Calculus II Labs

The College of Wooster

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Contents

Intro to Ximera

SAGE
```
caseInsensitive = function(a,b) {
   return a.toLowerCase() == b.toLowerCase();
 };
```

Julia: Hi, I'm Julia. Is anyone sitting here?

James: Nope, just me! I'm James by the way. Let's get started on this lab!

Multiple Choice and Select All

First things first - let's answer an easy multiple choice question! Simply click the correct box and then "Check Your Work"!

Question 1 *Are you ready to learn how to use Ximera for your Calculus course?*

Multiple Choice:

(a) *Never!*

(b) *No!*

(c) *Heck yeah!* ✓

(d) *No way!*

Feedback (`correct`): *Well great news for you! That's just what we'll do!*

Dylan: Ah! What was that?

James: Quit yelling in here! That was a feedback box, they usually give you a little more information on the question you answered.

Dylan: Oh, alright. Well, I'm Dylan! It's a pleasure.

Learning outcomes:
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Julia: I'm Julia, and this is James.

Let's look at another type of question here: select all. These allow you to pick multiple boxes before checking your answer, and you need to get all of them to get the right answer! These choices will not always be made clear in these labs, so if you think you see two or more right answers, click away!

Question 2 *Who have we met in this lab?*

Select All Correct Answers:

(a) *Jim*

(b) *Jeff*

(c) *Julia* ✓

(d) *Jennifer*

(e) *James* ✓

(f) *Dillon*

(g) *Dylan* ✓

(h) *Don*

Fill in the Blanks!

Dylan: Woah, what's this blank box?

Julia: Looks like we put our answer in it? But how do I know how to format it?

James: Don't worry you two! Ximera is pretty smart, so as long as what you put in is equivalent to the answer Ximera knows, it should work fine! Check it out down here!

Question 3 *Go ahead and put in $2x^2$ into the following blank, using*

`2x^2`

to raise x to the power of two:

$\boxed{2x^2}$

Now, the answer to this box is $2x^2$ as well, but try $2*(x)*(x)$ or $2x*x$!

$\boxed{2x^2}$

Feedback* (`correct`)*: *Look! It all works the same! Isn't Ximera great?*

Dylan: Well that's cool and all, but what if I need a square root?

James: That's easy!

There are two ways to enter a square root in Ximera; sqrt() and raising to the one-half power.

Question 4 *Using what we learned in the last example, use*

```
2^(1/2)
```

to input $\sqrt{2}$. You'll have to use parentheses on the power, so that Ximera knows you want everything to the power. $\boxed{\sqrt{2}}$

Question 5 *Now, use sqrt(2) to input it here!*

$\boxed{\sqrt{2}}$

Feedback* (`correct`)*: *Notice that Ximera gives you what it thinks you're inputting as you fill in the box! If you keep getting the wrong answer but think you're right, make sure to see if Ximera is interpretting your input correctly!*

Hints

Julia: Ximera is cool, but I'm a little worried. What if I get stuck and I'm doing it outside of class? I can't exactly ask the professor then!

James: That is true Julia, but the people who made this thought of just that! When a problem can be tough or confusing, they sometimes drop you a **hint**. Look down below, and click the show hint button to see what they can do!

Let's put some tough questions down, and use hints to answer them!

Question 6 ***Hint:*** *It's one of the characters we've seen so far, and the only one who doesn't have a J in their name!*

Who wrote this lab? $\boxed{\textit{Dylan}}$

Question 7 ***Hint:*** *This was three years before 2020.*

What year was this lab written? $\boxed{2017}$

Sometimes, a single question block can have multiple hints - if you're stuck, and there's a hint box, it's always worth clicking it again to see if another hint will appear!

Question 8 ***Hint:*** *I don't think you need a hint here.*

This question is easy, just click yes!

Multiple Choice:

(a) *No*

(b) *I refuse*

(c) *yes!* ✓

(d) *Yes*

Hint: *My favorite number is* $\sqrt{4}$.

What is my favorite number? $\boxed{2}$

Desmos

Dylan: Hey, this question wants me to graph something. Do I just put it on to paper?

Julia: Well, there's a box here that looks like a coordinate plane, but I'm not quite sure how to go about putting anything onto it.

James: This is a Desmos graph, and graphing with it is so easy! Just click the arrow on the left side, and put your equation in!

In the following graph, input $x^3 + 4x$.

Graph of

It should look like this:

Graph of $x^3 + 4x$

Julia: Wow, that was easy!

Dylan: And sometimes I guess we'll just be given the graph!

James: I guess it all depends on the question! You can also change your window size on the right side of the graph, either with the "+" and "-" buttons, or by directly modifying the maximum and minimum x and y values by going into the window which opens with the wrench!

Play with the following Desmos graph to see everything the wrench menu can allow you to do!

Graph of $2x^3 + 4x - 8$

Julia: Well, this looks like it's going to be a fun year!

James: Let's make it a great one!

Dylan: And let's dive in to Calculus!

Introduction to Sage

SageMath is a computer algebra system which uses python, throughout these labs sage cells will be used for certain problems. This lab introduces you to the basics of using SageMath via Sage Cells.

Introduction

If you ever want to use a sage cell when one is not provided, or would like to experiment with Sage Cells, you can follow this link.

Functions

To define a function you use the notation in the following sage cell:

SAGE
```
f(x)=x^5+3*x+4
```

Question 1 *What output did you get from evaluating the sage cell?*

Multiple Choice:

(a) *None* ✓

(b) $f(x) = x^5 + 3x + 4$

(c) $x^5 + 3x + 4$

Feedback (attempt): *All we did was define a function, to see the function definition type f(x).*

Evaluate the function at $x = 3$ by typing f(3) in the sage cell, what did you get? $\boxed{256}$

Learning outcomes:
Author(s): The College of Wooster
See link at https://sagecell.sagemath.org/

Question 2 *Define $f(x) = \sin(x)^2$ in the following cell evaluate at $x = 4\pi$*

Hint: *In sage, you type pi for pi and remember to use the carrot for powers and * for multiplication!*

SAGE

```
#To stop something from being evaluated put it in a comment using the hashtag
```

What did you get? $\boxed{0}$

If you don't use function notation, or want to define a function of multiple variables you must define your variables before using them, as in the following Sage Cell. The following sage cell defines the equation $4x + y = 1$, and then solves it for y.

SAGE

```
var('x y')
eqn=4*x+y==1
solve(eqn,y)
```

Question 3 *From the sage cell above, what can you say about "=" vs "=="?*

Multiple Choice:

(a) *"=" is used for assignment and "==" is used to signify equality* ✓

(b) *"=" is used to signify equality and "==" is used for assignment*

Feedback (attempt): *Note that you need to include the * operator, go back and take out the * to see how Sage Does error messages and debugging.*

The solve command is also shown above, it's fairly intuitive to use, the thing you want to solve is the first parameter and what you're solving for is the second parameter.

Question 4 *Using the solve command, find the roots for $f(x) = x^2 + 3x + 2$*

Hint: *You should be solving $f(x)$ for x*

SAGE

```
```

Copy paste what you got in your sage cell here: $\boxed{[x == -2, x == -1]}$

Limits

Limits are also fairly intuitive to use in Sage. This is shown in the following Sage Cell to find $\lim_{x\to\infty} 2x + 3$

SAGE

```
f(x)=2*x+3
limit(f(x),x=infinity)
```

Question 5 *Using the commands shown above, find the limit of* $\lim_{x\to 4} \frac{x^2 - 2x - 8}{x - 4}$

SAGE

```
```

What did you get? $\boxed{6}$

Differentiation

To differentiate in sage, use the diff command. This is shown below. It takes in the function you are differentiating and the variable you're differentiating with respect to.

SAGE

```
f(x)=2*x+3
diff(f(x),x)
```

Question 6 *Using the diff command find* $\frac{d}{dx}\frac{x^2 - 2x - 8}{x - 4}$

SAGE

```
```

Copy paste your answer from Sage here: $\boxed{2*(x-1)/(x-4)-(x^2-2*x-8)/(x-4)^2}$

Integration

The integral command uses the same parameters as the diff command, try it below for $f(x) = 2*x+3$

SAGE

```
```

Question 7 *Copy paste your answer from Sage here:* $\boxed{x^2+3*x}$

Getting Help

If you ever get stuck trying to use a command, there is built in documentation (as well as Google). Type the command followed directly by "?" to get extensive documentation on how to use it with examples. Try this for the solve command in the following cell.

SAGE

```
```

Euler's Method

Julia: I know Wooster has oil, but this is kind of ridiculous don't you think?

Dylan: What are you talking about Julia?

Julia: My professor keeps talking about Oiler's Method. Like, what is that? This is calculus, not geology.

Dylan: Actually, it's *Euler's* Method. He was a Swiss mathematician who came up with a way of approximating solutions to differential equations when we start with a given value!

James: That's right Dylan! Euler did a lot more than just that though; he's considered to be one the greatest mathematicians of all time!

Introduction

Euler's Method is a simple method of approximating the solution to a differential equation given an initial value, y_0, at a point t_0, or $y(t_0) = y_0$. Additionally, $F(t, y)$ is given, which is equivalent to $\frac{dy}{dt}y$. From here, the user chooses a step size, h, and uses

$$y_k = y_{k-1} + h \cdot F(t_{k-1}, y_{k-1})$$

to approximate the value at a point t_1, which is h units away from t_0, or $t_1 = t_0 + h$. At this point, we repeat the process, evaluating $F(t, y)$ at our new point, and moving another h units along the t-axis. By continuing this process, it is possible to approximate the solution at a point other than that which we are given.

Question 1 *What alteration to h might produce a more accurate estimation?*

Hint: *Consider a function with a rapidly changing derivative. How might a larger step-size approximate the rapid changes? A smaller one?*

Multiple Choice:

(a) *Increase the size of h to ignore minor jumps that would make the prediction less accurate.*

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(b) *Decrease the size of h to take into account very minor alterations in the function's derivative.* ✓

(c) *Use an h equivalent to the functional value at the point.*

(d) *Use an h equivalent to the value of the derivative at that point.*

When will this approximation be the best? When will it be the worst?

Hint: *Think about the derivative of the graph here, and how it affects the shape.*

Multiple Choice:

(a) *The approximation will be the best at rapid changes and worst where minimal change occurs.*

(b) *The approximation will be equally good at all points.*

(c) *The approximation will be best where the graph stays positive or negative, and worst where the parity changes.*

(d) *The approximation will be best where little change occurs, and worst where the most change occurs.* ✓

Guided Example

Given

$$F(t, y) = t + 2y$$

and the initial condition

$$y(0) = 0,$$

we will approximate the value of the solution at $t = 1$ using various step sizes.

Using a step size of $h = 0.5$, we find $t_1 = h + t_0 = 0.5 + 0 = 0.5$. Next, we see that $y_1 = y_0 + h \cdot F(t_0, y_0)$, or $y_1 = 0 + 0.5(t_0 + 2y_0) = 0 + (0 + 2 \cdot 0) = 0$. Thus, $y(t_1) = y(0.5) = 0$.

On step two, we see $t_2 = 0.5 + 0.5 = 1$, and $y_2 = 0 + 0.5(0.5 + 2 \cdot 0) = 0.25$. Thus $y(t_2) = y(1) = 0.25$.

Let's check our estimation - the actual solution to our differential equation was

$$y = 0.25 \cdot e^{2t} - 0.5t - 0.25.$$

Don't worry about how we found this; just note that at $t = 1$, $y = 1.097$.

Clearly, our estimation is not very good. But look at our step size! We moved an entire unit in only two steps - but that's an easy fix.

Let's look at the result when we use $h = 0.02$, using Sage! While an example has been provided below, click here for the documentation on how to use eulers_method!

SAGE

```
from sage.calculus.desolvers import eulers_method#imports the Euler's Method function from S
t,y = PolynomialRing(QQ,2,"ty").gens()#Defines our two variables
eulers_method(t+2*y,0,0,0.02,1,algorithm="table")#Produces a table of the t and y values.
```

Clearly a much better approximation! Note that the x column is simply our t, Sage uses an x instead of t. By simply decreasing h, we can increase the accuracy of Euler's Method greatly, at the cost of much harder work if done by hand.

On Your Own

(a) For the following, use step sizes of 0.5, 0.25, and 0.1 in combination to approximate the given point.

Remark 1. *Euler's Method does not require each step to be the same size.*

(i) $F(t, y) = t^2 - y$, $y(2) = 3$ at $y(3.5)$.

(ii) $F(t, y) = y + t$, $y(0) = 1$ at $y(3.85)$.

(iii) $F(t, y) = t\sin(y)$, $y(1) = 2$ at $y(2.4)$.

Dylan: Euler's Method is cool and all, but the approximation is so bad if I want it done in a reasonable amount of time without a computer.

James: Well, we typically will use a computer with Euler's Method, but there is a modification of Euler's Method that is much more accurate! It's known as *Euler's Midpoint Method*, which uses the derivative at the midpoint of the step, so the change is better approximated.

Julia: How much better is it?

James: Let's take a look!

The equation for Euler's Midpoint Method is

$$y_k = y_{k-1} + h \cdot m_{k-1},$$
$$\text{where } m_{k-1} = F\left(t_{k-1} + \frac{h}{2}, y_{k-1} + \frac{h}{2}F(t_{k-1}, y_{k-1})\right).$$

(a) Using both Euler's Method and Euler's Midpoint Method, approximate the solution $y(t)$ at the given point.

(i) $F(t, y) = y + t$, $y(0) = 1$, $h = 0.1$ at $y(0.5)$.

(ii) $F(t, y) = t^2 - y$, $y(1) = 3$, $h = 0.2$ at $y(2)$.

Julia: Wow! Euler's Method is pretty cool!

Dylan: Yeah, it means I don't have to always mess around with integrating if I'm given the derivative of a function and have to find a point!

James: Let's make sure we remember what we learned today, okay?

In Summary

Definition 1. ***Euler's Method*** *is a system which approximates solutions of first order differential equations by using the rate of change over a small distance to approximate the actual change. The basic method uses the equation*

$$y_k = y_{k-1} + h \cdot F(t_{k-1}, y_{k-1}),$$

$$\textit{where } \frac{dy}{dt}y = F(t, y),$$

h is step size, and $F(x_{n-1}, y_{n-1})$ is the derivative at the previous point.

Definition 2. ***Euler's Midpoint Method*** *is a modified version of Euler's Method, which uses the derivative at the midpoint between the end and start of the step to better approximate the rate of change over the step. This method uses a slightly modified equation,*

$$y_k = y_{k-1} + h \cdot m_{k-1},$$

$$\textit{where } m_{k-1} = F\left(t_{k-1} + \frac{h}{2}, y_{k-1} + \frac{h}{2}F(t_{k-1}, y_{k-1})\right).$$

Exponentials

Introduction

Dylan: Hey Julia, can you help me with this derivative?

Julia: Sure, which one is it? They've been pretty easy so far.

Dylan: I can't figure out 2^x.

Julia: Oh, I just did $x \cdot 2^{x-1}$.

Let's look at what Julia did and see if it makes sense.

Question 1 *Below are 2^x and $x \cdot 2^{x-1}$ graphed on the same set of axes.*

Graph of $2^x, x * 2^{x-1}$

Does it seem like $x \cdot 2^{x-1}$ is really the graph of the derivative?

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Guided Example

Dylan: Maybe we could go to office hours and get some help with this? I really don't understand what I'm supposed to do.

Julia: What if we called James? He always knows what to do!

James: Y'all need help?

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Julia and Dylan: James! How did you get here?

Julia: I didn't even call you yet...

James: Don't worry about it y'all. Anyway, let's look at the limit definition of the derivative for this one.

$$\frac{d}{dx}(2^x) = \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h}$$

Question 2 *Manipulate the definition James gave to factor out 2^x from the limit. Which of the following is the result?*

Multiple Choice:

(a) $2 \cdot x \lim_{h\to 0} \frac{2^h - 1}{h}$

(b) $2 \cdot x \lim_{h\to 0} \frac{2^h - 2}{h}$

(c) $2^x \cdot \lim_{h\to 0} \frac{2^h - 2}{h}$

(d) $2^x \lim_{h\to 0} \frac{2^h - 1}{h}$ ✓

Convince yourself that this limit exists. You may zoom in on the graph at the y-axis, or use progressively smaller values of h to prove this to yourself.

Graph of

SAGE

Notice that the derivative is a constant times $f(x)$. Create a graph with y equal to the constant you found, and on the same axes plot $\ln(x)$. Where is the constant?

Multiple Choice:

(a) 0.712

(b) 0.693 ✓

(c) 0.684

(d) 0.671

Because the intersection is there, what is your constant equivalent to?

Multiple Choice:

(a) 0.5^2

(b) $\log_{10}(2)$

(c) $\frac{1}{2}$

(d) $\ln(2)$ ✓

Repeat this process for 3^x and see if you obtain similar results.
Where is the constant located?

Multiple Choice:

(a) 1.0986 ✓

(b) 1.0934

(c) 1.0094

(d) 1.0731

Because the intersection is there, what is your constant equivalent to?

Multiple Choice:

(a) 0.5^3

(b) $\log_{10}(3)$

(c) $\frac{1}{3}$

(d) $\ln(3)$ ✓

On Your Own

Question 3 *Based on your results from the previous section, what is* $\dfrac{d}{dx}(a^x)$ *for any* $a > 0$*?*

Multiple Choice:

(a) a^x

(b) $\ln(h) \cdot \lim_{h \to 0} \dfrac{a^x - 1}{h}$

(c) $\ln(a) \cdot \lim_{h \to 0} \dfrac{a^h - 1}{h}$

(d) $a^x \cdot \lim_{h \to 0} \dfrac{a^h - 1}{h}$ ✓

Now, we would like to see a value for which $\lim_{h \to 0} \dfrac{a^h - 1}{h} = 1$*. What would this mean* $\dfrac{d}{dx}(a^x)$ *would equal?*

Multiple Choice:

(a) a^x ✓

(b) $\ln(a)$

(c) $\ln(x)$

(d) x^a

Using Sage, numerically evaluate the limit at $a = 2$ *and* $a = 3$*. How do they relate to the value we're looking for (where* $\lim_{h \to 0} \dfrac{a^h - 1}{h} = 1$*)?*

SAGE

```
```

Multiple Choice:

(a) *Both 2 and 3 are too large.*

(b) *Both 2 and 3 are too small.*

(c) *The value is between 2 and 3.* ✓

Using what you just noticed, use Sage, along with trial and error, to attempt to find the a for which the limit will be one.

SAGE

What value do you find?

Multiple Choice:

(a) *2.3*

(b) *2.1*

(c) *2.69*

(d) *2.71* ✓

(e) *3.14*

(f) *1.8*

Dylan: *Hey, this looks familiar...*

Julia: *I swear I've seen that before!*

James: *That's e! Euler discovered this constant, and its unique properties have made it a natural choice for a logarithmic base, leading to a plethora of names for it! e itself is also known as Euler's number and the Naperian base, and when used as a logarithmic base, it is shown as $\ln(x)$ and known as the natural log!*

To confirm this is the case use Sage to evaluate $\dfrac{d}{dx}(e^x)$.

SAGE

What result do you get? $\boxed{e^x}$

Julia: Well, I guess we found something pretty cool!

Dylan: I guess it's cool that we found something another mathematician did, but what's the point? Like, that's neat that it is its own derivative, but is there any other reason to know it?

James: The constant e is extremely common in mathematics Dylan! Right now, the money in your savings account is being affected by it!

Dylan: What?! What are you talking about?!

A Simple Application

When money is put into a savings account with a growth rate of r, it grows by a factor of $1 + r$ at the end of each year. This means that, at the end of each year, your funds will be

$$P_n = P_{n-1} + P_{n-1} \cdot r = P_{n-1}(1 + r),$$

where P_0 is your initial balance, or principal, and P_n is your balance after n years.

Now, imagine if, for whatever reason, your bank wanted to apply half that rate to your account, twice per year, i.e., at the end of the year your balance would be

$$P_n = P_{n-1}\left(1 + \frac{r}{2}\right)\left(1 + \frac{r}{2}\right) = P_{n-1}\left(1 + \frac{r}{2}\right)^2.$$

In general, the change in balance when compounded n times per year is

$$P_n = P_{n-1}\left(1 + \frac{r}{n}\right)^n.$$

Question 4 *For all $r > 0$, what is the relationship between $\left(1 + \frac{r}{2}\right)^2$ and $(1+r)$?*

Multiple Choice:

(a) $(1 + r) \leq \left(1 + \frac{r}{2}\right)^2$ ✓

(b) $\left(1 + \frac{r}{2}\right)^2 \leq (1 + r)$

(c) $(1 + r) = \left(1 + \frac{r}{2}\right)^2$

(d) $\left(1 + \frac{r}{2}\right)^2 < (1 + r)$

Determine the factor your balance grows by for the following intervals.

- *Quarterly*

  Multiple Choice:

  (a) $\left(1 + \frac{r}{4}\right)^4$ ✓

  (b) $\left(1 + \frac{r}{48}\right)^{48}$

(c) $\left(1+\frac{r}{3}\right)^{3}$

(d) $\left(1+\frac{r}{25}\right)^{25}$

- *Monthly*

  Multiple Choice:

  (a) $\left(1+\frac{r}{38}\right)^{38}$

  (b) $\left(1+\frac{r}{48}\right)^{48}$

  (c) $\left(1+\frac{r}{12}\right)^{12}$ ✓

  (d) $\left(1+\frac{r}{35}\right)^{35}$

- *Daily*

  Multiple Choice:

  (a) $\left(1+\frac{r}{36}\right)^{36}$

  (b) $\left(1+\frac{r}{365}\right)^{365}$ ✓

  (c) $\left(1+\frac{r}{380}\right)^{380}$

  (d) $\left(1+\frac{r}{24}\right)^{24}$

- *Hourly*

  Multiple Choice:

  (a) $\left(1+\frac{r}{8760}\right)^{8760}$ ✓

  (b) $\left(1+\frac{r}{525600}\right)^{525600}$

  (c) $\left(1+\frac{r}{365}\right)^{365}$

  (d) $\left(1+\frac{r}{8640}\right)^{8640}$

As the number of compoundings gets larger and large, the multiplication factor becomes

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}.$$

Substitute $r = 1$ into the factor, and evaluate using your the following Sage Cell. What is your result?

SAGE

1

Multiple Choice:

(a) ∞

(b) 1

(c) π

(d) e ✓

Evaluate the limit for the following values of r:

- $r = 0.3$

  Multiple Choice:

  (a) 1.42

  (b) $e^{0.3}$ ✓

  (c) $\frac{e}{3}$

  (d) 1.33

- $r = 0.1$

  Multiple Choice:

  (a) $\frac{e}{10}$

  (b) 1

  (c) 1.12

  (d) $e^{0.1}$ ✓

- $r = 0.7$

  Multiple Choice:

  (a) $e^{0.7}$ ✓

  (b) $\frac{e}{7}$

  (c) 1.023

  (d) e^{7}

- *r, the general case*

Multiple Choice:

(a) $\frac{1}{10} \cdot r$

(b) $\frac{e}{r}$

(c) e^r ✓

(d) r

Parametric Equations

Introduction

Julia: Ugh, I hate when they use stuff other than x and y. I'm used to them! Why do they need to change them?

Dylan: It looks like there's a lot more going on here than usual. There are x and y, but they're in different equations, and there's a t that's all over the place!

James: These are what are known as *parametric equations*. Rather than x and y being defined in terms of each other, they are defined by their relationship to a common variable, which here is t.

Dylan: Why is it called parametric? And why should we bother with it?

James: Well, they're called parametric equations because they are *parameterized* by t, meaning they're represented in terms of t. Parameters show underlying factors to better model data. Think about this: as more families make chili, fewer drownings are recorded. Does it make sense that the chili is causing this? We could write something like

$$\text{drownings} = 10 - \sqrt{\text{chili}}.$$

Julia: But that doesn't make sense! Those two things don't affect each other at all!

James: That's right! But temperature would affect both; it's cold out, so I make chili, and I don't want to go swimming! By using a parameter of temperature, we could make two equations which don't assume some non-existent relationship.

Examining Parametric Graphs

Dylan: So, how exactly do we input parametric equations in Desmos?

Learning outcomes:
Author(s): The College of Wooster

James: It's easy! First, we put a parenthesis, then the equation for our x-value. After that, we put a comma, our y-value equation, then close our parentheses!

Julia: But how do we work with our t then?

James: Desmos will pop up a range for t just under your equations!

For the following questions, graph the given equations, and give a short explanation of why each graph looks the way it does.

Question 1 $x = \sin(2t)$, $y = 2t^2$, $t = [-2, 2]$

Graph of

Free Response:

$x = t + \sin(3t)$, $y = 7t + \sin(2t)$, $t = [-6, 6]$

Graph of

Free Response:

$x = 3\sin(2t)$, $y = 2\cos(t)$, $t = [0, 2\pi]$

Graph of

Free Response:

$$x(t) = 11\cos(t) - 6\cos\left(\frac{11}{6}t\right), y(t) = 11\sin(t) - 6\sin\left(\frac{11}{6}t\right),\ t = [0, 50]$$

Graph of

Free Response:

On Your Own

Question 2

Your friend Joe is beyond excited about his new car, and wants to see just what it can do, despite your requests to be careful. He has set up a large wooden ramp designed to cause his car to do three barrel rolls before landing. These barrel rolls will be perfect circles, his car is 1.5 meters wide, and will take off exactly at the same angle it will land at.

How could the position of his right headlight be modeled, if the center of his front bumper is the origin?

Multiple Choice:

(a) $x(t) = \sin(t)$
$y(t) = \cos(t)$

(b) $x(t) = \cos(1.5t)$
$y(t) = \sin(1.5t)$

(c) $x(t) = 1.5 \cdot \cos(t)$
$y(t) = 1.5 \cdot \sin(t)$ ✓

(d) $x(t) = 1.5 \cdot \cos(t)$
$y(t) = \sin(t)$

What interval of t should be used to replicate the spinning of Joe's car, assuming he lands the jump?

Multiple Choice:

(a) $t = [0, 4\pi]$

(b) $t = [0, \frac{10\pi}{3}]$

(c) $t = [0, 6\pi]$ ✓

(d) $t = [0, 2\pi]$

Using the arc length formula, determine the distance traveled by his right headlight. Do not use a full period, or Sage will return zero; instead, use half of a period and remember to multiply your final result by two!

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

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$\boxed{9\pi}$ m

Question 3 *After his successful jump, Joe has become even more daring, deciding to jump across the Grand Canyon. Choosing the narrowest point along the canyon, in Marble Canyon, he needs to jump "only" 185 meters to safely land on the other side. Consider the base of the canyon directly under the jump to be the origin. This point lies 140 meters below the ramp.*

First, design position equations for Joe's car, starting with acceleration and working your way to position. You will not have the values to solve it, but you will end up with a skeleton for the final equation.

Multiple Choice:

(a) $p_y(t) = a_g t^2 + v_0 t + 140$
$p_x(t) = v_0 t$ ✓

(b) $p_y(t) = v_0 t + 140$
$p_x(t) = a_g t + v_0$

(c) $p_y(t) = v_0 t$
$p_x(t) = a_g t^2 + v_0 t$

(d) $p_y(t) = a_g t^2 + 140$
$p_x(t) = v_0 t + 140$

For acceleration due to gravity, use $9.8\text{m}\backslash s^2$. *Joe's car leaves the ramp at* $60\text{m}\backslash s$, *at an angle of* $30°$. *What are the initial velocities in the* x *and* y *directions?*

Multiple Choice:

(a) $v_{x0} = 30\text{m}/s$
$v_{y0} = 51.96\text{m}/s$

(b) $v_{x0} = 60\text{m}/s$
$v_{y0} = 60\text{m}/s$

(c) $v_{x0} = 51.96\text{m}/s$
$v_{y0} = 30\text{m}/s$ ✓

(d) $v_{x0} = 42.42m/s$
$v_{y0} = 42.42m/s$

Feedback (`correct`): *What is the equation for v_y?*

Multiple Choice:

(a) $v_y = 30$
(b) $v_y = 30 + a_g t$ ✓
(c) $v_y = 30 + a_g t^2$
(d) $v_y = 30t + a_g t^2$

What is the equation for v_x?

Multiple Choice:

(a) $v_x = 51.96t$
(b) $v_y = 51.96 + a_g t$
(c) $v_x = 51.96$ ✓
(d) $v_x = 51.96t + a_g t^2$

How long will it take Joe to reach the same altitude he started at?

Multiple Choice:

(a) 6.1224 s ✓

(b) 3.0612 s

(c) 5.3020 s

(d) 10.6040 s

How far will Joe travel before he returns to 160 meters off the base of the canyon?

Multiple Choice:

(a) 183.672 m

(b) 145.09 m

(c) 159.06 m

(d) 318.12 m ✓

Does Joe make it across?

Multiple Choice:

(a) *Yes* ✓

(b) *No*

What is the necessary initial velocity for him to perfectly make the jump?

Multiple Choice:

(a) $v_0 = 45.092m/s$

(b) $v_0 = 40.3672m/s$ ✓

(c) $v_0 = 35.673m/s$

(d) $v_0 = 3.5673m/s$

Question 4 *Just before Joe started to accelerate towards the ramp, a young spider crawled onto one of his tires! Your friend noticed just before Joe started to move, and was able to give the position of the spider in both the x and y directions with respect to time:*

$$x = \frac{3}{\pi} \cdot (t - \sin(t))$$
$$y = \frac{3}{\pi} \cdot (1 - \cos(t)).$$

Unfortunately, your friend didn't see what happened after Joe reached the ramp, and was unable to model everything which followed.

If Joe's tires have a radius of 0.4 *meters and he must travel 131.85 meters to reach the base of the jump, how much distance will the spider have covered in this time?*

Note: We are not looking for the area under the curve here. Think of the distance around the tire, and the number of rotations the tires will experience.

$\boxed{131.85}\,m$

How does this distance relate to the distance the car itself moved?

Multiple Choice:

(a) *The spider traveled a greater distance than the car.*

(b) *The spider traveled the same distance as the car.* ✓

(c) *The spider traveled a lesser distance than the car.*

For a parametric curve, the area under the curve may be represented by the integral

$$A = \int_{t_0}^{t_1} y(t)x'(t)dt.$$

What is the area under the curve for one full period? $\boxed{.48\pi}$

How does this relate to the area of the circle which created the cycloid?

The area of the cycloid is $\boxed{3}$ *times as much as the area of the circle which traces it.*

Logarithms

Introduction

Dylan: $\log_b(x)$? Why are they talking about trees on this paper?

Julia: Well, that doesn't seem quite right... it probably isn't talking about forests or anything. Is it?

James: Come on you guys, the lecture *just* went over this! It's the inverse exponential function!

Examining Log Rules

Dylan: Alright, so it isn't about trees, and maybe I wasn't paying attention during the lecture. So, what do I need to know before I do the lab?

Julia: Well, at least you're admitting it! I think I remember us going over a few rules for logarithms, but I can't quite seem to remember how they went...

James: Let's do a refresher then!

For the following multiple choice questions, you'll be given the left hand side of the equation. Match it up with the right hand side!

Question 1 $log_b\left(\dfrac{x}{y}\right)$

Multiple Choice:

(a) $log_b(y)$ - $log_b(x)$

(b) $\dfrac{log_b(x)}{log_b(y)}$

(c) $log_b(x)$ - $log_b(y)$ ✓

Learning outcomes:
Author(s): The College of Wooster

(d) $\dfrac{log_b(y)}{log_b(x)}$

$log_b(x * y)$

Multiple Choice:

(a) $log_b(y) + log_b(x)$

(b) $log_b(x) \cdot log_b(y)$

(c) $log_b(y) \cdot log_b(x)$

(d) $log_b(x) + log_b(y)$ ✓

$log_b(x^y)$

Multiple Choice:

(a) $y \cdot log_b(x)$ ✓

(b) $x \cdot log_b(y)$

(c) $log_b(x^y)$

(d) $log_b(y^x)$

Free Response:

Beyond the Basics

James: Now that we've gotten past the basic stuff, let's talk about the meaty stuff - calculus with logs!

Dylan: I'm not going to like this, am I?

Julia: Sometimes you're way too into this James...

Example 1. *Lets work together to determine the value of* $\dfrac{d}{dx} ln(x)$

Explanation. *First, lets think about a number that might make it easier for us to determine the value.*

Multiple Choice:

(a) x^2

(b) x

(c) e^x ✓

(d) π

Feedback (attempt): *We want e^x because it is its own derivative - that will make our differentiation more than a bit easier!*

Now, lets consider a general equation $y = ln(e^y)$.

What is the derivative of the left hand side? $\boxed{1}$

On the right hand side, we apply the chain rule to see

$$\frac{d}{dy} ln(e^y) = \frac{dx}{dy} \cdot \frac{d}{dx} ln(x), \text{ where } x = e^y$$

Now, because we know that $\frac{d}{dy} e^y = e^y$, we can change our $\frac{dx}{dy} \cdot \frac{d}{dx} ln(x) = 1$ to what?

Multiple Choice:

(a) x

(b) $\frac{d}{dz} = x$

(c) $\frac{1}{\frac{d}{dx}}$

(d) $x * \frac{d}{dx} ln(x)$ ✓

Feedback (attempt): *And thus, we see that we have $\frac{d}{dx} ln(x) = \frac{1}{x}$. So in general, the derivative of $ln(x)$ is $\frac{1}{x}$!*

Dylan: Well, that was quite a bit James!

James: Its good to know!

Julia: Do you think you could give us a little practice James? I wanna be sure I understand how to use it to get an A on this coming exam!

James: It would be my pleasure!

Practice with Logarithmic Differentiation

Take the derivative of the following functions, without using any technology (except to enter your answer!).

Problem 2 $\ln(\ln(x)^3) = \boxed{\dfrac{3}{x \cdot \ln(x)}}$

Problem 3 $\ln(\cot(x)) = \boxed{-\csc(x) \cdot \sec(x)}$

Now, use your knowledge of Sage (and if necessary, a quick glance back at the Intro to Sage lab) to evaluate the following derivatives at the indicated point.

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Problem 4 $\ln(\sin(\cos(\ln(x))))$ *at* $x = 10$

$\boxed{-1/10 \cdot \sin \cdot \ln(10) \cdot \cot(\cos(\ln(10)))}$

Problem 5 $16^{\ln(\csc(x)^2)}$ *at* $x = 1$

$\boxed{-3 \cdot \ln(16) \cdot \cot(1) \cdot 16^{3\ln(\csc(1))}}$

Finally, we'll look at the general form of the derivative of a logarithmic function with any base b. Here's a hint to get you started:

Use your knowledge of $\dfrac{d}{dx}\ln(x)$ and the change of base formula, $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$ to find the derivative for $\log_b(x)$.

Problem 6 *What is* $\dfrac{d}{dx}\log_b(x)$*?* $\boxed{\dfrac{1}{x \cdot \ln(b)}}$