Differentiation Rules Part Two

SAGE
```
caseInsensitive = function(a,b) {
   return a.toLowerCase() == b.toLowerCase();
 };
```

Julia: You know, some of those rules we learned were pretty useful, but some of these derivatives still suck! There **HAS** to be a better way!

Dylan: I'm sure there is, and I'm sure I know who could help us!

James: Did I hear my name?

Dylan: Not yet!

Julia: James!

James: There are more rules for differentiation that can make your life just a little bit easier!

The Product Rule

James: From the last time we did this, what rule do you think would exist for the product of two functions?

Julia: Well, last time we added or subtracted the derivative of both functions, so I bet we multiply the derivative of both!

Dylan: Let's check!

Consider the functions $f(x) = 2x$ and $g(x) = 3x^3 + x^2$.

Graph of $f(x) = 2x, g(x) = 3x^3 + x^2$

Question 1 *Use Julia's guess to find the derivative of $f(x) \cdot g(x)$.*

$\boxed{18x^2 + 4x}$

Learning outcomes:

Definition 1. *The **derivative** of $f(x)$ at a is defined by the following limit:*

$$\left[\frac{d}{dx} f(x)\right]_{x=a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Use the limit definition of the derivative to find the derivative of $f(x)\cdot g(x)$.

$\boxed{24x^3 + 6x^2}$

Was Julia right?

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Julia: Darn! It didn't work!

Dylan: It must be a little harder than that...

James: That's right Dylan, but it is easier than the limit definition! All we have to do is use

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + f'(x)\cdot g(x).$$

This is called the **Product Rule**.

Question 2 *Using the Product Rule, differentiate the products of the following functions:*

$f(x) = 6x^3$, $g(x) = 7x^4$

$\boxed{294x^6}$

$f(x) = \cos(x) + 4x$, $g(x) = 3x^2$

$\boxed{-3x^2\sin(x) + 6x\cos(x)}$

$f(x) = x^2$, $g(x) = 3x^3 - 3x$

$\boxed{15x^4 - 9x^2}$

$f(x) = x^7$, $g(x) = 2x^{32}$

$\boxed{78x^{38}}$

The Quotient Rule

Dylan: Wow! That's gonna save a ton of time with products! Is there anything like it we can do with quotients?

James: There is! It's even called **the Quotient Rule**!

Julia: I bet it's a pain too though, just like the product rule.

James: Well, why don't you try using your intuition first rather than guessing?

Dylan: Alright, well, I guess I would divide the derivative of the numerator by the derivative of the denominator.

Question 3 *Consider the functions* $f(x) = x^3 + 1$ *and* $g(x) = x$.

Graph of $f(x) = x^3 + 1, g(x) = x$

Use Dylan's guess to find the derivative of $\dfrac{f(x)}{g(x)}$.

$\boxed{3x^2/1}$

Use the limit definition of the derivative to find the derivative of $\dfrac{f(x)}{g(x)}$.

$\boxed{(2x^3 - 1)/x^2}$

Was Dylan right?

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Julia: I knew it! It's never that easy!

James: Now calm down Julia, this rule is worse than the last one, but it's much better than going through by the limit definition:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Question 4 *Using the Quotient Rule, differentiate the products of the following functions to find* $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]$:

$f(x) = 3x - 1$, $g(x) = 2x + 1$

$\boxed{5/(2x+1)^2}$

$f(x) = 1$, $g(x) = x + 10$

$\boxed{-1/(x+10)^2}$

$f(x) = x^2$, $g(x) = 3x^3 - 3x$

$\boxed{(-x^2-1)/(3(x^2-1)^2)}$

$f(x) = x^7$, $g(x) = 2x^{32}$

$\boxed{-25/(2x^{26})}$

The Chain Rule

James: There's one last rule to learn today; the **Chain Rule**.

Dylan: That rule sounds pretty cool! When do we use it though? I thought we already covered the functions we need to know...

Julia: Yeah, what else is there?

James: We use the chain rule in composition of functions, like when we have $\sin(2x)$ - $2x$ is a function, and so is $\sin(x)$

Julia: And how bad is the rule?

James: This one is a little more tricky -

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Dylan and Julia: That's so gross.

James: Well, let's give it a try and see if you like it more than the limit definition!

Question 5 *Consider* $f(x) = \sqrt{(}x)$ *and* $g(x) = \dfrac{1}{x}$

Graph of $sqrt(x), 1/x$

Using the limit definition of derivative, evaluate the derivative of $f(g(x))$.

$\boxed{-1/2x^{3/2}}$

Now, evaluate the same limit using the chain rule. Notice you get the same answer. Yay.

Question 6 *Find the compostition $f(g(x))$, then using the Chain Rule, differentiate $f(g(x))$ for the following functions:*

$f(x) = 3x + x^2$, $g(x) = x^4 + 7x$

$$f(g(x)) = \boxed{3(x^4+7x)+(x^4+7x)^2} \quad \frac{d}{dx}\left[f(g(x))\right] = \boxed{8x^7+70x^4+12x^3+98x+21}$$

$f(x) = \cos(x)$, $g(x) = \sin(x)$

$$f(g(x)) = \boxed{\cos(\sin(x))} \quad \frac{d}{dx}\left[f(g(x))\right] = \boxed{-\cos(x)\sin(\sin(x))}$$

$f(x) = \cos(x)$, $g(x) = x^3$

$$f(g(x)) = \boxed{\cos(x^3)} \quad \frac{d}{dx}\left[f(g(x))\right] = \boxed{-3x^2\sin(x^3)}$$

Question 7 *Using the Chain Rule, differentiate the compositions $g(f(x))$ for the following functions:*

$f(x) = 3x + x^2$, $g(x) = x^4 + 7x$

$$g(f(x)) = \boxed{(3x+x^2)^4+7(3x+x^2)} \quad \frac{d}{dx}\left[g(f(x))\right] = \boxed{(2x+3)(4x^3(x+3)^3+7)}$$

$f(x) = \cos(x)$, $g(x) = \sin(x)$

$$g(f(x)) = \boxed{\sin(\cos(x))} \quad \frac{d}{dx}\left[g(f(x))\right] = \boxed{-\cos(\cos(x))\sin(x)}$$

$f(x) = \cos(x)$, $g(x) = x^3$

$$g(f(x)) = \boxed{\cos(x)^3} \quad \frac{d}{dx}\left[g(f(x))\right] = \boxed{-3\cos^2(x)\sin(x)}$$

In Summary

We've covered a lot of differentiation rules in this lab, to help you out we've summarized the theorems below:

Theorem 1 (The Product Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$

Theorem 2 (The Quotient Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}$$

Theorem 3 (The Chain Rule)**.**

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$