(Unit - 3) (Section - B)

(Typical Question)

① (a) Discuss the maxima and minimal of $f(x,y) = x^3y^2(1-x-y)$.

Solution :- Let $f(x,y) = x^3y^2(1-x-y)$

$$= 3x^2y^3$$

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 2x^3y - 2x^4y - 3x^3y^2$$

Then $f_x = 0$ and $f_y = 0$

$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$ and

$3x^2y^2 - 4x^3y^2 - 3x^3y^2 = 0$

$2x^3y - 2x^4y$

i.e $x^2y^2(3-4x-3y) = 0$ and $x^3y(2-2x-3y) = 0$

$\therefore x = 0$ or $y = 0$ or $4x+3y = 3$ and

$x = 0$ or $y = 0$ or $2x+3y = 2$

solving $4x+3y = 3$ and $2x+3y = 2$

we get $x = \frac{1}{2}$ $y = \frac{1}{3}$

Hence the critical points are $(0,0)$ and $\left(\frac{1}{2}, \frac{1}{3}\right)$.

Further, $A = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$

$$= 6xy^2(1-2x-y)$$

$B = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$

$$= x^2y(6-8x-9)$$

$C = f_{yy} = 2x^3 - 2x^4 - 4x^3y$

$$= 2x^3(1-x-3y)$$

(i) At the point $(0,0)$, $A=0$, $B=0$

$C=0$, $AC-B^2=0$ and futher investigation required.

(iii) At the poin $\left(\frac{1}{2}, \frac{1}{3}\right)$

$A = -\frac{1}{9}$, $B = \frac{-1}{12}$, $C = -\frac{1}{8}$

Now $AC - B^2 = \left(\frac{-1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2 = \frac{1}{144} > 0$

and $A = -\frac{1}{9} < 0$

$\therefore f(x,y)$ attains its maximum at $\left(\frac{1}{2}, \frac{1}{3}\right)$

Maximum $f(x,y) = f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{8} \cdot \frac{1}{9}\left[1 - \frac{1}{2} - \frac{1}{3}\right]$

$= \frac{1}{432}$

(b) Discuss the minium and maximum values for $f(x,y) = x^3 + y^3 - 3axy$

Sol:- let $f(x,y) = 3x^3 + y^3 - 3axy$
At the critical values of the function, both the first derivatives are zero.

$\Rightarrow f(x) = 3x^2 - 3ay = 0$ and $f_y = 3y^2 - 3ax = 0$

$\Rightarrow y = \frac{x^2}{a}$ $\Rightarrow 3\left(\frac{x^2}{a}\right)^2 - 3ax = 0$

$\Rightarrow \frac{x^4}{a^2} - ax = 0$ $\Rightarrow x^4 - a^3x = 0$

$\Rightarrow x(x^3 - a^3) = 0$.

$\Rightarrow x = 0$ or $x = a$

when $x = 0, y = 0$ and when $x = a$; $y = a$

$f_{xx} = 6x$, $f_{yy} = 6y$ and $f_{xy} = -3a$

At $(x,y) = (0,0)$

$f_{xx} f_{yy} - f_{xy}^2 = 36xy - 9a^2 = -9a^2 < 0$

$\Rightarrow (x, y) = (0, 0)$ is a saddle point

At $(x, y) = (a, a)$

$f_{xx} f_{yy} - f^2_{xy} = 36xy - 9a^2 = 27a^2 > 0$

Further $f_{xx} = 6a > 0$ if $a > 0$ and $f_{xx} = 6a < 0$ if $a < 0$

$\Rightarrow (x, y) = (a, a)$ represents a minimum if $a > 0$ and represents a maximum if $a < 0$

when $(x, y) = (a, a)$, $f(x, y) = x^3 + y^3 - 3axy = -a^3$.

If $a = 0$, the function becomes $x^3 + y^3$, which does not have any maximum or minimum and has a saddle point at $(x, y) = (0, 0)$.

(2) Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solut: let $P = (x, y, z)$ be a point on the ellipsoid with $x, y, z > 0$ Take the eight different points with $P_i (\pm x, \pm y, \pm z)$

These points are the vertices of a parallelepiped with the side length $2x, 2y$ and $2z$. Then the volume parallelepiped is:

$V = 2x \cdot 2y \cdot 2z = 8 \cdot x \cdot y \cdot z$.

let $h(x)$ = maximum value $= 8xyz + f(x^2/a^2 + y^2/b^2 + b^2 + z^2/c^2 + 1) \longrightarrow (1)$

where f = langragian multiplier.

The differentiate partialy with respect to x

$$8yz + 2fx/a^2 = 0 \longrightarrow (2)$$

$$1/a^2 = (-8yz/2fx)$$

Similarly

$$1/b^2 = (-8x/2fy)$$

$$1/c^2 = (-8xy/2fz)$$

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

$$\Rightarrow f = -12xyz$$

Put f value in eqn (2)

$$x = \frac{a}{\sqrt{3}}$$

Similarly $y = \frac{b}{\sqrt{3}}$, $z = \frac{c}{\sqrt{3}}$

$\Rightarrow$ the largest volume of parallelopiped inscribed in ellipsoid

$$= 8xyz = 8(a/\sqrt{3})(b/\sqrt{3})(c/\sqrt{3})$$

$$= \frac{8abc}{3\sqrt{3}} //$$

3(a) Let x, y, z respectively be the length, breadth and height of the rectangular box. Since it is open at the top, the surface area (S) is given by

$$S = xy + 2xz + 2yz = 432$$

volume (V) $= xyz$

We need to find x, y, z such that V is maximum subject to the condition that

$$xy + 2xz + 2yz = 432$$

Let $F = xyz + \lambda(xy + 2xz + 2yz)$

we form the Equation $F_x = 0, F_y = 0, F_z = 0$

i.e, $yz + \lambda(y + 2z) = 0$ or $\lambda = -yz/(y+2y)$

$xz + \lambda(x + 2z) = 0$ (or) $\lambda = -xz/(x+2z)$

$xy + \lambda(2x + 2y) = 0$ (or) $\lambda = -xy/2(x+y)$

Now $$\frac{-yz}{y+2z} = \frac{-xz}{x+2z} = \frac{-xy}{2(x+y)}$$

(or) $$\frac{y}{y+2z} = \frac{x}{x+2z}$$ Gives $x = y$

Also $$\frac{z}{x+2z} = \frac{y}{2x+2y}$$ Gives $y = 2z$

Hence, $x = y = 2z$

But $xy + 2x + 2yz = 432$

$x^2 + x^2 + x^2 = 432$

(or) $3x^2 = 432$ or $x^2 = 144$

We have $x = 12$ and hence $y = 12, z = 6$

Thus the required dimensions are 12, 12, 6

③ ⓑ Sol: Suppose the (say, positive) sum of three numbers is S, and we suppose that each of the summands is required to be positive.

If x, y, z are three numbers whose sum is S. then $z = S - x - y$, so the product of three numbers is

$$P = xyz = xy(S - x - y)$$

which we hence regard as a function of (x, y) in the first quadrant $\{x, y > 0\}$

where this function has a local extremum we have

$$0 = \frac{\partial P}{\partial x} = Sy - 2xy - y^2,$$

by symmetry

$$0 = \frac{\partial P}{\partial y} = Sx - 2xy - x^2$$

sloving this system gives that the unique extremum for which $x, y, z = S - x - y$ are all positive is

$$x = y = \frac{S}{3}$$

and hence

$$z = S - x - y = \frac{S}{3}$$