AN INTRODUCTION TO SCHUR FUNCTIONS

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1. **Symmetric Functions.** We consider polynomials in n variables $x_1, \ldots, x_n$. Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, x^α denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ A symmetric polynomial in n variables $x_1, \ldots, x_n$ is a polynomial of the form

$$f(x_1, \ldots, x_n) = \sum_\alpha c_\alpha x^\alpha,$$

where, for any permutation $w \in S_n$,

$$c_{(\alpha_1, \ldots, \alpha_n)} = c_{(\alpha_{w(1)}, \ldots, \alpha_{w(n)})}.$$

We call the integer partition λ obtained by sorting the coordinates of α the shape of α and write $\lambda = \lambda(\alpha)$. The most obvious example of a symmetric polynomial in n variables is the *monomial symmetric function*, defined for each integer partition λ:

$$m_\lambda = \sum_{\lambda(\alpha)=\lambda} c_\alpha x^\alpha.$$

Note that m_λ is homogeneous of degree $|\lambda|$ (the sum of the parts of λ).

Exercise 1.1. *Take $n = 4$. Compute the monomial symmetric function $m_{(3)}$, $m_{(2,1)}$, and $m_{(1^3)}$.*

Theorem 1.2. *The polynomials $m_\lambda(x_1, \ldots, x_n)$, as λ runs over all the integer partition of d, form a basis for the space of homogeneous symmetric polynomials of degree d in n variables.*

2. **Complete and Elementary Symmetric Polynomials.** Recall that the coefficients of a polynomial are symmetric polynomials in its roots:

$$(t - x_1)(t - x_2) \cdots (t - x_n) = t^n - e_1(x_1, \ldots, x_n)t^{n-1} + \cdots + (-1)^n e_n(x_1, \ldots, x_n), \tag{1}$$

where coefficient $e_i(x_1, \ldots, x_n)$ of t^{n-i} is given by:

$$e_i(x_1, \ldots, x_n) = \sum_{1 \le j_1 < \cdots < j_i \le n} x_{j_1} x_{j_2} \cdots x_{j_i}. \tag{2}$$

The polynomial e_i is called the ith *elementary symmetric polynomial.* By convention, write $e_i(x_1, \dots, x_n) = 0$, for $i > n$.

The identity (1) can be written more elegantly as:

$$(1 + tx_1) \cdots (1 + tx_n) = \sum_{i=0}^{n} e_i(x_1, \cdots, x_n)t^i.$$

Dually[1], the *complete symmetric polynomials* are defined by the formal identity:

$$\frac{1}{(1 - x_1 t) \cdots (1 - x_n t)} = \sum_{i=0}^{\infty} h_i(x_1, \cdots, x_n)t^i.$$

Example 2.1. *In three variables, we have:*

$$\begin{aligned} e_2(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + x_2x_3, \\ h_2(x_1, x_2, x_3) &= x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_2^3. \end{aligned}$$

Exercise 2.2. *Show that*

$$h_i(x_1, \cdots x_n) = \sum_{1 \leq j_1 \leq \cdots \leq j_i \leq n} x_{j_1} \cdots x_{j_i}.$$

More generally, for any integer partition $\lambda = (\lambda_1, \dots, \lambda_l)$, define:

$$\begin{aligned} h_\lambda &= h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}, \\ e_\lambda &= e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}. \end{aligned}$$

Theorem 2.3. *Given integer partitions $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \cdots, \mu_m)$ of and integer d, let $M_{\lambda\mu}$ denote the number of integer matrices (a_{ij}) with non-negative entries whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j. Then*

$$h_\lambda = \sum_\mu M_{\lambda\mu} m_\mu.$$

Dually, let $= N_{\lambda\mu}$ denote the number of integer matrices (a_{ij}) with entries 0 or 1, whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j.

$$e_\lambda = \sum_\mu N_{\lambda\mu} m_\mu.$$

[1]We will refer to the replacing of $(1 + u)$ by $(1 - u)^{-1}$ in a formal identity as *dualization.*

Proof. We first prove the second identity involving elementary symmetric functions. A monomial in the expansion:

$$e_\lambda = \prod_{i=1}^{l} \sum_{j_1 < \cdots < j_{\lambda_j}} x_{j_1} \cdots x_{j_{\lambda_i}}$$

is a product of summands, one chosen from each of the l factors. Construct an $l \times m$ matrix (a_{ij}) corresponding to such a choice as follows: if the summand $x_{j_1} \cdots x_{j_{\lambda_i}}$ is chosen from the ith factor, then set the entries $a_{i,j_1}, \ldots, a_{i,j_{\lambda_j}}$ to be 1 (the remaining entries of the ith row are 0). Clearly the ith row of such a matrix sums to λ_i. The monomial corresponding to this choice is x^μ if, for each j, the the number of i for which x_j appears in $a_{i,j_1}, \ldots, a_{i,j_{\lambda_j}}$, which is the sum of the jth column of the matrix (a_{ij}). It follows that the coefficient of x^μ, and hence the coefficient of m_μ in the expansion of e_λ in the basis of monomial symmetric functions of degree n, is $N_{\lambda\mu}$.

A similar proof can be given for the first identity involving complete symmetric functions. The only difference is that a variables are repeated in the monomials that appear in h_i. Counting the number of repetitions (instead of just recording 0 or 1) gives non-negative integer matrices. □

3. **Alternating Polynomials.** An *alternating polynomial* in $x_1, \ldots, x_n$ is of the form:

$$f(x_1, \ldots, x_n) = \sum_\alpha c_\alpha x_\alpha, \tag{3}$$

where, $c_{w(\alpha)} = \epsilon(w) c_\alpha$ for every multiindex α as in Section 1. Here $\epsilon : S_n \to \{\pm 1\}$ denotes the sign function. Equivalently, an alternating polynomial is one whose sign is reversed upon the interchange of any two variables.

Exercise 3.1. *If α is a multiindex where $\alpha_i = \alpha_j$ for some $i \neq j$, then $c_\alpha = 0$.*

In particular, every monomial in an alternating polynomial must be composed of distinct powers. Moreover, the polynomial is completely determined by the coefficients c_α of strictly decreasing multiindices, namely, multiindices of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 > \cdots > \alpha_n$.

Exercise 3.2. *Let δ denote the strictly increasing multiindex $(n-1, n-2, \ldots, 1, 0)$ of lowest degree. Given an integer partition with at most n parts, we will pad it with 0's so that it can be regarded as a weakly decreasing multiindex. Then $\lambda \mapsto \lambda + \delta$ is a bijection from the set* *of integer partitions with at most n onto the set of strictly decreasing multiindices.*

Example 3.3. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a weakly decreasing multiindex. The polynomial:*

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j+n-j})$$

is alternating, with unique strictly decreasing monomial $x^{\lambda+\delta}$.

Exercise 3.4. *The alternating polynomial of the form (3) is equal to*

$$\sum_\lambda c_\lambda a_{\lambda+\delta},$$

the sum being over all weakly decreasing multiindices.

4. **Cauchy's bialternant form of a Schur function.** The simplest polynomial of the form $a_{\lambda+\delta}$ arises when $\lambda = 0$; a_δ is the Vandermonde determinant:

$$a_\delta = \prod_{1\leq i<j\leq n} (x_i - x_j).$$

Exercise 4.1. *Show that, for every weakly decreasing multiindex λ, $a_{\lambda+\delta}$ is divisible by a_δ in the ring of polynomials in $x_1, \dots, x_n$.*

Exercise 4.2. *Show that $f \mapsto fa_\delta$ is an isomorphism of the space of symmetric polynomials in $x_1, \dots, x_n$ of degree d onto the space of alternating polynomials of degree $d + \binom{n}{2}$.*

The above exercise allows us to give the historically oldest definition of Schur functions—*Cauchy's bialternant formula*:

$$s_\lambda(x_1, \dots, x_n) = a_{\lambda+\delta}/a_\delta, \tag{4}$$

for any partition λ with at most n parts. If λ has more than n parts, set $s_\lambda(x_1, \dots, x_n) = 0$. This is clearly a symmetric function of degree $|\lambda|$. When λ has more than n parts, we shall write $s_\lambda(x_1, \dots, x_n) = 0$

Theorem 4.3. *As λ runs over all integer partitions of d with at most n parts, the Schur functions $s_\lambda(x_1, \dots, x_n)$ form a basis of the space of all homogeneous symmetric functions in $x_1, \dots, x_n$ of degree d.*

Proof. This follows from Exercises 3.4 and 4.2. □

Exercise 4.4. *Show that substituting $x_n = 0$ in the Schur function $s_\lambda(x_1, \dots, x_n)$ with n variables gives the corresponding Schur function $s_\lambda(x_1, \dots, x_{n-1})$ with $n - 1$ variables.*

Exercise 4.5. *From Cauchy's bialternant form (4), deduce that*

(1) $s_{(i)} = h_i$, *and*
(2) $s_{(1^i)} = e_i$.