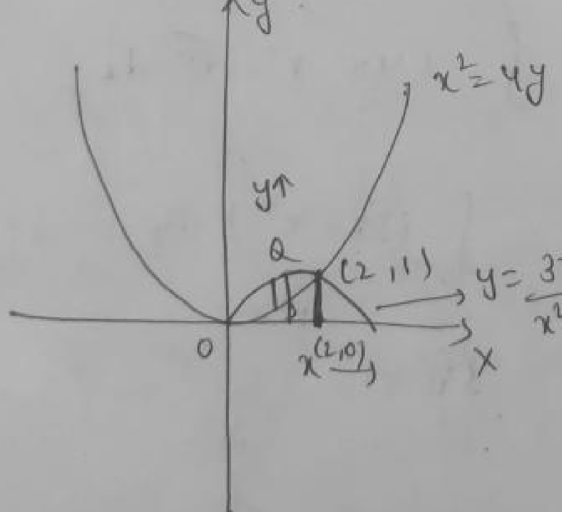
Sol: The two curves intersect at points whose abscissae are given by $\frac{3x}{x^2+2} = \frac{x^2}{4}$

$$x^3 + 2x - 12 = 0$$

one of the root is 2 and the remaining two roots are imaginary.

Using vertical strips, the required area lies between $x=0$, $x=2$ and $y = \frac{x^2}{4}$, $y = \frac{3x}{x^2+2}$

$\therefore$ Required area $= \int_0^2 \int_{x^2/4}^{3x/x^2+2} dy\, dx$



$$= \int_0^2 y \Big|_{x^2/4}^{3x/x^2+2} dx$$

$$= \int_0^2 \left(\frac{3x}{x^2+2} - \frac{x^2}{4} \right) dx$$

$$= \frac{3}{2} \int_0^2 \frac{2x}{x^2+2} dx - \frac{1}{4} \int_0^2 x^2 dx$$

$$= \frac{3}{2} \log(x^2+2) \Big|_0^2 - \frac{1}{4} \frac{x^3}{3} \Big|_0^2$$

$$= \frac{3}{2} (\log 6 - \log 2) - \frac{8}{12}$$

x	0	2	1	3
y	0	1	1	0.81

$= \frac{3}{2}\log 3 - \frac{2}{3}$

(6) Find by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$

sol: Eliminating r between the equations of two curves

$a\sin\theta = a(1-\cos\theta)$

$\sin\theta = 1-\cos\theta$

$1 = \sin\theta + \cos\theta$

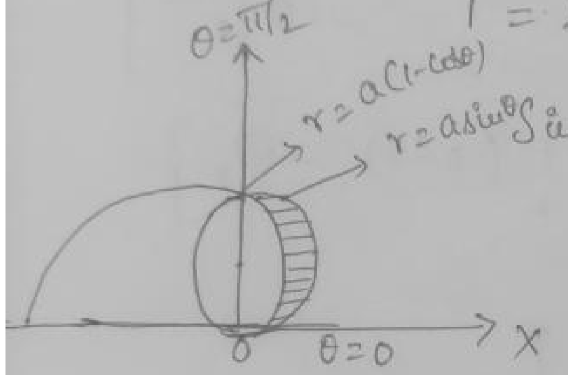
$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$

$1 + \sin 2\theta = 1$

$\sin 2\theta = 0$

$2\theta = 0$ (or) π

$\theta = 0$ (or) $\pi/2$



$r = a(1-\cos\theta)$

$a = 1$

$\theta = 0,\ r = 0$

$\theta = \pi/2,\ r = 1$

$\theta = \pi,\ r = 2$

$r = a\sin\theta$

$r^2 = ay$

$x^2 + y^2 - ay = 0$

$(0, \frac{a}{2})$

$r = a/2$

For the required area, r varies from $a(1-\cos\theta)$ to $a\sin\theta$ and θ varies from 0 to $\pi/2$

$\therefore$ Required area $= \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\, dr\, d\theta$

$= \int_0^{\pi/2} \frac{r^2}{2}\Big|_{a(1-\cos\theta)}^{a\sin\theta} d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \left[a^2\sin^2\theta - a^2(1-\cos\theta)^2\right] d\theta$

$= \frac{a^2}{2}\int_0^{\pi/2} (\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta)$

$$= \frac{a^2}{2} \int_0^{\pi/2} (-2\cos^2\theta + 2\cos\theta)\, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} (2\cos\theta - 2\cos^2\theta)\, d\theta$$

$$= \frac{a^2}{2} \left[\int_0^{\pi/2} (2\cos\theta + \cos 2\theta - 1)\, d\theta \right]$$

$2\cos^2\theta - 1 = \cos 2\theta$

$\cos 2\theta - 1 = -2\cos^2\theta$

$$= \frac{a^2}{2} \left[\cdot 2\sin\theta \Big|_0^{\pi/2} + \frac{\sin 2\theta}{2} \Big|_0^{\pi/2} - \theta \Big|_0^{\pi/2} \right]$$

$$= \frac{a^2}{2} \left[2\cdot(1) + \frac{1}{2}(0-0) - \pi/2 \right]$$

$$= \frac{a^2}{2} \left[2 - \pi/2 \right]$$

$$= a^2 \left[1 - \pi/4 \right]$$

(7) Find the area lying inside the Cardioid $r = 1+\cos\theta$ and outside the parabola $r(1+\cos\theta) = 1$.

Sol: The points of intersection of the Cardioid $r = 1+\cos\theta$ and the parabola $r = \frac{1}{1+\cos\theta}$ are obtained as

$r = 1+\cos\theta$

when $\theta = 0$, $r = 2$

$\theta = \pi/2$, $r = 1$

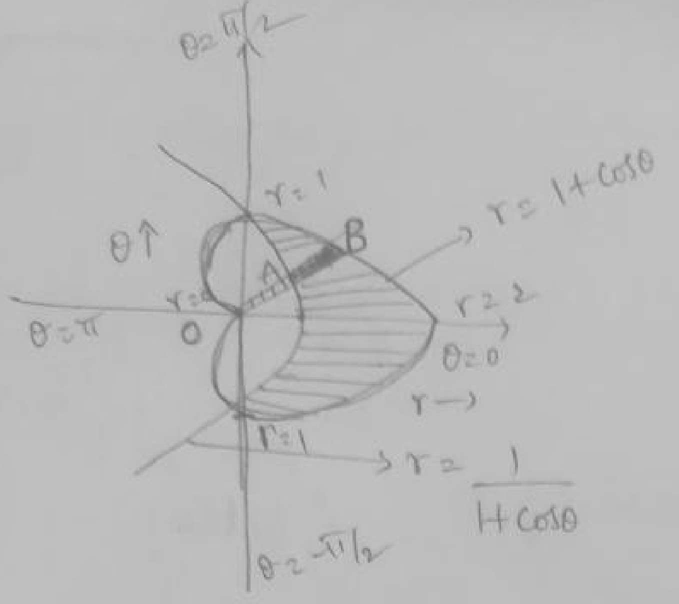
$\theta = \pi$, $r = 0$

$\theta = -\pi/2$, $r = 1$

$$1+\cos\theta = \frac{1}{1+\cos\theta}$$

$$(1+\cos\theta)^2 = 1$$

$$\cos\theta = 0 \Rightarrow \theta = \pm \pi/2$$



The region is symmetrical about the line $\theta = 0$. Draw an elementary radius vector OAB from the origin in the region above the initial line $\theta = 0$. OAB enters in the region from the parabola $r = \frac{1}{1+\cos\theta}$ and leaves at the Cardioid $r = 1 + \cos\theta$.

$\therefore$ The limits of r: $\frac{1}{1+\cos\theta}$ to $1+\cos\theta$

θ: 0 to $\pi/2$.

$\therefore$ Total Area $= 2$ (Area above the initial line)

$r = \frac{1}{1+\cos\theta}$

when $\theta = 0$, $r = 1/2 = 0.5$

$\theta = \pi/2$, $r = 1$

$\theta = -\pi/2$, $r = 1$

$\theta = \pi$, $r = \infty$

$$= 2\int_0^{\pi/2}\int_{1/(1+\cos\theta)}^{1+\cos\theta} r\, d\theta\, dr$$

$$= \cancel{2}\int_0^{\pi/2} \frac{r^2}{\cancel{2}}\Bigg|_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} d\theta$$

$$= \int_0^{\pi/2}\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right] d\theta$$

$$= \int_0^{\pi/2}\left[(1+\cos^2\theta + 2\cos\theta) - \frac{1}{(2\cos^2\theta/2)^2}\right] d\theta$$

$$= \int_0^{\pi/2} \left\{ 1 + 2\cos\theta + \frac{1+\cos 2\theta}{2} - \frac{1}{4}\left(1+\tan^2\frac{\theta}{2}\right)\sec^2\frac{\theta}{2} \right\} d\theta$$

$$= \int_0^{\pi/2} \left\{ \frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2} - \frac{1}{4}\sec^2\frac{\theta}{2} - \frac{1}{2}\tan^2\frac{\theta}{2} \left(\frac{1}{2}\sec^2\frac{\theta}{2}\right) \right\} d\theta$$

$$= \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\frac{\sin 2\theta}{2} - \frac{1}{4}\left(2\tan\frac{\theta}{2}\right) - \frac{1}{2}\frac{\tan^3\frac{\theta}{2}}{3} \right]_0^{\pi/2}$$

$$= \left[\frac{3}{2}\cdot\frac{\pi}{2} + 2\cdot 1 + \frac{1}{4}(0) - \frac{1}{4}\cdot 2\cdot 1 - \frac{1}{6}\cdot 1 \right]$$

$$= \frac{3\pi}{4} + 2 - \frac{1}{2} - \frac{1}{6}$$

$$= \frac{3\pi}{4} + \frac{3}{2} - \frac{1}{6}$$

$$= 3\pi/4 + 4/3$$

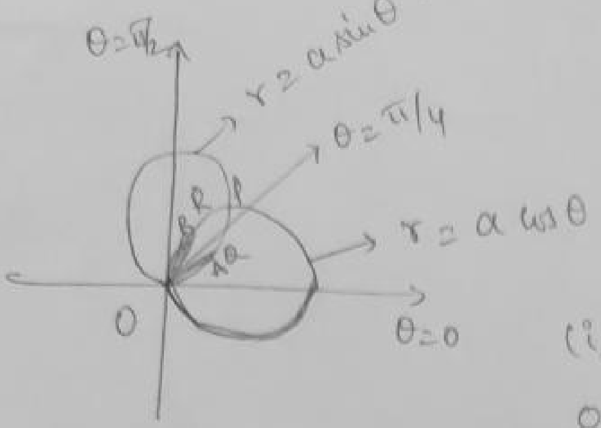
(8) Find the area common to the circles $r = a\cos\theta$, $r = a\sin\theta$ by double integration.

Sol: The points of intersection of the circles $r = a\cos\theta$ and $r = a\sin\theta$ is obtained as

$$a\sin\theta = a\cos\theta$$

$$\tan\theta = 1$$

$$\theta = \pi/4$$



Divide the region on OQPR into two subregions OQP and ORP. Draw an elementary radius vector in each subregion.

(i) In subregion OQP, radius vector OA starts from the origin and terminates on the circle $r = a\sin\theta$.

$\therefore$ Limits of r: $r = 0$ to $a\sin\theta$

θ: $\theta = 0$ to $\pi/4$

(ii) In Subregion ORP, radius vector OB starts from the origin and terminates on the circle $r = a\cos\theta$

$\therefore$ Limits of r: $r = 0$ to $a\cos\theta$

θ: $\theta = \pi/4$ to $\pi/2$

$$\therefore \text{Area} = \int_0^{\pi/4}\int_0^{a\sin\theta} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{a\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \frac{r^2}{2}\Big|_0^{a\sin\theta} d\theta + \int_{\pi/4}^{\pi/2} \frac{r^2}{2}\Big|_0^{a\cos\theta} d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}(a^2\sin^2\theta)\,d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(a^2\cos^2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\int_0^{\pi/4} \frac{(1-\cos2\theta)}{2}\,d\theta + \int_{\pi/4}^{\pi/2} \frac{(1+\cos2\theta)}{2}\,d\theta\right]$$

$$= \frac{a^2}{4}\left[\left(\theta - \frac{\sin2\theta}{2}\right)\Big|_0^{\pi/4} + \left(\theta + \frac{\sin2\theta}{2}\right)\Big|_{\pi/4}^{\pi/2}\right]$$

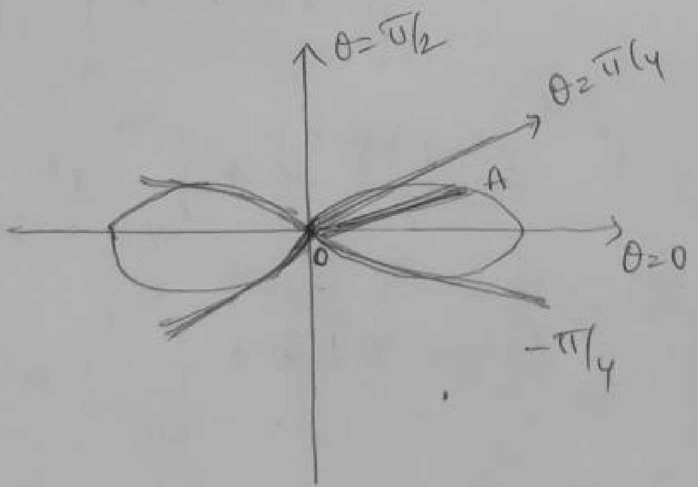
$$= \frac{a^2}{4}\left[\frac{\pi}{4} - \frac{1}{2}\cdot 1 + \left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \frac{1}{2}\cdot(0-1) \right]$$

$$= \frac{a^2}{4}\left[\cancel{\pi/4} - 1 + \pi/2 - \pi/4 \right]$$

$$= \frac{a^2}{4}\left(\pi/2 - 1 \right)$$

(5) Find by double integration, the area of the lemniscate $r^2 = a^2\cos 2\theta$.

sol: The region of integration R is covered by radial strips whose ends are $r=0$ and $r = a\sqrt{\cos 2\theta}$, the strips starting from $\theta = -\pi/4$ and ending at $\theta = \pi/4$.



$$\therefore \text{Area} = 2\int_{-\pi/4}^{\pi/4}\int_{0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= \cancel{2}\int_{-\pi/4}^{\pi/4} \frac{r^2}{\cancel{2}}\Big|_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} a^2\cos 2\theta\, d\theta$$

$$= 2a^2\int_0^{\pi/4} \cos 2\theta\, d\theta$$

$$= 2a^2 \frac{\sin 2\theta}{2}\Big|_0^{\pi/4}$$

$$= a^2[1] = \underline{\underline{a^2}}$$