Linear Programming in Combinatorics

Amritanshu Prasad

The Institute of Mathematical Sciences, Chennai.

Homi Bhabha National Institute, Mumbai.
Email address: amri@imsc.res.in

CHAPTER 1

Introduction to Linear Programming

1. Feasibility and Optimization

A *linear program in equational form* consists of a set of variables $\mathbf{x} = (x_1, \ldots, x_n)$, an $m \times n$ matrix A with real entries, a bound vector $\mathbf{b} = (b_1, \ldots, b_m)$, and an objective vector $\mathbf{c} = (c_1, \ldots, c_n)$. The *linear program* is the problem:

$$\text{(LP)} \qquad \text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } \mathbf{x} \geq 0 \text{ and } A\mathbf{x} = \mathbf{b}.$$

The set

$$P(A, \mathbf{b}) = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{x} \geq 0,\ A\mathbf{x} = \mathbf{b}\}$$

is called the polytope of all *feasible solutions* to (LP). The function $\mathbf{x} \mapsto \mathbf{c}^T\mathbf{x}$ is called the *objective function.* An *optimal solution* is a vector $\mathbf{x}_0 \in P(A, \mathbf{b})$ such that $\mathbf{c}^T\mathbf{x} \leq \mathbf{c}^T\mathbf{x}_0$ for every $\mathbf{x} \in P(A, \mathbf{b})$. Sometimes we will only be interested in the set $P(A, \mathbf{b})$ of feasible solutions, which does not depend on the objective vector.

Assume that $\mathbf{b}$ lies in the column space of A (for otherwise, we would have $P(A, \mathbf{b}) = \emptyset$), and that the rows of A are linearly independent (if not, we could discard redundant rows to achieve this). For a subset $B \subset [n]$, let A_B denote the submatrix of A consisting of columns from B. We say that B is a *basic set* if B has m elements and A_B has rank m. For $\mathbf{x} \in \mathbf{R}^n$ define:

$$\operatorname{supp}(\mathbf{x}) = \{1 \leq j \leq n \mid x_j \neq 0\}.$$

Definition 1.1 (Basic feasible solution). A *basic feasible solution* to (LP) is a feasible solution $\mathbf{x} \in P(A, \mathbf{b})$ such that $\operatorname{supp}(\mathbf{x})$ is contained in a basic set.

Clearly, a feasible solution $\mathbf{x}$ is basic if and only if the submatrix $A_{\operatorname{supp}(\mathbf{x})}$ has linearly independent columns.

Example 1.2 (The Birkhoff polytope). Take $n = d^2$, indexing the d^2 variables as $\mathbf{x} = (x_{ij})_{1 \leq i,j \leq d}$, a square array of size d. As constraints, say that the row sums and column sums of $\mathbf{x}$ are all equal to 1, i.e.,

$$\text{(R)} \qquad \sum_j x_{ij} = 1 \text{ for } j = 1, \ldots, d,$$

$$\text{(C)} \qquad \sum_i x_{ij} = 1 \text{ for } j = 1, \ldots, d.$$

These $2d$ constraints are not independent–the sum of the row sum constraints (R) is equal to the sum of the column sum constraints (C); it is the constraint that all entries of the matrix add up to d. Removing, for instance, the last column constraint gives a $(2d - 1) \times d^2$ matrix A of rank $2d - 1$.

When $d = 2$, the constraints can be expressed in matrix form as

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{1}$$

Every submatrix of A with three columns is non-singular. Thus there are four possible basic sets. It is easy to see, however, that there are only two basic solutions, given by the permutation matrices:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

EXERCISE 1.3. Let B be a basic subset of $[d] \times [d]$. By definition, B has cadrinality $2d-1$. Given B, define a bipartite graph Γ_B on the set $\{1, 2, 3, 1', 2', 3'\}$ by joining i to j' if $(i, j) \in B$. Show that Γ_B is a spanning tree for the complete bipartite graph $K_{3,3}$. Show that this construction gives rise to a bijection from the set of basic subsets of $[d] \times [d]$ onto the set of spanning trees of $K_{n,n}$.

Let $\sigma \in S_d$ and suppose $\mathbf{x}$ is the permutation matrix $x_{ij} = \delta_{i\sigma(i)}$. Then $\operatorname{supp}(\mathbf{x}) = \{(i, \sigma_i) \mid i \in [d]\}$. The first d rows of the corresponding column vectors of A are just the coordinate vectors of $\mathbf{R}^d$. Therefore each permutation matrix is a basic feasible solution. Which spanning trees does it correspond to?

LEMMA 1.4. *For each basic subset $B \subset [d]$, there exists at most one basic feasible solution $\mathbf{x}$ with $\operatorname{supp}(\mathbf{x}) \subset B$.*

In the situation described in the preceding lemma, we say that $\mathbf{x}$ *is the basic solution corresponding to B.*

PROOF. Let $\mathbf{x}_B$ denote the vector $(x_i)_{i \in B}$. The matrix A_B is non-singular, so the equation $A_B\mathbf{x}_B = \mathbf{b}$ has at most one solution. Solutions $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ with $\operatorname{supp}(\mathbf{x}) \subset B$ are in bijection with solutions of $A_B\mathbf{x}_B = \mathbf{b}$ (set $x_j = 0$ for $j \notin B$ to get $\mathbf{x}$ from $\mathbf{x}_B$). Therefore $A\mathbf{x} = \mathbf{b}$ also has at most one solution. $\square$

REMARK 1.5. The same basic feasible solution could be obtained from different basic sets. For example, each basic solution for (1) corresponds to two basic sets. Also not every basic set B admits a basic feasible solution. For example, in Example 1.2, $B = \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1)\}$ is a basic set with no feasible solution.

THEOREM 1.6 (Existence of basic optimal solutions). *For a linear program in equational form:*

$$\textit{maximize } \mathbf{c}^t\mathbf{x} \textit{ subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0$$

if there is at least one feasible solution, and the objective function is bounded above on $P(A, \mathbf{b})$, then there exists at least one optimal solution. Among the optimal solutions there is at least one basic solution.

PROOF. We claim that, for any feasible solution $\mathbf{x}_0$, there exists a basic feasible solution $\mathbf{x}$ with $\mathbf{c}^T\mathbf{x} \geq \mathbf{c}^T\mathbf{x}_0$. This implies that an optimal solution, if it exists, will be basic. Suppose $\mathbf{x}$ is a feasible solution. Among all feasible solutions $\mathbf{x}$ with $\mathbf{c}^T\mathbf{x} \geq \mathbf{c}^T\mathbf{x}_0$ choose one with support of minimal cardinality and call it $\tilde{\mathbf{x}}$. We will show that $A_{\operatorname{supp}(\tilde{\mathbf{x}})}$ has linearly independent columns, so that $\tilde{\mathbf{x}}$ is basic.

Suppose this is not the case. Then there exists a vector $\mathbf{y} \in \mathbf{R}^n$ with $\operatorname{supp}(\mathbf{y}) \subset \operatorname{supp}(\mathbf{x})$ such that $A\mathbf{y} = 0$. Replacing $\mathbf{y}$ by $-\mathbf{y}$ if necessary, assume that $\mathbf{c}^T\mathbf{y} \geq 0$.

We claim that we may further assume that $\mathbf{y}$ has at least one *negative* coordinate. Suppose that all the coordinates of $\mathbf{y}$ are non-negative. If $\mathbf{c}^T\mathbf{y} = 0$, then we can replace $\mathbf{y}$ with $-\mathbf{y}$. If $\mathbf{c}^T\mathbf{y} > 0$ and all coordinates of $\mathbf{y}$ are positive, then $\tilde{\mathbf{x}} + t\mathbf{y}$ is a feasible solution for all $t > 0$. The objective function $\mathbf{c}^T(\tilde{\mathbf{x}} + t\mathbf{y})$ grows unboundedly as t grows, contradicting its boundedness.

Thus $\mathbf{y}$ has at least one negative coordinate, hence it is possible to choose a value $t > 0$ such that $\tilde{\mathbf{x}} + t\mathbf{y}$ is a feasible solution with $\operatorname{supp}(\tilde{\mathbf{x}} + t\mathbf{y}) \subsetneq \operatorname{supp}(\tilde{\mathbf{x}})$. Since $\mathbf{c}^T\mathbf{y} \geq 0$, we have $\mathbf{c}^T(\tilde{\mathbf{x}} + t\mathbf{y}) \geq \mathbf{c}^T\tilde{\mathbf{x}}$ and $\operatorname{supp}(\tilde{\mathbf{x}} + t\mathbf{y})$. This contradicts the minimality condition on the cardinality of $\operatorname{supp}(\tilde{\mathbf{x}})$.

The set of basic feasible solutions is finite. The element of this set that maximizes the objective function must therefore be an optimal solution. □

Definition 1.7 (Vertex). Let $P \subset \mathbf{R}^n$ be convex closed set. An element $\mathbf{v} \in P$ is said to be a *vertex* of P if there exists $\mathbf{c} \in \mathbf{R}^n$ such that $\mathbf{c}^T\mathbf{x}$ attains its maximum *uniquely* at $\mathbf{v}$.

Theorem 1.6 says that every vertex of $P(A, \mathbf{b})$ is a basic feasible solution. The converse is also true:

Theorem 1.8. *The basic feasible solutions to* (LP) *are precisely the vertices of* $P(A, \mathbf{b})$.

Proof. Let $B \subset [n]$ be a basic subset $\mathbf{v}$ be the basic feasible solution to (LP) with respect to B. Define $\mathbf{c}$ to be the vector with $c_j = 0$ for $j \in B$, and $c_j = -1$ (or any strictly negative number) otherwise. Then $\mathbf{c}^T\mathbf{v} = 0$, and by Lemma 1.4, $\mathbf{c}^T\mathbf{x} < 0$ for every $\mathbf{x} \in P(A, \mathbf{b})$ different from $\mathbf{x}_0$. □

Definition 1.9 (General form of a linear program). A more general form of a linear program involves linear inequalities and equalities. As before take A to be an $m \times n$ matrix with real entries, $\mathbf{b} \in \mathbf{R}^m$, and $\mathbf{c} \in \mathbf{R}^n$. A general linear program has the form:

$$\text{(GLP)} \qquad \text{optimize } \mathbf{c}^T\mathbf{x} \text{ subject to } a_{i1}x_1 + \cdots + a_{in}x_n \; R_i \; b_i \text{ for } i = 1, \ldots, m,$$

where R_i is one of the three symbols $\leq$, $\geq$, and $=$, and the word optimize is replaced by either maximize, or minimize. A basic feasible solution is one that is defined by equalities in n linearly independent constraints (which could be equality or inequality to begin with).

Example 1.10 (Standard equational form of the simplex). Consider the linear program in n variables with just one linear equation:

$$\mathbf{x} \geq 0;\ x_1 + \cdots + x_n = 1.$$

The matrix A in this case has a single row, and rank one. The polytope $P(A, 1)$ is called the standard $(n-1)$-simplex. Every singleton subset of $[n]$ is basic. The basic solution corresponding to $B = \{j\}$ is the ith coordinate vector e_j. Given an objective vector $\mathbf{c} \in \mathbf{R}^n$, the optimal solution is e_j where j is any of the indices for which c_j is maximal among the coordinates of $\mathbf{c}$.

Example 1.11. The cube can be defined by the inequalities:

$$0 \leq x_i \leq 1, \text{ for } i = 1, 2, 3.$$

The inequality $x_i \leq 1$ can be turned into an equality by introducing *slack variables* y_i, and writing:

$$x_i \geq 0,\ y_i \geq 0,\ x_i + y_i \leq 1 \text{ for } i = 1, 2, 3.$$

The linear program in equational form is equivalent to the original, more general one, in the sense that there is a bijection amongst their feasible solutions that maps vertices to vertices (why?). What are the basic subsets? What are the basic feasible solutions?

EXERCISE 1.12 (The simplex in terms of inequalities). The n-simplex can also be expressed in terms of inequalities:

$$\Delta_n = \{(x_1, \dots, x_n) \mid 0 \leq x_1 \leq x_2 \leq \cdots x_n \leq 1\}.$$

Rewrite this in equational form. Determine the basic sets and basic solutions.

EXERCISE 1.13. Express the hyperoctahedron:

$$H_n = \{\mathbf{x} \mid -1 \leq x_1 + \cdots + x_n \leq 1\}$$

in equational form.

2. Simplex Tableaus

The simplex method begins with a basic set B for which there exists a basic feasible solution. Let $\bar{B} = [n] - B$, the complement of B in $[n]$. Using the relations imposed by $A\mathbf{x} = \mathbf{b}$, each of the basic variables x_j, $j \in B$, can be expressed in terms of the non-basic variables x_j, $j \in \bar{B}$. Using this, the objective function can also be expressed in terms of the non-basic variables.

To do this explicitly, note that the system of equations $A\mathbf{x} = \mathbf{b}$ can be rearranged as:

$$A_B \mathbf{x}_B = \mathbf{b} - A_{\bar{B}} \mathbf{x}_{\bar{B}}. \tag{2}$$

Here $\mathbf{x}_B \in \mathbf{R}^m$ and $\mathbf{x}_{\bar{B}} \in \mathbf{R}^{m-n}$ are vectors whose coordinates are those coordinates of $\mathbf{x}$ whose indices lie in B and $\bar{B}$ respectively. Since A_B is invertible, the basic variables can be expressed in terms of the non-basic ones:

$$\mathbf{x}_B = A_B^{-1}(\mathbf{b} - A_{\bar{B}} \mathbf{x}_{\bar{B}}).$$

Indeed, the basic feasible solution is computed by setting $\mathbf{x}_{\bar{B}} = 0$ in the above equation. If $A_B^{-1}\mathbf{b} \geq 0$, then it is the basic feasible solution corresponding to B. Othewise there is no basic feasible solution corresponding to B. Since each basic variable is expressed in terms of the non-basic variables in (2), the objective function can be expressed in terms of the non-basic variables only:

$$\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_{\bar{B}}^T \mathbf{x}_{\bar{B}} = \mathbf{c}_B^T A_B^{-1}(\mathbf{b} - A_{\bar{B}} \mathbf{x}_{\bar{B}}) + \mathbf{c}_{\bar{B}}^T \mathbf{x}_{\bar{B}}.$$

Thus (LP) is represented in terms of a *simplex tableau*:

$$\begin{array}{ccc} \mathbf{x}_B & = & \mathbf{d} - D\mathbf{x}_{\bar{B}} \\ \hline \mathbf{c}^T\mathbf{x} & = & e - \mathbf{e}^T\mathbf{x}_{\bar{B}}, \end{array} \tag{T}$$

the equations above the line being the constraints, and the objective function $\mathbf{c}^T\mathbf{x}$ below the line to be maximized. Here:

$$\begin{aligned}\mathbf{d} &= A_B^{-1}\mathbf{b}\\ D &= A_B^{-1}A_{\bar{B}}\\ e &= \mathbf{c}_B^T\mathbf{d}\\ \mathbf{e}^T &= \mathbf{c}_B^T D - \mathbf{c}_{\bar{B}}^T.\end{aligned}$$

The information contained in (T) is equivalent to the information in (LP). But (T) gives a parametrization of $P(A,b)$ in terms of a subset of $\mathbf{R}^{n-m}$.

THEOREM 2.1. *For every basic subset $B \subset [n]$ in the linear program* (LP),

$$P(A,\mathbf{b}) = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{x}_B = A_B^{-1}(\mathbf{b} - A_{\bar{B}}\mathbf{x}_{\bar{B}}) \geq 0,\ \mathbf{x}_{\bar{B}} \geq 0\}. \tag{$*$}$$

In other words, (T) *gives a parametrization of $P(A,\mathbf{b})$ in terms of the polytope:*

$$P_B(A,\mathbf{b}) = \{\mathbf{x}_{\bar{B}} \in \mathbf{R}^{n-m}_{\geq 0} \mid A_B^{-1}(\mathbf{b} - A_{\bar{B}}\mathbf{x}_{\bar{B}}) \geq 0\} \subset \mathbf{R}^{n-m}.$$

PROOF. The conditions $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_B = A_B^{-1}(\mathbf{b} - A_{\bar{\mathbf{x}}}x_{\bar{B}})$ are equivalent. □

EXAMPLE 2.2. Consider the Birkhoff polytope (Example 1.2) for $d = 3$. For convenience we abbreviate the variable indices (i,j) to ij. The matrix A has columns indexed by pairs in $[d] \times [d]$ written in increasing lexicographic order:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Fix as objective function $x_{11} + x_{22} + x_{33}$. A basic subset is $B = \{11, 12, 13, 22, 31\}$ with basic solution given by $x_{13} = x_{22} = x_{31} = 1$, and all other coordinates zero. The corresponding simplex tableau is:

$$\begin{array}{ccc} x_{11} & = & -x_{21} + x_{32} + x_{33}\\ x_{12} & = & x_{21} + x_{23} - x_{32}\\ x_{13} & = & 1 - x_{23} - x_{33}\\ x_{22} & = & 1 - x_{21} - x_{23}\\ x_{31} & = & 1 - x_{32} - x_{33}\\ \hline \mathbf{c}^T\mathbf{x} & = & 1 - 2x_{21} - x_{23} + x_{32} + 2x_{33}. \end{array}$$

In the above example, the objective function can be increased by increasing x_{32} or x_{33}, the variables with positive coefficients in the last row of the tableau. However, this increase should respect the constraints that all the variables are non-negative. The condition $x_{12} \geq 0$ (using the second equation, and leaving the values of x_{21} and x_{23} unchanged at 0) gives $x_{32} \leq 0$. Therefore, it is not feasible to increase x_{32}. However, it is feasible to increase x_{33}. The conditions $x_{13} \geq 0$ and $x_{31} \geq 0$ give $x_{33} \leq 1$. So we set $x_{33} = 1$, and recalculate all the basic variables, getting $x_{11} = x_{22} = x_{33} = 1$, and all other variables 0. We move x_{33} to the set of basic variables, and move x_{13} to the set of non-basic (which has now become 0), and use the equation:

$$x_{33} = 1 - x_{13} - x_{23}.$$

Using this we get a new tableau:

$$\begin{array}{rcl} x_{11} & = & 1 - x_{21} + x_{32} + x_{23} - x_{13} \\ x_{12} & = & x_{21} + x_{23} - x_{32} \\ x_{22} & = & 1 - x_{21} - x_{23} \\ x_{31} & = & x_{13} + x_{23} - x_{32} - x_{33} \\ x_{33} & = & 1 - x_{23} - x_{13} \\ \hline \mathbf{c}^T\mathbf{x} & = & 3 - 2x_{13} - 2x_{21} - 3x_{23} + x_{32}. \end{array}$$

All the non-basic variables have negative coefficients, except x_{32}. However, the constraint $x_{12} \geq 0$ still does not allow us to increase x_{32} without changing any other non-basic variable. This suggests that we may have arrived at a maximum value for the objective function. Indeed, $x_{12} = x_{21} + x_{23} - x_{32} \geq 0$ implies that $x_{32} \leq x_{21} + x_{23}$ for every $\mathbf{x} \in P(A, \mathbf{b})$, whence

$$\mathbf{c}^T\mathbf{x} = 3 - 2x_{13} - 2x_{21} - 3x_{23} + x_{32} \leq 3 - 2x_{13} - x_{21} - 2x_{23} \leq 3.$$

Therefore 3 is indeed a global maximum for the objective function, and is obtained uniquely at $x_{ij} = \delta_{ij}$.

An alternative approach would be to induct x_{32} into the set of basic variables, and remove x_{12}. Now the basic set is changed to $\{11, 22, 31, 32, 33\}$, but the basic feasible solution remains unchanged. This will result in the tableau:

$$\begin{array}{rcl} x_{11} & = & 1 - x_{12} + x_{32} + 2x_{23} - x_{13} \\ x_{22} & = & 1 - x_{21} - x_{23} \\ x_{31} & = & x_{12} + x_{13} - x_{21} - x_{33} \\ x_{32} & = & x_{21} + x_{23} - x_{12} \\ x_{33} & = & 1 - x_{23} - x_{13} \\ \hline \mathbf{c}^T\mathbf{x} & = & 3 - 2x_{13} - x_{21} - 2x_{23} - x_{12}. \end{array}$$

In this case, all the coefficients of the objective function, when expressed in terms of non-basic variables, are negative. Theorem 2.1 then implies that $x_{ij} = \delta_{ij}$ is the unique global maximum.

More generally Theorem 2.1 gives:

THEOREM 2.3. *Let B be a basic set for* (LP). *Suppose that the objective function, when expressed in terms of the non-basic variables as in the last line of the tableau* (T) *has all coefficients negative. Then the basic feasible solution $\mathbf{x}_0$ corresponding to B is a solution to* (LP). *If all the non-basic variables occur with strictly negative coefficients, then $\mathbf{x}_0$ is the unique solution to* (LP).

3. Pivot Rules

Given a basic subset B for the linear program (LP), consider the corresponding tableau (T). Assume also that B admits a basic feasible solution $\mathbf{x}$. Suppose that $e_k > 0$ for some $k \in \bar{B}$. For each i in B, we have:

$$x_i = d_i - \sum_{j \in \bar{B}} D_{ij} x_j. \tag{3}$$

The constraint $x_i \geq 0$ can be rewritten as:

$$D_{ik} x_k \leq d_i - \sum_{j \in \bar{B},\ j \neq k} D_{ij} x_j.$$

Since B admits a basic feasible solution where all the basic variables are zero, we have $d_i \geq 0$. If $D_{ik} \leq 0$, then x_k can be increased without bound without violating (3). If, on the other hand, $D_{ik} > 0$, then x_k can be increased up to d_i/D_{ik}. Therefore, we can increase x_k up to $m = \min\{d_i/D_{ik} \mid i \in B\}$. In general $m \geq 0$, and it is quite possible (as we have seen in Example 2.2) that $m = 0$. Let l be any index for which $m = d_l/D_{il}$. Let $B' = B \cup \{k\} - \{l\}$.

We have an expression for x_k in terms of the variables $\{x_j \mid j \in \bar{B}'\}$:

$$x_k = D_{lk}^{-1}(d_l - x_l - \sum_{j \in \bar{B}\ j \neq k} D_{lj} x_j). \tag{4}$$

This can be substituted into the equations (3) for all $i \in B - \{l\}$ to write x_i in terms of the variables $\{x_j \mid j \in \bar{B}'\}$. Thus each variable x_i, for $i \in B'$ can be expressed in terms of the variables $\{x_j \mid j \in \bar{B}'\}$. Also, the substitution (4) can be used to express the objective function in terms of the variables $\{x_j \mid j \in \bar{B}'\}$. Let $\mathbf{x}'$ be the vector obtained from $\mathbf{x}$ by changing x_k to m, and x_l to 0.

LEMMA 3.1. *If B is a basic subset with basic feasible vector $\mathbf{x}$ and B' and $\mathbf{x}'$ are obtained from B and $\mathbf{x}$ as described above, then B' is also a basic subset, and $\mathbf{x}'$ is a basic feasible solution corresponding to B'.*

PROOF. Since B is a basic subset, the analysis at the beginning of Section 2 shows that the equations:

$$A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x}_B = \mathbf{d} - D\bar{\mathbf{x}}_{\bar{B}}$$

are equivalent. Our choices of k and l above ensure that these equations are further equivalent to

$$\mathbf{x}_{B'} = \mathbf{d}' - D'\bar{\mathbf{x}}_{\bar{B}'},$$

for some vector $\mathbf{d}'$ and some matrix D'. Setting $\mathbf{x}_{\bar{B}'} = 0$ says that there is a unique value for $\mathbf{x}_{B'}$ such that, if $\mathbf{x}'' \in \mathbf{R}^n$ is obtained from $\mathbf{x}_{B'}$ by setting the coordinates in $\bar{B}'$ to 0, then $A\mathbf{x}'' = \mathbf{b}$. In other words, $A_{B'}\mathbf{x}_{B'} = \mathbf{b}$ has a unique solution. It follows that $A_{B'}$ is non-singular, and so B' basic. By Lemma 1.4, the vector $\mathbf{x}''$ constructed in this proof coincides with the vector $\mathbf{x}'$ constructed earlier, so the rest of the lemma also follows. $\square$