Implicit Differentiation - Finish solutions

Dylan: Woah! What's up with this?

Julia: I didn't know functions were explicit!

Dylan: The x and y are on the same side of the equation! I can't deal with this.

James: Functions can be explicit or implicit! And it not the way you're thinking Julia...

Introduction

So far we have dealt only with explicitly defined functions, where $y = f(x)$. Functions where there are both x and y on one side or both sides of the equation are called **implicit functions**.

Guided Example

Question 1 *Which of the following equations are defined implicitly?*

Select All Correct Answers:

(a)
$$y = x^2 + 5x - 7$$

(b)
$$y = \sin(x)$$

(c)
$$x^2 + y^2 = 1$$

✓

(d)
$$y = \sqrt{(}x - 3)$$

Learning outcomes:

(e) $x^2y^3 + y = 5x + 8y$ ✓

Graph the following implicitly defined function below,

$$x^2 + y^2 = 1$$

Graph of

Now, in the following sage cell, solve the function for y. For help using the solve command refer to the documentation here.

SAGE

```
x,y = var("x, y")
#eqn = x**2+y**2==1, this sets eqn to the unit circle
#use the solve command to solve eqn for y
```

Graph the two explicit equations on the same axis below.

Graph of

Question 2 *Which of the following are true?*

Select All Correct Answers:

(a) $x^2 + y^2 = 1$ *is a function*

(b) $-\sqrt{1-x^2}$ *is a function* ✓

(c) $\sqrt{1-x^2}$ *is a function* ✓

Question 3 *Using the functions you found, differentiate to find the slope of the tangent lines at the point* $(\frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2})$*. You may do this in the above sage cell or by hand.* $\boxed{-1}$

Unfortunately not all implicit equations can be easily solved for y, which is why we use implicit differentiation!

Explanation. *Starting with*

$$x^2 + y^2 = 1$$

we first differentiate each term using $\frac{d}{dx}$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$$

You can already fill in 2 of the terms

$$\boxed{2x} + \frac{d}{dx}y^2 = \boxed{0}$$

For the term $\frac{d}{dx}y^2$ *you can imagine* $y = f(x)$, *and hence by the chain rule*

$$\frac{d}{dx}y^2 = \frac{d}{dx}(f(x))^2$$

$$= 2 \cdot f(x) \cdot f'(x)$$

$$= 2y\frac{dy}{dx}$$

Putting this together we have

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ *we get*

$$\boxed{\frac{-x}{y}}$$

On Your Own

Consider the equation $y^4 + xy = x^3 - x + 2$.

Question 4 *Using the method shown in the previous section, evaluate the function for* y.

SAGE
```
x,y = var("x, y")
```

Does this equation look easy to differentiate?

$\boxed{No}$

Instead, let's treat our equation as an expression writing it instead as $y^4 + xy - x^3 + x - 2 = 0$ *Now consider* y *as* $y(x)$, *a function of* x, *and differentiate with respect to* x. *Each* y *term will gain* $\frac{dy}{dx}$. *Then, set the expression equal to zero, and solve for* $\frac{dy}{dx}$. *What does this represent?*

Free Response:

Question 5 *Using your result in the previous section, evaluate* $\frac{dy}{dx}$ *at* $x = 3$ *and* $x = 7$.

$x = 3 : \square$

$x = 7 : \square$

SAGE

```
var("x, y")

```

Question 6

Now use Sage Math to find the slope of $\sin(x^2) = \cos(xy^2)$ *at any point.* *Look here for information on implicit differentiation in Sage* $\square$

Perpendicular at a Point

Julia: Wow, implicit differentiation is rough.

Dylan: You're telling me... I've been doing this for hours! I wish we could at least do a little more with it if I have to learn it.

James: Did I hear that you guys want to know more about using implicit differentiation?

Julia and Dylan: James! Tell us more!

James: Alright guys, you can use implicit differentiation with implicit functions to tell if two functions are perpendicular at a point!

Julia: But how?

Dylan: Yeah, I don't see how that helps.

James: It's easy - all we have to do is see if the tangent lines are perpendicular at that point, and if they are, then so are the curves!

Question 7 *Graph* $3x - 2y + x^3 - x^2y = 0$ *and* $x^2 - 2x + y^2 - 3y = 0$ *on the same set of axes.*

Graph of

Do they look perpendicular anywhere? $\boxed{Yes}$

Question 8 ***Hint:*** *Remember, two perpendicular lines will have their slopes be the negative inverse of one another.*

Prove the two curves are (or are not) perpendicular at the origin.

Free Response:

In Summary

There are two main methods to solve implicit equations

(a) Solve for y and then differentiate.

(b) Treat y as $y(x)$ and differentiate for x, eventually solving for $\frac{dy}{dx}$ to give the value of the derivative at any point.