# AN INTRODUCTION TO SCHUR FUNCTIONS

AMRITANSHU PRASAD

1. **Symmetric Functions.** Consider polynomials in $n$ variables $x_1, \ldots, x_n$. Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, let $x^\alpha$ denote the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. A *symmetric polynomial* is a polynomial of the form

$$f(x_1, \ldots, x_n) = \sum_\alpha c_\alpha x^\alpha,$$

where, for any permutation $w \in S_n$,

$$c_{(\alpha_1,\ldots,\alpha_n)} = c_{(\alpha_{w(1)},\ldots,\alpha_{w(n)})}.$$

The integer partition $\lambda$ obtained by sorting the coordinates of $\alpha$ is called the *shape* of $\alpha$, denoted $\lambda(\alpha)$. The most obvious example of a symmetric polynomial in $n$ variables is the *monomial symmetric function*, defined for each integer partition $\lambda$:

$$m_\lambda = \sum_{\lambda(\alpha)=\lambda} c_\alpha x^\alpha.$$

Note that $m_\lambda$ is homogeneous of degree $|\lambda|$ (the sum of the parts of $\lambda$).

**Exercise 1.1.** *Take $n = 4$. Compute the monomial symmetric functions $m_{(3)}$, $m_{(2,1)}$, and $m_{(1^3)}$.*

**Theorem 1.2.** *The polynomials $m_\lambda(x_1, \ldots, x_n)$, as $\lambda$ runs over all the integer partition of $d$, form a basis for the space of homogeneous symmetric polynomials of degree $d$ in $n$ variables.*

2. **Complete and Elementary Symmetric Polynomials.** Recall that the coefficients of a polynomial are symmetric polynomials in its roots:

$$\begin{aligned}(1)\quad & (t - x_1)(t - x_2) \cdots (t - x_n) \\ & \qquad = t^n - e_1(x_1, \ldots, x_n)t^{n-1} + \cdots + (-1)^n e_n(x_1, \ldots, x_n),\end{aligned}$$

where coefficient $e_i(x_1, \ldots, x_n)$ of $t^{n-i}$ is given by:

$$e_i(x_1, \ldots, x_n) = \sum_{1 \leq j_1 < \cdots < j_i \leq n} x_{j_1} x_{j_2} \cdots x_{j_i}. \tag{2}$$

The polynomial $e_i$ is called the $i$th *elementary symmetric polynomial*. By convention, $e_i(x_1, \ldots, x_n) = 0$, for $i > n$.

The identity (1) can be written more elegantly as:

$$(1+tx_1)\cdots(1+tx_n) = \sum_{i=0}^{n} e_i(x_1,\cdots,x_n)t^i.$$

Dually[1], the *complete symmetric polynomials* are defined by the formal identity:

$$\frac{1}{(1-x_1t)\cdots(1-x_nt)} = \sum_{i=0}^{\infty} h_i(x_1,\cdots,x_n)t^i.$$

**Example 2.1.** *In three variables:*

$$e_2(x_1,x_2,x_3) = x_1x_2 + x_1x_3 + x_2x_3,$$
$$h_2(x_1,x_2,x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_2^3.$$

**Exercise 2.2.** *Show that*

$$h_i(x_1,\cdots x_n) = \sum_{1\leq j_1\leq\cdots\leq j_i\leq n} x_{j_1}\cdots x_{j_i}.$$

More generally, for any integer partition $\lambda = (\lambda_1,\ldots,\lambda_l)$, define:

$$h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_l},$$
$$e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_l}.$$

**Theorem 2.3.** *Given partitions $\lambda = (\lambda_1,\ldots,\lambda_l)$ and $\mu = (\mu_1,\cdots,\mu_m)$ of $d$, let $M_{\lambda\mu}$ denote the number of matrices $(a_{ij})$ with non-negative integer entries whose $i$th row sums to $\lambda_i$ for each $i$, and whose $j$th column sums to $\mu_j$ for each $j$. Then*

$$h_\lambda = \sum_\mu M_{\lambda\mu}m_\mu.$$

*Dually, let $= N_{\lambda\mu}$ denote the number of integer matrices $(a_{ij})$ with entries $0$ or $1$, whose $i$th row sums to $\lambda_i$ for each $i$, and whose $j$th column sums to $\mu_j$ for each $j$.*

$$e_\lambda = \sum_\mu N_{\lambda\mu}m_\mu.$$

*Proof.* To prove the second identity involving elementary symmetric functions, note that a monomial in the expansion of

$$e_\lambda = \prod_{i=1}^{l}\sum_{j_1<\cdots<j_{\lambda_j}} x_{j_1}\cdots x_{j_{\lambda_i}}$$

is a product of summands, one chosen from each of the $l$ factors. Construct an $l\times m$ matrix $(a_{ij})$ corresponding to such a choice as follows: if the summand $x_{j_1}\cdots x_{j_{\lambda_i}}$ is chosen from the $i$th factor, then set the entries $a_{i,j_1},\ldots,a_{i,j_{\lambda_j}}$ to be 1 (the remaining entries of the $i$th row are 0). Clearly

[1]We will refer to the replacing of $(1+u)$ by $(1-u)^{-1}$ in a formal identity as *dualization*.

the $i$th row of such a matrix sums to $\lambda_i$. The monomial corresponding to this choice is $x^\mu$ if, for each $j$, the the number of $i$ for which $x_j$ appears in the monomial corresponding to the $i$gth row is $\mu_j$. This is just the sum of the $j$th column of the matrix $(a_{ij})$. It follows that the coefficient of $x^\mu$, and hence the coefficient of $m_\mu$ in the expansion of $e_\lambda$ in the basis of monomial symmetric functions of degree $n$, is $N_{\lambda\mu}$.

A similar proof can be given for the first identity involving complete symmetric functions. The only difference is that variables may be repeated in the monomials that appear in $h_i$. Counting the number of repetitions (instead of just recording 0 or 1) gives non-negative integer matrices. $\square$

3. **Alternating Polynomials.** An *alternating polynomial* in $x_1, \dots, x_n$ is of the form:

$$f(x_1, \dots, x_n) = \sum_\alpha c_\alpha x_\alpha, \tag{3}$$

where, $c_{w(\alpha)} = \epsilon(w) c_\alpha$ for every multiindex $\alpha$ as in Section 1. Here $\epsilon : S_n \to \{\pm 1\}$ denotes the sign function. Equivalently, an alternating polynomial is one whose sign is reversed upon the interchange of any two variables.

**Exercise 3.1.** *If $\alpha$ is a multiindex where $\alpha_i = \alpha_j$ for some $i \neq j$, then $c_\alpha = 0$.*

In particular, every monomial in an alternating polynomial must be composed of distinct powers. Moreover, the polynomial is completely determined by the coefficients $c_\alpha$ of strictly decreasing multiindices, namely, multiindices of the form $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_1 > \cdots > \alpha_n$.

**Exercise 3.2.** *Let $\delta = (n-1, n-2, \dots, 1, 0)$. Given an integer partition with at most $n$ parts, we will pad it with 0's so that it can be regarded as a weakly decreasing multiindex of length $n$. Then $\lambda \mapsto \lambda + \delta$ is a bijection from the set of integer partitions with at most $n$ onto the set of strictly decreasing multiindices.*

**Example 3.3.** *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a weakly decreasing multiindex. The polynomial:*

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + n - j})$$

*is alternating, with unique strictly decreasing monomial $x^{\lambda+\delta}$.*

**Exercise 3.4.** *The alternating polynomial of the form* (3) *is equal to*

$$\sum_\lambda c_\lambda a_{\lambda+\delta},$$

*the sum being over all weakly decreasing multiindices.*

4. **Interpretation of Alternants with Labeled Abaci.** A labeled abacus with $n$ beads is a word $w = (w_k; k \geq 0)$ such that the subword of non-zero letters is a permutation of $1, 2, \ldots, n$. The sign $\epsilon(w)$ of the abacus is the sign of this permutation, the support is the set $\mathrm{supp}(w) = \{k \mid w_k > 0\}$, and the weight is defined as:

$$\mathrm{wt}(w) = \prod_k x_{w_k}^k.$$

The shape of the abacus, shape$(w)$ is the unique partition $\lambda$ such that the components of $\lambda + \delta$ form the support of $w$.

**Example 4.1.** *Consider the labeled abacus* $w = 510032046000\cdots$. *Its underlying permutation is* $513246$*, which has sign* $-1$*, so* $\epsilon(w) = -1$. *Also,* $\mathrm{supp}(w) = \{0, 1, 4, 5, 7, 8\}$, $\mathrm{shape}(w) = (3, 3, 2, 2)$ *(indeed,* $(3, 3, 2, 2, 0, 0) + (5, 4, 3, 2, 1, 0) = (8, 7, 5, 4, 1, 0)$*) and* $\mathrm{wt}(w) = x_5^0 x_1^1 x_3^4 x_2^5 x_4^7 x_6^8$. *We visualize the abacus* $w$ *as a condifuration of beads on a single runner, with possible positions of beads numbered* $1, 2, 3, \ldots$. *If* $w_k = i$ *where* $i > 0$*, then a bead labeled* $i$ *is placed in position* $k$ *on the runner. If* $w_k = 0$*, then the position* $k$ *is unoccupied. In the running example the visualization is:*

| *0* | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | *9* |
|---|---|---|---|---|---|---|---|---|---|
| ⑤ | ① | • | • | ③ | ② | • | ④ | ⑥ | • |

*The first row shows the positions* $k = 0, 1, \ldots$ *on the runner and the decond row shows the beads.*

**Theorem 4.2.** *For every partition* $\lambda$ *the alternant in* $n$ *variables,*

$$a_{\lambda+\delta} = (-1)^{\lfloor n/2 \rfloor} \sum_w \epsilon(w)\mathrm{wt}(w),$$

*the sum being over all labeled abaci with* $n$ *beads and shape* $\lambda$.

*Proof.* The theorem follows from the expansion of the determinant. □

5. **Cauchy's Bialternant Form of a Schur Function.** The simplest polynomial of the form $a_{\lambda+\delta}$ arises when $\lambda = 0$; $a_\delta$ is the Vandermonde determinant:

$$a_\delta = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

**Exercise 5.1.** *Show that, for every weakly decreasing multiindex* $\lambda$, $a_{\lambda+\delta}$ *is divisible by* $a_\delta$ *in the ring of polynomials in* $x_1, \ldots, x_n$.

**Exercise 5.2.** *Show that* $f \mapsto f a_\delta$ *is an isomorphism of the space of symmetric polynomials in* $x_1, \ldots, x_n$ *of degree* $d$ *onto the space of alternating polynomials of degree* $d + \binom{n}{2}$.

The above exercise gives the historically oldest definition of Schur functions—*Cauchy's bialternant formula*:

$$s_\lambda(x_1, \ldots, x_n) = a_{\lambda+\delta}/a_\delta, \tag{4}$$

for any partition $\lambda$ with at most $n$ parts. If $\lambda$ has more than $n$ parts, set $s_\lambda(x_1, \ldots, x_n) = 0$. This is clearly a symmetric function of degree $|\lambda|$.

**Theorem 5.3.** *As $\lambda$ runs over all integer partitions of $d$ with at most $n$ parts, the Schur functions $s_\lambda(x_1, \ldots, x_n)$ form a basis of the space of all homogeneous symmetric functions in $x_1, \ldots, x_n$ of degree $d$.*

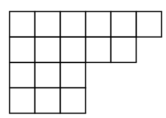
*Proof.* This follows from Exercises 3.4 and 5.2. □

**Exercise 5.4** (Stability of Schur functions)**.** *Show that substituting $x_n = 0$ in the Schur function $s_\lambda(x_1, \ldots, x_n)$ with $n$ variables gives the corresponding Schur function $s_\lambda(x_1, \ldots, x_{n-1})$ with $n-1$ variables.*

6. **Pieri's rule.** The set of integer partitions is endowed with the *containment order.* We say that a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ *contains* a partition $\mu = (\mu_1, \ldots, \mu_m)$ if $l \geq m$, and $\lambda_i \geq \mu_i$ for every $i = 1, \cdots, m$. We write $\lambda \supset \mu$ or $\mu \subset \lambda$. Recall that the Young diagram of the partition $\lambda$ is the set of points

$$\{(i,j) \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}.$$

Visually, each node $(i,j)$ of the Young diagram is replaced by a box, and the box corresponding to $(i,j)$ is placed in the $i$th row and $j$th column (matrix notation). Thus, the Young diagram of $\lambda = (6,5,3,3)$ is depicted by:



Note that containment of partitions is nothing but the containment relation on their Young diagrams. By abuse of notation, we will also use $\lambda$ to denote the Young diagram of $\lambda$.

By a skew-shape, we mean a difference of Young diagrams $\lambda \setminus \mu$, where $\lambda \supset \mu$. We write $\lambda/\mu$ for this skew-shape. A skew-shape is called a *horizontal strip* (respectively, a *vertical strip*) if it has at most one element in each vertical column (respectively, horizontal row).

**Theorem 6.1.** *For every partition $\lambda$, and every positive integer $k$,*

$$s_\lambda h_k = \sum_\mu s_\mu,$$

*where the sum runs over all partitions $\mu \supset \lambda$ such that $\mu/\lambda$ is a horizontal strip of size $k$. Dually,*

$$s_\lambda e_k = \sum_\mu s_\mu,$$

*where the sum runs over all partitions $\mu \supset \lambda$ such that $\mu/\lambda$ is a vertical strip of size $k$.*

*Proof.* We reproduce the proof from Loehr [1]. Let $\mathrm{Abc}(\lambda)$ denote the set of all $n$-bead labeled abaci (see Section 4) of shape $\lambda$. Let $M(n,k)$ denote the set of all vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$ with non-negative integer coordinates and sum $k$. Set $\mathrm{wt}(\alpha) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The first identity is equivalent to showing that:

$$\sum_{w \in \mathrm{Abc}(\lambda)} \epsilon(w) \sum_{\alpha \in M(n,k)} \mathrm{wt}(\alpha) = \sum_{\mu} \sum_{w \in \mathrm{Abc}(\mu)} \epsilon(w)\mathrm{wt}(w),$$

the sum on the right being over all partitions $\mu \supset \lambda$ such that $\mu/\lambda$ is a horizontal strip. We will define an involution $I$ on the $\mathrm{Abc}(\lambda) \times M(n,k)$ whose fixed points correspond to elements of $\coprod_{\mu/\lambda \text{ is a horiz. strip of size } k} \mathrm{Abc}(\mu) \times M(n,k)$ under a bijection that preserves weights and signs, and such that if $I(w,\alpha) = (w',\alpha')$ then $\mathrm{wt}(w)\mathrm{wt}(\alpha) = \mathrm{wt}(w')\mathrm{wt}(\alpha')$ and $\epsilon(w') = -\epsilon(w)$. Then all terms on the left hand side, except for those which do not correspond to fixed points, will cancel, and the surviving terms will give the right hand side.

To construct $I$, scan the abacus from left to right. Upon encountering a bead numbered $j$, move the bead $\alpha_j$ steps to the right, one step at a time. If this process completes without this bead colliding with another bead, $(w,\alpha)$ is a fixed point of $I$. The new abacus $w^*$ has $\epsilon(w^*) = \epsilon(w)$ (the underlying permutation remains unchanged), and $\mathrm{shape}(w^*)/\mathit{shape}(w)$ is a horizontal strip of size $k$.

However, suppose a collision does occur, say the first collision is when bead $j$ hits bead $k$ that is located $p \leq \alpha_j$ position to the right of its initial position. Define $I(w,\alpha) = (w',\alpha')$, where $w'$ is $w$ with the beads $i$ and $j$ interchanged, $\alpha'_j = \alpha_j - p$, $\alpha'_k = \alpha_k + p$ and all other coordinates of $\alpha$ and $\alpha'$ are equal. Clearly $w'$ has the opposite sign from $w$, and $\mathrm{wt}(w)\mathrm{wt}(\alpha) = \mathrm{wt}(w')\mathrm{wt}(\alpha')$. One checks that $I(w',\alpha') = (w,\alpha)$.

**Example 6.2.** *Let $n = 6$, $\lambda = (3,3,2,2,0,0)$, $k = 3$, and*

$$(w,\alpha) = (51003204600\cdots, (2,1,0,0,0,0)).$$

*Reading the abacus from left to right, the first bead encountered is numbered $1$, which can be moved $2$ places to the right without any collisions. After that the bead numbered $2$ can be moved $1$ place to the right, again without collisions. So $(w,\alpha)$ is a fixed point for $I$. The new abacus $50013024600\cdots$ has shape $(3,3,3,2,2,0)$ obtained by adding a horizontal $3$-strip to $(3,3,2,2,0,0)$.*

*On the other hand, if $\alpha = (1,1,1,0,0,0)$, then the first collision is of the bead numbered $3$ with the bead numbered $2$ in the very first step. We have $I(w,\alpha) = (51002304600\cdots, (1,2,0,0,0,0))$.*

Let $N(n,k)$ denote the set of vectors $\alpha = (\alpha_1,\ldots,\alpha_n)$ such that $\alpha_i \in \{0,1\}$ for each $i$, and $\alpha_1 + \cdots + \alpha_n = k$. For elementary symmetric functions, we wish to prove:

$$\sum_{w \in \mathrm{Abc}(\lambda)} \epsilon(w) \sum_{\alpha \in N(n,k)} \mathrm{wt}(\alpha) = \sum_{\mu} \sum_{w \in \mathrm{Abc}(\mu)} \epsilon(w)\mathrm{wt}(w),$$

where $\mu$ runs over all partition such that $\mu/\lambda$ is a vertical strip of size $k$.

We construct an involution $I$ on $\mathrm{Abc}(\lambda) \times N(n,k)$ as follows: scan the abacus from *right to left*. Upon encountering a bead numbered $j$, if $\alpha_j = 1$, try to move the bead one step to the right. If this process completes without collisions, then $(w, \alpha)$ is a fixed point of $I$. Otherwise, if the first collision occurs with bead numbered $j$ colliding with bead numbered $k$, then define $w'$ to be $w$ with beads $j$ and $k$ interchanged. Also, since the $k$th bead was adjacent to the $j$th bead, it could not have been moved in its turn. So $\alpha_k = 0$. Let $\alpha'$ be obtained from $\alpha$ by interchanging $\alpha_k$ and $\alpha_j$. $\square$

**Example 6.3.** *The pair* $(51003204600\cdots, (1,1,1,0,0,0))$ *is a fixed point for* $I$*, and the shifted abacus is* $(50100324600\cdots)$ *of shape* $(3,3,3,3,1,0)$*. On the other hand*

$$I(51003204600\cdots, (0,0,1,0,1,1)) = (51002304600\cdots, (0,1,0,0,1,1)).$$

The following is a special case of Pieri's rule:

**Corollary 6.4.** *For every positive integer* $k$*,*

$$s_{(k)} = h_k, \text{ and } s_{(1^k)} = e_k.$$

## 7. **Schur to Complete and Elementary via Tableaux.**

Pieri's rule allows us to compute the complete and elementary symmetric functions $h_\lambda$ and $e_\lambda$ in terms of Schur functions.

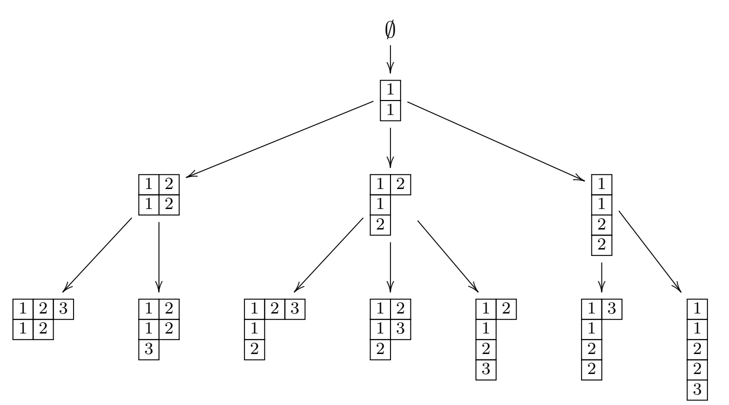
**Example 7.1.** *Repeated application of Pieri's rule gives an expansion of* $e_{(2,2,1)} = e_2e_2e_1$ *as:*

$$\begin{tikzcd}[column sep=small]
 & & & e_2e_2e_1 \arrow[d] & & & \\
 & & & s_{(1^2)}e_2e_1 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & s_{(2^2)}e_1 \arrow[dl] \arrow[d] & & s_{(2,1^2)}e_1 \arrow[dl] \arrow[d] \arrow[dr] & & s_{(1^4)}e_1 \arrow[d] \arrow[dr] & \\
s_{(3,2)} & s_{(2^2,1)} & s_{(3,1^2)} & s_{(2^2,1)} & s_{(2,1^3)} & s_{(2,1^3)} & s_{(1^5)}
\end{tikzcd}$$

*giving:*

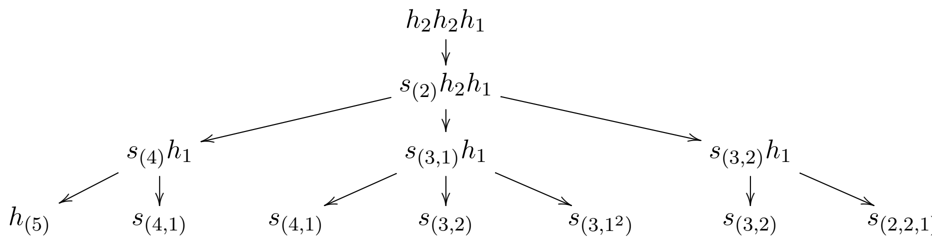
$$e_{(2^2,1)} = s_{(3,2)} + 2s_{(2^2,1)} + s_{(3,1^2)} + 2s_{(2,1^3)} + s_{(1^5)}.$$

*The steps going from the first line of the above calculation to each term of the last line can be recorded by putting numbers into Young diagrams:*



*The boxes in the vertical strip added at the* $i$*th stage are filled with* $i$*.*

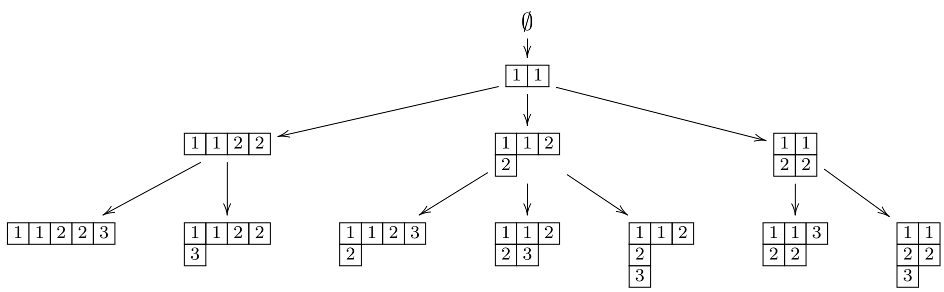
**Example 7.2.** *Repeated application of Pieri's rule gives an expansion of* $h_{(2,2,1)} = h_2h_2h_1$ *as:*



*giving:*

$$h_{(2^2,1)} = s_{(5)} + 2s_{(4,1)} + 2s_{(3,2)} + s_{(3,1^2)} + s_{(2,2,1)}.$$

*The steps going from the first line of the above calculation to each term of the last line can be recorded by putting numbers into Young diagrams:*



*The boxes in the horizontal strip added at the $i$th stage are filled with $i$.*

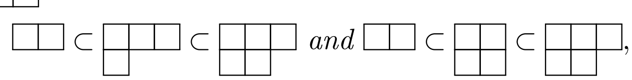
**Definition 7.3** (Semistandard tableau)**.** A semistandard tableau of shape $\lambda = (\lambda_1, \dots, \lambda_l)$ and type $\mu = (\mu_1, \dots, \mu_m)$ is the Young diagram of $\lambda$ filled with numbers $1, \dots, m$ such that the number $i$ appears $\mu_i$ times, the numbers weakly increase along rows, and strictly increase along columns.

**Exercise 7.4.** *Semistandard tableaux of shape $\lambda$ and type $\mu$ correspond to chains of integer partitions*

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(m)} = \lambda$$

*where $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip of size $\mu_i$.*

**Example 7.5.** *The semistandard tableau of type* $(3,2)$ *and type* $(2,2,1)$ *are* $\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2&3\\ \cline{1-2}\end{array}$ *and* $\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&2\\ \cline{1-2}\end{array}$*. They correspond to the chains:*



*respectively. As illustrated in Example 7.2, the coefficient of $s_{(3,2)}$ in the complete symmetric function $h_{(2,2,1)}$ is the number of semistandard tableau of shape $(3,2)$ and type $(2,2,1)$.*

**Definition 7.6** (Kostka number)**.** Given two partitions $\lambda$ and $\mu$, the Kostka number $K_{\lambda\mu}$ is the number of semistandard tableau of shape $\lambda$ and type $\mu$.

**Exercise 7.7.** *For every partition $\lambda$, show that $K_{\lambda\lambda} = 1$.*

**Exercise 7.8** ($f$-number)**.** *The $f$-number of a partition $\lambda$ of $n$ is defined to be the Kostka number $K_{\lambda,(1^n)}$, and is denoted $f_\lambda$.*

**Exercise 7.9.** *For a partition $\lambda$, let $\lambda^-$ denote the set of all partitions whose Young diagram can be obtained by removing one box from the Young diagram of $\lambda$. Show that $f_\lambda = \sum_{\mu\in\lambda^-} f_\mu$.*

**Exercise 7.10.** *A hook is a partition of the form $h(a,b) = (a+1, 1^b)$. Show that $f_{h(a,b)} = \binom{a+b}{a}$.*

In order to understand the expansion of elementary symmetric functions we would need a variant of semistandard tableaux, one where the difference between successive shapes are vertical strips, rather than horizontal strips. However, it has become common practice to *conjugate* partitions instead:

**Definition 7.11** (Conjugate of a partition)**.** The *conjugate* of a partition $\lambda$ is the partition $\lambda'$ whose Young diagram is given by:

$$\lambda' = \{(j,i) \mid (i,j) \in \lambda\}.$$

In other words, the Young diagram of $\lambda'$ is the reflection of the Young diagram of $\lambda$ about its principal diagonal.

Clearly $\lambda \mapsto \lambda'$ is an involution. For example, if $\lambda = (2,2,1)$, then $\lambda' = (3,2)$.

**Exercise 7.12.** *Semistandard tableaux of shape $\lambda'$ and type $\mu$ correspond to chains of integer partitions*

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(m)} = \lambda$$

*where $\lambda^{(i)}/\lambda^{(i-1)}$ is a* vertical *strip of size $\mu_i$.*

**Theorem 7.13.** *The expansion of complete symmetric functions in terms of Schur functions is given by:*

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda.$$

*Dually, the extension of elementary symmetric functions in terms of Schur functions is given by:*

$$e_\mu = \sum_\lambda K_{\lambda'\mu} s_\lambda.$$

8. **Triangularity of Kostka Numbers.** In this section we give a necessary and sufficient condition for the positivity of Kostka number. As usual, take $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_m)$. Suppose $K_{\lambda\mu} > 0$. Then there exists a semistandard tableau $t$ of shape $\lambda$ and type $\mu$. Since the columns of $t$ are strictly increasing, all the 1's in $t$ must occur in its first row, so $\lambda_1 \geq \mu_1$. Also, all the 2's must occur in the first two rows (along with all the 1's), so

$\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. More generally, all the numbers $1, \dots, i$ for $i = 1, \dots, m$ should occur in the first $m$ rows of $t$. We have:

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots \mu_i \text{ for } i = 1, \dots, m. \tag{5}$$

**Definition 8.1.** We say that an integer partition $\lambda = (\lambda_1, \dots, \lambda_l)$ *dominates* the integer partition $\mu = (\mu_1, \dots, \mu_m)$ $|\lambda| = |\mu|$ and if the condition (5) holds. When this happens we write $\lambda \rhd \mu$. This relation defines a partial order on the set of all integer partitions of $n$ for any non-negative integer $n$.

**Exercise 8.2.** *Show that $(n)$ is maximal and $(1^n)$ is minimal among all the integer partitions of $n$. What is the smallest integer $n$ for which the dominance order on partitions of $n$ is not a linear order?*

**Theorem 8.3** (Triangularity of Kostka Numbers)**.** *Given partition $\lambda$ and $\mu$ of an integer $n$, $K_{\lambda\mu} > 0$ if and only if $\lambda \rhd \mu$.*

*Proof.* We have already seen that if $K_{\lambda\mu} > 0$, then $\lambda \rhd \mu$. While reading the proof of the converse, it is helpful to keep in mind Example 8.4 below. Suppose that $\lambda \rhd \mu$. Then $\lambda_1 \geq \mu_1 \geq \mu_m$. Therefore, the Young diagram of $\lambda$ has at least $\mu_m$ cells in its first row, or in other words, it has at least $\mu_m$ columns. Choose the smallest integer $i$ for which $\lambda_i \geq \mu_m$. Fill the bottom-most box in the $\lambda_{i+1}$ leftmost columns with $m$. Also, from the $i$th row, fill the rightmost $\mu_m - \lambda_{i+1}$ boxes with $m$. The remaining (unfilled) boxes in the Young diagram of $\lambda$ now form the Young diagram of the partition

$$\eta = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - \mu_m + \lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_l),$$

a partition with $l - 1$ parts. Writing $(\eta_1, \dots, \eta_{l-1})$ for the parts of $\eta$, note that, since the first $i - 1$ parts of $\eta$ are the same as those of $\lambda$, we have:

$$\eta_1 + \cdots + \eta_j \geq \mu_1 + \cdots + \mu_j$$

for $j \leq i - 1$. For $j \geq i$, we have

$$\begin{aligned} \eta_1 + \cdots + \eta_j &= \lambda_1 + \cdots + \lambda_{j+1} - \mu_m \\ &\geq \mu_1 + \cdots + \mu_j + \mu_{j+1} - \mu_m \\ &\geq \mu_1 + \cdots + \mu_j. \end{aligned}$$

It follows that $\eta \rhd (\mu_1, \dots, \mu_{m-1})$. The result now follows by induction in $m$. $\square$

**Example 8.4.** *Consider the case where $\lambda = (7, 3, 2)$ and $\mu = (4, 4, 4)$. Then the smallest integer $i$ such that $\lambda_i \geq 4$ is 1. Accordingly, we enter 3 into the bottom-most boxes in the three leftmost columns, and also into one rightmost box in the first row:*

$$\begin{ytableau} \ & \ & \ & \ & \ & \ & 3 \\ \ & \ & 3 \\ 3 & 3 \end{ytableau}$$

*We are left with the problem of finding a semistandard tableau of shape $(6, 2)$ and type $(4, 4)$. Recursively applying our process to this smaller problem* *gives:*

$$\begin{array}{ccccccc} \square & \square & \square & \square & 2 & 2 & 3 \\ 2 & 2 & 3 & & & & \\ 3 & 3 & & & & & \end{array},$$

*and finally the desired tableau*

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & & & & \\ 3 & 3 & & & & & \end{array}.$$

**Theorem 8.5.** *The complete symmetric functions:*

$$\{h_\mu \mid \mu \textit{ is a partition of } d \textit{ with at most } n \textit{ parts}\}$$

*and the elementary symmetric functions:*

$$\{e_\mu \mid \mu \textit{ is a partition of } d \textit{ with } \mu_1 \leq n\}$$

*form bases of the space of homogeneous symmetric polynomials of degree $d$ in variables $x_1, \ldots, x_n$.*

*Proof.* In view of the triangularity of Kostka numbers (Theorem 8.3) and the fact that $K_{\lambda\lambda} = 1$ (Exercise 7.7) the theorem follows from Theorem 7.13. □

9. **Schensted's insertion algorithm.** Let $t$ be a semistandard tableau, and $x$ be a positive integer. Schensted's insertion algorithm is a method of inserting a box with the number $x$ into $t$, resulting in a new tableau INSERT$(t, x)$. Applied repeatedly, it gives a way to convert any word into a tableau. This tableau succinctly expresses some combinatorial properties of the original word.

First consider the case where $t$ has a single row, with entries $a_1 \leq \cdots \leq a_k$. Use $\emptyset$ to denote the empty word. The algorithm ROWINS takes as input the single row $t$ and a letter $x$, and returns a pairt $(b, t')$, where $b'$ is either the empty word, or a single letter, and $t'$ is a row:

$$\mathrm{ROWINS}(a_1 a_2 \cdots a_k, x) = \begin{cases} (\emptyset, a_1 \cdots a_k x) & \text{if } x \geq a_k, \\ (a_j, a_1 \cdots a_{j-1} x a_j \cdots a_k) & \text{if } j = \min\{r \mid a_r > x\}. \end{cases}$$

In the second case, one says that $x$ *has been inserted into* $t = a_1 \cdots a_k$*, obtaining* $t' = a_1 \cdots a_{j-1} x a_j \cdots a_k$*, and* ***bumping out*** $a_j$. Also, it is notationally convenient to write ROWINS$(t, \emptyset) = (\emptyset, t)$ (when nothing is inserted, $t$ remains unchanged, and nothing is bumped out).

Now suppose $t$ is a tableau, with rows $r_1, r_2, \ldots, r_l$. The reading word of $t$ is $r_l r_{l-1} \cdots r_1$. Suppose that ROWINS$(r_1, x) = (y, r_1')$. Recursively define:

$$\mathrm{INSERT}(t, x) = \mathrm{INSERT}(r_l \cdots r_2, y) r_1'$$

**Example 9.1.** *Consider the insertion of* 3 *into the tableau:*

$$t = \begin{array}{ccccc} 1 & 3 & 3 & 5 & 8 \\ 2 & 4 & 6 & 6 & \\ 3 & 5 & 8 & & \\ 4 & & & & \end{array}.$$

*We have* ROWINS$(13358,3) = (5,13338)$*;* ROWINS$(2466,5) = (6,2456)$*;* ROWINS$(358,6) = (8,356)$*;* ROWINS$(4,8) = (\emptyset,48)$*. Thus,* INSERT$(t,3)$ *is the tableau:*

$$\begin{array}{ccccc} 1 & 3 & 3 & 3 & 8 \\ 2 & 4 & 5 & 6 & \\ 3 & 5 & 6 & & \\ 4 & 8 & & & \end{array}.$$

*In general, it is not possible to recover $t$ and $x$ from* INSERT$(t,x)$*, even if we know $x$. For example, the above tableau can be obtained by inserting* $3$ *into a different tableau:*

$$\mathrm{INSERT}\left(\begin{array}{ccccc} 1 & 3 & 3 & 6 & 8 \\ 2 & 4 & 5 & & \\ 3 & 5 & 6 & & \\ 4 & 8 & & & \end{array}, 3\right) = \begin{array}{ccccc} 1 & 3 & 3 & 3 & 8 \\ 2 & 4 & 5 & 6 & \\ 3 & 5 & 6 & & \\ 4 & 8 & & & \end{array}.$$

Clearly, the shape of INSERT$(t,x)$ can be obtained by adding one box to the shape of $t$. If we know the row into which the new box was added, and the value of $x$, then $t$ can be recovered from INSERT$(t,x)$. This recovery is based on the fact that ROWINS can be inverted: define

$$\mathrm{ROWDEL}(a, a_1a_2\cdots a_k) = (a_1\cdots a_{j-1}aa_{j+1}\cdots a_k, a_j),$$

where $j = k$ if $a_r \leq a$ for all $r = 1,\ldots,k$ and $j = \min\{r \mid a_{r+1} > a\}$. To recover $t$ and $x$ from $s = \mathrm{INSERT}(t,x)$ and $r$, delete the last entry of the $r$th row of $s$, say $x_r$. Let $u_{r-1}$ denote the $(r-1)$st row of $s$. Suppose ROWDEL$(x_r, u_{r-1}) = (v_{r-1}, x_{r-1})$, replace the $(r-1)$st row of $s$ with $v_{r-1}$. Continue this process until ROWDEL$(x_2, u_1) = (v_1, x_1)$ is obtained and the first row of $s$ is replaced with $v_1$. The tableau obtained at the end of this process is $t$, and $x = x_1$. Write DELETE$(t,r) = (s,x)$. The preceding discussion shows:

**Theorem 9.2.** *If* DELETE$(t,r) = (s,x)$*, then* INSERT$(s,x) = t$*, and* shape$(t)$ *is obtained from* shape$(s)$ *by adding a cell to its $r$th row.*

**Exercise 9.3.** *Verify Theorem 9.2 for the insertions in Example 9.1.*

10. **The Tableau Associated to a Word.** Let $A_n^*$ denote the concatenation monoid of all words in the alphabet $\{1,\ldots,n\}$. For any $w = a_1\cdots a_k \in A_n^*$, Schensted's insertion algorithm allows us to associate a unique semistandard tableau $P(w)$ as follows:

- If $w = a$ has only one letter, then $P(a)$ is the single-cell tableau with entry $a$.
- If $w = ua$, where $u \in A_n^*$ and $a \in \{1,\ldots,n\}$, then $P(w) = \mathrm{INSERT}(P(u), a)$.

**Example 10.1.** *If $w = 1374433254$, then $P(w)$ is the tableau:*

$$\begin{array}{ccccc} 1 & 2 & 3 & 3 & 4 \\ 3 & 4 & 5 & & \\ 4 & & & & \\ 7 & & & & \end{array}$$

11. **Tableau as words.** Given a semistandard tableau $t$, its reading word $w$ is defined to be the sequence of numbers obtained from reading its rows from left to right, starting with the bottom row, and moving up sequentially to the top row. Since the first entry of each row is strictly smaller than the last entry of the row below it, the tableau $t$ can be recovered from $w$ by chopping it up into segments with a cut after each $a_i$ with $a_{i+1} < a_i$ (we say that $w$ has a descent at $i$). The resulting segments, taken from right to left, form the rows of $t$.

**Example 11.1.** *The reading word of the tableau $t$ formed at the end of Example 8.4 is:*

$$w = 332231111223.$$

*The tableau $t$ is recovered by marking off the descents $w = 333|223|1111223$, and then rearranging the segments into a tableau.*

Let $A_n^*$ denote the set of all words in the alphabet $\{1, \ldots, n\}$. We shall call the word $w = a_1 \cdots a_k$ a *row* if $a_1 \leq \cdots \leq a_k$. We call it a *column* if $a_1 > \cdots > a_k$. We write $x^w$ for the monomial $x_{a_1} x_{a_2} \cdots x_{a_k}$. Not every word comes from a tableau; for example the word 132, when broken up at descents gives rise to $\begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$. We shall say that a word is a tableau if it is the reading word of a semistandard tableau.

**Exercise 11.2.** *Show that, for every positive integer $i$,*

$$h_i(x_1, \ldots, x_n) = \sum_{w \,\in\, A_n^* \text{ is a row}} x^w,$$

*and*

$$e_i(x_1, \ldots, x_n) = \sum_{w \,\in\, A_n^* \text{ is a column}} x^w.$$

The set of words $A_n^*$ comes endowed with an additional structure—that of a monoid—which can be lifted to the algebra of polynomials. If $w_1$ and $w_2$ are words, and $w_1 w_2$ is their concatenation, then

$$x^{w_1} x^{w_2} = x^{w_1 w_2}.$$

This gives rise to an algebra homomorphism $\mathbf{Z}[A_n^*] \to \mathbf{Z}[x_1, \ldots, x_n]$. Obviously, this homomorphism has a large kernel; the algebra on the left is the free algebra, whereas the algebra on the right is commutative. The algebra on the right contains our primamry object of interest—the algebra $\mathbf{Z}[x_1, \ldots, x_n]^{S_n}$ of symmetric polynomials. In the next few sections, we shall learn about an equivalence relation "$\equiv$" on $A_n^*$ such that the resulting quotient monoid $\mathrm{Pl}_n := A_n^*/\equiv$ (called the *plactic monoid*) has the property that the subalgebra of $\mathbf{Z}[\mathrm{Pl}_n]$ generated by the elements

$$\mathbf{S}_{(n)} := \sum_{w \,\in\, A_n^* \text{ is a row}} w$$

or the elements

$$\mathbf{S}_{(1^n)} = \sum_{w \in A_n* \text{ is a column}} w$$

is isomorphic to $\mathbf{Z}[x_1, \ldots, x_n]^{S_n}$ under the map $w \mapsto x^w$.

## 12. The Plactic Monoid.

## 13. Tableaux Correspondences.

## 14. Monomials in the Schur function.

**15. The Lindström-Gessel-Viennot Lemma.** Let $R$ be a commutative ring. Let $S$ be any set of points, and $v : S \times S \to R$ be any function (we think of $w$ as a *weight function.* Given $s, t \in S$, a path in $S$ from $s$ to $t$ is is a sequence $\omega = (s = s_0, s_1, \ldots, s_k = t)$ of distinct points in $S$. We denote this by $\omega : s \to t$. The weight of the path $\omega$ is defined to be:

$$v(\omega) = v(s_0, s_1)v(s_1, s_2) \cdots v(s_{k-1}, s_k).$$

**Definition 15.1** (non-crossing paths)**.** Two paths $\omega = (s_0, \ldots, s_k)$ and $\eta = (t_0, \ldots, t_l)$ are said to be non-crossing if $s_i \neq t_j$ for all $0 \leq i \leq k$ and $0 \leq t \leq l$.

Fix points $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ in $S$, and define an $n \times n$ matrix $(a_{ij})$ by:

$$a_{ij} = \sum_{\omega : A_i \to B_j} v(\omega).$$

**Theorem 15.2** (Lindström-Gessel-Viennot Lemma)**.** *The determinant of the matrix $(a_{ij})$ defined above is given by:*

$$\det(a_{ij}) = \sum_{\omega_i : A_i \to B_i} v(\omega_1) \cdots v(\omega_n),$$

*where the sum is over all $n$-tuples $(\omega_1, \ldots, \omega_n)$ of pairwise non-crossing paths $\omega_i : A_i \to B_i$.*

**16. The Jacobi-Trudi Identities.** We have seen that the Kostka numbers can be used to express complete and elementary symmetric functions in terms of Schur functions. The reverse operation—that of expressing Schur functions in terms of complete or elementary symmetric functions—is done by the Jacobi-Trudi identities:

**Theorem 16.1** (Jacobi-Trudi identities)**.** *For every integer partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ form the $l \times l$ matrices with $(i, j)$th entry $h_{\lambda_i - i + j}$ and $e_{\lambda_i - i + j}$ respectively. Then*

$$s_\lambda = \det(h_{\lambda_i - i + j}), \quad s_{\lambda'} = \det(e_{\lambda_i - i + j}).$$

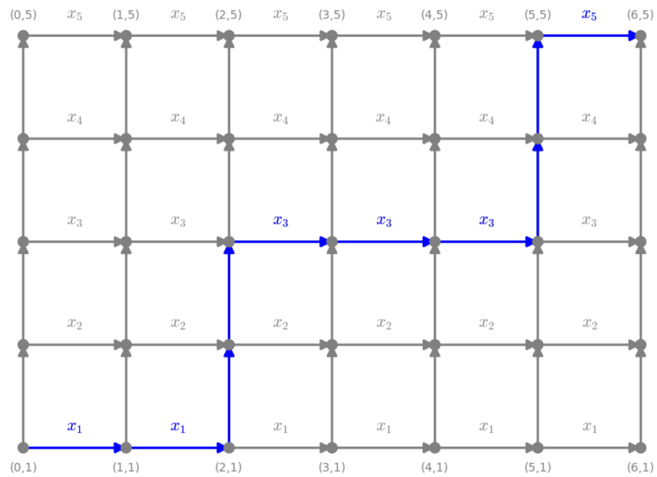



FIGURE 1. A path from $(0,5)$ to $(6,1)$ whose weight is the monomial $x_1^2x_3^3x_5$ in $h_6(x_1,\ldots,x_5)$.

*Proof.* The Jacobi-Trudi identities can be proved using the Lindström-Gessel-Viennot lemma (Theorem 15.2). For the first identity take $S$ to be the positive cone in the the integer lattice:

$$S = \{(i,j) \mid i \geq 0,\ j > 0 \text{ are integers}\}.$$

Set the weight $v((i,j),(i+1,j))$ of each rightward horizontal edge to be $x_j$ for $j = 1,\ldots,n$, the weight of each upward vertical edge $v((i,j),(i,j+1))$ to be 1 for all $j = 1,\ldots,n-1$. The remaining weights are all zero.

**Lemma 16.2.** *For all integers $i > 0$ and $j \geq 0$, we have:*

$$\sum_{\omega:(i,1)\to(i+j,n)} v(\omega) = h_j(x_1,\ldots,x_n).$$

*Proof.* Only rightward or upward steps have non-zero weights. So every path with non-zero weight is composed of unit upward and rightward steps. A path with non-zero weight from $(i,1)$ to $(i+j,n)$ must have exactly $j$ rightward steps, say in rows $1 \leq i_1 \geq i_2 \cdots \leq i_j \leq n$. The weight of such a path is $x_{i_1}\cdots x_{i_j}$, and hence, the sum of the weights of all such paths is $h_j(x_1,\ldots,x_n)$. For an example, see Fig. 1. □

Gien $\lambda = (\lambda_1,\ldots,\lambda_l)$, and working with $n$ variables $x_1,\ldots,x_n$, let $A_i = (l-i,1)$ and $B_i = (\lambda_i + l - i, n)$ for $i = 1,\ldots,l$. Then by Lemma 16.2,

$$\sum_{\omega:A_i\to B_j} v(\omega) = h_{\lambda_j+i-j}.$$

So the left-hand-side of the first Jacobi-Trudi identity is the left-hand-side of the Lindström-Gessel-Viennot lemma. The right hand side of the Linstróm-Gessel-Viennot lemma consists of a sequence of non-crossing paths

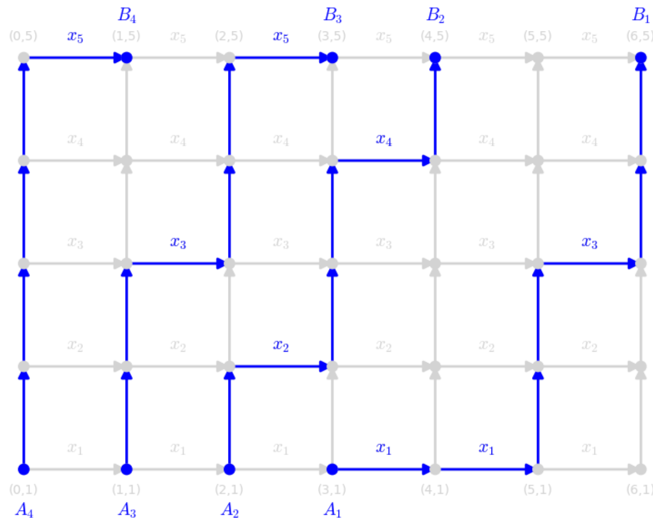



FIGURE 2. Non-crossing paths corresponding to the tableau $\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&4\\ \cline{1-2} 3&5\\ \cline{1-2} 5\\ \cline{1-1}\end{array}$

$(\omega_1, \dots, \omega_n)$, where $\omega_i : A_i \to B_i$. Reading the row numbers of the horizontal steps in $\omega_i$ gives a weakly increasing sequence of integers $1 \leq k_1 \leq \cdots \leq k_{\lambda_i} \leq n$. Enter these numbers into the $i$th row of the Young diagram of $\lambda$ for $i = 1, \dots, n$. Since the paths are non-crossing, the $j$th rightward step of $\omega_i$ must be strictly higher than the $j$th rightward step of $\omega_{i+1}$. This means that the columns of the resulting numbering are strictly increasing, resulting in a semistandard tableau of shape $\lambda$.

Figure 2 shows the non-crossing path configuration corresponding to $n = 5$, $\lambda = ()$ which corresponds to the semistandard tableau $\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&4\\ \cline{1-2} 3&5\\ \cline{1-2} 5\\ \cline{1-1}\end{array}$. Thus, it follows from the Lindstöm-Gessel-Viennot lemma that

$$\det(h_{\lambda_j+i-j}) = \sum_{t \in \mathrm{Tab}(\lambda)} x^t.$$

For the second Jacobi-Trudi identity take

$$S = \{(i, j) \mid i \geq 0,\ j \geq 0\}.$$

Define $v((i, j), (i + 1, j)) = 1$ (as before) and $v((i - 1, j), (i, j + 1)) = x_i$; all other weights are zero. For the new weights, the analog of Lemma 16.2 is:

**Lemma 16.3.** *For all integers $i > 0$ and $j > 0$, we have:*

$$\sum_{\omega:(i,0)\to(i+j,n)} v(\omega) = e_j(x_1, \dots, x_n).$$

*Proof.* Every path with non-zero weights consists of unit upward or upper-rightward diagonal steps. A path with non-zero weight from $(i, 0)$ to $(i + j, n)$ must have $n$

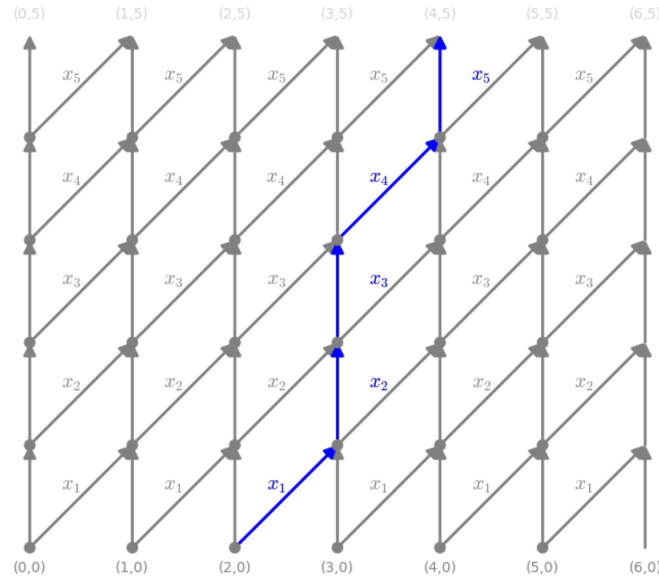



FIGURE 3. A path from $(2, 0)$ to $(4, 5)$ whose weight is the monomial $x_1x_4$ in $e_2(x_1, \dots, x_5)$.

such steps, of which $j$ must be diagonal. If the steps numbered $i_1, \dots, i_j$ are the diagonal steps, then the path has weight $x_{i_1} \cdots x_{i_j}$. For an example of such a path, see Fig. 3. Summing over all possible paths gives $e_j(x_1, \dots, x_n)$. $\square$

Suppose that the conjugate partition $\lambda' = (\lambda'_1, \dots, \lambda'_k$. In order to apply the Lindström-Gessel-Viennot lemma to obtain the second Jacobi-Trudi identity, take $A_i = (k - i, 0)$ and $B_i = (\lambda'_i + k - i, n)$ for $i = 1, \dots, k$. Then by Lemma 16.3,

$$\sum_{\omega_i : A_i \to B_j} v(\omega) = e_{\lambda_j + i - j}.$$

So the left-hand-side of the second Jacobi-Trudi identity is the left-hand-side of the Lindström-Gessel-Viennot lemma.

The right hand side of the Linstróm-Gessel-Viennot lemma consists of a sequence of non-crossing paths $(\omega_1, \dots, \omega_n)$, where $\omega_i : A_i \to B_i$. Reading the row numbers where the upper-rightward steps in $\omega_i$ originate gives a strictly increasing sequence of integers $1 \leq k_1 \leq \cdots \leq k_{\lambda'_i} \leq n$. Enter these numbers into the $i$th column of the Young diagram of $\lambda$. Since the paths are non-crossing, the $j$th upper-rightward step of $\omega_i$ must be no lower than the $j$th upper-rightward step of $\omega_{i+1}$. This means that the rows of the resulting numbering are weakly increasing, resulting in a semi-standard tableau of shape $\lambda$. Figure 2 shows the non-crossing path configuration corresponding to $n = 5$, $\lambda = ()$ which corresponds to the semistandard tableau $\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 3 & 5 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$. Thus, it follows from the Lindstöm-Gessel-Viennot lemma that

$$\det(e_{\lambda_j + i - j}) = \sum_{t \in \mathrm{Tab}(\lambda')} x^t.$$

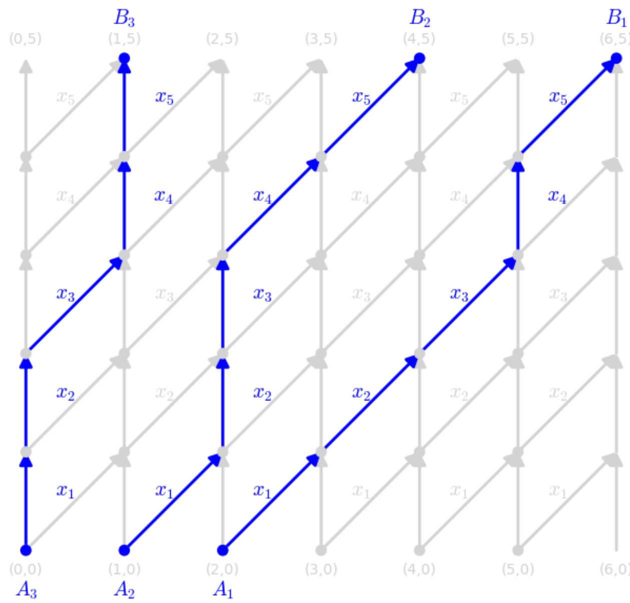



FIGURE 4. Non-crossing paths corresponding to the tableau $\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&4\\\cline{1-2} 3&5\\\cline{1-2} 5\\\cline{1-1}\end{array}$.

$\square$

## REFERENCES

[1] N. A. Loehr. Abacus proofs of Schur function identities. *SIAM J. Discrete Math.*, 24, 2010.