(UNIT-2) Section--B

① (a) verify Rolle's theorem for $f(x)=(x-a)^m(x-b)^n$ in $[a,b]$ where m and n are positive integers

Solution: $f(x) = (x-a)^m (x-b)^n$

$f(a) = 0$

$f(b) = 0$

$f'(x) = 0 \quad x \in [a,b]$

$f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m n(x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)] = 0$

$= m[x-b] + n[x-a] = 0$

$= mx - mb + nx - na = 0$

$= (m+n)x = mb + na$

$$\boxed{x = \frac{mb+na}{m+n}}$$

$x \in [a,b]$

B) Using Rolle's theorem for $f(x) = x^{2m-1}(a-x)^{2n}$, find the value of x between 0 and a, where $f'(x) = 0$.

<u>Solution</u>:- Every Polynomial $f(x)$ is Continuous and differentiable Every where in R

$\therefore f(b) = 0 \Rightarrow f(a) = f(b)$

$$f'(c) = (2m-1)x^{2m-2}(a-x)^{2n} + (2n)(a-x)^{2n-1}x^{2m-1}$$

$$0 = x^{2m-2}(a-x)^{2n-1}[2m-1(a-x) + 2n(x)]$$

$$0 = 2ma - a - 2mx + x + 2nx$$

$$a - 2ma = x(2n+1-2m)$$

$$\boxed{x = \frac{a-2ma}{2n+1-2m}}$$

(a)

(2) Verify Rolle's theorem for $\frac{\sin x}{e^x}$ in $[0, \pi]$

Solution :- $f(x) = e^{-x} \sin x$, $x \in (0, \pi)$

for Rolle's Theorem, $f(0) = f(\pi)$ and $f(x)$ must be Continuous and differentiable over $[0, \pi]$

Let us check if $f(0) = f(\pi)$

$\Rightarrow f(0) = e^{-0} \sin(0) = 0$

$\Rightarrow f(\pi) = e^{-\pi} \sin(\pi) = 0$

There fore, $f(0) = f(\pi)$

The function e^{-x} and $\sin x$ are both continuous and differentiable $[0, \pi]$

There fore, Rolle's theorem can be applied for the function Given.

There exists 'c' such that $f'(c) = 0$

$f'(x) = -e^{-x} \sin x + e^{-x} \cos x$

$f'(c) = -e^{-c} \sin(c) + e^{-c} \cos(c) = 0$

$\Rightarrow e^{-c} [\cos c - \sin c] = 0$

$\Rightarrow \cos c = \sin c \Rightarrow \tan c = 1$

$\Rightarrow \boxed{c = \frac{\pi}{4}}$

Hence, Rolle's Theorem is verified.

(b) In the Lagrange's mean value theorem, determine 'c' lying between 'a' and b, if $f(x) = x(x-1)(x-2)$, where $a=0$ and $b=\frac{1}{2}$.

Solution :- From mean value theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$a=0, f(a)=0$

$\Rightarrow b = \frac{1}{2}, f(b) = \frac{3}{8}$

$f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$

$f'(c) = (c-1)(c-2) + c(c-2) + c(c-1)$

$= c^2 - 3c + 2 + c^2 - 2c + c^2 - c$

That is

$f'(c) = 3c^2 - 6c + 2$

According to mean value theorem, we have

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{(3/8)-0}{(1/2)-0} = \frac{3}{4}$$

$$\Rightarrow 3c^2 - 6c + \frac{5}{4} = 0$$

$$c = \frac{6 \pm \sqrt{36-15}}{2 \times 3} = \frac{6 \pm \sqrt{21}}{6}$$

$$= \frac{1 \pm \sqrt{21}}{6}$$

13) ⓐ Verify lagrange's mean Value Theorem and find the Appropriate Value of C for the function in $(0,4)$
$f(x) = (x-1)(x-2)(x-3)$

Sol:- $f(x) = (x-1)(x-2)(x-3)$
$= (x^3 - 6x^2 + 11x - 6)$

$f(x)$ is a Cubic polynomial and Continuous at $(0,4)$

$f'(x) = 3x^2 - 12x + 11$ Exists

So $f'(x)$ is differentiable at $(0,4)$

$f'(c) = 3c^2 - 12c + 11$

Now $f'(c) = \frac{f(4) - f(0)}{4-0}$

$3c^2 - 12c + 11 = \frac{6-(-6)}{4} = 3$

Solving we get $c = 2 \pm \frac{2}{\sqrt{3}}$
$= 3.154$ or 0.84

So c lies between $(0,4)$.

(b) Using Lagrange's theorem show that

$x > \log(1+x) > \frac{x}{1+x}$, for $x > 0$.

Solution :- Let $f(x) = \log(1+x)$ in $[0, x]$

Since $f(x)$ satisfies the condition of L.M.V theorem in $[0, x]$, there exist $\theta \; (0 < \theta < 1)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\theta x)$$

$$\Rightarrow \frac{\log(1+x)}{x} = \frac{1}{1+\theta x}$$

Now, $0 < \theta < 1$, $x > 0 \Rightarrow \theta x < x$

$$\Rightarrow 1 + \theta x < 1 + x$$

$$\Rightarrow \frac{1}{1+\theta x} > \frac{1}{1+x}$$

$$\Rightarrow \frac{x}{1+\theta x} > \frac{x}{1+x} \rightarrow (1)$$

Again $0 < \theta < 1$, $x > 0$

$$\Rightarrow \theta x > 0$$

$$\Rightarrow 1 + \theta x > 1$$

$$\Rightarrow \frac{1}{1+\theta x} < 1$$

$$\Rightarrow \frac{x}{1+\theta x} < x \rightarrow (2)$$

from (1) and (2) we get.

$$\frac{x}{1+x} < \log(1+x) < x.$$