AN INTRODUCTION TO SCHUR FUNCTIONS

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1. **Symmetric Functions.** We consider polynomials in n variables $x_1, \dots, x_n$. Given a tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, x^α denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ A symmetric polynomial in n variables $x_1, \dots, x_n$ is a polynomial of the form

$$f(x_1, \dots, x_n) = \sum_\alpha c_\alpha x^\alpha,$$

where, for any permutation $w \in S_n$,

$$c_{(\alpha_1, \dots, \alpha_n)} = c_{(\alpha_{w(1)}, \dots, \alpha_{w(n)})}.$$

We call the integer partition λ obtained by sorting the coordinates of α the shape of α and write $\lambda = \lambda(\alpha)$. The most obvious example of a symmetric polynomial in n variables is the *monomial symmetric function*, defined for each integer partition λ:

$$m_\lambda = \sum_{\lambda(\alpha)=\lambda} c_\alpha x^\alpha.$$

Note that m_λ is homogeneous of degree $|\lambda|$ (the sum of the parts of λ).

Exercise 1.1. *Take $n = 4$. Compute the monomial symmetric function $m_{(3)}$, $m_{(2,1)}$, and $m_{(1^3)}$.*

Theorem 1.2. *The polynomials $m_\lambda(x_1, \dots, x_n)$, as λ runs over all the integer partition of d, form a basis for the space of homegeneous symmetric polynomials of degree d in n variables.*

2. **Complete and Elementary Symmetric Polynomials.** Recall that the coefficients of a polynomial are symmetric polynomials in its roots:

$$(t - x_1)(t - x_2) \cdots (t - x_n) = t^n - e_1(x_1, \dots, x_n)t^{n-1} + \cdots + (-1)^n e_n(x_1, \dots, x_n), \tag{1}$$

where coefficient $e_i(x_1, \dots, x_n)$ of t^{n-i} is given by:

$$e_i(x_1, \dots, x_n) = \sum_{1 \leq j_1 < \cdots < j_i \leq n} x_{j_1} x_{j_2} \cdots x_{j_i}. \tag{2}$$

The polynomial e_i is called the ith *elementary symmetric polynomial.* By convention, write $e_i(x_1, \ldots, x_n) = 0$, for $i > n$.

The identity (1) can be written more elegantly as:

$$(1 + tx_1) \cdots (1 + tx_n) = \sum_{i=0}^{n} e_i(x_1, \cdots, x_n)t^i.$$

Dually[1], the *complete symmetric polynomials* are defined by the formal identity:

$$\frac{1}{(1 - x_1t) \cdots (1 - x_nt)} = \sum_{i=0}^{\infty} h_i(x_1, \cdots, x_n)t^i.$$

Example 2.1. *In three variables, we have:*

$$\begin{aligned}
e_2(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + x_2x_3, \\
h_2(x_1, x_2, x_3) &= x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_2^3.
\end{aligned}$$

Exercise 2.2. *Show that*

$$h_i(x_1, \cdots x_n) = \sum_{1 \leq j_1 \leq \cdots \leq j_i \leq n} x_{j_1} \cdots x_{j_i}.$$

More generally, for any integer partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, define:

$$\begin{aligned}
h_\lambda &= h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}, \\
e_\lambda &= e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}.
\end{aligned}$$

Theorem 2.3. *Given integer partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \cdots, \mu_m)$ of and integer d, let $M_{\lambda\mu}$ denote the number of integer matrices (a_{ij}) with non-negative entries whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j. Then*

$$h_\lambda = \sum_\mu M_{\lambda\mu} m_\mu.$$

Dually, let $= N_{\lambda\mu}$ denote the number of integer matrices (a_{ij}) with entries 0 or 1, whose ith row sums to λ_i for each i, and whose jth column sums to μ_j for each j.

$$e_\lambda = \sum_\mu N_{\lambda\mu} m_\mu.$$

[1]We will refer to the replacing of $(1 + u)$ by $(1 - u)^{-1}$ in a formal identity as *dualization.*

Proof. We first prove the second identity involving elemenatry symmetric functions. A monomial in the expansion of

$$e_\lambda = \prod_{i=1}^{l} \sum_{j_1 < \cdots < j_{\lambda_j}} x_{j_1} \cdots x_{j_{\lambda_i}}$$

corresponds to the choice of a summand from each factor. Construct a matrix (a_{ij}) corresponding to such a choice as follows: if the factor $x_{j_1} \cdots x_{j_{\lambda_i}}$ is chosen from the ith factor, then set the entries $a_{i,j_1}, \ldots, a_{i,j_j}$ to be 1. Clearly the ith row of such a matrix sums to λ_i. The monomial corresponding to this choice is x^μ if, for each j, the the number of i for which x_j appears in $a_{i,j_1}, \ldots, a_{i,j_j}$, which is the sum of the jth column of the matrix (a_{ij}). It follows that the coefficient of x^μ, and hence the coefficient of m_μ in the expansion of e_λ in the basis of monomial symmetric functions of degree n, is $N_{\lambda\mu}$. $\square$