Calculus II Labs

College of Wooster

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Contents

Introduction to Sage and Ximera

SageMath is a computer algebra system which uses python, throughout these labs sage cells will be used for certain problems. This lab introduces you to the basics of using SageMath via Sage Cells.

Introduction

If you ever want to use a sage cell when one is not provided, or would like to experiment with Sage Cells, you can follow this link.

Functions

To define a function you use the notation in the following sage cell:

SAGE
```
f(x)=x^5+3*x+4
```

Question 1 *What output did you get from evaluating the sage cell?*

Multiple Choice:

(a) *None* ✓

(b) $f(x) = x^5 + 3x + 4$

(c) $x^5 + 3x + 4$

Feedback (attempt): *All we did was define a function, to see the function definition type f(x).*

Evaluate the function at $x = 3$ by typing f(3) in the sage cell, what did you get? $\boxed{256}$

Question 2 *Define $f(x) = \sin(x)^2$ in the following cell evaluate at $x = 4\pi$*

Learning outcomes:
See link at https://sagecell.sagemath.org/

Hint: *In sage, you type pi for pi and remember to use the carrot for powers and * for multiplication!*

SAGE

```
#To stop something from being evaluated put it in a comment using the hashtag
```

What did you get? $\boxed{0}$

If you don't use function notation, or want to define a function of multiple variables you must define your variables before using them, as in the following Sage Cell. The following sage cell defines the equation $4x + y = 1$, and then solves it for y.

SAGE

```
var('x y')
eqn=4*x+y==1
solve(eqn,y)
```

Question 3 *From the sage cell above, what can you say about "=" vs "=="?*

Multiple Choice:

(a) *"=" is used for assignment and "==" is used to signify equality* ✓

(b) *"=" is used to signify equality and "==" is used for assignment*

Feedback (attempt): *Note that you need to include the * operator, go back and take out the * to see how Sage Does error messages and debugging.*

The solve command is also shown above, it's fairly intuitive to use, the thing you want to solve is the first parameter and what you're solving for is the second parameter.

Question 4 *Using the solve command, find the roots for $f(x) = x^2 + 3x + 2$*

Hint: *You should be solving $f(x)$ for x*

SAGE

```

```

Copy paste what you got in your sage cell here: $\boxed{[x == -2, x == -1]}$

Limits

Limits are also fairly intuitive to use in Sage.

Getting Help

If you ever get stuck trying to use a command, there is built in documentation (as well as Google). Type the command followed directly by “?” to get extensive documentation on how to use it with examples. Try this for the solve command in the following cell.

SAGE

```

```

Euler's Method

Julia: I know Wooster has oil, but this is kind of ridiculous don't you think?

Dylan: What are you talking about Julia?

Julia: My professor keeps talking about Oiler's Method. Like, what is that? This is calculus, not geology.

Dylan: Actually, it's *Euler's* Method. He was a Swiss mathematician who came up with a way of approximating solutions to differential equations when we start with a given value!

James: That's right Dylan! Euler did a lot more than just that though; he's considered to be one the greatest mathematicians of all time!

Introduction

Euler's Method is a simple method of approximating the solution to a differential equation given an initial value, y_0, at a point t_0, or $y(t_0) = y_0$. Additionally, $F(t, y)$ is given, which is equivalent to $\frac{dy}{dt}y$. From here, the user chooses a step size, h, and uses

$$y_k = y_{k-1} + h \cdot F(t_{k-1}, y_{k-1})$$

to approximate the value at a point t_1, which is h units away from t_0, or $t_1 = t_0 + h$. At this point, we repeat the process, evaluating $F(t, y)$ at our new point, and moving another h units along the t-axis. By continuing this process, it is possible to approximate the solution at a point other than that which we are given.

Question 1 *What alteration to h might produce a more accurate estimation?*

Hint: *Consider a function with a rapidly changing derivative. How might a larger step-size approximate the rapid changes? A smaller one?*

Multiple Choice:

(a) *Increase the size of h to ignore minor jumps that would make the prediction less accurate.*

Learning outcomes:

(b) *Decrease the size of h to take into account very minor alterations in the function's derivative.* ✓

(c) *Use an h equivalent to the functional value at the point.*

(d) *Use an h equivalent to the value of the derivative at that point.*

When will this approximation be the best? When will it be the worst?

Hint: *Think about the derivative of the graph here, and how it affects the shape.*

Multiple Choice:

(a) *The approximation will be the best at rapid changes and worst where minimal change occurs.*

(b) *The approximation will be equally good at all points.*

(c) *The approximation will be best where the graph stays positive or negative, and worst where the parity changes.*

(d) *The approximation will be best where little change occurs, and worst where the most change occurs.* ✓

Guided Example

Given

$$F(t, y) = t + 2y$$

and the initial condition

$$y(0) = 0,$$

we will approximate the value of the solution at $t = 1$ using various step sizes.

Using a step size of $h = 0.5$, we find $t_1 = h + t_0 = 0.5 + 0 = 0.5$. Next, we see that $y_1 = y_0 + h \cdot F(t_0, y_0)$, or $y_1 = 0 + 0.5(t_0 + 2y_0) = 0 + (0 + 2 \cdot 0) = 0$. Thus, $y(t_1) = y(0.5) = 0$.

On step two, we see $t_2 = 0.5 + 0.5 = 1$, and $y_2 = 0 + 0.5(0.5 + 2 \cdot 0) = 0.25$. Thus $y(t_2) = y(1) = 0.25$.

Let's check our estimation - the actual solution to our differential equation was

$$y = 0.25 \cdot e^{2t} - 0.5t - 0.25.$$

Don't worry about how we found this; just note that at $t = 1$, $y = 1.097$.

Clearly, our estimation is not very good. But look at our step size! We moved an entire unit in only two steps - but that's an easy fix.

Let's look at the result when we use $h = 0.02$, using Sage! While an example has been provided below, click herefor the documentation on how to use eulers_method!

SAGE

```
from sage.calculus.desolvers import eulers_method#imports the Euler's Method function from S
t,y = PolynomialRing(QQ,2,"ty").gens()#Defines our two variables
eulers_method(t+2*y,0,0,0.02,1,algorithm="table")#Produces a table of the t and y values.
```

Clearly a much better approximation! Note that the x column is simply our t, which Sage uses an x for. By simply decreasing h, we can increase the accuracy of Euler's Method greatly, at the cost of much harder work if done by hand.

On Your Own

(a) For the following, use step sizes of 0.5, 0.25, and 0.1 in combination to approximate the given point.

Remark 1. *Euler's Method does not require each step to be the same size.*

(i) $F(t, y) = t^2 - y$, $y(2) = 3$ at $y(3.5)$.

(ii) $F(t, y) = y + t$, $y(0) = 1$ at $y(3.85)$.

(iii) $F(t, y) = t\sin(y)$, $y(1) = 2$ at $y(2.4)$.

Dylan: Euler's Method is cool and all, but the approximation is so bad if I want it done in a reasonable amount of time without a computer.

James: Well, we typically will use a computer with Euler's Method, but there is a modification of Euler's Method that is much more accurate! It's known as *Euler's Midpoint Method*, which uses the derivative at the midpoint of the step, so the change is better approximated.

Julia: How much better is it?

James: Let's take a look!

The equation for Euler's Midpoint Method is

$$y_k = y_{k-1} + h \cdot m_{k-1},$$

$$\text{where } m_{k-1} = F\left(t_{k-1} + \frac{h}{2}, y_{k-1} + \frac{h}{2} F(t_{k-1}, y_{k-1})\right).$$

(a) Using both Euler's Method and Euler's Midpoint Method, approximate the solution $y(t)$ at the given point.

(i) $F(t, y) = y + t$, $y(0) = 1$, $h = 0.1$ at $y(0.5)$.

(ii) $F(t, y) = t^2 - y$, $y(1) = 3$, $h = 0.2$ at $y(2)$.

Julia: Wow! Euler's Method is pretty cool!

Dylan: Yeah, it means I don't have to always mess around with integrating if I'm given the derivative of a function and have to find a point!

James: Let's make sure we remember what we learned today, okay?

In Summary

Definition 1. ***Euler's Method*** *is a system which approximates solutions of first order differential equations by using the rate of change over a small distance to approximate the actual change. The basic method uses the equation*

$$y_k = y_{k-1} + h \cdot F(t_{k-1}, y_{k-1}),$$

$$\textit{where } \frac{dy}{dt}y = F(t, y),$$

h is step size, and $F(x_{n-1}, y_{n-1})$ is the derivative at the previous point.

Definition 2. ***Euler's Midpoint Method*** *is a modified version of Euler's Method, which uses the derivative at the midpoint between the end and start of the step to better approximate the rate of change over the step. This method uses a slightly modified equation,*

$$y_k = y_{k-1} + h \cdot m_{k-1},$$

$$\textit{where } m_{k-1} = F\left(t_{k-1} + \frac{h}{2}, y_{k-1} + \frac{h}{2}F(t_{k-1}, y_{k-1})\right).$$

Exponentials

Introduction

Dylan: Hey Julia, can you help me with this derivative?

Julia: Sure, which one is it? They've been pretty easy so far.

Dylan: I can't figure out 2^x.

Julia: Oh, I just did $x \cdot 2^{x-1}$.

Let's look at what Julia did and see if it makes sense.

Question 1 *Below are 2^x and $x \cdot 2^{x-1}$ graphed on the same set of axes.*

Graph of $2^x, x * 2^{x-1}$

Does it seem like $x \cdot 2^{x-1}$ is really the graph of the derivative?

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Guided Example

Dylan: Maybe we could go to office hours and get some help with this? I really don't understand what I'm supposed to do.

Julia: What if we called James? He always knows what to do!

James: Y'all need help?

Julia and Dylan: James! How did you get here?

Learning outcomes:

Julia: I didn't even call you yet...

James: Don't worry about it guys. Anyway, let's look at the limit definition of the derivative for this one.

$$\frac{d}{dx}(2^x) = \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h}$$

Question 2 *Manipulate the definition James gave to factor out 2^x from the limit.*

Multiple Choice:

(a) $2 \cdot x \lim_{h \to 0} \frac{2^h - 1}{h}$

(b) $2 \cdot x \lim_{h \to 0} \frac{2^h - 2}{h}$

(c) $2^x \cdot \lim_{h \to 0} \frac{2^h - 2}{h}$

(d) $2^x \lim_{h \to 0} \frac{2^h - 1}{h}$ ✓

Convince yourself that this limit exists. You may zoom in on the graph at the y-axis, or use progressively smaller values of h to prove this to yourself.

Graph of

SAGE

1

Notice that the derivative is a constant times $f(x)$. Create a graph with y equal to the constant you found, and on the same axes plot $\ln(x)$. Where is the constant?

Multiple Choice:

(a) 0.712

(b) 0.693 ✓

(c) 0.684

(d) 0.671

Because the intersection is there, what is your constant equivalent to?

Multiple Choice:

(a) 0.5^2

(b) $\log_{10}(2)$

(c) $\frac{1}{2}$

(d) $\ln(2)$ ✓

Repeat this process for 3^x and see if you obtain similar results.

Where is the constant located?

Multiple Choice:

(a) 1.0986 ✓

(b) 1.0934

(c) 1.0094

(d) 1.0731

Because the intersection is there, what is your constant equivalent to?

Multiple Choice:

(a) 0.5^3

(b) $\log_{10}(3)$

(c) $\frac{1}{3}$

(d) $\ln(3)$ ✓

On Your Own

Question 3 *Based on your results from the previous section, what is $\frac{d}{dx}(a^x)$ for any $a > 0$?*

Multiple Choice:

(a) a^x

(b) $\ln(h) \cdot \lim_{h \to 0} \frac{a^x - 1}{h}$

(c) $\ln(a) \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$

(d) $a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$ ✓

Now, we would like to see a value for which $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$. *What would this mean* $\frac{d}{dx}(a^x)$ *would equal?*

Multiple Choice:

(a) a^x ✓

(b) $\ln(a)$

(c) $\ln(x)$

(d) x^a

Using Sage, numerically evaluate the limit at $a = 2$ *and* $a = 3$. *How do they relate to the value we're looking for (where* $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$*)?*

SAGE

```
```

Multiple Choice:

(a) *Both 2 and 3 are too large.*

(b) *Both 2 and 3 are too small.*

(c) *The value is between 2 and 3.* ✓

Using what you just noticed, use Sage, along with trial and error, to attempt to find the a for which the limit will be one.

SAGE

```
```

What value do you find?

Multiple Choice:

(a) *2.3*

(b) *2.1*

(c) *2.69*

(d) *2.71* ✓

(e) *3.14*

(f) *1.8*

Dylan: *Hey, this looks familiar...*

Julia: *I swear I've seen that before!*

James: *That's e! Euler discovered this constant, and its unique properties have made it a* natural *choice for a logarithmic base, leading to a plethora of names for it! e itself is also known as Euler's number and the Naperian base, and when used as a logarithmic base, it is shown as $\ln(x)$ and known as the natural log!*

To confirm this is the case use Sage to evaluate $\frac{d}{dx}(e^x)$.

SAGE

1

What result do you get? $\boxed{e^x}$

Julia: Well, I guess we found something pretty cool!

Dylan: I guess it's cool that we found something another mathematician did, but what's the point? Like, that's neat that it is its own derivative, but is there any other reason to know it?

James: e is extremely common in mathematics Dylan! Right now, the money in your savings account is being affected by it!

Dylan: What?! What are you talking about?!

A Simple Application

When money is put into a savings account with a growth rate of r, it grows by a factor of $1+r$ at the end of each year. This means that, at the end of each year, your funds will be

$$P_n = P_{n-1} + P_{n-1} \cdot r = P_{n-1}(1+r),$$

where P_0 is your initial balance, or principal, and P_n is your balance after n years.

Now, imagine if, for whatever reason, your bank wanted to apply half that rate to your account, twice per year, i.e., at the end of the year your balance would be

$$P_n = P_{n-1}\left(1+\frac{r}{2}\right)\left(1+\frac{r}{2}\right) = P_{n-1}\left(1+\frac{r}{2}\right)^2.$$

In general, the change in balance when compounded n times per year is

$$P_n = P_{n-1}\left(1+\frac{r}{n}\right)^n.$$

Question 4 *For all $r > 0$, what is the relationship between $\left(1+\frac{r}{2}\right)^2$ and $(1+r)$?*

Multiple Choice:

(a) $(1+r) \leq \left(1+\frac{r}{2}\right)^2$ ✓

(b) $\left(1+\frac{r}{2}\right)^2 \leq (1+r)$

(c) $(1+r) = \left(1+\frac{r}{2}\right)^2$

(d) $\left(1+\frac{r}{2}\right)^2 < (1+r)$

Determine the factor your balance grows by for the following intervals.

- *Quarterly*

  Multiple Choice:

  (a) $\left(1+\frac{r}{4}\right)^4$ ✓

  (b) $\left(1+\frac{r}{48}\right)^{48}$

(c) $\left(1+\dfrac{r}{3}\right)^{3}$

(d) $\left(1+\dfrac{r}{25}\right)^{25}$

- *Monthly*

  Multiple Choice:

  (a) $\left(1+\dfrac{r}{38}\right)^{38}$

  (b) $\left(1+\dfrac{r}{48}\right)^{48}$

  (c) $\left(1+\dfrac{r}{12}\right)^{12}$ ✓

  (d) $\left(1+\dfrac{r}{35}\right)^{35}$

- *Daily*

  Multiple Choice:

  (a) $\left(1+\dfrac{r}{36}\right)^{36}$

  (b) $\left(1+\dfrac{r}{365}\right)^{365}$ ✓

  (c) $\left(1+\dfrac{r}{380}\right)^{380}$

  (d) $\left(1+\dfrac{r}{24}\right)^{24}$

- *Hourly*

  Multiple Choice:

  (a) $\left(1+\dfrac{r}{8760}\right)^{8760}$ ✓

  (b) $\left(1+\dfrac{r}{525600}\right)^{525600}$

  (c) $\left(1+\dfrac{r}{365}\right)^{365}$

  (d) $\left(1+\dfrac{r}{8640}\right)^{8640}$

As the number of compoundings gets larger and large, the multiplication factor becomes

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}.$$

Substitute $r = 1$ into the factor, and evaluate using your the following Sage Cell. What is your result?

SAGE

1

Multiple Choice:

(a) ∞

(b) 1

(c) π

(d) e ✓

Evaluate the limit for the following values of r:

- $r = 0.3$

  Multiple Choice:

  (a) 1.42

  (b) $e^{0.3}$ ✓

  (c) $\frac{e}{3}$

  (d) 1.33

- $r = 0.1$

  Multiple Choice:

  (a) $\frac{e}{10}$

  (b) 1

  (c) 1.12

  (d) $e^{0.1}$ ✓

- $r = 0.7$

  Multiple Choice:

  (a) $e^{0.7}$ ✓

  (b) $\frac{e}{7}$

  (c) 1.023

  (d) e^{7}

- *r, the general case*

Multiple Choice:

(a) $\frac{1}{10} \cdot r$

(b) $\frac{e}{r}$

(c) e^r ✓

(d) r

Parametric Equations

Introduction

Julia: Ugh, I hate when they use stuff other than x and y. I'm used to them! Why do they need to change them?

Dylan: It looks like there's a lot more going on here than usual. There are x and y, but they're in different equations, and there's a t that's all over the place!

James: These are what are known as *parametric equations*. Rather than x and y being defined in terms of each other, they are defined by their relationship to a common variable, which here is t.

Dylan: Why is it called parametric? And why should we bother with it?

James: Well, they're called parametric equations because they are *parameterized* by t, meaning they're represented in terms of t. Parameters show underlying factors to better model data. Think about this: as more families make chili, fewer drownings are recorded. Does it make sense that the chili is causing this? We could write something like

$$\text{drownings} = 10 - \sqrt{\text{chili}}.$$

Julia: But that doesn't make sense! Those two things don't affect each other at all!

James: That's right! But temperature would affect both; it's cold out, so I make chili, and I don't want to go swimming! By using a parameter of temperature, we could make two equations which don't assume some non-existent relationship.

Examining Parametric Graphs

For the following questions, use your favorite CAS to graph the given equations, and give a short explanation of why each graph looks the way it does.

Learning outcomes:

(a) $x = \sin(2t)$, $y = 2t^2$, $t = [-2, 2]$

(b) $x = t + \sin(3t)$, $y = 7t + \sin(2t)$, $t = [-6, 6]$

(c) $x = 3\sin(2t)$, $y = 2\cos(t)$, $t = [0, 2 * \pi]$

(d) $x(t) = 11\cos(t) - 6\cos(\frac{11}{6}t), y(t) = 11\sin(t) - 6\sin(\frac{11}{6}t)$, $t = [0, 50]$

Problems

(a) Your friend Joe is beyond excited about his new car, and wants to see just what it can do, despite your requests to be careful. He has set up a large wooden ramp designed to cause his car to do three barrel rolls before landing. These barrel rolls will be perfect circles, his car is 1.5 meters wide, and will take off exactly at the same angle it will land at.

(i) How could the position of his right headlight be modeled, if the center of his front bumper is the origin?

(ii) What interval of t should be used to replicate the spinning of Joe's car, assuming he lands the jump?

(iii) Using the arc length formula, determine the distance traveled by his right headlight.

(b) After his successful jump, Joe has become even more daring, deciding to jump across the Grand Canyon. Choosing the narrowest point along the canyon, in Marble Canyon, he needs to jump "only" 185 meters to safely land on the other side. Consider the base of the canyon directly under the jump to be the origin. This point lies 140 meters below the ramp.

(i) First, design a position equation for Joe's car, starting with acceleration and working your way to position. You will not have the values to solve it, but you will end up with a skeleton for the final equation.

(ii) For acceleration due to gravity, use 9.8mi\s^2. Joe's car leaves the ramp at 60mi\s, at an angle of 30°. What are the initial velocities in the x and y directions?

(iii) What is the equation for v_y?

(iv) What is the equation for v_x?

(v) How long will it take Joe to reach the ground?

(vi) How far will Joe travel before he returns to 160 meters off the base of the canyon?

(vii) Does Joe make it across? What is the necessary speed for him to perfectly make the jump?

(c) Just before Joe started to accelerate towards the ramp, a young spider crawled onto one of his tires! Your friend noticed just before Joe started to move, and was able to give the position of the spider in both the x and y directions with respect to time:

$$x = \frac{3}{\pi} \cdot (t - \sin(t)$$
$$y = \frac{3}{\pi} \cdot (1 - \cos(t).$$

Unfortunately, your friend didn't see what happened after Joe reached the ramp, and was unable to model everything which followed.

(i) If Joe's tires have a radius of 0.4 meters and he must travel 131.85 meters to reach the base of the jump, how much distance will the spider have covered in this time? *Note: We are not looking for the area under the curve here. Think of the distance around the tire, and the number of rotations the tires will experience.*

(ii) For a parametric curve, the area under the curve may be represented by the integral
$$A = \int_{t_0}^{t_1} y(t)x'(t)dt.$$

i. What is the area under the curve for one full period?

ii. How does this relate to the area of the circle which created the cycloid?