THE TIMED PLACTIC MONOID

AMRITANSHU PRASAD

Abstract. The plactic monoid is a simple algebraic structure that lies at the crossroads of the theory of symmetric polynomials, enumerative geometry, representation theory, and combinatorics. It was introduced by Lascoux and Schützenberger, who used it to prove the Littlewood-Richardson rule. It is the quotient of the concatenation monoid of words in a totally ordered langauge modulo a pair of relations discovered by Knuth. Timed words were introduced by Adur and Dill in the context of timed automata, which model the behavior of real-time computer systems. Timed words too form a monoid under concatenation. This monoid contains the monoid of words as a submonoid. In this article, we extend the definition of the plactic monoid to include timed words by presenting timed analogs of Knuth relations. We extend some basic results from the theory of plactic monoids, such as Greene's theorem and the RSK correspondence to timed words.

1. Tableaux, Insertion, and Greene's Theorem

1.1. **Tableaux.** Recall that a partition is a tuple $\lambda = (\lambda_1, \dots, \lambda_l)$ of integers such that $\lambda_1 \geq \cdots \geq \lambda_l > 0$. The Young diagram of the partition λ is defined as the array of points

$$Y(\lambda) = \{(i,j) \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}$$

drawn in matrix notation, so that the point (i,j) lies in the ith row and jth column of $Y(\lambda)$. Let $A_n = \{1, \dots, n\}$.

Definition 1.1.1. A semistandard Young tableau[1] in A_n of shape λ is an assignment $t : Y(\lambda) \to A_n$ such that the numbers increase weakly from left to right along each row, and increase strictly from top to bottom along each column. The weight of t is the tuple $(m_1, \dots, m_n)$, where m_i is the number of times that i occurs in the image of t.

For brevity, a semistandard Young tableaux will be referred to as a tableau in the rest of this article.

[1]It is customary to write the plural of *tableau* as *tableaux*, following French conventions.

Example 1.1.2. The following is a tableau of shape $(5,2,1)$ and weight $(2,1,4,1)$ in A_4:

$$t = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 3 & 3 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

We denote by Tab_n the set of all tableaux in A_n, $\mathrm{Tab}_n(\lambda)$ the set of all tableaux of shape λ in A_n, and by $\mathrm{Tab}(\lambda,\mu)$ the set of all tableaux of shape λ and weight μ.

1.2. **Row Insertion.** A row of length k is defined to be a weakly increasing sequence $u = a_1a_2\cdots a_k$ in A_n. Let $R(A_n)$ denote the set of all rows in A_n. Each row of a tableau is a row in the sense of this definition. For each $u = a_1\cdots a_k \in R(A_n)$ and $a \in A_n$, define:

$$\mathrm{ROWINS}(u,a) = \begin{cases} (\emptyset, a_1\cdots a_k a) & \text{if } a_k \leq a, \\ (a_j, a_1\cdots a_{j-1}aa_{j+1}\cdots a_k) & \text{otherwise, with} \\ & j = \min\{i \mid a < a_i\}. \end{cases}$$

Here $\emptyset$ should be thought of as an empty row of length zero.

Example 1.2.1. $\mathrm{ROWINS}(11333,3) = (\emptyset, 113333)$, $\mathrm{ROWINS}(11333,2) = (3, 11233)$.

It is clear from the construction that, for any $u \in R(A_n)$ and $a \in A_n$, if $(a',u') = \mathrm{ROWINS}(u,a)$, then u' is again a row. For convenience set $\mathrm{ROWINS}(u,\emptyset) = (\emptyset, u)$.

1.3. **Tableau Insertion.** Let t be a tableau with rows $u_1, u_2, \ldots, u_l$. Then $\mathrm{INSERT}(t,a)$, the insertion of a into t, is defined as follows: first a is inserted into u_1; if $\mathrm{ROWINS}(u_1,a) = (a'_1, u'_1)$, then u_1 is replaced by u'_1. Then a'_1 is inserted into u_2; if $\mathrm{ROWINS}(u_2, a'_1) = (a'_2, u_3)$, then u_2 is replaced by u'_2, and so on. This process continues, generating $a'_1, a'_2, \ldots, a'_k$ and $u'_1, \ldots, u'_k$. The tableau $t' = \mathrm{INSERT}(t,a)$ has rows $u'_1, \ldots, u'_k$, and a last row (possibly empty) consisting of a'_k. It turns out that $\mathrm{INSERT}(t,a)$ is a tableau [3].

Example 1.3.1. For t as in Example 1.3.1, we have

$$\mathrm{INSERT}(t,2) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2}\end{array},$$

since $\mathrm{ROWINS}(11333,2) = (3,11233)$, $\mathrm{ROWINS}(24,3) = (4,23)$, and $\mathrm{ROWINS}(3,4) = (\emptyset, 34)$.

1.4. **Insertion Tableau of a Word.** An arbitrary sequence $a_1 \cdots a_k$ in A_n will be called a word in A_n. The set of all words in A_n is denoted by A_n^*. This set may be regarded as a monoid under concatenation, with identity element as the empty word, denoted by $\emptyset$.

Definition 1.4.1. The insertion tableau $P(w)$ of a word w is defined recursively as:

$$P(\emptyset) = \emptyset \tag{1}$$

$$P(a_1 \cdots a_k) = \mathrm{INSERT}(P(a_1 \cdots a_{k-1}), a_k). \tag{2}$$

Example 1.4.2. Take $w = 133324132$. Then using Definition 1.4.1, then sequentially inserting the terms of w into the empty tableau $\emptyset$ gives the sequence of tableaux:

$$\begin{ytableau} 1 \end{ytableau}, \begin{ytableau} 1 & 3 \end{ytableau}, \begin{ytableau} 1 & 3 & 3 \end{ytableau}, \begin{ytableau} 1 & 3 & 3 & 3 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 3 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 3 & 4 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 1 & 3 & 3 & 4 \\ 2 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 1 & 3 & 3 & 3 \\ 2 & 4 \\ 3 \end{ytableau},$$

and finally, the insertion tableau $P(w) = \begin{ytableau} 1 & 1 & 2 & 3 & 3 \\ 2 & 3 \\ 3 & 4 \end{ytableau}$.

1.5. **Greene's Theorem.** Given a word $w = a_1 a_2 \cdots a_l$, a subword is a word of the form

$$v = a_{i_1} a_{i_2} \cdots a_{i_k},$$

for some $1 \leq i_1 < i_2 < \cdots < i_k$. We say that the subword v is a row if $a_{i_1} \leq a_{i_2} \leq a_{i_k}$. The subword v as above is said to be disjoint from a subword $u = a_{j_1} a_{j_2} \cdots a_{j_h}$ if the sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_h\}$ of indices are disjoint.

Given a word w, its kth Greene invariant *Greene invariant* $a_k(w)$ are defined as the maximal cardinality of a union of k pairwise disjoint row subwords.

Schensted [5] showed that the first Greene invariant $a_1(w)$ of a word is the length of the first row of its insertion tableau $P(w)$. For instance, the word w from Example 1.4.2 has longest increasing row subword of length 5, and its insertion tableau has first row of length 5. His theorem was generalized by Curtis Greene:

Theorem 1.5.1 (Greene [2])**.** *For any $w \in A_k$, suppose that the insertion tableau $P(w)$ has l rows of length $\lambda_1, \ldots, \lambda_l$. Then, for each $k = 1, \ldots, l$, $a_l(w) = \lambda_1 + \cdots + \lambda_k$.*

1.6. **Knuth Relations and the Plactic Monoid.** The most elegant proof of Greene's theorem (Theorem 1.5.1) proceeds via the notions of Knuth equivalence and the plactic monoid (see [4]).

The plactic monoid $\mathrm{pl}_0(A_n)$ is the quotient of the monoid A_n^* by the submonoid generated by the Knuth relations:

$$xzy \equiv zxy \text{ if } x \leq y < z, \tag{K1}$$

$$yxz \equiv yzx \text{ if } x < y \leq z. \tag{K2}$$

On a more concrete level, it is the set of words $w \in A_n$ modulo Knuth equivalence, were words v and w are said to be Knuth equivelent if w can be obtained from v by a sequence of moves of the form $(K1)$ and $(K2)$ involving any three letters of the words obtained at each stage. For example,

$$113\mathbf{332} \equiv_{K2} 11\mathbf{332}3 \equiv_{K2} 1\mathbf{132}33 \equiv_{K1} \mathbf{131}233 \equiv_{K1} 311233.$$

At each stage, the letters to which the Knuth moves will be applied to obtain the next stage are highlighted.

Definition 1.6.1. If t is a tableau with rows $u_1, \ldots, u_l$, then its reading word is obtained by concatenating its rows, starting from bottom to top: $\mathrm{read}(t) = u_l u_{l-1} \cdots u_1$.

Example 1.6.2. The reading word of the tableau t from Example 1.1.2 is 32411333.

A tableau is easily recovered from the reading word. Line breaks are inserted after each letter that is followed by a strictly smaller one. For example, 32411333 is broken as 3/24/1133, recovering the rows of the tableau t. However, it is easy to construct examples of words which are not reading words of tableau. Thus $\mathrm{read} : \mathrm{Tab}_n \to A_n^*$ is an injective function. Following Lascoux, Leclerc and Thibon [4] tableaux are identified with their reading words, and a word in A_n^* is called a tableau if it lies in the image of read.

Theorem 1.6.3. *Every word $w \in A_n^*$ is Knuth equivalent to the reading word of $P(w)$. Moreover, if $t, t' \in \mathrm{Tab}_n$ have $\mathrm{read}(t) \equiv \mathrm{read}(t')$, then $t = t'$. Consequently, $P(w)$ is the unique tableau in the Knuth-equivalence class of w.*

The plactic proof of Greene's theorem proceeds via Theorem 1.6.3—it is easy so see that Greene invariants are unchanged under Knuth moves, and that if $t \in \mathrm{Tab}_n(\lambda)$, where $\lambda = (\lambda_1, \ldots, \lambda_l)$, then

$$a_k(\mathrm{read}(t)) = \lambda_1 + \cdots + \lambda_k \text{ for } k = 1, \ldots, l.$$

Theorems 1.5.1 and 1.6.3 are special cases of their timed versions which will be proved in Section 2, inspired by the proof in [4].

2. A Timed Version of Greene's Theorem

2.1. Timed Tableaux.

Definition 2.1.1 (Timed Word)**.** A timed word of length r in the alphabet A is a peicewise-constant right-continuous function $w : [0, r) \to A_n$. We write $l(w) = r$. In other words, for some finite sequence $0 = r_0 < r_1 < \ldots < r_k = r$ of transition points, and letters $c_1, \ldots, c_k$ in A, $w(x) = c_i$ if $r_{i-1} \leq x < r_i$. Given such a function, we write

$$w = c_1^{t_1} c_2^{t_2} \cdots c_k^{t_k}, \ , \tag{3}$$

where $t_i = r_i - r_{i-1}$. We call this the *exponential string* for w.

The exponential string, as defined above, is not unique; if two successive letters c_i and c_{i+1} are equal, then we can merge them, replacing $c_i^{t_i} c_{i+1}^{t_{i+1}} = c_i^{t_i + t_{i+1}}$.

The above definition is a finite variant of Definition 3.1 of Alur and Dill [1], where $r = \infty$, and there is an infinite increasing sequence of transition points.

Given timed words w_1 and w_2, their *concatenation* is defined in the most obvious manner—their exponential strings are concatenated (and if necessary, successive equal values merged). The monoid formed by all timed words in an alphabet A, with product defined by concatenation, will be denoted by $A^\dagger$. We take A to be $A_n = \{1, \ldots, n\}$. The submonoid of $A_n^\dagger$ consisting of timed words where the exponents $t_1, t_2, \ldots, t_k$ in exponential string (3) are integers is the free monoid A_n^* from Section 1. In fact, all definitions and theorems in this section will specialize to those of Section 1 when the exponents are integral.

Definition 2.1.2 (Timed Subword)**.** Given a timed word $w : [0, r) \to A_n$, and $S \subset [0, r)$ a finite disjoint union of intervals of the form $[a, b) \subset [0, r)$, the timed subword of w with respect to S is defined as the timed word:

$$w_S(t) = w(\inf\{u \in [0, r) \mid \mathrm{meas}([0, u) \cap S) \geq t\}) \text{ for } 0 \leq t < \mathrm{meas}(S).$$

Given two words v and w, v is said to be a subword of w if there exists $S \subset [0, r)$ as above such that $v = w_S$. Subwords $v_1, \ldots, v_k$ of w are said to be pairwise disjoint if there exist pairwise disjoint subsets $S_1, \ldots, S_k$ as above such that $v_i = w_{S_i}$ for $i = 1, \ldots, k$.

A *timed row* is, by definition, a weakly increasing timed word. Every timed word w has a unique decomposition into rows:

$$w = u_l u_{l-1} \cdots u_1,$$

such u_i is a row for each $i = 1, \dots, l$, and $u_i u_{i-1}$ is not a row for any $i = 2, \dots, l$. We shall refer to such a decomposition as the row decomposition of w. Given two rows u and v, say that u is dominated by v (denoted $u \lhd v$) if $l(u) \geq l(v)$ and $u(t) < v(t)$ for all $0 \leq t < l(v)$.

Definition 2.1.3 (Timed Tableau)**.** A timed tableau in A_n is a timed word w in A_n with row decomposition $w = u_l u_{l-1} \cdots u_1$ such that $u_1 \lhd \cdots \lhd u_l$. The shape of w is the weakly decreasing tuple $(l(u_1), l(u_2), \dots, l(u_l))$ of positive real numbers (henceforth called a *real partition*), and the wieght of w is the vector:

$$\mathrm{wt}(w) = (m_1, \dots, m_n),$$

where m_i is the Lebesgue measure of the pre-image of i under t, i.e., $m_i = \mathrm{meas}(w^{-1}(i))$.

The above is a direct generalization of the notion of the reading word of a tableau in the sense of Section 1.

Example 2.1.4. $w = 3^{0.8}4^{1.1}1^{1.4}2^{1.6}3^{0.7}$ is a timed tableau in A_5 of shape $(3.7, 1.9)$ and weight $(1.4, 1.6, 1.5, 1.1, 0)$.

2.2. **Timed Insertion.** Given a timed word w and $0 \leq a < b \leq l(w)$, write $w_{[a,b)}$ for the timed word of length $b - a$ such that

$$w_{[a,b)}(t) = w(a + t) \text{ for } 0 \leq t < b - a.$$

We call $w_{[a,b)}$ a segment of w. If $a = 0$, then $w_{[a,b)}$ is called an initial segment of w.

Definition 2.2.1 (Timed row insertion)**.** Given a timed row w, define the insertion $\mathrm{ROWINS}(w, c^{t_c})$ of c^{t_c} into w as follows: if $w(t) \leq c$ for all $0 \leq t < l(u)$, then

$$\mathrm{ROWINS}(w, c^{t_c}) = (\emptyset, wc^{t_c}).$$

Otherwise, there exists $0 \leq t < l(u)$ such that $w(t) > c$. Let

$$t_0 = \min\{0 \leq t < u(l) \mid w(t) > c\}.$$

Define

$$\mathrm{ROWINS}(w, c^{t_c}) = \begin{cases} (w_{[t_0, t_0+t_c)}, w_{[0,t_0)} c^{t_c} w_{[t_0+t_c, l(w))}) & \text{if } l(u) - t_0 > t_c, \\ (w_{[t_0, l(u))}, w_{[0,t_0)} c^{t_c}) & \text{if } l(u) - t_0 \leq t_c. \end{cases}$$

If u is a row of the form $c_1^{t_1} \cdots c_l^{t_l}$. Define $\mathrm{ROWINS}(w, u)$ by induction on l as follows: Having defined $(v', w') = \mathrm{ROWINS}(w, c_1^{t_1} \cdots c_{l-1}^{t_{l-1}})$, let $(v'', w'') = \mathrm{ROWINS}(w', c_l^{t_l})$. Then define

$$\mathrm{ROWINS}(w, u) = (v'v'', w'').$$

Example 2.2.2. $\mathrm{ROWINS}(1^{1.4}2^{1.6}3^{0.7}, 1^{0.7}2^{0.2}) = (2^{0.7}3^{0.2}, 1^{2.1}2^{1.1}3^{0.5})$.

Definition 2.2.3 (Timed Tableau Insertion)**.** Let w be a timed tableau with row decomposition $u_l \dots u_1$, and let v be a timed row. Then $\mathrm{INSERT}(w, v)$, the insertion of v into w is defined as follows: first v is inserted into u_1. If $\mathrm{ROWINS}(u_1, v) = (v'_1, u'_1)$, then v'_1 is inserted into u_2; if $\mathrm{ROWINS}(u_2, v'_1) = (v'_2, u'_2)$, then v'_2 is inserted in u_3, and so on. This process continues, generating $v'_1, \dots, v'_l$ and $u'_1, \dots, u'_l$. $\mathrm{INSERT}(t, v)$ is defined to be $v'_l u'_l \cdots u'_1$. Note that it is quite possible that $v'_l = \emptyset$.

Example 2.2.4. If w is the timed tableau from Example 2.1.4, then

$$\mathrm{INSERT}(w, 1^{0.7}2^{0.2}) = 3^{0.7}4^{0.2}2^{0.7}3^{0.3}4^{0.9}1^{2.1}2^{1.1}3^{0.5}.$$

Definition 2.2.5. Given two real partition $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_{l-1})$, we say that μ *interleaves* λ if the inequalities

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{l-1} \geq \mu_{l-1} \geq \mu_l.$$

In other words, the successive parts of μ lie in-between the successive parts of λ.

Theorem 2.2.6. *For any timed tableau w in A_n and any timed row v in A_n,* $\mathrm{INSERT}(w, v)$ *is again a timed tableau in A_n. We have*

$$\mathrm{wt}(\mathrm{INSERT}(w, v)) = \mathrm{wt}(w) + \mathrm{wt}(v),$$

and $\mathrm{shape}(w)$ *interleaves* $\mathrm{shape}(\mathrm{INSERT}(w, v))$.

The proof of Theorem 2.2.6 can be broken down into three preparatory lemmas:

Lemma 2.2.7. *Suppose u and v are rows, and* $(v', u') = \mathrm{ROWINS}(u, v)$. *Then $v \lhd v'$.*

Proof. Going through the cases in Definition 2.2.1 shows that v' is a concatenation of segments of u, each of which is displaced by a term in the exponential string of an initial segment of v. Moreover, the value of the term in v is stricly less than the minimum value of the segment that it displaces. This shows that $v' \lhd v$. □

Lemma 2.2.8. *Suppose u and v are rows, and* $(v', u') = \mathrm{ROWINS}(u, v)$. *Then $u' \lhd v$.*

Proof. From the argument in the proof of Lemma 2.2.7, it follows that the row u' contains a subword v'' that equals the initial segment of v of length $l(v')$. By Lemma 2.2.7, $v''(t) < v(t)$ for all $0 \leq t < l(v')$, and since u' is a row, and v'' is a subword of u', $v''(t) \leq u'(t)$ for all $0 \leq t < l(v')$. It follows that $u' \lhd v$. □

Lemma 2.2.9. *Suppose* $u_1 \lhd u_2$ *and* $v_1 \lhd v_2$, $(v_1', u_1') = \mathrm{ROWINS}(u_1, v_1)$, *and* $(v_2', u_2') = \mathrm{ROWINS}(u_2, v_2)$. *Then* $u_1' \lhd u_2'$ *and* $v_1' \lhd v_2'$.

Proof of Theorem 2.2.6. Suppose w has row decomposition $u_l \cdots u_1$. Let $(v_1', u_1') = \mathrm{ROWINS}(u_1, v)$. By Lemma 2.2.8, $v \lhd v_1'$. Now suppose $(v_2', u_2') = \mathrm{ROWINS}(u_2, v_1')$. By Lemma 2.2.9, $v_1' \lhd v_2'$, and $u_1' \lhd u_2'$. Continuing in this manner, since $(v_{i+1}', u_{i+1}') = \mathrm{ROWINS}(u_{i+1}, v_i')$, it follows that $u_i' \lhd u_{i+1}'$ for $i = 1, \ldots, l-1$ by induction on i. Finally, it remains to show that $u_l' \lhd v_l'$, but this follows from Lemma 2.2.7. $\square$

Definition 2.2.10 (Insertion Tableau of a Timed Word)**.** Let w be a timed word with row decompositon $u_1 \cdots u_l$. The insertion tableau of w is defined as:

$$P(w) = \mathrm{INSERT}(\cdots \mathrm{INSERT}(\mathrm{INSERT}(u_1, u_2), u_3), \ldots, u_l).$$

Example 2.2.11. If $w = 3^{0.8}1^{0.5}4^{1.1}1^{0.9}2^{1.6}3^{0.7}1^{0.7}2^{0.2}$ has four rows in its row decomposition. $P(w)$ is calculated via the following steps:

w	$P(w)$
$3^{0.8}$	$3^{0.8}$
$3^{0.8}1^{0.5}4^{1.1}$	$3^{0.5}1^{0.5}3^{0.3}4^{1.1}$
$3^{0.8}1^{0.5}4^{1.1}1^{0.9}2^{1.6}3^{0.7}$	$3^{0.8}4^{1.1}1^{1.4}2^{1.6}3^{0.7}$
$3^{0.8}1^{0.5}4^{1.1}1^{0.9}2^{1.6}3^{0.7}1^{0.7}2^{0.2}$	$3^{0.7}4^{0.2}2^{0.7}3^{0.3}4^{0.9}1^{2.1}2^{1.1}3^{0.5}$

Definition 2.2.12 (Schuetzenberger Involution on Timed Words)**.** Given $w = c_1^{t_1} \cdots c_k^{t_k} \in A_n^\dagger$, define

$$w^\sharp = (n - c_k + 1)^{t_k} \cdots (n - c_1 + 1)^{t_1}, \tag{4}$$

in effect, reversing both the order on the alphabet, and the positional order of letters in the timed word.

Lemma 2.2.13. *Suppose* $\mathrm{ROWINS}(w, u) = (u', w')$, *and* $l(u') = l(u)$. *Then* $\mathrm{ROWINS}(w'^\sharp, u'^\sharp) = (u^\sharp, w^\sharp)$.

Corollary 2.2.14. *The timed row insertion algorithm gives rise to a bijection:*

$$\mathrm{ROWINS} : \mathrm{tR}_n(r) \times \mathrm{tR}_n(s) \tilde{\to} \{(u', w') \in R_n(s') \times R_n(r + s - s') \mid 0 \leq s' \leq s,\ u' \rhd w'\}.$$

Theorem 2.2.15 (Timed Pieri Rule)**.** *The timed insertion algorithm gives rise to a bijection:*

$$\mathrm{INSERT} : \mathrm{tt}_n(\lambda) \times \mathrm{tR}_n(r) \tilde{\to} \coprod_{\substack{\lambda \text{ interleaves } \mu \\ l(\lambda)+r=l(\mu)}} \mathrm{tt}_n(\mu)$$

2.3. **Timed Knuth Equivalence and the Timed Plactic Monoid.**

Definition 2.3.1 (Timed Knuth Relations)**.** The first timed Knuth relation is, given timed rows x, y, and z:

$$xzy \equiv zxy, \tag{κ_1}$$

if xyz is a row, $l(z) = l(y)$, and the last letter of y is strictly less than the first letter of z.

The second timed Knuth relation is, given timed rows x, y, and z:

$$yxz \equiv yzx, \tag{κ_2}$$

if xyz is a row, $l(x) = l(y)$, and the last letter of x is strictly less than the first letter of y.

The timed plactic monoid $\mathrm{pl}(A_n)$ is the quotient $A^\dagger/\equiv$, where $\equiv$ is the congruence generated by the timed Knuth relations (κ_1) and (κ_2).

In other words, two elements of $A^\dagger$ are said to differ by a Knuth move if they are of the form uv_1w and uv_2w, where v_1 and v_2 are terms on opposite sides of one of the timed Knuth relations (κ_1) and (κ_2). Knuth equivalence $\equiv$ is the equivalence relation generated by Knuth moves. Since this equivalence is stable under left and right multiplication in $A^\dagger$, the concatenation product on $A^\dagger$ descends to a product on the set $\mathrm{pl}(A)$ of Knuth equivalence classes, giving it the structure of a monoid.

Lemma 2.3.2. *Then v and w differ by a Knuth move (κ_1) if and only if $v^\sharp$ and $w^\sharp$ (see Definition 2.2.12) differ by a Knuth move (κ_2).*

Proof. When the involution $w \mapsto w^\sharp$ is applied to the Knuth relation (κ_1), the Knuth relation (κ_2) is obtained. □

2.4. **Greene's Invariants for Timed Words.**

Definition 2.4.1 (Greene's Invariants for Timed Words)**.** Given a word $w \in A_n^\dagger$, its kth Greene's invariant $a_k(w)$ is defined to be the maximum possible sum of lengths of a set of k pairwise disjoint row subwords of w:

$$\begin{aligned} a_k(w) = \max\{l(u_1) + \cdots + l(u_k) \mid u_1, \ldots, u_k \text{ are pairwise disjoint} \\ \text{subwords and each } u_i \text{ is a row } \} \end{aligned}$$

Lemma 2.4.2. *If w is a timed tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_l)$, then for each $1 \le k \le l$,*

$$a_k(w) = \lambda_1 + \cdots + \lambda_k.$$

Proof. This proof is very similar to the proof of the corresponding result in [4]. Indeed, $u_1, \dots, u_k$ are pairwise disjoint subwords that are rows, so

$$a_k(w) \geq \lambda_1 + \cdots + \lambda_l.$$

Conversely, any row subword of w will cannot consist of overlapping segments from two different rows u_i and u_j of w, because if $i > j$, then $u_i(t) > u_j(t)$, but in the row decomposition of w, u_i occurs before u_j. Therefore, k disjoint subwords can have length at most the sum of lengths of the largest k rows of w, which is $\lambda_1 + \ldots + \lambda_k$. □

Lemma 2.4.3. *If two timed words are Knuth equivalent, then they have the same Greene invariants.*

Proof. It suffices to prove that if two words differ by a Knuth move they have the same Greene invariants. For the Knuth move (κ_1), suppose that xyz is a timed row with $l(z) = l(y)$, and the last letter of y is strictly less than the first letter of z. For any timed words w and u, we wish to show that Greene's invariants coincide for $wxzyu$ and $wzxyu$. Now suppose that $v_1, \dots, v_k$ are pairwise disjoint row subwords of $wxzyu$ whose lengths add up to $a_k(wxzyu)$. We may write $v_i = w_i x_i z_i y_i u_i$ for each i, where w_i, x_i, z_i, y_i and u_i are row subwords of w, x, z, y and u respectively.

Since the last letter of y is strictly smaller than the first letter of z, it cannot be that $y_i \neq \emptyset$ and $z_i \neq \emptyset$ simultaneously for the same i. Renumber $v_1, \dots, v_k$ in such a way that $v_i = w_i x_i z_i u_i$ for $i = 1, \dots, r$ and $v_i = w_i x_i y_i u_i$ for $i = r+1, \dots, k$. If $x_i = \emptyset$ for $i = 1, \dots, r$, then all the v_i's remain row subwords of $wzxyu$, and so $a_k(wzxyu) \geq a_k(wxzyu)$. If $y_i = \emptyset$ for $i = r+1, \dots, k$, then set

$$\begin{aligned} v_1' &= w_1 x_1 y u_1, \\ v_i' &= w_i x_i u_i \text{ for } i = 2, \dots, k. \end{aligned}$$

Then v_i' are pairwise disjoint row subwords of $wzxyu$, and $l(v_1') + \cdots + l(v_k') \geq l(v_1) + \cdots l(v_k)$, since $l(y) = l(z) \geq l(z_1) + \cdots + l(z_r)$. It follows that $a_k(wzxyu) \geq a_k(wxzyu)$.

If at least one $x_i \neq \emptyset$ and one $y_i \neq \emptyset$, assume that x_1 has the least first letter among $x_1, \dots, x_r$, and that y_k has the largest last letter among $y_{r+1}, \dots, y_k$. Let x_0 be the row subword of x obtained by concatenating the segements of $x_1, \dots, x_r$ in the order in which they appear in x (so x_0 is a row of length $l(x_1) + \cdots + l(x_r)$). Let y_0 be the row subword of y obtained by concatenating the segments of $y_{r+1}, \dots, y_k$ in the order in which they appear in y (so $y_0$is a row of lenght $l(y_{r+1}) + \cdots + l(y_k)$).

Define

$$\begin{aligned}
v_1' &= w_1 x_0 y_0 u_k \\
v_i' &= w_i z_i u_i \text{ for } i = 2, \ldots, r, \\
v_i' &= w_i x_i u_i \text{ for } i = r+1, \ldots, k, \\
v_k' &= w_k z_1 u_1.
\end{aligned}$$

Then $v_1', \ldots, v_k'$ are pairwise disjoint row subwords of $wzxyu$ with total length $l(v_1) + \cdots + l(v_k)$, so $a_k(wzxyu) \geq a_k(wxzyu)$. The reverse inequality $a_k(wzxyu) \geq a_k(wxzyu)$ is obvious, since every row subword of $wzxyu$ is also a row subword of $wxzyu$. It follows that $a_k(wzxyu) = a_l(wxzyu)$ for all k.

For the Knuth move (κ_2) a similar argument can be given, however, a more elegant method is to use Lemma 2.3.2, noting that $a_k(w) = a_k(w^\sharp)$ for all $k \geq 1$ and all $w \in A_n^\dagger$. □

2.5. **Timed Tableaux and Timed Knuth Equivalence.**

Lemma 2.5.1. *Every timed word is Knuth equivalent to its timed insertion tableau.*

Proof. It suffices to show that, for every timed row u, and every term c^t, if $(v, u') = \mathrm{ROWINS}(u, c^t)$, then $uc^t \equiv vu'$. If $u(t) \leq c$ for all $0 \leq t < l(u)$, there is nothing to show. Otherwise, a segment v of u, beginning at t_0, and of length $t_1 = \min(l(u) - t_0, t)$ is displaced by the segment c^{t_1} of c^t. Write $u = u'vu''$. It suffices to show $u'vu''c^{t_1} \equiv vu'c^{t_1}u''$. But this can be done in two Knuth moves as follows (the segment to which the Knuth move is applied is underlined):

$$u'\underline{vu''c^t} \equiv_{\kappa_2} u'\underline{vc^tu''} = \underline{u'vc^t}u'' \equiv_{\kappa_1} \underline{vu'c^t}u'',$$

completing the proof of the lemma. □

For any real partition λ, let $\mathrm{tt}_n(\lambda)$ denote the set of all timed tableaux in A_n of shape λ. Let $\mathrm{tR}_n(r)$ denote the set of timed rows in A_n of length r. Given $w \in A_n^\dagger$, let $\bar{w}$ denote the word in $A_{n-1}^\dagger$ whose exponential string is obtained by removing all terms of the form n^t with $t > 0$ from the exponential string of w. The word $\bar{w}$ is called the restriction of w to A_{n-1}.

Lemma 2.5.2. *For every timed tableau $w \in A_n^\dagger$,* $\mathrm{shape}(\bar{w})$ *interleaves* $\mathrm{shape}(w)$.

Proof. Suppose w has row decomposition $u_l u_{l-1} \cdots u_1$. Since n is the largest element of A_n, we meay write $u_i = u_i' n^{t_i}$ for some $t_i \geq 0$. Clearly $l(u_i) \geq l(u_i')$. Since w is semistandard, $l(u_i') \geq l(u_{i+1})$ for

$i = 1, \ldots, l-1$. It follows that the shape of w', which is $(l(u'_1), \ldots, l(u'_l))$ interleaves the shape of w, which is $(l(u_1), \ldots, l(u_l))$. $\square$

Lemma 2.5.3. *If $v, w \in A_n^\dagger$ are Knuth equivalent, then their restrictions to A_{n-1}, $\bar{v}$ and $\bar{w}$ are Knuth equivalent in A_{n-1}.*

Proof. Applying the restriction to A_{n-1} map $w \mapsto \bar{w}$ to both sides of the Knuth relation (κ_1) gives: $x\bar{z}y$ and $\bar{z}xy$, which are still Knuth equivalent. $\square$

Lemma 2.5.4. *If timed tableaux v and w in $A_n^\dagger$ are Knuth equivalent, then $v = w$.*

Proof. We prove this by induction on n. The base case, where $n = 1$, is trivial. Now suppose v and w are Knuth equivalent tableaux in $A_n^\dagger$. Since they are Knuth equivalent, they have the same Greene invariants (Theorem 2.4.3). Therefore by Lemma 2.4.2, v and w have the same shape. By Lemma 2.5.3, $\bar{v}$ and $\bar{w}$ are Knuth equivalent tableaux. By induction $\bar{v} = \bar{w}$. But v and w are obtained from $\bar{v} = \bar{w}$ by adding segments of the form n^t to the rows of the latter to obtain the common shape of v and w, and so must be equal too. $\square$

Theorem 2.5.5. *For any $w \in A_n^\dagger$, $P(w)$ is the unique timed tableau in its timed Knuth equivalence class.*

Proof. By Lemma 2.5.1, $P(w)$ lies in the timed Knuth equivalence class of w. By Lemma 2.5.4, $P(w)$ is the only timed tableau in this class. $\square$

2.6. The Greene Invariants of a Timed Tableau.

Lemma 2.6.1. *Let w be a timed tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_l)$. Then the Greene invariants of w are given by:*

$$a_k(w) = \lambda_1 + \cdots + \lambda_k \text{ for } k = 1, \ldots, l.$$

Proof. The proof is similar to that of Theorem 6.2.1 in [4].

If w has row decomposition $u_l u_{l-1} \cdots u_1$, then the rows $u_1, \ldots, u_k$ form a set of pairwise disjoint subwords of w which are all rows, and so $a_k(w) \geq \lambda_1 + \ldots + \lambda_k$.

For the converse, suppose that $v_1, \ldots, v_k$ are pairwise disjoint row subwords. Each v_i in turn can be written as a concatenation $v_{il} \cdots v_{i1}$ where v_{ij} is a subword of u_j. In the notation of Definition 2.1.2, suppose that $v_{ij} = (u_j)_{S_{ij}}$. Then the fact that v_i is a row and the tableau condition (Definition 2.1.3) implies that all of S_{ij} lies strictly to the left of $S_{ij'}$ for $j > j'$. Thus $l(v_i)$ cannot exceed the length of any single row of w. Similarly, the total lengths of $v_1, \ldots, v_k$ cannot exceed the lengths of k longest rows of w. $\square$

3. The Real RSK Correspondence

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The Institute of Mathematical Sciences, Chennai

Homi Bhabha National Institute, Mumbai