Differentiation Rules! Again!

Julia: You know, some of those rules we learned were pretty useful, but some of these derivatives still suck! There **HAS** to be a better way!

Dylan: I'm sure there is, and I'm sure I know who could help us!

James: Did I hear my name?

Dylan: Not yet!

Julia: James!

James: There are more rules for differentiation that can make your life just a little bit easier!

The Product Rule

James: From the last time we did this, what rule do you think would exist for the product of two functions?

Julia: Well, last time we added or subtracted the derivative of both functions, so I bet we multiply the derivative of both!

Dylan: Let's check!

Consider the functions $f(x) = 2x$ and $g(x) = 3x^3 + x^2$.

Graph of $2x, 3x^3 + x^2$

Question 1 *Use Julia's guess to find the derivative of* $f(x) * g(x)$. $\boxed{18x^2 + 4x}$
Use the limit definition of the derivative to find the derivative of $f(x) * g(x)$.

□

Was Julia right?

$\boxed{No}$

Julia: Darn! It didn't work!

Dylan: It must be a little harder than that...

James: That's right Dylan, but it is easier than the limit definition! All we have to do is use

$$\frac{d}{dx} f(x) * g(x) = f(x) * g'(x) + f'(x) * g(x).$$

This is called the **Product Rule**.

Question 2 *Using the Product Rule, derive the products of the following functions:*

$f(x) = \sin(x) + x^2$, $g(x) = 3x^3 + x$

□

$f(x) = \cos(x) + 4x$, $g(x) = 3x^2 + x$

□

$f(x) = x^2$, $g(x) = 3x^3 - 3x$

□

$f(x) = x^7$, $g(x) = 2x^3 2$

□

The Quotient Rule

Dylan: Wow! That's gonna save a ton of time with products! Is there anything like it we can do with quotients?

James: There is! It's even called **the Quotient Rule**!

Julia: I bet it's a pain too though, just like the product rule.

James: Well, why don't you try using your intuition first rather than guessing?

Dylan: Alright, well, I guess I would divide the derivative of the numerator by the derivative of the denominator.

Question 3 *Consider the functions* $f(x) = x^3$ *and* $g(x) = \cos(x)$.

Graph of $x^3, cos(x)$

Use Dylan's guess to find the derivative of $\frac{f(x)}{g(x)}$.

$\boxed{3x^2/sin(x)}$

Use the limit definition of the derivative to find the derivative of $\frac{f(x)}{g(x)}$.

□

Was Dylan right?

$\boxed{No}$

Julia: I knew it! It's never that easy!

James: Now calm down Julia, this rule is worse than the last one, but it's much better than going through by hand:

$$\frac{d}{dx} * \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Question 4 *Using the Product Rule, derive the products of the following functions:*

$f(x) = \sin(x) + x^2$, $g(x) = 3x^3 + x$

□

$f(x) = \cos(x) + 4x$, $g(x) = 3x^2 + x$

□

$f(x) = x^2$, $g(x) = 3x^3 - 3x$

□

$f(x) = x^7$, $g(x) = 2x^3 2$

□

The Chain Rule

James: There's one last rule to learn today; the **Chain Rule**.

Dylan: That rule sounds pretty cool! When do we use it though? I thought we already covered the functions we need to know...

Julia: Yeah, what else is there?

James: We use the chain rule in composition of functions, like when we have $\sin(2x)$ - $2x$ is a function, and so is sin()!

Julia: And how bad is the rule?

James: This one is a little more tricky -

$$\frac{d}{dx} f(g(x)) = f'(g(x)) * g'(x).$$

Dylan and Julia: That's so gross.

James: Well, let's give it a try and see if you like it more than the limit definition!

Question 5 *Consider* $f(x) = \cos(x)$ *and* $g(x) = x^3$

Graph of $cos(x), x^3$

Using the limit definition of derivative, evaluate the derivative of $f(g(x))$.

□

Now, evaluate the same limit using the chain rule. Was it any better?

$\boxed{Yes}$

Question 6 *Using the Chain Rule, derive the compositions* $f(g(x))$ *for the following functions:*

$f(x) = 3x + x^2$, $g(x) = x^4 + 7x$

□

$f(x) = \cos(x)$, $g(x) = \sin(x)$

□

$f(x) = x^2 - 5x$, $g(x) = \sqrt{x+3}$

□

$f(x) = x^7$, $g(x) = \sin(x) - x^3 + 3$

□
