I

unit - 2 section (B)

(Moderate Questions)

① If $a<b$, prove that $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$ by using L.M.V theorem. Hence deduce the following.

(i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$

(ii) $\frac{5\pi+4}{20} < \tan^{-1}(2) < \frac{\pi+2}{4}$.

solution:- Take $f(x) = \tan^{-1} x$

then $f'(x) = \frac{1}{1+x^2}$

By LMV theorem, we have

$(b-a) f'(c) = f(b) - f(a)$, for some $c \in (a,b)$

$$\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a}$$

for this $a<c<b$, we have

$1+a^2 < 1+c^2 < 1+b^2$

$$\therefore \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

i.e $\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$

$$\boxed{\therefore \frac{(b-a)}{(1+b^2)} < \tan^{-1} b - \tan^{-1} a < \frac{(b-a)}{(1+a^2)}}$$

(ii) In particular, if $a = 1, b = \frac{4}{3}$ then

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{\frac{4}{3}-1}{1+1}$$

$$\frac{\pi}{4} + \frac{3}{35} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

(iii) Put $a = 1, b = 2$, $\frac{2-1}{1+4} < \tan^{-1}(2) - \tan^{-1}(1) < \frac{2-1}{1+1}$

$$\frac{1}{5} < \tan^{-1}(2) - \frac{\pi}{4} < \frac{1}{2}$$

$$\frac{1}{5} + \frac{\pi}{4} < \tan^{-1}(2) < \frac{1}{2} + \frac{\pi}{4}$$

$$\frac{5\pi+4}{20} < \tan^{-1}(2) < \frac{\pi+2}{4}.$$

(b) Find 'c' of Cauchy's mean value theorem for $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in (a, b), where $0 < a < b$.

<u>solution</u>:- Clearly f, g are continuous on $[a, b] \subseteq R^+$

we have $f'(x) = \frac{1}{2\sqrt{x}}$ and $g'(x) = \frac{-1}{2x\sqrt{x}}$ which Exist on (a, b).

$\therefore f, g$ are differentiable on $(a, b) \subseteq R^+$.

$\therefore$ Conditions of Cauchy's mean value theorem are satisfied on (a, b).

$\therefore$ There Exisists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\therefore \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} \Rightarrow \frac{\sqrt{b} - \sqrt{a}}{\frac{\sqrt{a} - \sqrt{b}}{\sqrt{ab}}} = \frac{-2c\sqrt{c}}{2\sqrt{c}}$$

$$\Rightarrow \sqrt{ab} = c.$$

(2) (b) If $f(x) = \sin^{-1} x$, $0 < a < b < 1$, use mean value theorem to prove that $\frac{b-a}{\sqrt{(1-a^2)}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$

Solution: Take $f(x) = \sin^{-1} x$ then by LMV Theorem on (a, b)

$$f(b) - f(a) = (b-a) f'(c), \text{ for some } c \in (a, b)$$

$$\boxed{\sin^{-1} b - \sin^{-1} a = (b-a) \frac{1}{\sqrt{1-c^2}}} \text{ for some } c \in (a, b)$$

Now, $a < c < b$

$$\Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow 1 - a^2 > 1 - c^2 > 1 - b^2$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \frac{b-a}{\sqrt{1-c^2}} < \frac{b-a}{\sqrt{1-b^2}}$$

$$\therefore \frac{b-a}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{b-a}{\sqrt{1-b^2}}$$

③ ⓐ verify Cauchy's mean value theorem for

(a) $f(x) = e^{x}$ and $g(x) = e^{-x}$ in $[a, b]$, where $0 < a < b$

Sol:- $f(x) = e^{x}$

$g(x) = e^{-x}$

$g(x) = e^{-x}$

$g(x) \neq 0 \ \forall \ x \in (a, b)$

By Cauchy's mean value theorem

$$\frac{e^{b} - e^{a}}{e^{-b} - e^{-a}} = \frac{e^{c}}{-e^{-c}} = -e^{2c}$$

$$\frac{(e^{b} - e^{a})}{(e^{a} - e^{b})} e^{a+b} = -e^{2c}$$

$$\therefore e^{a+b} = e^{2c}$$

$$\boxed{\therefore \ c = \frac{a+b}{2} \in (a, b}$$

(3)(b) $f(x) = \sin x$ and $g(x) = \cos x$ in $\left[0, \frac{\pi}{2}\right]$

Sol: $f(x) = \sin x$; $g(x) = \cos x$

If $g(x) \neq 0 \ \forall \ x \in [a, b]$, Then by Cauchy's theorem

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \text{ ; for some } a \in (a, b)$$

i.e $$\frac{\cos c}{-\sin c} = \frac{\sin b - \sin a}{\cos b - \cos a} = \frac{2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)}{2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)}$$

$$\therefore \cot c = \cot\left(\frac{a+b}{2}\right)$$

$$\boxed{\therefore c = \frac{a+b}{2} \in (a, b)}$$