Newton's Methods

Introduction

Dylan: I'm so tired of having to solve roots by hand. It's a real drag.

Julia: Yeah, some of these roots are rough. I wish there was a better way!

James: There's always a better way!

Dylan and Julia: Show us!!!

James: Maybe you've heard of Sir Isaac Newton? He got tired of solving roots too, and made a whole method to approximate them!

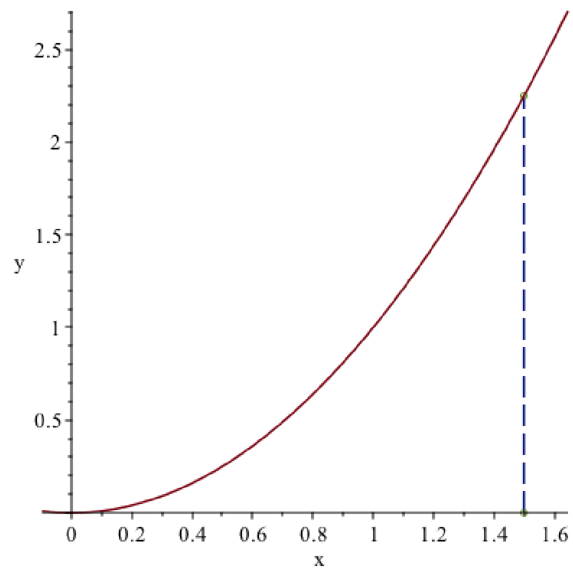
Dylan: Wow! I'm just like him except worse in every way!

Newton's Method is a system of approximating roots of polynomials by using tangent lines from an initial estimate. While this method is extremely accurate when used properly, it is possible to have a very inaccurate estimate when used improperly.

Guided Example

In the following figure we have an initial guess x_0, then we have the blue tangent line with respect to the point x_0

Learning outcomes:



Question 1 *What is the slope, in general, for the tangent line of $y = f(x)$ at x_0?*

Multiple Choice:

(a) $f'(x_0)$ ✓

(b) $f'(x)$

(c) $f(x)$

(d) $f(f(x))$

What is the equation of the tangent line for the point $(x_0, f(x_0)$?

Multiple Choice:

(a) $y = f(x) \cdot x + b$

(b) $y = f'(x_0) \cdot x_0 + b$ ✓

(c) $y = f'(x) \cdot b + x_0$

(d) $y = f'(x) \cdot b + f(x)$

How would you use the tangent line you found above to estimate the value of the root?

Multiple Choice:

(a) *Solve for $x = 0$ to find a point near the actual root.*

(b) *Solve for $y = 0$ to find a point near the actual root.* ✓

(c) *Derivate the tangent line to find where the second derivative is zero.*

(d) *Evaluate the tangent line at a y-value which was the initial estimate.*

On Your Own

Question 2 *Consider the function $f(x) = x^2 - 1$.*

Graph of $x^2 - 1$

Find the equation of the tangent line at an initial estimate of $x_0 = 3$.

$y = \boxed{6x - 12}$

Plot the tangent line and function on the same axes. Does the x-intercept of the tangent line seem more or less accurate than your initial estimate?

Multiple Choice:

(a) *More Accurate* ✓

(b) *Less Accurate*

What is the x-intercept of the tangent line?

$\boxed{2.5}$

Continue this process until the x-intercepts change by less than .0001 on each iteration. How many iterations did this take?

$\boxed{6}$

Question 3 *Consider the function $g(x) = x^3 - 4x^2 - 1$.*

Graph of $x^3 - 4x^2 - 1$

Explain why the function has only one solution with the help of a graph.

Graph of

Free Response:

Using $g(x)$ from the previous problem, use an initial guess of 2. After 5 iterations, what result do you get? $\boxed{Areallybadone}$

Why is it important to use caution with Newton's method?

Free Response:

Question 4 *Consider the function $h(x) = 4x^3 - 12x^2 + 2x + 1$.*

Graph of $4x^3 - 12x^2 + 2x + 1$

Use an initial guess of $x = 3$ to estimate a root of $h(x)$. What do you find?

$\square$

Look at the graph, and attempt to estimate another root using $x = 0$. Did you find the root to the right or the left of this point?

Multiple Choice:

(a) *Left* ✓

(b) *Right*

Increment the initial guess by 0.02 and use Newton's method until you find the other root. What value of x is the first to work? $\square$

In Summary

Julia: Wow! Newton's Method is awesome!

Dylan: Yeah, it's way more accurate than just guessing! If you're too far off on that initial guess though...

James: Things can go downhill quickly. While Newton's Method can be handy, it's important to remember how important an accurate initial estimate is!

Dylan and Julia: Thanks James!