L'Hôpital's Rule Practice

Introduction

Julia: I have a limit and it's just $\frac{0}{0}$... is that one? Infinity? Zero?

Dylan: Well... I would probably go with zero? Like, at least they were zeroes before, right?

James: Woah, hold on! We have a method of figuring these things out, you don't need to just guess!

Recall that L'Hôpital's Rule says the following:

Theorem 1. *Suppose we have that the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ has indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided $g'(x)\neq 0$ around a and that $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists or is infinite.

Note that

- This rule is valid if you replace a with $\pm\infty$.
- This rule is valid for one-sided limits as well.
- The *indeterminate form* $\frac{0}{0}$ means $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$.
- The *indeterminate form* $\frac{\infty}{\infty}$ means $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$.
- You can apply L'Hôpital's Rule more than once!! As long as the hypothesis regarding the indeterminate form is satisfied, you can apply the rule again and again.
- *Never do the quotient rule!!!* It's the derivative of the top over the derivative of the bottom - NOT the derivative of the quotient!!

Learning outcomes:
Author(s):

Indeterminate Forms

The following are all *indeterminate forms* for limits that you might encounter:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, 1^\infty$$

- For the form $0 \cdot \infty$, use algebra to make the limit be in $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.
- For the form $\infty - \infty$, you'll generally need to combine fractions to get it into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.
- For either of 0^0 or 1^∞, you need to use logarithms.

Problems for Practice

In your group, try to use L'Hôpital's Rule to determine the limit. Make sure you check first if L'Hôpital's Rule applies. If it doesn't, you might have to change the limit with some algebra to be able to use L'Hôpital's Rule.

Problem 1 $\displaystyle\lim_{x\to\infty} \frac{x}{e^x} = \boxed{0}$

Problem 2 $\displaystyle\lim_{x\to 0} \frac{\tan(x)}{x} = \boxed{1}$

Problem 3 $\displaystyle\lim_{x\to 0} \frac{2e^x + x - 2}{\sin(x)} = \boxed{3}$

Problem 4 $\displaystyle\lim_{x\to \pi} \frac{\sin^2(x)}{1 + \cos(x)} = \boxed{2}$

Problem 5 ***Hint:*** $\displaystyle x\ln(x) = \frac{\ln(x)}{\frac{1}{x}}$

$\displaystyle\lim_{x\to 0^+} x\ln(x) = \boxed{0}$

Problem 6 $\lim_{x \to -\infty} x \sin\left(\frac{1}{x}\right) = \boxed{1}$

Problem 7 ***Hint:*** *Make quotients with Trig!*

$\lim_{x \to \frac{\pi}{2}} \sec(x) - \tan(x) = \boxed{0}$

Problem 8 *To compute* $\lim_{x \to 0^+} x^x$*, we use logs. Let* $y = x^x$*. Then* $\ln(y) = x\ln(x)$*. Now compute:*

$$\lim_{x \to 0^+} \ln(y)$$

$$\boxed{0}$$

Using continuity, we have $\lim_{x \to 0^+} \ln(y) = \ln\left(\lim_{x \to 0^+} y\right)$*. Use your answer above and your knowledge of exponential functions to determine now* $\lim_{x \to 0^+} x^x$*.*

$$\boxed{1}$$

Problem 9 *Use the same tricks as the last problem to compute:*

$$\lim_{x \to 1} (1 + \ln(x))^{\frac{1}{x-1}}$$

$$\boxed{e}$$

Problem 10 *Apply L'Hôpital's Rule to the following limit:* $\lim_{x \to 4} \frac{x^3 - 64}{x^2 + 16}$*. Then, conclude why this is wrong.*

Problem 11 ***Hint:*** *Use L'Hôpital's Rule more than once!!*

Evaluate $\lim_{x \to \infty} \frac{x^3}{e^x}$*.*
