AN INTRODUCTION TO SCHUR FUNCTIONS

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1. **Symmetric Functions.** We consider polynomials in n variables $x_1, \dots, x_n$. Given a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, x^α denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ A symmetric polynomial in n variables $x_1, \dots, x_n$ is a polynomial of the form

$$f(x_1, \dots, x_n) = \sum_\alpha c_\alpha x^\alpha,$$

where, for any permutation $w \in S_n$,

$$c_{(\alpha_1,\dots,\alpha_n)} = c_{(\alpha_{w(1)},\dots,\alpha_{w(n)})}.$$

We call the integer partition λ obtained by sorting the coordinates of α the shape of α and write $\lambda = \lambda(\alpha)$. The most obvious example of a symmetric polynomial in n variables is the *monomial symmetric function*, defined for each integer partition λ:

$$m_\lambda = \sum_{\lambda(\alpha)=\lambda} c_\alpha x^\alpha.$$

Note that m_λ is homogeneous of degree $|\lambda|$ (the sum of the parts of λ).

Exercise 1.1. *Take $n = 4$. Compute the monomial symmetric functions $m_{(3)}$, $m_{(2,1)}$, and $m_{(1^3)}$.*

Theorem 1.2. *The polynomials $m_\lambda(x_1, \dots, x_n)$, as λ runs over all the integer partition of d, form a basis for the space of homogeneous symmetric polynomials of degree d in n variables.*

2. **Complete and Elementary Symmetric Polynomials.** Recall that the coefficients of a polynomial are symmetric polynomials in its roots:

$$(t - x_1)(t - x_2) \cdots (t - x_n) \\ = t^n - e_1(x_1, \dots, x_n)t^{n-1} + \cdots + (-1)^n e_n(x_1, \dots, x_n), \tag{1}$$

where coefficient $e_i(x_1, \dots, x_n)$ of t^{n-i} is given by:

$$e_i(x_1, \dots, x_n) = \sum_{1 \le j_1 < \cdots < j_i \le n} x_{j_1} x_{j_2} \cdots x_{j_i}. \tag{2}$$

The polynomial e_i is called the ith *elementary symmetric polynomial*. By convention, write $e_i(x_1, \dots, x_n) = 0$, for $i > n$.

The identity (1) can be written more elegantly as:

$$(1 + tx_1) \cdots (1 + tx_n) = \sum_{i=0}^{n} e_i(x_1, \cdots, x_n)t^i.$$

Dually[1], the *complete symmetric polynomials* are defined by the formal identity:

$$\frac{1}{(1 - x_1t) \cdots (1 - x_nt)} = \sum_{i=0}^{\infty} h_i(x_1, \cdots, x_n)t^i.$$

Example 2.1. *In three variables, we have:*

$$e_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3,$$
$$h_2(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_2^3.$$

Exercise 2.2. *Show that*

$$h_i(x_1, \cdots x_n) = \sum_{1 \leq j_1 \leq \cdots \leq j_i \leq n} x_{j_1} \cdots x_{j_i}.$$

More generally, for any integer partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, define:

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l},$$
$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}.$$

Theorem 2.3. *Given integer partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \cdots, \mu_m)$ of and integer d, let $M_{\lambda\mu}$ denote the number of integer matrices (a_{ij}) with non-negative entries whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j. Then*

$$h_\lambda = \sum_\mu M_{\lambda\mu} m_\mu.$$

Dually, let $= N_{\lambda\mu}$ denote the number of integer matrices (a_{ij}) with entries 0 or 1, whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j.

$$e_\lambda = \sum_\mu N_{\lambda\mu} m_\mu.$$

Proof. We first prove the second identity involving elementary symmetric functions. A monomial in the expansion:

$$e_\lambda = \prod_{i=1}^{l} \sum_{j_1 < \cdots < j_{\lambda_j}} x_{j_1} \cdots x_{j_{\lambda_i}}$$

is a product of summands, one chosen from each of the l factors. Construct an $l \times m$ matrix (a_{ij}) corresponding to such a choice as follows: if the summand $x_{j_1} \cdots x_{j_{\lambda_i}}$ is chosen from the ith factor, then set the entries $a_{i,j_1}, \ldots, a_{i,j_{\lambda_j}}$ to be 1 (the remaining entries of the ith row are 0). Clearly

[1]We will refer to the replacing of $(1+u)$ by $(1-u)^{-1}$ in a formal identity as *dualization*.

the ith row of such a matrix sums to λ_i. The monomial corresponding to this choice is x^μ if, for each j, the the number of i for which x_j appears in $a_{i,j_1}, \ldots, a_{i,j_{\lambda_j}}$, which is the sum of the jth column of the matrix (a_{ij}). It follows that the coefficient of x^μ, and hence the coefficient of m_μ in the expansion of e_λ in the basis of monomial symmetric functions of degree n, is $N_{\lambda\mu}$.

A similar proof can be given for the first identity involving complete symmetric functions. The only difference is that a variables are repeated in the monomials that appear in h_i. Counting the number of repetitions (instead of just recording 0 or 1) gives non-negative integer matrices. □

3. **Alternating Polynomials.** An *alternating polynomial* in $x_1, \ldots, x_n$ is of the form:

$$f(x_1, \ldots, x_n) = \sum_\alpha c_\alpha x_\alpha, \tag{3}$$

where, $c_{w(\alpha)} = \epsilon(w) c_\alpha$ for every multiindex α as in Section 1. Here $\epsilon : S_n \to \{\pm 1\}$ denotes the sign function. Equivalently, an alternating polynomial is one whose sign is reversed upon the interchange of any two variables.

Exercise 3.1. *If α is a multiindex where $\alpha_i = \alpha_j$ for some $i \neq j$, then $c_\alpha = 0$.*

In particular, every monomial in an alternating polynomial must be composed of distinct powers. Moreover, the polynomial is completely determined by the coefficients c_α of strictly decreasing multiindices, namely, multiindices of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 > \cdots > \alpha_n$.

Exercise 3.2. *Let δ denote the strictly increasing multiindex $(n-1, n-2, \ldots, 1, 0)$ of lowest degree. Given an integer partition with at most n parts, we will pad it with 0's so that it can be regarded as a weakly decreasing multiindex. Then $\lambda \mapsto \lambda + \delta$ is a bijection from the set of integer partitions with at most n onto the set of strictly decreasing multiindices.*

Example 3.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a weakly decreasing multiindex. The polynomial:*

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + n - j})$$

is alternating, with unique strictly decreasing monomial $x^{\lambda+\delta}$.

Exercise 3.4. *The alternating polynomial of the form* (3) *is equal to*

$$\sum_\lambda c_\lambda a_{\lambda+\delta},$$

the sum being over all weakly decreasing multiindices.

4. **Cauchy's Bialternant Form of a Schur Function.** The simplest polynomial of the form $a_{\lambda+\delta}$ arises when $\lambda = 0$; a_δ is the Vandermonde determinant:

$$a_\delta = \prod_{1\leq i<j\leq n} (x_i - x_j).$$

Exercise 4.1. *Show that, for every weakly decreasing multiindex λ, $a_{\lambda+\delta}$ is divisible by a_δ in the ring of polynomials in $x_1, \ldots, x_n$.*

Exercise 4.2. *Show that $f \mapsto fa_\delta$ is an isomorphism of the space of symmetric polynomials in $x_1, \ldots, x_n$ of degree d onto the space of alternating polynomials of degree $d + \binom{n}{2}$.*

The above exercise allows us to give the historically oldest definition of Schur functions—*Cauchy's bialternant formula*:

$$s_\lambda(x_1, \ldots, x_n) = a_{\lambda+\delta}/a_\delta, \tag{4}$$

for any partition λ with at most n parts. If λ has more than n parts, set $s_\lambda(x_1, \ldots, x_n) = 0$. This is clearly a symmetric function of degree $|\lambda|$. When λ has more than n parts, we shall write $s_\lambda(x_1, \ldots, x_n) = 0$

Theorem 4.3. *As λ runs over all integer partitions of d with at most n parts, the Schur functions $s_\lambda(x_1, \ldots, x_n)$ form a basis of the space of all homogeneous symmetric functions in $x_1, \ldots, x_n$ of degree d.*

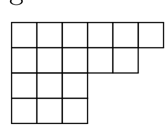
Proof. This follows from Exercises 3.4 and 4.2. □

Exercise 4.4 (Stability of Schur functions)**.** *Show that substituting $x_n = 0$ in the Schur function $s_\lambda(x_1, \ldots, x_n)$ with n variables gives the corresponding Schur function $s_\lambda(x_1, \ldots, x_{n-1})$ with $n - 1$ variables.*

5. **Pieri's rule.** The set of integer partitions is endowed with the *containment order*. We say that a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ *contains* a partition $\mu = (\mu_1, \ldots, \mu_m)$ if $l \geq m$, and $\lambda_i \geq \mu_i$ for every $i = 1, \cdots, m$. We write $\lambda \supset \mu$ or $\mu \subset \lambda$. Recall that the Young diagram of the partition λ is the set of points

$$\{(i, j) \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}.$$

Visually, each node (i, j) of the Young diagram is replaced by a box, and the box corresponding to (i, j) is placed in the ith row and jth column (matrix notation). Thus, the Young diagram of $\lambda = (6, 5, 3, 3)$ is depicted by:



Note that containment of partitions is nothing but the containment relation on their Young diagrams. By abuse of notation, we will also use λ to denote the Young diagram of λ.

By a skew-shape, we mean a difference of Young diagrams $\lambda \setminus \mu$, where $\lambda \supset \mu$. We write λ/μ for this skew-shape. A skew-shape is called a *horizontal* *strip* (respectively, a *vertical strip*) if it has at most one element in each vertical column (respectively, horizontal row).

Theorem 5.1. *For every partition λ, and every positive integer k,*

$$s_\lambda h_k = \sum_\mu s_\mu,$$

where the sum runs over all partitions $\mu \supset \lambda$ such that μ/λ is a horizontal strip of size k. Dually,

$$s_\lambda e_k = \sum_\mu s_\mu,$$

where the sum runs over all partitions $\mu \supset \lambda$ such that μ/λ is a vertical strip of size k.

Proof. The first identity is equivalent to showing that:

$$a_{\lambda+\delta} \sum_{\alpha_1+\cdots+\alpha_n=k} x_1^{\alpha_1}\cdots x_n^{\alpha_n} = \sum_\mu a_{\mu+\delta},$$

the sum on the right being over all partitions $\mu \supset \lambda$ such that μ/λ is a horizontal strip.

Writing $\alpha = (\alpha_1, \dots, \alpha_n)$, the sum on the left hand side can be regarded as a sum of determinants:

$$a_{\lambda+\delta} \sum_{|\alpha|=k} x_1^{\alpha_1}\cdots x_n^{\alpha_n} = \sum_{|\alpha|=k} a_{\lambda+\alpha+\delta}. \tag{5}$$

Suppose there exists an integer α_i such that $\alpha_i > \lambda_i - \lambda_{i+1}$ (in other words, $(\lambda+\alpha)/\lambda$ is not a horizontal strip), then define $\beta = (\beta_1, \dots, \beta_n)$ by $\beta_i = \alpha_{i+1} - (\lambda_i - \lambda_{i+1} + 1)$, $\beta_{i+1} = \alpha_i + (\lambda_i - \lambda_{i+1} + 1)$, and $\beta_j = \alpha_j$ for all $j \notin \{i, i+1\}$. Then $a_{\lambda+\alpha+\delta} = -a_{\lambda+\beta+\delta}$. So the only terms that survive on the right hand side of (5) are of the form $a_{\mu+\delta}$, where μ/λ is a horizontal strip.

The proof of the second identity in the theorem is similar (in fact, a little simpler) and is left to the reader as an exercise. □

As a special case of Pieri's rule, we have:

Corollary 5.2. *For every positive integer k,*

$$s_{(k)} = h_k, \text{ and } s_{(1^k)} = e_k.$$

6. **Schur to Complete and Elementary via Tableaux.** Pieri's rule allows us to compute the complete and elementary symmetric functions h_λ and e_λ in terms of Schur functions.

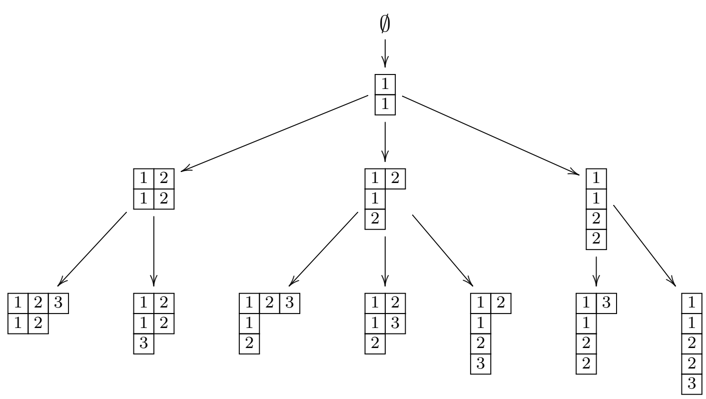
Example 6.1. *Repeated application of Pieri's rule gives an expansion of $e_{(2,2,1)} = e_2e_2e_1$ as:*

$$\begin{tikzcd}
 & & & e_2e_2e_1 \arrow[d] & & & \\
 & & & s_{(1^2)}e_2e_1 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & s_{(2^2)}e_1 \arrow[dl] \arrow[d] & & s_{(2,1^2)}e_1 \arrow[dl] \arrow[d] \arrow[dr] & & s_{(1^4)}e_1 \arrow[d] \arrow[dr] & \\
s_{(3,2)} & s_{(2^2,1)} & s_{(3,1^2)} & s_{(2^2,1)} & s_{(2,1^3)} & s_{(2,1^3)} & s_{(1^5)}
\end{tikzcd}$$

giving:

$$e_{(2^2,1)} = s_{(3,2)} + 2s_{(2^2,1)} + s_{(3,1^2)} + 2s_{(2,1^3)} + s_{(1^5)}.$$

The steps going from the first line of the above calculation to each term of the last line can be recorded by putting numbers into Young diagrams:



The boxes in the vertical strip added at the ith stage are filled with i.

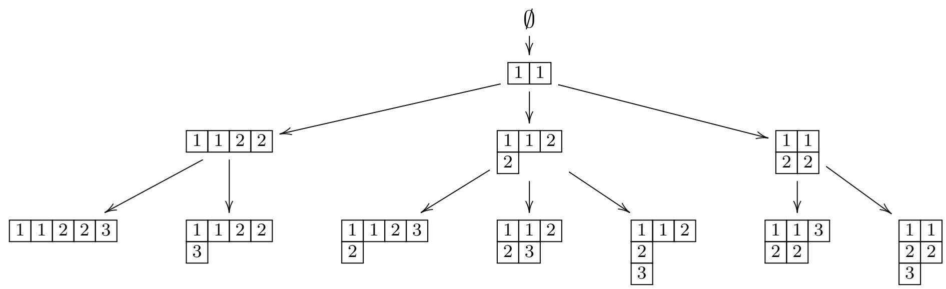
Example 6.2. *Repeated application of Pieri's rule gives an expansion of $h_{(2,2,1)} = h_2h_2h_1$ as:*

$$\begin{tikzcd}
 & & & h_2h_2h_1 \arrow[d] & & & \\
 & & & s_{(2)}h_2h_1 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & s_{(4)}h_1 \arrow[dl] \arrow[d] & & s_{(3,1)}h_1 \arrow[dl] \arrow[d] \arrow[dr] & & s_{(3,2)}h_1 \arrow[d] \arrow[dr] & \\
h_{(5)} & s_{(4,1)} & s_{(4,1)} & s_{(3,2)} & s_{(3,1^2)} & s_{(3,2)} & s_{(2,2,1)}
\end{tikzcd}$$

giving:

$$h_{(2^2,1)} = s_{(5)} + 2s_{(4,1)} + 2s_{(3,2)} + s_{(3,1^2)} + s_{(2,2,1)}.$$

The steps going from the first line of the above calculation to each term of the last line can be recorded by putting numbers into Young diagrams:



The boxes in the horizontal strip added at the ith stage are filled with i.

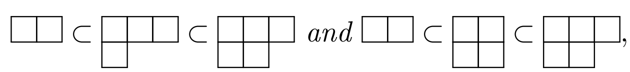
Definition 6.3 (Semistandard tableau)**.** A semistandard tableau of shape $\lambda = (\lambda_1, \dots, \lambda_l)$ and type $\mu = (\mu_1, \dots, \mu_m)$ is the Young diagram of λ filled with numbers $1, \dots, m$ such that the number i appears μ_i times, the numbers weakly increase along rows, and strictly increase along columns.

Exercise 6.4. *Semistandard tableaux of shape λ and type μ correspond to chains of integer partitions*

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \dots \subset \lambda^{(m)} = \lambda$$

where $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip of size μ_i.

Example 6.5. *The semistandard tableau of type $(3,2)$ and type $(2,2,1)$ are $\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2}\end{array}$. They correspond to the chains:*



respectively. As illustrated in Example 6.2, the coefficient of $s_{(3,2)}$ in the complete symmetric function $h_{(2,2,1)}$ is the number of semistandard tableau of shape $(3,2)$ and type $(2,2,1)$.

Definition 6.6 (Kostka number)**.** Given two partitions λ and μ, the Kostka number $K_{\lambda\mu}$ is the number of semistandard tableau of shape λ and type μ.

Exercise 6.7. *For every partition λ, show that $K_{\lambda\lambda} = 1$.*

Exercise 6.8 (f-number)**.** *The f-number of a partition λ of n is defined to be the Kostka number $K_{\lambda,(1^n)}$, and is denoted f_λ.*

Exercise 6.9. *For a partition λ, let λ^- denote the set of all partitions whose Young diagram can be obtained by removing one box from the Young diagram of λ. Show that $f_\lambda = \sum_{\mu \in \lambda^-} f_\mu$.*

Exercise 6.10. *A hook is a partition of the form $h(a,b) = (a+1, 1^b)$. Show that $f_{h(a,b)} = \binom{a+b}{a}$.*

In order to understand the expansion of elementary symmetric functions we would need a variant of semistandard tableaux, one where the difference between successive shapes are vertical strips, rather than horizontal strips. However, it has become common practice to *conjugate* partitions instead:

Definition 6.11 (Conjugate of a partition)**.** The *conjugate* of a partition λ is the partition λ' whose Young diagram is given by:

$$\lambda' = \{(j,i) \mid (i,j) \in \lambda\}.$$

In other words, the Young diagram of λ' is the reflection of the Young diagram of λ about its principal diagonal.

Clearly $\lambda \mapsto \lambda'$ is an involution. For example, if $\lambda = (2,2,1)$, then $\lambda' = (3,2)$.

Exercise 6.12. *Semistandard tableaux of shape λ' and type μ correspond to chains of integer partitions*

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(m)} = \lambda$$

where $\lambda^{(i)}/\lambda^{(i-1)}$ is a vertical *strip of size μ_i.*

Theorem 6.13. *The expansion of complete symmetric functions in terms of Schur functions is given by:*

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda.$$

Dually, the extension of elementary symmetric functions in terms of Schur functions is given by:

$$e_\mu = \sum_\lambda K_{\lambda'\mu} s_\lambda.$$

7. **Triangularity of Kostka Numbers.** In this section we give a necessary and sufficient condition for the positivity of Kostka number. As usual, take $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$. Suppose $K_{\lambda\mu} > 0$. Then there exists a semistandard tableau t of shape λ and type μ. Since the columns of t are strictly increasing, all the 1's in t must occur in its first row, so $\lambda_1 \geq \mu_1$. Also, all the 2's must occur in the first two rows (along with all the 1's), so $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. More generally, all the numbers $1, \ldots, i$ for $i = 1, \ldots, m$ should occur in the first m rows of t. We have:

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots \mu_i \text{ for } i = 1, \ldots, m. \tag{6}$$

Definition 7.1. We say that an integer partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ *dominates* the integer partition $\mu = (\mu_1, \ldots, \mu_m)$ $|\lambda| = |\mu|$ and if the condition (6) holds. When this happens we write $\lambda \rhd \mu$. This relation defines a partial order on the set of all integer partitions of n for any non-negative integer n.

Exercise 7.2. *Show that (n) is maximal and (1^n) is minimal among all the integer partitions of n. What is the smallest integer n for which the dominance order on partitions of n is not a linear order?*

Theorem 7.3 (Triangularity of Kostka Numbers)**.** *Given partition λ and μ of an integer n, $K_{\lambda\mu} > 0$ if and only if $\lambda \rhd \mu$.*

Proof. We have already seen that if $K_{\lambda\mu} > 0$, then $\lambda \rhd \mu$. While reading the proof of the converse, it is helpful to keep in mind Example 7.4 below. Suppose that $\lambda \rhd \mu$. Then $\lambda_1 \geq \mu_1 \geq \mu_m$. Therefore, the Young diagram of λ has at least μ_m cells in its first row, or in other words, it has at least μ_m columns. Choose the smallest integer i for which $\lambda_i \geq \mu_m$. Fill the bottom-most box in the λ_{i+1} leftmost columns with m. Also, from the ith row, fill the rightmost $\mu_m - \lambda_{i+1}$ boxes with m. The remaining (unfilled) boxes in the Young diagram of λ now form the Young diagram of the partition

$$\eta = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - \mu_m + \lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_l),$$

a partition with $l-1$ parts. Writing $(\eta_1, \ldots, \eta_{l-1})$ for the parts of η, note that, since the first $i-1$ parts of η are the same as those of λ, we have:

$$\eta_1 + \cdots + \eta_j \geq \mu_1 + \cdots + \mu_j$$

for $j \leq i-1$. For $j \geq i$, we have

$$\begin{aligned} \eta_1 + \cdots + \eta_j &= \lambda_1 + \cdots + \lambda_{j+1} - \mu_m \\ &\geq \mu_1 + \cdots + \mu_j + \mu_{j+1} - \mu_m \\ &\geq \mu_1 + \cdots + \mu_j. \end{aligned}$$

It follows that $\eta \rhd (\mu_1, \ldots, \mu_{m-1})$. The result now follows by induction in m. $\square$

Example 7.4. *Consider the case where $\lambda = (7,3,2)$ and $\mu = (4,4,4)$. Then the smallest integer i such that $\lambda_i \geq 4$ is 1. Accordingly, we enter 3 into the bottom-most boxes in the three leftmost columns, and also into one rightmost box in the first row:*

$$\ytableaushort{{}{}{}{}{}{}3,{}{}3,33}$$

We are left with the problem of finding a semistandard tableau of shape $(6,2)$ and type $(4,4)$. Recursively applying our process to this smaller problem gives:

$$\ytableaushort{{}{}{}{}223,223,33},$$

and finally the desired tableau

$$\ytableaushort{1111223,223,33}.$$

Theorem 7.5. *The complete symmetric functions:*

$$\{h_\mu \mid \mu \text{ is a partition of } d \text{ with at most } n \text{ parts}\}$$

and the elementary symmetric functions:

$$\{e_\mu \mid \mu \text{ is a partition of } d \text{ with } \mu_1 \leq n\}$$

form bases of the space of homogeneous symmetric polynomials of degree d in variables $x_1, \dots, x_n$.

Proof. In view of the triangularity of Kostka numbers (Theorem 7.3) and the fact that $K_{\lambda\lambda} = 1$ (Exercise 6.7) the theorem follows from Theorem 6.13. □

8. **The Lindström-Gessel-Viennot Lemma.** Let R be a commutative ring. Let S be any set of points, and $v : S \times S \to R$ be any function (we think of w as a *weight function.* Given $s, t \in S$, a path in S from s to t is is a sequence $\omega = (s = s_0, s_1, \dots, s_k = t)$ of distinct points in S. We denote this by $\omega : s \to t$. The weight of the path ω is defined to be:

$$v(\omega) = v(s_0, s_1)v(s_1, s_2) \cdots v(s_{k-1}, s_k).$$

Definition 8.1 (non-crossing paths)**.** Two paths $\omega = (s_0, \dots, s_k)$ and $\eta = (t_0, \dots, t_l)$ are said to be non-crossing if $s_i \neq t_j$ for all $0 \leq i \leq k$ and $0 \leq t \leq l$.

Fix points $A_1, \dots, A_n$ and $B_1, \dots, B_n$ in S, and define an $n \times n$ matrix (a_{ij}) by:

$$a_{ij} = \sum_{\omega : A_i \to B_j} v(\omega).$$

Theorem 8.2 (Lindström-Gessel-Viennot Lemma)**.** *The determinant of the matrix (a_{ij}) defined above is given by:*

$$\det(a_{ij}) = \sum_{\omega_i : A_i \to B_i} v(\omega_1) \cdots v(\omega_n),$$

where the sum is over all n-tuples $(\omega_1, \dots, \omega_n)$ of pairwise non-crossing paths $\omega_i : A_i \to B_i$.

9. **The Jacobi-Trudi Identities.**

$$s_\lambda = \det(h_{\lambda_i - i + j}), \quad s_{\lambda'} = \det(e_{\lambda_i - i + j}).$$

For the first Jacobi-Trudi identity take S to be the positive cone in the the integer lattice:

$$S = \{(i, j) \mid i \geq 0, \ j > 0 \text{ are integers}\}.$$

Set the weight $v((i, j), (i + 1, j))$ of each rightward horizontal edge to be x_j for $j = 1, \dots, n$, the weight of each upward vertical edge $v((i, j), (i, j + 1))$ to be 1 for all $j = 1, \dots, n - 1$. The remaining weights are all zero.

Lemma 9.1. *For all integers $i > 0$ and $j \geq 0$, we have:*

$$\sum_{\omega : (i,n) \to (i+j,1)} v(\omega) = h_j(x_1, \dots, x_n).$$

Proof. Only rightward or upward steps have non-zero weights. So every path with non-zero weight is composed of unit upward and rightward steps. A path with non-zero weight from $(i, 1)$ to $(i + j, n)$ must have exactly j rightward steps, say in rows $1 \leq i_1 \geq i_2 \cdots \leq i_j \leq n$. The weight of such

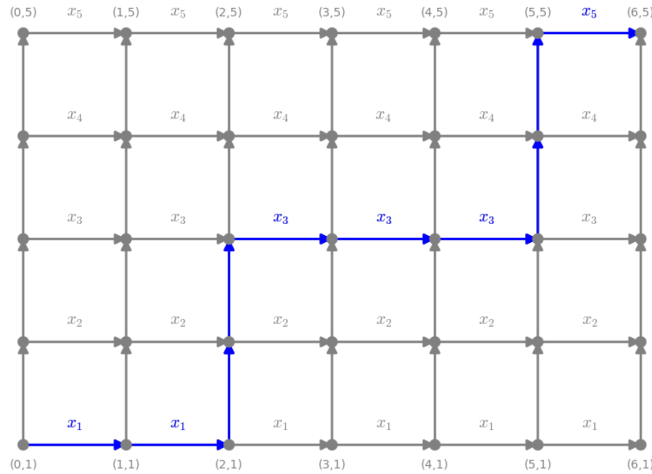


FIGURE 1. A path from $(0,5)$ to $(6,1)$ whose weight is the monomial $x_1^2 x_3^3 x_5$ in $h_6(x_1, \ldots, x_5)$.

a path is $x_{i_1} \cdots x_{i_j}$, and hence, the sum of the weights of all such paths is $h_j(x_1, \ldots, x_n)$. For an example, see Fig. 1. □

Let λ be a partition of d with at most n parts. By appending 0's to the right of λ, think of λ as an n-tuple $(\lambda_1, \ldots, \lambda_n)$ with weakly decreasing coordinates. Let $A_i = (n-i, 1)$ and $B_i = (\lambda_i + n - i, n)$ for $i = 1, \ldots, n$. Then by Lemma 9.1,

$$\sum_{\omega: A_i \to B_j} v(\omega) = h_{\lambda_j + i - j}.$$

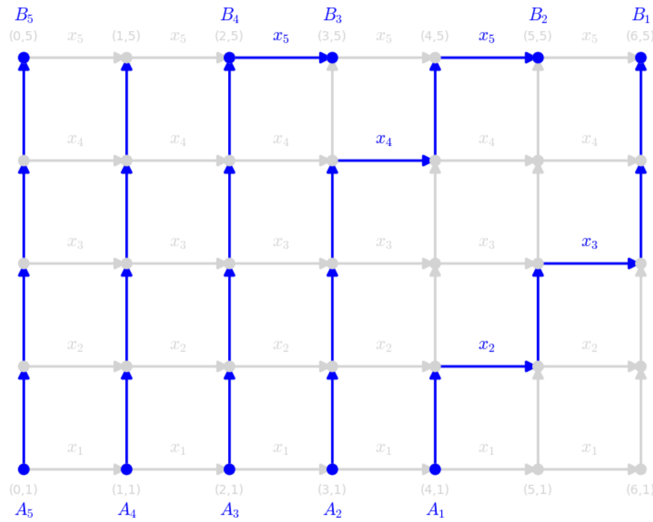


FIGURE 2. Non-intersecting paths corresponding to the tableau $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$