AN INTRODUCTION TO SCHUR FUNCTIONS

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1. **Symmetric Functions.** We consider polynomials in n variables $x_1, \dots, x_n$. Given a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, x^α denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ A symmetric polynomial in n variables $x_1, \dots, x_n$ is a polynomial of the form

$$f(x_1, \dots, x_n) = \sum_\alpha c_\alpha x^\alpha,$$

where, for any permutation $w \in S_n$,

$$c_{(\alpha_1,\dots,\alpha_n)} = c_{(\alpha_{w(1)},\dots,\alpha_{w(n)})}.$$

We call the integer partition λ obtained by sorting the coordinates of α the shape of α and write $\lambda = \lambda(\alpha)$. The most obvious example of a symmetric polynomial in n variables is the *monomial symmetric function*, defined for each integer partition λ:

$$m_\lambda = \sum_{\lambda(\alpha)=\lambda} c_\alpha x^\alpha.$$

Note that m_λ is homogeneous of degree $|\lambda|$ (the sum of the parts of λ).

Exercise 1.1. *Take $n = 4$. Compute the monomial symmetric functions $m_{(3)}$, $m_{(2,1)}$, and $m_{(1^3)}$.*

Theorem 1.2. *The polynomials $m_\lambda(x_1, \dots, x_n)$, as λ runs over all the integer partition of d, form a basis for the space of homogeneous symmetric polynomials of degree d in n variables.*

2. **Complete and Elementary Symmetric Polynomials.** Recall that the coefficients of a polynomial are symmetric polynomials in its roots:

$$(1)\quad (t - x_1)(t - x_2) \cdots (t - x_n) \\ = t^n - e_1(x_1, \dots, x_n)t^{n-1} + \cdots + (-1)^n e_n(x_1, \dots, x_n),$$

where coefficient $e_i(x_1, \dots, x_n)$ of t^{n-i} is given by:

$$(2)\qquad e_i(x_1, \dots, x_n) = \sum_{1 \leq j_1 < \cdots < j_i \leq n} x_{j_1} x_{j_2} \cdots x_{j_i}.$$

The polynomial e_i is called the ith *elementary symmetric polynomial*. By convention, write $e_i(x_1, \dots, x_n) = 0$, for $i > n$.

The identity (1) can be written more elegantly as:

$$(1+tx_1)\cdots(1+tx_n) = \sum_{i=0}^{n} e_i(x_1,\cdots,x_n)t^i.$$

Dually[1], the *complete symmetric polynomials* are defined by the formal identity:

$$\frac{1}{(1-x_1t)\cdots(1-x_nt)} = \sum_{i=0}^{\infty} h_i(x_1,\cdots,x_n)t^i.$$

Example 2.1. *In three variables, we have:*

$$\begin{aligned} e_2(x_1,x_2,x_3) &= x_1x_2 + x_1x_3 + x_2x_3, \\ h_2(x_1,x_2,x_3) &= x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_2^3. \end{aligned}$$

Exercise 2.2. *Show that*

$$h_i(x_1,\cdots x_n) = \sum_{1\leq j_1\leq\cdots\leq j_i\leq n} x_{j_1}\cdots x_{j_i}.$$

More generally, for any integer partition $\lambda = (\lambda_1,\ldots,\lambda_l)$, define:

$$\begin{aligned} h_\lambda &= h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_l}, \\ e_\lambda &= e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_l}. \end{aligned}$$

Theorem 2.3. *Given integer partitions $\lambda = (\lambda_1,\ldots,\lambda_l)$ and $\mu = (\mu_1,\cdots,\mu_m)$ of and integer d, let $M_{\lambda\mu}$ denote the number of integer matrices (a_{ij}) with non-negative entries whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j. Then*

$$h_\lambda = \sum_\mu M_{\lambda\mu}m_\mu.$$

Dually, let $= N_{\lambda\mu}$ denote the number of integer matrices (a_{ij}) with entries 0 or 1, whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j.

$$e_\lambda = \sum_\mu N_{\lambda\mu}m_\mu.$$

Proof. We first prove the second identity involving elementary symmetric functions. A monomial in the expansion:

$$e_\lambda = \prod_{i=1}^{l} \sum_{j_1<\cdots<j_{\lambda_j}} x_{j_1}\cdots x_{j_{\lambda_i}}$$

is a product of summands, one chosen from each of the l factors. Construct an $l\times m$ matrix (a_{ij}) corresponding to such a choice as follows: if the summand $x_{j_1}\cdots x_{j_{\lambda_i}}$ is chosen from the ith factor, then set the entries $a_{i,j_1},\ldots,a_{i,j_{\lambda_j}}$ to be 1 (the remaining entries of the ith row are 0). Clearly

[1] We will refer to the replacing of $(1+u)$ by $(1-u)^{-1}$ in a formal identity as *dualization*.

the ith row of such a matrix sums to λ_i. The monomial corresponding to this choice is x^μ if, for each j, the the number of i for which x_j appears in $a_{i,j_1}, \dots, a_{i,j_{\lambda_j}}$, which is the sum of the jth column of the matrix (a_{ij}). It follows that the coefficient of x^μ, and hence the coefficient of m_μ in the expansion of e_λ in the basis of monomial symmetric functions of degree n, is $N_{\lambda\mu}$.

A similar proof can be given for the first identity involving complete symmetric functions. The only difference is that a variables are repeated in the monomials that appear in h_i. Counting the number of repetitions (instead of just recording 0 or 1) gives non-negative integer matrices. $\square$

3. **Alternating Polynomials.** An *alternating polynomial* in $x_1, \dots, x_n$ is of the form:

$$f(x_1, \dots, x_n) = \sum_\alpha c_\alpha x_\alpha, \tag{3}$$

where, $c_{w(\alpha)} = \epsilon(w) c_\alpha$ for every multiindex α as in Section 1. Here $\epsilon : S_n \to \{\pm 1\}$ denotes the sign function. Equivalently, an alternating polynomial is one whose sign is reversed upon the interchange of any two variables.

Exercise 3.1. *If α is a multiindex where $\alpha_i = \alpha_j$ for some $i \neq j$, then $c_\alpha = 0$.*

In particular, every monomial in an alternating polynomial must be composed of distinct powers. Moreover, the polynomial is completely determined by the coefficients c_α of strictly decreasing multiindices, namely, multiindices of the form $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_1 > \dots > \alpha_n$.

Exercise 3.2. *Let δ denote the strictly increasing multiindex $(n-1, n-2, \dots, 1, 0)$ of lowest degree. Given an integer partition with at most n parts, we will pad it with 0's so that it can be regarded as a weakly decreasing multiindex of length n. Then $\lambda \mapsto \lambda + \delta$ is a bijection from the set of integer partitions with at most n onto the set of strictly decreasing multiindices.*

Example 3.3. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a weakly decreasing multiindex. The polynomial:*

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + n - j})$$

is alternating, with unique strictly decreasing monomial $x^{\lambda+\delta}$.

Exercise 3.4. *The alternating polynomial of the form* (3) *is equal to*

$$\sum_\lambda c_\lambda a_{\lambda+\delta},$$

the sum being over all weakly decreasing multiindices.

4. **Cauchy's Bialternant Form of a Schur Function.** The simplest polynomial of the form $a_{\lambda+\delta}$ arises when $\lambda = 0$; a_δ is the Vandermonde determinant:

$$a_\delta = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Exercise 4.1. *Show that, for every weakly decreasing multiindex λ, $a_{\lambda+\delta}$ is divisible by a_δ in the ring of polynomials in $x_1, \ldots, x_n$.*

Exercise 4.2. *Show that $f \mapsto f a_\delta$ is an isomorphism of the space of symmetric polynomials in $x_1, \ldots, x_n$ of degree d onto the space of alternating polynomials of degree $d + \binom{n}{2}$.*

The above exercise allows us to give the historically oldest definition of Schur functions—*Cauchy's bialternant formula*:

$$s_\lambda(x_1, \ldots, x_n) = a_{\lambda+\delta}/a_\delta, \tag{4}$$

for any partition λ with at most n parts. If λ has more than n parts, set $s_\lambda(x_1, \ldots, x_n) = 0$. This is clearly a symmetric function of degree $|\lambda|$. When λ has more than n parts, we shall write $s_\lambda(x_1, \ldots, x_n) = 0$

Theorem 4.3. *As λ runs over all integer partitions of d with at most n parts, the Schur functions $s_\lambda(x_1, \ldots, x_n)$ form a basis of the space of all homogeneous symmetric functions in $x_1, \ldots, x_n$ of degree d.*

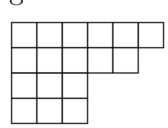
Proof. This follows from Exercises 3.4 and 4.2. □

Exercise 4.4 (Stability of Schur functions)**.** *Show that substituting $x_n = 0$ in the Schur function $s_\lambda(x_1, \ldots, x_n)$ with n variables gives the corresponding Schur function $s_\lambda(x_1, \ldots, x_{n-1})$ with $n - 1$ variables.*

5. **Pieri's rule.** The set of integer partitions is endowed with the *containment order*. We say that a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ *contains* a partition $\mu = (\mu_1, \ldots, \mu_m)$ if $l \geq m$, and $\lambda_i \geq \mu_i$ for every $i = 1, \cdots, m$. We write $\lambda \supset \mu$ or $\mu \subset \lambda$. Recall that the Young diagram of the partition λ is the set of points

$$\{(i, j) \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}.$$

Visually, each node (i, j) of the Young diagram is replaced by a box, and the box corresponding to (i, j) is placed in the ith row and jth column (matrix notation). Thus, the Young diagram of $\lambda = (6, 5, 3, 3)$ is depicted by:



Note that containment of partitions is nothing but the containment relation on their Young diagrams. By abuse of notation, we will also use λ to denote the Young diagram of λ.

By a skew-shape, we mean a difference of Young diagrams $\lambda \setminus \mu$, where $\lambda \supset \mu$. We write λ/μ for this skew-shape. A skew-shape is called a *horizontal* *strip* (respectively, a *vertical strip*) if it has at most one element in each vertical column (respectively, horizontal row).

Theorem 5.1. *For every partition λ, and every positive integer k,*

$$s_\lambda h_k = \sum_\mu s_\mu,$$

where the sum runs over all partitions $\mu \supset \lambda$ such that μ/λ is a horizontal strip of size k. Dually,

$$s_\lambda e_k = \sum_\mu s_\mu,$$

where the sum runs over all partitions $\mu \supset \lambda$ such that μ/λ is a vertical strip of size k.

Proof. The first identity is equivalent to showing that:

$$a_{\lambda+\delta} \sum_{\alpha_1+\cdots+\alpha_n=k} x_1^{\alpha_1}\cdots x_n^{\alpha_n} = \sum_\mu a_{\mu+\delta},$$

the sum on the right being over all partitions $\mu \supset \lambda$ such that μ/λ is a horizontal strip.

Writing $\alpha = (\alpha_1, \ldots, \alpha_n)$, the sum on the left hand side can be regarded as a sum of determinants:

$$a_{\lambda+\delta} \sum_{|\alpha|=k} x_1^{\alpha_1}\cdots x_n^{\alpha_n} = \sum_{|\alpha|=k} a_{\lambda+\alpha+\delta}. \tag{5}$$

Suppose there exists an integer α_i such that $\alpha_i > \lambda_i - \lambda_{i+1}$ (in other words, $(\lambda+\alpha)/\lambda$ is not a horizontal strip), then define $\beta = (\beta_1, \ldots, \beta_n)$ by $\beta_i = \alpha_{i+1} - (\lambda_i - \lambda_{i+1} + 1)$, $\beta_{i+1} = \alpha_i + (\lambda_i - \lambda_{i+1} + 1)$, and $\beta_j = \alpha_j$ for all $j \notin \{i, i+1\}$. Then $a_{\lambda+\alpha+\delta} = -a_{\lambda+\beta+\delta}$. So the only terms that survive on the right hand side of (5) are of the form $a_{\mu+\delta}$, where μ/λ is a horizontal strip.

The proof of the second identity in the theorem is similar (in fact, a little simpler) and is left to the reader as an exercise. □

As a special case of Pieri's rule, we have:

Corollary 5.2. *For every positive integer k,*

$$s_{(k)} = h_k, \text{ and } s_{(1^k)} = e_k.$$

6. **Schur to Complete and Elementary via Tableaux.** Pieri's rule allows us to compute the complete and elementary symmetric functions h_λ and e_λ in terms of Schur functions.

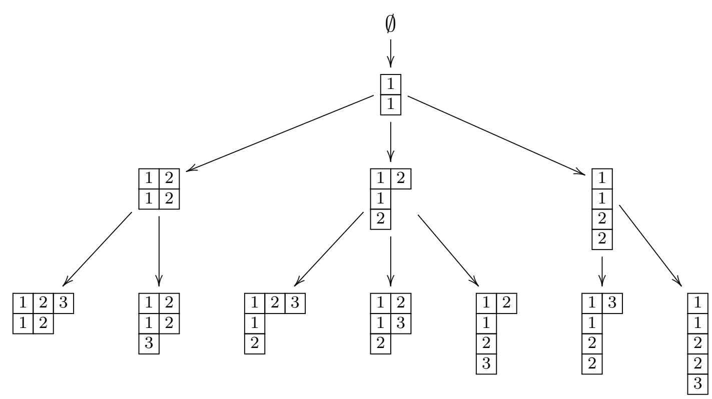
Example 6.1. *Repeated application of Pieri's rule gives an expansion of* $e_{(2,2,1)} = e_2e_2e_1$ *as:*

$$\begin{tikzcd}
& & & e_2e_2e_1 \arrow[d] & & & \\
& & & s_{(1^2)}e_2e_1 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
& s_{(2^2)}e_1 \arrow[dl] \arrow[d] & & s_{(2,1^2)}e_1 \arrow[dl] \arrow[d] \arrow[dr] & & s_{(1^4)}e_1 \arrow[d] \arrow[dr] & \\
s_{(3,2)} & s_{(2^2,1)} & s_{(3,1^2)} & s_{(2^2,1)} & s_{(2,1^3)} & s_{(2,1^3)} & s_{(1^5)}
\end{tikzcd}$$

giving:

$$e_{(2^2,1)} = s_{(3,2)} + 2s_{(2^2,1)} + s_{(3,1^2)} + 2s_{(2,1^3)} + s_{(1^5)}.$$

The steps going from the first line of the above calculation to each term of the last line can be recorded by putting numbers into Young diagrams:



The boxes in the vertical strip added at the ith stage are filled with i.

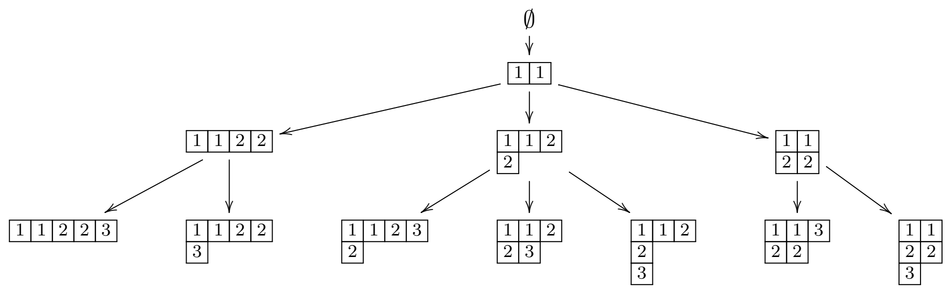
Example 6.2. *Repeated application of Pieri's rule gives an expansion of* $h_{(2,2,1)} = h_2h_2h_1$ *as:*

$$\begin{tikzcd}
& & & h_2h_2h_1 \arrow[d] & & & \\
& & & s_{(2)}h_2h_1 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
& s_{(4)}h_1 \arrow[dl] \arrow[d] & & s_{(3,1)}h_1 \arrow[dl] \arrow[d] \arrow[dr] & & s_{(3,2)}h_1 \arrow[d] \arrow[dr] & \\
h_{(5)} & s_{(4,1)} & s_{(4,1)} & s_{(3,2)} & s_{(3,1^2)} & s_{(3,2)} & s_{(2,2,1)}
\end{tikzcd}$$

giving:

$$h_{(2^2,1)} = s_{(5)} + 2s_{(4,1)} + 2s_{(3,2)} + s_{(3,1^2)} + s_{(2,2,1)}.$$

The steps going from the first line of the above calculation to each term of the last line can be recorded by putting numbers into Young diagrams:



The boxes in the horizontal strip added at the ith stage are filled with i.

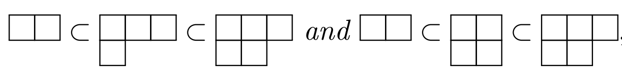
Definition 6.3 (Semistandard tableau)**.** A semistandard tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_l)$ and type $\mu = (\mu_1, \ldots, \mu_m)$ is the Young diagram of λ filled with numbers $1, \ldots, m$ such that the number i appears μ_i times, the numbers weakly increase along rows, and strictly increase along columns.

Exercise 6.4. *Semistandard tableaux of shape λ and type μ correspond to chains of integer partitions*

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(m)} = \lambda$$

where $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip of size μ_i.

Example 6.5. *The semistandard tableau of type $(3,2)$ and type $(2,2,1)$ are $\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2&3\\ \cline{1-2}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&2\\ \cline{1-2}\end{array}$. They correspond to the chains:*



respectively. As illustrated in Example 6.2, the coefficient of $s_{(3,2)}$ in the complete symmetric function $h_{(2,2,1)}$ is the number of semistandard tableau of shape $(3,2)$ and type $(2,2,1)$.

Definition 6.6 (Kostka number)**.** Given two partitions λ and μ, the Kostka number $K_{\lambda\mu}$ is the number of semistandard tableau of shape λ and type μ.

Exercise 6.7. *For every partition λ, show that $K_{\lambda\lambda} = 1$.*

Exercise 6.8 (f-number)**.** *The f-number of a partition λ of n is defined to be the Kostka number $K_{\lambda,(1^n)}$, and is denoted f_λ.*

Exercise 6.9. *For a partition λ, let λ^- denote the set of all partitions whose Young diagram can be obtained by removing one box from the Young diagram of λ. Show that $f_\lambda = \sum_{\mu \in \lambda^-} f_\mu$.*

Exercise 6.10. *A hook is a partition of the form $h(a,b) = (a+1, 1^b)$. Show that $f_{h(a,b)} = \binom{a+b}{a}$.*

In order to understand the expansion of elementary symmetric functions we would need a variant of semistandard tableaux, one where the difference between successive shapes are vertical strips, rather than horizontal strips. However, it has become common practice to *conjugate* partitions instead:

Definition 6.11 (Conjugate of a partition)**.** The *conjugate* of a partition λ is the partition λ' whose Young diagram is given by:

$$\lambda' = \{(j,i) \mid (i,j) \in \lambda\}.$$

In other words, the Young diagram of λ' is the reflection of the Young diagram of λ about its principal diagonal.

Clearly $\lambda \mapsto \lambda'$ is an involution. For example, if $\lambda = (2,2,1)$, then $\lambda' = (3,2)$.

Exercise 6.12. *Semistandard tableaux of shape λ' and type μ correspond to chains of integer partitions*

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(m)} = \lambda$$

where $\lambda^{(i)}/\lambda^{(i-1)}$ is a vertical *strip of size μ_i.*

Theorem 6.13. *The expansion of complete symmetric functions in terms of Schur functions is given by:*

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda.$$

Dually, the extension of elementary symmetric functions in terms of Schur functions is given by:

$$e_\mu = \sum_\lambda K_{\lambda'\mu} s_\lambda.$$

7. **Triangularity of Kostka Numbers.** In this section we give a necessary and sufficient condition for the positivity of Kostka number. As usual, take $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_m)$. Suppose $K_{\lambda\mu} > 0$. Then there exists a semistandard tableau t of shape λ and type μ. Since the columns of t are strictly increasing, all the 1's in t must occur in its first row, so $\lambda_1 \geq \mu_1$. Also, all the 2's must occur in the first two rows (along with all the 1's), so $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. More generally, all the numbers $1, \dots, i$ for $i = 1, \dots, m$ should occur in the first m rows of t. We have:

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots \mu_i \text{ for } i = 1, \dots, m. \tag{6}$$

Definition 7.1. We say that an integer partition $\lambda = (\lambda_1, \dots, \lambda_l)$ *dominates* the integer partition $\mu = (\mu_1, \dots, \mu_m)$ $|\lambda| = |\mu|$ and if the condition (6) holds. When this happens we write $\lambda \rhd \mu$. This relation defines a partial order on the set of all integer partitions of n for any non-negative integer n.

Exercise 7.2. *Show that (n) is maximal and (1^n) is minimal among all the integer partitions of n. What is the smallest integer n for which the dominance order on partitions of n is not a linear order?*

Theorem 7.3 (Triangularity of Kostka Numbers)**.** *Given partition λ and μ of an integer n, $K_{\lambda\mu} > 0$ if and only if $\lambda \rhd \mu$.*

Proof. We have already seen that if $K_{\lambda\mu} > 0$, then $\lambda \rhd \mu$. While reading the proof of the converse, it is helpful to keep in mind Example 7.4 below. Suppose that $\lambda \rhd \mu$. Then $\lambda_1 \geq \mu_1 \geq \mu_m$. Therefore, the Young diagram of λ has at least μ_m cells in its first row, or in other words, it has at least μ_m columns. Choose the smallest integer i for which $\lambda_i \geq \mu_m$. Fill the bottom-most box in the λ_{i+1} leftmost columns with m. Also, from the ith row, fill the rightmost $\mu_m - \lambda_{i+1}$ boxes with m. The remaining (unfilled) boxes in the Young diagram of λ now form the Young diagram of the partition

$$\eta = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - \mu_m + \lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_l),$$

a partition with $l-1$ parts. Writing $(\eta_1, \ldots, \eta_{l-1})$ for the parts of η, note that, since the first $i-1$ parts of η are the same as those of λ, we have:

$$\eta_1 + \cdots + \eta_j \geq \mu_1 + \cdots + \mu_j$$

for $j \leq i-1$. For $j \geq i$, we have

$$\begin{aligned} \eta_1 + \cdots + \eta_j &= \lambda_1 + \cdots + \lambda_{j+1} - \mu_m \\ &\geq \mu_1 + \cdots + \mu_j + \mu_{j+1} - \mu_m \\ &\geq \mu_1 + \cdots + \mu_j. \end{aligned}$$

It follows that $\eta \rhd (\mu_1, \ldots, \mu_{m-1})$. The result now follows by induction in m. $\square$

Example 7.4. *Consider the case where $\lambda = (7,3,2)$ and $\mu = (4,4,4)$. Then the smallest integer i such that $\lambda_i \geq 4$ is 1. Accordingly, we enter 3 into the bottom-most boxes in the three leftmost columns, and also into one rightmost box in the first row:*

$$\begin{ytableau} \ & \ & \ & \ & \ & \ & 3 \\ \ & \ & 3 \\ 3 & 3 \end{ytableau}$$

We are left with the problem of finding a semistandard tableau of shape $(6,2)$ and type $(4,4)$. Recursively applying our process to this smaller problem gives:

$$\begin{ytableau} \ & \ & \ & \ & 2 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 \end{ytableau},$$

and finally the desired tableau

$$\begin{ytableau} 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 \end{ytableau}.$$

Theorem 7.5. *The complete symmetric functions:*

$$\{h_\mu \mid \mu \text{ is a partition of } d \text{ with at most } n \text{ parts}\}$$

and the elementary symmetric functions:

$$\{e_\mu \mid \mu \text{ is a partition of } d \text{ with } \mu_1 \leq n\}$$

form bases of the space of homogeneous symmetric polynomials of degree d in variables $x_1, \ldots, x_n$.

Proof. In view of the triangularity of Kostka numbers (Theorem 7.3) and the fact that $K_{\lambda\lambda} = 1$ (Exercise 6.7) the theorem follows from Theorem 6.13. □

8. **The Lindström-Gessel-Viennot Lemma.** Let R be a commutative ring. Let S be any set of points, and $v : S \times S \to R$ be any function (we think of w as a *weight function.* Given $s, t \in S$, a path in S from s to t is is a sequence $\omega = (s = s_0, s_1, \ldots, s_k = t)$ of distinct points in S. We denote this by $\omega : s \to t$. The weight of the path ω is defined to be:

$$v(\omega) = v(s_0, s_1)v(s_1, s_2) \cdots v(s_{k-1}, s_k).$$

Definition 8.1 (non-crossing paths)**.** Two paths $\omega = (s_0, \ldots, s_k)$ and $\eta = (t_0, \ldots, t_l)$ are said to be non-crossing if $s_i \neq t_j$ for all $0 \leq i \leq k$ and $0 \leq t \leq l$.

Fix points $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ in S, and define an $n \times n$ matrix (a_{ij}) by:

$$a_{ij} = \sum_{\omega : A_i \to B_j} v(\omega).$$

Theorem 8.2 (Lindström-Gessel-Viennot Lemma)**.** *The determinant of the matrix (a_{ij}) defined above is given by:*

$$\det(a_{ij}) = \sum_{\omega_i : A_i \to B_i} v(\omega_1) \cdots v(\omega_n),$$

where the sum is over all n-tuples $(\omega_1, \ldots, \omega_n)$ of pairwise non-crossing paths $\omega_i : A_i \to B_i$.

9. **The Jacobi-Trudi Identities.** We have seen that the Kostka numbers can be used to express complete and elementary symmetric functions in terms of Schur functions. The reverse operation—that of expressing Schur functions in terms of complete or elementary symmetric functions—is done by the Jacobi-Trudi identities:

Theorem 9.1 (Jacobi-Trudi identities)**.** *For each n, and every integer partition λ with at most n parts*[2]*, form the $n \times n$ matrices with (i, j)th entry $h_{\lambda_i - i + j}$ and $e_{\lambda_i - i + j}$ respectively. Then*

$$s_\lambda = \det(h_{\lambda_i - i + j}), \quad s_{\lambda'} = \det(e_{\lambda_i - i + j}).$$

Proof. The Jacobi-Trudi identities can be proved using the Lindström-Gessel-Viennot lemma (Theorem 8.2). For the first identity take S to be the positive cone in the the integer lattice:

$$S = \{(i, j) \mid i \geq 0,\ j > 0 \text{ are integers}\}.$$

[2]If λ has fewer than n parts, it is padded with zeros on the right to make it have exactly n parts

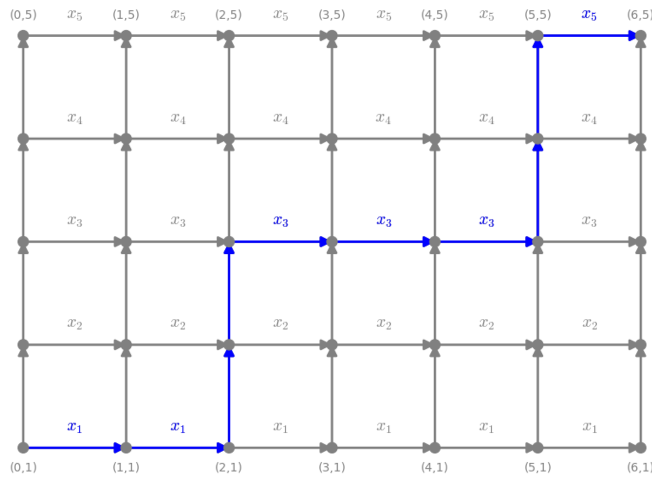


FIGURE 1. A path from $(0, 5)$ to $(6, 1)$ whose weight is the monomial $x_1^2x_3^3x_5$ in $h_6(x_1, \dots, x_5)$.

Set the weight $v((i, j), (i + 1, j))$ of each rightward horizontal edge to be x_j for $j = 1, \dots, n$, the weight of each upward vertical edge $v((i, j), (i, j + 1))$ to be 1 for all $j = 1, \dots, n - 1$. The remaining weights are all zero.

Lemma 9.2. *For all integers $i > 0$ and $j \geq 0$, we have:*

$$\sum_{\omega:(i,1)\to(i+j,n)} v(\omega) = h_j(x_1, \dots, x_n).$$

Proof. Only rightward or upward steps have non-zero weights. So every path with non-zero weight is composed of unit upward and rightward steps. A path with non-zero weight from $(i, 1)$ to $(i + j, n)$ must have exactly j rightward steps, say in rows $1 \leq i_1 \geq i_2 \cdots \leq i_j \leq n$. The weight of such a path is $x_{i_1} \cdots x_{i_j}$, and hence, the sum of the weights of all such paths is $h_j(x_1, \dots, x_n)$. For an example, see Fig. 1. □

Let λ be a partition of d with at most n parts. By appending 0's to the right of λ, think of λ as an n-tuple $(\lambda_1, \dots, \lambda_n)$ with weakly decreasing coordinates. Let $A_i = (n - i, 1)$ and $B_i = (\lambda_i + n - i, n)$ for $i = 1, \dots, n$. Then by Lemma 9.2,

$$\sum_{\omega:A_i\to B_j} v(\omega) = h_{\lambda_j+i-j}.$$

The right hand side of the Linstróm-Gessel-Viennot lemma consists of a sequence of non-crossing paths $(\omega_1, \dots, \omega_n)$, where $\omega_i : A_i \to B_i$. Reading the row numbers of the horizontal steps in ω_i gives a weakly increasing sequence of integers $1 \leq k_1 \leq \cdots \leq k_{\lambda_i} \leq n$. Enter these numbers into the ith row of the Young diagram of λ for $i = 1, \dots, n$. Since the paths are non-crossing, the jth rightward step of ω_i must be strictly higher than the jth rightward step of ω_{i+1}. This means that the columns of the resulting

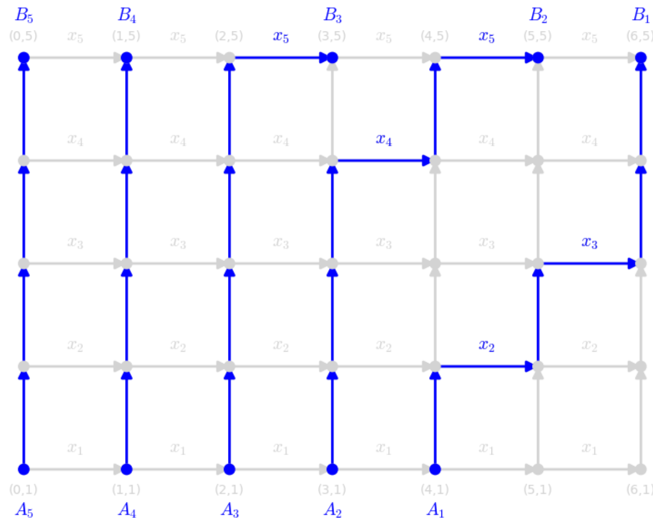


FIGURE 2. Non-crossing paths corresponding to the tableau $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 5 \\ \cline{1-1}\end{array}$

numbering are strictly increasing, resulting in a semistandard tableau of shape λ.

Figure 2 shows the non-crossing path configuration corresponding to $n = 5$, $\lambda = (2, 2, 1, 0, 0)$ which corresponds to the semistandard tableau $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 5 \\ \cline{1-1}\end{array}$.

For the second Jacobi-Trudi identity take

$$S = \{(i, j) \mid i \geq 0,\ j \geq 0\}.$$

Define $v((i, j), (i + 1, j)) = 1$ (as before) and $v((i - 1, j), (i, j + 1)) = x_i$; all other weights are zero. For the new weights, the analog of Lemma 9.2 is:

Lemma 9.3. *For all integers $i > 0$ and $j > 0$, we have:*

$$\sum_{\omega:(i,0)\to(i+j,n)} v(\omega) = e_j(x_1, \ldots, x_n).$$

Proof. Every path with non-zero weights consists of unit upward or upper-rightward diagonal steps. A path with non-zero weight from $(i, 0)$ to $(i+j, n)$ must have n such steps, of which j must be diagonal. If the steps numbered $i_1, \ldots, i_j$ are the diagonal steps, then the path has weight $x_{i_1} \cdots x_{i_j}$. Summing over all possible paths gives

$$\sum_{1\leq i_1<\cdots<i_j\leq n} x_{i_1} \cdots x_{i_j} = e_j(x_1, \ldots, x_n),$$

as required. □

□

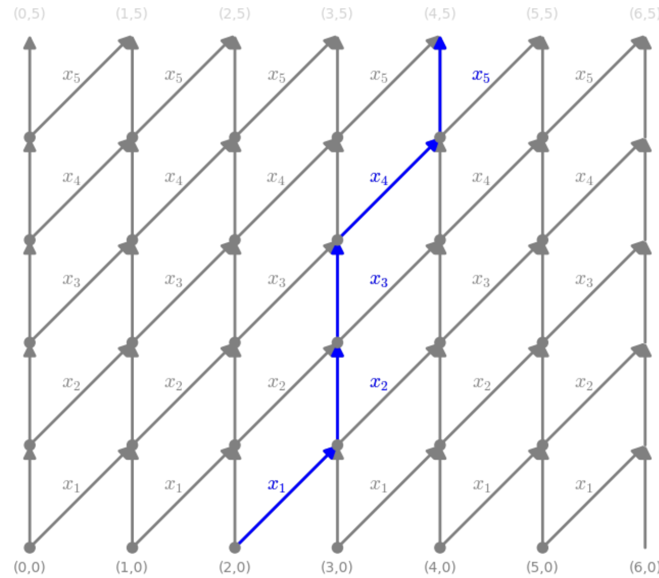


FIGURE 3. A path from $(2, 0)$ to $(4, 5)$ whose weight is the monomial $x_1 x_4$ in $e_2(x_1, \ldots, x_5)$.