GREENE'S DUALITY THEOREM FOR TIMED WORDS

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Definition 1 (Vertical and Horizontal strips)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_l) \subset \mu = (mu_1, \ldots, \mu_m)$ be real partitions. Then, we say that λ/μ is a *vertical strip* of size r if $\lambda_i - \mu_i \leq 1$ for each $i = 1, \ldots, l$, and $\mu_i \leq 1$ for $i = l+1, \ldots, m$, and $r = l(\lambda) - l(\mu)$. We say that λ/μ is a *horizontal strip* of size r if λ and μ can be padded with 0's in such a way that $l = m+1$, and $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all $i = 1, \ldots, m$, and $l(\lambda) - l(\mu) = r$.

Definition 2. A *timed row* in A_n is a timed word of the form

$$u = 1^{t_1} \cdots n^{t_n}, \text{ with } t_i \geq 0 \text{ for } i = 1, \ldots, n.$$

The set of all timed rows of length r is denoted $R^\dagger(r)$. A *timed column* in A_n is a timed word of the form

$$u = n^{t_n} \cdots 1^{t_1}, \text{ with } 0 \leq t_i \leq 1 \text{ for } i = 1, \ldots, n.$$

The set of all timed columns of length r is denoted $C^\dagger(r)$.

Theorem 3 (Pieri Rule and its Dual)**.** *For every real partition λ, the function*

$$(t, u) \mapsto P(tu)$$

defines a bijection

$$\mathrm{Tab}_n(\lambda) \times R_n^\dagger(r) \tilde{\to} \coprod_\mu \mathrm{Tab}_n(\mu),$$

the union being over all real partitions μ such that μ/λ is a horizontal strip of size r.

Similarly, the function

$$(t, u) \mapsto P(tu)$$

defines a bijection

$$\mathrm{Tab}_n(\lambda) \times C_n^\dagger(r) \tilde{\to} \coprod_\mu \mathrm{Tab}(\mu),$$

the union being over all real partitions μ such that μ/λ is a vertical strip of size r.

Proof. Suppose that $u = 1^{s_1} \cdots n^{s_n}$ is a timed row. Suppose $i, j \in A_n$ are such that $i > j$, and $s, t \in [0,1]$. Let

$$(v_1, u_1) = \mathrm{RINS}(u, i^s),$$
$$(v_2, u_2) = \mathrm{RINS}(u_1, j^t).$$

Then v_1 is a prefix of $(i+1)^{s_{i+1}} \cdots n^{s_n}$ of length $\min(s, s_{i+1} + \cdots + s_n)$. Since $j < i$, □

Definition 4 (Dual Timed Tableau)**.** A dual timed tableau of shape $\lambda = (\lambda_1, \dots, \lambda_l)$ is a sequence $(u_1, \dots, u_l)$ of timed rows such that

(1) $l(u_i) = \lambda_i$ for $i = 1, \dots, l$,
(2) $u_i(t+1) > u_i(t)$ for all $t \in [0, l(u_i) - 1)$ for all $i = 1, \dots, l$,
(3) $u_i(t) \leq u_{i+1}(t)$ for all $i = 0, \dots, l-1$, and $t \in [0, \lambda_{i+1})$.

The weight of a dual timed tableau is the sum of the weight vectors of its rows. The set of dual timed tableau of shape λ and weight μ is denoted by $\mathrm{Tab}^*(\lambda, \mu)$.

Given a dual timed tableau $t = (u_1, \dots, u_l)$ of shape λ, let $\lambda_i^{(j)}$ be the length of the restriction of u_i to the alphabet $\{1, \dots, j\}$ for $j = 1, \dots, n$. Then setting $\lambda^{(j)} = (\lambda_1^{(j)}, \dots, \lambda_l^{(j)})$, its dual Gelfand-Tsetlin pattern is defined to be the chain of real partitions:

$$\mathrm{GT}^*(t) := (\emptyset = \lambda^{(0)} \subset \cdots \subset \lambda^{(n)} = \lambda), \tag{1}$$

such that

$$|\lambda^{(j)}| - |\lambda^{(j-1)}| = \mu_j \text{ for } j = 1, \dots, n, \tag{2}$$

$$\lambda_i^{(j-1)} \leq \lambda_i^{(j)} \leq \lambda_i^{(j-1)} + 1 \text{ for } i = 1, dotsc, l \text{ and } j = 1, dotsc, n. \tag{3}$$

Definition 5 (Dual Gelfand-Tsetlin Patterns)**.** A dual Gelfand-Tsetlin pattern of shape λ and weight μ is a sequence of real partitions of the form (1) satisfying the conditions (2) and (3).

Lemma 6. *The map* GT^* *is a bijection from the set of dual timed tableau of shape* λ *and weight* μ *onto the set of dual Gelfand-Tsetlin patterns of weight* μ*.*

Theorem 7. *There exists a dual timed tableau of shape* λ *and weight* μ *if and only if*

$$\mu_1 + \cdots + \mu_j \leq \min(\lambda_1, j) + \cdots + \min(\lambda_l, j), \tag{4}$$

for all $j = 1, \dots, n$*.*

Proof. Suppose $t = (u_1, \dots, u_l)$ is a dual timed tableau. Then each letter has weight at most one in the rows u_i, $w_1(u_i) + \cdots + w_j(u_i) \leq \min(l(u_i), j)$ for each i. Adding over all i gives (4).

For the converse, suppose the □

The following algorithm takes as input an $m \times n$ matrix $A = (a_{ij})$ such that $0 \leq a_{ij} \leq 1$ for all i, j.

Dual RSK Algorithm

- $P \leftarrow \emptyset$, $Q \leftarrow \emptyset$
- For $j = 1, \ldots, n$ repeat the following steps:
  - $Q \leftarrow \mathrm{INS}(Q, n^{a_{nj}} \cdots 2^{a_{n2}} 1^{a_{n1}})$
  - $P \leftarrow \mathrm{INFL}(P, \mathrm{shape}(Q), j)$
- Return P, Q.

If the dual RSK algorithm return (P, Q) on input A, we write:

$$\mathrm{RSK}^*(A) = (P, Q).$$

Theorem 8. *Let $\mu = (\mu_1, \ldots, \mu_m)$ and $\nu = (\nu_1, \ldots, \nu_n)$ be real partitions such that $|\mu| = |\nu|$. Let $N_{\mu\nu}$ denote the set of all matrices $A = (a_{ij})_{m \times n}$ with $0 \leq a_{ij} \leq 1$. Then the dual RSK algorithm defines a bijection:*

$$\mathrm{RSK}^* : N_{\mu\nu} \tilde{\to} \coprod_{\lambda} \mathrm{Tab}^*(\lambda, \nu) \times \mathrm{Tab}(\lambda, \mu).$$

Recall also the following theorem:

Theorem 9. *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be real partitions, with $|\lambda| = |\mu|$. There exists a timed tableau of shape λ and content μ if and only if $l \leq m$, $\mu_1 + \ldots + \mu_j \leq \lambda_1 + \cdots + \lambda_j$ for $j = 1, \ldots, l$.*

Proof. Suppose that $\lambda_i < \mu_m \leq \lambda_{i-1}$. Then fill m into the last $\lambda_i - \lambda_{i+1}$ of rows $i, i+1, \ldots, l$. Totally this uses up λ_i amount of m. Also fill m into the last $\mu_m - \lambda_i$ of the $(i-1)$st row. Thus the feasibility problem is reduced to $(\mu_1, \ldots, \mu_{m-1})$, and $(\lambda_1, \ldots, \lambda_{i-2}, \lambda_{i-1} - \mu_m + \lambda_i, \lambda_{i+1}, \ldots, \lambda_l, 0)$. Note that $\lambda_{i-1} > \lambda_{i-1} - (\mu_m - \lambda_i)$, so the $(i-1)$st row of lambda is reduced in length, but $({}_{i-1} - \mu_m) + \lambda_i \geq \lambda_i \geq \lambda_{i+1}$, so the new vector is still a partition. For $j = 1, \ldots, i-2$, the feasibility condition $\mu_1 + \cdots + \mu_j \leq \lambda_1 + \cdots + \lambda_j$ remains unchanged. For $j = i-1$, we have $\mu_1 + \cdots + \mu_{i-1} \leq \lambda_1 + \lambda_{i-1} - \mu_m + \lambda_i$, □

Combining Theorems 7, 8, and 9, we get:

Theorem 10 (Real Gale-Ryser Theorem)**.** *Let μ and ν be real partitions with the same sum. Then $N_{\mu\nu}$ is non-empty if and only if there exists a real partition $(\lambda = (\lambda_1, \ldots, \lambda_l)$ such that both the conditions*

$$\mu_1 + \cdots + \mu_j \leq \min(\lambda_1, j) + \cdots + \min(\lambda_l, j) \qquad \text{for } j = 1, \ldots, m,$$
$$\nu_1 + \cdots + \nu_j \leq \lambda_1 + \cdots + \lambda_j \qquad \text{for } j = 1, \cdots, l.$$

Lemma 11. *Suppose λ and μ are real partitions with $|\lambda| = |\mu|$. Then*

$$\mu_1 + \cdots + \mu_j \leq \min(\lambda_1, j) + \cdots + \min(\lambda_l, j) \text{ for } j = 1, \cdots + m$$

if and only if

$$\lambda_1 + \cdots + \lambda_j \leq \min(\mu_1, j) + \cdots + \min(\mu_m, j) \text{ for } j = 1, \cdots + l.$$