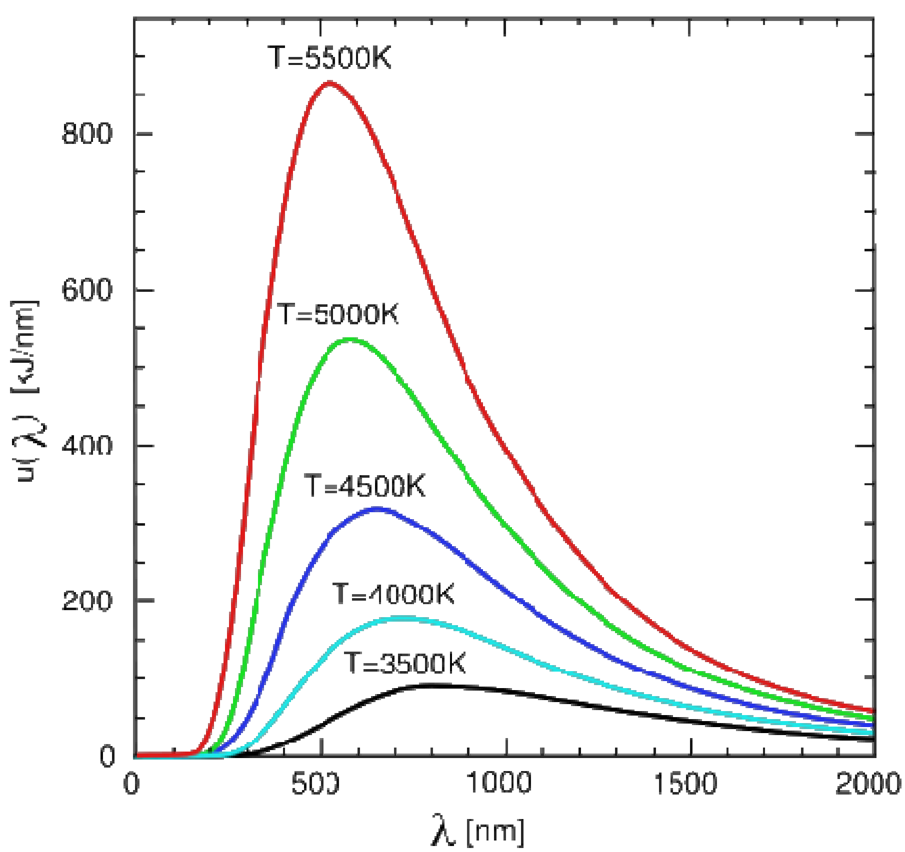
Modern Physics

Blackbody radiation spectrum:

A body that is capable of absorbing all radiation incident on it is called a perfectly black body. It also emits all radiations when maintained at a constant temperature. Radiation emitted by the black body is called blackbody radiation. Black body radiation depends on the temperature of the body.

Important features of blackbody spectrum:

At thermal equilibrium (i.e. at constant temperature), **t**he distribution of energy density of blackbody radiation among different wavelength varies as shown in the figure. It is called black body spectrum.



The important features of blackbody spectrum may be summarized as follows;

- At a given temperature, the energy density initially increases with increase in wavelength and becomes maximum at λ_m. With further increase in wavelength the energy density of radiation decreases.

- As the temperature of the black body increases, λ_m shifts towards shorter wavelength region. Here, λ_m is the wavelength at which maximum emission of energy takes place.

  i.e. $$\lambda_m \propto \frac{1}{T}$$

  (or) $$\lambda_m T = \text{constant} = 2.898 \times 10^{-3}\,\text{mK}$$

  This result is known as **Wien's displacement law**.

- An increase in temperature results in an increase in the total energy emitted at all wavelengths. The area under the curve is a measure of the total energy of radiation at that temperature and it is proportional to the fourth power of the absolute temperature of black body. This is known as **Stefan's law.**

Laws of radiation:

1. Wien's law of radiation:

Wien deduced the relation between the wavelength of emitted radiation and temperature of the black body based on classical physics. According to <u>Wien's law of radiation</u>, the energy density of black body for wavelength in the range λ and $\lambda+d\lambda$ is given by,

$$\boxed{U(\lambda)\,d\lambda = \frac{C_1}{\lambda^5}\frac{1}{e^{C_2/\lambda T}}\,d\lambda}$$

where C_1 and C_2 are constants.

Limitation: Wien's law holds good only for shorter wavelengths region (i.e. for $\lambda < \lambda_m$) and at high temperature of the source. It failed to explain gradual drop in the energy density at higher wavelengths longer than λ_m.

2. Rayleigh -Jean's law of radiation:

In a different approach, Rayleigh and Jean treated the radiations inside a cavity as standing electromagnetic waves and obtained the following formula for energy density.

The number of modes of vibration/unit volume whose wavelength in the range λ and $\lambda+d\lambda$.is given by $8\pi\lambda^{-4}d\lambda$.

By the law of equipartition of energy, the average thermal energy per degree of freedom is (kT), where k is Boltzmann's constant.

Then, The Energy emitted per unit volume per second of a black body is given by,

$U_\lambda d\lambda =$ (Number of standing waves formed in the cavity) × (Average thermal energy)

$$\mathbf{U(\lambda)\ d\lambda = 8\pi kT\lambda^{-4}\ d\lambda}$$

This is Rayleigh -Jean's law of radiation.

Limitation: Rayleigh -Jean's law correctly predicts the fall of intensity of the radiation towards the longer wavelength side (i.e. for $\lambda > \lambda_m$). However, according to Rayleigh -Jean's law, the radiant energy increases enormously with the decreasing wavelength. Thus the black body must radiate practically all the energy at very short wave length side. This is not in agreement with the experimental observation.

Note: Rayleigh -Jean's law fail to explain the lower wavelength side of the black body spectrum. This failure of Rayleigh Jean's law to explain emission of radiation below the violet region towards the lower wavelength side of the spectrum is referred to as **"Ultraviolet Catastrophe"**.

Note: Wien's formula agreed with the experimental curves for shorter wavelengths while Rayleigh formula agrees for longer wavelengths. Thus both the laws of classical physics failed to explain the entire spectrum of the black body radiation.

3. Planck's Law of radiation:

Max Planck derived an equation based on quantum theory which successfully explained the entire spectrum of blackbody radiation. The atoms in the walls of black body behave like simple harmonic oscillators. According to Planck's quantum theory, the atomic oscillators absorb or emit radiation in discrete packets called quanta or photons. Energy of each photon is, E = hν, where 'ν' is the frequency of radiation and is same as that of frequency of atomic oscillator, 'h' is the Planck's constant. ($h=6.63X10^{-34}Js$)

Based on these assumptions, Planck obtained an empirical relation to explain the energy distribution of black body spectrum as,

$$U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{\left[e^{hc/\lambda kT}-1\right]}d\lambda$$

This relation is known as **Planck's radiation law**. This law agrees well with the experimental data.

Reduction of Planck's radiation law to Wien's law and Rayleigh-Jean's law:

<u>**Case I**:</u> For <u>shorter wavelength regions</u>, $e^{\frac{hc}{\lambda kT}}$ is very large

i.e. $e^{\frac{hc}{\lambda kT}} >> 1$

Therefore in eqn. (1), denominator is $\left(e^{\frac{hc}{\lambda kT}}-1\right) \approx e^{\frac{hc}{\lambda kT}}$

Substituting the above quantity in Planck's radiation law,

$$U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{\left[e^{hc/\lambda kT}-1\right]}d\lambda$$

$$U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{\left[e^{hc/\lambda kT}\right]}d\lambda$$

$$U_\lambda d\lambda = \frac{C_1}{\lambda^5}\frac{1}{\left[e^{C_2/\lambda T}\right]}d\lambda$$

where $C_1 = 8\pi hc$ and $C_2 = hc/k$

This is the Wien's distribution law.

Case II : for longer wavelengths, the quantity $\frac{hc}{\lambda kT}$ is small.

Hence $e^{\frac{hc}{\lambda kT}}$ is also small.

Expanding $e^{\frac{hc}{\lambda kT}}$ by exponential series

$$e^{\frac{hc}{\lambda kT}} = 1 + \frac{hc}{\lambda kT} + \frac{(hc/\lambda kT)^2}{2!} + \frac{(hc/\lambda kT)^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Since $\frac{hc}{\lambda kT}$ is very small neglecting the higher order terms.

$$e^{\frac{hc}{\lambda kT}} = 1 + \frac{hc}{\lambda kT} \qquad \text{or} \qquad e^{\frac{hc}{\lambda kT}} - 1 = \frac{hc}{\lambda kT}$$

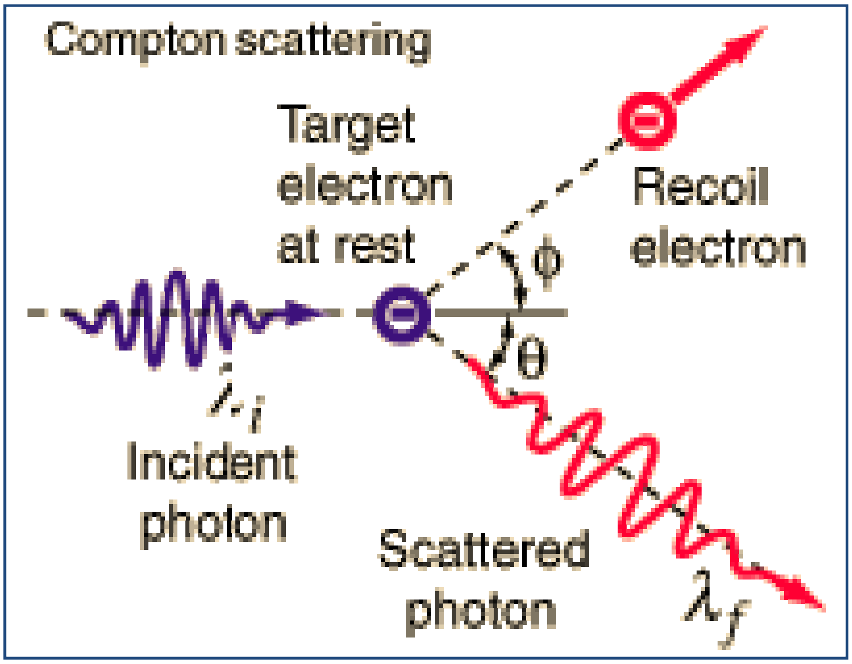
Substituting above quantity in the Planck's radiation law, we get

$$U(\lambda)\, d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$$ This is the Rayleigh-Jeans law.

Thus Wien's law and Rayleigh-Jeans law come out as special cases of Planck's law of radiation.

Compton Effect:

When a beam of monochromatic X-rays is scattered by a suitable target, the scattered radiation consists of two components, one of the same wavelength (coherent) and another of longer wavelength (incoherent) than that of incident wavelength. The change in wavelength of scattered beam is independent of the target material but depends on scattering angle. This phenomenon is called Compton Effect and the change in wavelength is called Compton shift.



Explanation: The incident beam of X-rays is assumed to be consisting of a stream of photons. The scattering process is analyzed as a collision between two particles, the incident photon of energy $E=hc/\lambda$ and the electron of the target of rest energy m_0c^2. As a result, the photon is scattered at an angle of θ to the incident direction and its energy reduces from $E=hc/\lambda$ to $E'=hc/\lambda'$. Due to this impact, the electron recoil at an angle ϕ with the incident direction of photon.

Expression for Compton shift (Δλ):

Applying law of conservation of energy for X-ray scattering,

$$E_{initial} = E_{final}$$

i.e. $$\frac{hc}{\lambda} + m_0c^2 = \frac{hc}{\lambda'} + mc^2 \qquad \text{where } m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Here m_0 is rest mass of electron and m is relativistic mass of electron.

$$\frac{h}{\lambda} - \frac{h}{\lambda'} + m_0c = \frac{m_0c}{\sqrt{1-\frac{v^2}{c^2}}}$$

Let us take β=v/c, then we can write above equation as

$$\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right) + m_0c = \frac{m_0c}{\sqrt{1-\beta^2}}$$

Squaring on both sides

$$\left(\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2h^2}{\lambda\lambda'}\right) + m_0^2c^2 + 2m_0ch\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right) = \frac{m_0^2c^2}{1-\beta^2} \qquad \text{(or)}$$

$$\left(\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2h^2}{\lambda\lambda'}\right) + 2m_0ch\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right) = \frac{m_0^2c^2}{1-\beta^2} - m_0^2\ c^2 \ \ldots\ldots\ldots\ldots \ (1)$$

Now, using law of conservation of momentum along X-direction

$$(P_x)_{initial} = (P_x)_{final}$$

$$\frac{h}{\lambda} + 0 = mv\ \cos\phi + \frac{h}{\lambda'}\cos\theta$$

$$\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta = \frac{m_0v\ \cos\phi}{\sqrt{1-\beta^2}} \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

Similarly using momentum conservation along Y-direction,

$$(P_y)_{initial} = (P_y)_{final}$$

$$0 = mv \ \sin\phi - \frac{h}{\lambda'}\sin\theta$$

$$\frac{h}{\lambda'}\sin\theta = \frac{m_0 v \ \sin\phi}{\sqrt{1-\beta^2}} \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

Squaring and adding equations (2) and (3), we get

$$\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2}\cos^2\theta - \frac{2h^2}{\lambda\lambda'}\cos\theta + \frac{h^2}{\lambda'^2}\sin^2\theta = \frac{m_0^2 v^2}{1-\beta^2}$$

$$\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2h^2}{\lambda\lambda'}\cos\theta = \frac{m_0^2\beta^2 c^2}{1-\beta^2} \quad \left(\because \beta = \frac{v}{c}\right) \qquad \text{(or)}$$

$$\frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2h^2}{\lambda\lambda'}\cos\theta = \frac{m_0^2 c^2}{1-\beta^2} - m_0^2 \ c^2 \qquad \ldots\ldots\ldots\ldots(4) \quad)$$

Subtracting equation (4) from (1), we get

$$-\frac{2h^2}{\lambda\lambda'} + 2m_0 ch\left(\frac{\lambda' - \lambda}{\lambda\lambda'}\right) + \frac{2h^2}{\lambda\lambda'}\cos\theta = 0$$

$$\frac{m_0 c}{h}\left(\lambda' - \lambda\right) = 1 - \cos\theta$$

$$\boxed{\left(\lambda' - \lambda\right) = \frac{h}{m_0 c}\left(1 - \cos\theta\right)}$$

This change in wavelength of scattered X-rays is called Compton shift ($\Delta\lambda = \lambda' - \lambda$). Compton shift depends only on the scattering angle ‘θ’ and is independent of the wavelength of the incident X-rays and the nature of the target material. This is experimentally verified.

<u>Significance of Compton Effect:</u>

In the Compton Effect, the elastic collisions between photon and electron take place. Collision and momentum are characteristics of particle behavior. Thus Compton Effect confirms the particle behavior of radiation.

Note 1: When $\theta = 90^o$, $\Delta\lambda = \frac{h}{m_o c} = 2.42\times10^{-12}$ m . This constant is called <u>Compton wavelength</u>.

Note 2: When $\theta = 180^o$, $\Delta\lambda$ is maximum and will be twice the Compton wavelength.

de-Broglie's hypothesis (Matter waves):

Matter and radiation are both different forms of energy and are inconvertible, as established by Einstein's mass energy equivalence relationship, $E=mc^2$. Radiation has wavelike properties (interference, diffraction and polarization) and in certain cases (black body radiation, photoelectric effect and Compton scattering) behaves like particles. Thus radiation exhibits dual nature.

Louis de-Broglie put a suggestion that the wave-particle dualism need not be special feature of radiation alone but material particles must also exhibit dual behavior. His suggestion was based on the fact that ***"nature loves symmetry".*** If radiation can behave as particles, then entities such as electrons, protons neutrons etc. which ordinarily behaves as particles should also exhibit wave properties under appropriate circumstances.

According to de-Broglie's hypothesis, every moving material particle is associated with a wave whose wavelength is given by

$$\lambda = \frac{h}{mv} = \frac{h}{p}$$

where 'm' is mass, 'v' is velocity and 'p' is the momentum of the particle. The waves associated with moving material particles are called **matter waves** (or) de-Broglie waves. λ is called de-Broglie wavelength.

Note 1: For ordinary objects (macroscopic), the wave like behavior cannot be observed because the wavelength is very small ($\approx 10^{-34}$m). In subatomic scale (microscopic), the momentum can be sufficiently small to bring the de-Broglie wavelength into observable range.

Note 2: If E_k is the kinetic energy of the particle, then

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}\frac{m^2v^2}{m} = \frac{1}{2}\frac{p^2}{m}$$

or $$p = \sqrt{2mE_K}$$

$\therefore$ De-Broglie wavelength $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_K}}$

Note 3: Also if charge 'q' is accelerated through a potential difference of 'V' volts, then

K.E.= $E_K = qV$

Hence de-Broglie wavelength of an accelerated electron, $\lambda = \frac{h}{\sqrt{2mqV}} = \frac{1.226}{\sqrt{V}}$ nm

Phase velocity:

Definition: Phase velocity is the velocity with which a phase point located on a progressive wave is transported. Phase velocity is same as wave velocity. $v_p = \nu\lambda$ (ν is the frequency).

Let P be a phase point located on a travelling wave represented by the equation,

$$y = A \sin(kx - \omega t)$$

where, ω is the angular frequency ($\omega = 2\pi\nu$) and k is the angular wave number ($k = 2\pi/\lambda$).

The velocity of the phase point will be the same as wave velocity and is given by

$$V_p = \nu\lambda = \frac{\omega}{2\pi}.\frac{2\pi}{k} = \frac{\omega}{k}$$

Note:

1) $V_P = \nu\lambda = \frac{\omega}{2\pi}.\frac{2\pi}{k} = \frac{\omega}{k}$ …………(i)

Also, $\nu = E/h$ and $E = mc^2$ ……… (ii)

$\lambda = h/p$ and $p = mv$ …………(iii)

Substituting (ii) and (iii) in (i), we obtain $V_P = \frac{E}{p} = c^2/v$

Here, v is the velocity of the particle associated with the wave.

2) For photons, v = c . Hence phase velocity is equal to velocity of light.

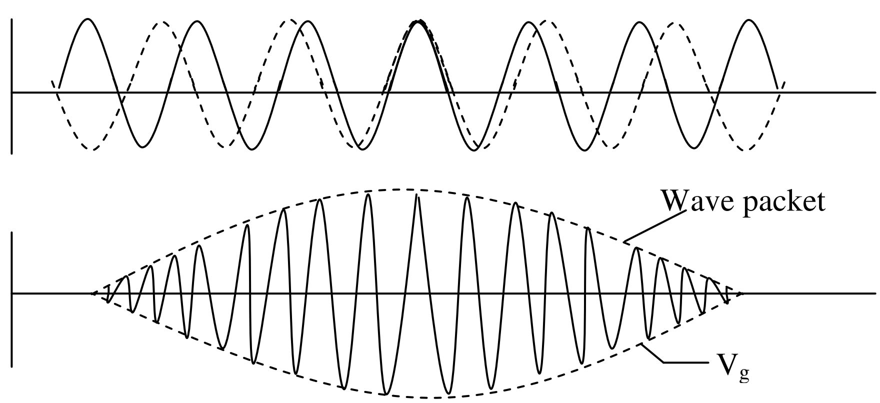
3) The velocity of material particles is always less than the velocity of light. (i.e. v < c). So, phase velocity of matter waves becomes greater than speed of light (i.e. $V_P > c$) which has no physical significance. Hence for the physical representation of matter waves, we require to consider group of waves (wave packet) to be associated with the moving particle.

Group velocity:

When two or more waves of slightly different wavelengths moving in the same direction overlaps, a wave group (wave packet) is formed in which the amplitude modulation occurs.

Group velocity is defined as the rate at which the amplitude is modulated in the resultant pattern **(OR)** the rate at which energy is transported by the group of waves. Group velocity is the velocity with which the entire group of waves (wave packet) would travel.

$$v_g = \frac{d\omega}{dk}$$



The super position of waves of nearly equal wavelengths to give the wave packet.

Expression for Group velocity:

Consider two waves of same amplitude 'A', but of slightly different wave numbers and angular frequencies represented by the equations,

$$y_1 = A \sin (kx-\omega t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$y_2 = A \sin [(k+\Delta k)x- (\omega+\Delta\omega)t] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

The resultant displacement y due to the superposition of two waves is given by,

$$y = y_1 + y_2$$

$$\therefore \qquad y = A \sin (kx-\omega t) + A \sin [(k+\Delta k)x- (\omega+\Delta\omega)t]$$

Using the relation, $\quad \text{Sin A} + \text{Sin B} = 2 \cos\left(\frac{A-B}{2}\right) \sin\left(\frac{A+B}{2}\right)$ we can write

$$y = 2A\cos\left[\left(\frac{\Delta k}{2}\right)x - \left(\frac{\Delta\omega}{2}\right)t\right]\sin\left[\left(\frac{2k+\Delta k}{2}\right)x - \left(\frac{2\omega+\Delta\omega}{2}\right)t\right]$$

since Δk and $\Delta\omega$ are very small, we can take $2k+\Delta k \approx 2k$ and $2\omega+\Delta\omega \approx 2\omega$

$$y = 2A\cos\left[\left(\frac{\Delta k}{2}\right)x - \left(\frac{\Delta\omega}{2}\right)t\right]\sin(kx-\omega t) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Comparing equations (1) and (3), it is possible to treat the coefficient of sin $\{wt—kx\}$ in both the equations to be the amplitude of the representative waves. In this sense, in eq. (1), the amplitude will be A, which is constant. But in eq. (3) the amplitude becomes,

$$2\,A \cos \left[\left(\frac{\Delta\omega}{2}\right) t — \left(\frac{\Delta k}{2}\right) x\right]$$

This is not a constant, but varies as a wave. Thus above equation (3) represents a sine wave whose amplitude is modulated with the angular frequency $\Delta\omega/2$ and wave number $\Delta k/2$.

By definition, the velocity with which the variation in amplitude is transmitted in the resultant wave is the group velocity. Hence from equation (3), rate of modulation of amplitude is

$$v_{group} = \frac{\left(\frac{\Delta\omega}{2}\right)}{\left(\frac{\Delta k}{2}\right)} = \frac{\Delta\omega}{\Delta k}$$

In the limit, $\left(\frac{\Delta\omega}{\Delta k}\right) \rightarrow \left(\frac{d\omega}{dk}\right)$

$$\therefore \quad \boxed{v_{g} = \frac{d\omega}{dk}}$$

Note: The wave packet consists of regions of constructive and destructive interference. The probability of finding the particle in a given region depends on the amplitude of the wave group in that region. The wave group representation enables the localization of the particle. It is the motion of wave group, not the motion of individual waves that makes up the wave group, corresponds to the motion of the particle.

Relation between Group velocity (V_{group}) and Particle velocity ($V_{Particle}$)

We have the equation for group velocity as,

$$v_g = \frac{d\omega}{dk} \quad \text{……………..} \quad (1)$$

$$\text{But, } \omega = 2\pi\nu = 2\pi \frac{E}{h} \qquad (\because E = h\nu)$$

$$\therefore \ d\omega = \left(\frac{2\pi}{h}\right) dE \quad \text{....................} \ (2)$$

$$\text{Also, we have,} \quad k = \frac{2\pi}{\lambda} = 2\pi \frac{p}{h} \quad (\because \lambda = \frac{h}{p})$$

$$\therefore \ dk = \left(\frac{2\pi}{h}\right) dp \quad \text{............................}(3)$$

Substituting eq. (2) and (3) in eq. (1) we get,

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp}$$

But we know that $E = \frac{p^2}{2m}$

where, p is the momentum of the particle.

$$\therefore dE = \frac{2p\, dp}{2m}$$

$$\therefore \quad \mathbf{v_g = \frac{dE}{dp} = \frac{2p}{2m} = \frac{p}{m} = \frac{mv}{m} = v} \quad \text{(Here 'v' is particle velocity).}$$

i.e. $\boxed{v_{group} = v_{particle}}$

Thus, the particle and the associated wave packet move together.

Relation between Phase velocity (V_{phase}) and Group velocity (V_{group})

We obtain the relationship between group and phase velocities in a dispersive medium in the following way:

Phase velocity is given by,

$$\mathbf{V_P} = \frac{\boldsymbol{\omega}}{\mathbf{k}} \text{ (or) } \mathbf{k.V_P} = \boldsymbol{\omega}$$

and $$V_g = \frac{d\omega}{dk}$$

$$\therefore V_g = \frac{d}{dk}(\mathbf{k.V_P}) = V_P + k\frac{dV_P}{dk} \quad \text{...........(1)}$$

But, $$k = \frac{2\pi}{\lambda}$$

$$\therefore dk = -\left(\frac{2\pi}{\lambda^2}\right)d\lambda = -\left(\frac{k}{\lambda}\right)d\lambda$$

substituting this in equation (1) we obtain,

$$\mathbf{V_g} = \mathbf{V_P} - \boldsymbol{\lambda}\left(\frac{\mathbf{dV_P}}{\mathbf{d\lambda}}\right)$$

This equation relates phase and group velocities in a dispersive medium.

Note 1: In a **non-dispersive medium**, $\left(\frac{\mathbf{dV_P}}{\mathbf{d\lambda}}\right) = \mathbf{0}.$ Hence $\mathbf{V_g} = \mathbf{V_P}$

Note 2: Rrelation between group velocity, phase velocity and particle velocity:

Phase velocity, $\boldsymbol{V_P} = \frac{\boldsymbol{\omega}}{\boldsymbol{k}} = \frac{\boldsymbol{E}}{\boldsymbol{p}} = \frac{\boldsymbol{mc^2}}{\boldsymbol{mv}} = \frac{\boldsymbol{c^2}}{\boldsymbol{v}}$ where, v is the velocity of the particle

$$\text{Now, } V_g = \frac{d\omega}{dk} = \frac{dE}{dp} = v$$

$$\therefore V_P = c^2/v = c^2/V_g$$

$$\text{(OR)} \quad \mathbf{V_P.V_g} = \mathbf{c^2}$$