

Differentiation Rules! Again!

```
1 caseInsensitive = function(a,b) {
2     return a.toLowerCase() == b.toLowerCase();
3 }
```

Julia: You know, some of those rules we learned were pretty useful, but some of these derivatives still suck! There **HAS** to be a better way!

Dylan: I'm sure there is, and I'm sure I know who could help us!

James: Did I hear my name?

Dylan: Not yet!

Julia: James!

James: There are more rules for differentiation that can make your life just a little bit easier!

The Product Rule

James: From the last time we did this, what rule do you think would exist for the product of two functions?

Julia: Well, last time we added or subtracted the derivative of both functions, so I bet we multiply the derivative of both!

Dylan: Let's check!

Consider the functions $f(x) = 2x$ and $g(x) = 3x^3 + x^2$.

Graph of $2x, 3x^3 + x^2$

Question 1 Use Julia's guess to find the derivative of $f(x) \cdot g(x)$.

$18x^2 + 4x$

Learning outcomes:

Differentiation Rules! Again!

Definition 1. The **derivative** of $f(x)$ at a is defined by the following limit:

$$\left[\frac{d}{dx} f(x) \right]_{x=a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Use the limit definition of the derivative to find the derivative of $f(x) \cdot g(x)$.

$$24x^3 + 6x^2$$

Was Julia right?

No

Julia: Darn! It didn't work!

Dylan: It must be a little harder than that...

James: That's right Dylan, but it is easier than the limit definition! All we have to do is use

$$\frac{d}{dx} (f(x) \cdot g(x)) = f(x) \cdot g'(x) + f'(x) \cdot g(x).$$

This is called the **Product Rule**.

Question 2 Using the Product Rule, differentiate the products of the following functions:

$$f(x) = 6x^3, g(x) = 7x^4$$

$$294x^6$$

$$f(x) = \cos(x) + 4x, g(x) = 3x^2 + x$$

$$3x^2 \sin(x) + 36x^2 + x \sin(x) + 6x \cos(x) + 8x + \cos(x)$$

$$f(x) = x^2, g(x) = 3x^3 - 3x$$

$$15x^4 - 9x^2$$

$$f(x) = x^7, g(x) = 2x^{32}$$

$$78x^{38}$$

The Quotient Rule

Dylan: Wow! That's gonna save a ton of time with products! Is there anything like it we can do with quotients?

James: There is! It's even called **the Quotient Rule!**

Julia: I bet it's a pain too though, just like the product rule.

James: Well, why don't you try using your intuition first rather than guessing?

Dylan: Alright, well, I guess I would divide the derivative of the numerator by the derivative of the denominator.

Question 3 Consider the functions $f(x) = x^3$ and $g(x) = \cos(x)$.

Graph of $x^3, \cos(x)$

Use Dylan's guess to find the derivative of $\frac{f(x)}{g(x)}$.

$3x^2 / \sin(x)$

Use the limit definition of the derivative to find the derivative of $\frac{f(x)}{g(x)}$.

$(\cos(x)3x^2 - x^3 \sin(x)) / \cos(x)^2$

Was Dylan right?

No

Julia: I knew it! It's never that easy!

James: Now calm down Julia, this rule is worse than the last one, but it's much better than going through by the limit definition:

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Question 4 Using the Quotient Rule, differentiate the products of the following functions:

$$f(x) = \sin(x) + x^2, g(x) = 3x^3 + x$$

$(-3x^4 + 3x^3 \cos(x) + x^2 - 9x^2 \sin(x) - x \cos(x) - \sin(x)) / (x^2(3x^2 + 1)^2)$

$$f(x) = \cos(x) + 4x, g(x) = 3x^2 + x$$

$(-x(12x + (3x + 1) \sin(x)) + (6x + 1) \cos(x)) / (x^2(3x + 1)^2)$

$$f(x) = x^2, g(x) = 3x^3 - 3x$$

$(-x^2 - 1) / (3(x^2 - 1)^2)$

$$f(x) = x^7, g(x) = 2x^{32}$$

$$\boxed{-25/(2x^{26})}$$

The Chain Rule

James: There's one last rule to learn today; the **Chain Rule**.

Dylan: That rule sounds pretty cool! When do we use it though? I thought we already covered the functions we need to know...

Julia: Yeah, what else is there?

James: We use the chain rule in composition of functions, like when we have $\sin(2x)$ - $2x$ is a function, and so is $\sin(x)$

Julia: And how bad is the rule?

James: This one is a little more tricky -

$$\frac{d}{dx}f(g(x)) = f'(g(x)) * g'(x).$$

Dylan and Julia: That's so gross.

James: Well, let's give it a try and see if you like it more than the limit definition!

Question 5 Consider $f(x) = \cos(x)$ and $g(x) = 2x$

Graph of $\cos(x)$, $2x$

Using the limit definition of derivative, evaluate the derivative of $f(g(x))$.

$$\boxed{-2 \sin(2x)}$$

Now, evaluate the same limit using the chain rule. Was it any better?

\boxed{Yes}

Question 6 Using the Chain Rule, differentiate the compositions $f(g(x))$ for the following functions:

$$f(x) = 3x + x^2, g(x) = x^4 + 7x$$

Differentiation Rules! Again!

$$\boxed{8x^7 + 70x^4 + 12x^3 + 98x + 21}$$

$$f(x) = \cos(x), g(x) = \sin(x)$$

$$\boxed{-\cos(x) \sin(\sin(x))}$$

$$f(x) = \cos(x), g(x) = x^3$$

$$\boxed{3x^2 \sin(x^3)}$$

$$f(x) = x^7, g(x) = \sin(x) - x^3 + 3$$

$$\boxed{7(-x^3 + \sin(x))^6 (\cos(x) - 3x^2)}$$

Question 7 Using the Chain Rule, differentiate the compositions $g(f(x))$ for the following functions:

$$f(x) = 3x + x^2, g(x) = x^4 + 7x$$

$$\boxed{(2x + 3)(4x^3(x + 3)^3 + 7)}$$

$$f(x) = \cos(x), g(x) = \sin(x)$$

$$\boxed{\sin(x)(-\cos(\cos(x)))}$$

$$f(x) = \cos(x), g(x) = x^3$$

$$\boxed{-3 \cos(x)^2 \sin(x)}$$

$$f(x) = x^7, g(x) = \sin(x) - x^3 + 3$$

$$\boxed{7x^6 (\cos(x^7) - 3x^{14})}$$
