Vibrations

The study of vibrations is concerned with the oscillating motion of elastic bodies and the force associated with them. All bodies possessing mass and elasticity are capable of vibrations. The characteristic feature of vibration is its periodicity i.e. there is a movement or displacement or variation in the value of a physical quantity that repeats over and over again. In mechanical systems the periodicity refers to displacement while in the electrical system, it is related to current and voltage. At the microscopic level, atoms and molecules execute periodic vibrations in the solid state. The propagation of light involves the vibrations of electric and magnetic fields while that of sound, the periodic motion of atoms and molecules. Most engineering machines and structures experience vibrations to some degree and their design generally requires consideration of their oscillatory motions.

Vibrations are classified in to three types: (i) **Free vibrations (ii) Damped Vibrations (iii) Forced Vibrations.**

- ### Free Vibrations:

The vibrations in which the body vibrates with its natural frequency when left free to itself, without decrease in its amplitude are called free vibrations.

In free vibrations only restoring force acts on the body. This restoring force is proportional to the displacement and acts always towards the mean position. For example vibrations of simple pendulum and simple spring mass system. In practice, it is not possible to eliminate friction completely. Actually the amplitude of the vibrating body gradually decreases to zero as a result of friction. In the above examples, friction is less and can be considered as the examples of free vibrations.

Theory / Analysis of free vibrations:

Let us consider the motion of a particle of mass m acted upon by a restoring force proportional to its displacement. The restoring force may be expressed by $-\mu y$ where μ is constant of proportionality and negative sign indicates that the restoring force acts in opposite direction to the displacement.

Now according to Newton's law

Force= mass ×acceleration= $m\frac{d^2y}{dt^2}$

$$m\frac{d^2y}{dt^2} = -\mu y$$

$$\frac{d^2y}{dt^2} = -\frac{\mu}{m}y = -\omega^2 y$$

where $\omega^2 = \frac{\mu}{m}$

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \text{……………...(1)}$$

Equation (1) is homogeneous differential equation of second order, and its solution is of the form,

$$y = e^{\alpha t} \quad \text{……………(2)}$$

Then $\frac{dy}{dt} = \alpha.\ e^{\alpha t}$

and $\frac{d^2y}{dt^2} = \alpha^2 e^{\alpha t}$

Substituting these values in eq (1), we get

$$e^{\alpha t}\alpha^2 + \omega^2 e^{\alpha t} = 0$$

$$e^{\alpha t}(\alpha^2 + \omega^2) = 0$$

Here, $e^{\alpha t} \neq 0$, because it leads to trivial solution.

$$\alpha^2 + \omega^2 = 0$$
$$\therefore \alpha^2 = -\omega^2$$
$$\alpha = \pm i\omega$$

Thus, the general solution of eq (1) is given by

$$y = A_1 e^{i\omega t} + A_2 e^{-i\omega t} \qquad \text{…………..(3)}$$

where A_1 and A_2 are some constants

$$y = A_1\left[\cos(\omega t) + i\ Sin(\omega t)\right] + A_2\left[\cos(\omega t) - i\ Sin(\omega t)\right]$$
$$y = (A_1 + A_2)\cos(\omega t) + i(A_1 - A_2)Sin(\omega t)$$

Let $(A_1+A_2)=A\ \sin\phi$

and $i(A_1-A_2)=A\ \cos\phi$

$$y = A\ \sin\phi.\cos(\omega t) + A\ \cos\phi.\sin(\omega t)$$

$$y = A\ \sin(\omega t + \phi) \qquad \ldots\ldots\ldots\ldots\ldots\ldots.. \quad (4)$$

Equation (4) represents the displacement of the body in free vibrations and A is maximum displacement called amplitude.

The value of 'y' repeats when 't' is increased by $(2\pi/\omega)$. In eqn (4), if 't' is replaced $(t+\frac{2\pi}{\omega})$, we get

$$y = A\ \sin\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right]$$
$$y = A\ \sin\left[\omega t + 2\pi + \phi\right]$$
$$y = A\ \sin(\omega t + \phi)$$

Hence the period of vibrations, $T = \frac{2\pi}{\omega}$ and frequency of vibration, $f = \frac{1}{T} = \frac{\omega}{2\pi}$

Thus, in case of free vibrations, the amplitude (A) remains constant with time. In other words, the total energy of the vibrating body remains constant in free vibrations.

- ## <u>Damped Vibrations:</u>

Free vibrations are purely ideal case. In practice, for a body executing vibrations, the amplitude keeps on deceasing because of resistive and frictional forces. Hence the energy of vibrating body decreases and vibrations die out after some time. The motion is said to be damped by friction and such vibrations are called damped vibrations.

Theory of damped vibrations:

In damped vibrations, the body is subjected to,

(i) A restoring force, which is proportional to displacement but oppositely directed. This is written as -μy, where μ is a constant of proportionality or force constant.

(ii) A damping force (may be resistive or frictional), which is proportional to the velocity but oppositely directed. This may be written as -r dy/dt, where r is the damping force per unit velocity.

Since, Force = mass × acceleration = $m\frac{d^2y}{dt^2}$

Therefore the equation of motion of the particle in damped vibration is given by

$$m\frac{d^2y}{dt^2} = -\mu\ y - r\ \frac{dy}{dt}$$

or
$$\frac{d^2y}{dt^2} + \frac{r}{m}\ \frac{dy}{dt} + \frac{\mu}{m} y = 0$$

or
$$\frac{d^2y}{dt^2} + 2b\ \frac{dy}{dt} + \omega^2 y = 0 \quad \text{……………..(1)}$$

where r/m=2b and $\mu/m=\omega^2$

Eq.(1) is a homogeneous differential equation of second order.

Let its solution be $y = e^{\alpha t}$ …………….. (2)

where α is arbitrary constant.

Differentiating eq.(2) with respect to *t*, we get

$$\frac{dy}{dt} = \alpha .\ e^{\alpha t}$$

$$\frac{d^2y}{dt^2} = \alpha^2 e^{\alpha t}$$

Substituting these values in eq.(1), we have

$$\alpha^2 e^{\alpha t} + 2b\ \alpha e^{\alpha t} + \omega^2 e^{\alpha t} = 0$$

or $$e^{\alpha t}(\alpha^2 + 2b\ \alpha + \omega^2) = 0$$

As $e^{\alpha t} \neq 0, \quad \therefore \alpha^2 + 2b\ \alpha + \omega^2 = 0$

This gives $$\alpha = -b \pm \sqrt{\left(b^2 - \omega^2\right)}$$

$\therefore$ The general solution of eq.(1) is given by

$$y = A_1 \exp\left[-b + \sqrt{\left(b^2 - \omega^2\right)}\right]t + A_2 \exp\left[-b - \sqrt{\left(b^2 - \omega^2\right)}\right]t \quad \ldots\ldots\ldots(3)$$

where A_1 and A_2 are arbitrary constants.

Depending upon the relative values of b and ω following three cases are possible.

- **Case 1: Over damping (**When $b^2 > \omega^2$) .

In this case $\sqrt{\left(b^2 - \omega^2\right)}$ is real and less than b. Hence the exponential terms in eq.(3) are both negative. Thus the displacement y consists of two terms, both dying off exponentially to zero without performing any oscillations. The rate of decrease of displacement is governed by the term $\left[-b + \sqrt{\left(b^2 - \omega^2\right)}\right]t$ as the other term reduced to zero quickly relative to it. This type of motion is called as over-damped or dead beat. Example:- pendulum moving in thick oil.

➢ **Case 2:: Under damping (when** $b^2<\omega^2$).

In this case, $\sqrt{(b^2-\omega^2)}$ is imaginary. Let us write $\sqrt{(b^2-\omega^2)} = i\sqrt{(\omega^2-b^2)} = i\,\beta$

where $\beta = \sqrt{(\omega^2-b^2)}$ *and* $i = \sqrt{(-1)}$

Eq.(3) now becomes

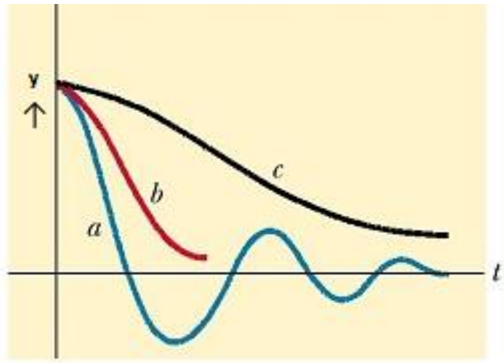


Figure: Graphs of displacement versus time for (a) an underdamped oscillator, (b) a critically damped oscillator, and (c) an overdamped oscillator.

$$y = A_1 \exp(-b+i\,\beta)t + A_2 \exp\,(-b-i\,\beta)t$$

$$= e^{-bt}\left[A_1\, e^{i\beta t} + A_2 e^{-i\beta t}\right]$$

$$= e^{-bt}\left[A_1\,(\cos\beta t + i\,\sin\beta t) + A_2(\cos\beta t - i\,\sin\beta t)\right]$$

$$= e^{-bt}\left[(A_1+A_2)\cos\beta t + i\,(A_1-A_2)\sin\beta t\right]$$

$$= e^{-bt}\left[a\sin\phi\cos\beta t + a\cos\phi\sin\beta t\right]$$

where $a\sin\phi = (A_1+A_2)$ *and* $a\cos\phi = i\,(A_1-A_2)$

$$= e^{-bt}\left[a\,\sin\,(\beta t+\phi)\right]$$

$$= a\,e^{-bt}\sin\left[\,\beta t + \phi\,\right]$$

$$= A\sin\left[\,\sqrt{(\omega^2-b^2)}\,t + \phi\,\right]$$

where $A = a\,e^{-bt}$ and this represents the amplitude of damped oscillator. The amplitude of the motion is continuously decreasing due the factor e^{-bt} which is called damping factor.

The value of $\sin\left[\,\sqrt{(\omega^2-b^2)}\,t+\phi\,\right]$ varies between +1 and -1, therefore, the amplitude also varies between $a\,e^{-bt}$ and $-a\,e^{-bt}$. The decay of the amplitude depends upon damping coefficient b. It is called "under damped" motion. Examples:- Motion of pendulum in air, Electric oscillations of L-C-R circuit.

Note: The time period of such oscillations is, $T = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\omega^2-b^2}}$, In this case the period is slightly increased or frequency decreased because the period is now $\frac{2\pi}{\sqrt{\omega^2-b^2}}$ while in the absence of damping it was $2\pi/\omega$.

Case 3: Critical damping ($b^2=\omega^2$)

If we put $b^2=\omega^2$ in eq. (3), then this solution does not satisfy the differential eq. (1). Let us consider $\sqrt{(b^2-\omega^2)}=h\to 0.$ Now eq. (3) reduces to

$$y=A_1\exp(-b+h)t+A_2\exp(-b-h)t$$

$$=e^{-bt}\left[A_1e^{ht}+A_2e^{-ht}\right]$$

$$=e^{-bt}\left[A_1(1+ht+.....)+A_2(1-ht+.....)\right]$$

$$=e^{-bt}\left[(A_1+A_2)+ht(A_1-A_2)+.....\right]$$

$$y=e^{-bt}\left[p+qt\right] \qquad ....(4)$$

where p=(A_1+A_2) and q=$h(A_1-A_2)$

Eq.(4) represents possible form of solution. It is clear that as t increases, the factor $(p+qt)$ increases but the factor e^{-bt} decreases. In this case, displacement y first increases due to factor $(p+qt)$ but at the same time reversal occurs due to exponential term e^{-bt} and the displacement approaches zero as t increases. In this case, particle tends to acquire its position of equilibrium more rapidly than in case -1. Such a motion is called critically damped motion. This type of motion is exhibited by many pointer instruments such as voltmeter, ammeter etc. in which the pointer moves to the correct position and comes to rest without any oscillation.

- ## Attenuation coefficient of vibrating system:

The characteristics of vibrating system under damping are described by three parameters namely (i) Logarithmic decrement (λ), (ii) Relaxation time (τ) and (iii) Quality factor (Q)

(i) Logarithmic decrement (λ):

Logarithmic decrement measures the rate at which the amplitude dies away. The amplitude of damped harmonic oscillator is given by $A = a\,e^{-bt}$, where b is damping constant. The amplitude gradually decreases with time. Let A_0, A_1, A_2, A_3... be the amplitudes at time t=0, 1T, 2T , 3T .... respectively, where T= period of oscillation. Then A_0=a, $A_1 = a\,e^{-bT}$, $A_2 = a\,e^{-b(2T)}$, $A_3 = a\,e^{-b(3T)}$......

Now $\frac{A_0}{A_1} = \frac{A_1}{A_2} = \frac{A_2}{A_3} = ... = e^{-bT} = e^{\lambda}$

where λ=bT. Here, λ is known as logarithmic decrement.

Taking the natural logarithms, we get

$$\lambda = \log_e \frac{A_0}{A_1} = \log_e \frac{A_1}{A_2} = \log_e \frac{A_2}{A_3}$$

Thus logarithmic decrement is defined as the natural logarithmic ratio of the two successive maximum amplitudes which are separated by one period.

(ii) Relaxation time (τ) :

The total energy of a damped oscillator at an instant 't' is given by

$$E = \frac{1}{2}\mu A^2$$

$$= \frac{1}{2}\mu (a\,e^{-bt})^2$$

$$= \frac{1}{2}\mu a^2\, e^{-2bt}$$

$$E = E_0\, e^{-2bt} \quad \textbf{................(1)}$$

Relaxation time is defined as the time duration in which the total energy of the vibrating system reduces to $(\frac{1}{e})$ times of its original value.

Let τ be the relaxation time. i.e. at t=τ, $E = \frac{E_0}{e}$, then from eq. (1), we get

$$\frac{E_0}{e} = E_0\, e^{-2b\tau}$$

$$\frac{1}{e} = \frac{1}{e^{2b\tau}}$$

$$2b\tau = 1$$

$$\therefore\ \tau = \frac{1}{2b} \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

This shows that, greater the damping, lower is the relaxation time.
From eqs. (1) and (2), we get

$$E = E_0\, e^{-t/\tau}$$

(iii) Quality factor (Q):

The quality factor is defined as 2π times the ratio of the energy stored in the vibrating system to the energy lost per period.

$$Q = 2\pi \frac{energy\ stored\ in\ the\ system}{energy\ lost\ per\ period} = 2\pi \frac{E}{PT}$$

where P is power dissipated and T is time period.

But $P = \frac{dE}{dt} = \frac{d}{dt}(E_0\, e^{-2bt})$

$$P = E_0\,(-2b).e^{-2bt} = -2bE$$

$$\therefore\ Q = 2\pi\left[\frac{E}{2bET}\right]$$

$$Q = \left(\frac{2\pi}{T}\right).\left(\frac{1}{2b}\right)$$

$$\therefore Q = \frac{\omega}{2b} \quad or \quad Q = \omega.\tau$$

This shows that, for low damping, the quality factor of the vibrating system is high.

- **<u>Forced Vibrations:</u>**

The loss of energy in a damped oscillator can be compensated by an external periodic force so that the vibrations are sustained.

"The vibrations in which the body vibrates with a frequency other than its natural frequency under the action of an external periodic force are called forced vibrations."

- **Theory of forced vibrations:**

In forced vibrations, the body is acted upon by,

(i) A restoring force proportional to the displacement but oppositely directed, given by (-μy), where μ is known as force constant.

(ii) A damping force proportional to velocity but oppositely directed, given by (-r dy/dt), where r is the damping force per unit velocity.

(iii) An external force, represented by F sin pt, where F is the maximum value of this force and p/2π is its frequency.

So the total force acting on the particle is given by $-\mu y - r\dfrac{dy}{dt} + F\sin pt$

By Newton's second law of motion this must be equal to the product of mass *m* of the particle and its instantaneous acceleration i.e. $m\dfrac{d^2y}{dt^2}$, hence

$$m\frac{d^2y}{dt^2} = -\mu y - r\frac{dy}{dt} + F\sin pt$$

or
$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + \mu y = F\sin pt$$

or
$$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{\mu}{m}y = \frac{F}{m}\sin pt$$

or
$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = f\sin pt \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

where $\frac{r}{m} = 2b$, $\frac{\mu}{m} = \omega^2$ and $\frac{F}{m} = f$

$b \rightarrow damping\ factor$

$\frac{\omega}{2\pi} \rightarrow natural\ frequency$

The equation (1) is differential equation of motion of the particle. When the steady state is set up, the particle vibrates with the frequency of applied force and not with its own natural frequency. The solution of this eq.(1) must be of the form,

$$y = A\sin(pt - \theta) \qquad \ldots\ldots\ldots\ldots\ldots \qquad \mathbf{(2)}$$

where A is the steady amplitude of vibrations and θ is the angle by which the displacement y lags behind the applied force F sin pt. A and θ being arbitrary constants.

Differentiating eq. (2), we have

$$\frac{dy}{dt} = A\,p\cos(pt - \theta)$$

$$\frac{d^2y}{dt^2} = -A\,p^2\sin(pt - \theta)$$

Substituting these values in eq. (1), we get

$$-A\,p^2\sin(pt-\theta) + 2b\,Ap\cos(pt-\theta) + \omega^2 A\sin(pt-\theta) = f\sin pt = f\sin\{(pt-\theta)+\theta\}$$

or $A(\omega^2 - p^2)\sin(pt-\theta) + 2b\,Ap\cos(pt-\theta) = f\sin(pt-\theta)\cos\theta + f\cos(pt-\theta)\sin\theta$

If this equation holds good for all values of 't', then the coefficients of $\sin(pt-\theta)$ and $\cos(pt-\theta)$ terms on both sides of this equation must be equal, Hence we have

$$A(\omega^2 - p^2) = f\cos\theta \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

$$and \qquad 2b\,A\,p = f\sin\theta \qquad \ldots\ldots\ldots\ldots\ldots..(4)$$

squaring and adding eqs. (3) and (4), we get

$$A^2(\omega^2 - p^2)^2 + 4b^2A^2p^2 = f^2$$

$$A = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2p^2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

while dividing eq.(4) by eq.(3), we get

$$\tan\theta = \frac{2b\,p}{(\omega^2 - p^2)}$$

$$\theta = \tan^{-1}\left(\frac{2b\,p}{\omega^2 - p^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots(6)$$

Eq.(5) gives the amplitude of forced vibrations while (6) its phase. Depending upon the relative values of p and ω**,** the following three cases are possible:

Case(i): when driving frequency is low i.e. p<<ω . In this case the amplitude of vibration is given by

$$A = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2\,p^2}}$$

$$A \approx \frac{f}{\omega^2} = \text{constant}$$

and $\theta = \tan^{-1}\left(\frac{2b\,p}{\omega^2 - p^2}\right) = \tan^{-1}(0) = 0$

This shows that the amplitude of vibration is independent of frequency of force. This amplitude depends on the magnitude of applied force and force constant. Also, the force and displacement are always in phase.

Case (ii): when frequency of applied force is very high (p>>ω)

In this case $A = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2\,p^2}}$

$$A = \frac{f}{\sqrt{p^4 + 4b^2\,p^2}} \approx \frac{f}{p^2}$$

and $\theta = \tan^{-1}\left(\frac{2b\,p}{\omega^2 - p^2}\right) = \tan^{-1}\left(-\frac{2b}{p}\right) = \tan^{-1}(-0) = \pi$

Thus in this case the amplitude of vibration decreases with increase in frequency of applied force and phase difference tends to π.

Case (iii): when p=ω i.e. frequency of force is equal to natural frequency of body.

In this case, the amplitude of vibration is given by

$$A = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2\ p^2}}$$

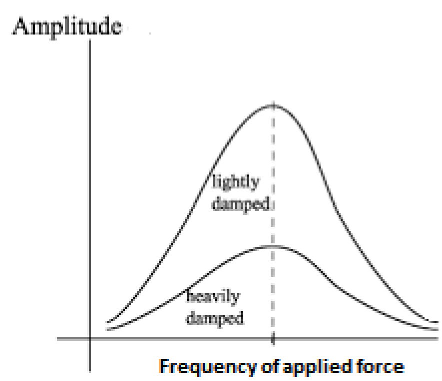
$$A = \frac{f}{2b\,p} = \frac{f}{r\,\omega} \qquad\qquad [\because \quad f = \frac{F}{m},\ 2b = \frac{r}{m}\ \ and\ \ p = \omega]$$

Also $\theta = \tan^{-1}\left(\frac{2b\,p}{\omega^2 - \omega^2}\right) = \tan^{-1}(\infty) = \pi/2$

This shows amplitude of vibration is maximum when p=ω. Then vibrating body is said to be in **resonance.** Also the amplitude of vibration is governed by damping and for small damping forces, the amplitude of vibration will be quite large. The displacement lags behind the force by phase π/2.

- **Resonance:**

In forced vibrations, the amplitude of vibration will be maximum when the frequency of applied force is equal to the natural frequency.

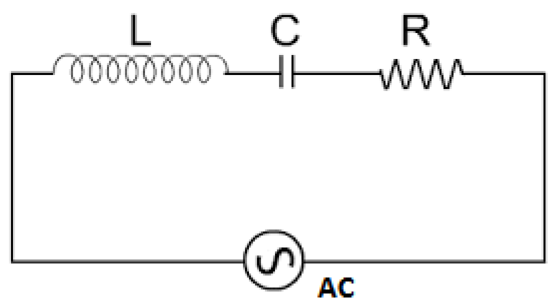


"The phenomenon of making a body to vibrate with maximum amplitude under the influence of an external force whose frequency is equal to the natural frequency of the vibrating body is called resonance".

A plot of amplitude of vibration versus the frequency of applied force is shown below. It is called resonance curve. For low damping, the sharpness of resonance is high.

- **<u>Forced vibrations in LCR series circuit:</u>**

Consider LCR series circuit connected to an AC source. Let the applied AC voltage be



$V = V_0 \sin(pt)$

where, $(\frac{p}{2\pi}) = frequency\ of\ applied\ AC$

At any instant of time,

voltage across the capacitor, $V_C = \frac{q}{C}$

voltage across the inductor, $V_L = L\left(\frac{dI}{dt}\right) = L\frac{d^2q}{dt^2}$

voltage across the resistor, $V_R = iR = R\frac{dq}{dt}$

Now, applying Kirchoff's Voltage law in the circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \sin(pt)$$

Or $$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{CL} = \frac{V_0}{L}\sin(pt) \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

Comparing this eq. with the eq. of motion in forced vibrations,

$$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{\mu}{m}y = \frac{F}{m}\sin pt \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From eq. (1) and (2), we get the correspondence between quantities in <u>mechanical</u> and <u>electrical</u> oscillators.

Mechanical Oscillator	Electrical Oscillator
Displacement, y	charge, q
Mass, m	Inductance, L
Velocity, v= $\frac{dy}{dt}$	current, I= $\frac{dq}{dt}$
Damping constant, r	Resistance, R
Force constant, µ	Reciprocal of C, $\frac{1}{C}$
Maximum applied force, F	Maximum voltage, V_0

Also, $\omega^2 = \frac{1}{LC}$ and $\frac{r}{m} = \frac{R}{L} = 2b$

Natural frequency of vibrating system=$= \frac{\omega}{2\pi}$

In LCR circuit, $\omega^2 = \frac{1}{LC}$

so $$\omega = \frac{1}{\sqrt{LC}}$$

$\therefore$ Natural frequency of LCR circuit= $f = \frac{1}{2\pi\sqrt{LC}}$

Quality factor=$Q = \frac{\omega}{2b}$

$\therefore$ In LCR series circuit,

$$Q = \frac{1}{2\pi\sqrt{LC}\left(\frac{R}{L}\right)} \qquad \because \omega = \frac{1}{\sqrt{LC}} \quad and \quad 2b = \frac{R}{L}$$

$$\therefore Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$