Labs

College of Woooster

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Contents

Continuity and Discontinuity

Julia: What does it mean for a graph to be discontinuous? I don't get it!

Dylan: I think it's like when there's a hole in the graph or something.

James: Actually there are different kinds of discontinuities, but it's hard to visualize so let's take a look!

Altogether: LET'S DIVE IN!

Introduction

Question 1 *A function f is said to be continuous at a point $x = a$ if which three conditions are satisfied?*

Select All Correct Answers:

(a) $f(a)$ *is defined* ✓

(b) $f(a) \neq 0$

(c) $\lim_{x \to a} f(x)$ *exists* ✓

(d) $\lim_{x \to a} f(x) = f(a)$ ✓

(e) $f(x)$ *is linear*

(f) $f(x) \neq f(a)$

Example

Take the function $f(x) = \dfrac{(1-x)^2}{1-x}$.

Graph of $\dfrac{(1-x)^2}{1-x}$

Learning outcomes:

Through some simple elimination, we can easily see that this function is equivalent to $1 - x$, where $x \neq 1$. Thus, there is one point on the original function we should pay close attention to: $x = 1$.

Using the simple trick of squaring the denominator to create our numerator, we were able to easily pick a point where we will have a discontinuous function, without using a jump or infinite discontinuity. Jump discontinuities can easily be made using piecewise functions, and infinite discontinuities are often best made with rational functions, like fractions of polynomials! Don't worry if you haven't discussed these discontinuities yet; we'll see plenty in this lab!

Problems

Question 2 *Create a function with the left handed and right handed limits not equal. What kind of discontinuity have you made here? Is there any kind of discontinuity that can't be created like this? Is there another that can? Consider the function*

$$f(x) = \frac{x^3 + 6x^2 + 12x + 8}{x + 2},$$

Describe the continuity of this function. If there is a discontinuity, where is it present? Is it possible to modify the function to remove this discontinuity? If so, how?

Free Response:

Question 3 *Consider the function*

$$g(x) = \frac{5x + 2}{2x - 3},$$

Describe the continuity of this function. If there is a discontinuity, where is it present? Is it possible modify the function to remove this discontinuity? If so, how?

Free Response:

Question 4 *Design a function with a removable discontinuity at 2, and a jump discontinuity at 0.*

Free Response:

Question 5 *Design a function with an infinite discontinuity and at least one other type of discontinuity*

Free Response:

Julia: Whenever I see people talking about jump discontinuities, they always use piecewise functions. Do you think it's possible to make one without the function being piecewise?

Dylan: If there's one thing that I've learned in math, it's that there are usually two ways to do anything! I'm not really sure how you would make something like that though...

James: I know one function that would work! Here's a hint - my function has one value on the positives, the opposite of that on the negatives, and is undefined at 0.

Question 6 *Can you create a function which has a jump discontinuity, but is not piecewise?*

Free Response:

Julia: Hey y'all, I was looking at our continuous graphs and noticed something.

Dylan: What did you see? They all look like pretty normal functions to me.

James: Yeah, I don't really know what you mean.

Julia: Well, discontinuities mean there are a chunk of the graph where you can skip over a value, right? Like, we can jump right from 1 to 5, or have a hole where some value isn't attained.

Dylan and James: Right. And?

Julia: I think if we picked a continuous function and looked at the functional values on each end of a range, we could say something about all the values in between those two!

Question 7 *Can you create a function which has a jump discontinuity, but is not piecewise?*

Free Response:

Question 8 ***Hint:*** *Can we skip any of the values?*

What can we say about every value in a range $[f(a), f(b)]$ on a continuous graph?

Free Response:

Derivative

Julia: Ah, this sucks!

Dylan: What's up?

Julia: I'm supposed to find the slope of a parabola at a point, and I'm not sure how!

Dylan: Well, what if we just make a secant line on the function?

Julia: Secant line? What's that?

Dylan: *A secant line is just a line which connects two points on a function*!

Guided Example

Consider the function

$$f(x) = x^2$$

Question 9 *Find the slope between $x = 2$ and $x = 7$. Does this seem to be a good approximation for the rate of change at $x = 2$? Why or why not?* $\boxed{9}$

Question 10 *Dylan thinks we can solve the problem by just picking something closer than 10. What is the slope between $x = 2$ and $x = 3$?* $\boxed{5}$

Julia: Dylan, this still isn't a great approximation...

Dylan: Well, I think we need to get even closer. Like, infinitesimally close! But how would we do that....

James: You guys need some help?

Julia and Dylan: James! How do we find the slope of a line at a point?

James: It isn't too tough! Before, you were considering a certain point as your comparison. What if instead, you used the point you want to evaluate at plus something really small? Let's call it h.

How can you make the h in

$$\frac{f(a+h) - f(a)}{(a+h) - a}$$

become a value closer and closer to zero when we evaluate it? Using the method you determined in the previous question, approximate the rate of change at the point x=2.

James: The value at that point is the slope of the tangent line!

Dylan: What's a tangent line?

James: *A tangent line is a line which intersects a differentiable curve at a point where the slope of the curve equals the slope of the line.*. Want to know something really cool?

Julia and Dylan: What James?

James: The function you just discovered is how you determine a function's derivative! Using that process, you can find the rate of change at any point on a function!

Julia and Dylan: Wow! So cool!

On Your Own

Using what you've learned, find the derivative of the following functions at the given point.

Question 1 $g(x) = x^5 - 5x^4 - x^2 + 2x + 1$, $x = 2$ $\boxed{-82}$

Question 2 $h(x) = \frac{1}{x}$, $x = 2$ $\boxed{-0.25}$

By replacing the a in our formula for the derivative with x, we may determine the derivative at any point on the function. Determine the derivative for the following functions.

Question 3 $m(x) = x^3$ $\boxed{3x^2}$

Question 4 $n(x) = 3x + 2$ $\boxed{3}$

In Summary

Julia: So why is it called a secant line?

James: It comes from the Latin word secare which means to cut.

Dylan: Ohh, I get it now! Because a secant line is any line that connects two points on a function!

In this lab we've (hopefully) learned the function for finding a derivative using the limit as h approaches 0. We also learned what secant and tangent lines are. For your convenience, the important definitions are listed below.

Definition 1. *A **secant line** is any line that connects any two points on a curve.*

Definition 2. *A **tangent line** is a line which intersects a differentiable curve at a point where the slope of the curve equals the slope of the line.*

Definition 3. *The **derivative** $f'(a)$ is defined by the following limit:*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Application of the Derivative: Change answers to final problem prior to rollout

Julia: I love class, but I keep wondering why I'm even learning this stuff. I'm not a math major.

Dylan: It isn't like we're ever going to use this stuff in our lives. It's all just theoretical.

James: Hold on guys! Actually, we use derivatives all the time - it is a way of measuring change after all.

Dylan: No way man, I can forget all this after class. Give me one time I'd use a derivative other than class.

James: I'll give you three!

The Great Molasses Flood

On January 15, 1919, a molasses storage tank in Boston burst, sending molasses rushing down the streets at 35 miles per hour. [2]

Let's pretend something similar happens in Wooster! Imagine you're on the street, walking by our newly installed molasses tank when it begins to burst. Unfortunately, you're by Born, and the molasses is rushing down the hill towards you with its position modeled by

$$\frac{1}{5}t^2 + t,$$

Your position can be modeled by

$$3t + 45.$$

In both cases, t is measured in seconds, with each equation reporting a position in meters.

Question 1 *What is your speed at any point?* $\boxed{3m/s}$

What about the speed of the molasses? $\boxed{2/5tm/s}$

Question 2 *What is your acceleration?* $\boxed{0m/s^2}$

The acceleration of the molasses? $\boxed{2/5m/s^2}$

[2] https://www.scientificamerican.com/article/molasses-flood-physics-science/

Question 3 *How quickly is the molasses travelling after one minute?* $\boxed{24m/s}$

Question 4 ***Hint:*** *Make sure to take into account the distance the molasses will need to travel to each location.*

If you want to survive the flood, you'll need to get off the street and into a tall, sturdy building. Born is only 10 meters away, but there is a group of people trying to get in, meaning once you are there, it will take 20 seconds to reach the inside of the building. Bissman has very little foot traffic, but you'll take exactly 20 seconds to get there and inside. Which building should you go to?

Multiple Choice:

(a) *It makes no difference*

(b) *Born*

(c) *Bissman* ✓

Marginal Profit

A company that makes peanut butter has a profit of

$$P(x) = -0.0027x^3 + 0.05x^2 + 18x - 125,$$

where x is the units produced. One unit of peanut butter contains 10,000 jars and the profit is in thousands of dollars.

Question 1 *Compute the marginal profit,* $P'(x)$. $\boxed{-.0081x^2 + .1x + 18}$
What is meant by marginal profit?

Free Response:

Question 2 *Use the marginal profit function to approximate the increase in profit when production is increased from 20 units to 21 units.* $\boxed{16.5297}$

Question 3 *Use the marginal profit function to approximate the increase in profit when production is increased from 65 to 66 units.* $\boxed{-10.6836}$

Question 4 *Graph the marginal profit function:*

Graph of

How would you change production based on this graph if the company was currently producing 20 units?

Multiple Choice:

(a) *Increase Production* ✓

(b) *Maintain Current Production*

(c) *Decrease Production*

What about 65 units?

Multiple Choice:

(a) *Increase Production*

(b) *Maintain Current Production*

(c) *Decrease Production* ✓

Dorm Room Froyo

You've opened up a Froyo franchise in your dorm room! It's a little cramped, but people are hearing about it and enjoying your generous pricing and the convenient location. We can model how many people hear about your franchise with the equation

$$p(t) = \frac{1}{1000}t^2 + 2t.$$

We can also model the profit of your location with the equation

$$t(p) = (p^2 + \frac{10}{3}p - 7000)^{\frac{1}{4}},$$

where t is time in days, and p is the population of people who are willing to come to your franchise each day.

Question 1 *If you start with no customers, how many days will it take you to start profiting?* $\boxed{\textit{Lots}}$

Question 2 *Using the Chain Rule, how will your profit be changing 35 days from now? Explain exactly what your answer means.* $\boxed{Whoknows}$

Free Response: