QUESTION BANK UNIT-1

SECTION-A

1) Rank of matrix $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix} \simeq 2$
[MODEL paper 2]

2) Solve equations $x+2y+3z=0$; $3x+4y+4z=0$; $7x+10y+12z=0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \simeq \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow x+2y+3z$$

$-2y-5z=0$

$\boxed{z=0}$

$\Rightarrow -2y-0=0$

$\boxed{y=0}$

$\Rightarrow x+0+0=0$

$\boxed{x=0}$

so $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the only solution.

3) $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \simeq \lambda^2 - 7\lambda + 6 = 0$

$\lambda^2 - 6\lambda - \lambda + 6 = 0$

$\lambda(\lambda-6) - 1(\lambda-6) = 0$

$\lambda = 1, 6$

4) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} \simeq$ $|A-\lambda I|=0$ $= \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}$

$= \lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$

$S_2 = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 6 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} = 6 + 10 + 15 = 31\lambda$

$S_3 = 3(10) - 1(0) + 4(0) = 30$

$$\Rightarrow 2 \left| \begin{array}{cccc} 1 & -10 & 31 & -30 \\ 0 & 2 & -16 & 30 \\ \hline 1 & -8 & 15 & 0 \end{array} \right.$$

$\simeq x^2 - 8x + 15 = 0 \Rightarrow x^2 - 3x - 5x + 15$

$x(x-3) - 5(x-3) = 0$

$x = 5, 3$

$\therefore$ eigen values $= 2, 3, 5$

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⑤ two eigen values of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are 3, 15

$\simeq \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \simeq \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$S_2 = \begin{bmatrix} 8 & -6 \\ -6 & 7 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -4 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 2 \\ 2 & 3 \end{bmatrix} = 20 + 5 + 20 = 45$

$S_3 = 8(21-16) + 6(-18+8) + 2(24-14) = 40 - 60 + 20$
$= \underline{0}$

$$\begin{array}{r|ccc} 3 & 1 & -18 & 45 \\ & 0 & 3 & -45 \\ \hline & 1 & -15 & 0 \end{array}$$

$x^2 - 15x = 0$
$x(x-15) = 0$
$\underline{\underline{x = 0}}$

⇒ Moderate Type

① CHT for $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \Rightarrow |A - \lambda I| = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{vmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{vmatrix} = \begin{vmatrix} 9 & 16 \\ 8 & 17 \end{vmatrix}$

$4\lambda = 4\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 16 \\ 8 & 12 \end{vmatrix} \Rightarrow \begin{vmatrix} 9 & 16 \\ 8 & 12 \end{vmatrix} - \begin{vmatrix} 4 & 16 \\ 8 & 12 \end{vmatrix} - \begin{vmatrix} 5 & 0 \\ 0 & 5 \end{vmatrix} = 0$

Hence, verified.

② Characteristic equation $\begin{bmatrix} 3 & 10 & 15 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} \simeq \begin{bmatrix} 3-\lambda & 10 & 15 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{bmatrix}$

$\lambda^3 - 7\lambda^2 + 14\lambda - 2 = 0$

$S_2 = \begin{bmatrix} 3 & 10 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ 5 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 15 \\ 3 & 7 \end{bmatrix} = 11 - 1 - 24 = -14$

$S_3 = 3(-21+20) - 10(-14+12) + 15(-10+9) = -3 + 20 - 15 = 2$

4) index and signature of quad. form $x_1^2+2x_2^2-3x_3^2$

index = no. of +ve square terms in canonical form
$= 2$

signature = diff. b/w +ve & -ve terms
$= 2-1 = 1$

5) $2xy + 2xz - 2yz = 0$

5) $x^2 - y^2 + 2z^2 + 2xy - 4yz + 6xz$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & -2 \\ 3 & -2 & 2 \end{bmatrix}$$

SECTION-B

1 a) $R_2 \leftrightarrow R_1$, $R_2 - 4R_1$, $R_3 - 15R_1$

$$\begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{bmatrix}$$

$$R_3 - 3R_2 \simeq \begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & -6 & 18 & 12 \\ & 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2, \quad \rho(A:B) = 2$$

consistent with unique solution as $\rho(A) = \rho(A/B) = n$

b) $R_2 - R_1$, $R_3 - R_1$; $R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

i) no solution: $\rho(A) \neq \rho[A|B]$
λ = any value but 3
$\mu = 10$

ii) unique solution
$\rho(A) = \rho[A|B] = n$
$\lambda \neq 3, \mu \neq 10$

iii) infinitely many solutions
$\rho(A) = \rho(A|B) < n$
$\lambda = 3, \mu = 10$

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$R_2 \leftrightarrow R_1$, $R_2 - 3R_1$; $R_3 - 6R_1$

2) $\begin{bmatrix} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{bmatrix} \simeq \begin{bmatrix} 1 & 2 & -3 & -2 \\ 3 & -1 & 4 & 3 \\ 6 & 5 & \lambda & -3 \end{bmatrix} \simeq \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & -7 & \lambda+18 & 9 \end{bmatrix}$

$R_3 - R_2$ $\begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & 0 & \lambda+5 & 0 \end{bmatrix}$

$\boxed{\text{infinite}}$

$P(A) = P(A/B) < n$

let $\boxed{\lambda = -5}$ so

$2 = 2 < 3$

$\begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$ $3x - y + 4z = 3$
$x + 2y - 3z = -2$
$6x + 5y - 5z = -3$

$\Rightarrow$

$R_2 - R_1$; $R_3 - 4R_1$; $R_4 - 5R_1$

3) $\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 10 & -16 \end{bmatrix}$

$R_2 \div R_1$ $\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -12 & 10 & -16 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -6 & 5 & -8 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $2R_3 - R_2$

$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -6 & 5 & -8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$ $x + y - 2z = 3$
$-6y + 5z = -8 \rightarrow -6y = -8$

$\boxed{z = 0}$ $\boxed{y = \frac{4}{3}}$

$x + \frac{4}{3} = 3$

$x = 3 - \frac{4}{3} = \boxed{\frac{5}{3} = x}$

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eigen values of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are 3, 15, 0

$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$ let $\lambda = 0$ $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ $R_3 \leftrightarrow R_1$, $R_2 + 3R_1$, $R_3 - 4R_1$

$$\begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix} \simeq \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix} \simeq \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$2x - 4y + 3z = 0$

$-5y + 5z = 0$

$\boxed{y = z}$

$\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ for $\boxed{\lambda = 0}$

let $\lambda = 3$ $\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \simeq \begin{bmatrix} 1 & -2 & 0 \\ -3 & 2 & -2 \\ 5 & -6 & 2 \end{bmatrix} \simeq \begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -2 \\ 0 & 4 & 2 \end{bmatrix}$

$R_3 \leftrightarrow R_1$, $\frac{R_3}{2}$

$R_3 - 5R_1$; $R_2 + R_3$ $\simeq \begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_2 + 3R_1$

$x - 2y = 0$

$-4y + 2z = 0$

$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ for $\lambda = 3$

for $\lambda = 15$, $\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \simeq \begin{bmatrix} -7 & -6 & 2 \\ -3 & -4 & -2 \\ 1 & -2 & -6 \end{bmatrix}$ $R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & -2 & -6 \\ -3 & -4 & -2 \\ -7 & -6 & 2 \end{bmatrix} \simeq \begin{bmatrix} 1 & -2 & -6 \\ 0 & -10 & -20 \\ 0 & 0 & 0 \end{bmatrix} \simeq \begin{bmatrix} 1 & -2 & -6 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$R_3 + 7R_1$

$R_2 + 3R_1$

$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$x - 2y - 6z = 0$

$-y - 2z = 0$

$\boxed{-2z = y}$
$\boxed{x = 2z}$

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2) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$S_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3 + 3 + 3 = 9$

$S_3 = 2(4-1) + 1(-2+1) + 1(1-2) = 6 - 1 - 1 = \underline{4}$

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 9 & -4 \\ & 0 & 1 & -5 & 4 \\ \hline & 1 & -5 & 4 & 0 \end{array} \Rightarrow x^2 - 5x + 4 = 0$$

$x^2 - 4x - x + 4$

$x(x-4) - 1(x-4) = 0$

$x = \underline{1, 4}$

Multiply with A^{-2}

$\lambda - 6I + 9A^{-1} - 4A^{-2} = 0 \Rightarrow A^{-2} = \frac{1}{4}\left[A - 6I + 9A^{-1}\right]$

$A^{-1} = \frac{\text{adj}A}{|A|}$ $\text{adj}A = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}^T \simeq \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -2 & -1 & 3 \end{bmatrix} \frac{1}{4}$

$A^{-2} = \frac{1}{4}\left[\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} + \right.$

$A^{-2} = \frac{1}{4}\begin{bmatrix} \frac{11}{4} & \frac{-13}{4} & \frac{-5}{4} \\ \frac{-13}{4} & \frac{11}{4} & \frac{-13}{4} \\ \frac{-5}{4} & \frac{-13}{4} & \frac{11}{4} \end{bmatrix}$

3) $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ $\lambda^3 - 4\lambda^2 - \lambda - 4 = 0$ — ①

$S_2 = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -1 & -3 \end{bmatrix} = 6 - 1 - 6 = -1$

$S_3 = 4(-9+8) - 6(-3+2) + 6(-4+3) = -4 + 6 - 6$

Multiply with A^{-1}

$A^2 - 4A - I - 4A^{-1} = 0$

$\rightarrow A^{-1} = \frac{1}{4}(A^2 - 4A - I)$

$$A^2 = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} = \begin{matrix} 16 & 18 & 18 \\ 5 & 7 & 6 \\ 5 & -6 & -5 \end{matrix}$$

$$A^{-1} = \frac{1}{4}\left[\begin{bmatrix} 16 & 18 & 18 \\ 5 & 7 & 6 \\ 5 & -6 & -5 \end{bmatrix} - \begin{bmatrix} 16 & 24 & 24 \\ 4 & 12 & 8 \\ -4 & -16 & -12 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} -1 & -6 & -6 \\ 1 & -6 & -2 \\ 1 & 10 & 6 \end{bmatrix}$$

⇒ TYPICAL QUESTIONS

1) Reduce $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ to diagonal form.

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = \quad \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$S_2 = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} = 14 + 8 + 14 = 36$$

$6(9-1) + 2(-6+2) + 2(-6+2) = 48 - 8 - 8 = 32$

$$\begin{array}{c|cccc} & 1 & -12 & 36 & -32 \\ 2 & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

$= x^2 - 10x + 16 = 0$

$x^2 - 8x - 2x + 16 = 0$

$x(x-8) - 2(x-8) = 0$

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$x = 2, 2, 8$ $\quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

for $x = 2$, $\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \simeq \begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \simeq \begin{bmatrix} -2 & 1 & -1 \\ 2 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_1$

$R_1 + R_3$

$R_2 + 2R_1$

$R_1 + R_2$

$-2x + y - z = 0$ $\quad \simeq \begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$y = k_1, z = k_2$

$-2x + k_1 - k_2 = 0$ $\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} \frac{k_1}{2} \\ k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{k_2}{2} \\ 0 \\ k_2 \end{bmatrix}$

$x = \frac{k_1 - k_2}{2}$

$= k_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$

Put $\lambda = 8$ $\quad A = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}$

$R_1 + R_3$

$R_1 - R_2$

$\simeq \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \simeq \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad R_2 - R_3$

$-2x - 2y - 2z = 0$ $\quad -2x - 2y + 2y = 0$

$-3y - 3z = 0$ $\quad \boxed{x = 0}$ $\quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$-y = z$

$X = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ $\quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

$D = N^T . AN \Rightarrow N = \begin{bmatrix} 0 & \frac{1}{\sqrt3} & -\frac{1}{\sqrt3} \\ -\frac{1}{\sqrt2} & \frac{2}{\sqrt3} & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{2}{\sqrt3} \end{bmatrix}$ $\quad N^T = \begin{bmatrix} 0 & -1/\sqrt2 & 1/\sqrt2 \\ 1/\sqrt3 & 2/\sqrt3 & 0 \\ -1/\sqrt2 & 0 & 2/\sqrt3 \end{bmatrix}$

$\begin{bmatrix} 0 & 1/\sqrt3 & -\frac{1}{\sqrt3} \\ -\frac{1}{\sqrt3} & 2/\sqrt3 & 0 \\ 1/\sqrt2 & 0 & 2/\sqrt3 \end{bmatrix} \begin{bmatrix} 0 & -1/\sqrt2 & 1/\sqrt2 \\ 1/\sqrt3 & 2/\sqrt3 & 0 \\ -\frac{1}{\sqrt2} & 0 & 2/\sqrt3 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & -3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$x^2 + 3y^2 + 3z^2 - 2yz$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \Rightarrow \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

$$S_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = 3 + 8 + 3$$

$S_3 = 1(9-1) - 0(\ ) + 0(\ )$

$$\begin{array}{c|cccc} 1 & 1 & -7 & 14 & -8 \\ & 0 & 1 & -6 & 8 \\ \hline & 1 & -6 & 8 & 0 \end{array}$$

$\Rightarrow x^2 - 6x + 8 = 0$

$x^2 - 4x - 2x + 8 = 0$

$x(x-4) - 2(x-4) = 0$

$x = 4, 2$

$\Rightarrow \lambda = 1, 4, 2$

$\Rightarrow$ for $\lambda = 1$, eigen vector $\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

$2y - z = 0$

$y + z = 0$

$y = -z$

$x = 0$

$2y - z = 0$

$-y + 2z = 0$

$2z = y$

$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

UNIT-II

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State Rolle's theorem.

If $f:[a,b] \to R$ is a function such that

i) $f(x)$ is continuous in closed interval $[a,b]$

ii) $f'(x)$ exists for every value of x in the open interval (a,b)

iii) $f(a)=f(b)$ then there is atleast one value c, in (a,b) such that $\boxed{f'(c)=0}$

State lagrange's MVT

If $f:[a,b] \to R$ is a function such that

i) $f(x)$ is continuous in closed interval $[a,b]$

ii) $f'(x)$ exists for every value of x in the open interval and then there is atleast one value in c in (a,b) such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

Cauchy's MVT,

If $f,g:[a,b] \to R$ are continuous functions such that

i) $f(x)$ and $g(x)$ are continuous on $[a,b]$

ii) $f'(x)$ and $g'(x)$ exist in (a,b)

iii) $g'(x) \neq 0$ for any value of x in (a,b)

then there is atleast one value of x in (a,b) such that.

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'c}}$$

Q) $f(x)=x^3-4x$ in $(-2,2)$

i) every poly. func. is continuous & differentiable every where in R

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$\therefore f(-2)=0=f(2)=0$

$\frac{1}{\cos^2 x} = 0$

Now, $f'(x) = \frac{d}{dx}[x^3 - 4x] = 3x^2 - 4 = 0$

$3x^2 = 4$

$\boxed{x = \frac{2}{\sqrt{3}}}$

(5) $f(x) = \tan x$ in $(0, \pi)$

A) since $\frac{\pi}{2} \in (0, \pi)$ and $f(x)$ is not continuous at $x = \frac{\pi}{2}$

$\therefore$ condition of continuity of $f(x)$ in $[0, \pi]$ is not satisfied. So, Rolle's theorem is not applicable

MODERATE TYPE

(1) Lagrange's mean value theorem for $f(x) = |x|$ in $[-1, 1]$

A) $f(x)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$

But it is known that $f(x) = |x|$ is not differentiable at $x = 0 \in (-1, 1)$.

Thus, Lagrange's theorem is not applicable for the given function

$9 - 9 + 2$

$4 + 6 + 2$

(2) Lagrange's mean value theorem for $f(x) = x^2 - 3x + 2$ in $[-2, 3]$

A) $f(x)$ is continuous in closed interval $[-2, 3]$

$f'(x)$ exists for every value of x in open interval $(-2, 3)$

$f'(c) = \frac{f(3) - f(-2)}{5} = \frac{-10}{5} = -2 = 2x - 3$

$\boxed{x = \frac{1}{2}}$

$f(3) = 2$

$f(-2) = 12$

e^x and e^{-x} in (a,b)

exponential function are continuous on $[a,b]$ and differentiable on (a,b)

$\Rightarrow f(x) = e^x \Rightarrow f'(x) = e^x$

$\Rightarrow g(x) = e^{-x} \Rightarrow g'(x) = -e^{-x}$

By Cauchy's MVT, $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

$\Rightarrow \dfrac{e^b - e^a}{e^{-b} - e^{-a}} = \dfrac{e^c}{-e^{-c}} = e^{-2c}$

$\Rightarrow \dfrac{(e^b - e^a)}{(e^a - e^b)} e^{a+b} = -e^{2c} \Rightarrow e^{a+b} = e^{2c}$

$\Rightarrow c = \dfrac{a+b}{2} \in (a,b)$

(2) Maclaurin's series expansion of $f(x) = e^x$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$f''(x) = e^x \Rightarrow f''(0) = 1$

$f(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(\theta x)}$$

(3) Maclaurin's series expansion of the function $f(x) = \sin x$

$f(x) = \sin x \; ; \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{iv}(x) = +\sin x \Rightarrow f^{iv}(0) = 0$

$f^{v}(x) = \cos x \Rightarrow f^{v}(0) = 1$

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(\theta x)}$$

$\Rightarrow f(x) = 0 + x(1) + \dfrac{x^2}{2!}(0) + \dfrac{x^3}{3!}(-1) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots + \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$

Teacher's Signature

SECTION - B

1. a) Rolle's theorem for $f(x) = (x-a)^m (x-b)^n$ in $[a,b]$

A) every polynomial fn. is continuous and differentiable every where in R. $\therefore f(a) = 0 ; f(b) = 0 \therefore f(a) = f(b)$

$$f'(c) = m(x-a)^{m-1}(x-b)^n + n(x-b)^{n-1}(x-a)^m$$

$$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\left[m(x-b) + n(x-a)\right] = 0$$

$$\Rightarrow mx - bm + nx - na = 0 \Rightarrow x(m+n) = mb + na$$

$$\boxed{x = \frac{mb+na}{m+n}}$$

b) Rolle's theorem for $f(x) = x^{2m-1}(a-x)^{2n}$, find x between 0 and a, where $f'(x) = 0$.

A) every polynomial fn. is continuous and differentiable every where in R. $\therefore f(a) = 0 ; f(b) = 0 \rightarrow f(a) = f(b)$

$$f'(c) = (2m-1)x^{2m-2}(a-x)^{2n} + (2n)(a-x)^{2n-1}x^{2m-1}$$

$$0 = x^{2m-2}(a-x)^{2n-1}\left(2m-1(a-x) + 2n(x)\right]$$

$$0 = 2ma - a - 2mx + x + 2nx$$

$$a - 2ma = x(2n+1-2m)$$

$$\boxed{x = \frac{a-2ma}{2n+1-2m}}$$

a) Rolle's theorem for $\frac{\sin x}{e^x}$ in $[0,\pi]$

$f(x)$ is continuous and differentiable in $[0,\pi]$
[$\sin x$ & e^x is every where continuous) and $e^x \neq 0 \ \forall x \in \mathbb{R}$

$f(0) = 0$; $f(\pi) = 0 \Rightarrow f(0) = f(\pi)$

Now, $f'(x) = \frac{e^x \cos x - \sin x e^x}{e^{2x}} = \frac{e^x(\cos x - \sin x)}{e^{2x}}$

$\Rightarrow \sin x = \cos x \Rightarrow x = \frac{\pi}{4} \in (0,\pi)$

$f(x) = x(x-1)(x-2)$ where $a=0, b=\frac{1}{2}$

$f(a) = 0 \Rightarrow f(b) = -\frac{1}{8} \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{-\frac{1}{8}}{\frac{1}{2}} = -\frac{1}{4}$

$\Rightarrow 3x^2 - 6x + 2 = -\frac{1}{4} \Rightarrow 12x^2 - 24x + 8 = -1$

$\Rightarrow 12x^2 - 24x + 9 = 0$

$\Rightarrow 4x^2 - 8x + 3 = 0$

$4x^2 - 2x - 6x + 3 = 0$

$2x(2x-1) - 3(2x-1) = 0$

$x = \frac{3}{2}, \frac{1}{2}$

$\therefore c = \frac{1}{2}$

3) a) $f(x) = (x-1)(x-2)(x-3)$ where $(0,4)$

~~$\Rightarrow f(a) = 0$; $f(b) = 24 \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{24}{4} = 6$~~

~~$3x^2 - 6x + 2 = 6$~~ $\Rightarrow f(a) = -6$

~~$3x^2 - 6x - 4 = 0$~~ $f(b) = f(4) = +6$

$x^3 - 6x^2 - 2x - 6$

$f'(c) = \frac{6-(-6)}{4} = \frac{3}{2}$ $\Rightarrow 3c^2 - 12c + 11 = 3$

$\Rightarrow 3c^2 - 12c + 8 = 0$

$\boxed{c = 2 \pm \frac{2}{\sqrt{3}}}$

Teacher's Signature ____________

b) lagrange's MVT, $x > \log(1+x) > \frac{x}{1+x}$ for $x>0$.

A) Let $\log(1+x) = f(x)$ in $[0,x]$

$\theta\ (0<\theta<1)$

$$\frac{f(x)-f(0)}{x-0} = f'(\theta x)$$

$$\frac{\log(1+x)}{x} = \frac{1}{1+\theta x}$$

Now $0<\theta<1,\ x>0 \Rightarrow \theta x < x$

$\Rightarrow 1+\theta x < 1+x$

$\Rightarrow \frac{1}{1+\theta x} > \frac{1}{1+x}$

$\Rightarrow \frac{x}{1+\theta x} > \frac{x}{1+x}$ — ①

Again, $0<\theta<1,\ x>0$

$\theta x > 0$

$\Rightarrow 1+\theta x > 1$

$\Rightarrow \frac{1}{1+\theta x} < 1 \Rightarrow \frac{x}{1+\theta x} < x$ — ②

from ① & ②

$$\frac{x}{1+x} < \log(1+x) < x$$

MODERATE

① take $f(x) = \tan^{-1}x$, then $f'(x) = \frac{1}{1+x^2}$

By lagrange's theorem, we have $f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow$ for some $c \in (a,b)$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$$

for this, $a<c<b$, we have $\Rightarrow 1+a^2 < 1+c^2 < 1+b^2$

$\therefore \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$ i.e. $\frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$

(i) $\left(\frac{b-a}{1+b^2}\right) < \tan^{-1} b - \tan^{-1} a < \frac{(b-a)}{(1+a^2)}$

Hence proved.

Put $a=1,\ b=\frac{4}{3}$, $\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1}$

$= \frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$

$\Rightarrow \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$

ii) Put $a=1,\ b=2$, $\frac{2-1}{1+4} < \tan^{-1}(2) - \tan^{-1}(1) < \frac{2-1}{1+1}$

$\frac{1}{5} < \tan^{-1}(2) - \frac{\pi}{4} < \frac{1}{2}$

$\frac{1}{5} + \frac{\pi}{4} < \tan^{-1}(2) < \frac{1}{2} + \frac{\pi}{4}$

$\frac{5\pi+4}{20} < \tan^{-1}(2) < \frac{\pi+2}{4}$

(2) a) $f(x) = \sin^{-1} x,\ 0<a<b<1$, P.T $\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$

A) $f(x) = \sin^{-1} x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

$\Rightarrow \frac{1}{\sqrt{1-x^2}} = \frac{\sin^{-1} b - \sin^{-1} a}{b-a}$ - for this, $a<x<b$ we have

$\Rightarrow \sqrt{1-a^2} < \sqrt{1-x^2} < \sqrt{1-b^2} \Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-x^2}} < \frac{1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$

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b) Cauchy's MVT for $f(x)=\sqrt{x}$ & $g(x)=\frac{1}{\sqrt{x}}$ in (a,b) where $0<a<b$

A) $\frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$; $g'(x) = -\frac{1}{2x^{3/2}}$

$\Rightarrow \frac{\frac{1}{2\sqrt{x}}}{-\frac{1}{2}x^{-3/2}} = -\frac{1}{2x^{1/2}} \cdot 2x^{3/2} = x = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}$

$\Rightarrow x = -\sqrt{ab}$

3) Cauchy's MVT, for

a) $f(x)=e^x$ and $g(x)=e^{-x}$ in $[a,b]$ where $0<a<b$

A) $f'(x)=e^x$; $g'(x)=-e^{-x} \Rightarrow \frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$\Rightarrow \frac{e^x}{-e^{-x}} = \frac{e^b-e^a}{-e^{-b}-e^{-a}} \Rightarrow -e^{2x} = \frac{e^b-e^a}{\frac{-1}{e^b}-\frac{+1}{e^a}} = \frac{e^b-e^a}{\frac{+e^a-e^b}{e^ae^b}}$

$\Rightarrow e^{a+b} = -e^{2c} \Rightarrow \frac{a+b}{2} = c \in (a,b)$

b) $f(x)=\sin x$ and $g(x)=\cos x$ in $[0,\frac{\pi}{2}]$

A) $f'(x)=\cos x \Rightarrow g'(x) = -\sin x$

$\Rightarrow \frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)} \Rightarrow -\cot x = \frac{\sin\frac{\pi}{2}-\sin 0}{\cos\frac{\pi}{2}-\cos 0}$

$\Rightarrow -\cot x = \frac{1}{-1} \Rightarrow \cot x = 1$

$\Rightarrow \boxed{\frac{\pi}{4} = x}$

c) $x > \log(1+x) > x - \frac{x^2}{2}$

A) $f(x) = \log(1+x)$

$f'(x) = \frac{1}{1+x}$; $f''(x) = \frac{-1}{(1+x)^2}$, $f'''(x) = \frac{2}{(1+x)^3}$

By machlaurin's theorem, $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$

$= 0 + x\cdot 1 + \frac{x^2}{2!}\frac{(-1)}{(1+\theta x)^2}$

$\log(1+x) = x - \frac{x^2}{2!}\cdot\frac{1}{(1+\theta x)^2}$ for some $0<\theta<1$

since $0<\theta<1$, $0<\theta x<x$

$\therefore 1 < 1+\theta x < 1+x$

$\Rightarrow \frac{1}{(1+x)^2} < \frac{1}{(\theta x+1)^2} < 1 \Rightarrow \frac{-x^2}{2!(x+1)^2} > \frac{-x^2}{2!(1+\theta x)^2} > \frac{-x^2}{2}$

$\Rightarrow x - \frac{x^2}{2!(x+1)^2} > x - \frac{x^2}{2!(1+\theta x)^2} > x - \frac{x^2}{2!}$

$\Rightarrow x \geq \log(1+x) > x - \frac{x^2}{2!}$

b) Taylor's theorem, P.T $x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}$, for $x>0$

A) $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow f''(x) = -\sin x \Rightarrow f'''(x) = -\cos x$

$f^{iv}(x) = \sin x \Rightarrow f^{v}(x) = \cos x$

$\Rightarrow f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^5}{5!} f^{v}(\theta x)$

$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}\cos\theta x \Rightarrow \sin x + \frac{x^3}{6} = x + \frac{x^5}{5!}\cos\theta x$

since $0<\theta<1$, $x>0$, we have $0<\cos\theta x<1$

$\therefore x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}$

3 a) $f(x) = (1-x)^{5/2}$ Maclaurin's theorem with remainder up to 3 terms, where $x=1$

A) $f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$

$f'(x) = \frac{-5}{2}(1-x)^{3/2} \Rightarrow f'(0) = \frac{-5}{2}$

$f''(x) = \frac{+15}{4}(1-x)^{1/2} \Rightarrow f''(0) = \frac{15}{4}$

$f'''(x) = \frac{-15}{8}(1-x)^{-1/2} \Rightarrow f'''(0) = \frac{-15}{8}$

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By machlaurin's theorem,

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x)$

for some $0 < \theta < 1 \Rightarrow 1 - \frac{5x}{2} + \frac{15}{8}x^2 - \frac{15}{8}(1-\theta x)^{-1/2}$

$\Rightarrow f(1) = 1 - \frac{5}{2} + \frac{15}{8} - \frac{15}{48}(1-\theta)^{-1/2}$

$\Rightarrow (1-\theta)^{1/2} = \frac{5}{6} \Rightarrow 1-\theta = \frac{25}{36} \Rightarrow \theta = \frac{11}{36} = 0.305$

b) Taylor's series expansion of $\log_e x$ in powers of $(x-1)$

A) $f(x) = \log_e x \rightarrow f(1) = 0$

$f'(x) = \frac{1}{x}$; $f'(1) = 1$

$f''(x) = \frac{-1}{x^2}$; $f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$, $f'''(1) = 2$

$(1+x)^{-1}$

$-1(x+1)^0$

Taylor series $\Rightarrow f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$

$\rightarrow \log x = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \dots$

Q) Machlaurin's series, a) $\log(1+x)$

b) $e^x \cos x$

a) $\log(1+x)$

$f(x) = \log(1+x)$; $f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x}$; $f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$; $f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$; $f'''(0) = 2$

By machlaurin's theorem

$f(x) = f(0) + \frac{x}{1!}(f'(0)) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0)$

$\log(1+x) = 0 + x + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2)$

$\Rightarrow \log(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!}$

ii) $e^x \cos x$

$f(x) = e^x \cos x$; $f(0) = e^0 \cos 0 = 1$

$f'(x) = -e^x \sin x + \cos x e^x = e^x(\cos x - \sin x)$; $f'(0) = 1$

$f''(x) = e^x(-\sin x - \cos x) + (\cos x - \sin x)e^x$

$= e^x(-2\sin x) = -2e^x \sin x$; $f''(0) = 0$

$f'''(x) = e^x(-2\cos x) - 2\sin x e^x = -2e^x(\cos x + \sin x)$

$f'''(0) = -2$

$\Rightarrow e^x \cos x = 1 + x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2)$

$e^x \cos x = 1 + x - \frac{2x^3}{3!}$

UNIT-IV

$\Rightarrow$ SECTION-A [EASY]

1) Evaluate $\int_{x=0}^{2}\int_{y=0}^{y=x} y \, dy \, dx$

A) $y = x$ to 0 $\Rightarrow \int_0^2 \left[\frac{y^2}{2}\right]_0^x dx = \int_0^2 \frac{x^2}{2} dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{4}{3}$

$x = 0$ to 2

Teacher's Signature

2) Evaluate $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

A) $I = \int_{x=0}^{1}\int_{y=0}^{1} \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}}\left[\int_{y=0}^{1} \frac{dy}{\sqrt{1-y^2}}\right] dx$

let $\sin u = \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x)$

$I = \int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}} \left|\sin^{-1}(y)\right|_0^1 dx = \int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}} \left|\sin^{-1}(1) - \sin^{-1}(0)\right|$

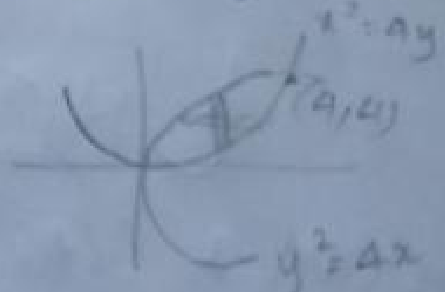
$= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left(\frac{\pi}{2}\right) dx = \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\left|\sin^{-1}(x)\right|_0^1$

$= \frac{\pi}{2}\left\{\sin^{-1}(1) - \sin^{-1}(0)\right\} = \frac{\pi}{2}\left\{\frac{\pi}{2}\right\} = \frac{\pi^2}{4}$

$\therefore \int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \frac{\pi^2}{4}$ //

③ $\iint_R f(x,y)\,dx\,dy$ where R = region bounded by parabolas $y^2 = 4x$ and $x^2 = 4y$

A) $x = \frac{y^2}{4} \Rightarrow \frac{y^4}{16} = 4y$



$y^2 = 4x$
$16 = 4x$
$\boxed{x = 4}$

$y^4 = 64y$
$y^3 = 64$
$\boxed{y = 4}$

$\Rightarrow I = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} \Rightarrow \int_0^4 \left[2\sqrt{y} - \frac{y^2}{4}\right] dy$

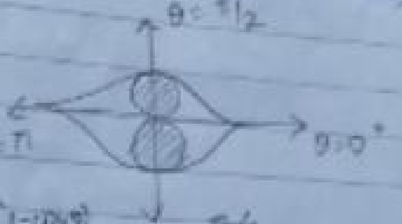
$$= 2y^{3/2} - \frac{y^3}{12}\Big]_0^4 = 2(2^3) - \frac{4^3}{12} = 16 - \frac{64}{12} = \frac{4}{3}$$

1) Evaluate $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}\, r\, d\theta\, dr$

$$= \int_0^{\pi/2}\left[-\frac{e^{-r^2}}{2}\right]_0^{\infty} d\theta = \frac{1}{2}[\theta]_0^{\pi/2} = \frac{\pi}{4}$$ //

2) limits of $\iint_R f(r,\theta)\,dr\,d\theta$ over the cardiod $r = a(1-\cos\theta)$ and $r = a(1+\cos\theta)$

$r = a(1-\cos\theta)$; $r = a(1+\cos\theta)$



$$A = 4\int_{0\,(\theta)}^{\pi/2}\int_{0\,(r)}^{a(1-\cos\theta)} r\cdot dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left.\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta = 2\int_0^{\pi/2} a^2(1-\cos\theta)^2\cdot d\theta$$

$$= 2a^2\int_0^{\pi/2}(1+\cos^2\theta - 2\cos\theta)\,d\theta = 2a^2\int_0^{\pi/2}\left[1+\frac{1+\cos 2\theta}{2} - 2\cos\theta\right]d\theta$$

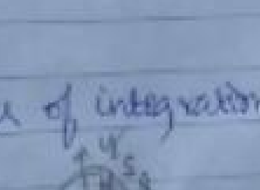
$$= 2a^2\left[\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} - 2\sin\theta\right]_0^{\pi/2} = 2a^2\left[\frac{\pi}{2} + \frac{\pi}{4} - 2\right]$$

$$= a^2\left[\frac{3\pi - 4}{2}\right]$$ //

⇒ MODERATE TYPE:-

1) new limits ⇒ after changing the order of integration

$$\int_{-a\,(y)}^{a}\int_{0\,(x)}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy \qquad x^2+y^2=a^2$$



let RS be new strip, limit varies from

$y = -a$ to a $\qquad x = 0$ to $x = a$

$x = 0$ to $\sqrt{a^2-y^2}$ $\qquad y = -\sqrt{a^2-x^2}$ to $y = \sqrt{a^2-x^2}$

$$\boxed{x^2+y^2=a^2}$$

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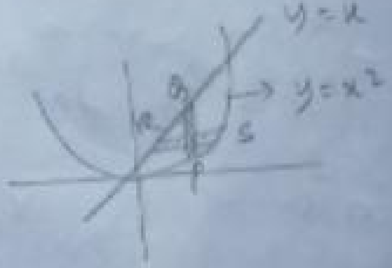
On changing the strip,

$$\rightarrow \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$$

Q) new limits of int. after changing order of integration

$$\int_0^1 \int_{x^2}^{x} f(x,y)\,dx\,dy$$



A) $y = x^2$ to x $\Rightarrow$ $\boxed{x=0}$ $y=0$

$x = 0$ to 1 $\rightarrow$ $\boxed{x=1}$ $y=1$

limits of new strip (RS) :-

$y=0$ to $y=1$

and $x=0$ to $x=\sqrt{y}$

$$\Rightarrow \int_0^1 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy = \int_0^1 \sqrt{y}\,dy = \frac{y^{3/2}}{3/2} = \frac{2}{3}$$

Q) Convert into polar coordinates & evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx$

A) $x=0$ to $x=a$ consider,

$y=0$ to $y=\sqrt{a^2-x^2}$ $x = r\cos\theta$

$y = r\sin\theta \Rightarrow x^2+y^2=1$

$dx\,dy = r\,dr\,d\theta$

$r\cos\theta = \sqrt{1-r^2\sin^2\theta}$

$r^2\cos^2\theta = 1-r^2\sin^2\theta$

$r^2=1$

$\boxed{r=1}$

$$I = \int_0^{\pi/2} \int_0^a r^3 \cdot dr\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} r^4 \Big]_0^a d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} a^4 d\theta = \frac{\pi}{8} a^4$$

$\iint_R x^2y^3\,dx\,dy$, R = region bounded by rectangle $0 \le x \le 1$ and $0 \le y \le 3$

$$\int_0^1\int_0^3 x^2y^3\,dy\,dx = \left.\frac{y^4}{4}\right]_0^3 \left[\frac{x^3}{3}\right]_0^1 = \frac{81}{4}\cdot\frac{1}{3} = \frac{27}{4}$$

$\iint_R f(r,\theta)\,dr\,d\theta$ bound by circles $r = 2\sin\theta$ & $r = 4\sin\theta$

$r = 2\sin\theta$ & $r = 4\sin\theta$, angle θ varies from 0 to π.

$\therefore$ $$\iint_R f(r,\theta)\,dr\,d\theta = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} dr\,d\theta = \int_0^{\pi} r\Big|_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \int_0^{\pi}(4-2)\sin\theta\,d\theta = \int_0^{\pi} 2\sin\theta\,d\theta = 2\int_0^{\pi}\sin\theta\,d\theta$$

$$\Rightarrow 2(-\cos\theta)\Big]_0^{\pi} = 2[-\cos\pi - \cos 0]$$

$$= 2[1-1] = 0$$

SECTION-B

⇒ EASY

3) a) Evaluate $\int_0^1\int_x^{\sqrt{x}}(x^2+y^2)\,dx\,dy$

$$\int_0^1\left(x^2y + \frac{y^3}{3}\right)\Big]_x^{\sqrt{x}} dx = \int_0^1\left[\left(x^2\sqrt{x} + \frac{x^{3/2}}{3}\right) - \left(x^3 + \frac{x^3}{3}\right)\right]dx$$

$$= \int_0^1\left[\left(\frac{3x^{5/2}+x^{3/2}}{3}\right) - \left(\frac{4x^3}{3}\right)\right]dx = \left(\frac{2x^{7/2}}{7} + \frac{2x^{5/2}}{15}\right) - \left(\frac{x^4}{3}\right)\Big]_0^1$$

$$= \left(\frac{2}{7} + \frac{2}{15}\right) - \frac{1}{3} = \frac{44}{105} - \frac{1}{3} = 3/35$$

b) Evaluate $\int_0^4 \int_0^{x^2} e^{y/x} \, dy \cdot dx$

A) $\int_0^4 \left[\frac{e^{y/x}}{1/x} \right]_0^{x^2} \cdot dx = \int_0^4 x e^{y/x} \Big]_0^{x^2} = \int_0^4 x \cdot (e^{x} - e^{0/x}) \, dx$

$= \int_0^4 x(e^x - 1) \, dx = \int_0^4 (xe^x - x) \, dx$

$\int_0^4 xe^x = \left[x \int e^x dx - \int \left[\frac{d}{dx}(x) \int e^x dx \right] \right] = xe^x - \int e^x dx = xe^x - e^x$

$\Rightarrow I = xe^x \Big]_0^4 - e^x \Big]_0^4 - \frac{x^2}{2} \Big]_0^4$

$= 4e^4 - e^4 + e^0 - 8 = \underline{\underline{3e^4 - 7}}$

2) a) Evaluate $\iint xy \, dx \, dy$ over positive quadrant of the circle $x^2 + y^2 = a^2$

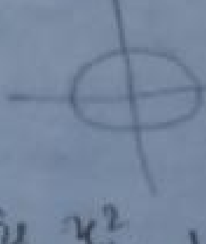
A) $I = \iint xy \cdot dy \cdot dx = \int_0^a \int_0^{\sqrt{a^2 - x^2}} xy \cdot dx \cdot dy$

$= \int_0^a x \left[\frac{y^2}{2} \right]_0^{\sqrt{a^2 - x^2}} \cdot dx = \frac{1}{2} \left[\int_0^a x(a^2 - x^2) \, dx = \frac{1}{2} \left[\frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a \right.$

$= \frac{1}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right] = \frac{a^4}{8}$

b) Evaluate $\iint (x+y)^2 \, dx \, dy$ over the area bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

A) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{(a^2 - x^2)}{a^2} b^2 = y^2$



$y = 0$ to $\boxed{y = \frac{b}{a} \sqrt{a^2 - x^2}}$ $\frac{x^2}{a} = 1 \Rightarrow \boxed{x = \pm a}$

$$I = \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x+y)^2\,dx\,dy = 4\int_0^a\int_0^{b/a\sqrt{a^2-x^2}} (x+y)^2\,dx\,dy$$

$$= 4\int_0^a\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dx\,dy$$

x y

$$= \int_0^a \left[x^2y + \frac{y^3}{3} + \frac{2xy^2}{2} \right]_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}}$$

$$\Rightarrow \int_{-a}^{a} \left[x^2\left(\frac{2b}{a}\sqrt{a^2-x^2}\right) + \frac{1}{3}\left[\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^3 - \left(-\frac{b}{a}\sqrt{a^2-x^2}\right)^3\right] + x\left[\cancel{\frac{b^2}{a^2}(a^2-x^2)} - \cancel{\frac{b^2}{a^2}(a^2-x^2)}\right]\right]$$

$$\Rightarrow \int_{-a}^{a} x^2\left(\frac{2b}{a}\sqrt{a^2-x^2}\right) + \frac{1}{3}\left[\frac{2b^3}{a^3}(a^2-x^2)^{3/2}\right] dx$$

$$= 2\int_0^a \left[x^2\frac{b}{a}\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right] dx$$

$$= 4\int_0^a \left[x^2\frac{b}{a}\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^3)^{3/2}\right] dx$$

$\because$ function is even.

put $x = a\sin\theta$; $dx = a\cos\theta\, d\theta$

$$= 4\int_0^{\pi/2} \left[a^2\sin^2\theta\,\frac{b}{a}\sqrt{a^2\cos^2\theta} + \frac{b^3}{3a^3}(a^2-a^2\sin^2\theta)^{3/2}\right] a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2} \left(\frac{a^3}{a} b\sin^2\theta\cos\theta + \frac{b^3}{3a^3}\cdot a^3\cos^3\theta\right] a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2} a^3b\sin^2\theta\cos^2\theta + \frac{b^3}{3}a\cos^4\theta)\, d\theta$$

$$= 4a^3b\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta + \frac{4ab^3}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta$$

Teacher's Signature

We know that $\int_0^{\pi/2} \sin^m\theta\cos^n\theta\, d\theta = \dfrac{(m-1)(m-3)\ldots \times (n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \times \pi/2$

$\therefore \int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta = \dfrac{(2-1)(2-1)}{4(4-2)} \times \dfrac{\pi}{2} = \dfrac{\pi}{16}$

So $\int_0^{\pi/2} \cos^n\theta\, d\theta = \dfrac{(n-1)(n-3)\ldots}{n(n-2)(n-4)\ldots} \times \dfrac{\pi}{2}$ (if n is even)

$= \int_0^{\pi/2} \cos^4\theta\, d\theta = \dfrac{(4-1)(4-3)}{4(4-2)} \times \dfrac{\pi}{2} = \dfrac{3\pi}{16}$//

$\therefore \iint_R (x+y)^2\, dx\,dy = 4a^3b\left(\dfrac{\pi}{16}\right) + \dfrac{4b^3a}{3}\left(\dfrac{3\pi}{16}\right)$

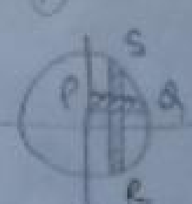
$= \dfrac{4\pi}{16}\left(a^3b + ab^3\right) = \dfrac{\pi}{4}ab\left(a^2+b^2\right)$

3) By changing order of integration, evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3y\, dx\,dy$

(y) (x)

A) $x = 0$ to $x = \sqrt{1-y^2}$ $\Rightarrow x^2+y^2 = 1$

$y = 0$ to $y = 1$



Let RS be new strip, $x = 0$ to $x = 1$

On changing strip, $y = \quad$ to $y = \sqrt{1-x^2}$

$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^3y)\, dy \cdot dx = \int_0^1 \int_0^{\sqrt{1-x^2}} x^3y \cdot dy\, dx$

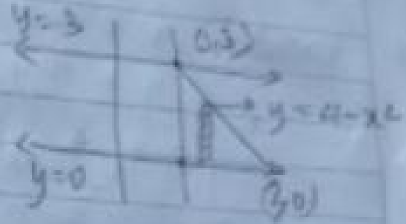
(x) (y)

$\Rightarrow \int_0^1 \left[x^3\dfrac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \dfrac{1}{2}\int_0^1 x^3(1-x^2)\,dx = \dfrac{1}{2}\int_0^1 (x^3 - x^5)$

$= \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_1^1 = \frac{1}{2}\left[\frac{1}{4} - \frac{1}{6}\right] = \frac{1}{24}$

b) By changing order of integration evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$

$x = 0$ to $x = \sqrt{4-y}$ $\Rightarrow x^2 = 4-y$
$y = 0$ to $y = 3$



$I = \int_1^2 \int_0^{4-x^2} (x+y)\,dx\,dy.$

$I = \int_1^2 \left[xy + \frac{y^2}{2}\right]_0^{4-x^2} dx = \int_1^2 (4x - x^3) + \frac{(4-x^2)^2}{2}$

$\Rightarrow \frac{4x^2}{2} - \frac{x^4}{4} + \frac{64x}{2} + \frac{x^5}{10} - \frac{8x^3}{6}\Big]_1^2 = \frac{-2267}{60}$

$= -37.7$

MODERATE

1) a) By changing order of integration evaluate

$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy.$

$x = r\cos\theta,\ y = r\sin\theta$

$r\sin^2\theta = \frac{r\cos\theta}{a}$

$x = 0$ to a

$y = \frac{x}{a}$ to $y = \sqrt{\frac{x}{a}}$

$\int_0^1 \int_{ay}^{ay^2} (x^2+y^2)$

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5) By changing order of integration $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dx\cdot dy$.

A) Consider, $x=0$ to $x=\infty$
$y=x$ to $y=\infty$

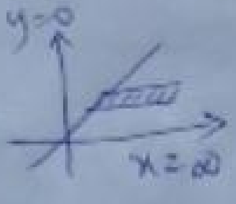
$$I = \int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\cdot dx\, dy$$

$$= \int_0^{\infty} \frac{e^{-y}}{y}[x]_0^y dy = \int_0^{\infty} e^{-y}\cdot dy = \left.\frac{e^{-y}}{-1}\right]_0^{\infty} = \frac{0-1}{-1} = 1$$

(2)

Q) a) By changing order of integration, $\int_0^{\infty}\int_0^{x} x e^{-x^2/y} dx\cdot dy$.

A) we have, $x=0$, $x=\infty$
$y=0$, $y=x$



$$I = \int_0^{\infty}\int_y^{\infty} x\cdot e^{-x^2/y}\cdot dx dy \Rightarrow \text{Put } x^2=t$$
$$2x\cdot dx = dt$$

$$= \int_0^{\infty}\int_{y^2}^{\infty} e^{-t/y}\cdot\frac{dt}{2}\cdot dy = \frac{1}{2}\int_0^{\infty} \left.\frac{e^{-t/y}}{-1/y}\right]_{y^2}^{\infty}\cdot dy = \frac{1}{2}\int_0^{\infty} -ye^{-t/y}\Big]_{y^2}^{\infty}$$

(y) (t)

$$= \frac{1}{2}\int_0^{\infty} 0+ye^{-y}\Big] dy = \frac{1}{2}\left[y\cdot\frac{e^{-y}}{-1} - 1\cdot\frac{e^{-y}}{(-1)^2}\right]_0^{\infty} =$$

$$= \frac{1}{2}\left[(0-0)-(0-1)\right] = 1/2$$

5. T $\iint r^2 \sin\theta \, dr \, d\theta = \frac{2a^3}{3}$ over the region R, where R is the semicircle $r = 2a\cos\theta$ above the initial line

let the given limits are 0 to $\frac{\pi}{2}$, r limit are 0 to $2a\cos\theta$

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2 \sin\theta \, dr \, d\theta = \int_0^{\pi/2} \frac{8a^3}{3}\cos^3\theta \sin\theta \, d\theta$$

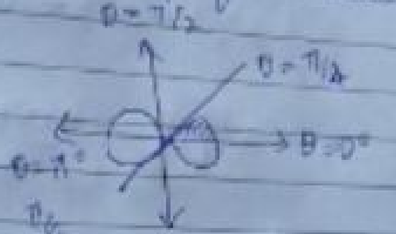
$$= \frac{8a^3}{3}\left[\frac{t^4}{4}\right]_0^1 = \frac{2a^3}{3}$$

put $\cos\theta = t$
$-\sin\theta \, d\theta = dt$
$t \to 1$ to 0

a) Evaluate $\iint \frac{r}{\sqrt{a^2+r^2}} \, dr \, d\theta$ over one loop of lemniscates

$r^2 = a^2\cos 2\theta$

$$I = \int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{a^2+r^2}} \, dr \, d\theta$$



$$I = \int_0^{\pi/4}\left[\sqrt{a^2+r^2}\right]_0^{a\sqrt{\cos 2\theta}} \cdot d\theta = \int_0^{\pi/4}\left[\sqrt{a^2+a^2\cos 2\theta} - a\right] d\theta$$

$$= \int_0^{\pi/4} (a\sqrt{2}\cos\theta - a) \, d\theta = a\left(\sqrt{2}\sin\theta - \theta\right)_0^{\pi/4} = a\left[1 - \frac{\pi}{4}\right]$$

$(1 + \cos 2\theta = 2\cos^2\theta)$ $\quad 1 - 2\cos^2\theta = \cos 2\theta$

6) Evaluate $\iint r^3 \, dr \, d\theta$ over the area bound by circles $r = 2\cos\theta$ and $r = 4\cos\theta$

The given circle, $r = 2\cos\theta$ & $r = 4\cos\theta$

$x^2+y^2-2x=0$ $\quad$ $x^2+y^2-4x=0$

$C_1(1,0)$ $\quad$ $C_2(2,0)$

$$\Rightarrow \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3 \, dr \, d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta = 60\int_{-\pi/2}^{\pi/2}\cos^4\theta \, d\theta$$

$$= 120\int_0^{\pi/2}\cos^4\theta \, d\theta = 120\left[\frac{3\pi}{16}\right] = \frac{45\pi}{2}$$

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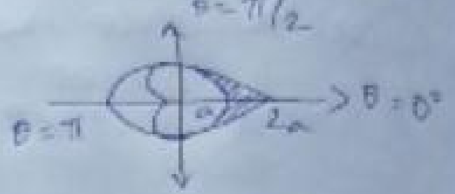
→ TYPICAL

1) Find area inside the cardioid ~~$r\theta$~~ $r = a(1+\cos\theta)$ and outside the circle $r = a$

A)



$$A = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

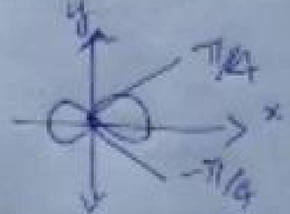
$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta = \int_0^{\pi/2} a^2(1+\cos\theta)^2 - a^2]\,d\theta$$

$$= \int_0^{\pi/2} a^2(1+\cos^2\theta + 2\cos\theta - 1]\,d\theta = a^2\int_0^{\pi/2} 1 + \frac{\cos 2\theta}{2} + 2\cos\theta$$

$$= a^2\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi/2} = a^2\left[\frac{\pi}{4} + 2\right]$$

b) area of lemniscate $= r^2 = a^2\cos 2\theta$.

A)



$r \to 0$ to $a\sqrt{\cos 2\theta}$

$\theta \to -\pi/4$ to $\pi/4$

$$\text{Area} = 2\iint r\,dr\,d\theta = 2\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta = 2\int_{-\pi/4}^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}}$$

$$= 2\int_{-\pi/4}^{\pi/4}\frac{a^2\cos 2\theta}{2}\,d\theta = 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = a^2$$

a) Area common to the circle $r = a\cos\theta$ and $r = a\sin\theta$

$r = a\cos\theta$, $r = a\sin\theta$

$x^2+y^2-ax=0$ $\quad x^2+y^2-ay=0$

$C\left(\frac{a}{2}, 0\right)$ $\quad C\left(0, \frac{a}{2}\right)$

$$A = \int_0^{\pi/4}\int_0^{a\sin\theta} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{a\cos\theta} r\,dr\,d\theta$$

θ r θ r

$$A = 2\int_0^{\pi/4}\int_0^{a\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta = \int_0^{\pi/4} a^2\sin^2\theta\,d\theta$$

$$= a^2\int_0^{\pi/4}\frac{1-\cos 2\theta}{2}\,d\theta = \frac{a^2}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \frac{a^2}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

Area lying the cardiod $r = 1+\cos\theta$ and outside the parabola $r(1+\cos\theta) = 1$

$$I = 2\int_0^{\pi/2}\int_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} r\,dr\,d\theta \Rightarrow 2\left[\int_0^{\pi/2}\frac{1}{2}r^2\,d\theta\right)_{r=a(1+\cos\theta)} + \int_{\pi/2}^{\pi}(1+\cos\theta)^2\,d\theta\Big)_{r=a(1+\cos\theta)}$$

$$= a^2\left[\int_0^{\pi/2}(1-2\cos\theta+\cos^2\theta)\,d\theta + \int_{\pi/2}^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta\right.$$

$$= a^2\int_0^{\pi}(1+\cos^2\theta)\,d\theta - 2\int_0^{\pi/2}\cos\theta\,d\theta + 2\int_{\pi/2}^{\pi}\cos\theta\,d\theta$$

$$= a^2\left[\int_0^{\pi}\left(1+\frac{1+\cos 2\theta}{2}\right)d\theta - 2\left|\sin\theta\right)_0^{\pi/2} + 2\left|\sin\theta\right|_{\pi/2}^{\pi}\right.$$

$$= a^2\left(\frac{3}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} - 2(1-0) + 2(0-1)$$

$$= \left(\frac{3\pi}{2} - 4\right)a^2$$

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3) a) Find area lying b/w parabola $y = 4x - x^2$ & line $y = x$

A) first, find intersection $\rightarrow 4x - x^2 = x$

$x^2 - 4x + x = 0$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0$ and $x = 3$

$$\rightarrow \int_0^3 \int_x^{4x-x^2} dx\,dy$$

$$\rightarrow \int_0^3 4x - x^2 - x\,dx = \int_0^3 3x - x^2\,dx = \int_0^3 \frac{3x^2}{2} - \frac{x^3}{3}$$

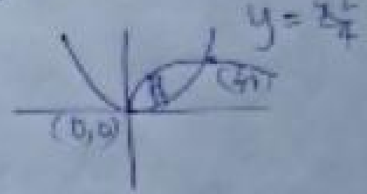
$$\Rightarrow \frac{3}{2}(9) - \frac{1}{3}(27) = \frac{27}{2} - \frac{27}{3} = \frac{81-54}{6} = \frac{9}{2}$$

b) Find area enclosed by $y = \frac{3x}{x^2+2}$ & $4y = x^2$

A) $y = 0,\ x = 0 \quad (0,0)$

$y = 1,\ \frac{3x}{x^2+2} = 1 \Rightarrow x^2 - 3x + 2 = 0$

$x = (1, 2)$



$(1,1)(2,1)$

$(0,0)\ (1,1)\ (2,1) \quad \Rightarrow y = \frac{x^2}{4}$ to $\frac{3x}{x^2+2}$

$x \rightarrow 0$ to 2

$$A = \int_0^2 \int_{x^2/4}^{\frac{3x}{x^2+2}} dx\,dy = \int_0^2 y\Big]_{x^2/4}^{\frac{3x}{x^2+2}} = \int_0^2 \frac{3x}{x^2+2} - \frac{x^2}{4}$$

$$\Rightarrow \frac{3}{2}\left(\log(x^2+2) - \frac{x^3}{12}\right)_0^2 = \frac{3}{2}\log 6 - \frac{8}{12} - \frac{3}{2}\log 2$$

UNIT-5

SECTION-A

EASY

1) Evaluate $\int_0^1\int_1^2\int_2^3 x^2y^3z^2\,dx\,dy\,dz$

$\int_0^1\int_1^2 \left.\frac{x^3}{3}\right]_2^3 y^3z^2\,dy\,dz = \int_0^1\int_1^2 \frac{19}{3}y^3z^2\,dy\,dz$

$= \int_0^1 \frac{19}{3}\left.\frac{y^4}{4}\right]_1^2 z^2\,dy\,dz = \int_0^1 \frac{19}{3}\frac{(15)}{4} = \frac{95}{4}\left.\frac{z^3}{3}\right]_0^1 dz = \frac{95}{12}$

2) Evaluate $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx$

$\int_0^1\int_0^1 \left.\frac{z^3}{3}+x^2z+y^2z\right]_0^1 dy\,dx = \int_0^1\int_0^1 \frac{1}{3}+x^2+y^2\,dy\,dx$

$= \int_0^1 \left.\frac{y}{3}+x^2y+\frac{y^3}{3}\,dx\right]_0^1 = \int_0^1 \frac{1}{3}+x^2+\frac{1}{3} = \int_0^1 \frac{2}{3}+x^2\,dx$

$= \left[\frac{2}{3}x+\frac{x^3}{3}\right]_0^1 dx = \frac{2}{3}+\frac{1}{3} = 1$

3) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dx\,dy\,dz$ into spherical polar coordinates

A) x y z we have, $z = \sqrt{x^2+y^2}$, $z=1$ | $x=0, x=1$

$z^2 = x^2+y^2$

$y = \sqrt{1-x^2}$

$x^2+y^2=1, y=0$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta} f(x,y,z)\,dx\,dy\,dz$

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$x = u(v+1)$

4) Using the transformation $x+y=u$, $y=uv$ transform

$$\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dy\,dx$$

A) $x=0 \mid y=0$
$x=1 \mid y=1-x \quad (x+y=1)$

At $x=0$:- $u=0, v=1 \quad [x=u(v+1)]$

At $y=0$:- $\Rightarrow u=0, v=0$

At $x+y=1$:- $u=1$

At $x=1$, $u=1, v=0$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$I = \int_0^1\int_0^1 e^{\frac{uv}{u}} \cdot |J| \cdot du\,dv$$

$$I = \int_0^1\int_0^1 e^{v} \cdot u \cdot du\,dv = (e-1)\int_0^1 u\,du$$

$$I = \frac{1}{2}(e-1)$$

5) The volume bound by the xy-plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$ is given by $\iint_R (3-x-y)\,dx\,dy$. Find x, y limits

A) $$V = \int_0^{\pi/2}\int_0^1 (3-x-y)\,dx\,dy = \int_0^{\pi/2}\int_0^1 (3 - r\cos\theta - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{3r^2}{2} - \frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta\right]_0^1 d\theta = \int_0^{\pi/2}\left(\frac{3}{2} - \frac{\cos\theta}{3} - \frac{\sin\theta}{3}\right)d\theta$$

$$= \left[\frac{3\theta}{2} - \frac{\sin\theta}{2} + \frac{\cos\theta}{3}\right]_0^{\pi/2} = \left(\frac{3\pi}{4} - \frac{1}{3}\right) - \left(0 - 0 + \frac{1}{3}\right) = \frac{3\pi}{4} - \frac{1}{3}$$

UNIT-5
SECTION-B A

EASY

Q Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$

Q ② If $\int_0^k \int_1^2 \int_2^3 xyz\, dx\, dy\, dz = \frac{15}{8}$, find k

A) $\int_0^k \int_1^2 \left[\frac{x^2}{2}\right]_2^3 yz\, dy\, dz \Rightarrow \int_0^k \int_1^2 \frac{5}{2} yz\, dy\, dz = \int_0^k \left[\frac{5}{2}\frac{y^2}{2}\right]_1^2 z\, dz$

$\frac{5}{2}\int_0^k \frac{3}{2} z\, dz = \frac{15}{4}\int_0^k z\, dz = \frac{15}{4}\left[\frac{z^2}{2}\right]_0^k = \frac{15}{4}\left(\frac{k^2}{2}\right) = \frac{15}{8}$

$\frac{k^2}{2} = \frac{1}{2} \Rightarrow k = 1$

Q Express the triple integral $\iiint_R f(x,y,z)\, dx\, dy\, dz$ in spherical polar coordinates

A) $x = r\sin\theta\cos\phi$ $\Rightarrow \int_0^{\pi/2}\int_0^{\pi/4}\int_0^1 |J|\, dr\, d\theta\, d\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Q Transform the $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ into polar form

A) Let $x = r\cos\theta$; $y = r\sin\theta$; $|J| = r$, $x^2+y^2 = r^2$

$\theta \to 0$ to $\pi/2$

$r = 0$ to ∞ $\rightarrow \iint_R f(x,y)\, dx\, dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

$\Rightarrow \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\, dr\, d\theta$

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let $x^2 = t \Rightarrow 2dx = dt$

$t \to 0$ to ∞

$$\int_0^{\pi/2}\int_0^{\infty} e^{-t}\frac{dt}{2}\,d\theta = \frac{1}{2}\int_0^{\pi}\left[e^{-t}\right]_0^{\infty} d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{2}(\theta)_0^{\pi/2} = \frac{\pi}{4}$$

$$\int_0^{\infty}\int_0^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy = \frac{\pi}{4} \qquad \text{let } x = y$$

$$\int_0^{\infty} e^{-x^2}dx \int_0^{\infty} e^{-x^2}dx = \frac{\pi}{4}$$

$$\Rightarrow \left(\int_0^{\infty} e^{-x^2}dx\right)^2 = \frac{\pi}{4} \Rightarrow \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

$\Rightarrow$ MODERATE

① Evaluate $\int_0^2\int_1^2\int_0^{yz} xyz\,dx\,dy\,dz$

A)
$$\int_1^2\int^2 \left[\frac{x^2}{2}\right]_0^{yz} dy\,dz = \int_0^2\int_0^2 \frac{y^2z^2}{2}\,dy\,dz = \frac{1}{2}\int_0^2 \left[\frac{y^3}{3}\right]_1^2 z^2\,dz$$

$$= \frac{1}{6}\int_0^2 (z^3-1)z^2\,dz = \frac{1}{6}\int_0^2 z^5 - z^2\,dz = \frac{1}{6}\left(\frac{z^6}{6} - \frac{z^3}{3}\right]_0^2 dz$$

$$= \frac{1}{6}\left(\frac{64}{6} - \frac{8}{3}\right)dz = \frac{1}{6}\left(\frac{32}{3} - \frac{8}{3}\right) = \frac{1}{6}\left(\frac{24}{3}\right) = \frac{4}{3}$$

② The volume of cylinder $x^2+y^2=a^2$ & $x^2+z^2=a^2$ is given by double integrals $2\int_{k_1}^{k_2}\int_{f(x)}^{g(x)} z\,dx\,dy$, find $k_1, k_2, f(x)$ & $g(x)$

A) Consider, $x^2+y^2=a^2$

$x^2+z^2=a^2$ $\quad V = 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} z\,dx\,dy$

$K_1 = -a \quad f(x) = -\sqrt{a^2 \ldots}$

$K_2 = a \quad g(x) = \sqrt{a^2 \ldots}$

SECTION - B

1) Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$.

$$\int_0^a \int_0^x \left[e^x e^y e^z\right]_0^{x+y} = \int_0^a \int_0^x \left[e^x e^y e^{x+y} - 1\right] dy\,dx$$

$$= \int_0^a \int_0^x \left(e^{2x} \cdot e^{2y} - e^x \cdot e^y\right) dy\,dx = \int_0^a \left\{\int_0^x e^{2x} \cdot e^{2y} - \int_0^x e^x e^y\,dy\right\} dx$$

$$= \int_0^a \left[e^{2x}\left[\frac{e^{2y}}{2}\right]_0^x - e^x\left[e^y\right]_0^x\right] dx = \int_0^a \left\{\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^x\right\} dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{4} - \frac{e^{2x}}{2} + e^x\right]_0^a = \frac{e^{4a}}{8} - \frac{e^{2a}}{4} - \frac{e^{2a}}{2} + e^a - \left[\frac{1}{8} - \frac{1}{4} - \frac{1}{2} + 1\right]$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8} = \frac{1}{8}\left[e^{4a} - 6e^{2a} + 8e^a - 3\right]$$

2) Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\,dz\,dx\,dy$.

$$\int \log z = z\log z - z$$

$$\int_1^e \int_1^{\log y} (z\log z - z)_1^{e^x}\,dx\,dy = \int_1^e \int_1^{\log y} (e^x x - e^x + 1)\,dx\,dy$$

$$= \int_1^e \left(xe^x - e^x - e^x + x\right)_1^{\log y} dx = \int_1^e y\log y - y - y + \log y - e + e + e + 1$$

$$= \left[\frac{y^2}{2}\log y - \frac{y^2}{4} - y^2 + y\log y - y + (e+1)y\right]_1^e$$

$$= \frac{e^2}{2} - \frac{e^2}{4} - e^2 + e - e + (e+1)e + \frac{1}{4} + 1 + 1 - (e+1)$$

$$= \frac{1}{4}\left(e^2 - 8e + 13\right)$$

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③ Evaluate $\iiint_V \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ by changing into spherical co-ordinates where V is the volume of the sphere $x^2+y^2+z^2=a^2$

A) consider, $I = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} \frac{r^2 \sin\theta \, dr\,d\theta\,d\phi}{r^2}$
$(\phi) \quad (\theta) \quad (r)$

By taking the spheric coordinates where $x = r\cos\theta\sin\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$
$|J| = r^2\sin\theta$

$r \to 0$ to a
$\theta \to 0$ to $\frac{\pi}{2}$
$\phi \to 0$ to $\pi/2$

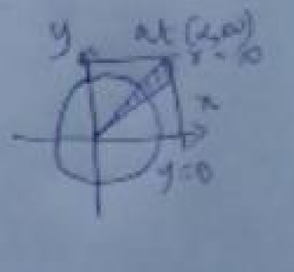
$$\Rightarrow 8\int_0^{a}\int_0^{\pi/2}\int_0^{\pi/2} \frac{r^2\sin\theta}{r^2}\, dr\,d\theta\,d\phi$$

$$= 8\int_0^{a}\int_0^{\pi/2} (-\cos\theta)_0^{\pi/2}\, dr\,d\phi = 8\int_0^{a}\int_0^{\pi/2} 1\, dr\,d\theta$$

$$= 8\int_0^{a} \left(\frac{\pi}{2}\right) dr = 8\frac{\pi}{2}a = \underline{4\pi a}$$

→ MODERATE

① Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$ by changing into polar coordinates
Hence show that $\int_0^{\infty} e^{-x^2}dx = \sqrt{\frac{\pi}{4}}$



A) let $x = r\cos\theta$, $y = r\sin\theta$; $|J| = r$, $x^2+y^2=1$
$\theta \to 0$ to $\pi/2$
$r \to 0$ to ∞

$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$\Rightarrow \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

let $r^2 = t \Rightarrow 2r\,dr = dt$

$$\Rightarrow \int_0^{\pi/2}\int_0^\infty e^{-t}\frac{dt}{2}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(e^{-t}\right)_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{2}(\theta)_0^{\pi/2} = \frac{\pi}{4}$$

$$\int_0^\infty\int_0^\infty e^{-x^2} e^{-y^2}\,dx\,dy = \frac{\pi}{4}$$

let $x = y \Rightarrow \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-x^2}dx = \frac{\pi}{4}$

$$= \left(\int_0^\infty e^{-x^2}dx\right)^2 = \frac{\pi}{4} = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Q) Find volume of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

A) $$V = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\,dy\,dz$$

$$= 8c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right] dx\,dy$$

$\left[\text{let } \sqrt{1-\frac{x^2}{a^2}} = \frac{\alpha}{b}\right]$

$$= \frac{8c}{b}\int_0^a\int_0^\alpha \sqrt{\frac{\alpha^2}{b^2}-\frac{y^2}{b^2}}\cdot dy\,dx = \frac{8c}{b}\int_0^a\int_0^\alpha \sqrt{\alpha^2-y^2}\,dy\,dx$$

$$= \frac{8c}{b}\int_0^a\int_0^\alpha \sqrt{\alpha^2-y^2}\,dy\,dx = \frac{8b}{c}\int_0^a \left[\frac{y}{2}\sqrt{\alpha^2-y^2} + \frac{\alpha^2}{2}\sin^{-1}\left(\frac{y}{\alpha}\right)\right]_0^\alpha dx$$

$$= \frac{8c}{b}\int_0^a 0 + \frac{\alpha^2}{2}\sin^{-1}(1)\cdot dx = \frac{8c}{b}\times\frac{\pi}{2}\times\frac{1}{2}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

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$= \frac{4c}{b} \times \pi \times b^2 \left[x^2 - \frac{x^3}{3} \times \frac{1}{a^2} \right]_0^a = 4c\pi b \left[a - \frac{a}{3} \right]$

$= \frac{4}{3} \pi abc$

[Q1]

(2) $V = \iiint dx\,dy\,dz$ for the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$,
we can take $V = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$

By transformation,
$x = ar\sin\theta\cos\phi$, $y = br\sin\theta\sin\phi$, $z = cr\cos\theta$
$dx\,dy\,dz = abc\, r^2 \sin\theta\, dr\, d\theta\, d\phi$
r limits are 0 to 1, ϕ limits are 0 to $\frac{\pi}{2}$, θ limits are 0 to $\frac{\pi}{2}$

$$V = 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 abc\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$
(θ) (ϕ) (r)

$$= 8abc \left[\frac{r^3}{3}\right]_0^1 (-\cos\theta)_0^{\pi/2} (\phi)_0^{\pi/2}$$

$$= \frac{4\pi abc}{3}$$

Q3) Volume of sphere $x^2+y^2+z^2 = a^2$

A) Consider, $x^2+y^2+z^2 = a^2 \Rightarrow r^2 = a^2$
spherical coordinates, $x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$
$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

$$V = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^2 \sin\theta \, dr\, d\theta\, d\phi = 8\int_0^{\pi/2}\int_0^{\pi/2} \frac{r^3}{3}\Big]_0^a \sin\theta \, d\theta\, d\phi$$
$(\phi)\ (\theta)\ (r)$

$$= \frac{8a^3}{3}\int_0^{\pi/2} \left(-\cos\theta\right]_0^{\pi/2} d\phi = \frac{8a^3}{3}\int_0^{\pi/2} d\phi$$

$$= \frac{8a^3}{3}\cdot\frac{\pi}{2} = \frac{4\pi a^3}{3}$$

TYPICAL

1) Find volume bounded by the xy plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$.

$$V = \underbrace{\iint}_{\text{circle}}\int_0^{3-y-x} dx\,dy\,dz = \int_0^{\pi/2}\int_0^1 (3 - r\cos\theta - r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{3r^2}{2} - \frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta \Big]_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{3}{2} - \frac{\cos\theta}{3} - \frac{\sin\theta}{3}\Big] d\theta = \frac{3\theta}{2} - \frac{\sin\theta}{3} + \frac{\cos\theta}{3}\Big]_0^{\pi/2}$$

$$= \frac{3\pi}{4} - \frac{1}{3} - \left(0 - 0 + \frac{1}{3}\right) = \frac{3\pi}{4} - \frac{2}{3}$$

2) Find volume common to cylinders $x^2+y^2=a^2$ & $x^2+z^2=a^2$

$$V = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2}\, dy\,dx$$

$$= 2\int_{-a}^{a} \sqrt{a^2-x^2}\left(2\sqrt{a^2-x^2}\right) dx = 4\int_{-a}^{a} (a^2-x^2)\,dx$$

$$= 4\left(a^2x - \frac{x^3}{3}\right]_{-a}^{a} = 4\left[a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3}\right]$$

$$= \frac{16a^3}{3}$$

Teacher's Signature ____________

(3) Find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ay$

A) $V=\iiint dx\,dy\,dz$ for the given region,

z limits are $-\sqrt{a^2-x^2-y^2}$ to $\sqrt{a^2-x^2-y^2}$

x, y limits are from the project $x^2-y^2=0$;

$$V = 2\iint_{x^2+y^2=ay} \sqrt{a^2-x^2-y^2}\,dx\,dy$$

for the projection region, $\theta \to 0$ to π

$r \to 0$ to $a\sin\theta$ $\left(\begin{array}{l} x^2+y^2=ay \\ r^2=ar\sin\theta \\ r=a\sin\theta \end{array}\right.$

$$V = 2\int_0^{\pi}\int_0^{a\sin\theta} \sqrt{a^2-r^2}\,r\,dr\,d\theta$$

$$= -\frac{2}{3}\int_0^{\pi} \left[(a^2-r^2)^{3/2}\right]_0^{a\sin\theta} d\theta = -\frac{2}{3}\int_0^{\pi} a^3\left[\cos^3\theta-1\right]d\theta$$

$$= +\frac{2a^3}{3}\left[\theta-\frac{\sin 3\theta}{12}-\frac{3\sin\theta}{4}\right]_0^{\pi} = 2\frac{a^3}{3}\pi$$

$\sin\frac{\pi}{4}$