Implicit Differentiation

```
SAGE
caseInsensitive = function(a,b) {
   return a.toLowerCase() == b.toLowerCase();
 };
```

Dylan: Woah! What's up with this?

Julia: I didn't know functions were explicit!

Dylan: The x and y are on the same side of the equation! I can't deal with this.

James: Functions can be explicit or implicit! And it not the way you're thinking Julia...

Introduction

So far we have dealt only with explicitly defined functions, where $y = f(x)$. Here y is dependent variable and it is given in terms of the independent variable x. Functions given in terms of both independent and dependent variables are called *implicit* functions.

Guided Example

Question 1 *Which of the following equations defined y as a function of x implicitly?*

Select All Correct Answers:

(a)
$$y = x^2 + 5x - 7$$

(b)
$$y = \sin(x)$$

Learning outcomes:

(c)
$$x^2 + y^2 = 1$$
✓

(d)
$$y = \sqrt{(x-3)}$$

(e) $x^2y^3 + y = 5x + 8y$ ✓

Graph the curve defined by this equation:

$$x^2 + y^2 = 1$$

Graph of

Now, in the following Sage cell, solve for y. For help using the solve command refer to the documentation here.

SAGE

```
x,y = var("x, y")
#eqn = x^2+y^2==1, this sets eqn to the unit circle
#use the solve command to solve eqn for y
```

Graph the two explicit equations on the same axis below.

Graph of

Question 2 *Which of the following are true?*

Select All Correct Answers:

(a) $x^2 + y^2 = 1$ *is a function of* x.

(b) $y = -\sqrt{1-x^2}$ *is a function of* x. ✓

(c) $y = \sqrt{1-x^2}$ *is a function of* x. ✓

SAGE

Question 3

Using the functions you found, differentiate to find the slope of the tangent lines at the point $\left(\left(\frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}\right)\right)$. *You may do this in the above Sage cell or by hand.*

$\boxed{-1}$

Unfortunately not all implicit equations can be easily solved for y, which is why we use implicit differentiation!

Explanation. *Starting with*

$$x^2 + y^2 = 1$$

we first differentiate each term using $\frac{d}{dx}$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$$

You can already fill in 2 of the terms

$$\boxed{2x} + \frac{d}{dx}y^2 = \boxed{0}$$

For the term $\frac{d}{dx}y^2$ *you can imagine* $y = f(x)$, *and hence by the chain rule*

$$\begin{aligned}\frac{d}{dx}y^2 &= \frac{d}{dx}(f(x))^2 \\ &= 2 \cdot f(x) \cdot f'(x) \\ &= 2y\frac{dy}{dx}\end{aligned}$$

Thus we have the following:

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ *we get*

$$\boxed{\frac{-x}{y}}$$

Question 4 *Use the equation obtained from the above explanation to find* $\frac{dy}{dx}$ at $\left(\left(\frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}\right)\right)$ $\boxed{-1}$

Feedback (attempt): *Using both methods you can obtain the same answer, but for many equations the first method is much more work!*

Question 5 *We can fairly easily use Sage to do this process for us, to illustrate the process evaluate the following Sage cell.*

SAGE

```
var('x,y')
y(x)=function('y')(x)
eq=x^2+y^2==1
eq.substitute(y =y(x))
diff(eq,x)
solve(diff(eq,x),diff(y(x)))
```

What did you get as output from your Sage cell? (copy just the answer portion after the "==" and before the "]") $\boxed{-x/y(x)}$

Feedback (attempt): *Notice that Sage uses $y(x)$ for y in the output.*

On Your Own

In each of the following problems, evaluate the derivative by hand, and use the Sage cells as a check.

Consider the equation $y^4 + xy = x^3 - x + 2$. Using the following Sage cell implicitly differentiate to find $\frac{dy}{dx}$ using the same commands as shown in the previous question.

SAGE

```
var('x,y')
```

Question 6 *What did you get as output from your Sage cell? (copy just the answer portion after the "==" and before the "]")*

$\boxed{(3 * x^2 - y(x) - 1)/(4 * y(x)^3 + x)}$

Question 7 *Using your result in the previous section, evaluate* $\dfrac{dy}{dx}$ *at the point* $(1,1)$. $\boxed{1/5}$

Question 8 *Now use Sage Math again to find* $\dfrac{dy}{dx}$ *for* $\sin(x^2) = \cos(xy^2)$, *copy your answer in the same way as indicated in the previous section.*

SAGE

```
var('x,y')
```

$\boxed{-1/2*(sin(x*y(x)^2)*y(x)^2+2*x*cos(x^2))/(x*sin(x*y(x)^2)*y(x))}$

Perpendicular at a Point

Julia: Wow, implicit differentiation is rough.

Dylan: You're telling me... I've been doing this for hours! I wish we could at least do a little more with it if I have to learn it.

James: Did I hear that you guys want to know more about using implicit differentiation?

Julia and Dylan: James! Tell us more!

James: Alright guys, you can use implicit differentiation with implicit functions to tell if two functions are perpendicular at a point!

Julia: But how?

Dylan: Yeah, I don't see how that helps.

James: It's easy - all we have to do is see if the tangent lines are perpendicular at that point, and if they are, then so are the curves!

Question 9 *Graph* $3x - 2y + x^3 - x^2y = 0$ *and* $x^2 - 2x + y^2 - 3y = 0$ *on the same set of axes.*

Graph of

Do they look perpendicular anywhere?

Multiple Choice:

(a) *Yes* ✓

(b) *No*

Hint: *To show two lines are perpendicular you must show that the slope of one is the opposite inverse of the other*

Show the two curves are (or are not) perpendicular at the origin. You can do this in the Sage cell provided or by hand.

SAGE

```
```

Slope of $3x - 2y + x^3 - x^2y = 0$ *at the origin:*

$\boxed{3/2}$

Slope of $x^2 - 2x + y^2 - 3y = 0$ *at the origin:*

$\boxed{-2/3}$

Are the lines perpendicular at the origin?

In Summary

There are two main methods to differentiate implicit equations

(a) Solve for y and then differentiate.

(b) Treat y as $y(x)$ and differentiate with respect to the variable x, eventually solving for $\frac{dy}{dx}$ to give the value of the derivative at any point (x, y).