AN INTRODUCTION TO SCHUR FUNCTIONS

AMRITANSHU PRASAD

1. **Symmetric Functions.** Consider polynomials in n variables $x_1, \ldots, x_n$. Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, let x^α denote the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. A *symmetric polynomial* is a polynomial of the form

$$f(x_1, \ldots, x_n) = \sum_\alpha c_\alpha x^\alpha,$$

where, for any permutation $w \in S_n$,

$$c_{(\alpha_1,\ldots,\alpha_n)} = c_{(\alpha_{w(1)},\ldots,\alpha_{w(n)})}.$$

The integer partition λ obtained by sorting the coordinates of α is called the *shape* of α, denoted $\lambda(\alpha)$. The most obvious example of a symmetric polynomial in n variables is the *monomial symmetric function*, defined for each integer partition λ:

$$m_\lambda = \sum_{\lambda(\alpha)=\lambda} c_\alpha x^\alpha.$$

Note that m_λ is homogeneous of degree $|\lambda|$ (the sum of the parts of λ).

Exercise 1.1. *Take $n = 4$. Compute the monomial symmetric functions $m_{(3)}$, $m_{(2,1)}$, and $m_{(1^3)}$.*

Theorem 1.2. *The polynomials $m_\lambda(x_1, \ldots, x_n)$, as λ runs over all the integer partition of d, form a basis for the space of homogeneous symmetric polynomials of degree d in n variables.*

2. **Complete and Elementary Symmetric Polynomials.** Recall that the coefficients of a polynomial are symmetric polynomials in its roots:

$$(t - x_1)(t - x_2) \cdots (t - x_n) = t^n - e_1(x_1, \ldots, x_n)t^{n-1} + \cdots + (-1)^n e_n(x_1, \ldots, x_n), \tag{1}$$

where coefficient $e_i(x_1, \ldots, x_n)$ of t^{n-i} is given by:

$$e_i(x_1, \ldots, x_n) = \sum_{1\leq j_1<\cdots<j_i\leq n} x_{j_1} x_{j_2} \cdots x_{j_i}. \tag{2}$$

The polynomial e_i is called the ith *elementary symmetric polynomial*. By convention, $e_i(x_1, \ldots, x_n) = 0$, for $i > n$.

The identity (1) can be written more elegantly as:

$$(1+tx_1)\cdots(1+tx_n) = \sum_{i=0}^{n} e_i(x_1,\cdots,x_n)t^i.$$

Dually[1], the *complete symmetric polynomials* are defined by the formal identity:

$$\frac{1}{(1-x_1t)\cdots(1-x_nt)} = \sum_{i=0}^{\infty} h_i(x_1,\cdots,x_n)t^i.$$

Example 2.1. *In three variables:*

$$\begin{aligned} e_2(x_1,x_2,x_3) &= x_1x_2 + x_1x_3 + x_2x_3, \\ h_2(x_1,x_2,x_3) &= x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_2^3. \end{aligned}$$

Exercise 2.2. *Show that*

$$h_i(x_1,\cdots x_n) = \sum_{1\leq j_1\leq\cdots\leq j_i\leq n} x_{j_1}\cdots x_{j_i}.$$

More generally, for any integer partition $\lambda = (\lambda_1,\ldots,\lambda_l)$, define:

$$\begin{aligned} h_\lambda &= h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_l}, \\ e_\lambda &= e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_l}. \end{aligned}$$

Theorem 2.3. *Given partitions $\lambda = (\lambda_1,\ldots,\lambda_l)$ and $\mu = (\mu_1,\cdots,\mu_m)$ of d, let $M_{\lambda\mu}$ denote the number of matrices (a_{ij}) with non-negative integer entries whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j. Then*

$$h_\lambda = \sum_\mu M_{\lambda\mu}m_\mu.$$

Dually, let $= N_{\lambda\mu}$ denote the number of integer matrices (a_{ij}) with entries 0 or 1, whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j.

$$e_\lambda = \sum_\mu N_{\lambda\mu}m_\mu.$$

Proof. To prove the second identity involving elementary symmetric functions, note that a monomial in the expansion of

$$e_\lambda = \prod_{i=1}^{l} \sum_{j_1<\cdots<j_{\lambda_j}} x_{j_1}\cdots x_{j_{\lambda_i}}$$

is a product of summands, one chosen from each of the l factors. Construct an $l\times m$ matrix (a_{ij}) corresponding to such a choice as follows: if the summand $x_{j_1}\cdots x_{j_{\lambda_i}}$ is chosen from the ith factor, then set the entries $a_{i,j_1},\ldots,a_{i,j_{\lambda_j}}$ to be 1 (the remaining entries of the ith row are 0). Clearly

[1]We will refer to the replacing of $(1+u)$ by $(1-u)^{-1}$ in a formal identity as *dualization*.

the ith row of such a matrix sums to λ_i. The monomial corresponding to this choice is x^μ if, for each j, the the number of i for which x_j appears in the monomial corresponding to the $i$gth row is μ_j. This is just the sum of the jth column of the matrix (a_{ij}). It follows that the coefficient of x^μ, and hence the coefficient of m_μ in the expansion of e_λ in the basis of monomial symmetric functions of degree n, is $N_{\lambda\mu}$.

A similar proof can be given for the first identity involving complete symmetric functions. The only difference is that variables may be repeated in the monomials that appear in h_i. Counting the number of repetitions (instead of just recording 0 or 1) gives non-negative integer matrices. $\square$

3. **Alternating Polynomials.** An *alternating polynomial* in $x_1, \ldots, x_n$ is of the form:

$$f(x_1, \ldots, x_n) = \sum_\alpha c_\alpha x_\alpha, \tag{3}$$

where, $c_{w(\alpha)} = \epsilon(w)c_\alpha$ for every multiindex α as in Section 1. Here $\epsilon : S_n \to \{\pm 1\}$ denotes the sign function. Equivalently, an alternating polynomial is one whose sign is reversed upon the interchange of any two variables.

Exercise 3.1. *If α is a multiindex where $\alpha_i = \alpha_j$ for some $i \neq j$, then $c_\alpha = 0$.*

In particular, every monomial in an alternating polynomial must be composed of distinct powers. Moreover, the polynomial is completely determined by the coefficients c_α of strictly decreasing multiindices, namely, multiindices of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 > \cdots > \alpha_n$.

Exercise 3.2. *Let $\delta = (n-1, n-2, \ldots, 1, 0)$. Given an integer partition with at most n parts, we will pad it with 0's so that it can be regarded as a weakly decreasing multiindex of length n. Then $\lambda \mapsto \lambda + \delta$ is a bijection from the set of integer partitions with at most n onto the set of strictly decreasing multiindices.*

Example 3.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a weakly decreasing multiindex. The polynomial:*

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + n - j})$$

is alternating, with unique strictly decreasing monomial $x^{\lambda+\delta}$.

Exercise 3.4. *The alternating polynomial of the form* (3) *is equal to*

$$\sum_\lambda c_\lambda a_{\lambda+\delta},$$

the sum being over all weakly decreasing multiindices.

4. **Cauchy's Bialternant Form of a Schur Function.** The simplest polynomial of the form $a_{\lambda+\delta}$ arises when $\lambda = 0$; a_δ is the Vandermonde determinant:

$$a_\delta = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Exercise 4.1. *Show that, for every weakly decreasing multiindex λ, $a_{\lambda+\delta}$ is divisible by a_δ in the ring of polynomials in $x_1, \ldots, x_n$.*

Exercise 4.2. *Show that $f \mapsto f a_\delta$ is an isomorphism of the space of symmetric polynomials in $x_1, \ldots, x_n$ of degree d onto the space of alternating polynomials of degree $d + \binom{n}{2}$.*

The above exercise gives the historically oldest definition of Schur functions—*Cauchy's bialternant formula*:

$$s_\lambda(x_1, \ldots, x_n) = a_{\lambda+\delta}/a_\delta, \tag{4}$$

for any partition λ with at most n parts. If λ has more than n parts, set $s_\lambda(x_1, \ldots, x_n) = 0$. This is clearly a symmetric function of degree $|\lambda|$.

Theorem 4.3. *As λ runs over all integer partitions of d with at most n parts, the Schur functions $s_\lambda(x_1, \ldots, x_n)$ form a basis of the space of all homogeneous symmetric functions in $x_1, \ldots, x_n$ of degree d.*

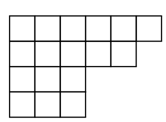
Proof. This follows from Exercises 3.4 and 4.2. □

Exercise 4.4 (Stability of Schur functions)**.** *Show that substituting $x_n = 0$ in the Schur function $s_\lambda(x_1, \ldots, x_n)$ with n variables gives the corresponding Schur function $s_\lambda(x_1, \ldots, x_{n-1})$ with $n - 1$ variables.*

5. **Pieri's rule.** The set of integer partitions is endowed with the *containment order*. We say that a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ *contains* a partition $\mu = (\mu_1, \ldots, \mu_m)$ if $l \geq m$, and $\lambda_i \geq \mu_i$ for every $i = 1, \cdots, m$. We write $\lambda \supset \mu$ or $\mu \subset \lambda$. Recall that the Young diagram of the partition λ is the set of points

$$\{(i, j) \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}.$$

Visually, each node (i, j) of the Young diagram is replaced by a box, and the box corresponding to (i, j) is placed in the ith row and jth column (matrix notation). Thus, the Young diagram of $\lambda = (6, 5, 3, 3)$ is depicted by:



Note that containment of partitions is nothing but the containment relation on their Young diagrams. By abuse of notation, we will also use λ to denote the Young diagram of λ.

By a skew-shape, we mean a difference of Young diagrams $\lambda \setminus \mu$, where $\lambda \supset \mu$. We write λ/μ for this skew-shape. A skew-shape is called a *horizontal strip* (respectively, a *vertical strip*) if it has at most one element in each vertical column (respectively, horizontal row).

Theorem 5.1. *For every partition λ, and every positive integer k,*

$$s_\lambda h_k = \sum_\mu s_\mu,$$

where the sum runs over all partitions $\mu \supset \lambda$ such that μ/λ is a horizontal strip of size k. Dually,

$$s_\lambda e_k = \sum_\mu s_\mu,$$

where the sum runs over all partitions $\mu \supset \lambda$ such that μ/λ is a vertical strip of size k.

Proof. The first identity is equivalent to showing that:

$$a_{\lambda+\delta} \sum_{\alpha_1+\cdots+\alpha_n=k} x_1^{\alpha_1}\cdots x_n^{\alpha_n} = \sum_\mu a_{\mu+\delta},$$

the sum on the right being over all partitions $\mu \supset \lambda$ such that μ/λ is a horizontal strip.

Writing $\alpha = (\alpha_1, \dots, \alpha_n)$, the sum on the left hand side can be regarded as a sum of determinants:

$$a_{\lambda+\delta} \sum_{|\alpha|=k} x_1^{\alpha_1}\cdots x_n^{\alpha_n} = \sum_{|\alpha|=k} a_{\lambda+\alpha+\delta}. \tag{5}$$

Suppose there exists an integer α_i such that $\alpha_i > \lambda_i - \lambda_{i+1}$ (in other words, $(\lambda+\alpha)/\lambda$ is not a horizontal strip), then define $\beta = (\beta_1, \dots, \beta_n)$ by $\beta_i = \alpha_{i+1} - (\lambda_i - \lambda_{i+1} + 1)$, $\beta_{i+1} = \alpha_i + (\lambda_i - \lambda_{i+1} + 1)$, and $\beta_j = \alpha_j$ for all $j \notin \{i, i+1\}$. Then $a_{\lambda+\alpha+\delta} = -a_{\lambda+\beta+\delta}$. So the only terms that survive on the right hand side of (5) are of the form $a_{\mu+\delta}$, where μ/λ is a horizontal strip.

The proof of the second identity in the theorem is similar (in fact, a little simpler) and is left to the reader as an exercise. □

As a special case of Pieri's rule, we have:

Corollary 5.2. *For every positive integer k,*

$$s_{(k)} = h_k, \text{ and } s_{(1^k)} = e_k.$$

6. **Schur to Complete and Elementary via Tableaux.** Pieri's rule allows us to compute the complete and elementary symmetric functions h_λ and e_λ in terms of Schur functions.

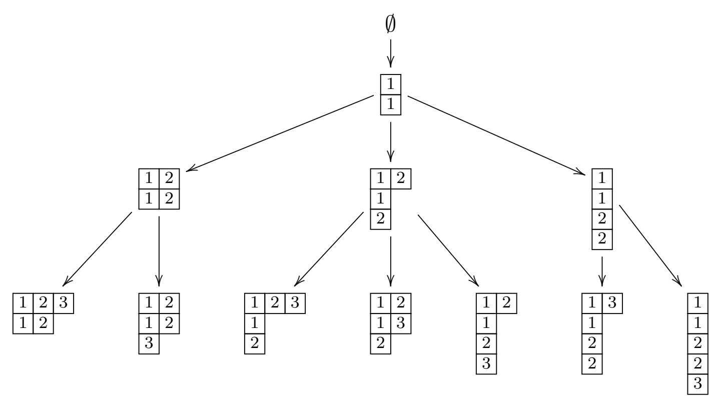
Example 6.1. *Repeated application of Pieri's rule gives an expansion of* $e_{(2,2,1)} = e_2e_2e_1$ *as:*

$$\begin{tikzcd}
& & & e_2e_2e_1 \arrow[d] & & & \\
& & & s_{(1^2)}e_2e_1 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
& s_{(2^2)}e_1 \arrow[dl] \arrow[d] & & s_{(2,1^2)}e_1 \arrow[dl] \arrow[d] \arrow[dr] & & s_{(1^4)}e_1 \arrow[d] \arrow[dr] & \\
s_{(3,2)} & s_{(2^2,1)} & s_{(3,1^2)} & s_{(2^2,1)} & s_{(2,1^3)} & s_{(2,1^3)} & s_{(1^5)}
\end{tikzcd}$$

giving:

$$e_{(2^2,1)} = s_{(3,2)} + 2s_{(2^2,1)} + s_{(3,1^2)} + 2s_{(2,1^3)} + s_{(1^5)}.$$

The steps going from the first line of the above calculation to each term of the last line can be recorded by putting numbers into Young diagrams:



The boxes in the vertical strip added at the i*th stage are filled with* i*.*

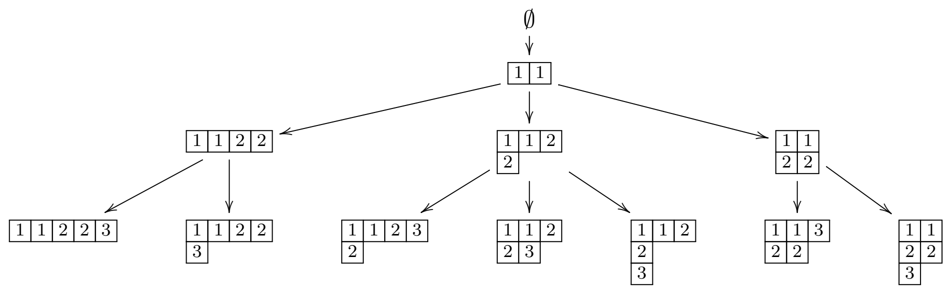
Example 6.2. *Repeated application of Pieri's rule gives an expansion of* $h_{(2,2,1)} = h_2h_2h_1$ *as:*

$$\begin{tikzcd}
& & & h_2h_2h_1 \arrow[d] & & & \\
& & & s_{(2)}h_2h_1 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
& s_{(4)}h_1 \arrow[dl] \arrow[d] & & s_{(3,1)}h_1 \arrow[dl] \arrow[d] \arrow[dr] & & s_{(3,2)}h_1 \arrow[d] \arrow[dr] & \\
h_{(5)} & s_{(4,1)} & s_{(4,1)} & s_{(3,2)} & s_{(3,1^2)} & s_{(3,2)} & s_{(2,2,1)}
\end{tikzcd}$$

giving:

$$h_{(2^2,1)} = s_{(5)} + 2s_{(4,1)} + 2s_{(3,2)} + s_{(3,1^2)} + s_{(2,2,1)}.$$

The steps going from the first line of the above calculation to each term of the last line can be recorded by putting numbers into Young diagrams:



The boxes in the horizontal strip added at the ith stage are filled with i.

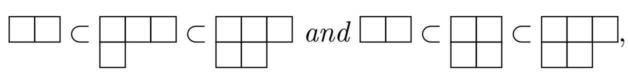
Definition 6.3 (Semistandard tableau)**.** A semistandard tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_l)$ and type $\mu = (\mu_1, \ldots, \mu_m)$ is the Young diagram of λ filled with numbers $1, \ldots, m$ such that the number i appears μ_i times, the numbers weakly increase along rows, and strictly increase along columns.

Exercise 6.4. *Semistandard tableaux of shape λ and type μ correspond to chains of integer partitions*

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(m)} = \lambda$$

where $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip of size μ_i.

Example 6.5. *The semistandard tableau of type $(3, 2)$ and type $(2, 2, 1)$ are* $\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$ *and* $\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$*. They correspond to the chains:*



respectively. As illustrated in Example 6.2, the coefficient of $s_{(3,2)}$ in the complete symmetric function $h_{(2,2,1)}$ is the number of semistandard tableau of shape $(3, 2)$ and type $(2, 2, 1)$.

Definition 6.6 (Kostka number)**.** Given two partitions λ and μ, the Kostka number $K_{\lambda\mu}$ is the number of semistandard tableau of shape λ and type μ.

Exercise 6.7. *For every partition λ, show that $K_{\lambda\lambda} = 1$.*

Exercise 6.8 (f-number)**.** *The f-number of a partition λ of n is defined to be the Kostka number $K_{\lambda,(1^n)}$, and is denoted f_λ.*

Exercise 6.9. *For a partition λ, let λ^- denote the set of all partitions whose Young diagram can be obtained by removing one box from the Young diagram of λ. Show that $f_\lambda = \sum_{\mu \in \lambda^-} f_\mu$.*

Exercise 6.10. *A hook is a partition of the form $h(a, b) = (a+1, 1^b)$. Show that $f_{h(a,b)} = \binom{a+b}{a}$.*

In order to understand the expansion of elementary symmetric functions we would need a variant of semistandard tableaux, one where the difference between successive shapes are vertical strips, rather than horizontal strips. However, it has become common practice to *conjugate* partitions instead:

Definition 6.11 (Conjugate of a partition)**.** The *conjugate* of a partition λ is the partition λ' whose Young diagram is given by:

$$\lambda' = \{(j,i) \mid (i,j) \in \lambda\}.$$

In other words, the Young diagram of λ' is the reflection of the Young diagram of λ about its principal diagonal.

Clearly $\lambda \mapsto \lambda'$ is an involution. For example, if $\lambda = (2,2,1)$, then $\lambda' = (3,2)$.

Exercise 6.12. *Semistandard tableaux of shape λ' and type μ correspond to chains of integer partitions*

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(m)} = \lambda$$

where $\lambda^{(i)}/\lambda^{(i-1)}$ is a vertical *strip of size μ_i.*

Theorem 6.13. *The expansion of complete symmetric functions in terms of Schur functions is given by:*

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda.$$

Dually, the extension of elementary symmetric functions in terms of Schur functions is given by:

$$e_\mu = \sum_\lambda K_{\lambda'\mu} s_\lambda.$$

7. **Triangularity of Kostka Numbers.** In this section we give a necessary and sufficient condition for the positivity of Kostka number. As usual, take $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$. Suppose $K_{\lambda\mu} > 0$. Then there exists a semistandard tableau t of shape λ and type μ. Since the columns of t are strictly increasing, all the 1's in t must occur in its first row, so $\lambda_1 \geq \mu_1$. Also, all the 2's must occur in the first two rows (along with all the 1's), so $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. More generally, all the numbers $1, \ldots, i$ for $i = 1, \ldots, m$ should occur in the first m rows of t. We have:

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots \mu_i \text{ for } i = 1, \ldots, m. \tag{6}$$

Definition 7.1. We say that an integer partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ *dominates* the integer partition $\mu = (\mu_1, \ldots, \mu_m)$ $|\lambda| = |\mu|$ and if the condition (6) holds. When this happens we write $\lambda \rhd \mu$. This relation defines a partial order on the set of all integer partitions of n for any non-negative integer n.

Exercise 7.2. *Show that (n) is maximal and (1^n) is minimal among all the integer partitions of n. What is the smallest integer n for which the dominance order on partitions of n is not a linear order?*

Theorem 7.3 (Triangularity of Kostka Numbers). *Given partition λ and μ of an integer n, $K_{\lambda\mu} > 0$ if and only if $\lambda \rhd \mu$.*

Proof. We have already seen that if $K_{\lambda\mu} > 0$, then $\lambda \rhd \mu$. While reading the proof of the converse, it is helpful to keep in mind Example 7.4 below. Suppose that $\lambda \rhd \mu$. Then $\lambda_1 \geq \mu_1 \geq \mu_m$. Therefore, the Young diagram of λ has at least μ_m cells in its first row, or in other words, it has at least μ_m columns. Choose the smallest integer i for which $\lambda_i \geq \mu_m$. Fill the bottom-most box in the λ_{i+1} leftmost columns with m. Also, from the ith row, fill the rightmost $\mu_m - \lambda_{i+1}$ boxes with m. The remaining (unfilled) boxes in the Young diagram of λ now form the Young diagram of the partition

$$\eta = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - \mu_m + \lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_l),$$

a partition with $l-1$ parts. Writing $(\eta_1, \dots, \eta_{l-1})$ for the parts of η, note that, since the first $i-1$ parts of η are the same as those of λ, we have:

$$\eta_1 + \cdots + \eta_j \geq \mu_1 + \cdots + \mu_j$$

for $j \leq i-1$. For $j \geq i$, we have

$$\begin{aligned} \eta_1 + \cdots + \eta_j &= \lambda_1 + \cdots + \lambda_{j+1} - \mu_m \\ &\geq \mu_1 + \cdots + \mu_j + \mu_{j+1} - \mu_m \\ &\geq \mu_1 + \cdots + \mu_j. \end{aligned}$$

It follows that $\eta \rhd (\mu_1, \dots, \mu_{m-1})$. The result now follows by induction in m. ☐

Example 7.4. *Consider the case where $\lambda = (7, 3, 2)$ and $\mu = (4, 4, 4)$. Then the smallest integer i such that $\lambda_i \geq 4$ is 1. Accordingly, we enter 3 into the bottom-most boxes in the three leftmost columns, and also into one rightmost box in the first row:*

$$\begin{array}{|c|c|c|c|c|c|c|} \hline \ & \ & \ & \ & \ & \ & 3 \\ \hline \ & \ & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} \end{array}$$

We are left with the problem of finding a semistandard tableau of shape $(6, 2)$ and type $(4, 4)$. Recursively applying our process to this smaller problem gives:

$$\begin{array}{|c|c|c|c|c|c|c|} \hline \ & \ & \ & \ & 2 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} \end{array},$$

and finally the desired tableau

$$\begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} \end{array}.$$

Theorem 7.5. *The complete symmetric functions:*

$$\{h_\mu \mid \mu \text{ is a partition of } d \text{ with at most } n \text{ parts}\}$$

and the elementary symmetric functions:

$$\{e_\mu \mid \mu \text{ is a partition of } d \text{ with } \mu_1 \leq n\}$$

form bases of the space of homogeneous symmetric polynomials of degree d in variables $x_1, \dots, x_n$.

Proof. In view of the triangularity of Kostka numbers (Theorem 7.3) and the fact that $K_{\lambda\lambda} = 1$ (Exercise 6.7) the theorem follows from Theorem 6.13. $\square$

8. **Tableau as words.** Given a semistandard tableau t, its reading word w is defined to be the sequence of numbers obtained from reading its rows from left to right, starting with the bottom row, and moving up sequentially to the top row. Since the first entry of each row is strictly smaller than the last entry of the row below it, the tableau t can be recovered from w by chopping it up into segments with a cut after each a_i with $a_{i+1} < a_i$ (we say that w has a descent at i). The resulting segments, taken from right to left, form the rows of t.

Example 8.1. *The reading word of the tableau t formed at the end of Example 7.4 is:*

$$w = 332231111223.$$

The tableau t is recovered by marking off the descents $w = 333|223|1111223$, and then rearranging the segments into a tableau.

Let A_n^* denote the set of all words in the alphabet $\{1, \dots, n\}$. We shall call the word $w = a_1 \cdots a_k$ a *row* if $a_1 \leq \cdots \leq a_k$. We call it a *column* if $a_1 > \cdots > a_k$. We write x^w for the monomial $x_{a_1} x_{a_2} \cdots x_{a_k}$. Not every word comes from a tableau; for example the word 132, when broken up at descents gives rise to $\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}$. We shall say that a word is a tableau if it is the reading word of a semistandard tableau.

Exercise 8.2. *Show that, for every positive integer i,*

$$h_i(x_1, \dots, x_n) = \sum_{w \,\in\, A_n^* \text{ is a row}} x^w,$$

and

$$e_i(x_1, \dots, x_n) = \sum_{w \,\in\, A_n^* \text{ is a column}} x^w.$$

The set of words A_n^* comes endowed with an additional structure—that of a monoid—which can be lifted to the algebra of polynomials. If w_1 and w_2 are words, and $w_1 w_2$ is their concatenation, then

$$x^{w_1} x^{w_2} = x^{w_1 w_2}.$$

This gives rise to an algebra homomorphism $\mathbf{Z}[A_n^*] \to \mathbf{Z}[x_1, \dots, x_n]$. Obviously, this homomorphism has a large kernel; the algebra on the left is the free algebra, whereas the algebra on the right is commutative. The algebra on the right contains our primamry object of interest—the algebra $\mathbf{Z}[x_1, \dots, x_n]^{S_n}$ of symmetric polynomials. In the next few sections, we shall learn about an equivalence relation "$\equiv$" on A_n^* such that the resulting quotient monoid $\mathrm{Pl}_n := A_n^*/\equiv$ (called the *plactic monoid*) has the property that the natural homomorphism $\mathbf{Z}[\mathrm{Pl}_n] \to \mathbf{Z}[x_1, \ldots, x_n]^{S_n}$ is an algebra homomorphism.

9. **Schensted's insertion algorithm.** Let t be a semistandard tableau, and x be a positive integer. Schensted's insertion algorithm is a method of inserting a box with the number x into t, resulting in a new tableau $t \leftarrow x$. Applied repeatedly, it gives a way to convert any word into a tableau. This tableau succinctly expresses some combinatorial properties of the original word.

We shall identify tableaux with their reading words, as explained in Section 8. We first consider the case where t has a single row, so its reading word $t = a_1 \cdots a_k$ is a row in the sense of Section 8.

$$a_1 a_2 \cdots a_k \leftarrow x = \begin{cases} a_1 \cdots a_k x & \text{if } x \geq a_k, \\ a_j\ a_1 \cdots a_{j-1} x a_j \cdots a_k & \text{if } j = \min\{k \mid a_k > x\}. \end{cases}$$

In the second case, the resulting word is a tableau with two rows, the second row a_j has length one. We shall describe the situation by saying that x *has been inserted into* $a_1 \cdots a_k$*, obtaining* $a_1 \cdots a_{j-1} x a_j \cdots a_k$*, and* ***bumping out*** a_j.

10. **The Plactic Monoid.**

11. **Tableaux Correspondences.**

12. **Monomials in the Schur function.**

13. **The Lindström-Gessel-Viennot Lemma.** Let R be a commutative ring. Let S be any set of points, and $v : S \times S \to R$ be any function (we think of w as a *weight function*. Given $s, t \in S$, a path in S from s to t is is a sequence $\omega = (s = s_0, s_1, \ldots, s_k = t)$ of distinct points in S. We denote this by $\omega : s \to t$. The weight of the path ω is defined to be:

$$v(\omega) = v(s_0, s_1) v(s_1, s_2) \cdots v(s_{k-1}, s_k).$$

Definition 13.1 (non-crossing paths)**.** Two paths $\omega = (s_0, \ldots, s_k)$ and $\eta = (t_0, \ldots, t_l)$ are said to be non-crossing if $s_i \neq t_j$ for all $0 \leq i \leq k$ and $0 \leq t \leq l$.

Fix points $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ in S, and define an $n \times n$ matrix (a_{ij}) by:

$$a_{ij} = \sum_{\omega : A_i \to B_j} v(\omega).$$

Theorem 13.2 (Lindström-Gessel-Viennot Lemma)**.** *The determinant of the matrix* (a_{ij}) *defined above is given by:*

$$\det(a_{ij}) = \sum_{\omega_i : A_i \to B_i} v(\omega_1) \cdots v(\omega_n),$$

where the sum is over all n-tuples $(\omega_1, \ldots, \omega_n)$ of pairwise non-crossing paths $\omega_i : A_i \to B_i$.

14. **The Jacobi-Trudi Identities.** We have seen that the Kostka numbers can be used to express complete and elementary symmetric functions in terms of Schur functions. The reverse operation—that of expressing Schur functions in terms of complete or elementary symmetric functions—is done by the Jacobi-Trudi identities:

Theorem 14.1 (Jacobi-Trudi identities)**.** *For every integer partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ form the $l \times l$ matrices with (i,j)th entry $h_{\lambda_i - i + j}$ and $e_{\lambda_i - i + j}$ respectively. Then*

$$s_\lambda = \det(h_{\lambda_i - i + j}), \quad s_{\lambda'} = \det(e_{\lambda_i - i + j}).$$

Proof. The Jacobi-Trudi identities can be proved using the Lindström-Gessel-Viennot lemma (Theorem 13.2). For the first identity take S to be the positive cone in the the integer lattice:

$$S = \{(i,j) \mid i \geq 0,\ j > 0 \text{ are integers}\}.$$

Set the weight $v((i,j),(i+1,j))$ of each rightward horizontal edge to be x_j for $j = 1, \ldots, n$, the weight of each upward vertical edge $v((i,j),(i,j+1))$ to be 1 for all $j = 1, \ldots, n-1$. The remaining weights are all zero.

Lemma 14.2. *For all integers $i > 0$ and $j \geq 0$, we have:*

$$\sum_{\omega:(i,1)\to(i+j,n)} v(\omega) = h_j(x_1, \ldots, x_n).$$

Proof. Only rightward or upward steps have non-zero weights. So every path with non-zero weight is composed of unit upward and rightward steps. A path with non-zero weight from $(i,1)$ to $(i+j,n)$ must have exactly j rightward steps, say in rows $1 \leq i_1 \geq i_2 \cdots \leq i_j \leq n$. The weight of such a path is $x_{i_1} \cdots x_{i_j}$, and hence, the sum of the weights of all such paths is $h_j(x_1, \ldots, x_n)$. For an example, see Fig. 1. □

Gien $\lambda = (\lambda_1, \ldots, \lambda_l)$, and working with n variables $x_1, \ldots, x_n$, let $A_i = (l-i, 1)$ and $B_i = (\lambda_i + l - i, n)$ for $i = 1, \ldots, l$. Then by Lemma 14.2,

$$\sum_{\omega:A_i\to B_j} v(\omega) = h_{\lambda_j + i - j}.$$

So the left-hand-side of the first Jacobi-Trudi identity is the left-hand-side of the Lindström-Gessel-Viennot lemma. The right hand side of the Linstróm-Gessel-Viennot lemma consists of a sequence of non-crossing paths $(\omega_1, \ldots, \omega_n)$, where $\omega_i : A_i \to B_i$. Reading the row numbers of the horizontal steps in ω_i gives a weakly increasing sequence of integers $1 \leq k_1 \leq \cdots \leq k_{\lambda_i} \leq n$. Enter these numbers into the ith row of the Young diagram of λ for $i = 1, \ldots, n$. Since the paths are non-crossing, the jth rightward step of ω_i must be strictly higher than the jth rightward step of ω_{i+1}. This means

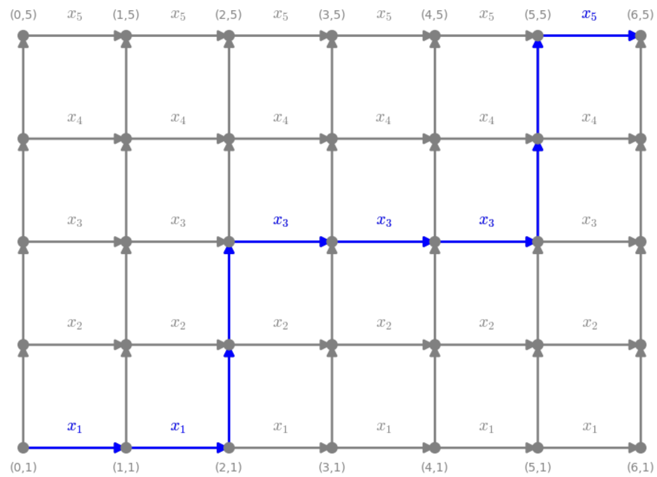


FIGURE 1. A path from $(0, 5)$ to $(6, 1)$ whose weight is the monomial $x_1^2 x_3^3 x_5$ in $h_6(x_1, \ldots, x_5)$.

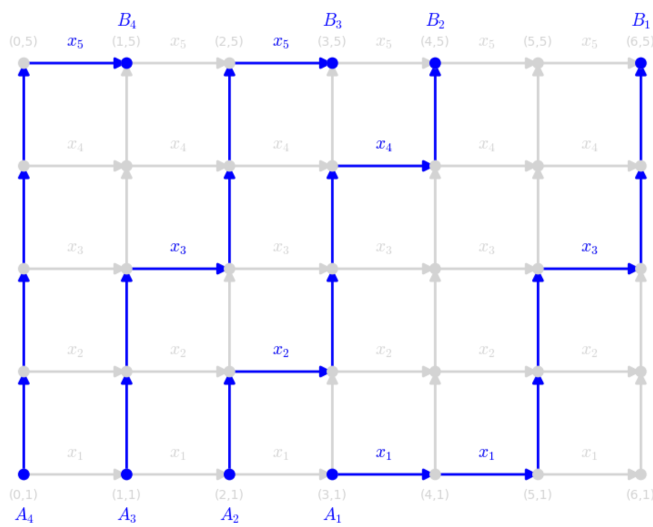


FIGURE 2. Non-crossing paths corresponding to the tableau $\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&4\\ \cline{1-2} 3&5\\ \cline{1-2} 5\\ \cline{1-1}\end{array}$

that the columns of the resulting numbering are strictly increasing, resulting in a semistandard tableau of shape λ.

Figure 2 shows the non-crossing path configuration corresponding to $n = 5$, $\lambda = ()$ which corresponds to the semistandard tableau $\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&4\\ \cline{1-2} 3&5\\ \cline{1-2} 5\\ \cline{1-1}\end{array}$. Thus, it

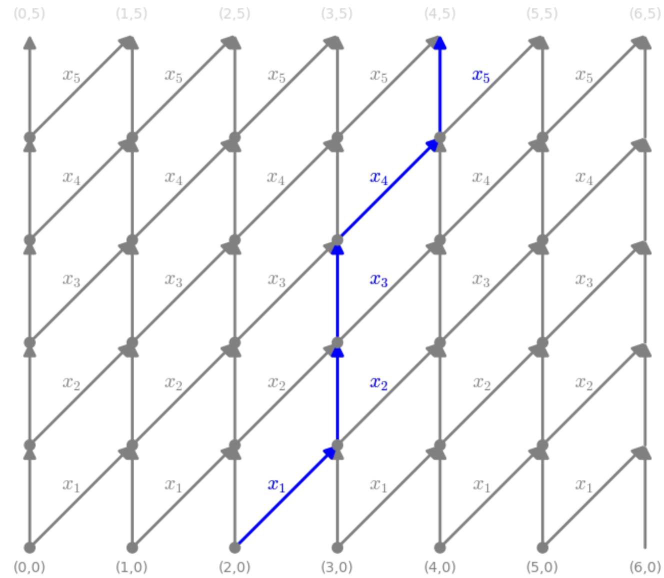


FIGURE 3. A path from $(2,0)$ to $(4,5)$ whose weight is the monomial x_1x_4 in $e_2(x_1,\ldots,x_5)$.

follows from the Lindstöm-Gessel-Viennot lemma that

$$\det(h_{\lambda_j+i-j}) = \sum_{t\in \mathrm{Tab}(\lambda)} x^t.$$

For the second Jacobi-Trudi identity take

$$S = \{(i,j) \mid i \geq 0,\ j \geq 0\}.$$

Define $v((i,j),(i+1,j)) = 1$ (as before) and $v((i-1,j),(i,j+1)) = x_i$; all other weights are zero. For the new weights, the analog of Lemma 14.2 is:

Lemma 14.3. *For all integers $i > 0$ and $j > 0$, we have:*

$$\sum_{\omega:(i,0)\to(i+j,n)} v(\omega) = e_j(x_1,\ldots,x_n).$$

Proof. Every path with non-zero weights consists of unit upward or upper-rightward diagonal steps. A path with non-zero weight from $(i,0)$ to $(i+j,n)$ must have n such steps, of which j must be diagonal. If the steps numbered $i_1,\ldots,i_j$ are the diagonal steps, then the path has weight $x_{i_1}\cdots x_{i_j}$. For an example of such a path, see Fig. 3. Summing over all possible paths gives $e_j(x_1,\ldots,x_n)$. $\square$

Suppose that the conjugate partition $\lambda' = (\lambda'_1,\ldots,\lambda'_k$. In order to apply the Lindström-Gessel-Viennot lemma to obtain the second Jacobi-Trudi identity, take $A_i = (k-i,0)$ and $B_i = (\lambda'_i + k - i, n)$ for $i = 1,\ldots,k$. Then by Lemma 14.3,

$$\sum_{\omega_i:A_i\to B_j} v(\omega) = e_{\lambda_j+i-j}.$$

So the left-hand-side of the second Jacobi-Trudi identity is the left-hand-side of the Lindström-Gessel-Viennot lemma.

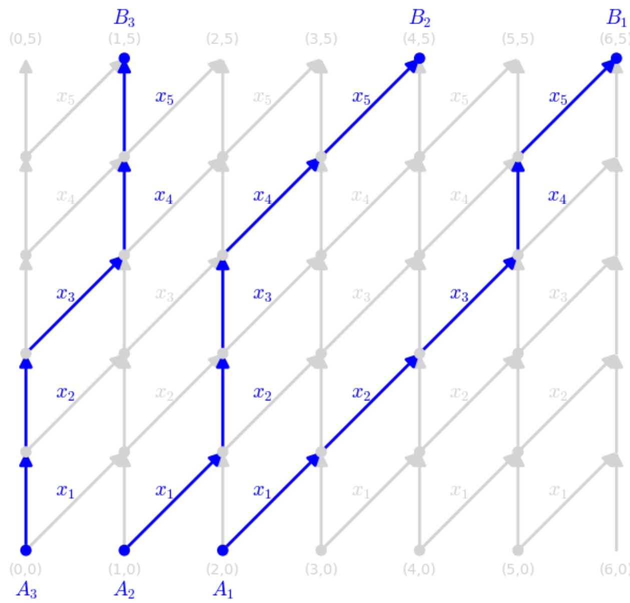


FIGURE 4. Non-crossing paths corresponding to the tableau $\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 3 & 5 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$.

The right hand side of the Linstróm-Gessel-Viennot lemma consists of a sequence of non-crossing paths $(\omega_1, \ldots, \omega_n)$, where $\omega_i : A_i \to B_i$. Reading the row numbers where the upper-rightward steps in ω_i originate gives a strictly increasing sequence of integers $1 \leq k_1 \leq \cdots \leq k_{\lambda'_i} \leq n$. Enter these numbers into the ith column of the Young diagram of λ. Since the paths are non-crossing, the jth upper-rightward step of ω_i must be no lower than the jth upper-rightward step of ω_{i+1}. This means that the rows of the resulting numbering are weakly increasing, resulting in a semi-standard tableau of shape λ. Figure 2 shows the non-crossing path configuration corresponding to $n = 5$, $\lambda = ()$ which corresponds to the semistandard tableau $\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 3 & 5 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$. Thus, it follows from the Lindstöm-Gessel-Viennot lemma that

$$\det(e_{\lambda_j+i-j}) = \sum_{t \in \mathrm{Tab}(\lambda')} x^t.$$

$\square$