Calculus I Labs

The College of Wooster

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Contents

Intro to Ximera

An introduction to the adventures of Ximera with Julia, Dylan and James

SAGE
```
caseInsensitive = function(a,b) {
   return a.toLowerCase() == b.toLowerCase();
 };
```

Julia: Hi, I'm Julia. Is anyone sitting here?

James: Nope, just me! I'm James by the way. Let's get started on this lab!

Multiple Choice and Select All

First things first - let's answer an easy multiple choice question! Simply click the correct box and then "Check Your Work"!

Question 1 *Are you ready to learn how to use Ximera for your Calculus course?*

Multiple Choice:

(a) *Never!*

(b) *No!*

(c) *Heck yeah!* ✓

(d) *No way!*

Feedback (`correct`): *Well great news for you! That's just what we'll do!*

Dylan: Ah! What was that?

James: Quit yelling in here! That was a feedback box, they usually give you a little more information on the question you answered.

Learning outcomes:
Author(s): The College of Wooster

Dylan: Oh, alright. Well, I'm Dylan! It's a pleasure.

Julia: I'm Julia, and this is James.

Let's look at another type of question here: select all. These allow you to pick multiple boxes before checking your answer, and you need to get all of them to get the right answer! These choices will not always be made clear in these labs, so if you think you see two or more right answers, click away!

Question 2 *Who have we met in this lab?*

Select All Correct Answers:

(a) *Jim*

(b) *Jeff*

(c) *Julia* ✓

(d) *Jennifer*

(e) *James* ✓

(f) *Dillon*

(g) *Dylan* ✓

(h) *Don*

Fill in the Blanks

Dylan: Woah, what's this blank box?

Julia: Looks like we put our answer in it? But how do I know how to format it?

James: Don't worry you two! Ximera is pretty smart, so as long as what you put in is equivalent to the answer Ximera knows, it should work fine! Check it out down here!

Question 3 *Go ahead and put in* $2x^2$ *into the following blank, using* `2x^2` *to raise* x *to the power of two:* $\boxed{2x^2}$

Question 4 *Now, the answer to this box is $2x^2$ as well, but try $2 * (x) * (x)$ or $2x * x$!*

$\boxed{2x^2}$

Feedback (`correct`): *Look! It all works the same! Isn't Ximera great?*

Dylan: Well that's cool and all, but what if I need a square root?

James: That's easy!

There are two ways to enter a square root in Ximera; sqrt() and raising to the one-half power.

Question 5 *Using what we learned in the last example, use*

`2^(1/2)`

to input $\sqrt{2}$. You'll have to use parentheses on the power, so that Ximera knows you want everything to the power. $\boxed{\sqrt{2}}$

Question 6 *Now, use sqrt(2) to input it here!*

$\boxed{\sqrt{2}}$

Feedback (`correct`): *Notice that Ximera gives you what it thinks you're inputting as you fill in the box! If you keep getting the wrong answer but think you're right, make sure to see if Ximera is interpreting your input correctly!*

Hints

Julia: Ximera is cool, but I'm a little worried. What if I get stuck and I'm doing it outside of class? I can't exactly ask the professor then!

James: That is true Julia, but the people who made this thought of just that! When a problem can be tough or confusing, they sometimes drop you a **hint**. Look down below, and click the show hint button to see what they can do!

Let's put some tough questions down, and use hints to answer them!

Question 7 ***Hint:*** *It's one of the characters we've seen so far, and the only one who doesn't have a J in their name!*

Who wrote this lab? $\boxed{\textit{Dylan}}$

Question 8 ***Hint:*** *This was three years before 2020.*

What year was this lab written? $\boxed{2017}$

Sometimes, a single question block can have multiple hints - if you're stuck, and there's a hint box, it's always worth clicking it again to see if another hint will appear!

Question 9 ***Hint:*** *I don't think you need a hint here.*

This question is easy, just click yes!

Multiple Choice:

(a) *No*

(b) *I refuse*

(c) *yes!* ✓

(d) *Yes*

Hint: *My favorite number is* $\sqrt{4}$.

What is my favorite number? $\boxed{2}$

Desmos

Dylan: Hey, this question wants me to graph something. Do I just put it on to paper?

Julia: Well, there's a box here that looks like a coordinate plane, but I'm not quite sure how to go about putting anything onto it.

James: This is a Desmos graph, and graphing with it is so easy! Just click the arrow on the left side, and put your equation in!

In the following graph, input $x^3 + 4x$.

Graph of

It should look like this:

Graph of $x^3 + 4x$

Julia: Wow, that was easy!

Dylan: And sometimes I guess we'll just be given the graph!

James: I guess it all depends on the question! You can also change your window size on the right side of the graph, either with the "+" and "-" buttons, or by directly modifying the maximum and minimum x and y values by going into the window which opens with the wrench!

Play with the following Desmos graph to see everything the wrench menu can allow you to do!

Graph of $2x^3 + 4x - 8$

Julia: Well, this looks like it's going to be a fun year!

James: Let's make it a great one!

Dylan: And let's dive in to Calculus!

Introduction to Sage

SageMath is a computer algebra system which uses Python. Throughout these labs Sage Cells will be used for certain problems. This lab introduces you to the basics of using SageMath via Sage Cells.

Introduction

If you ever want to use a sage cell when one is not provided, or would like to experiment with Sage Cells, you can follow this link.

Functions

To define a function you use the notation in the following sage cell:

SAGE
```
f(x)=x^5+3*x+4
```

Question 1 *What output did you get from evaluating the sage cell?*

Multiple Choice:

(a) *None* ✓

(b) $f(x) = x^5 + 3x + 4$

(c) $x^5 + 3x + 4$

Feedback (attempt): *All we did was define a function. To see the function definition type f(x).*

Evaluate the function at $x = 3$ by typing f(3) in the sage cell, what did you get? $\boxed{256}$

Learning outcomes:
Author(s): The College of Wooster
See link at https://sagecell.sagemath.org/

Question 2 *Define $f(x) = \sin(x)^2$ in the following cell, then (on the next line) evaluate at $x = 4\pi$*

Hint: *In sage, you type pi for π. Remember to use the carrot $\wedge$ for powers and * for multiplication!*

SAGE

```
#To stop something from being evaluated put it in a comment using the hashtag
```

What did you get? $\boxed{0}$

If you don't use function notation, or want to define a function of multiple variables you must define your variables before using them, as in the following Sage Cell. The following sage cell defines the equation $4x + y = 1$, and then solves it for y.

SAGE

```
var('x y')
eqn=4*x+y==1
solve(eqn,y)
```

Question 3 *From the sage cell above, what can you say about "=" vs "=="?*

Multiple Choice:

(a) *"=" is used for assignment and "==" is used to signify equality* ✓

(b) *"=" is used to signify equality and "==" is used for assignment*

Feedback (attempt): *Note that you need to include the * operator, go back and take out the * to see how Sage Does error messages and debugging.*

The solve command is also shown above, it's fairly intuitive to use, the thing you want to solve is the first parameter and what you're solving for is the second parameter.

Question 4 *Using the solve command, find the roots for $f(x) = x^2 + 3x + 2$*

Hint: *You should be solving $f(x)$ for x*

SAGE
```
#
```

Copy paste what you got in your sage cell here: $\boxed{[x == -2, x == -1]}$

Getting Help

If you ever get stuck trying to use a command, there is built in documentation (as well as Google). Type the command followed directly by "?" to get extensive documentation on how to use it with examples. Try this for the solve command in the following cell.

SAGE
```

```

Transformations of Functions

SAGE

```
caseInsensitive = function(a,b) {
   return a.toLowerCase() == b.toLowerCase();
 };
```

Julia: Ugh!

Dylan: What's up Julia?

Julia: I have these functions I have to graph, and they're *so* close to functions I know really well, but they're a little bit different and it makes it so I have to calculate a bunch of points before I can confidently graph it!

James: Sounds like you could use some help Julia!

Julia and Dylan: James!

James: There are a ton of ways to transform functions, so let's get going and look at how we can modify our favorite functions!

Introduction

While you work with many different functions, there are a few basic types of functions. These include polynomials, rational functions, trigonometric functions, exponential functions, and logarithmic functions. In this lab we will explore different variations on these basic functions called **transformations**.

Guided Example

Consider the function $f(x) = x^2$.

Graph of $f(x) = x^2$

Learning outcomes:
Author(s): The College of Wooster

Question 1 *On the same axis graph $g(x) = f(x) + 2$. What change happened from $f(x)$ to $g(x)$?*

The graph shifted $\boxed{2}$ *units* $\boxed{up}$.

What can you infer about $f(x) - 2$?

The graph would shift $\boxed{2}$ *units* $\boxed{down}$.

Consider the function $f(x+2) = (x+2)^2$. How do you think this graph will be different from the graph of $f(x)$?

Free Response:

Graph the function $f(x+2)$ in the desmos window above, was your prediction correct? What can you infer about the function $f(x-2)$? Graph this function to verify your prediction.

Free Response:

What rule can you write about a general function $f(x+c)$ where c is a positive constant? The function will shift $\boxed{c}$ *units* $\boxed{left}$

Why do you think the graph moves in the direction it does when using the rule you determined in the last question? Hint: Think about the x-intercept and how it changes when you add or subtract a constant from the x value

Free Response:

How do you think the graph of $f(x) = x^2$ be affected when you multiply the whole function by some constant c?

Free Response:

Graph the function $c \cdot f(x)$ for the following values of $c = 2, \frac{1}{2}, -2, \frac{-1}{2}$

Graph of

Describe what is happening to the function based on the value of c, what can you generalize from this? It may be helpful to make a table with the x and y values to understand why this change happens.

Free Response:

On your own

Question 2 *Using $f(x) = x^2$ as your base function create a new function that will shift the graph up 4 units, to the right 3 units, reflect it across the x-axis and stretch it vertically by a factor of 2 and graph it below*

Graph of

Graph the function $f(2x)$

Graph of

What constant does this stretch or compress x^2 by?

$\boxed{1/2}$

Graph $f(2x + 6)$ on the same axis above, what transformation occurred?

Free Response:

Note the following expansion of the general function $f(x) = (ax + b)^2$:

$$f(x) = (ax + b)^2 = \left(a\left(x + \frac{b}{a}\right)\right)^2 = a^2\left(x + \frac{b}{a}\right)^2$$

From this expansion, how is a function in the form $f(x) = (ax+b)^2$ being shifted and stretched/compressed in terms of a and b?

Free Response:

In Summary

For the following questions, pick in which way the general graph $f(x)$ would change under certain transformations.

Question 3

$$c \cdot f(x)$$

When $c > 1$

Multiple Choice:

(a) *Shrink $f(x)$ vertically by c*

(b) *Stretch $f(x)$ vertically by c* ✓

(c) *Shrink $f(x)$ horizontally by c*

(d) *Stretch $f(x)$ horizontally by c*

(e) *Flip $f(x)$ over the x axis*

When $c < -1$

Multiple Choice:

(a) *Flip $f(x)$ over the x axis*

(b) *Shrink $f(x)$ horizontally by c*

(c) *Flip $f(x)$ over the y axis and stretch horizontally by c*

(d) *Flip $f(x)$ over the x axis and stretch vertically by c* ✓

(e) *Flip $f(x)$ over the x axis and stretch horizontally by c*

When $0 < c < 1$

Multiple Choice:

(a) *Stretch $f(x)$ horizontally by c*

(b) *Shrink $f(x)$ vertically by c* ✓

(c) *Shrink $f(x)$ horizontally by c*

(d) *Stretch $f(x)$ horizontally by c*

(e) *Flip $f(x)$ over the x axis*

Question 4

$$f(x+c)$$

When $c > 0$

Multiple Choice:

(a) *Shift $f(x)$ left by $|c|$.* ✓

(b) *Flip $f(x)$ over the x-axis.*

(c) *Shift* $f(x)$ *right by* $|c|$

(d) *Flip* $f(x)$ *over the x-axis and shift it up by* $|c|$.

(e) *No change occurs to* $f(x)$.

When $c < 0$

Multiple Choice:

(a) *Shift* $f(x)$ *left by* $|c|$.

(b) *Flip* $f(x)$ *over the x-axis.*

(c) *Shift* $f(x)$ *right by* $|c|$ ✓

(d) *Flip* $f(x)$ *over the x-axis and shift it up by* $|c|$.

(e) *No change occurs to* $f(x)$.

When $c = 0$

Multiple Choice:

(a) *Shift* $f(x)$ *left by* $|c|$.

(b) *Flip* $f(x)$ *over the x-axis.*

(c) *Shift* $f(x)$ *right by* $|c|$

(d) *Flip* $f(x)$ *over the x-axis and shift it up by* $|c|$.

(e) *No change occurs to* $f(x)$. ✓

Question 5

$$f(x) + c$$

When $c > 0$

Multiple Choice:

(a) *Shift* $f(x)$ *down by* $|c|$.

(b) *Stretch* $f(x)$ *vertically by* $|c|$.

(c) *Flip* $f(x)$ *over the x-axis.*

(d) *Shift $f(x)$ up by $|c|$.* ✓

(e) *No change will occur.*

When $c = 0$

Multiple Choice:

(a) *Shift $f(x)$ down by $|c|$.*

(b) *Stretch $f(x)$ vertically by $|c|$.*

(c) *Flip $f(x)$ over the x-axis.*

(d) *Shift $f(x)$ up by $|c|$.*

(e) *No change will occur.* ✓

When $c < 0$

Multiple Choice:

(a) *Shift $f(x)$ down by $|c|$.* ✓

(b) *Stretch $f(x)$ vertically by $|c|$.*

(c) *Flip $f(x)$ over the x-axis.*

(d) *Shift $f(x)$ up by $|c|$.*

(e) *No change will occur.*

SAGE

```
caseInsensitive = function(a,b) {
   return a.toLowerCase() == b.toLowerCase();
 };
```

Continuity and Discontinuity

Julia: What does it mean for a graph to be discontinuous? I don't get it!

Dylan: I think it's like when there's a hole in the graph or something.

James: Actually there are different kinds of discontinuities, but they can be hard to visualize so let's take a look!

Altogether: Let's dive in!

Introduction

Question 1 *A function f is said to be continuous at a point $x = a$ if which three conditions are satisfied?*

Select All Correct Answers:

(a) $f(a)$ *is defined* ✓

(b) $f(a) \neq 0$

(c) $\lim_{x \to a} f(x)$ *exists* ✓

(d) $\lim_{x \to a} f(x) = f(a)$ ✓

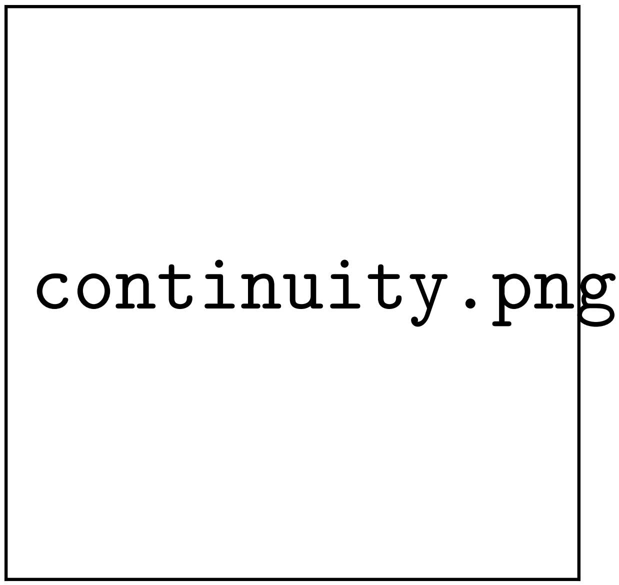
(e) $f(x)$ *is linear*

(f) $f(x) \neq f(a)$

Learning outcomes:
Author(s): The College of Wooster

Example

Consider the function $f(x) = \dfrac{(1-x)^2}{1-x}$.



Through some simple elimination, we can easily see that this function is equivalent to $1 - x$, where $x \neq 1$. Thus, there is one point on the original function we should pay close attention to: $x = 1$.

Using the simple trick of squaring the denominator to create our numerator, we were able to easily pick a point where we will have a discontinuous function, without using a jump or infinite discontinuity. Jump discontinuities can easily be made using piecewise functions, and infinite discontinuities are often best made with rational functions, like fractions of polynomials! Don't worry if you haven't discussed these discontinuities yet; we'll see plenty in this lab!

Problems

Question 2 *Consider the function $f(x)$:*

Select all points which have a discontinuity.

Select All Correct Answers:

(a) $x = -2$ ✓

(b) $x = 0$

(c) $x = -5$

(d) $x = 5$

(e) $x = 1$

(f) $x = -1$

What kind of discontinuity is present? Select all which apply.

Select All Correct Answers:

(a) *Removable Discontinuity* ✓

(b) *Jump Discontinuity*

(c) *Infinite Discontinuity*

Using the format (x-value, type of discontinuity), indicate the x-values with their corresponding type of discontinuity. If multiple discontinuities exist, list them in ascending x-value order. Make sure to put commas following each ordered pair when necessary.

$\boxed{(-2, \textit{Removable})}$

Question 3 *Consider the function $f(x)$:*

Select all points which have a discontinuity.

Select All Correct Answers:

(a) $x = 2$

(b) $x = 0$

(c) $x = -2$

(d) $x = 5$

(e) $x = 1.5$ ✓

(f) $x = -1.5$

What kind of discontinuity is present? Select all which apply.

Select All Correct Answers:

(a) *Removable Discontinuity*

(b) *Jump Discontinuity*

(c) *Infinite Discontinuity* ✓

Using the format (x-value, type of discontinuity*), indicate the* x*-values with their corresponding type of discontinuity. If multiple discontinuities exist, list them in ascending* x*-value order. Make sure to put commas following each ordered pair when necessary.*

$\boxed{(1.5, Infinite)}$

Question 4 *Consider the function* $f(x)$*:*

Select all points which have a discontinuity.

Select All Correct Answers:

(a) $x = 2$

(b) $x = 0$ ✓

(c) $x = -5$

(d) $x = 5$

(e) $x = 1$ ✓

(f) $x = -1$

What kind of discontinuity is present? Select all which apply.

Select All Correct Answers:

(a) *Removable Discontinuity*

(b) *Jump Discontinuity* ✓

(c) *Infinite Discontinuity* ✓

Using the format (x-value, type of discontinuity*), indicate the* x*-values with their corresponding type of discontinuity. If multiple discontinuities exist, list them in ascending* x*-value order. Make sure to put commas following each ordered pair when necessary.*

$\boxed{(0, Jump), (1, Infinite)}$

Question 5 *Consider the function* $f(x)$*:*

Select all points which have a discontinuity.

Select All Correct Answers:

(a) $x = 2$ ✓

(b) $x = 0$ ✓

(c) $x = -2$

(d) $x = 5$

(e) $x = 1.5$

(f) $x = -1.5$

What kind of discontinuity is present? Select all which apply.

Select All Correct Answers:

(a) *Removable Discontinuity* ✓

(b) *Jump Discontinuity* ✓

(c) *Infinite Discontinuity*

Using the format (x-value, type of discontinuity*), indicate the* x*-values with their corresponding type of discontinuity. If multiple discontinuities exist, list them in ascending* x*-value order. Make sure to capitalize the type of discontinuity, and put commas following each ordered pair when necessary.*

$\boxed{(0, Jump), (2, Removable)}$

Question 6 *Consider the function* $f(x)$*:*

Select all points which have a discontinuity.

Select All Correct Answers:

(a) $x = 2$ ✓

(b) $x = 0$ ✓

(c) $x = -2$ ✓

(d) $x = 5$ ✓

(e) $x = 1.5$

(f) $x = -1.5$

What kind of discontinuity is present? Select all which apply.

Select All Correct Answers:

(a) *Removable Discontinuity*

(b) *Jump Discontinuity* ✓

(c) *Infinite Discontinuity*

Question 7 ***Hint:*** *Think of the different types of numbers - Rationals, Irrationals, Integers, Natural Numbers, Real Numbers, etc. If need be, look up what each of these are to refresh your memory.*

Indicate for what kind of numbers the function is discontinuous. $\boxed{Integers}$

Julia: Whenever I see people talking about jump discontinuities, they always use piecewise functions. Do you think it's possible to make one without the function being piecewise?

Dylan: If there's one thing that I've learned in math, it's that there are usually two ways to do anything! I'm not really sure how you would make something like that though...

James: I know one function that would work!

Question 8 ***Hint:*** *James says the function has a constant value on the positive x values, a different constant value for negative x values, and is undefined at $x = 0$.*

What function is James talking about? $\boxed{|x|/x}$

Julia: Hey y'all, I was looking at our continuous graphs and noticed something.

Dylan: What did you see? They all look like pretty normal functions to me.

James: Yeah, I don't really know what you mean.

Julia: Well, discontinuities mean there is a chunk of the graph where you can skip over a value, right? Like, we can jump right from 1 to 5, or have a hole where some value isn't attained.

Dylan and James: Right. And?

Question 9 ***Hint:*** *Can we skip any of the values?*

What does Julia want to say about every value in a range of y values $[f(a), f(b)]$ on a continuous graph?

Multiple Choice:

(a) *Every value between $f(a)$ and $f(b)$ will be attained at some point on the interval $[a, b]$* ✓

(b) *Only normal looking functions are continuous.*

(c) *No values that are not between $f(a)$ and $f(b)$ will be attained over the interval.*

(d) *No functional values are repeated over the interval.*

Feedback (`correct`): *Congrats! You found what's called the Intermediate Value Theorem!*

Theorem 1 (Intermediate Value)**.** *If f is a continuous function for all x in the closed interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$.*

Rational Functions

Introduction

James: Hey guys, I slept through class yesterday... could you fill me in on what a rational function is?

Julia: See, class didn't make a lot of sense to me because I was thinking, "Functions can be rational?"

Dylan: They don't mean rational like me or you, Julia! It means *the function can be represented as a fraction where the numerator and denominator are both polynomials.*

Julia and James: Oh!

Dylan: Rational functions are pretty neat, because they can have two different types of discontinuities!

Altogether: LET'S DIVE IN!

Guided Example

Consider the function $f(x) = \dfrac{(x-2)(x+4)}{(x-3)(x+3)(x+4)}$

Question 1 *Describe the graph. What strange things do you notice?*

Free Response:

The "hole" present in the graph is called a **removable discontinuity**.

The vertical lines that the curve approaches at $x = 3$ and $x = -3$ is called a **vertical asymptote**, another type of discontinuity.

Learning outcomes:
Author(s): The College of Wooster

On Your Own

Find and report the locations of discontinuities in the following functions, note that at this time Desmos does not show removable discontinuities. You will need to find those by hand:

Question 2 $a(x) = \dfrac{x^2+1}{x-2}$

Graph of

Select All Correct Answers:

(a) $x = 2$ ✓

(b) $x = -1$

(c) $x = 1$

(d) $x = 0$

(e) $x = 4$

(f) *None*

$b(x) = \dfrac{x^2-5x+7}{x^2-x-6}$

Graph of

Select All Correct Answers:

(a) $x = 2$

(b) $x = -2$ ✓

(c) $x = 1$

(d) $x = 0$

(e) $x = 3$ ✓

(f) *None*

$c(x) = \dfrac{x^2-x}{x}$

Graph of

Select All Correct Answers:

(a) $x = 2$

(b) $x = 5$

(c) $x = 1$

(d) $x = 0$ ✓

(e) $x = 3$

(f) *None*

$d(x) = \dfrac{x+2}{x^2 - x - 6}$

Graph of

Select All Correct Answers:

(a) $x = -6$

(b) $x = 3$ ✓

(c) $x = 2$

(d) $x = 0$

(e) $x = -2$ ✓

(f) *None*

$f(x) = \dfrac{2x^2 + 5}{x^2 - 25}$

Graph of

Select All Correct Answers:

(a) $x = -1$

(b) $x = -5$ ✓

(c) $x = 10$

(d) $x = 0$

(e) $x = 5$ ✓

(f) *None*

$g(x) = \dfrac{x^3 - x^2 - 15x - 9}{x + 3}$

Graph of

Select All Correct Answers:

(a) $x = 12$

(b) $x = -4$

(c) $x = 3$

(d) $x = 5$

(e) $x = -3$

(f) *None* ✓

$h(x) = \dfrac{1}{3x^2 - x}$

Graph of

Select All Correct Answers:

(a) $x = -\sqrt{3}$

(b) $x = \dfrac{1}{3}$ ✓

(c) $x = 3$

(d) $x = 0$ ✓

(e) $x = -3$

(f) *None*

Question 3 *How can you tell if a rational function has a vertical asymptote or a removable discontinuity?*

Multiple Choice:

(a) *Vertical asymptotes occur where only the denominator approaches zero, and removable discontinuities*

occur where both the numerator and denominator approach zero. ✓

(b) *Vertical asymptotes occur where only the numerator approaches zero, and removable discontinuities*

occur where both the numerator and denominator approach zero.

(c) *Vertical asymptotes occur where both the numerator and denominator approach zero, and removable*

discontinuities occur where only the denominator approaches zero.

(d) *Vertical asymptotes occur where only the numerator approaches zero, and removable*

discontinuities occur where only the denominator approaches zero.

In Summary

James: These functions are pretty neat! What were they called again?

Dylan: They're called **rational functions**, *fractions where the numerator and denominator are both polynomials*!

Julia: So, when exactly does a **vertical asymptote** occur?

James: I know this one! **Vertical asymptotes** *occur at points where the denominator of the function will be zero, but the numerator is non-zero*!

Julia: That makes sense! But when do removable discontinuities occur then?

Dylan: **Removable discontinuities** *occur where the numerator and denominator are both zero.*

Derivative

Julia: Ah, this sucks!

Dylan: What's up?

Julia: I'm supposed to find the slope of a parabola at a point, and I'm not sure how!

Dylan: Well if we had two points we could make a secant line to approximate it!

Julia: Secant line? What's that?

Dylan: A *secant line* is just a line which connects two points on a function!

Julia: But isn't the *tangent* line one that skims a curve at one point? So the slope of the tangent line is the slope at that point! See?

Graph of $f(x) = x^2, g(x) = 2(x-1)+1$

Dylan: Well do you know how to find the equation for a line with just one point?

Julia: ...

James: Come on guys we can approximate the tangent line using the secant line!

Altogether: Let's dive in!

Guided Example

Consider the function $f(x) = x^2$. In green is $g(x) = 4x - 4$ the tangent line at the point $x = 2$. Thus the slope of the tangent line at $x = 2$ is 4.

Graph of $f(x) = x^2, g(x) = 4x - 4$

Question 1 *Find the slope of the secant line between $x = 2$ and $x = 7$.*

$\boxed{9}$

Does this seem to be a good approximation for the slope of the tangent line at $x = 2$?

Learning outcomes:
Author(s): The College of Wooster

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Dylan thinks we can solve the problem by just picking something closer than 7. Find the slope of the secant line between $x = 2$ and $x = 3$.

$\boxed{5}$

Is this a good approximation for the slope of the tangent line at $x = 2$?

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Is it better than the last attempt?

Multiple Choice:

(a) *Yes* ✓

(b) *No*

Julia: Dylan, this still isn't a great approximation...

Dylan: Well, I think we need to get even closer. Like, infinitesimally close! But how would we do that....

James: You guys need some help?

Julia and Dylan: James! How do we find the slope of a line at a point?

James: It isn't too tough! Before, you were considering a certain point as your comparison. What if instead, you used the point you want to evaluate at plus something really small? Let's call it h.

Question 2 *How can you make the h in*

$$\frac{f(2+h) - f(2)}{(2+h) - 2}$$

approach 0?

Multiple Choice:

(a) *Use* $\lim_{h \to 0}$. ✓

(b) *Use* $\lim_{h \to \infty}$.

(c) *Divide the fraction by* h.

(d) *Pick a function* $f(x)$ *so that* $f(2)$ *is 0.*

Using the method you determined, approximate the slope of the tangent line at the point x=2.

$\boxed{4}$

James: Want to know something really cool?

Julia and Dylan: What James?

James: The function we just discovered is how you determine a function's derivative! Using that process, you can find the instantaneous rate of change at any point on a function!

Julia and Dylan: Wow! So cool!

On Your Own

Using what you've learned, find the derivative of the following functions at the given point.

Question 1 ***Hint:*** *Remember you can enter* $\sqrt{x}$ *as either*

```
sqrt(...) or (...)^(1/2)
```

$g(x) = x^2 + 1$, $x = 2$

$\boxed{4}$

$h(x) = \frac{1}{x}$, $x = 2$

$\boxed{-0.25}$

$f(x) = 3x^2 + 4x + 2$, $x = -1$

$\boxed{-2}$

$f(x) = \sqrt{x^2+1}$, $x = 3$

$\boxed{\dfrac{3}{\sqrt{10}}}$

$f(x) = x + x^{-1}$,$x = 4$

$\boxed{15/16}$

By replacing the point in our formula for the derivative with x, we may determine the derivative at any point on the function. Determine the derivative for the following functions.

Question 2 $m(x) = x^3$

$\boxed{3x^2}$

$n(x) = 3x + 2$

$\boxed{3}$

$f(x) = 4 - x^2$

$\boxed{-2x}$

$f(x) = 12 + 7x$

$\boxed{7}$

$f(x) = \dfrac{4}{x+1}$

$\boxed{\dfrac{-4}{(x+1)^2}}$

In Summary

Julia: So why is it called a secant line?

James: It comes from the Latin word secare, which means 'to cut'.

Dylan: Ohh, I get it now! Because a secant line is a line that 'cuts' a function!

In this lab we've (hopefully) learned the function for finding a derivative using the limit as h approaches 0. We also learned what secant and tangent lines are. For your convenience, the important definitions are listed below.

Definition 1. *A **secant line** is any line that connects any two points on a curve.*

Definition 2. *A **tangent line** is a line which intersects a differentiable curve at a point where the slope of the curve equals the slope of the line.*

Definition 3. *The **derivative** of $f(x)$ at a is defined by the following limit:*

$$\frac{d}{dx}f(x)\Big|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

tikz

Differentiation Rules!

Julia: Hmm...I don't think differentiation rules, it takes so long and I hate using that long limit definition!

Dylan: No no Julia, it's differentiation *rules*!

Julia: Ohhhh, that makes more sense!

The Power Rule

Julia: I hate how long it takes to differentiate powers!

Dylan: Yeah, it takes forever! I feel like there was some sort of pattern to it, but I couldn't figure anything out.

James: Sounds like you guys need my help again?

Julia and Dylan: Help us James!

James: There *is* a pattern! Check out this table I made!

$f(x)$	$f(x)$
x^2	$2x^1$
x^3	$3x^2$
x^3	$4x^3$

Question 1 *What pattern do you notice in James' table? Generalize this pattern in terms of x^n.*

Multiple Choice:

(a) $n \cdot x^{n-1}$ ✓

(b) $n - 1 \cdot x^{n-1}$

(c) $n \cdot x^n$

Learning outcomes:
Author(s): The College of Wooster

(d) $n - 1 \cdot x^n$

***Feedback* (`correct`):** *Congrats! You've found what's known as the Power Rule!*

Definition 4. *The **derivative** of $f(x)$ at a is defined by the following limit:*

$$\frac{d}{dx} f(x)\bigg|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Question 2 *Using the limit definition of a derivative, compute the derivative for x^3.*

$\frac{d}{dx} x^3 = \boxed{3x^2}$

***Feedback* (`correct`):** *Notice that your answer fits the same pattern as before!*

Question 3 *Use the power rule to differentiate the following functions.*

$f(x) = x^{10}$ $\quad f(x) = \boxed{10x^9}$

$f(x) = 3x^2$ $\quad f(x) = \boxed{6x}$

Hint: *The value $\frac{1}{x}$ can be represented by x^{-1}.*

$f(x) = \frac{5}{x}$ $\quad \frac{d}{dx} f(x) = \boxed{-5x^{-2}}$

The Constant Rule

Dylan: Wow! That's neat!

Julia: I wish we could use rules like this all over the place though, it would really save me time.

James: There are plenty of places with rules like this! Why don't we look at a function like $y = 3$?

Consider $y = c$, where c is some arbitrary constant.

Question 4

Differentiate this function using the limit definition.

$\frac{d}{dx}c = \boxed{0}$

What can you generalize about the derivative of $y = c$ based on this?

Multiple Choice:

(a) $\frac{d}{dx}c = 2c$

(b) $\frac{d}{dx}c = 0$ ✓

(c) $\frac{d}{dx}c = x$

(d) $\frac{d}{dx}c = \frac{c}{2}$

Feedback (*`correct`*): *Congrats! You've found what's known as the Constant Rule!*

Question 5 *Using what you found in the previous question, compute the following derivatives:*

$f(x) = 2$ $\qquad \frac{d}{dx}f(x) = \boxed{0}$

$f(x) = 100$ $\qquad \frac{d}{dx}f(x) = \boxed{0}$

$f(x) = 0$ $\qquad \frac{d}{dx}f(x) = \boxed{0}$

The Constant Multiple Rule

Julia: James! Show us more! These things are going to save me so much time on my homework!

James: Alright alright, calm down Julia. We can look at a function like $y = 3x$ next.

Consider $y = kx$, where k is some arbitrary constant.

Question 6 *Differentiate this function using the limit definition:* $\frac{d}{dx}(kx) = \boxed{k}$
What can you generalize about the derivative of $y = kx$ based on this?

Multiple Choice:

(a) $\frac{d}{dx}kx = 2k$

(b) $\frac{d}{dx}kx = kx^2$

(c) $\frac{d}{dx}kx = k$ ✓

(d) $\frac{d}{dx}kx = x$

Feedback (`correct`): *Congrats! You've found what's known as the Constant Multiple Rule!*

Question 7

Using what you found in the previous problem, compute the following derivatives:

$f(x) = 4x$ $\qquad \frac{d}{dx}f(x) = \boxed{4}$

$f(x) = 10x$ $\qquad \frac{d}{dx}f(x) = \boxed{10}$

$f(x) = \frac{1}{5}x$ $\qquad \frac{d}{dx}f(x) = \boxed{\frac{1}{5}}$

The Sum and Difference Rules

Dylan: Wow, this stuff is awesome! Is there any way to put it all together? Like, is there an easy way to tell what the derivative of $f(x) = 3x + 4$ is?

James: There is Dylan!

Question 8 *Consider the differentiable functions $f(x)$ and $g(x)$. Let $F(x) = f(x) + g(x)$, use the limit definition to find $F'(x)$.*

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0} \frac{\boxed{f(x+h)-f(x)}}{h} + \frac{\boxed{g(x+h)-g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}
\end{aligned}$$

What can you generalize based on this?

Multiple Choice:

(a) $(f(x)+g(x))' = f'(x) - g'(x)$

(b) $(f(x)+g(x))' = g'(x) \cdot f(x) + f'(x)$

(c) $(f(x)+g(x))' = f(x) \cdot g'(x) - g(x) \cdot f'(x)$

(d) $(f(x)+g(x))' = f'(x) + g'(x)$ ✓

Feedback (`correct`): *Congrats! You've found what's known as the Sum Rule!*

Question 9 *Using what you found in the previous problem, compute the following derivatives where $F(x) = f(x) + g(x)$:*

$f(x) = 3x^2 - 5x + 2,\ g(x) = x^2 + 3x$ $\qquad \frac{d}{dx}F(x) = \boxed{8x - 2}$

$f(x) = x^2 - 4x + 2,\ g(x) = -4x^2 + 3$ $\qquad \frac{d}{dx}F(x) = \boxed{-6x - 4}$

$f(x) = 5x^3 + 3x,\ g(x) = 2x^2 - 13x$ $\qquad \frac{d}{dx}F(x) = \boxed{15x^2 + 4x - 10}$

Question 10 *Julia wonders if a similar rule exists for $m(x) = f(x) - g(x)$. Using the limit definition of derivative, determine if there is a pattern. Then, if there is a rule, use it to solve the following problems. If there is not, do them using the limit definition.*

$f(x) = 3x^2 - 5x + 2,\ \ g(x) = x^2 + 3x$ $\qquad \frac{d}{dx}m(x) = \boxed{4x - 8}$

$f(x) = x^2 - 4x + 2,\ \ g(x) = -4x^2 + 3$ $\qquad \frac{d}{dx}m(x) = \boxed{10x - 4}$

$f(x) = 5x^3 + 3x,\ \ g(x) = 2x^2 - 13x \qquad \dfrac{d}{dx}m(x) = \boxed{15x^2 - 4x + 16}$

In Summary

We've covered a lot of differentiation rules in this lab, to help you out we've made the following table for you to fill out.

Question 11

Power Rule	$x^n = \boxed{n} \cdot \boxed{x^{n-1}}$
Constant Rule	$c = \boxed{0}$
Constant Multiple Rule	$c \cdot f(x) = \boxed{c} \cdot \boxed{f(x)}$
Sum Rule	$(f(x) + g(x)) = \boxed{f(x)} + \boxed{g(x)}$
Difference Rule	$(f(x) - g(x)) = \boxed{f(x)} - \boxed{g(x)}$

Differentiation Rules Part Two

SAGE
```
caseInsensitive = function(a,b) {
   return a.toLowerCase() == b.toLowerCase();
 };
```

Julia: You know, some of those rules we learned were pretty useful, but some of these derivatives still suck! There **HAS** to be a better way!

Dylan: I'm sure there is, and I'm sure I know who could help us!

James: Did I hear my name?

Dylan: Not yet!

Julia: James!

James: There are more rules for differentiation that can make your life just a little bit easier!

The Product Rule

James: From the last time we did this, what rule do you think would exist for the product of two functions?

Julia: Well, last time we added or subtracted the derivative of both functions, so I bet we multiply the derivative of both!

Dylan: Let's check!

Consider the functions $f(x) = 2x$ and $g(x) = 3x^3 + x^2$.

Graph of $f(x) = 2x, g(x) = 3x^3 + x^2$

Question 1 *Use Julia's guess to find the derivative of $f(x) \cdot g(x)$.*

$\boxed{18x^2 + 4x}$

Learning outcomes:
Author(s): The College of Wooster

Definition 5. *The **derivative** of $f(x)$ at a is defined by the following limit:*

$$\frac{d}{dx} f(x)\bigg|_{x=a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Use the limit definition of the derivative to find the derivative of $f(x)\cdot g(x)$.

$\boxed{24x^3+6x^2}$

Was Julia right?

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Julia: Darn! It didn't work!

Dylan: It must be a little harder than that...

James: That's right Dylan, but it is easier than the limit definition! All we have to do is use

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + f'(x)\cdot g(x).$$

This is called the **Product Rule**.

Question 2 *Using the Product Rule, differentiate the products of the following functions:*

$f(x) = 6x^3$, $g(x) = 7x^4$

$\boxed{294x^6}$

$f(x) = \cos(x) + 4x$, $g(x) = 3x^2$

$\boxed{-3x^2\sin(x) + 6x\cos(x)}$

$f(x) = x^2$, $g(x) = 3x^3 - 3x$

$\boxed{15x^4 - 9x^2}$

$f(x) = x^7$, $g(x) = 2x^{32}$

$\boxed{78x^{38}}$

The Quotient Rule

Dylan: Wow! That's gonna save a ton of time with products! Is there anything like it we can do with quotients?

James: There is! It's even called **the Quotient Rule**!

Julia: I bet it's a pain too though, just like the product rule.

James: Well, why don't you try using your intuition first rather than guessing?

Dylan: Alright, well, I guess I would divide the derivative of the numerator by the derivative of the denominator.

Question 3 *Consider the functions $f(x) = x^3 + 1$ and $g(x) = x$.*

Graph of $f(x) = x^3 + 1, g(x) = x$

Use Dylan's guess to find the derivative of $\dfrac{f(x)}{g(x)}$.

$\boxed{3x^2/1}$

Use the limit definition of the derivative to find the derivative of $\dfrac{f(x)}{g(x)}$.

$\boxed{(2x^3 - 1)/x^2}$

Was Dylan right?

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Julia: I knew it! It's never that easy!

James: Now calm down Julia, this rule is worse than the last one, but it's much better than going through by the limit definition:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Question 4 *Using the Quotient Rule, differentiate the products of the following functions to find $\dfrac{d}{dx}\dfrac{f(x)}{g(x)}$:*

$f(x) = 3x - 1$, $g(x) = 2x + 1$

$\boxed{5/(2x+1)^2}$

$f(x) = 1$, $g(x) = x + 10$

$\boxed{-1/(x+10)^2}$

$f(x) = x^2$, $g(x) = 3x^3 - 3x$

$\boxed{(-x^2-1)/(3(x^2-1)^2)}$

$f(x) = x^7$, $g(x) = 2x^{32}$

$\boxed{-25/(2x^{26})}$

The Chain Rule

James: There's one last rule to learn today; the **Chain Rule**.

Dylan: That rule sounds pretty cool! When do we use it though? I thought we already covered the functions we need to know...

Julia: Yeah, what else is there?

James: We use the chain rule in composition of functions, like when we have $\sin(2x)$ - $2x$ is a function, and so is $\sin(x)$

Julia: And how bad is the rule?

James: This one is a little more tricky -

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Dylan and Julia: That's so gross.

James: Well, let's give it a try and see if you like it more than the limit definition!

Question 5 *Consider* $f(x) = \sqrt(x)$ *and* $g(x) = \dfrac{1}{x}$

Graph of $sqrt(x), 1/x$

Using the limit definition of derivative, evaluate the derivative of $f(g(x))$.

$\boxed{-1/2x^{3/2}}$

Now, evaluate the same limit using the chain rule. Notice you get the same answer. Yay.

Question 6 *Find the compostition $f(g(x))$, then using the Chain Rule, differentiate $f(g(x))$ for the following functions:*

$f(x) = 3x + x^2$, $g(x) = x^4 + 7x$

$f(g(x)) = \boxed{3(x^4 + 7x) + (x^4 + 7x)^2}$ $\frac{d}{dx}f(g(x)) = \boxed{8x^7 + 70x^4 + 12x^3 + 98x + 21}$

$f(x) = \cos(x)$, $g(x) = \sin(x)$

$f(g(x)) = \boxed{\cos(\sin(x))}$ $\frac{d}{dx}f(g(x)) = \boxed{-\cos(x)\sin(\sin(x))}$

$f(x) = \cos(x)$, $g(x) = x^3$

$f(g(x)) = \boxed{\cos(x^3)}$ $\frac{d}{dx}f(g(x)) = \boxed{-3x^2\sin(x^3)}$

Question 7 *Using the Chain Rule, differentiate the compositions $g(f(x))$ for the following functions:*

$f(x) = 3x + x^2$, $g(x) = x^4 + 7x$

$g(f(x)) = \boxed{(3x + x^2)^4 + 7(3x + x^2)}$ $\frac{d}{dx}g(f(x)) = \boxed{(2x + 3)(4x^3(x + 3)^3 + 7)}$

$f(x) = \cos(x)$, $g(x) = \sin(x)$

$g(f(x)) = \boxed{\sin(\cos(x))}$ $\frac{d}{dx}g(f(x)) = \boxed{-\cos(\cos(x))\sin(x)}$

$f(x) = \cos(x)$, $g(x) = x^3$

$g(f(x)) = \boxed{\cos(x)^3}$ $\frac{d}{dx}g(f(x)) = \boxed{-3\cos^2(x)\sin(x)}$

In Summary

We've covered a lot of differentiation rules in this lab, to help you out we've summarized the theorems below:

Theorem 2 (The Product Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Theorem 3 (The Quotient Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,[f(x)] - f(x)\,[g(x)]}{[g(x)]^2}$$

Theorem 4 (The Chain Rule)**.**

$$\frac{d}{dx}\,[f(x)g(x)] = f(x)\frac{d}{dx}\,[g(x)] + g(x)\frac{d}{dx}\,[f(x)]$$

Implicit Differentiation

SAGE

```
caseInsensitive = function(a,b) {
   return a.toLowerCase() == b.toLowerCase();
 };
```

Dylan: Woah! What's up with this?

Julia: I didn't know functions were explicit!

Dylan: The x and y are on the same side of the equation! I can't deal with this.

James: Functions can be explicit or implicit! And it not the way you're thinking Julia...

Introduction

So far we have dealt only with explicitly defined functions, where $y = f(x)$. Here y is dependent variable and it is given in terms of the independent variable x. Functions given in terms of both independent and dependent variables are called *implicit* functions.

Guided Example

Question 1 *Which of the following equations defined y as a function of x implicitly?*

Select All Correct Answers:

(a)
$$y = x^2 + 5x - 7$$

(b)
$$y = \sin(x)$$

Learning outcomes:
Author(s): The College of Wooster

(c)

$$x^2 + y^2 = 1$$

✓

(d)

$$y = \sqrt{(x-3)}$$

(e) $x^2y^3 + y = 5x + 8y$ ✓

Graph the curve defined by this equation:

$$x^2 + y^2 = 1$$

Graph of

Now, in the following Sage cell, solve for y. For help using the solve command refer to the documentation here.

SAGE

```
x,y = var("x, y")
#eqn = x^2+y^2==1, this sets eqn to the unit circle
#use the solve command to solve eqn for y
```

Graph the two explicit equations on the same axis below.

Graph of

Question 2 *Which of the following are true?*

Select All Correct Answers:

(a) $x^2 + y^2 = 1$ *is a function of* x.

(b) $y = -\sqrt{1-x^2}$ *is a function of* x. ✓

(c) $y = \sqrt{1-x^2}$ *is a function of* x. ✓

SAGE

1 **Question 3**

Using the functions you found, differentiate to find the slope of the tangent lines at the point $\left(\left(\frac{\sqrt{2}}{2}\right),\left(\frac{\sqrt{2}}{2}\right)\right)$*. You may do this in the above Sage cell or by hand.*

$\boxed{-1}$

Unfortunately not all implicit equations can be easily solved for y, which is why we use implicit differentiation!

Explanation. *Starting with*

$$x^2 + y^2 = 1$$

we first differentiate each term using $\frac{d}{dx}$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$$

You can already fill in 2 of the terms

$$\boxed{2x} + \frac{d}{dx}y^2 = \boxed{0}$$

For the term $\frac{d}{dx}y^2$ *you can imagine* $y = f(x)$*, and hence by the chain rule*

$$\begin{aligned}\frac{d}{dx}y^2 &= \frac{d}{dx}(f(x))^2 \\ &= 2 \cdot f(x) \cdot f'(x) \\ &= 2y\frac{dy}{dx}\end{aligned}$$

Thus we have the following:

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ *we get*

$$\boxed{\frac{-x}{y}}$$

Question 4 *Use the equation obtained from the above explanation to find* $\frac{dy}{dx}$ at $\left(\left(\frac{\sqrt{2}}{2}\right),\left(\frac{\sqrt{2}}{2}\right)\right)$ $\boxed{-1}$

Feedback (attempt): *Using both methods you can obtain the same answer, but for many equations the first method is much more work!*

Question 5 *We can fairly easily use Sage to do this process for us, to illustrate the process evaluate the following Sage cell.*

SAGE

```
var('x,y')
y(x)=function('y')(x)
eq=x^2+y^2==1
eq.substitute(y =y(x))
diff(eq,x)
solve(diff(eq,x),diff(y(x)))
```

What did you get as output from your Sage cell? (copy just the answer portion after the "==" and before the "]") $\boxed{-x/y(x)}$

Feedback (attempt): *Notice that Sage uses $y(x)$ for y in the output.*

On Your Own

In each of the following problems, evaluate the derivative by hand, and use the Sage cells as a check.

Consider the equation $y^4 + xy = x^3 - x + 2$. Using the following Sage cell implicitly differentiate to find $\frac{dy}{dx}$ using the same commands as shown in the previous question.

SAGE

```
var('x,y')
```

Question 6 *What did you get as output from your Sage cell? (copy just the answer portion after the "==" and before the "]")*

$\boxed{(3 * x^2 - y(x) - 1)/(4 * y(x)^3 + x)}$

Question 7 *Using your result in the previous section, evaluate* $\frac{dy}{dx}$ *at the point* $(1,1)$. $\boxed{1/5}$

Question 8 *Now use Sage Math again to find* $\frac{dy}{dx}$ *for* $\sin(x^2) = \cos(xy^2)$, *copy your answer in the same way as indicated in the previous section.*

SAGE
```
var('x,y')
```

$\boxed{-1/2 * (sin(x * y(x)^2) * y(x)^2 + 2 * x * cos(x^2))/(x * sin(x * y(x)^2) * y(x))}$

Perpendicular at a Point

Julia: Wow, implicit differentiation is rough.

Dylan: You're telling me... I've been doing this for hours! I wish we could at least do a little more with it if I have to learn it.

James: Did I hear that you guys want to know more about using implicit differentiation?

Julia and Dylan: James! Tell us more!

James: Alright guys, you can use implicit differentiation with implicit functions to tell if two functions are perpendicular at a point!

Julia: But how?

Dylan: Yeah, I don't see how that helps.

James: It's easy - all we have to do is see if the tangent lines are perpendicular at that point, and if they are, then so are the curves!

Question 9 *Graph* $3x - 2y + x^3 - x^2y = 0$ *and* $x^2 - 2x + y^2 - 3y = 0$ *on the same set of axes.*

Graph of

Do they look perpendicular anywhere?

Multiple Choice:

(a) *Yes* ✓

(b) *No*

Hint: *To show two lines are perpendicular you must show that the slope of one is the opposite inverse of the other*

Show the two curves are (or are not) perpendicular at the origin. You can do this in the Sage cell provided or by hand.

SAGE

```
```

Slope of $3x - 2y + x^3 - x^2y = 0$ *at the origin:*

$\boxed{3/2}$

Slope of $x^2 - 2x + y^2 - 3y = 0$ *at the origin:*

$\boxed{-2/3}$

Are the lines perpendicular at the origin?

In Summary

There are two main methods to differentiate implicit equations

(a) Solve for y and then differentiate.

(b) Treat y as $y(x)$ and differentiate with respect to the variable x, eventually solving for $\dfrac{dy}{dx}$ to give the value of the derivative at any point (x, y).

Mean Value Theorem

Introduction

Dylan: I don't know about this theorem...it seems pretty *mean*...

Julia: No no, they mean *mean* as in average!

Dylan: Oh, so were looking at the average value of a function?

James: Not quite, actually the **mean value theorem** states the following: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists at least one value c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

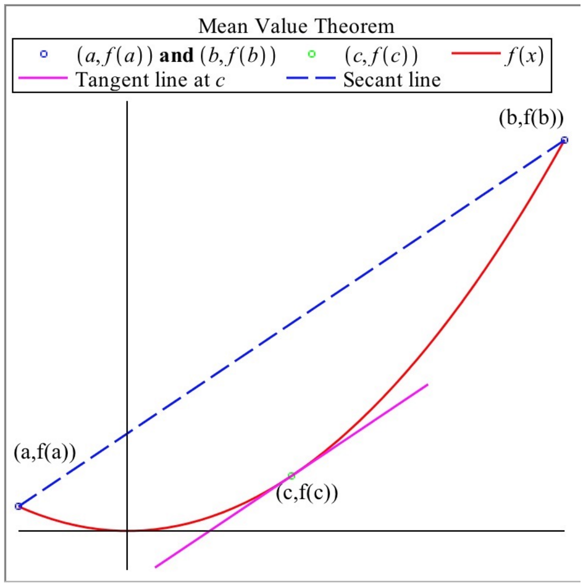
Dylan and Julia: Maybe we should do an example...that looks pretty confusing...

ALTOGETHER: Let's dive in!

Guided Example

Take a look at the following graph illustrating the Mean Value Theorem:

Learning outcomes:
Author(s): The College of Wooster



Question 1 *What do you notice about the tangent line at c with respect to the secant line from a to b?*

Multiple Choice:

(a) *They are parallel.* ✓

(b) *The slope of the tangent line is half that of the secant line.*

(c) *The tangent line is perpendicular to the secant line.*

(d) *The tangent line intersects the secant line.*

What does this mean the derivative of $f(x)$ is at c?

Multiple Choice:

(a) $\frac{1}{2}$

(b) $\frac{a-b}{f(a)-f(b)}$

(c) $\frac{f(b)-f(a)}{b-a}$ ✓

(d) $\frac{2a}{b}$

If $f'(x)$ was zero for all points in the interval, what could always be said about $f(x)$ on that interval?

Multiple Choice:

(a) *$f(x)$ is positive at all points on the interval.*

(b) *$f(x)$ is negative at all points on the interval.*

(c) *$f(x)$ is a constant on the interval.* ✓

(d) *$f(x)$ does not exist on the interval.*

Explanation. *Let $a < b$ be two points in I. Since f is continuous on $[a, b]$ and differentiable on (a, b), by the Mean Value Theorem we know*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c in the interval (a, b). Since $f'(c) = 0$ we see that $f(b) = f(a)$. Since a and b were arbitrarily chosen, $f(x)$ must be constant on the interval.

Question 2 *Use $f(x) = \sin(2x)$ on the interval $[0, 2\pi]$ for the following questions.*

Hint: *To graph $f(x) = \sin(2x)$ on the domain $[0, 2\pi]$ in desmos type: sin(2x)0¡=x¡=2pi.*

Graph $f(x)$

Graph of

What values for c satisfy the mean value theorem?(figure out where the derivative equals the slope of the secant line!)

Select All Correct Answers:

(a) $\frac{\pi}{4}$ ✓

(b) 1

(c) $\frac{3\pi}{5}$

(d) $\frac{3\pi}{4}$ ✓

(e) $\frac{5\pi}{4}$ ✓

(f) 0

(g) π

(h) $\dfrac{7\pi}{4}$ ✓

On Your Own

Let $f(x) = \left|x^2 - x - 2\right|$.

Graph of $f(x) = abs(x^2 - x - 2)$

Question 3 *Examine the graph. Does the function satisfy the hypothesis of the Mean Value Theorem on the interval $[a, b] = [0, 3]$?*

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Question 4 *Consider $f(x) = \dfrac{1}{x}$.*

Graph of $1/x$

Over which of the following regions does the function satisfy the hypothesis of the Mean Value Theorem?

Select All Correct Answers:

(a) $[-10, 10]$

(b) $[0, 10]$

(c) $[1, 10]$ ✓

(d) $[-10, -1]$ ✓

(e) $[-10, 0]$

Apply the Mean Value Theorem from [1, 4], determining what points experience the same instantaneous change as the entire interval. $\boxed{\frac{2}{\sqrt{3}}}$

Question 5 *Seeing a police officer on the side of the road, your friend Tom slows down to 35 mph. However, once the officer pulls over someone else for speeding, Tom speeds up to 70 mph. Half an hour and 35 miles later, Tom checks his navigation app and sees another police officer is up ahead, slowing himself down to the legal 35 mph. However, the police officer still pulls Tom over, saying he had been radioed by the first officer right when Tom passed, so he could prove that Tom was going 70 mph at some point in the last half hour. Tom is furious about the clearly faulty reasoning of the police officer. Let $g(x)$ be the position of Tom's car at time x.*

Thanks to the Mean Value Theorem, you know that the police officer is in the right. Using $g(x)$, explain to Tom why the officer had a valid reason to ticket him.

Hint: *What would Tom's average speed have to be to get 35 miles in a half hour?*

Free Response:

Talking to Tom, you find out that he accelerated to 70 mph in only 5 seconds after passing the officer. Prove that at some point, Tom had an acceleration of over 25,000 mi\h^2.

Hint: *What would Tom's average change of velocity have to be to go from 35 mph to 70 mph in 5 seconds(make sure to convert the seconds to hours)?*

Free Response:

In Summary

Theorem 5. *The **Mean Value Theorem** states that for any function f, if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists at least one value c such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This means that there is a point c such that the secant line from a, b has the same slope as the tangent line at c. It's important to note that this means if $f'(x) = 0$ for all x on (a, b), then f is constant on (a, b).

Curve Sketching

Introduction

Dylan: Using CAS systems to graph is great and all, but on a test where I don't have a calculator it's so hard to sketch a curve!

James: Well maybe we can use derivatives to figure out properties of the graph so it's easier to sketch!

Dylan: Oh! We'd be able to see where the graph was heading up or down, plus we'd be able to see extrema when the derivative at a point is zero!

Guided Problems

Question 1 *Consider the function $f(x) = x^3 + 12x^2 + 4$. Take the derivative and find the values where $f'(x) = 0$. Create a number line, and mark these points. Between them, mark the sign of the derivative of any point on that interval (Do this by hand). An example is shown below.*

On what interval(s) is the derivative positive?

Select All Correct Answers:

(a) $[-8, 0)$

(b) $(-\infty, -8)$ ✓

(c) $(-\infty, 0]$

(d) $(0, \infty)$ ✓

On what interval(s) is the derivative negative?

Learning outcomes:
Author(s): The College of Wooster

Select All Correct Answers:

(a) $[-4, \infty)$

(b) $(-8, 0)$ ✓

(c) $(-\infty, 0]$

(d) $(0, \infty)$

Select everything you can always say about the graph of a function at a point where the derivative is 0:

Select All Correct Answers:

(a) *The graph will be flat at that point.* ✓

(b) *The derivative will change from positive to negative at the point.*

(c) *The derivative will change from negative to positive at the point.*

(d) *The derivative will change sign going from one side of the point to the other.*

Feedback (`correct`): *If the derivative at a point is zero, we can only tell that the graph flattens out. It is possible that the sign does not change after crossing the point!*

Julia: So we have local maxima or minima when the derivative is 0, but what about the graph of x^3?

Dylan: Hmmm...I guess that means there are three different kinds of critical points! Two when the sign changes and one when it stays the same!

Julia: Wait... what's a critical point?

Dylan: A **critical point** is any point where the derivative is zero or does not exist! Because we know it's important, but we have to check to see what it means with our number line!

James: You guys are still figuring that out? I'm already determining concavity!

Dylan and Julia: Holy cat fur! What's concavity?!

James: A graph is **concave up** when its derivative is increasing, and **concave down** when its derivative is decreasing. The easiest way to tell is to look at the curve and think 'Would this hold water?' If it would, it's concave up, and if not, it's concave down!

Dylan and Julia: Wow! Thanks James!

Question 2 *Now determine the second derivative, $f''(x)$, of our original function $f(x) = x^3 + 12x^2 + 4$. What happens at the points where the second derivative is zero?*

Multiple Choice:

(a) *They show the function is at extrema as well.*

(b) *They indicate that concavity will remain the same.*

(c) *They indicate that concavity is changing.* ✓

(d) *They show that the function is at minimum change.*

Points where concavity changes are known as ***inflection points****.*

Draw another number line, this time for the second derivative, marking where the derivative equals zero. Evaluate $f''(x)$ on a point of each of the intervals created through this marking, and mark the sign. What might this mean in general for changing signs on each side of an inflection point?

Select All Correct Answers:

(a) *The graph will be flat at that point.*

(b) *The concavity will change from up to down at the point.*

(c) *The concavity will change from down to up at the point.*

(d) *The concavity will change in that point, but it is impossible to tell how without knowing the value of the*

second derivative around that point. ✓

Where is the graph concave up?

Multiple Choice:

(a) $[0,1]$

(b) $(-4,\infty)$ ✓

(c) $(-\infty,0)$

(d) $[0,260]$

(e) $[4,260]$

What about concave down?

Multiple Choice:

(a) $[0,1]$

(b) $[-4,\infty)$

(c) $(-\infty,-4)$ ✓

(d) $[0,260]$

(e) $[4,260]$

Inflection points are given as ordered pairs. Evaluate each inflection point you found using $f(x)$ to determine the ordered pairs, then select them below.

Select All Correct Answers:

(a) $(-4,132)$ ✓

(b) $(0,4)$

(c) $(-12.028,0)$

(d) $(-8,260)$

Based on what you've done until now, sketch the graph yourself. When you're done, click below to see the graph.

Multiple Choice:

(a) *I'm Done! Show me the graph!* ✓

Feedback (attempt):

Graph of x^3+12x^2+4

Julia: Wait but what about the graph of $f(x) = x^4$? $f''(x) = 0$ at the origin but the graph doesn't change concavity see?

Graph of $x^4, 12x^2$

Dylan: Oh jeez, does that mean that not every instance of $f''(x) = 0$ is an inflection point?

James: Not always! That's why you have to make another number line for the second derivative to see if the sign of $f''(x)$ changes on either side of the point where $f''(x) = 0$.

On Your Own

Question 3 *Now, consider the function*

$$f(x) = x^{2/3}(6 - x)^{1/3}.$$

Without graphing the function, complete the following:

Select each type of extrema present on the graph.

Select All Correct Answers:

(a) *Local Maximum* ✓

(b) *Local Minimum* ✓

(c) *Global Maximum*

(d) *Global Minimum*

Indicate the coordinates of any critical points.

Select All Correct Answers:

(a) *(-6,0)*

(b) *(6,0)* ✓

(c) *(3.175, 4)*

(d) *(4,3.175)* ✓

(e) *(1,3)*

(f) *(0,0)* ✓

(g) *(5,0)*

Indicate the coordinates of any inflection points.

Select All Correct Answers:

(a) *(-6,0)*

(b) *(6,0)* ✓

(c) *(3.175, 4)*

(d) *(4,3.175)*

(e) *(1,3)*

(f) *(0,0)*

(g) *(5,0)*

Select the intervals on which the function is concave up.

Select All Correct Answers:

(a) $(-\infty), 0)$

(b) $(0, 6)$

(c) $(0, \infty)$

(d) $(6, \infty)$ ✓

Select the intervals on which the function is concave down.

Select All Correct Answers:

(a) $(-\infty), 0)$ ✓

(b) $(0, 6)$ ✓

(c) $(0, \infty)$

(d) $(6, \infty)$

Now, create a sketch of your function on paper. When you're done, click below to see how your graph should look!

Multiple Choice:

(a) *I'm done waiting! Show me the graph!* ✓

(b) *I'm not ready yet, but I'm going to click regardless!*

Feedback (`correct`):

Graph of $x^{2/3}(6-x)^1/3$

Question 4 *Now, consider the function*

$$f(x) = x^3 - 3x + 5.$$

Without graphing the function, complete the following:

Select each type of extrema present on the graph.

Select All Correct Answers:

(a) *Local Maximum* ✓

(b) *Local Minimum* ✓

(c) *Global Maximum*

(d) *Global Minimum*

Indicate the coordinates of any critical points.

Select All Correct Answers:

(a) *(-2,0)*

(b) *(-1,7)* ✓

(c) *(-3,7)*

(d) *(7,-2)*

(e) *(1,3)* ✓

(f) *(0,5)*

(g) *(5,0)*

Indicate the coordinates of any inflection points.

Select All Correct Answers:

(a) $(-2,0)$

(b) $(-1, 7)$

(c) $(-3, 7)$

(d) $(7, -2)$

(e) $(1, 3)$

(f) $(0, 5)$ ✓

(g) $(5, 0)$

Select the intervals on which the function is concave up.

Select All Correct Answers:

(a) $[-5, 10]$

(b) $[1, 2]$

(c) $(0, \infty)$ ✓

(d) $(-\infty, 0)$

Select the intervals on which the function is concave down.

Select All Correct Answers:

(a) $[0, 10]$

(b) $[1, 2]$

(c) $[0, \infty)$

(d) $(-\infty, 0]$ ✓

Now, create a sketch of your function on paper. When you're done, click below to see how your graph should look!

Multiple Choice:

(a) *I'm done waiting! Show me the graph!* ✓

(b) *I'm not ready yet, but I'm going to click regardless!*

Feedback (`correct`):

Graph of $x^3 - 3x + 5$

Matching Graphs

Dylan: So if I wanted to match a graph with the graphs of its first and second derivative I can do that now!

Julia: Wait, really?? How?

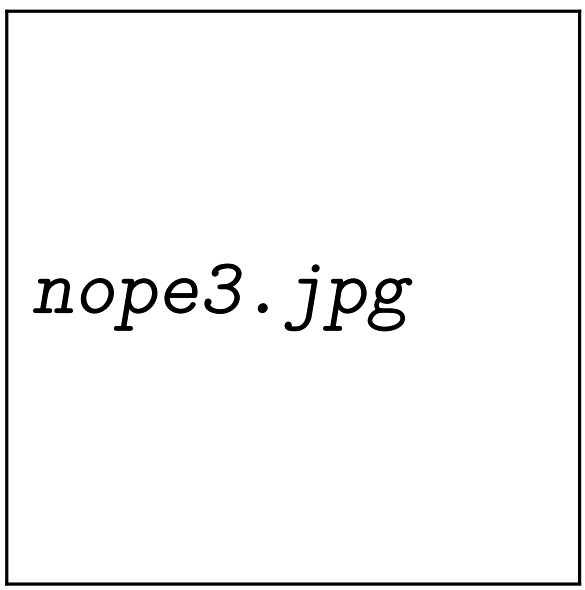
James: Well the y value of $f'(x)$ corresponds to the slope of $f(x)$, and the y value of $f''(x)$ corresponds to the slope of $f'(x)$ and is related to the concavity of $f(x)$.

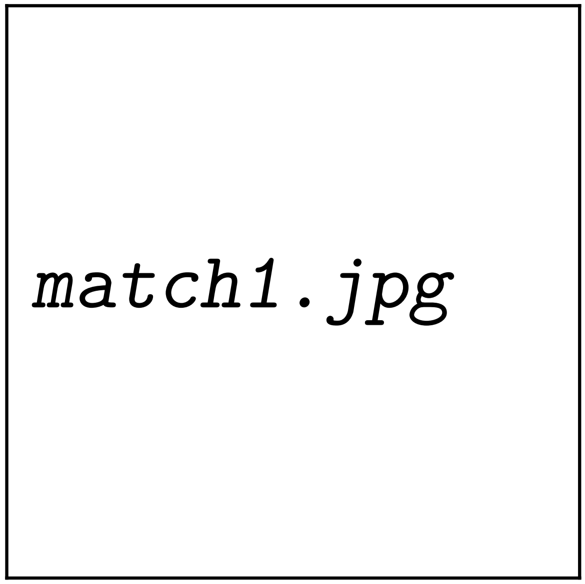
Julia: So we can match the graphs based on how all that information relates!

Dylan: Exactly, let's try it!

Question 5 *For the following problems, select the correct match of* $f(x), f'(x), andf''(x)$

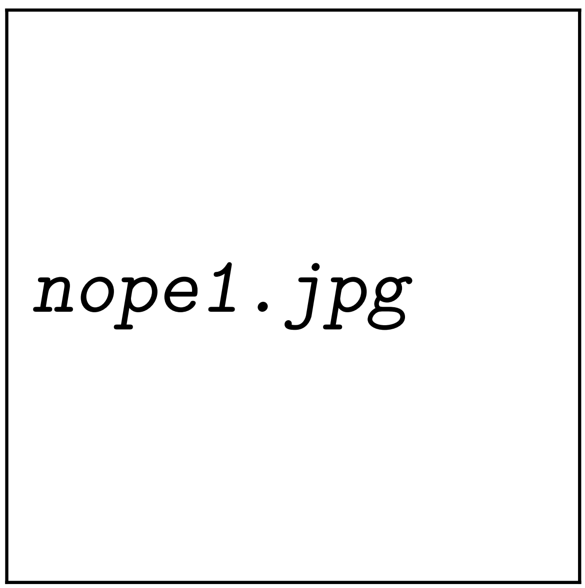
Multiple Choice:

(a) 

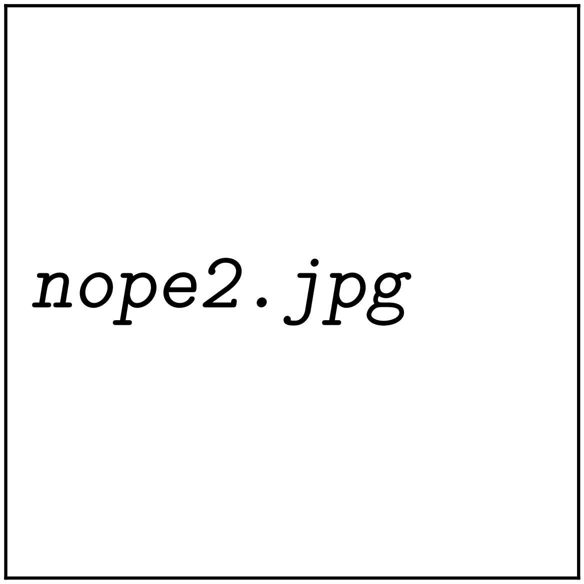


(b)

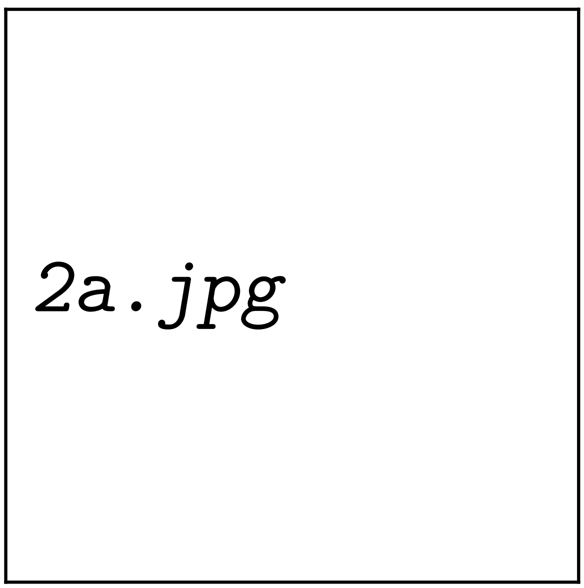
✓

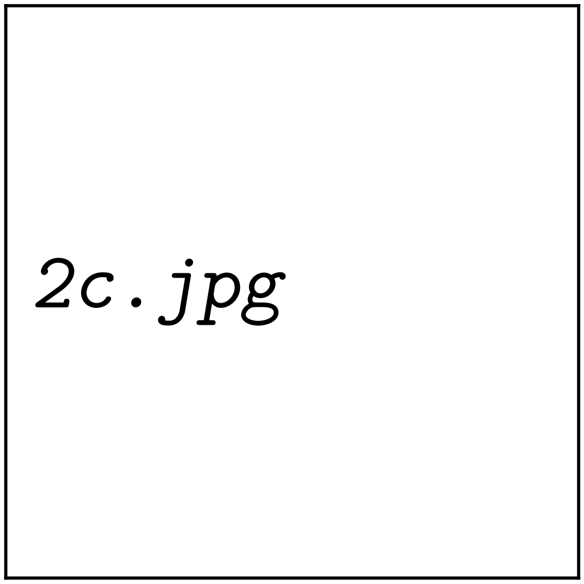


(c)

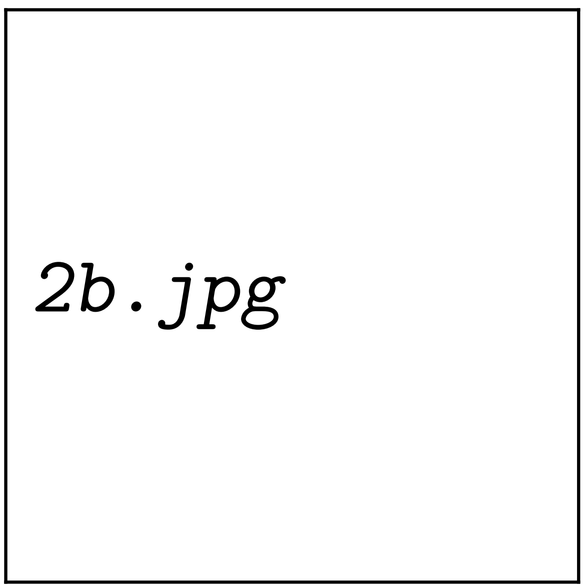
(d) 

Multiple Choice:

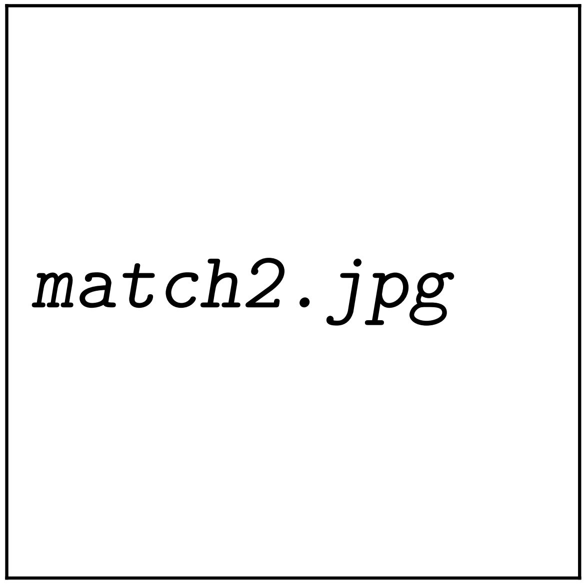
(a) 



(b)

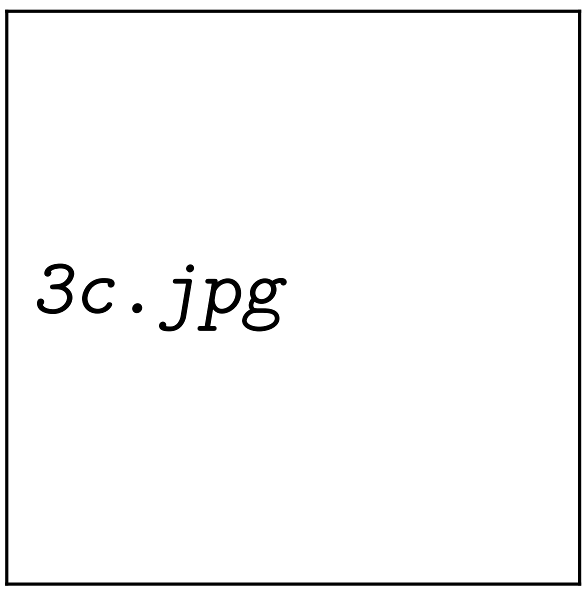


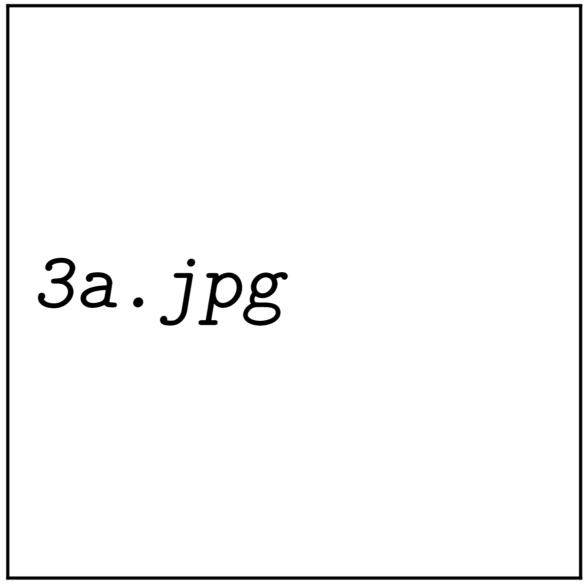
(c)

(d) 

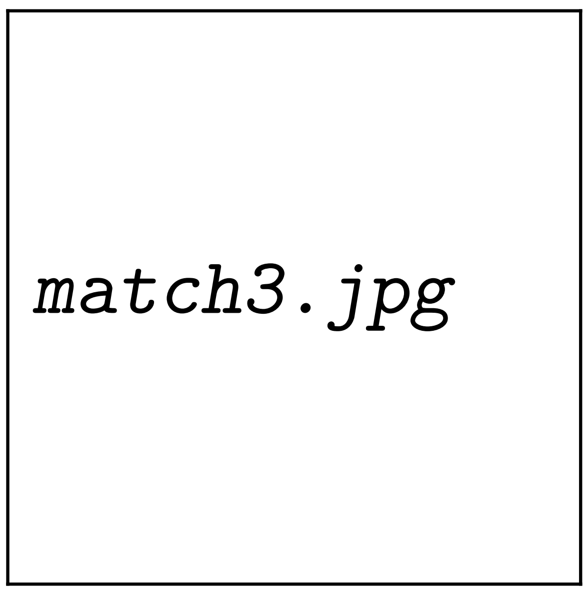
✓

Multiple Choice:

(a) 

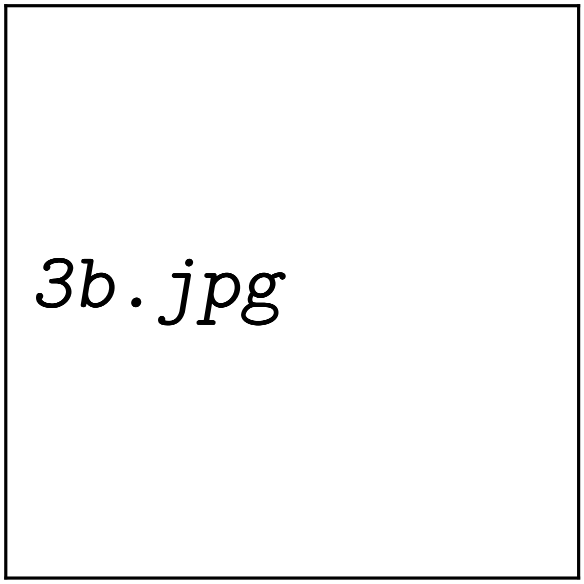


(b)

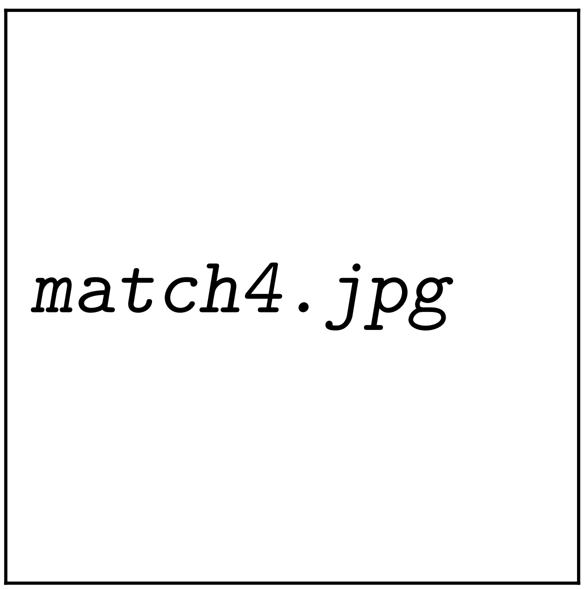


(c)

✓

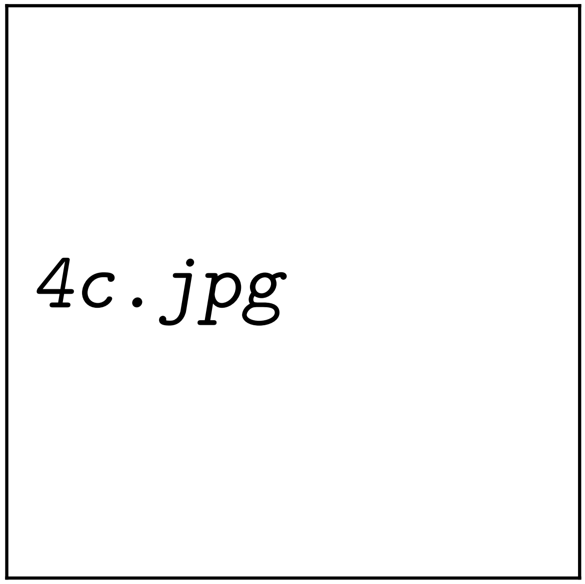
(d) 

Multiple Choice:

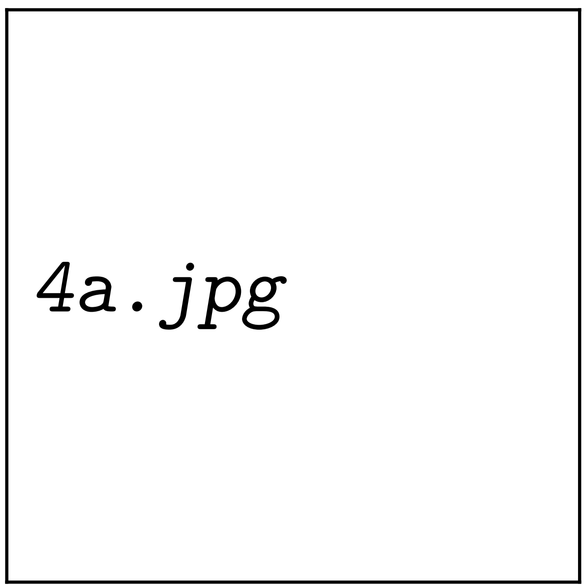
(a) 

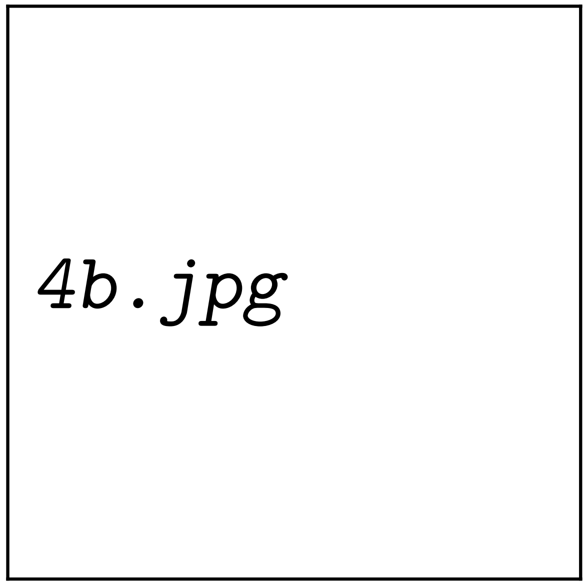
✓

(b)



(c)



(d) 

In Summary

In this lab, you've covered quite a bit. To help organize everything, we've put the important theorems below.

We've also included the second derivative test, which is another method to determine if a critical point is a maximum, minimum, or saddle point. To do this, you simply evaluate the second derivative at the critical point, and determine what that point is based on the value, as shown in the table.

Theorem 6 (First Derivative Test)**.** *Suppose that c is a critical number($f'(c) = 0$) on a continuous function f.*

(a) *If f' changes from positive to negative at c, then f has a local maximum at c.*

(b) *If f' changes from negative to positive at c, then f has a local minimum at c.*

(c) *If f' does not change sign at c, then f has a saddle point at c.*

Remark 1. *When $f'(c)$ does not exist:*

(a) *If c is in the domain of f, follow the same steps as when $f'(c) = 0$.*

(b) *If c is not in the domain, then c is not a critical number.*

Theorem 7 (Concavity Test)**.** *If $f''(x)$ exists for all $x \in (a, b)$*

(a) *If $f''(x) > 0$ for all $x \in (a, b)$, then f is concave up on (a, b)*

(b) *If $f''(x) < 0$ for all $x \in (a, b)$, then f is concave down on (a, b)*

Theorem 8 (Test for Inflection Points)**.** *If $f''(c) = 0$ or $f''(c)$ does not exist and $f''(x)$ changes sign at $x = c$, then f has a point of inflection at $x = c$*

Theorem 9 (Second Derivative Test)**.** *Let c be a critical point of $f(x)$. If $f''(c)$ exists, then*

(a) *If $f''(c) > 0$, then $f(c)$ is a local minimum.*

(b) *If $f''(c) < 0$, then $f(c)$ is a local maximum.*

(c) *If $f''(c) = 0$, then the test is inconclusive: $f(c)$ could be a local max, min, or neither.*

Application of the Derivative

Julia: I love class, but I keep wondering why I'm even learning this stuff. I'm not a math major.

Dylan: It isn't like we're ever going to use this stuff in our lives. It's all just theoretical.

James: Hold on guys! Actually, we use derivatives all the time - it is a way of measuring change after all.

Dylan: No way man, I can forget all this after class. Give me one time I'd use a derivative other than class.

James: I'll give you three!

Remark 2. *When possible, answer to three decimal places!*

The Great Molasses Flood

On January 15, 1919, a molasses storage tank in Boston burst, sending molasses rushing down the streets at 35 miles per hour.

Let's pretend something similar happens in Wooster! Imagine you're on the street, walking by our newly installed molasses tank when it begins to burst. Unfortunately, you're by Born, and the molasses is rushing down the hill towards you with its position modeled by

$$M(t) = \frac{1}{5}t^2 + t,$$

Your position can be modeled by

$$f(t) = 3t + 45.$$

In both cases, t is measured in seconds, with each equation reporting a position in meters.

Question 1 *What is your speed at any point?*

Learning outcomes:
Author(s): The College of Wooster
https://www.scientificamerican.com/article/molasses-flood-physics-science/

$\boxed{3}\, m/s$

What about the speed of the molasses?

$\boxed{(2/5)t + 1}\, m/s$

Question 2 *What is your acceleration?*

$\boxed{0}\, m/s^2$

The acceleration of the molasses?

$\boxed{2/5}\, m/s^2$

Question 3 *What is the speed of the molasses after one minute?*

$\boxed{25}\, m/s$

Question 4 ***Hint:*** *Make sure to take into account the distance the molasses will need to travel to each location.*

If you want to survive the flood, you'll need to get off the street and into a tall, sturdy building. Born is only 10 meters away, but there is a group of people trying to get in, meaning once you are there, it will take 20 seconds to reach the inside of the building. Bissman has very little foot traffic, but you'll take exactly 20 seconds to get there and inside. Which building should you go to?

Multiple Choice:

(a) *It makes no difference*

(b) *Born*

(c) *Bissman* ✓

Marginal Profit

A company that makes peanut butter has a profit of

$$P(x) = -0.0027x^3 + 0.05x^2 + 18x - 125,$$

where x is the number of units produced. One unit of peanut butter contains 10,000 jars and the profit is in thousands of dollars.

Question 1 *Compute the marginal profit, $P'(x)$.*

$P'(x) = \boxed{-.0081x^2 + .1x + 18}$

Explain what is meant by marginal profit?

Free Response:

Use the marginal profit function to approximate the increase in profit when production is increased from 20 units to 21 units.

$\boxed{16.76}$

Use the marginal profit function to approximate the increase in profit when production is increased from 65 to 66 units.

$\boxed{-9.723}$

Graph the marginal profit function:

Graph of

How would you change production based on this graph if the company was currently producing 20 units?

Multiple Choice:

(a) *Increase Production* ✓

(b) *Maintain Current Production*

(c) *Decrease Production*

What about 65 units?

Multiple Choice:

(a) *Increase Production*

(b) *Maintain Current Production*

(c) *Decrease Production* ✓

Dorm Room Froyo

You've opened up a Froyo franchise in your dorm room! It's a little cramped, but people are hearing about it and enjoying your generous pricing and the convenient location. We can model how many people hear about your franchise with the equation

$$p(t) = \sqrt[3]{50t^2 + \frac{100}{3}t + 10},$$

where t is time in days. We can also model the profit of your location with the equation

$$m(p) = .32p^2 - 30p - 7000,$$

where p is the population of people who are willing to come to your franchise each day.

Question 1 *Using the Chain Rule, at what rate will your profit be changing 35 days from now?*

$\boxed{-3.451}$

What will the rate of change be for your profit in 120 days?

$\boxed{13.665}$

Applications of Extrema

Julia: I love optimization, but I can't really imagine where we could use it in real life.

Dylan: Yeah, it seems great for graphs, but for real world problems? No way.

Julia and Dylan: .....

Julia and Dylan: .....

Julia: This is usually where James would chime in...

Dylan: Maybe he's running late?

James: Sorry guys, there was a traffic jam! I think it might be just perfect for our first illustration of the uses of optimization!

James' Traffic Jam

On the way back from Walmart, James ran into a traffic jam along the highway caused by an accident. While he was waiting in traffic, James decided to work on a function that roughly modeled the speed of the traffic over the day, using data from a surveyor who had been monitoring the accident. The equation he found was

$$v(t) = t^3 - 11t^2 + 25t + 45$$

where t is hours, $v(t)$ is miles per hour, and $t = 0$ at 7 AM. The function accurately models until 3 PM.

Question 1 *Use the graph of $v(t)$ to complete the following problems.*

Graph of $v(t) = t^3 - 11t^2 + 25t + 45$

At approximately what time is the traffic moving the slowest? Give your answer to the nearest hour. $\boxed{1}$

Multiple Choice:

(a) *AM*

Learning outcomes:
Author(s): The College of Wooster

(b) *PM* ✓

At what speed is the traffic moving at that time?

$\boxed{15}\, mi\backslash hr$

When is the traffic moving the most quickly?

$\boxed{8}$

Multiple Choice:

(a) *AM* ✓

(b) *PM*

At what speed, to the nearest mile per hour?

$\boxed{61}\, mi\backslash hr$

Dylan: Wow, I guess there are some uses for optimization!

Julia: Could we do something similar for the tree house I'm building for my cousin? It needs one side to be a screen to let air in and keep bugs out, but the rest should be wood. We want it to be 200 square feet.

James: Sure! Let's try and find the cheapest you could build it for.

Julia's Tree House

Julia is building a tree house for her younger cousin. She'd like two opposite sides to be large screens to give a great view and airflow, without letting bugs pour in. The rest will be made of wood, with windows (which we will not account for). Unfortunately, to have the screen be sturdy enough for Julia to be comfortable, it will cost \$18 per foot, while the wood will cost only \$7 per foot. The treehouse will have a wood floor and ceiling, and will be three times as long as it is wide. Given that she wants the volume to be 120 cubic feet, how should she design it to minimize the cost?

Question 2 *Determine the dimensions and cost of the cheapest tree house. Round cost to the nearest cent, and dimensions to 2 decimal places.*

You can use the following Sage cell to help with computations. You may find the diff and solve commands useful. Type diff? or solve? into the Sage cell for help using the commands.

SAGE

Length: $\boxed{10.008}$ *X Width:* $\boxed{3.336}$

Cost: $\boxed{1402.66}$ *$*

Julia: Wow, thanks James! That's going to be a real help!

James: Not a problem Julia.

Dylan: Could you help my little sister with her lemonade stand?

James: Sure, let's look at how she can maximize her profits!

Dylan's Lemonade Stand

Dylan's little sister is running a lemonade stand, selling a cup for 25 cents. Because all her lemonade is freshly squeezed, she never has a wasted cup. However, due to her success, she has hired numerous employees and runs smear campaigns against other lemonade stands, resulting in very steep running costs, modeled by

$$c(x) = x^3 + 11x^2 - 18x,$$

while her revenue can be modeled by

$$r(x) = 25x.$$

Both $r(x)$ and $c(x)$ are in thousands of dollars, and x is thousands of cups of lemonade.

Question 3 *You can use the following Sage cell to help with computations. You may find the diff and solve commands useful. Type diff? or solve? into the Sage cell for help using the commands.*

SAGE

How many thousands of cups of lemonade should Dylan's little sister sell? Please round your answer to two decimal places.

$\boxed{1.603}$

James: Excuse me for a second guys, I'm getting a call. Hello?... Yes, this is him... Sure, I'll get right on it!

Dylan: What was that about?

James: I just got a call from a small business that wanted me to help them figure out how to maximize their profits. Why don't you guys help me?

Julia and Dylan: Sounds good!

Handmade Paper Cups

James just got a call from a small company just north of Wooster which specializes in hand crafted paper cups. Every day, the company pays its workers \$2000, regardless of their productivity. Each thousand cups costs \$2.15 to produce, as a result of the "high quality" paper which is used. Every day, new materials are ordered for $\frac{\$1500}{x}$, where x is the number of cups in thousands produced in a single day.

Question 4 *You can use the following Sage cell to help with computations. You may find the diff and solve commands useful. Type diff? or solve? into the Sage cell for help using the commands.*

SAGE

How many thousands of cups should the company produce every day in order to minimize costs? Please round your answer to two decimal places.

$\boxed{26.414}$

Motion

Introduction

Dylan: I wonder where Julia and James are...

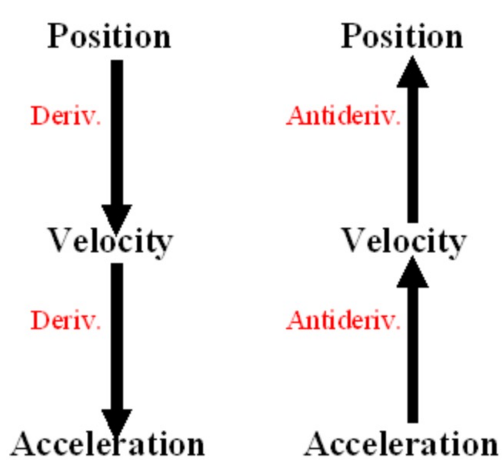
Julia: *(runs in panting and clutching side)* Ha! I win

James: *(enters, also catching breath)* I just don't get it, I was going faster than you at some point!!!

Dylan: Well don't you know that position, velocity, and acceleration are all related? Just because you were at a faster velocity at some point doesn't mean you got there first!

Julia and James: Oh gosh, please don't tell me this is more applications of derivatives...

There are three main aspects of motion that we will examine in this lab; position, velocity, and acceleration.



Learning outcomes:
Author(s): The College of Wooster

Guided Example

Question 1 *A banana is sliding across an ice hockey rink after being thrown in by an over-excited child. The position of the banana, in meters, can be given by*

$$p(t) = -\frac{1}{2}t^2 + 14t + 11,$$

where t is measured in seconds.

What does the slope of the graph mean in this context?

Multiple Choice:

(a) *The velocity of the banana in ft/s.*

(b) *The velocity of the banana in m/s.* ✓

(c) *The acceleration of the banana in m/s^2.*

(d) *The acceleration of the banana in ft/s^2.*

Graph this function.

Graph of

How would you determine the average velocity from $t = 3$ to $t = 6$?

Multiple Choice:

(a) $\dfrac{p(3) - p(6)}{3 - 6}$

(b) $\dfrac{p(6) - p(3)}{2}$

(c) $\dfrac{p(3) - p(6)}{2}$

(d) $\dfrac{p(6) - p(3)}{(6 - 3)}$ ✓

What is the average velocity over this interval?

$\boxed{9.5}$ *m/s*

With help from the formula you used in the previous question, determine the instantaneous velocity at any point.

$v(t) = \boxed{-t + 14}$

Graph the equation you found.

Graph of

Does this graph appear to model the rate of change of the original function?

Multiple Choice:

(a) *Yes* ✓

(b) *No*

If not, go back over your work from the previous problem.

What does the slope of this graph indicate?

Multiple Choice:

(a) *Acceleration of the banana.* ✓

(b) *Deceleration of the banana.*

(c) *The position of the banana over time.*

(d) *The velocity of the banana versus its initial velocity.*

Determine the average acceleration from $t = 3$ to $t = 6$.

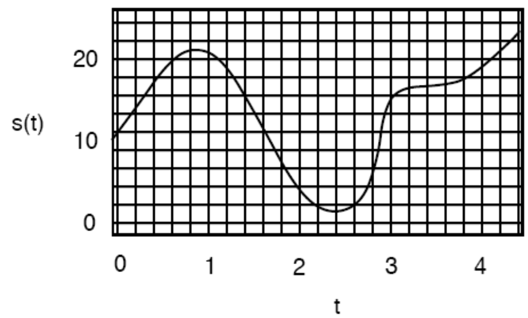
$\boxed{-1}\, m/s^2$

Now, create a function to determine the average acceleration at any point - the process will be extremely similar to that of problem 1 part d.

$a(t) = \boxed{-1}$

On Your Own

Question 2 *Examine the following graph of a particle's motion:*



At what time(s) does the particle return to its initial point?

Select All Correct Answers:

(a) 1.4 s

(b) 1.6 s✓

(c) 3 s

(d) 2.8 s✓

(e) 3 s

When, if ever, is the velocity of the particle zero?

Select All Correct Answers:

(a) 1 s

(b) 0.8 s✓

(c) 3 s

(d) 2.7 s

(e) 2.4 s✓

(f) *The particle never has zero velocity.*

If these points exist, does the object change direction each time?

Multiple Choice:

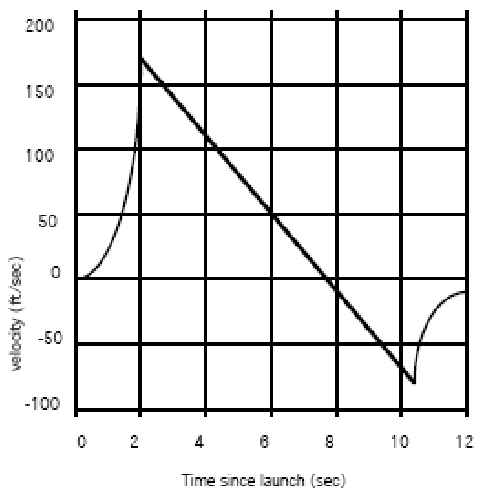
(a) *Yes*✓

(b) *No*

At approximately what time is the particle moving the most quickly?

$t = \square s$

Question 3 *Model rockets work through burning a propellant to completion, coasting on momentum for some time, and finally releasing a parachute when the rocket begins to fall in order to prevent the rocket and its components from being destroyed upon landing. Examine the following graph of one such rocket's motion:*



What was the maximum velocity obtained by the rocket?

$v = \square ft/s$

When did the rocket reach its highest point?

$t = \square s$

What was the velocity at that time?

$v = \boxed{0} ft/s$

When did the rocket's parachute deploy?

$t = \square s$

How fast was the rocket descending by that time?

$v = -\square ft/s$

Describe how long each phase of the rocket lasted.

Take-off $\square s$

Coasting $\square s$

Free-fall $\square s$

Parachute fall $\square s$

Question 4 *At the surface of the Earth, acceleration due to gravity is approximately* $9.8\text{m}\backslash s^2$. *Consider throwing a ball directly upward from atop a 160 meter building at* $35\text{m}\backslash s$.

Create an equation to express the acceleration of the ball at any time after it has been thrown.

$a(t) = \boxed{-9.8}$

Feedback (attempt): *Remember, up is the positive direction.*

Hint: *Don't forget to find the constant of integration - might it have something to do with the starting speed?*

Integrate the previously constructed equation to produce the equation for velocity at any time for the ball.

$v(t) = \boxed{-9.8t + 35}$

Hint: *Don't forget the constant here either - what is the other factor influencing height here?*

Now, integrate your new equation yet again to produce the equation for the position of the ball at any time.

$p(t) = \boxed{\frac{-9.8}{2}t^2 + 35t + 160}$

Question 5 *Consider a balloon which has been caught in a jet stream high above the ground. The horizontal position of the balloon at any time can be given by the equation*

$$p(t) = 3t^5 - 15t^3 + 13.$$

Hint: *Remember, acceleration is the derivative of velocity, and velocity is the derivative of position.*

Produce the velocity and acceleration equations for the balloon.

$v(t) = \boxed{15t^4 - 45t^2}$

$a(t) = \boxed{60t^3 - 90t}$

Hint: *Remember to check where the velocity is zero. Then you can check between these points for the sign of the velocity!*

Over what time period is the balloon moving in the positive direction?

$\boxed{\sqrt{3}} \leq t$

When is the velocity increasing?

$\boxed{\sqrt{\frac{3}{2}}} \leq t$

Hint: *Don't over complicate this - we only need to concern ourselves with position here!*

What was the displacement of the balloon over the interval [0, 2.25]? Displacement is distance from the initial position. Please answer to two decimal places.

$\boxed{2.13}\, m$

Question 6 ***Hint:*** *A great way to find the initial velocity is to find at what time the velocity became zero. This has to do with the 3.8 seconds!*

On a spring break trip with friends, you find yourself dared to stand upon George Washington's nose on Mount Rushmore. While on the dangerous climb down, you come up with an experiment, and request one of your friends go to the base of the mountain. When you're on the nose, you take out your phone and wallet, and toss the wallet into the air, starting the timer just as you release the wallet. Simultaneously on the ground, your friend starts a stopwatch on his phone. You stop the timer as the wallet passes you, with your friend stopping their's once the wallet smashes into the ground. Your stopwatch displays 3.8 seconds, and your friend's displays 13.72 seconds.

Determine the acceleration, velocity, and position functions for the wallet. You will need to use equations to determine each constant of integration. Don't worry about units here, and remember that we're only concerned about the vertical position of the wallet.

$p(t) = \boxed{666.902 + 18.62t - 4.9t^2}$ $v(t) = \boxed{18.62 - 9.8t}$ $a(t) = \boxed{-9.8}$

What is the wallet's initial velocity?

$v_0 = \boxed{18.62}\, m/s$

What is it's velocity as it hits the ground?

$v_f = -\boxed{115.836}\, m/s$

How far off the ground is George Washington's nose?

$h = \boxed{666.902}\, m$

In Summary

James: I guess there's more to position than just speed!

Julia: A *lot* more! Do you think you could run through the big points real quick Dylan?

Dylan: Sure Julia! When we derive position, we get velocity, and when we derive velocity, we get acceleration. Anti-differentiation will give us velocity from acceleration and position from velocity.

James: Okay, but how do we get the constant of integration?

Julia: I know this! It's whatever was the initial velocity or position in the problem!

Dylan: That's right Julia! When the initial isn't given, we can use knowledge of when an object returns to a position zero or stops for a moment to determine those constants.

Newton's Methods

Introduction

Dylan: I'm so tired of having to solve roots by hand. It's a real drag.

Julia: Yeah, some of these roots are rough. I wish there was a better way!

James: There's always a better way!

Dylan and Julia: Show us!!!

James: Maybe you've heard of Sir Isaac Newton? He got tired of solving roots too, and made a whole method to approximate them!

Dylan: Wow! I'm just like him except worse in every way!

Newton's Method is a system of approximating roots of polynomials by using tangent lines from an initial estimate. While this method is extremely accurate when used properly, it is possible to have a very inaccurate estimate when used improperly.

Answer all questions to at least three decimal places where possible!

Guided Example

In the following figure we have an initial guess x_0, then we have the blue tangent line with respect to the point x_0

Question 1 *What is the slope, in general, for the tangent line of $y = f(x)$ at x_0?*

Multiple Choice:

(a) $f'(x_0)$ ✓

Learning outcomes:
Author(s): The College of Wooster

(b) $f'(x)$

(c) $f(x)$

(d) $f(f(x))$

What is the equation of the tangent line for the point $(x_0, f(x_0)$?

Multiple Choice:

(a) $y = f(x) \cdot x + f(x_0)$

(b) $y = f'(x_0) \cdot (x - x_0) + f(x_0)$ ✓

(c) $y = f'(x) \cdot b + x_0$

(d) $y = f'(x) \cdot b + f(x)$

How would you use the tangent line you found above to estimate the value of the root?

Multiple Choice:

(a) *Solve for $x = 0$ to find a point near the actual root.*

(b) *Solve for $y = 0$ to find a point near the actual root.* ✓

(c) *Take the derivative of the tangent line to find where the second derivative is zero.*

(d) *Evaluate the tangent line at a y-value which was the initial estimate.*

On Your Own

Question 2 *Consider the function $f(x) = x^2 - 1$.*

Graph of $x^2 - 1$

Find the equation of the tangent line at an initial estimate of $x_0 = 3$.

$y = \boxed{6x - 10}$

Plot the tangent line and function on the same axes. Does the x-intercept of the tangent line seem more or less accurate than your initial estimate?

Multiple Choice:

(a) *More Accurate* ✓

(b) *Less Accurate*

What is the x-intercept of the tangent line?

$\boxed{\frac{5}{3}}$

SAGE

```
```

Continue this process until the x-intercepts change by less than .0001 on each iteration. How many iterations did this take?

$\boxed{5}$

Question 3 *Consider the function* $g(x) = x^3 - 4x^2 - 1$.

Graph of $x^3 - 4x^2 - 1$

Why does $g(x) = 0$ *have only one solution? Explain with the help of a graph.*

Free Response:

Fortunately, with the use of Sage Cells, we don't have to do all of this by hand! The following Sage Cell contains the commands through which we can perform Newton's Method, along with comments explaining just what it is that Sage is doing.

SAGE

```
x = var('x')#This line declares the variable we will be using
f(x)=cos(x)+5*x-5#Here we define the function whose root we're estimating
df=diff(f,x)#We then find the derivative of the function
NewtonsMethod(x)=x-(f/df)(x)#The formula for Newton's Method, which we are calling "NewtonsM
x0=1# initial guess
print x0#displays the initial guess in the output
x1=N(NewtonsMethod(x0))#first iteration of the method
print x1#displays first result
x2=N(NewtonsMethod(x1))#second iteration of Newtons Method
print x2#displays second result
```

The only way to run Newton's Method like this will be to take the first few lines of code and put them in each cell after this - the method is **not** defined as a function in Sage.

Question 4 *Using $g(x) = x^3 - 4x^2 - 1$ from before, use an initial guess of 2. After 5 iterations, what result do you get?*

SAGE

$\boxed{-0.395459530414480}$

Why is it important to use caution with your first guess when doing Newton's Method by hand? It might help you to understand why if you plot both the function and the tangent lines from Newton's method in the following Desmos graph.

Graph of

Free Response:

Question 5 *Consider the function $h(x) = 4x^3 - 12x^2 + 2x + 1$.*

Graph of $4x^3 - 12x^2 + 2x + 1$

SAGE

Use an initial guess of $x = 3$ to estimate a root of $h(x)$. What did you get after the fifth iteration?

$\boxed{2.78854472256823}$

Look at the graph, and attempt to estimate another root using $x = 0$. Did you find the root to the right or the left of this point?

Multiple Choice:

(a) *Left* ✓

(b) *Right*

Increment the x guess by 0.02 and use Newton's method until you find the root on the right. What value of x is the first to work?

SAGE

```
```

$\boxed{0.1}$

In Summary

Julia: Wow! Newton's Method is awesome!

Dylan: Yeah, it's way more accurate than just guessing! If you're too far off on that initial guess though...

James: Things can go downhill quickly. While Newton's Method can be handy, it's important to remember how important an accurate initial estimate is!

Dylan and Julia: Thanks James!

Riemann Sums

```
SAGE
caseInsensitive = function(a,b) {
   return a.toLowerCase() == b.toLowerCase();
 };
```

Dylan: Hey Julia, can you help me with this problem?

Julia: Yeah, of course! What do you need?

Dylan: I'm supposed to approximate area under a curve, and I don't really see what to do.

Julia: Actually, that's pretty easy! We'll just use Riemann sums.

Introduction

Riemann sums are a method of approximating area under a curve. In this lab, we will look at three varieties; left, right, and midpoint.

Please answer to at least three decimal places throughout this lab!

Obtained from mathforum.org

To create Riemann sums, you simply pick a number of desired subintervals, and then evenly divide the interval to produce the desired number. From here, we choose the height of what will be our rectangles differently for each version:

- **Left Riemann Sum:** The height is calculated using the left endpoint of the subinterval.
- **Right Riemann Sum:** The height is calculated using the right endpoint of the subinterval.
- **Midpoint Riemann Sum:** The height is calculated using the midpoint of the subinterval.

Learning outcomes:
Author(s): The College of Wooster

From here, we simply add the area of each rectangle to produce the area under the curve.

If we are approximating area with n rectangles, then

$$\begin{aligned}\text{Area} &\approx \sum_{k=1}^{n} (\text{height of } k\text{th rectangle}) \times (\text{width of } k\text{th rectangle}) \\ &= \sum_{k=1}^{n} f(x_k^*)\Delta x \\ &= f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \cdots + f(x_n^*)\Delta x.\end{aligned}$$

Definition 6. *A sum of the form:*

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a ***Riemann sum****.*

Increasing, Concave Up

Consider the function x^2 on the interval [1, 6].

Graph of x^2

Question 1 *The following qustion shows how to compute the Riemann sums using Sage cells, you will need to mimic this process in further questions so be sure to read the Sage code and understand how it represents the equation for Riemann sums.*

The left Riemann sum.

SAGE

```
f(x) = x^2
a=1.0
b=6.0
n=6
Dx=(b-a)/n
rsum=sum([f(a+i*Dx)*Dx for i in range(n)])
print rsum
```

What did you get for the left Reimann sum from evaluating the above Sage cell?
$\boxed{57.6620370370370}$

The right Riemann sum.

SAGE

```
f(x) = x^2
a=1.0
b=6.0
n=6
Dx=(b-a)/n
rsum=sum([f(a+(i+1)*Dx)*Dx for i in range(n)])
print rsum
```

What did you get for the right Reimann sum from evaluating the above Sage cell? $\boxed{86.8287037037037}$

Feedback (attempt): *Notice the change in the second to last line between the code for left and right Riemann sums.*

The midpoint Riemann sum.

SAGE

```
f(x) = x^2
a=1.0
b=6.0
n=6
Dx=(b-a)/n
rsum=sum([f(a+(i+0.5)*Dx)*Dx for i in range(n)])
print rsum
```

What did you get for the midpoint Reimann sum from evaluating the above Sage cell? $\boxed{71.3773148148148}$

Compute the integral numerically by evaluating the following Sage cell. Which of these estimates was most accurate? Show the percent error for each type of Riemann sum, rounding to two decimal places.

Remark 3. *Recall that percent error is calculated with the following formula:*

$$\left|\frac{Calculated - Actual}{Actual}\right| \cdot 100$$

SAGE

```
integral(x^2,x,0,6)
```

Most Accurate:

Multiple Choice:

(a) *Left Riemann Sum*

(b) *Right Riemann Sum*

(c) *Midpoint Riemann Sum* ✓

Left Percent Error: $\boxed{19.91}$%

Middle Percent Error: $\boxed{0.86}$%

Right Percent Error: $\boxed{20.60}$%

Decreasing, Concave Up

Question 2 *Consider the function x^2 on the interval [-7, 0].*

Graph of x^2

Calculate the following Riemann sums by modifying the code provided in the first question to have seven subintervals and reflect the new endpoints.

The left Riemann sum.

SAGE

```
1
```

$\boxed{140}$

The right Riemann sum.

SAGE

```
1
```

$\boxed{91}$

The midpoint Riemann sum.

SAGE

```
1
```

$\boxed{113.75}$

Compute the integral numerically. Which of these estimates was most accurate? Show the percent error for each type of Riemann sum, rounding to two decimal places.

SAGE

1

Most Accurate:

Multiple Choice:

(a) *Left Riemann Sum*

(b) *Right Riemann Sum*

(c) *Midpoint Riemann Sum* ✓

Left Percent Error: $\boxed{22.45}$%

Middle Percent Error: $\boxed{0.51}$%

Right Percent Error: $\boxed{20.41}$%

Increasing, Concave Down

Question 3 ***Hint:*** *Add .n() right after rsum to get a clean numerical approximation.*

Consider the function $\sin(x)$ *on the interval* $[0, \frac{\pi}{2}]$.

Graph of $sin(x)$

Calculate the following Riemann sums by modifying the code provided in the first question to have four subintervals and reflect the new endpoints and new function.

The left Riemann sum.

SAGE

1

$\boxed{0.790766260123413}$

The right Riemann sum.

SAGE

1

1.18346534182214

The midpoint Riemann sum.

SAGE

1

1.00645454279956

Compute the integral numerically. Which of these estimates was most accurate? Show the percent error for each type of Riemann sum, rounding to two decimal places.

SAGE

1

Most Accurate:

Multiple Choice:

(a) *Left Riemann Sum*

(b) *Right Riemann Sum*

(c) *Midpoint Riemann Sum* ✓

Left Percent Error: 20.92 %

Middle Percent Error: 0.65 %

Right Percent Error: 18.35 %

Decreasing, Concave Down

Question 4 *Consider the function* $\cos(x)$ *on the interval* $[0, \frac{\pi}{2}]$.

Graph of $\cos(x)$

Calculate the following Riemann sums by modifying the code provided in the first question to have four subintervals and reflect the new endpoints and new function.

The left Riemann sum.

SAGE

$\boxed{1.18346534182214}$

The right Riemann sum.

SAGE

$\boxed{0.790766260123413}$

The midpoint Riemann sum.

SAGE

$\boxed{1.00645454279956}$

Compute the integral numerically. Which of these estimates was most accurate? Show the percent error for each type of Riemann sum, rounding to two decimal places.

SAGE

Most Accurate:

Multiple Choice:

(a) *Left Riemann Sum*

(b) *Right Riemann Sum*

(c) *Midpoint Riemann Sum* ✓

Left Percent Error: $\boxed{18.35}$%

Middle Percent Error: $\boxed{0.65}$%

Right Percent Error: $\boxed{20.92}$%

Dylan: That's cool! Thanks Julia!

Julia: No problem!

Dylan: I wish I could make the sums more accurate though... some of them are pretty far off.

James: I think if you put your mind to it you could Dylan!

Julia and Dylan: James! You're late to class!

James: Haha no problem, I love helpi... that's not the point! Listen, just use sum notation, and try to make infinitely many subintervals. If you can do that, you'll have an accurate area.

Question 5 ***Hint:*** *Think of the start of the interval as a and the end as b.*

How would you represent the width of each rectangle when divided into n subintervals?

Multiple Choice:

(a) $\dfrac{b+a}{2}$

(b) $\dfrac{b-a}{2}$

(c) $\dfrac{b-a}{n}$ ✓

(d) $\dfrac{n}{b-a}$

Hint: *As n grows larger, will the rectangle be wider than a single point? What does that tell you about the height?*

How would you represent the height of each rectangle when divided into n subintervals?

Multiple Choice:

(a) $f(x_i)$ ✓

(b) $f(x_b) - f(x_a)$

(c) $\dfrac{f(x_b) - f(x_a)}{2}$

(d) $\displaystyle\sum_{i=a}^{b} f(x_i)$

How could you use sigma notation to represent the area under the curve from $i = 1$ to n, as n approaches infinity?

Multiple Choice:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n}$ ✓

(b) $\lim_{n\to 0} \sum_{i=1}^{n}$

(c) $\lim_{n\to\infty} \sum_{i=1}^{\infty}$

(d) $\sum_{i=1}^{n} \lim_{n\to\infty}$

Julia: Wow, that's just as accurate as asking our computers!

James: That's right Julia! You just found the integral of a function, with just a little guidance!

Julia and Dylan: Wow! Thanks James!

In Summary

We've learned a lot about Riemann sums today, and even the formula for a definite integral! So let's recap:

Definition 7. *A **Riemann sum** comes in three types, all of which first divide an interval into a number of subintervals:*

(a) ***Left endpoint Riemann sums*** *use the left endpoint of the subinterval to approximate the area.*

(b) ***Right endpoint Riemann sums*** *use the right endpoint of the subinterval to approximate the area.*

(c) ***Midpoint Riemann sums*** *use the midpoint of the subinterval to approximate the area.*

Following this, the area of each rectangle is added to approximate the area under the curve.

- The formula for the definite integral is

$$\int_a^b f(x) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x) \cdot \frac{b-a}{n}$$

where a and b are the endpoints of the interval.