Real Crystals

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1. Fractional Crystal Operators

We define operators $e_i^t, f_i^t, \sigma_i : A_n^\dagger \cup \{0\} \to A_n^\dagger \cup \{0\}$, for $i = 1, \ldots, n-1$, and $t \geq 0$. In order to define the operators for a given i on a timed word w, we will freeze a certain subwords of w. To begin with freeze every term of w which does not involve the letters i and $j = i+1$ of A_n. The unfrozen part of w is a subword of the form:

$$\bar{w} = j^{s_1} i^{r_1} j^{s_2} i^{r_2} \cdots j^{s_l} i^{r_l},$$

for some l and some positive real numbers s_i, r_i, except for the possibility that s_1 and r_l are zero. From the above subword, freeze segments $j^{\min(s_k, r_k)} i^{\min(s_k, r_k)}$ for each k. Since one of the terms 2^{s_k} and 1^{r_k} will be entirely frozen for each k, the unfrozen part of w will now have strictly fewer terms than before. Hence finitely many repetitions of this process will result in a word of the form

$$i^r j^s \text{ for some } r, s \geq 0$$

is obtained.

We call s the e_i-range of w, denoted range(e_i, w) and r the f_i-range of w, denoted range(f_i, w). For each $t \leq s$, define we_i^t to be the timed word obtained by transforming the unfrozen subword of w as

$$e_i^t : i^r j^s \mapsto i^{r+t} j^{s-t},$$

and leaving the frozen part unchanged. For $t > s$, define $we_i^t = 0$.

Similarly, for $t \leq r$, define wf_i^t to be the timed word obtained by transforming the unfrozen part of w as

$$f_i^t : i^r j^s \mapsto i^{r-t} j^{s+t},$$

and leaving the frozen part unchanged. Finally define $w\sigma_i$ to be the timed word obtained by transforming the unfrozien subword of w by

$$\sigma_i : i^r j^s \mapsto i^s j^r,$$

and leaving the frozen part unchanged.

Let $x \in A_{n-1}^\dagger$, say $x = c_1^{t_1} \cdots c_k^{t_k}$. Then define

$$we^x = we_{c_1}^{t_1} \cdots e_{c_k}^{t_k}$$
$$wf^x = wf_{c_1}^{t_1} \cdots f_{c_k}^{t_k}$$

The operators e^x and f^x, for $x \in A_{n-1}^\dagger$ are called *fractional crystal operators.*

Lemma 1. *Suppose w and w' are timed words that are in the same timed Knuth equivalence class, i.e., $w \equiv w'$. Then for any timed word $x \in A_{n-1}^\dagger$, if $we^x \neq 0$, then $w'e^x \neq 0$, and $we^x \equiv w'e^x$.*

Lemma 2. *If w is [the reading word of] a timed tableau, then we^x and wf^x, when not 0, are also [reading words of] timed taleaux. In other words, fractional crystal operators act on timed tableaux.*

Definition 3 (Yamanouchi Timed Word). A timed word $w \in A_n^\dagger$ is said to have *dominant valuation* if its weight vector is weakly decreasing. The timed word w is said to be Yamanouchi if the every suffix has a dominant valuation. The set of all Yamanouchi timed words of weight λ is denoted $\mathrm{Yam}^\dagger(\lambda)$.

Lemma 4. *A timed word $w \in A_n^\dagger$ has the property that* $\mathrm{range}(e_i, w) = 0$ *for $i = 1, \ldots, n-1$ if and only if w is a timed Yamanouchi word.*

Lemma 5. *For every $w \in A_n^\dagger$ there exists $x \in A_{n-1}^\dagger$ such that we^x is a timed Yamanouchi word. Moreover, if we^x and $we^{x'}$ are both Yamanouchi words, then they are equal.*

Proof. Both $P(w)e^x$ and $P(w)e^{x'}$ are timed tableaux of the same shape λ, by Lemma 2. By Lemma 1, $P(w)e^x \equiv we^x$ and $P(w)e^{x'} \equiv we^{x'}$. Since timed Knuth equivalence preserves Yamanouchiness, both $P(w)e^x$ and $P(w)e^{x'}$ are also Yamanouchi words. But the only timed tableaux of shape λ that is also a Yamanouchi word is the superstandard tableau of shape λ. So $P(w)e^x = P(w)e^{x'}$. Therefore we^x and $we^{x'}$ are Yamanouchi timed words of the same weight.??? ☐

Lemma 6. *Let $w \in A_n^\dagger$. Fix $1 \leq i < n$, and let $j = i+1$. Then w admits a unique decomposition of the form:*

$$w = xi^a uj^b v,$$

with $x, u, u^*, v^* \in \mathrm{Yam}_i$, $a > 0$, $b > 0$, x does not end with i, u does not begin with i or end with j, and v does not begin with j. In this case, for any $0 \leq t \leq b$, we have:

$$we_i^t = xi^{a+t} uj^{b-t} v,$$

and for any $0 \leq t \leq a$*, we have:*

$$wf_i^t = xi^{a-t}uj^{b+t}v.$$

2. Fractional Coplactic Classes

Definition 7 (Fractional Coplactic Class). Say that two words v and w are in the same fractional coplactic class if there exists a timed word $x \in A_{n-1}^\dagger$ such that $v = we^x$ or $v = wf^x$.

Definition 8 (Real Crystal). A real crystal of a timed word w is the coplactic class of w, together with families of relations e_i^t and f_i^t, for $i = 1, \dots, n-1$, and $t > 0$ defined by $we_i^t w'$ if $we_i^t = w'$ and $wf_i^t w'$ if $wf_i^t = w'$.

Each fractional coplactic class is a real crystal in the obvious manner. An isomorphism of real crystals is a bijection which preserves all the relations e_i^t and f_i^t.

3. The Robinson Correspondence

The following algorithm takes as input $w \in A_n^\dagger$, and returns a Yamanouchi timed word $Y(w)$ in the fractional coplactic class of w.

Robinson's Algorithm

- for i in $1, \dots, n-1$:
  - for j in $i, i-1, \dots, 1$:
    - $w \leftarrow we_j^{\mathrm{range}(e_j, w)}$

  Return w

Theorem 9 (Robinson's correspondence). *The map:*

$$R : w \mapsto (P(w), Y(w))$$

defines a bijection

$$\mathrm{Rob} : A_n^\dagger \tilde{\to} \coprod_\lambda \mathrm{Tab}_n^\dagger(\lambda) \times \mathrm{Yam}^\dagger(\lambda).$$

Proof. The inverse is obtained as follows: Given $(u, y) \in \mathrm{Tab}_n^\dagger(\lambda) \times \mathrm{Yam}^\dagger(\lambda)$, apply Robinson's algorithm to u until the superstandard tableau t_λ^0 of shape λ is obtained. This gives a word $x \in A_{n-1}^\dagger$ such that $ue^x = t_\lambda^0$. Let x^* be the opposite word to x. Recover $w = yf^{x^*}$. ☐

Definition 10 (Depth and rise). The i-depth of a timed word w is defined as:

$$\delta_i(w) = \sup\{t \geq 0 \mid we_i^t \neq 0\},$$

Define the depth vector of w to be:

$$\delta(w) = (\delta_1(w), \delta_2(w), \dots, \delta_{n-1}(w))$$

and the i-rise of a timed word w is defined as:

$$\epsilon_i(w) = \sup\{t \geq 0 \mid wf_i^t \neq 0\}.$$

Define the rise vector to w to be:

$$\epsilon(w) = (\epsilon_1(w), \epsilon_2(w), \dots, \epsilon_{n-1}(w)).$$

THEOREM 11. *A word w is Yamanouchi if and only if $\delta(w) = 0$.*

DEFINITION 12 (Difference operators). Recall Stembridge's difference operators:

$$\begin{aligned} \Delta_i \delta_j(w) &= \delta_j(we_i) - \delta_j(w) && \text{if } \delta_i(w) \geq 1, \\ \nabla_i \delta_j(w) &= \delta_j(w) - \delta_j(wf_i) && \text{if } \epsilon_i(w) \geq 1. \end{aligned}$$

and define their infinitesimal versions for timed words:

$$\begin{aligned} \frac{\partial \delta_j^+}{\partial e_i} &= \lim_{t \to 0^+} \frac{\delta(we_i^t) - \delta(w)}{t} && \text{if } \delta_i(w) > 0, \\ \frac{\partial \delta_j^-}{\partial e_i} &= \lim_{t \to 0^-} \frac{\delta(we_i^t) - \delta(w)}{t} && \text{if } \epsilon_i(w) > 0. \end{aligned}$$

THEOREM 13. *If y and y' are Yamanouchi timed words of weight λ, then their fractional coplactic classes are isomorphic as real crystals.*

Let y_λ^0 denote the unique timed tableau of shape λ and weight λ. Then y_λ^0 is also the only timed tableau of weight λ that is also Yamanouchi.

THEOREM 14. *The fractional coplactic class $\mathrm{fcop}(y_\lambda^0)$ consists of all timed tableaux of shape λ in $A_n^\dagger$.*

LEMMA 15. *Let $w \in A_n^\dagger$, and $i \in 1, \dots, n-1$. Then*

$$P(we_i^t) = P(w)e_i^t.$$

4. Littlewood-Richardson using crystals

We now recall the proof of the Littlewood-Richardson rule using crystal operators: For any word $w \in A_n^*$, let $C(w)$ denote the coplactic class of w. Then the Schur function is defined by:

$$s_\lambda(x_1, \dots, x_n) = \sum_{w \in C(y_\lambda)} \mathrm{wt}(w),$$

where y_λ is *any* Yamanouchi word of weight λ. We have:

$$s_\mu(x_1, \dots, x_n) s_\nu(x_1, \dots, x_n) = \sum_{u \in C(y_\mu^0), v \in C(y_\nu^0)} \mathrm{wt}(uv).$$

For every i, the effect $(uv)e_i$ of the (classical) crystal operator is either $(ue_i)v$ or $u(ve_i)$, so the sum on the right hand side is a union of coplactic classes. The number of coplactic classes containing a Yamanouchi word of weight λ is equal to the number of Yamanouchi words of weight λ in

$$\{uv \mid u \in C(y^0_\mu), v \in C(y^0_\nu)\},$$

Also, if uv is Yamanouchi, then v has to be Yamanouchi, hence $v = y^0_\nu$. We get the crystal version of the Littlewood-Richardson rule:

$$c^\lambda_{\mu\nu} = \#\{u \in C(y^0_\mu) \mid uy^0_\nu \in \mathrm{Yam}(\lambda)\}$$

The saturation theorem says:

THEOREM 16. *Given partition λ, μ, and ν, and a positive integer N, if $c^{N\lambda}_{N\mu N\nu} > 0$, then $c^\lambda_{\mu\nu} > 0$.*

The hypothesis posits the existence of $u \in C(y^0_{N\mu})$ such that $uy^0_{N\nu} \in \mathrm{Yam}(N\lambda)$. Consider the timed word $\bar{u} = u^{1/N} \in A^\dagger_n$. This world lies in $\mathrm{fcop}(y^0_\mu)$, and uy^0_ν is a Yamanouchi timed word of weight λ. We would like to replace $\bar{u}$ with $\tilde{u} \in A_n*$ such that:

(1) $\tilde{u}y^0_\nu \in \mathrm{Yam}_n(\lambda)$,
(2) $\tilde{u} \in \mathrm{cop}(y^0_\mu)$.

We have:

$$\bar{u}e^x = y^0_\mu,$$

where $x \in A^\dagger_n$. We have $\lambda - \nu = \mathrm{wt}(\tilde{u}) = \mu - \sum_{i=1}^{n-1} \mathrm{wt}_i(x)(e_i - e_{i+1})$. It follows that $\mathrm{wt}_i(x)$ is a nonnegative integer for every i. Moreover, we can assume that

$$x = 1^{t_{11}}\ 2^{t_{22}}1^{t_{21}} \cdots (n-1)^{t_{n-1,n-1}} \cdots 1^{t_{n-1,1}}.$$

It suffices to describe a rounding algorithm $x \mapsto r(x) \in A^*_n$ such $r(x)$ has the same weight as x, and the word $\tilde{u}$ determined by

$$\tilde{u}e^{r(x)} = y^0_\mu$$

has $\mathrm{range}(e_j, \tilde{u}) \le \mathrm{range}(e_j, \bar{u})$ for $j = 1, \ldots, n-1$.

LEMMA 17. *For any $u \in A^\dagger_n$, uy^0_ν is Yamanouchi if and only if*

$$\mathrm{range}(e_i, u) \le \langle \nu, \alpha_i \rangle$$

for $i = 1, \ldots, n-1$.

5. Stembridge axioms

THEOREM 18. *For any $w \in A_n^\dagger$, the following hold whenever the result of plactic operations that appear in them do not result in* 0.

(1) $\mathrm{range}(e_i, w) - \mathrm{range}(f_i, w) = \langle \mathrm{wt}(w), \alpha_i\rangle$.

(2) *We have:*
  (a) $\mathrm{range}(e_i, we_i^t) = \mathrm{range}(e_i, w) - t$,
  (b) $\mathrm{range}(e_i, w) \leq \mathrm{range}(e_i, we_{i+1}^t) \leq \mathrm{range}(e_i, w) + t$,
  (c) $\mathrm{range}(e_{i+1}, w) \leq \mathrm{range}(e_{i+1}, we_i^t) \leq \mathrm{range}(e_{i+1}, w) + t$,
  (d) *if* $|i - j| > 1$, *then* $\mathrm{range}(e_i, we_j) = \mathrm{range}(e_i, w)$.

(3) *Also,*
  (a) $\mathrm{range}(e_i, wf_i^t) = \mathrm{range}(e_i, w) + t$,
  (b) $\mathrm{range}(e_i, w) \geq \mathrm{range}(e_i, wf_{i+1}^t) \geq \mathrm{range}(e_i, w) - t$,
  (c) $\mathrm{range}(e_{i+1}, w) \geq \mathrm{range}(e_{i+1}, wf_i^t) \geq \mathrm{range}(e_{i+1}, w) - t$,
  (d) *if* $|i - j| > 1$, *then* $\mathrm{range}(e_i, wf_j) = \mathrm{range}(e_i, w)$.

PROOF. As in Section 1, let $j = i+1$. If $\mathrm{wt}(w) = (\mu_1, \dots, \mu_n)$, then $\langle \mathrm{wt}(w), \alpha_i\rangle = \mu_j - \mu_i$. When the freezing process of subwords described in the definition of e_i^t in Section 1 is conducted, the ith and and jth components of the weights of the frozen subwords are always equal. Thus, when the unfrozen word of the form $i^r j^s$ is finally obtained, $s - r = \mu_j - \mu_i$. Now, from the definition of e_i^t and f_i^t, $\mathrm{range}(e_i, w) = s$, while $\mathrm{range}(f_i, w) = r$. Thus

$$\begin{aligned}\mathrm{range}(e_i, w) - \mathrm{range}(f_i, w) &= s - r\\ &= \mu_j - \mu_i\\ &= \langle \mathrm{wt}(w), \alpha_i\rangle,\end{aligned}$$

proving (1).

The unfrozen part of we_i^t is of the form $i^{r+t} j^{s-t}$ for all $0 \leq t \leq s$, so

$$\begin{aligned}\mathrm{range}(e_i, we_i^t) &= s - t\\ &= \mathrm{range}(e_i, w) - t,\end{aligned}$$

proving part (a) of (2).

Now let $k = i + 2$.

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