Computing effective degrees of freedom ($N_{ef}^{\star}$) as a function of frequency

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Bretherton et al. (1998) gives

$$N_{ef}^{\star} = \frac{\left(\sum \lambda_i\right)^2}{\left(\sum \lambda_i^2\right)} = \frac{\mathrm{tr}\left(\mathbf{C}\right)^2}{\mathrm{tr}\left(\mathbf{C}^2\right)}$$

where λ_i are the eigenvalues of the covariance matrix

$$\mathbf{C} = \frac{1}{n-1}\mathbf{X}\mathbf{X}^{\top}$$

where n is the number of times and $\mathbf{X}$ is the data matrix. The λ_i are also the squared singular values of the matrix of model output $\mathbf{X}$, so that

$$\mathbf{X} = \mathbf{U}\Lambda^{\frac{1}{2}}\mathbf{V}^{\top}$$

where $\mathbf{\Lambda}$ is populated along its diagonal with the λ_i. Our goal is to compute a function

$$N_{ef}^{\star}\left(\nu\right)$$

that is a function of frequency, i.e. what are the degrees of freedom as a function of time scale?

The component of $\mathbf{X}$ that varies at a single frequency ν, $\mathbf{X}_{\nu}$, can be isolated by considering only the components of the principal components (the right singular vectors in the columns of $\mathbf{V}$) at that frequency,

$$\mathbf{X}_{\nu} = \mathbf{U}\Lambda^{\frac{1}{2}}\mathbf{V}_{\nu}^{\top}.$$

Our goal is to compute $N_{ef}^{\star}\left(\nu\right)$ using the trace form of the definition. The trace of a covariance matrix is equal to the sum of squared diagonal elements of any diagonalizing basis. Here we choose the EOF basis for the full $\mathbf{X}$; we will project $\mathbf{X}$ at varying frequencies onto this basis and compute the squared weights to get the trace. The squared projection of $\mathbf{X}_{\nu}$ is then

$$\lambda_{\nu\, i} = \lambda_i \left|\hat{\mathbf{v}}_i\left(\nu\right)\right|^2$$

where the latter term ($\left|\hat{\mathbf{v}}_i\left(\nu\right)\right|^2$) can be obtained via a power spectral density estimate. Note that the eigenvectors of $\mathbf{C}^2$ are the same as for $\mathbf{C}$ and the eigenvalues are the square. Thus we arrive at

$$N_{ef}^{\star}\left(\nu\right)=\frac{\left(\sum\lambda_i\left|\hat{\mathbf{v}}_i\left(\nu\right)\right|^2\right)^2}{\sum\lambda_i^2\left|\hat{\mathbf{v}}_i\left(\nu\right)\right|^4}.$$

For a simple example, consider the case where the field is 2x1 with a global mode with frequency .1 and a top-only mode with frequency .01...