Differentiation Rules! Again!

SAGE

```
caseInsensitive = function(a,b) {
  return a.toLowerCase() == b.toLowerCase();
 };
```

Julia: You know, some of those rules we learned were pretty useful, but some of these derivatives still suck! There **HAS** to be a better way!

Dylan: I'm sure there is, and I'm sure I know who could help us!

James: Did I hear my name?

Dylan: Not yet!

Julia: James!

James: There are more rules for differentiation that can make your life just a little bit easier!

The Product Rule

James: From the last time we did this, what rule do you think would exist for the product of two functions?

Julia: Well, last time we added or subtracted the derivative of both functions, so I bet we multiply the derivative of both!

Dylan: Let's check!

Consider the functions $f(x) = 2x$ and $g(x) = 3x^3 + x^2$.

Graph of $2x, 3x^3 + x^2$

Question 1 *Use Julia's guess to find the derivative of* $f(x) \cdot g(x)$.

$\boxed{18x^2 + 4x}$

Learning outcomes:

Definition 1. *The **derivative** of $f(x)$ at a is defined by the following limit:*

$$\left[\frac{d}{dx}f(x)\right]_{x=a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Use the limit definition of the derivative to find the derivative of $f(x)\cdot g(x)$.

$\boxed{24x^3+6x^2}$

Was Julia right?

$\boxed{No}$

Julia: Darn! It didn't work!

Dylan: It must be a little harder than that...

James: That's right Dylan, but it is easier than the limit definition! All we have to do is use

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + f'(x)\cdot g(x).$$

This is called the **Product Rule**.

Question 2 *Using the Product Rule, differentiate the products of the following functions:*

$f(x) = 6x^3$, $g(x) = 7x^4$

$\boxed{294x^6}$

$f(x) = \cos(x) + 4x$, $g(x) = 3x^2 + x$

$\boxed{3x^2\sin(x) + 36x^2 + x\sin(x) + 6x\cos(x) + 8x + \cos(x)}$

$f(x) = x^2$, $g(x) = 3x^3 - 3x$

$\boxed{15x^4 - 9x^2}$

$f(x) = x^7$, $g(x) = 2x^{32}$

$\boxed{78x^{38}}$

The Quotient Rule

Dylan: Wow! That's gonna save a ton of time with products! Is there anything like it we can do with quotients?

James: There is! It's even called **the Quotient Rule**!

Julia: I bet it's a pain too though, just like the product rule.

James: Well, why don't you try using your intuition first rather than guessing?

Dylan: Alright, well, I guess I would divide the derivative of the numerator by the derivative of the denominator.

Question 3 *Consider the functions $f(x) = x^3$ and $g(x) = \cos(x)$.*

Graph of $x^3, cos(x)$

Use Dylan's guess to find the derivative of $\dfrac{f(x)}{g(x)}$.

$\boxed{3x^2/\sin(x)}$

Use the limit definition of the derivative to find the derivative of $\dfrac{f(x)}{g(x)}$.

$\boxed{(\cos(x)3x^2 - x^3\sin(x))/\cos(x)^2}$

Was Dylan right?

$\boxed{No}$

Julia: I knew it! It's never that easy!

James: Now calm down Julia, this rule is worse than the last one, but it's much better than going through by the limit definition:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Question 4 *Using the Quotient Rule, differentiate the products of the following functions:*

$f(x) = \sin(x) + x^2$, $g(x) = 3x^3 + x$

$\boxed{(-3x^4 + 3x^3\cos(x) + x^2 - 9x^2\sin(x) - x\cos(x) - \sin(x))/(x^2(3x^2+1)^2)}$

$f(x) = \cos(x) + 4x$, $g(x) = 3x^2 + x$

$\boxed{(-x(12x + (3x+1)\sin(x)) + (6x+1)\cos(x))/(x^2(3x+1)^2)}$

$f(x) = x^2$, $g(x) = 3x^3 - 3x$

$\boxed{(-x^2 - 1)/(3(x^2-1)^2)}$

$f(x) = x^7$, $g(x) = 2x^{32}$

$\boxed{-25/(2x^{26})}$

The Chain Rule

James: There's one last rule to learn today; the **Chain Rule**.

Dylan: That rule sounds pretty cool! When do we use it though? I thought we already covered the functions we need to know...

Julia: Yeah, what else is there?

James: We use the chain rule in composition of functions, like when we have $\sin(2x)$ - $2x$ is a function, and so is $\sin(x)$

Julia: And how bad is the rule?

James: This one is a little more tricky -

$$\frac{d}{dx}f(g(x)) = f'(g(x)) * g'(x).$$

Dylan and Julia: That's so gross.

James: Well, let's give it a try and see if you like it more than the limit definition!

Question 5 *Consider* $f(x) = \cos(x)$ *and* $g(x) = 2x$

Graph of $cos(x), 2x$

Using the limit definition of derivative, evaluate the derivative of $f(g(x))$.

$\boxed{-2\sin(2x)}$

Now, evaluate the same limit using the chain rule. Was it any better?

$\boxed{Yes}$

Question 6 *Using the Chain Rule, differentiate the compositions* $f(g(x))$ *for the following functions:*

$f(x) = 3x + x^2$, $g(x) = x^4 + 7x$

$\boxed{8x^7 + 70x^4 + 12x^3 + 98x + 21}$

$f(x) = \cos(x)$, $g(x) = \sin(x)$

$\boxed{-\cos(x)\sin(\sin(x))}$

$f(x) = \cos(x)$, $g(x) = x^3$

$\boxed{3x^2 \sin(x^3)}$

$f(x) = x^7$, $g(x) = \sin(x) - x^3 + 3$

$\boxed{7(-x^3 + sin(x)_3)^6(cos(x) - 3x^2)}$

Question 7 *Using the Chain Rule, differentiate the compositions $g(f(x))$ for the following functions:*

$f(x) = 3x + x^2$, $g(x) = x^4 + 7x$

$\boxed{(2x + 3)(4x^3(x + 3)^3 + 7)}$

$f(x) = \cos(x)$, $g(x) = \sin(x)$

$\boxed{\sin(x)(-\cos(\cos(x))}$

$f(x) = \cos(x)$, $g(x) = x^3$

$\boxed{-3\cos(x)^2 \sin(x)}$

$f(x) = x^7$, $g(x) = \sin(x) - x^3 + 3$

$\boxed{7x^6(\cos(x^7) - 3x^{14})}$
