

## Differentiation Rules! Again!

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```
1 caseInsensitive = function(a,b) {
2     return a.toLowerCase() == b.toLowerCase();
3 }
```

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**Julia:** You know, some of those rules we learned were pretty useful, but some of these derivatives still suck! There **HAS** to be a better way!

**Dylan:** I'm sure there is, and I'm sure I know who could help us!

**James:** Did I hear my name?

**Dylan:** Not yet!

**Julia:** James!

**James:** There are more rules for differentiation that can make your life just a little bit easier!

## The Product Rule

**James:** From the last time we did this, what rule do you think would exist for the product of two functions?

**Julia:** Well, last time we added or subtracted the derivative of both functions, so I bet we multiply the derivative of both!

**Dylan:** Let's check!

Consider the functions  $f(x) = 2x$  and  $g(x) = 3x^3 + x^2$ .

Graph of  $2x, 3x^3 + x^2$

**Question 1** Use Julia's guess to find the derivative of  $f(x) \cdot g(x)$ .

$18x^2 + 4x$

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Learning outcomes:

## Differentiation Rules! Again!

**Definition 1.** The **derivative** of  $f(x)$  at  $a$  is defined by the following limit:

$$\left[ \frac{d}{dx} f(x) \right]_{x=a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Use the limit definition of the derivative to find the derivative of  $f(x) \cdot g(x)$ .

$$24x^3 + 6x^2$$

Was Julia right?

No

**Julia:** Darn! It didn't work!

**Dylan:** It must be a little harder than that...

**James:** That's right Dylan, but it is easier than the limit definition! All we have to do is use

$$\frac{d}{dx} (f(x) \cdot g(x)) = f(x) \cdot g'(x) + f'(x) \cdot g(x).$$

This is called the **Product Rule**.

**Question 2** Using the Product Rule, differentiate the products of the following functions:

$$f(x) = 6x^3, g(x) = 7x^4$$

$$294x^6$$

$$f(x) = \cos(x) + 4x, g(x) = 3x^2 + x$$

$$3x^2 \sin(x) + 36x^2 + x \sin(x) + 6x \cos(x) + 8x + \cos(x)$$

$$f(x) = x^2, g(x) = 3x^3 - 3x$$

$$15x^4 - 9x^2$$

$$f(x) = x^7, g(x) = 2x^{32}$$

$$78x^{38}$$

## The Quotient Rule

**Dylan:** Wow! That's gonna save a ton of time with products! Is there anything like it we can do with quotients?

*Differentiation Rules! Again!*

**James:** There is! It's even called **the Quotient Rule!**

**Julia:** I bet it's a pain too though, just like the product rule.

**James:** Well, why don't you try using your intuition first rather than guessing?

**Dylan:** Alright, well, I guess I would divide the derivative of the numerator by the derivative of the denominator.

**Question 3** Consider the functions  $f(x) = x^3$  and  $g(x) = \cos(x)$ .

*Graph of  $x^3, \cos(x)$*

Use Dylan's guess to find the derivative of  $\frac{f(x)}{g(x)}$ .

$3x^2 / \sin(x)$

Use the limit definition of the derivative to find the derivative of  $\frac{f(x)}{g(x)}$ .

$(\cos(x)3x^2 - x^3 \sin(x)) / \cos(x)^2$

Was Dylan right?

$No$

**Julia:** I knew it! It's never that easy!

**James:** Now calm down Julia, this rule is worse than the last one, but it's much better than going through by the limit definition:

$$\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

**Question 4** Using the Quotient Rule, differentiate the products of the following functions:

$f(x) = \sin(x) + x^2, g(x) = 3x^3 + x$

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$f(x) = \cos(x) + 4x, g(x) = 3x^2 + x$

☐

$f(x) = x^2, g(x) = 3x^3 - 3x$

☐

$f(x) = x^7, g(x) = 2x^{32}$

□

## The Chain Rule

**James:** There's one last rule to learn today; the **Chain Rule**.

**Dylan:** That rule sounds pretty cool! When do we use it though? I thought we already covered the functions we need to know...

**Julia:** Yeah, what else is there?

**James:** We use the chain rule in composition of functions, like when we have  $\sin(2x)$  -  $2x$  is a function, and so is  $\sin(x)$

**Julia:** And how bad is the rule?

**James:** This one is a little more tricky -

$$\frac{d}{dx}f(g(x)) = f'(g(x)) * g'(x).$$

**Dylan and Julia:** That's so gross.

**James:** Well, let's give it a try and see if you like it more than the limit definition!

**Question 5** Consider  $f(x) = \cos(x)$  and  $g(x) = 2x$

*Graph of  $\cos(x)$ ,  $2x$*

*Using the limit definition of derivative, evaluate the derivative of  $f(g(x))$ .*

□

*Now, evaluate the same limit using the chain rule. Was it any better?*

Yes

**Question 6** Using the Chain Rule, differentiate the compositions  $f(g(x))$  for the following functions:

$$f(x) = 3x + x^2, g(x) = x^4 + 7x$$

□

$$f(x) = \cos(x), g(x) = \sin(x)$$

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$$f(x) = x^2 - 5x, g(x) = \sqrt{x+3}$$

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$$f(x) = x^7, g(x) = \sin(x) - x^3 + 3$$

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**Question 7** Using the Chain Rule, differentiate the compositions  $g(f(x))$  for the following functions:

$$f(x) = 3x + x^2, g(x) = x^4 + 7x$$

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$$f(x) = \cos(x), g(x) = \sin(x)$$

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$$f(x) = x^2 - 5x, g(x) = \sqrt{x+3}$$

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$$f(x) = x^7, g(x) = \sin(x) - x^3 + 3$$

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