Real Crystals

Amritanshu Prasad

We define operators $e_i^t, f_i^t, \sigma_i : A_n^\dagger \to A_n^\dagger \cup \{0\}$, for $i = 1, \ldots, n-1$, and $t \geq 0$. Let $x \in A_{n-1}^\dagger$, say $x = c_1^{t_1} \cdots c_k^{t_k}$. Then define

$$we^x = we_{c_1}^{t_1} \cdots e_{c_k}^{t_k}$$
$$wf^x = wf_{c_1}^{t_1} \cdots f_{c_k}^{t_k}$$

The operators e^x and f^x, for $x \in A_{n-1}^\dagger$ are called *fractional crystal operators*.

Definition 1 (Fractional Coplactic Class). Say that two words v and w are in the same fractional coplactic class if there exists a timed word $x \in A_{n-1}^\dagger$ such that $v = we^x$ orr $v = wf^x$.

Definition 2 (Real Crystal). A real crystal is a set, together with families of relations e_i^t and f_i^t, for $i = 1, \ldots, n-1$, and $t > 0$.

Each fractional coplactic class is a real crystal in the obvious manner. An isomorphism of real crystals is a bijection which preserves all the relations e_i^t and f_i^t.

Definition 3 (Yamanouchi Timed Word). A timed word $w \in A_n^\dagger$ is said to have *dominant valuation* if its weight vector is weakly decreasing. The timed word w is said to be Yamanouchi if the every suffix has a dominant valuation. The set of all Yamanouchi timed words of weight λ is denoted $\mathrm{Yam}^\dagger(\lambda)$.

Theorem 4. *If y and y' are Yamanouchi timed words of weight λ, then their fractional coplactic classes are isomorphic as real crystals.*

Let y_λ^0 denote the unique timed tableau of shape λ and weight λ. Then y_λ^0 is also the only timed tableau of weight λ that is also Yamanouchi.

Theorem 5. *The fractional coplactic class $\mathrm{fcop}(y_\lambda^0)$ consists of all timed tableaux of shape λ in $A_n^\dagger$.*

LEMMA 6. *Let* $w \in A_n^\dagger$, *and* $i \in 1, \ldots, n-1$. *Then*

$$P(we_i^t) = P(w)e_i^t.$$