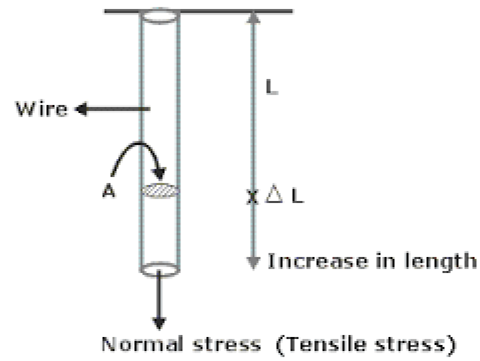
ELASTICITY

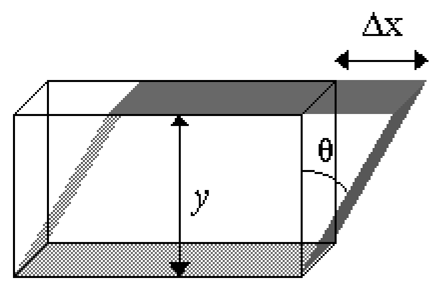
Stress and Strain:
When a body is subjected to deforming force, a restoring force is developed in the body. The restoring force per unit area is known as stress. This restoring force is equal in magnitude but opposite in direction to the applied force.
If F is the force applied and A is the area of cross section of the body, then Stress=F/A.
The SI unit of stress is Nm^{-2} or pascal (Pa) and its dimensional formula is $[ML^{-1}T^{-2}]$.
The ratio of change in dimension of a body to the original dimension is called strain. Strain has no units or dimensions.

There are two types of stress and 3 types of strains.

1) **Normal stress: If a force is applied normal to the cross section of the body, then the restoring force developed per unit area is called normal or longitudinal stress.** Longitudinal stress may be of two types i.e. tensile stress or compressive stress i.e. it can cause either extension (tensile stress) or compression (compressive stress) depending on the direction . In both cases, there is change in length of the body.



2) **Shearing Stress:** If a deforming force is applied parallel to the surface area of the body, there is relative displacement between opposite faces. **The restoring force per unit area developed due to applied tangential force is known as tangential or shearing stress.**



Strain:

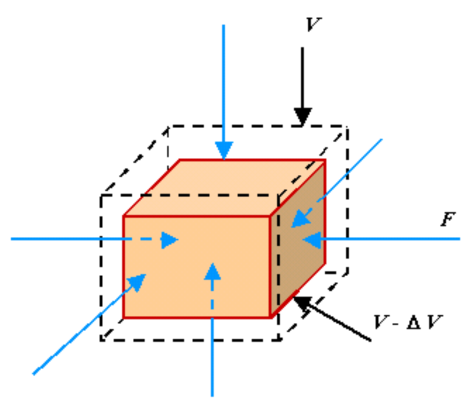
1. Longitudinal strain: The ratio of change in the length (Δl) to the original length (L) of the body is known as longitudinal strain.
i.e. Longitudinal $strain = \frac{\Delta l}{L}$
2. Shearing strain: As a result of applied tangential force, there is relative displacement Δx between opposite faces of the body as shown is figure. The strain so produced is known as shearing strain.

Shearing strain is defined as the ratio of relative displacement of the faces Δx to the height 'h'.

i.e. Shearing strain= $\boldsymbol{tan\theta = \frac{\Delta x}{y}}$

whereθ is the angular displacement of the surface from the vertical (original position). Usually θ is very small and hence $tan\theta \approx \theta = \frac{\Delta x}{y}$

3) **Volumetric strain: If a forces are applied normal to suface of body in all directions it undergoes change in volume. The ratio of change in volume (ΔV) to the original volume (V), without any change in shape is called volumetric strain and stress producing is called normal stress.**
i.e.$Volumetric\ strain = \frac{\Delta V}{V}$



Hooke's Law: *It states that Stress is directly proportional to strain within elastic limit.*

i.e. stress α strain or $\frac{stress}{strain} = constant$

The ratio of stress to strain is called the modulus of elasticity or elastic constant. It is a characteristic property of the material.

1. **Young's Modulus (Y):**

Young's Modulus is defined as the ratio of longitudinal (linear) stress to longitudinal strain with in the elastic limit.

Since strain is dimensionless quantity, the unit of Young's Modulus is the same as that of stress i.e. Nm^{-2}.

$Y = \frac{F/A}{L/l} Nm^{-2}$

F represents the force applied normal to the area 'A 'of a wire of length ' L' and '*l*' is the change in length.

For metals like iron, steel, copper, aluminium etc., the Young's moduli are very large. Therefore, these materials require a large force to produce small change in length.

2. **Rigidity modulus(n):**

The ratio of shearing stress to the corresponding shearing strain is called the shear modulus or Rigidity modulus of the material.

$n = \frac{F/A}{\theta} \text{ Nm}^{-2}$

F is the tangential force and θ is the shear strain.

3. **Bulk Modulus (K):**

Bulk Modulus is defined as the ratio of normal stress to volumetric strain with in the elastic limit.

$K = \frac{F/A}{v/V} \text{ Nm}^{-2}$

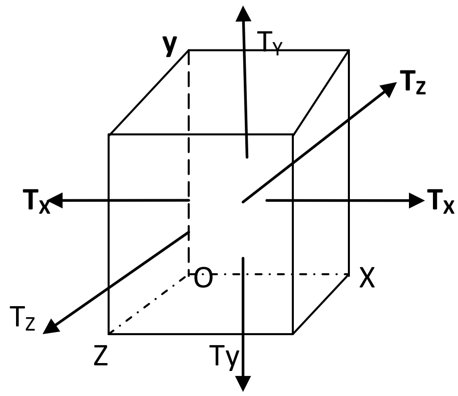
F/A is the normal stress and *v* is the change in volume and V is the original volume.

Poisson's ratio ,σ , is the ratio of lateral strain to longitudinal strain . $\sigma = \beta/\alpha$

α is increase in length per unit length per unit stress in the direction of the stress and **β** is decrease in length per unit length per unit stress in a direction perpendicular to the stress.

β is lateral strain per unit stress and α is the longitudinal strain per unit stress.

Expression for Bulk Modulus K in terms of α and β :



Bulk modulus is the ratio of normal stress to volume strain.
α represents the linear strain per unit stress and β is the lateral strain per unit stress.
Consider a unit cube (OX = OY =OZ = 1)
The initial volume of the unit cube = V = 1
Let the cube be subjected to tensile stresses T_x , T_Y , T_Z along the X,Y and Z axes. Each of these stresses is tensile along the direction in which it is applied and compressive in directions perpendicular to it.
T_X produces extension of the side OX and T_Y and T_Z produce compression of OX.
Due to these stresses the dimensions of OX, OY and OZ are altered and can be written as

$$OX = 1 + \alpha T_X - \beta T_Y - \beta T_Z$$
$$OY = 1 + \alpha T_Y - \beta T_X - \beta T_Z$$
$$OZ = 1 + \alpha T_Z - \beta T_X - \beta T_Y$$

Final Volume $= (1 + \alpha T_X - \beta T_Y - \beta T_Z) \times (1 + \alpha T_Y - \beta T_X - \beta T_Z) \times (1 + \alpha T_Z - \beta T_X - \beta T_Y)$
$= 1 + (\alpha - 2\beta)(T_x + T_Y + T_Z)$
$= 1 + (\alpha - 2\beta)(3\,T) \qquad (T_X = T_Y = T_Z = T)$

$$\text{Change in Volume} = [1 + (\alpha - 2\beta)(3\,T)] - 1 = (\alpha - 2\beta)\,3\,T$$

$$\text{Volume strain} = (\alpha - 2\beta)\,3\,T$$

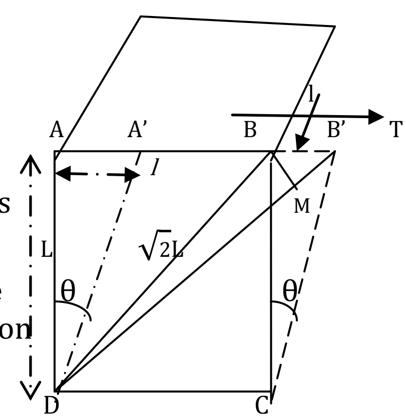
$$\text{Bulk Modulus}\ \ K = T/(\alpha - 2\beta)\,3\,T = 1/\,3\,(\alpha - 2\beta)$$

Expression for Youngs modulus :

$$Y = \text{longitudinal stress / longitudinal strain}$$

$$= T/\,T.\boldsymbol{\alpha} \;\; = 1/\boldsymbol{\alpha}$$

Expression for Rigidity modulus :



Consider a cube of dimension 'L' . The bottom face of the cube is fixed And a tangential stress 'T' is applied along the top face. Under the action of The tangential force, the cube gets deformed to A'B' C D. The shear strain is Equal to θ and is given by the ratio, θ = BB'/BC $= {}^{l}/_{L}$
The tangential stress is equal to a tensile stress T along BD and compressive stress T along AC. These tensile and compressive stresses produce extension of the diagonal BD. The increase in length of diagonal BD = MB'
If α is linear strain per unit stress and β is the lateral strain per unit stress , The strain produced along the diagonal BD = T (α + β)
= MB' /BD

MB' = BB' cos (BB'M) = BB' cos 45^0 = $l\,/\,\sqrt{2}$

$$T\,(\alpha + \beta) = \frac{l/\sqrt{2}}{L\sqrt{2}} = \frac{l}{2\,L} = \theta\,/2$$

Rigidity Modulus 'n' = shear stress /shear strain
$= T/\theta = 1/2(\alpha + \beta)$

Relation between Y, n and K

$$K = 1/3(\alpha - 2\beta) = \frac{1}{3\alpha(1-2\sigma)} = Y/3(1-2\sigma)$$

$$1 - 2\sigma = Y/3K \qquad \text{----------------(i)}$$

$$n = 1/2(\alpha + \beta) = 1/2\alpha\,(1+\sigma) = Y/2(1+\sigma)$$

$$2 + 2\sigma = Y/n \qquad \text{----------------(ii)}$$

Add (i) and (ii) and rearrange the terms to obtain the relation between the elastic modulii as

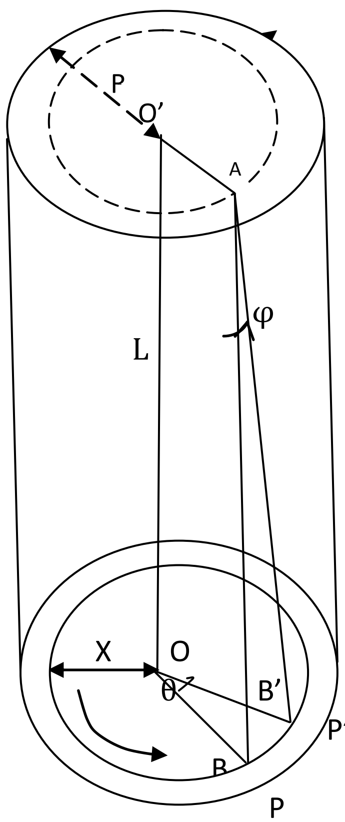
$$\frac{\mathbf{3}}{\boldsymbol{Y}} = \frac{\mathbf{1}}{\mathbf{3}\boldsymbol{K}} + \frac{\mathbf{1}}{\boldsymbol{n}}$$

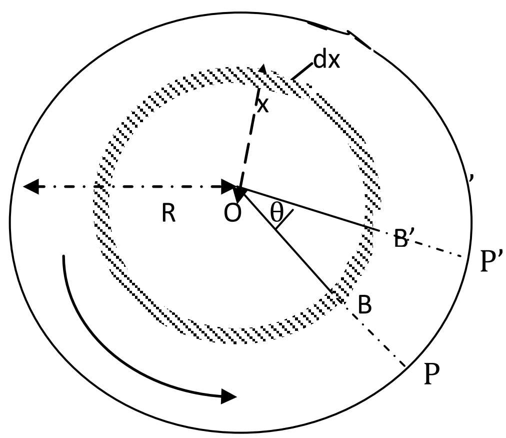
Theoretical limits of σ

The elastic modulii are positive, so $(1 - 2\sigma)$ is positive and it places a upper limit equal to 0.5 on the value of σ. Similarly we can deduce the lower limit to be equal to -1 from eqn. (ii).

Torsion of a Cylinder

Consider a cylinder of length 'L' and radius 'R'. Let the cylinder be clamped at the upper end a twisting couple be applied at the lower end. Let θ be the angle of twist.





The cylinder can be considered as made up of a number of co-axial hollow cylinders of varying radii (0 to R). At the bottom end, each of these radii are twisted through an angle, θ due to the applied external couple. As a result, OB is displaced to OB' and OP is displaced to OP' without change in dimensions.

A line AB parallel to OO' is displaced to AB' through an angle, φ, called the angle of shear. This is an example of pure shear as there is no change in either length or radius of the cylinder.

We can calculate the twisting couple on the co-axial cylinder of radius OB(OB = x) and integrate the expression between the limits, x=0 and x=R to obtain the magnitude of the twisting couple on the cylinder. We can write from the geometry of the figure,

$$BB' = x\,\theta = L\varphi$$

$$\text{Or,} \quad \varphi = x\,\theta/L \qquad \text{---------------(i)}$$

If δ F is the tangential force acting on a cylindrical shell of radius , 'x' and thickness, ' dx ',
The tangential stress = δF /area of the shell = δF / $2\pi x\, dx$ --------(ii)
The rigidity modulus = tangential stress/shear strain

$$n = (\delta F / 2\pi x\, dx) / \varphi \quad \text{------(iii)}$$

$$\delta F = 2\pi\, n\, x\, dx.\, \varphi \quad \text{---------(iv)}$$

Twisting couple on the hollow cylinder of radius 'x' and thickness, 'dx' can be written as

$$\boldsymbol{\delta C = tangential\ force\ \times distance = n.\varphi.2\pi x\ dx.x} \quad \text{----------(v)}$$

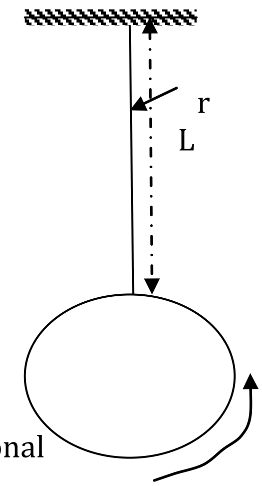
n , is the rigidity modulus of the material and is given by the ratio of tangential stress to shear strain.
Substitute for φ in eqn. (ii) to obtain

$$\boldsymbol{\delta C = 2\pi\, n\, \theta\, x^2 dx\, .\, x/L = 2\pi n\theta\, x^3 dx/L} \quad \text{------------------------------(vi)}$$

Twisting couple on the solid cylinder of radius 'R' ,

$$\boldsymbol{C = \frac{2\pi n\theta}{L} \int_0^R x^3 dx = \frac{2\pi n\theta}{L} [R^4/4] = \frac{\pi n R^4 \theta}{2L}}$$

Torsion Pendulum:



A rigid wire of length' L' and radius 'r' is fixed at one end and forms the axis of rotation for a regular or an irregular body attached to the free end. When the wire is twisted at the free end, the body is set into oscillations and these oscillations are called torsional oscillations.
Let I be the moment of inertia of the body about the given axis and T be the time period for torsional oscillations. Let the restoring couple per unit twist be 'C' and 'θ' be the angle of twist.

$$\boldsymbol{I \frac{d^2\theta}{dt^2} = -C\,\theta}$$

$$\boldsymbol{\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0}$$

represents the simple harmonic equation for torsional oscillations. From the above equation, we can get the time period for torsional oscillations

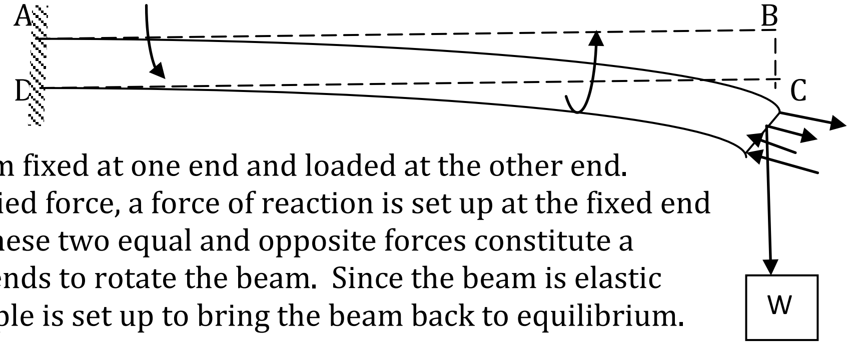
$$\boldsymbol{T = 2\pi\sqrt{I/C}}$$

Or, $\boldsymbol{\frac{I}{T^2} = \frac{C}{4\pi^2}}$ where $\boldsymbol{C = \frac{\pi n r^4 \theta}{2L}}$

For a given wire, C is constant and hence the ratio I/T^2 is constant irrespective of the body or the axis of rotation. This principle is used to determine the moment of inertia of irregular objects.

Bending Moment of a Beam:

A beam is a structural member whose length is very large compared to other dimensions. In the simple theory of bending of beams, the shear stresses are neglected and only tensile or compressive stresses are considered.



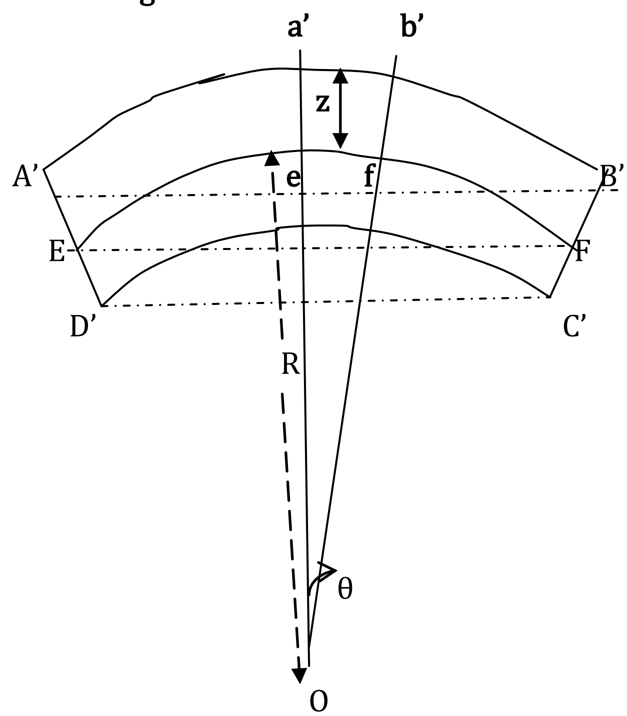
Consider a beam fixed at one end and loaded at the other end.
Due to the applied force, a force of reaction is set up at the fixed end of the beam. These two equal and opposite forces constitute a couple which tends to rotate the beam. Since the beam is elastic a restoring couple is set up to bring the beam back to equilibrium.

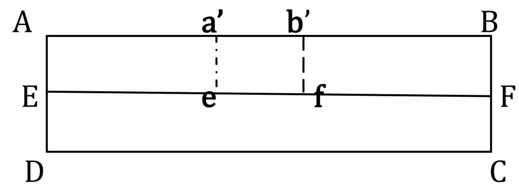
The beam can be imagined as made up of a number of longitudinal filaments. Under the applied load the Upper filaments will undergo extensions and assume a convex form and the lower filaments will undergo compressions and assume a concave form. The filament that does not undergo any change in dimension is the neutral axis. The magnitude of extension or compression will depend on the distance of the filament from the neutral axis.

The moment of the restoring couple is called bending moment of a beam.

*** Derivation for the expression of bending moment is not in syllabus***

***Expression for the bending moment of a beam** :





Let ABCD be a section of the beam. EF is the neutral axis. Under the action of the external couple, the section will bend into an arc without any change in length of neutral axis. The upper filaments will undergo extension and A'B' $>$ AB.
The lower filaments will be subjected to compressive stresses and C'D' $<$ CD

The beam is bent into an arc of a circle of radius 'R' with the center at O . Consider a small portion of the neutral axis, ef , subtending an angle θ at the center. a'b' is another small portion of the filament A'B' which is at a distance, z, from the neutral axis. In the absence of bending, a'b' =ef

ef $= R\theta$, and a'b' $= (R+z)\theta$

Strain in the filament, a'b' $= \{(R+z)\theta - R\theta\}/R\theta = z/R$

If δA is the area of the filament (a'b'), which is at a distance z from the neutral axis and Y is the Youngs modulus,

$$\boldsymbol{Y = \frac{\delta F/\delta A}{z/R}}$$

Force acting on the filament, $\delta F = Y\,z\,\delta A/R$ and

Moment of the force about neutral axis = $(\boldsymbol{Y\,z\,\delta A/R}).\boldsymbol{z}$

The bending moment of the beam is obtained by summing over the moments of all the filaments above and below the neutral axis

Bending Moment $\boldsymbol{= \sum \frac{Y}{R}\,\delta A\,z^2 = \frac{Y}{R}\sum \delta A z^2 = Y I_g/R}$

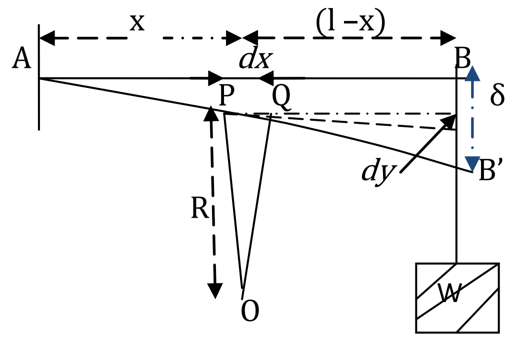
$\boldsymbol{I_g = \sum \delta A\ z^2 = A\,k^2}$ is the geometrical moment of inertia and depends on the area of cross section, $\boldsymbol{'A'}$ of the beam and the radius of gyration $\boldsymbol{k}$ of the area about the neutral axis.

For rectangular cross section , $\boldsymbol{I_g = b\,d^3/12}$, $\boldsymbol{b}$ is the breadth and $\boldsymbol{d}$ is the thickness

For circular cross section, $\boldsymbol{I_g = \pi\,r^4/4}$, $\boldsymbol{r}$ is the radius

Expression for Young's Modulus of a cantilever:

A cantilever is a beam fixed at one end and loaded at the other end. Consider a load, 'W' applied to the free end of the cantilever of length, '*l*'. Let δ be the deflection of the free end of the cantilever under the load. The deflection is maximum at the free end and is equal to zero at the fixed end.



Let PQ =dx ,be a small section of the neutral axis AB.
PQ is at a distance x from the fixed end.
Radius of curvature of neutral axis =R
Deflection of PQ =dy
The radius of curvature of the neutral axis is given by
The standard expression,

$$\frac{1}{R} = \left[\frac{d^2y/dx^2}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}\right] = \frac{d^2y}{dx^2} \quad \text{---------(i)}$$

dy/dx is very small and the higher power can be neglected.
The moment of applied force acting on the element PQ $= W(l\text{-}x)$ ------(ii)
The restoring couple due to the elasticity of the beam $= YI_g/R$ ---------(iii)
At equilibrium, these two are equal and opposite

$$\text{so} \quad YI_g/R = YI_g.\frac{d^2y}{dx^2} = W(l\text{-}x) \quad \text{-------------(iv)}$$

$$YI_g \int \frac{d^2y}{dx^2}\, dx = w\int (l-x)dx$$

On integration, we obtain the expression

$$YI_g \frac{dy}{dx} = w(lx - x^2/2) + C$$ where C is the constant of integration.

The deflection is zero at **x= *0*** and $dy/dx = 0$. Substituting this condition we get $C = 0$

$$YI_g \int_0^{\delta} dy = w\int_0^l \left\{lx - \frac{x^2}{2}\right\} dx$$

$$YI_g\, \delta = w\left[\frac{lx^2}{2} - \frac{x^3}{6}\right]_0^l = w\left[\frac{l^3}{2} - \frac{l^2}{6}\right] = \frac{wl^3}{3}$$

$$\delta = \frac{wl^3}{3YI_g} \quad \text{---------(v)}$$

For a rectangular cross section, $I_g = b\,d^3/12$, and substituting we obtain,

$$\delta = \frac{4wl^3}{Ybd^3}$$

The load $w = Mg$ and the Young's Modulus of the cantilever is given by

$$Y = 4Mgl^3/bd^3\,\delta$$