

# Engineering Mathematics – I

## (10 MAT11)

*LECTURE NOTES*  
(FOR I SEMESTER B E OF VTU)

[www.bookspar.com](http://www.bookspar.com)

# ENGINEERING MATHEMATICS – I

## Content

	CHAPTER
UNIT I	DIFFERENTIAL CALCULUS – I
UNIT II	DIFFERENTIAL CALCULUS – II
UNIT III	DIFFERENTIAL CALCULUS – III

## UNIT - I

### DIFFERENTIAL CALCULUS – I

#### Introduction:

The mathematical study of change like motion, growth or decay is calculus. The Rate of change of given function is derivative or differential.

The concept of derivative is essential in day to day life. Also applicable in Engineering, Science, Economics, Medicine etc.

#### Successive Differentiation:

Let  $y = f(x)$  be a real valued function.

The first order derivative of  $y$  denoted by  $\frac{dy}{dx}$  or  $y'$  or  $y_1$  or  $\Delta^1$

The Second order derivative of  $y$  denoted by  $\frac{d^2y}{dx^2}$  or  $y''$  or  $y_2$  or  $\Delta^2$

Similarly differentiating the function (1)  $n$ -times, successively,  
the  $n^{th}$  order derivative of  $y$  exists denoted by  $\frac{d^n y}{dx^n}$  or  $y^n$  or  $y_n$  or  $\Delta^n$

The process of finding 2<sup>nd</sup> and higher order derivatives is known as Successive Differentiation.

#### *$n^{th}$ derivative of some standard functions:*

1.  $y = e^{ax}$

Sol :  $y_1 = a e^{ax}$

$$y_2 = a^2 e^{ax}$$

*Differentiating Successively*

$$y_n = a^n e^{ax}$$

ie.  $D^n[e^{ax}] = a^n e^{ax}$

For,  $a = 1$   $D^n[e^x] = e^x$

---

2.  $y = \log(ax + b)$

**Solution :**  $y_1 = \frac{a}{ax+b}$

$$y_2 = \frac{(-1)a.a}{(ax+b)^2} = \frac{(-1)^1 a^2}{(ax+b)^2}$$

$$y_3 = \frac{(-1)(-2)a^2.a}{(ax+b)^3} = \frac{(-1)^2(1)(2)a^3}{(ax+b)^3} = \frac{(-1)^{3-1}(3-1)!a^3}{(ax+b)^3}$$

$$D^n [\log(ax + b)] = y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

Similarly  $D^n [\log x] = y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}$

3.  $y = (ax + b)^m$

**Solution:**  $y_1 = m(ax + b)^{m-1} a$

$$y_2 = m(m-1)(ax + b)^{m-2} a^2$$

$$y_3 = m(m-1)(m-2)(ax + b)^{m-3} a^3$$

Similarly

$$y_n = m(m-1)(m-2)\dots\dots(m-n+1)(ax + b)^{m-n} a^n$$

...(\*)

Case (i) :- If  $m = n$  in (\*)

$$D^n[(ax+b)^n] = n(n-1)(n-2)\dots\dots 3 \cdot 2 \cdot 1 \cdot a^n$$

$$= n! a^n$$

$$D^n[x^n] = n!$$

Case (ii) :- If  $m > n$  in (\*)

$$D^n[(ax+b)^m] = \frac{m(m-1)\dots\dots(m-n+1)(m-n)(m-n-1)\dots\dots 3 \cdot 2 \cdot 1}{(m-n)(m-n-1)\dots\dots 3 \cdot 2 \cdot 1} (ax+b)^{m-n} a^n$$

$$D^n[(ax+b)^m] = \frac{m!}{(m-n)!} (ax+b)^{m-n} a^n$$

$$D^n[x^m] = \frac{m!}{(m-n)!} x^{m-n} a^n$$

Case iii :- If  $m < n$  in (\*)

$$D^n[(ax+b)^m] = 0$$

Case iv :- If  $m = -1$  in (\*)

$$D^n\left[\frac{1}{ax+b}\right] = (-1)(-2)\dots\dots(-n)(ax+b)^{-1-n} a^n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$$

$$D^n\left[\frac{1}{(ax+b)^p}\right] = \frac{(-1)^n p(p+1)\dots\dots(p+n-1)a^n}{(ax+b)^{p+n}}$$

$$D^n\left[\frac{1}{(ax+b)^p}\right] = (-1)^n \frac{(p+n-1)!}{(p-1)!} \frac{a^n}{(ax+b)^{p+n}}$$

$$\text{If } a=1, \quad D^n\left[\frac{1}{x^p}\right] = (-1)^n \frac{(p+n-1)!}{(p-1)!} \frac{1}{x^{p+n}}$$

4.  $y = \cos(ax + b)$

$$y_1 = -\sin(ax + b) \cdot a = a \cos(ax + b + \pi/2)$$

$$y_2 = -\sin(ax + b + \pi/2) \cdot a^2 = a^2 \cos(ax + b + 2\pi/2)$$

$$y_n = D^n [\cos(ax + b)] = a^n \cos(ax + b + n\pi/2)$$

If  $a=1, b=0$

$$D^n [\cos x] = \cos(x + n\pi/2)$$

5.  $y = \sin(ax + b)$

$$y_n = D^n [\sin(ax + b)] = a^n \sin(ax + b + n\pi/2)$$

If  $a=1, b=0$

$$D^n [\sin x] = \sin(x + n\pi/2)$$

6.  $y = e^{ax} \sin(bx + c)$

$$y_1 = a e^{ax} \sin(bx + c) + b e^{ax} \cos(bx + c)$$

$$= e^{ax} [a \sin(bx + c) + b \cos(bx + c)]$$

Put  $a = r \cos \theta$   $b = r \sin \theta$  then  $r = \sqrt{a^2 + b^2}$ ,  $\theta = \tan^{-1} \frac{b}{a}$

$$y_1 = e^{ax} [r \cos \theta \sin(bx + c) + r \sin \theta \cos(bx + c)]$$

$$y_1 = r e^{ax} [\sin(bx + c + \theta)]$$

$$y_2 = r [e^{ax} a \sin(bx + c + \theta) + e^{ax} b \cos(bx + c + \theta)]$$

Put  $a = r \cos \theta$   $b = r \sin \theta$  then  $r = \sqrt{a^2 + b^2}$ ,  $\theta = \tan^{-1} \frac{b}{a}$

$$y_2 = r e^{ax} [r \cos \theta \sin(bx + c + \theta) + r \sin \theta \cos(bx + c + \theta)]$$

$$y_2 = r^2 e^{ax} [\sin(bx + c + 2\theta)]$$

Similarly,

$$y_n = r^n e^{ax} [\sin (bx + c + n\theta)]$$

$$y_n = D^n [e^{ax} \sin (bx + c)] = (a^2 + b^2)^{n/2} e^{ax} [\sin (bx + c + n \tan^{-1} \frac{b}{a})]$$

For  $a=b=1, c=0$

$$D^n [e^x \sin x] = (2)^{n/2} e^x [\sin (x + n\pi/4)]$$

$$7. y = e^{ax} [\cos (bx + c)]$$

$$y_n = D^n [e^{ax} \cos (bx + c)] = (a^2 + b^2)^{n/2} e^{ax} [\cos (bx + c + n \tan^{-1} \frac{b}{a})]$$

For  $a=b=1, c=0$

$$D^n [e^x \sin x] = (2)^{n/2} e^x [\sin (x + n\pi/4)]$$

$$8. \quad y = a^{mx}$$

$$y_1 = a^{mx} (\log a^m) = a^{mx} (m \log a)$$

$$y_2 = a^{mx} (m \log a)^2$$

Differentiating Successively

$$y_n = a^{mx} (m \log a)^n$$

$$\text{For } m=1, \quad D^n [a^x] = a^x (\log a)^n$$

**Leibnitz's Theorem :**

It provides a useful formula for computing the  $n^{\text{th}}$  derivative of a product of two functions.

**Statement :** If  $u$  and  $v$  are any two functions of  $x$  with  $u_n$  and  $v_n$  as their  $n^{\text{th}}$  derivative. Then the  $n^{\text{th}}$  derivative of  $uv$  is

$$(uv)_n = u_0 v_n + {}^n C_1 u_1 v_{n-1} + {}^n C_2 u_2 v_{n-2} + \dots + {}^n C_{n-1} u_{n-1} v_1 + u_n v_0$$

Note : We can interchange  $u$  &  $v$   $(uv)_n = (vu)_n$ ,

$${}^n C_1 = n, \quad {}^n C_2 = n(n-1)/2!, \quad {}^n C_3 = n(n-1)(n-2)/3! \dots$$

**1. Find the  $n^{\text{th}}$  derivations of  $e^{ax} \cos(bx + c)$** 

**Solution:**  $y_1 = e^{ax} - b \sin(bx + c) + a e^{ax} \cos(bx + c)$ , by product rule.

$$\text{i.e., } y_1 = e^{ax} [a \cos(bx + c) - b \sin(bx + c)]$$

Let us put  $a = r \cos \theta$ , and  $b = r \sin \theta$ .

$$\therefore a^2 + b^2 = r^2 \text{ and } \tan \theta = b/a$$

$$\text{i.e., } r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}(b/a)$$

$$\text{Now, } y_1 = e^{ax} [r \cos \theta \cos(bx + c) - r \sin \theta \sin(bx + c)]$$

$$\text{i.e., } y_1 = r e^{ax} \cos(\theta + bx + c)$$

where we have used the formula  $\cos A \cos B - \sin A \sin B = \cos(A + B)$

Differentiating again and simplifying as before,

$$y_2 = r^2 e^{ax} \cos(2\theta + bx + c).$$

$$\text{Similarly } y_3 = r^3 e^{ax} \cos(3\theta + bx + c).$$

.....

$$\text{Thus } y_n = r^n e^{ax} \cos(n\theta + bx + c)$$

$$\text{Where } r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}(b/a).$$

$$\text{Thus } D^n [e^{ax} \cos(bx + c)]$$

$$= \left[ (\sqrt{a^2 + b^2})^n e^{ax} \cos \left[ n \tan^{-1}(b/a) + bx + c \right] \right]$$

---



2. Find the  $n^{\text{th}}$  derivative of  $\log \sqrt{4x^2 + 8x + 3}$

**Solution :** Let  $y = \log \sqrt{4x^2 + 8x + 3} = \log (4x^2 + 8x + 3)^{\frac{1}{2}}$

ie.,  $y = \frac{1}{2} \log (4x^2 + 8x + 3) \because \log x^n = n \log x$

$y = \frac{1}{2} \log \{ (2x + 3) (2x+1) \}$ , by factorization.

$\therefore y = \frac{1}{2} \{ \log (2x + 3) + \log (2x + 1) \}$

Now  $y_n = \frac{1}{2} \left\{ \frac{(-1)^{n-1} (n-1)! 2^n}{(2x+3)^n} + \frac{(-1)^{n-1} (n-1)! 2^n}{(2x+1)^n} \right\}$

ie.,  $y_n = 2^{n-1} (-1)^{n-1} (n-1)! \left\{ \frac{1}{(2x+3)^n} + \frac{1}{(2x+1)^n} \right\}$

3. Find the  $n^{\text{th}}$  derivative of  $\log_{10} \{ (1-2x)^3 (8x+1)^5 \}$

**Solution :** Let  $y = \log_{10} \{ (1-2x)^3 (8x+1)^5 \}$

It is important to note that we have to convert the logarithm to the base e by the property:

$$\log_{10} x = \frac{\log_e x}{\log_e 10}$$

Thus  $y = \frac{1}{\log_e 10} \log_e \{ (1-2x)^3 (8x+1)^5 \}$

ie.,  $y = \frac{1}{\log_e 10} \{ 3 \log(1-2x) + 5 \log(8x+1) \}$

$\therefore y_n = \frac{1}{\log_e 10} \left\{ 3 \cdot \frac{(-1)^{n-1} (n-1)! (-2)^n}{(1-2x)^n} + 5 \frac{(-1)^{n-1} (n-1)! 8^n}{(8x+1)^n} \right\}$

ie.,  $y_n = \frac{(-1)^{n-1} (n-1)! 2^n}{\log_e 10} \left\{ \frac{3(-1)^n}{(1-2x)^n} + \frac{5(4)^n}{(8x+1)^n} \right\}$

4. Find the  $n^{\text{th}}$  derivative of  $e^{2x} \cos^2 x \sin x$

**Solution :**  $\gg$  let  $y = e^{2x} \cos^2 x \sin x = e^{2x} \left[ \frac{1 + \cos 2x}{2} \right] \sin x$

$$\text{ie., } y = \frac{e^{2x}}{2} (\sin x + \sin x \cos 2x)$$

$$= \frac{e^{2x}}{2} \left\{ \sin x + \frac{1}{2} [\sin 3x + \sin(-x)] \right\}$$

$$= \frac{e^{2x}}{4} (2 \sin x + \sin 3x - \sin x) \because \sin(-x) = -\sin x$$

$$\therefore y = \frac{e^{2x}}{4} (\sin x + \sin 3x)$$

Now  $y_n = \frac{1}{4} \{ D^n (e^{2x} \sin x) + D^n (e^{2x} \sin 3x) \}$

Thus  $y_n = \frac{1}{4} \left\{ (\sqrt{5})^n e^{2x} \sin [n \tan^{-1}(1/2) + x] + (\sqrt{13})^n e^{2x} \sin [n \tan^{-1}(3/2) + 3x] \right\}$

$$\therefore y_n = \frac{e^{2x}}{4} \left\{ (\sqrt{5})^n \sin [n \tan^{-1}(1/2) + x] + (\sqrt{13})^n \sin [n \tan^{-1}(3/2) + 3x] \right\}$$

5. Find the  $n^{\text{th}}$  derivative of  $e^{2x} \cos^3 x$

**Solution :** Let  $y = e^{2x} \cos^3 x = e^{2x} \cdot \frac{1}{4} (3 \cos x + \cos 3x)$

ie.,  $y = \frac{1}{4} (3 e^{2x} \cos x + e^{2x} \cos 3x)$

$$\therefore y_n = \frac{1}{4} \{ 3 D^n (e^{2x} \cos x) + D^n (e^{2x} \cos 3x) \}$$

$$y_n = \frac{1}{4} \left\{ 3 (\sqrt{5})^n e^{2x} \cos [n \tan^{-1}(1/2) + x] + (\sqrt{13})^n e^{2x} \cos [n \tan^{-1}(3/2) + 3x] \right\}$$

Thus  $y_n = \frac{e^{2x}}{4} \left\{ 3 (\sqrt{5})^n \cos [n \tan^{-1}(1/2) + x] + (\sqrt{13})^n \cos [n \tan^{-1}(3/2) + 3x] \right\}$

---

6. Find the  $n^{\text{th}}$  derivative of  $\frac{x^2}{(2x+1)(2x+3)}$

**Solution :**  $y = \frac{x^2}{(2x+1)(2x+3)}$  is an improper fraction because; the degree of the numerator being 2 is equal to the degree of the denominator. Hence we must divide and rewrite the fraction.

$$y = \frac{x^2}{4x^2 + 8x + 3} = \frac{1}{4} \cdot \frac{4x^2}{4x^2 + 8x + 3} \text{ for convenience.}$$

$$4x^2 + 8x + 3 \overline{) \begin{array}{r} 1 \\ 4x^2 \\ \underline{4x^2 + 8x + 3} \\ -8x - 3 \end{array}}$$

$$\therefore y = \frac{1}{4} \left[ 1 + \frac{-8x - 3}{4x^2 + 8x + 3} \right]$$

$$\text{ie., } y = \frac{1}{4} - \frac{1}{4} \left[ \frac{8x + 3}{4x^2 + 8x + 3} \right]$$

The algebraic fraction involved is a proper fraction.

$$\text{Now } y_n = 0 - \frac{1}{4} D^n \left[ \frac{8x + 3}{4x^2 + 8x + 3} \right].$$

$$\text{Let } \frac{8x + 3}{(2x+1)(2x+3)} = \frac{A}{2x+1} + \frac{B}{2x+3}$$

Multiplying by  $(2x+1)(2x+3)$  we have,  $8x + 3 = A(2x+3) + B(2x+1)$

.....(1)

By setting  $2x+1=0$ ,  $2x+3=0$  we get  $x = -1/2$ ,  $x = -3/2$ .

Put  $x = -1/2$  in (1):  $-1 - 1 + A(2) \Rightarrow A = -1/2$

Put  $x = -3/2$  in (1):  $-9 = B(-2) \Rightarrow B = 9/2$

$$\therefore y_n = -\frac{1}{4} \left\{ -\frac{1}{2} D^n \left[ \frac{1}{2x+1} \right] + \frac{9}{2} D^n \left[ \frac{1}{2x+3} \right] \right\}$$

$$= -\frac{1}{8} \left\{ (-1) \cdot \frac{(-1)^n n! 2^n}{(2x+1)^{n+1}} + 9 \cdot \frac{(-1)^n n! 2^n}{(2x+3)^{n+1}} \right\}$$

$$\text{ie., } y_n = \frac{(-1)^{n+1} n! 2^n}{8} \left\{ \frac{1}{(2x+1)^{n+1}} + \frac{9}{(2x+3)^{n+1}} \right\}$$

7. Find the  $n^{\text{th}}$  derivative of  $\frac{x^4}{(x+1)(x+2)}$

Solution :  $y = \frac{x^4}{(x+1)(x+2)}$  is an improper fraction.

(deg of nr. = 4 > deg. of dr. = 2)

On dividing  $x^4$  by  $x^2 + 3x + 2$ , We get

$$y = (x^2 - 3x + 7) + \left[ \frac{-15x - 14}{x^2 + 3x + 2} \right]$$

$$\therefore y_n = D^n (x^2 - 3x + 7) - D^n \left[ \frac{15x + 14}{x^2 + 3x + 2} \right]$$

$$\text{But } D = (x^2 - 3x + 7) = 2x - 3, D^2 (x^2 - 3x + 7) = 2$$

$$D^3 (x^2 - 3x + 7) = 0 \dots \dots \dots D^n (x^2 - 3x + 7) = 0 \text{ if } n > 2$$

$$\text{Hence } y_n = -D^n \left[ \frac{15x + 14}{(x+1)(x+2)} \right]$$

$$\text{Now, let } D^n \frac{15x + 14}{x^2 + 3x + 2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$\Rightarrow 15x + 14 = A(x+2) + B(x+1)$$

$$\text{Put } x = -1 ; -1 = A(1) \text{ or } A = -1$$

$$\text{Put } x = -2 ; -16 = B(-1) \text{ or } B = 16$$

$$Y_n = \left\{ -D^n \left[ \frac{1}{x+1} \right] + 16D^n \left[ \frac{1}{x+2} \right] \right\}$$

$$= \frac{(-1)^n n! 1^n}{(x+1)^{n+1}} - 16 \frac{(-1)^n n! 1^n}{(x+2)^{n+1}}$$

$$y_n = (-1)^n n! \left\{ \frac{1}{(x+1)^{n+1}} - \frac{16}{(x+2)^{n+1}} \right\} n > 2$$

8. Show that

$$\frac{d^n}{dx^n} \left( \frac{\log x}{x} \right) = \frac{(-1)^n n!}{x^{n+1}} \left\{ \log x - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{n} \right\}$$

**Solution :** Let  $y = \frac{\log x}{x} = \log x \cdot \frac{1}{x}$  and let  $u = \log x$ ,  $v = \frac{1}{x}$

We have Leibnitz theorem,

$$(uv)_n = uv_n + n_{C_1} u_1 v_{n-1} + n_{C_2} u_2 v_{n-2} + \dots + u_n v \quad \dots (1)$$

$$\text{Now, } u = \log x \quad \therefore u_n = \frac{(-1)^{n-1} (n-1)!}{x^n}$$

$$v = \frac{1}{x} \quad \therefore v_n = \frac{(-1)^n n!}{x^{n+1}}$$

Using these in (1) by taking appropriate values for n we get,

$$D_n = \left( \frac{\log x}{x} \right) = \log x \cdot \frac{(-1)^n n!}{x^{n+1}} + n \frac{1}{x} \cdot \frac{(-1)^{n-1} (n-1)!}{x^n}$$

$$+ \frac{n(n-1)}{1 \cdot 2} \left( -\frac{1}{x^2} \right) \frac{(-1)^{n-2} (n-2)!}{x^{n-1}}$$

$$+ \dots + \frac{(-1)^{n-1} (n-1)!}{x^n} \cdot \frac{1}{x}$$

$$\text{ie..} = \log x \frac{(-1)^n n!}{x^{n+1}} + \frac{(-1)^{n-1} n!}{x^{n+1}}$$

$$- \frac{(-1)^{n-2} n!}{2x^{n+1}} + \dots + \frac{(-1)^{n-1} (n-1)!}{x^{n+1}}$$

$$- \frac{(-1)^{n-2} n!}{x^{n+1}} \left[ \log x (-1)^{-1} - \frac{(-1)^{-2}}{2} + \dots + \frac{(-1)^{-1} (n-1)!}{n^1} \right]$$

$$\text{Note : } (-1)^{-1} = \frac{1}{-1} = -1; (-1)^{-2} = \frac{1}{(-1)^2} = 1$$

$$\text{Also } \frac{(n-1)!}{n!} = \frac{(n-1)!}{n(n-1)!} = \frac{1}{n}$$

$$\therefore \frac{d^n}{dx^n} \left[ \frac{\log x}{x} \right] = \frac{(-1)^n n!}{x^{n+1}} \left[ \log x - 1 - \frac{1}{2} - \frac{1}{3} \dots - \frac{1}{n} \right]$$

**9. If  $y_n = D^n (x^n \log x)$**

Prove that  $y_n = n y_{n-1} + (n-1)!$  and hence deduce that

$$y_n = n \left( \log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right)$$

**Solution :**  $y_n = D^n (x^n \log x) = D^{n-1} \{ D (x^n \log x) \}$

$$= D^{n-1} \left\{ x^n \cdot \frac{1}{x} + nx^{n-1} \log x \right\}$$

$$= D^{n-1} (x^{n-1}) + n D^{n-1} (x^{n-1} \log x)$$

$\therefore y_n = (n-1)! + n y_{n-1}$ . This proves the first part.

Now Putting the values for  $n = 1, 2, 3 \dots$  we get

$$y_1 = 0! + 1 y_0 = 1 + \log x = 1! (\log x + 1)$$

$$y_2 = 1! + 2 y_1 = 1 + 2 (\log x + 1)$$

$$\text{ie., } y_2 = 2! \log x + 3 = 2 (\log x + 3/2) = 2! \left( \log x + 1 + \frac{1}{2} \right)$$

$$y_3 = 2! + 3 y_2 = 2 + 3 (2 \log x + 3)$$

$$\text{ie., } y_3 = 6 \log x + 11 = 6 (\log x + 11/6) = 3! \left( \log x + 1 + \frac{1}{2} + \frac{1}{3} \right)$$

.....

$$y_n = n! \left( \log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right)$$

**10. If  $y = a \cos (\log x) + b \sin (\log x)$ , show that**

$x^2 y_2 + x y_1 + y = 0$ . Then apply Leibnitz theorem to differentiate this result  $n$  times.

or

If  $y = a \cos (\log x) + b \sin (\log x)$ , show that

$$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0.$$

[July-03]

**Solution :**  $y = a \cos (\log x) + b \sin (\log x)$

Differentiate w.r.t x

$$\therefore y_1 = -a \sin (\log x) \cdot \frac{1}{x} + b \cos (\log x) \cdot \frac{1}{x}$$

(we avoid quotient rule to find  $y_2$ ) .

$$\Rightarrow xy_1 = -a \sin (\log x) + b \cos (\log x)$$

Differentiating again w.r.t x we have,

$$xy_2 + 1 \cdot y_1 = -a \cos (\log x) + b \sin (\log x) \cdot \frac{1}{x}$$

$$\text{or } x^2y_2 + xy_1 = -[a \cos (\log x) + b \sin (\log x)] = -y$$

$$\therefore x^2y_2 + xy_1 + y = 0$$

Now we have to differentiate this result n times.

$$\text{ie., } D^n (x^2y_2) + D^n (xy_1) + D^n (y) = 0$$

We have to employ Leibnitz theorem for the first two terms.

Hence we have,

$$\left\{ x^2 \cdot D^n (y_2) + n \cdot 2x \cdot D^{n-1} (y_2) + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot D^{n-2} (y_2) \right\}$$

$$\left\{ x \cdot D^n (y_1) + n \cdot 1 \cdot D^{n-1} (y_1) \right\} + y_n = 0$$

$$\text{ie., } \{x^2y_{n+2} + 2n x y_{n+1} + n(n-1)y_n\} + \{xy_{n+1} + ny_n\} + y_n = 0$$

$$\text{ie., } x^2y_{n+2} + 2n x y_{n+1} + n^2y_n - ny_n + xy_{n+1} + ny_n + y_n = 0$$

$$\text{ie., } x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

**11. If  $\cos^{-1} (y/b) = \log (x/n)^n$ , then show that**

$$x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$$

**Solution :** By data,  $\cos^{-1} (y/b) = n \log (x/n) \therefore \log(a^m) = m \log a$

$$\Rightarrow \frac{y}{b} = \cos [n \log (x/n)]$$

$$\text{or } y = b \cdot \cos [n \log (x/n)]$$

Differentiating w.r.t x we get,

$$y_1 = -b \sin [n \log (x/n)] \cdot n \cdot \frac{1}{(x/n)} \cdot \frac{1}{n}$$

$$\text{or } xy_1 = -n b \sin [n \log (x/n)]$$

Differentiating w.r.t x again we get,

$$xy_2 + 1 \cdot y_1 = -n \cdot b \cos [n \log (x/n)] \cdot n \cdot \frac{1}{(x/n)} \cdot \frac{1}{n}$$

$$\text{or } x(xy_2 + y_1) = n^2 b \cos [n \log (x/n)] = -n^2 y, \text{ by using (1).}$$

$$\text{or } x^2 y_2 + xy_1 + n^2 y = 0$$

Differentiating each term n times we have,

$$D(x^2 y_2) + D^n(xy_1) + n^2 D^n(y) = 0$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{ x^2 y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n \right\} \\ + \{ xy_{n+1} + n \cdot 1 \cdot y_n \} + n^2 y_n = 0$$

$$\text{ie } x^2 y_{n+2} + 2x y_{n+1} + n^2 y_n + xy_{n+1} + ny_n + n^2 y_n = 0$$

$$\text{or } x^2 y_{n+2} + (2n+1) xy_{n+1} + 2n^2 y_n = 0$$

12. If  $y = \sin(\log(x^2 + 2x + 1))$ ,

or

[Feb-03]

If  $\sin^{-1} y = 2 \log(x+1)$ , show that

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2 + 4)y_n = 0$$

**Solution :** By data  $y = \sin \log(x^2 + 2x + 1)$

$$\therefore y_1 = \cos \log(x^2 + 2x + 1) \cdot \frac{1}{(x+1)^2} \cdot 2x + 2$$

$$\text{ie., } y_1 = \cos \log(x^2 + 2x + 1) \cdot \frac{1}{x^2 + 2x + 1} \cdot 2(x+1)$$

$$\text{ie., } y_1 = \frac{2 \cos \log(x^2 + 2x + 1)}{(x+1)}$$

$$\text{or } (x+1)y_1 = 2 \cos \log(x^2 + 2x + 1)$$

Differentiating w.r.t x again we get



$$(x+1)y_2 + 1 y_1 = -2 \sin \log (x^2 + 2x + 1) \frac{1}{(x+1)^2} \cdot 2(x+1)$$

$$\text{or } (x+1)^2 y_2 + (x+1) y_1 = -4y$$

$$\text{or } (x+1)^2 y_2 + (x+1) y_1 + 4y = 0 ,$$

Differentiating each term n times we have,

$$D^n [(x+1)^2 y_2] + D^n [(x+1) y_1] + D^n [y] = 0$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{ (x+1)^2 y_{n+2} + n \cdot 2(x+1) \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n \right\}$$

$$+ \{(x+1) y_{n+1} + n \cdot 1 \cdot y_n\} + 4y_n = 0$$

$$\text{ie., } (x+1)^2 y_{n+2} + 2n(x+1) y_{n+1}$$

$$+ n^2 y_n - n y_n + (x+1) y_{n+1} + n y_n + 4y_n = 0$$

$$\text{ie., } (x+1)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (n^2+4) y_n = 0$$

**13. If  $y = \log (x + \sqrt{1+x^2})$  prove that**

$$(1+x^2) y_{n+2} + (2n+1) x y_{n+1} + n^2 y_n = 0$$

$$\gg \text{By data, } y = \log (x + \sqrt{1+x^2})$$

$$\therefore y_1 = \frac{1}{(x + \sqrt{1+x^2})} \left\{ 1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x \right\}$$

$$\text{ie., } y_1 \frac{1}{(x + \sqrt{1+x^2})} \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\text{or } \sqrt{1+x^2} y_1 = 1$$

Differentiating w.r.t.x again we get

$$\sqrt{1+x^2} y_2 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x \cdot y_1 = 0$$

$$\text{or } (1+x^2) y_2 + x y_1 = 0$$

$$\text{Now } D^n [(1+x^2) y_2] + D^n [x y_1] = 0$$

Applying Leibnitz theorem to each term we get,

$$\left\{ (1+x^2)y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot y_n \right\} \\ + [x \cdot y_{n+1} + n \cdot 1 \cdot y_n] = 0 \\ \text{ie., } (1+x^2)y_{n+2} + 2n x y_{n+1} + n^2 y_n - n y_n + x y_{n+1} + n y_n = 0 \\ \text{or } (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$$

**14. If  $x = \sin t$  and  $y = \cos mt$ , prove that**

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0. \quad [\text{Feb-04}]$$

**Solution :** By data  $x = \sin t$  and  $y = \cos mt$

$$x = \sin t \Rightarrow t = \sin^{-1} x \text{ and } y = \cos mt \text{ becomes}$$

$$y = \cos [m \sin^{-1} x]$$

Differentiating w.r.t.  $x$  we get

$$y_1 = -\sin (m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

$$\text{or } \sqrt{1-x^2} y_1 = -m \sin (m \sin^{-1} x)$$

Differentiating again w.r.f.  $x$  we get,

$$\sqrt{1-x^2} y_2 + \frac{1}{2\sqrt{1-x^2}} (-2x) y_1 = -m \cos (m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

$$\text{or } (1-x^2)y_2 - xy_1 = -m^2 y$$

$$\text{or } (1-x^2)y_2 - xy_1 + m^2 y = 0$$

$$\text{Thus } (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

**15. If  $x = \tan (\log y)$ , find the value of**

$$(1+x^2)y_{n+1} + (2nx-1) y_n + n(n-1)y_{n-1} \quad [\text{July-04}]$$

**Solution :** By data  $x = \tan(\log y) \Rightarrow \tan^{-1} x = \log y$  or  $y = e^{\tan^{-1} x}$  Since the desired relation involves  $y_{n+1}$ ,  $y_n$  and  $y_{n-1}$  we can find  $y_1$  and differentiate  $n$  times the result associated with  $y_1$  and  $y$ .

$$\text{Consider } y = e^{\tan^{-1} x} \therefore y_1 = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

$$\text{or } (1+x^2)y_1 = y$$

Differentiating  $n$  times we have

$$D^n[(1+x^2)y_1] = D^n[y]$$

Applying Leibnitz theorem onto L.H.S, we have,

$$\{(1+x^2)D^n(y_1) + n \cdot 2x \cdot D^{n-1}(y_1) + \frac{n(n-1)}{1 \cdot 2} \cdot 2 \cdot D^{n-2}(y_1)\} = y_n$$

$$\text{i.e., } (1+x^2)y_{n+1} + 2nx y_n + n(n-1)y_{n-1} = 0$$

$$\text{Or } (1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$$

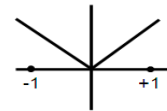
## Continuity & Differentiability

### Some Fundamental Definitions

A function  $f(x)$  is defined in the interval  $I$ , then it is said to be continuous at a point  $x = a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$

A function  $f(x)$  is said to be differentiable at  $x = a$  if  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(a)}{h} = f'(a)$  exists  $a \in I$

**Ex :** Consider a function  $f(x)$  is defined in the interval  $[-1,1]$  by  $f(x) = |x| = \begin{cases} -x & -1 \leq x \leq 0 \\ x & 0 \leq x \leq 1 \end{cases}$



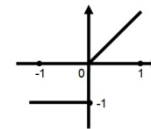
It is continuous at  $x = 0$

But not differentiable at  $x = 0$

**Note :** If a function  $f(x)$  is differentiable then it is continuous, but converse need not be true.  
Geometrically :

(1) If  $f(x)$  is Continuous at  $x=a$  means,  $f(x)$  has no breaks or jumps at the point  $x = a$

**Ex :**  $f(x) = \begin{cases} -1 & -1 \leq x \leq 0 \\ x & 0 < x \leq 1 \end{cases}$



Is discontinuous at  $x=0$

(2) If  $f(x)$  is differentiable at  $x = a$  means, the graph of  $f(x)$  has a unique tangent at the point or graph is smooth at  $x = a$

### 1. Give the definitions of Continuity & Differentiability:

**Solution:** A function  $f(x)$  is said to be continuous at  $x = a$ , if corresponding to an arbitrary positive number  $\epsilon$ , however small, there exists another positive number  $\delta$  such that.

$$|f(x) - f(a)| < \epsilon, \text{ where } |x - a| < \delta$$

It is clear from the above definition that a function  $f(x)$  is continuous at a point 'a'.

If (i) it exists at  $x = a$

(ii)  $\lim_{x \rightarrow a} f(x) = f(a)$

i.e, limiting value of the function at  $x = a$  is to the value of the function at  $x = a$

### Differentiability:

A function  $f(x)$  is said to be differentiable in the interval  $(a, b)$ , if it is differentiable at every point in the interval.

In Case  $[a, b]$  the function should possess derivatives at every point and at the end points  $a$  &  $b$  i.e.,  $Rf'(a)$  and  $Lf'(a)$  exists.

### 2. State Rolle's Theorem with Geometric Interpretation.

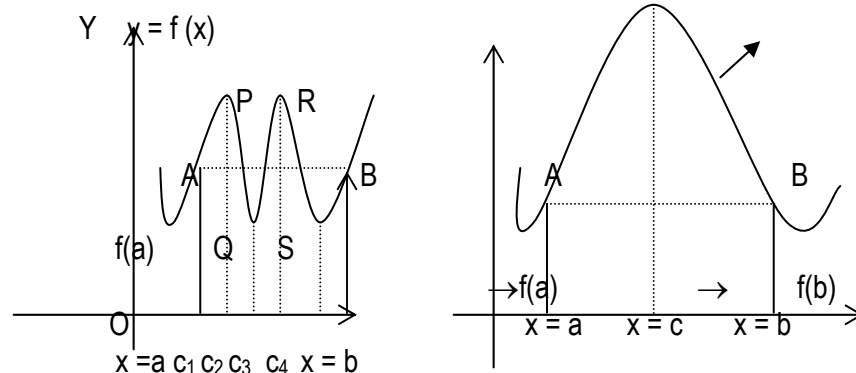
**Statement:** Let  $f(x)$  be a function is defined on  $[a, b]$  & it satisfies the following Conditions.

- (i)  $f(x)$  is continuous in  $[a, b]$
- (ii)  $f(x)$  is differentiable in  $(a, b)$
- (iii)  $f(a) = f(b)$

Then there exists at least a point  $C \in (a, b)$ , Here  $a < b$  such that  $f'(c) = 0$

### Proof:

#### Geometrical Interpretation of Rolle's Theorem:



Let us consider the graph of the function  $y = f(x)$  in  $xy$  – plane.  $A(a, f(a))$  and  $B(b, f(b))$  be the two points in the curve  $f(x)$  and  $a, b$  are the corresponding end points of  $A$  &  $B$  respectively. Now, explained the conditions of Rolle's theorem as follows.

- (i)  $f(x)$  is continuous function in  $[a, b]$ , Because from figure without breaks or jumps in between  $A$  &  $B$  on  $y = f(x)$ .
- (ii)  $f(x)$  is differentiable in  $(a, b)$ , that means let us joining the points  $A$  &  $B$ , we get a line  $AB$ .

$\therefore$  Slope of the line  $AB = 0$  then  $\exists$  a point  $C$  at  $P$  and also the tangent at  $P$  (or  $Q$  or  $R$  or  $S$ ) is Parallel to  $x$  –axis.

$\therefore$  Slope of the tangent at  $P$  (or  $Q$  or  $R$  or  $S$ ) to be Zero even the curve  $y = f(x)$  decreases or increases, i.e.,  $f(x)$  is Constant.

$$f'(x) = 0$$

$$\therefore f'(c) = 0$$

(iii) The Slope of the line AB is equal to Zero, i.e., the line AB is parallel to x – axis.

$$\therefore f(a) = f(b)$$

### 3. Verify Rolle's Theorem for the function $f(x) = x^2 - 4x + 8$ in the interval $[1,3]$

**Solution:** We know that every Poly nominal is continuous and differentiable for all points and hence  $f(x)$  is continuous and differentiable in the interval  $[1,3]$ .

$$\text{Also } f(1) = 1 - 4 + 8 = 5, f(3) = 3^2 - 4 \cdot 3 + 8 = 5$$

$$\text{Hence } f(1) = f(3)$$

Thus  $f(x)$  satisfies all the conditions of the Rolle's Theorem. Now  $f'(x) = 2x - 4$  and  $f'(x) = 0 \Rightarrow 2x - 4 = 0$  or  $x = 2$ . Clearly  $1 < 2 < 3$ . Hence there exists  $\xi \in (1,3)$  such that  $f'(\xi) = 0$ . This shows that Rolle's Theorem holds good for the given function  $f(x)$  in the given interval.

### 4. Verify Rolle's Theorem for the function $f(x) = x(x+3)e^{-x/2}$ in the interval $[-3, 0]$

**Solution:** Differentiating the given function W.r.t 'x' we get

$$\begin{aligned} f'(x) &= (x^2 + 3x) \left( -\frac{1}{2} \right) e^{-x/2} + (2x + 3) e^{-x/2} \\ &= -\frac{1}{2} (x^2 - x - 6) e^{-x/2} \end{aligned}$$

$\therefore f'(x)$  exists (i.e finite) for all  $x$  and hence continuous for all  $x$ .

Also  $f(-3) = 0, f(0) = 0$  so that  $f(-3) = f(0)$  so that  $f(-3) = f(0)$ . Thus  $f(x)$  satisfies all the conditions of the Rolle's Theorem.

$$\text{Now, } f'(x) = 0$$

$$\Rightarrow -\frac{1}{2} (x^2 - x - 6) e^{-x/2} = 0$$

Solving this equation we get  $x = 3$  or  $x = -2$

Clearly  $-3 < -2 < 0$ . Hence there exists  $-2 \in (-3,0)$  such that  $f'(-2) = 0$

This proves that Rolle's Theorem is true for the given function.

---

**5. Verify the Rolle's Theorem for the function  $\sin x$  in  $[-\pi, \pi]$**

**Solution:** Let  $f(x) = \sin x$

Clearly  $\sin x$  is continuous for all  $x$ .

Also  $f'(x) = \cos x$  exists for all  $x$  in  $(-\pi, \pi)$  and  $f(-\pi) = \sin(-\pi) = 0$ ;  $f(\pi) = \sin(\pi) = 0$  so that  $f(-\pi) = f(\pi)$

Thus  $f(x)$  satisfies all the conditions of the Rolle's Theorem.

Now  $f'(x) = 0 \Rightarrow \cos x = 0$  so that

$$x = \pm \frac{\pi}{2}$$

Both these values lie in  $(-\pi, \pi)$ . These exist  $c = \pm \frac{\pi}{2}$

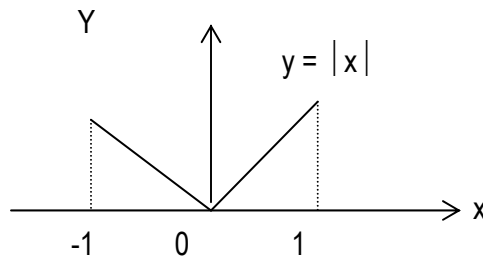
Such that  $f'(c) = 0$

Hence Rolle's theorem is verified.

**6. Discuss the applicability of Rolle's Theorem for the function  $f(x) = |x|$  in  $[-1, 1]$ .**

**Solution:** Now  $f(x) = |x| = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ -x & \text{for } -1 \leq x \leq 0 \end{cases}$

$f(x)$  being a linear function is continuous for all  $x$  in  $[-1, 1]$ .  $f(x)$  is differentiable for all  $x$  in  $(-1, 1)$  except at  $x = 0$ . Therefore Rolle's Theorem does not hold good for the function  $f(x)$  in  $[-1, 1]$ . Graph of this function is shown in figure. From which we observe that we cannot draw a tangent to the curve at any point in  $(-1, 1)$  parallel to the  $x$ -axis.



**Exercise:**

**7. Verify Rolle's Theorem for the following functions in the given intervals.**

a)  $x^2 - 6x + 8$  in  $[2, 4]$

b)  $(x - a)^3 (x - b)^3$  in  $[a, b]$

c)  $\log \left\{ \frac{x^2 + ab}{(a + b)x} \right\}$  in  $[a, b]$

**8. Find whether Rolle's Theorem is applicable to the following functions. Justify your answer.**

a)  $f(x) = |x - 1|$  in  $[0, 2]$

b)  $f(x) = \tan x$  in  $[0, \pi]$ .

**9. State & prove Lagrange's (1<sup>st</sup>) Mean Value Theorem with Geometric meaning.**

**Statement:** Let  $f(x)$  be a function of  $x$  such that

(i)  $f$  is continuous in  $[a, b]$

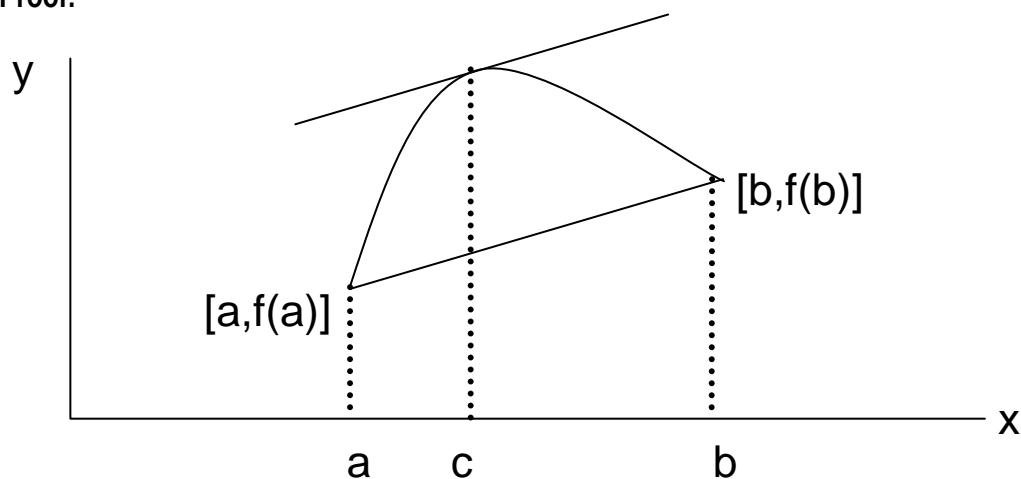
(ii)  $f$  is differentiable in  $(a, b)$

Then there exists atleast a point (or value)  $C \in (a, b)$  such that.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

i.e.,  $f(b) = f(a) + (b - a) f'(c)$

**Proof:**





Define a function  $g(x)$  so that  $g(x) = f(x) - Ax$  ----- (1)

Where  $A$  is a Constant to be determined.

So that  $g(a) = g(b)$

Now,  $g(a) = f(a) - Aa$

$G(b) = f(b) - Ab$

$\therefore g(a) = g(b) \Rightarrow f(a) - Aa = f(b) - Ab.$

$$\text{i.e., } A = \frac{f(b) - f(a)}{b - a} \text{ ----- (2)}$$

Now,  $g(x)$  is continuous in  $[a, b]$  as rhs of (1) is continuous in  $[a, b]$

$G(x)$  is differentiable in  $(a, b)$  as r.h.s of (1) is differentiable in  $(a, b)$ .

Further  $g(a) = g(b)$ , because of the choice of  $A$ .

Thus  $g(x)$  satisfies the conditions of the Rolle's Theorem.

$\therefore$  There exists a value  $x = c$  so that  $a < c < b$  at which  $g'(c) = 0$

$\therefore$  Differentiate (1) W.r.t 'x' we get

$$g'(x) = f'(x) - A$$

$$\therefore g'(c) = f'(c) - A (\because x = c)$$

$$\Rightarrow f'(c) - A = 0 (\because g'(c) = 0)$$

$$\therefore f'(c) = A \text{ ----- (3)}$$

From (2) and (3) we get

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ (or) } f(b) = f(a) + (b - a) f'(c) \text{ For } a < c < b$$

Corollary: Put  $b - a = h$

i.e.,  $b = a + h$  and  $c = a + \theta h$

Where  $0 < \theta < 1$

Substituting in  $f(b) = f(a) + (b - a) f'(c)$

$$\therefore f(a + h) = f(a) + h f'(a + \theta h), \text{ where } 0 < \theta < 1.$$


---

### Geometrical Interpretation:-

Since  $y = f(x)$  is continuous in  $[a, b]$ , it has a graph as shown in the figure below,

At  $x = a$ ,  $y = f(a)$

At  $x = b$ ,  $y = f(b)$

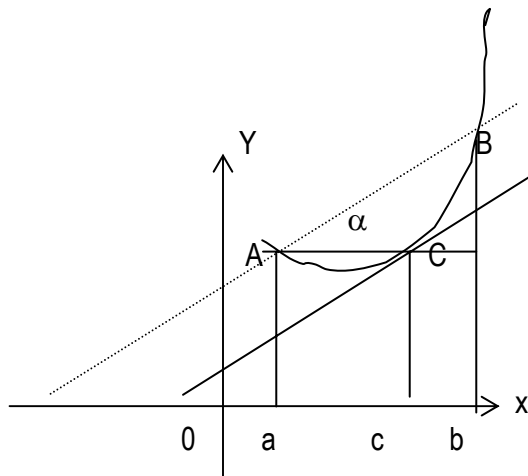
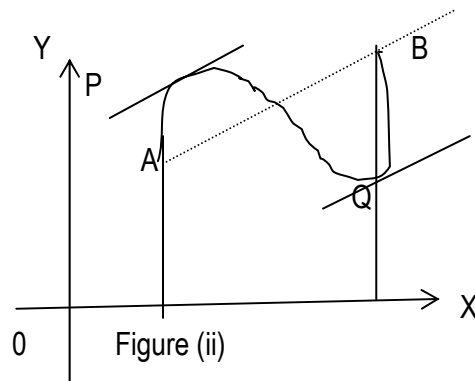


Figure (i)



Slope of the line joining the points A  $(a, f(a))$  and B  $(b, f(b))$

$$\text{Is } \frac{f(b) - f(a)}{b - a} \quad (\because \text{Slope} = m = \tan \theta)$$

$$= \tan \alpha$$

Where  $\alpha$  is the angle made by the line AB with x – axis

$$= \text{Slope of the tangent at } x = c$$

$$= f'(c), \text{ where } a < c < b$$

Geometrically, it means that there exists at least one value of  $x = c$ , where  $a < c < b$  at which the tangent will be parallel to the line joining the end points at  $x = a$  &  $x = b$ .

Note: There can be more than one value at which the tangents are parallel to the line joining points A & B (from Fig (ii)).

**10. Verify Lagrange's Mean value theorem for  $f(x) = (x - 1)(x - 2)(x - 3)$  in  $[0,4]$ .**

**Solution:** Clearly given function is continuous in  $[0,4]$  and differentiable in  $(0,4)$ , because  $f(x)$  is in polynomial.

$$f(x) = (x - 1)(x - 2)(x - 3)$$

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$\text{and } f(0) = 0^3 - 6(0)^2 + 11(0) - 6 = -6$$

$$f(4) = 4^3 - 6(4)^2 + 11(4) - 6 = 6$$

Differentiate  $f(x)$  W.r.t  $x$ , we get

$$f'(x) = 3x^2 - 6x + 11$$

$$\text{Let } x = c, f'(c) = 3c^2 - 6c + 11$$

By Lagrange's Mean value theorem, we have

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(0)}{(4 - 0)} \\ &= \frac{6 - (-6)}{4} = 3 \end{aligned}$$

$$\therefore 3c^2 - 6c + 11 = 3$$

$$\Rightarrow 3c^2 - 6c + 8 = 0$$

Solving this equation, we get

$$c = 2 \pm \frac{2}{\sqrt{3}} \in (0,4)$$

Hence the function is verified.

**11. Verify the Lagrange's Mean value theorem for  $f(x) = \log x$  in  $[1,e]$ .**

**Solution:** Now  $\log x$  is continuous for all  $x > 0$  and hence  $[1,e]$ .

Also  $f'(x) = \frac{1}{x}$  which exists for all  $x$  in  $(1,e)$

Hence  $f(x)$  is differentiable in  $(1,e)$

$\therefore$  by Lagrange's Mean Value theorem, we get

$$\frac{\text{Log} e - \text{Log} 1}{e - 1} = \frac{1}{c} \Rightarrow \frac{1}{e - 1} = \frac{1}{c}$$

$$\Rightarrow C = e - 1$$

$$\Rightarrow 1 < e - 1 < 2 < e$$

Since  $e \in (2, 3)$

$\therefore$  So that  $c = e - 1$  lies between 1 & e

Hence the Theorem.

**12. Find  $\theta$  for  $f(x) = Lx^2 + mx + n$  by Lagrange's Mean Value theorem.**

**Solution:**  $f(x) = Lx^2 + mx + n$

$$\therefore f'(x) = 2Lx + m$$

$$\text{We have } f(a + h) = f(a) + hf'(a + \theta h)$$

$$\text{Or } f(a + h) - f(a) = hf'(a + \theta h)$$

$$\text{i.e., } \{\ell(a + h)^2 + m(a + h) + n\} - \{\ell a^2 + ma + n\} = h\{2\ell(a + \theta h) + m\}$$

Comparing the Co-efficient of  $\ell h^2$ , we get

$$1 = 2\theta \quad \therefore \theta = \frac{1}{2} \in (0, 1)$$

**Exercise:**

**13. Verify the Lagrange's Mean Value theorem for  $f(x) = \sin^2 x$  in  $\left[0, \frac{\pi}{2}\right]$**

**14. Prove that,  $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$  if  $0 < a < b$  and reduce that**

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1} \frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

**15. Show that  $\frac{2}{\pi} < \frac{\sin x}{x} < 1$  in  $\left(0, \frac{\pi}{2}\right)$**

**16. Prove that  $\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$  Where  $a < b$ . Hence reduce**

$$\frac{\pi}{6} - \frac{1}{2\sqrt{3}} < \sin^{-1} \frac{1}{4} < \frac{\pi}{6} - \frac{1}{\sqrt{15}}$$

### 17. State & prove Cauchy's Mean Value Theorem with Geometric meaning.

**Proof:** The ratio of the increments of two functions called Cauchy's Theorem.

Statement: Let  $f(x)$  and  $g(x)$  be two functions of  $x$  such that,

- (i) Both  $f(x)$  and  $g(x)$  are continuous in  $[a,b]$
- (ii) Both  $f(x)$  and  $g(x)$  are differentiable in  $(a,b)$
- (iii)  $g'(x) \neq 0$  for any  $x \in (a,b)$

These three conditions exist at least one value  $x = c \in (a,b)$  at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof: Define a function,

$$\phi(x) = f(x) - A \cdot g(x) \quad \text{----- (1)}$$

So that  $\phi(a) = \phi(b)$  and  $A$  is a Constant to be determined.

$$\text{Now, } \phi(a) = f(a) - A \cdot g(a)$$

$$\phi(b) = f(b) - A \cdot g(b)$$

$$\therefore f(a) - A \cdot g(a) = f(b) - A \cdot g(b)$$

$$\Rightarrow A = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \text{----- (2)}$$

Now,  $\phi$  is continuous in  $[a,b]$  as r.h.s of (1) is continuous in  $[a,b]$  and  $\phi(x)$  is differentiable in  $(a,b)$  as r.h.s of (1) is differentiable in  $[a,b]$ .

$$\text{Also } \phi(a) = \phi(b)$$

Hence all the conditions of Rolle's Theorem are satisfied then there exists a value  $x = c \in (a,b)$  such that  $\phi'(c) = 0$ .

Now, Differentiating (1) W.r.t  $x$ , we get

$$\phi'(x) = f'(x) - A \cdot g'(x)$$

$$\text{at } x = c \in (a,b)$$

$$\therefore \phi^1(c) = f^1(c) - A g^1(c)$$

$$0 = f^1(c) - A g^1(c) \quad (\because g^1(x) \neq 0)$$

$$\Rightarrow A = \frac{f^1(c)}{g^1(c)} \quad \text{----- (3)}$$

Substituting (3) in (2), we get

$$\frac{f^1(c)}{g^1(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}, \text{ where } a < c < b$$

Hence the proof.

**18. Verify Cauchy's Mean Value Theorem for the function  $f(x) = x^2 + 3$ ,  $g(x) = x^3 + 1$  in  $[1, 3]$**

**Solution:** Here  $f(x) = x^2 + 3$ ,  $g(x) = x^3 + 1$

Both  $f(x)$  and  $g(x)$  are Polynomial in  $x$ . Hence they are continuous and differentiable for all  $x$  and in particular in  $[1, 3]$

$$\text{Now, } f^1(x) = 2x, g^1(x) = 3x^2$$

$$\text{Also } g^1(x) \neq 0 \text{ for all } x \in (1, 3)$$

Hence  $f(x)$  and  $g(x)$  satisfy all the conditions of the Cauchy's mean value theorem. Therefore

$$\frac{f(3) - f(1)}{g(3) - g(1)} = \frac{f^1(c)}{g^1(c)}, \text{ for some } c : 1 < c < 3$$

$$\text{i.e., } \frac{12 - 4}{28 - 2} = \frac{2c}{3c^2}$$

$$\text{i.e., } \frac{2}{13} = \frac{1}{3c} \Rightarrow C = \frac{13}{6} = 2\frac{1}{6}$$

Clearly  $C = 2\frac{1}{6}$  lies between 1 and 3.

Hence Cauchy's theorem holds good for the given function.

---

**19. Verify Cauchy's Mean Value Theorem for the functions  $f(x) = \sin x$ ,  $g(x) = \cos x$  in  $\left[0, \frac{\pi}{2}\right]$**

**Solution:** Here  $f(x) = \sin x$ ,  $g(x) = \cos x$  so that

$$f'(x) = \cos x, g'(x) = -\sin x$$

Clearly both  $f(x)$  and  $g(x)$  are continuous in  $\left[0, \frac{\pi}{2}\right]$ , and differentiable in  $\left(0, \frac{\pi}{2}\right)$

Also  $g'(x) = -\sin x \neq 0$  for all  $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore$  From Cauchy's mean value theorem we obtain

$$\frac{f\left(\frac{\pi}{2}\right) - f(0)}{g\left(\frac{\pi}{2}\right) - g(0)} = \frac{f'(c)}{g'(c)} \text{ for some } C : 0 < C < \frac{\pi}{2}$$

$$\text{i.e., } \frac{1-0}{0-1} = \frac{\cos c}{-\sin c} \quad \text{i.e., } -1 = -\cot c \text{ (or) } \cot c = 1$$

$$\therefore C = \frac{\pi}{4}, \text{ clearly } C = \frac{\pi}{4} \text{ lies between } 0 \text{ and } \frac{\pi}{2}$$

Thus Cauchy's Theorem is verified.

**Exercises:**

**20. Find C by Cauchy's Mean Value Theorem for**

a)  $f(x) = e^x$ ,  $g(x) = e^{-x}$  in  $[0,1]$

b)  $f(x) = x^2$ ,  $g(x) = x$  in  $[2,3]$

**21. Verify Cauchy's Mean Value theorem for**

a)  $f(x) = \tan^{-1} x$ ,  $g(x) = x$  in  $\left[\frac{1}{\sqrt{3}}, 1\right]$

b)  $f(x) = \log x$ ,  $g(x) = \frac{1}{x}$  in  $[1,e]$

---

## Generalized Mean Value Theorem:

### 22. State Taylor's Theorem and hence obtain Maclaurin's expansion (series)

**Statement:** If  $f(x)$  and its first  $(n - 1)$  derivatives are continuous in  $[a, b]$  and its  $n^{\text{th}}$  derivative exists in  $(a, b)$  then

$$f(b) = f(a) + (b - a) f'(a) + \frac{(b - a)^2}{2!} f''(a) + \dots + \frac{(b - a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b - a)^n}{n!} f^{(n)}(c)$$

Where  $a < c < b$

### Remainder in Taylor's Theorem:

We have

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!} f''(a) + \dots + \frac{(x - a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x - a)^n}{n!} f^{(n)}[a + (x - a)\theta]$$

$$f(x) = S_n(x) + R_n(x)$$

Where  $R_n(x) = \frac{(x - a)^n}{n!} f^{(n)}[a + (x - a)\theta]$  is called the Lagrange's form of the Remainder.

$$\text{Where } a = 0, R_n(x) = \frac{x^n}{n!} f^{(n)}(\theta x), 0 < \theta < 1$$

### Taylor's and Maclaurin's Series:

We have  $f(x) = S_n(x) + R_n(x)$

$$\therefore \lim_{n \rightarrow \infty} [f(x) - S_n(x)] = \lim_{n \rightarrow \infty} R_n(x)$$

$$\text{If } \lim_{n \rightarrow \infty} R_n(x) = 0 \text{ then } f(x) = \lim_{n \rightarrow \infty} S_n(x)$$

Thus  $\lim_{n \rightarrow \infty} S_n(x)$  converges and its sum is  $f(x)$ .

This implies that  $f(x)$  can be expressed as an infinite series.

$$\text{i.e., } f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!} f''(a) + \dots \text{ to } \infty$$

This is called Taylor's Series.



Putting  $a = 0$ , in the above series, we get

$$F(x) = f(0) + x f'(0) + \frac{(x)^2}{2!} f''(0) + \dots \text{to } \infty$$

This is called Maclaurin's Series. This can also denoted as

$$Y = y(0) + x y'(0) + \frac{(x)^2}{2!} y''(0) + \dots + \frac{(x)^n}{n!} y^{(n)}(0) \dots \text{to } \infty$$

Where  $y = f(x)$ ,  $y' = f'(x)$ ,  $\dots y^{(n)} = f^{(n)}(x)$

### 23. By using Taylor's Theorem expand the function $e^x$ in ascending powers of $(x - 1)$

**Solution:** The Taylor's Theorem for the function  $f(x)$  is ascending powers of  $(x - a)$  is

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!} f''(a) + \dots \quad (1)$$

Here  $f(x) = e^x$  and  $a = 1$

$$f'(x) = e^x \Rightarrow f'(a) = e$$

$$f''(x) = e^x \Rightarrow f''(a) = e$$

$\therefore$  (1) becomes

$$e^x = e + (x - 1) e + \frac{(x - 1)^2}{2} e + \dots$$

$$= e \left\{ 1 + (x - 1) + \frac{(x - 1)^2}{2} + \dots \right\}$$

### 24. By using Taylor's Theorem expand $\log \sin x$ in ascending powers of $(x - 3)$

**Solution:**  $f(x) = \log \sin x$ ,  $a = 3$  and  $f(3) = \log \sin 3$

$$\text{Now } f'(x) = \frac{\cos x}{\sin x} = \cot x, f'(3) = \cot 3$$

$$f''(x) = -\operatorname{cosec}^2 x, f''(3) = -\operatorname{cosec}^2 3$$

$$f'''(x) = -2\operatorname{cosec} x (-\operatorname{cosec} x \cot x) = 2\operatorname{cosec}^3 x \cot x$$

$$\therefore f'''(3) = 2\operatorname{cosec}^3 3 \cot 3$$

$$\therefore f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!} f''(a) + \frac{(x - a)^3}{3!} f'''(a) + \dots$$

$$\therefore \log \sin x = f(3) + (x - 3) f'(3) + \frac{(x - 3)^2}{2!} f''(3) + \frac{(x - 3)^3}{3!} f'''(3) + \dots$$

$$= \log \sin 3 + (x - 3) \cot 3 + \frac{(x - 3)^2}{2!} (-\operatorname{cosec}^2 3) + \frac{(x - 3)^3}{3!} 2 \operatorname{cosec}^3 3 \cot 3 + \dots$$

*Exercise:*

25. Expand  $\sin x$  in ascending powers of  $\left(x - \frac{\pi}{2}\right)$
26. Express  $\tan^{-1} x$  in powers of  $(x - 1)$  up to the term containing  $(x - 1)^3$
27. Apply Taylor's Theorem to prove

$$e^{x+h} = e^x \left[ 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots \right]$$

**Problems on Maclaurin's Expansion:**

28. Expand the  $\log(1+x)$  as a power series by using Maclaurin's theorem.

**Solution:** Here  $f(x) = \log(1+x)$ , Hence  $f(0) = \log 1 = 0$

We know that

$$f^n(x) = \frac{d^n}{dx^n} \{\log(1+x)\} = \frac{d^{n-1}}{dx^{n-1}} \left\{ \frac{1}{1+x} \right\}$$

$$= \frac{(-1)^{n-1} (n-1)!}{(1+x)^n}, \quad n = 1, 2, \dots$$

Hence  $f^n(0) = (-1)^{n-1} (n-1)!$

$$f^1(0) = 1, f^{11}(0) = -1, f^{111}(0) = 3!, f^{1111}(0) = -3!$$

Substituting these values in

$$f(x) = f(0) + x f^1(0) + \frac{x^2}{2!} f^{11}(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

$$\therefore \log(1+x) = 0 + x \cdot 1 + \frac{x^2}{2!} (-1) + \frac{x^3}{3!} 2! + \frac{x^4}{4!} - 3! + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

This series is called Logarithmic Series.

**29. Expand  $\tan^{-1} x$  by using Maclaurin's Theorem up to the term containing  $x^5$**

**Solution:** let  $y = \tan^{-1} x$ , Hence  $y(0) = 0$

We find that  $y_1 = \frac{1}{1+x^2}$  which gives  $y_1(0) = 1$

Further  $y_1(1+x^2) = 1$ , Differentiating we get

$$y_1 \cdot 2x + (1+x^2) y_2 = 0 \text{ (or) } (1+x^2) y_2 + 2xy_1 = 0$$

$$\text{Hence } y_2(0) = 0$$

Taking  $n^{\text{th}}$  derivative on both sides by using Leibniz's Theorem, we get

$$(1+x^2) y_{n+2} + n \cdot 2xy_{n+1} + \frac{n(n-1)}{1.2} \cdot 2 \cdot y_n + 2xy_{n-1} + n \cdot 2 \cdot y_n = 0$$

$$\text{i.e., } (1+x^2) y_{n+2} + 2(n+1)x y_{n+1} + n(n+1) y_n = 0$$

Substituting  $x = 0$ , we get,  $y_{n+2}(0) = -n(n+1) y_n(0)$

$$\text{For } n = 1, \text{ we get } y_3(0) = -2y_1(0) = -2$$

$$\text{For } n = 2, \text{ we get } y_4(0) = -2 \cdot 3 \cdot y_2(0) = 0$$

$$\text{For } n = 3, \text{ we get } y_5(0) = -3 \cdot 4 \cdot y_3(0) = 24$$

Using the formula

$$Y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots$$

$$\text{We get } \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Exercise:

**30. Using Maclaurin's Theorem prove the following:**

a)  $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \dots$

b)  $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots$

c)  $e^x \cos x = 1 + x - \frac{x^3}{3} + \dots$

d) Expand  $e^{ax} \cos bx$  by Maclaurin's Theorem as far as the term containing  $x^3$

**Exercise :** Verify Rolle's Theorem for

(i)  $f(x) = e^x(\sin x - \cos x)$  in  $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ ,

(ii)  $f(x) = x(x-2)e^{x/2}$  in  $[0, 2]$

(iii)  $f(x) = \frac{\sin 2x}{e^{2x}}$  in  $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ .

**Exercise :** Verify the Lagrange's Mean Value Theorem for

(i)  $f(x) = x(x-1)(x-2)$  in  $\left[0, \frac{1}{2}\right]$

(ii)  $f(x) = \tan^{-1}x$  in  $[0, 1]$

**Exercise :** Verify the Cauchy's Mean Value Theorem for

(i)  $f(x) = \sqrt{x}$  and  $g(x) = \frac{1}{\sqrt{x}}$  in  $\left[\frac{1}{4}, 1\right]$

(ii)  $f(x) = \frac{1}{x^2}$  and  $g(x) = \frac{1}{x}$  in  $[a, b]$

(iii)  $f(x) = \sin x$  and  $g(x) = \cos x$  in  $[a, b]$

## UNIT – II

### DIFFERENTIAL CALCULUS-II

**Give different types of Indeterminate Forms.**

If  $f(x)$  and  $g(x)$  be two functions such that  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  both exists, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

If  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = 0$  then

$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$  Which do not have any definite value, such an expression is called

indeterminate form. The other indeterminate forms are  $\frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, \infty^0$  and  $1^\infty$

#### 1. State & prove L' Hospital's Theorem (rule) for Indeterminate Forms.

L'Hospital rule is applicable when the given expression is of the form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$

**Statement:** Let  $f(x)$  and  $g(x)$  be two functions such that

$$(1) \lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = 0$$

$$(2) f'(a) \text{ and } g'(a) \text{ exist and } g'(a) \neq 0$$

$$\text{Then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f'(x)}{\lim_{x \rightarrow a} g'(x)}$$

Proof: Now  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \left[ \frac{1/g}{1/f} \right]$ , which takes the indeterminate form  $\frac{0}{0}$ . Hence applying the

L' Hospitals theorem, we get

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{-g'(x) / [g(x)]^2}{-f'(x) / [f(x)]^2} = \lim_{x \rightarrow a} \frac{g'(x)}{f'(x)} \left[ \frac{f(x)}{g(x)} \right]^2 \\ &= \left[ \lim_{x \rightarrow a} \frac{g'(x)}{f'(x)} \right] \left[ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \right]^2 \end{aligned}$$

If  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \neq 0$  and  $\neq \infty$  then

$$1 = \left[ \lim_{x \rightarrow a} \frac{g'(x)}{f'(x)} \right] \left[ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \right]$$

i.e.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

If  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 0$  or  $\infty$  the above theorem still holds good.

2. Evaluate  $\lim_{x \rightarrow a} \frac{\sin x}{x} = \frac{0}{0}$  form

**Solution:** Apply L'Hospital rule, we get

$$\lim_{x \rightarrow a} \frac{\cos x}{1} = \frac{\cos \theta}{1} = \frac{1}{1} = 1$$

$$\therefore \lim_{x \rightarrow a} \frac{\sin x}{x} = 1$$

3. Evaluate  $\lim_{x \rightarrow a} \frac{\log \sin x}{\cot x}$

**Solution:**  $\lim_{x \rightarrow a} \frac{\log \sin x}{\cot x} = \frac{\log \sin 0}{\cot 0} = \frac{\log 0}{\infty} = \frac{-\infty}{\infty}$  form

Apply L' Hospital rule

$$= \lim_{x \rightarrow a} \frac{-\cos x}{-2 \cot x \csc x} = \lim_{x \rightarrow a} \frac{-\cos x}{-2 \cot x \csc x}$$

$$= \lim_{x \rightarrow a} \frac{1}{2 \cot x} = 0$$

$$\therefore \lim_{x \rightarrow a} \frac{\log \sin x}{\cot x} = 0$$

### Exercise: 1

#### Evaluate

a)  $\lim_{x \rightarrow 0} \frac{\tan x}{x}$

b)  $\lim_{x \rightarrow 0} (1+x)^{1/x}$

c)  $\lim_{x \rightarrow \infty} \frac{a^x - 1}{x}$

d)  $\lim_{x \rightarrow 0} \frac{x^n - a^n}{x - a}$

#### 4. Explain $\infty - \infty$ and $0 \times \infty$ Forms:

**Solution:** Suppose  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = \infty$  in this case

$$\lim_{x \rightarrow a} f(x) - g(x) = 0 \times \infty, \text{ reduce this to } \frac{0}{0} \text{ or } \frac{\infty}{\infty} \text{ form}$$

$$\text{Let } \lim_{x \rightarrow a} [f(x).g(x)] = \lim_{x \rightarrow a} \left\{ \frac{f(x)}{1/g(x)} \right\} = \frac{0}{0} \text{ form}$$

$$\text{Or } \lim_{x \rightarrow a} [f(x).g(x)] = \lim_{x \rightarrow a} \left\{ \frac{g(x)}{1/f(x)} \right\} = \frac{\infty}{\infty} \text{ form}$$

L' Hospitals rule can be applied in either case to get the limit.

Suppose  $\lim_{x \rightarrow a} f(x) = \infty$  and  $\lim_{x \rightarrow a} g(x) = \infty$  in this case  $\lim_{x \rightarrow a} [f(x).g(x)] = \infty - \infty$  form, reduce

this  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  form and then apply L'Hospitals rule to get the limit

5. Evaluate  $\lim_{x \rightarrow 0} \left\{ \frac{1}{x} - \frac{\log(1+x)}{x^2} \right\}$

**Solution:** Given  $\lim_{x \rightarrow 0} \left\{ \frac{1}{x} - \frac{\log(1+x)}{x^2} \right\} = \infty - \infty$  form

$\therefore$  Required limit =  $\lim_{x \rightarrow 0} \left\{ \frac{x - \log(1+x)}{x^2} \right\} = \frac{0}{0}$  form

Apply L'Hospital rule.

$$= \lim_{x \rightarrow 0} \frac{1 - \frac{1}{1+x}}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{x}{1+x}}{2x} = \lim_{x \rightarrow 0} \frac{1}{2(1+x)} = \frac{1}{2}$$

6. Evaluate  $\lim_{x \rightarrow 0} \left\{ \frac{1}{x} - \cot x \right\}$

**Solution:** Given limit is  $\infty - \infty$  form at  $x = 0$ . Hence we have

$$\text{Required limit} = \lim_{x \rightarrow 0} \left\{ \frac{1}{x} - \frac{\cos x}{\sin x} \right\}$$

$$= \lim_{x \rightarrow 0} \left( \frac{\sin x - x \cos x}{x \sin x} \right) = \left( \frac{0}{0} \right) \text{ form}$$

Apply L' Hospital's rule

$$= \lim_{x \rightarrow 0} \frac{\cos x - \cos x + x \sin x}{x \cos x + \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{x \sin x}{x \cos x + \sin x} = \left( \frac{0}{0} \right) \text{ form}$$

Apply L' Hospitals rule



$$= \lim_{x \rightarrow 0} \frac{x \cos x + \sin x}{\cos x - x \sin x + \cos x}$$

$$= \frac{0}{2-0} = \frac{0}{2} = 0$$

**7. Evaluate**  $\lim_{x \rightarrow 0} \tan x \log x$

**Solution:** Given limit is  $(0 \times -\infty)$  form at  $x = 0$

$$\therefore \text{Required limit} = \lim_{x \rightarrow 0} \frac{\log x}{\cot x} = \left[ \frac{-\infty}{\infty} \right] \text{ form}$$

Apply L' Hospitals rule

$$= \lim_{x \rightarrow 0} \frac{1/x}{-\operatorname{Cosec}^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{-\sin^2 x}{x} \left( \frac{0}{0} \right) \text{ form}$$

Apply L Hospitals rule

$$= \lim_{x \rightarrow 0} \frac{-2 \sin x \cos x}{1} = 0$$

**8. Evaluate**  $\lim_{x \rightarrow 1} \sec\left(\frac{\pi x}{2}\right) \log x$

**Solution:** Given limit is  $(\infty \times 0)$  form at  $x = 1$

$$\therefore \text{Required limit} = \lim_{x \rightarrow 1} \frac{\log x}{\cos \frac{\pi x}{2}} \frac{0}{0} \text{ form}$$

Apply L' Hospitals rule

$$= \lim_{x \rightarrow 1} \frac{1/x}{-\sin \frac{\pi x}{2} \cdot \frac{\pi}{2}} = -\frac{2}{\pi}$$


---

**Exercise: 2**

**Evaluate**

a)  $\lim_{x \rightarrow 1} \left( \frac{x}{x-1} - \frac{1}{\log x} \right)$

b)  $\lim_{x \rightarrow 0} \left( \frac{a}{x} - \cot \left( \frac{x}{a} \right) \right)$

c)  $\lim_{x \rightarrow \frac{\pi}{2}} \left( \sec x - \frac{1}{1 - \sin x} \right)$

d)  $\lim_{x \rightarrow \infty} \left( a^{1/x} - 1 \right) x$

**9. Explain Indeterminate Forms  $0^0$ ,  $1^\infty$ ,  $\infty^0$ ,  $0^\infty$**

**Solution:** At  $x = a$ ,  $[f(x)]^{g(x)}$  takes the indeterminate form

(i)  $0^0$  if  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = 0$

(ii)  $1^\infty$  if  $\lim_{x \rightarrow a} f(x) = 1$  and  $\lim_{x \rightarrow a} g(x) = \infty$

(iii)  $\infty^0$  if  $\lim_{x \rightarrow a} f(x) = \infty$  and  $\lim_{x \rightarrow a} g(x) = 0$  and

(iv)  $0^\infty$  if  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = \infty$

In all these cases the following method is adopted to evaluate  $\lim_{x \rightarrow a} [f(x)]^{g(x)}$

Let  $L = \lim_{x \rightarrow a} [f(x)]^{g(x)}$  so that

$$\log L = \lim_{x \rightarrow a} [g(x) \log f(x)] = 0 \times \infty$$

Reducing this to  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  and applying L' Hospitals rule, we get  $\log L = a$  Or

$$L = e^a$$

**10. Evaluate**  $\lim_{x \rightarrow 0} x^{\sin x}$

**Solution:** let  $L = \lim_{x \rightarrow 0} x^{\sin x} \Rightarrow 0^0$  form at  $x = 0$

Hence  $\log L = \lim_{x \rightarrow 0} \sin x \log x \Rightarrow 0 \times \infty$  form

$$\therefore \log L = \lim_{x \rightarrow 0} \frac{\log x}{1/\sin x} = \lim_{x \rightarrow 0} \frac{\log x}{\cos x} \left( \frac{\infty}{\infty} \right) \text{ form}$$

Apply L' Hospital rule,

$$= \lim_{x \rightarrow 0} \frac{1/x}{-\cos x \cot x} = \lim_{x \rightarrow 0} \frac{-\sin x \cdot \tan x}{x} \left( \frac{0}{0} \right) \text{ form}$$

Apply L' Hospitals rule we get

$$= \lim_{x \rightarrow 0} \frac{\sin x \sec^2 x - \cos x \tan x}{1} = -\infty$$

$$\therefore \log L = -\infty \Rightarrow L = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$$\therefore L = 0$$

**11. Evaluate**  $\lim_{x \rightarrow 1} (x)^{1/(1-x)}$

**Solution:** let  $L = \lim_{x \rightarrow 1} (x)^{1/(1-x)}$  is  $1^\infty$  form

$$\therefore \log L = \lim_{x \rightarrow 1} \left( \frac{1}{1-x} \log x \right) = \left( \frac{0}{0} \right) \text{ form}$$

Apply L' Hospitals rule

$$= \lim_{x \rightarrow 1} \frac{1/x}{-1} = \lim_{x \rightarrow 1} \frac{1}{-x} = -1$$


---

$$\therefore \log L = -1$$

$$\Rightarrow L = e^{-1} = \frac{1}{e}$$

**12. Evaluate**  $\lim_{x \rightarrow 0} \left( \frac{\tan x}{x} \right)^{1/x^2}$

**Solution:** let  $L = \lim_{x \rightarrow 0} \left( \frac{\tan x}{x} \right)^{1/x^2} \equiv 1^\infty$  form

$$\therefore \log L = \lim_{x \rightarrow 0} \left\{ \frac{1}{x^2} \log \left( \frac{\tan x}{x} \right) \right\} \equiv (\infty \times 0) \text{ form}$$

$$\therefore \log L = \lim_{x \rightarrow 0} \left\{ \frac{\log(\tan x/x)}{x^2} \right\} \equiv \left( \frac{0}{0} \right) \text{ form}$$

Apply L' Hospitals rule

$$= \lim_{x \rightarrow 0} \left\{ \frac{\frac{1}{\tan x}}{\frac{x}{x^2}} \right\} \quad \left/ \begin{array}{l} x \sec^2 x - \tan x \\ 2x \end{array} \right.$$

$$\log L = \frac{1}{2} \lim_{x \rightarrow 0} \left[ \frac{x \sec^2 x - \tan x}{x^3} \right] = \left( \frac{0}{0} \right) \text{ form}$$

Apply L' Hospital rule, we get

$$= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sec^2 x + 2x \sec^2 x \tan x - \sec^2 x}{3x^2}$$

$$= \frac{1}{3} \lim_{x \rightarrow 0} (\sec^2 x) \left( \frac{\tan x}{x} \right)$$

$$\log L = \frac{1}{3}$$

$$\therefore L = e^{1/3}$$

**Exercise: 3**

Evaluate the following limits.

a)  $\lim_{x \rightarrow 0} (\sec x)^{\cot x}$

b)  $\lim_{x \rightarrow a} \left( 2 - \frac{x}{a} \right)^{\tan\left(\frac{\pi x}{2a}\right)}$

c)  $\lim_{x \rightarrow 0} \frac{(1+x)^{1/x} - e}{x}$

d)  $\lim_{x \rightarrow 0} (\cos ax)^{1/x^2}$

(i)  $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$

(ii)  $\lim_{x \rightarrow 0} \frac{\log(1+x^3)}{\sin^3 x}$

(iii)  $\lim_{x \rightarrow 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}$

(iv)  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\log \sin x}{\left(x - \frac{\pi}{2}\right)^2}$

(v)  $\lim_{x \rightarrow 0} \frac{\cosh x + \log(1-x) - 1 + x}{x^2}$

(vi)  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x \sin^{-1} x}{x^2}$

(vii)  $\lim_{x \rightarrow 0} \frac{e^{2x} - (1+x)^2}{x \log(1+x)}$

Evaluate the following limits.

(i)  $\lim_{x \rightarrow a} \left( 2 - \frac{x}{a} \right) \cot(x-a)$

(ii)  $\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$

(iii)  $\lim_{x \rightarrow \frac{\pi}{2}} \left[ x \tan x - \frac{\pi}{2} \sec x \right]$

(iv)  $\lim_{x \rightarrow 0} \left( \cot^2 x - \frac{1}{x^2} \right)$

(v)  $\lim_{x \rightarrow 0} \left[ \frac{1}{x^2} - \frac{1}{x \tan x} \right]$

(vi)  $\lim_{x \rightarrow \frac{\pi}{2}} [2x \tan x - \pi \sec x]$

(i)  $\lim_{x \rightarrow 0} (\cos ax)^{1/x^2}$

(ii)  $\lim_{x \rightarrow 0} \left( \frac{1 + \cos x}{2} \right)^{1/x^2}$

---

$$(iii) \lim_{x \rightarrow 1} (1 - x^2)^{\frac{1}{\log(1-x)}}$$

$$(iv) \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^{\frac{1}{x^2}}$$

$$(v) \lim_{x \rightarrow 0} (\sin x)^{\tan x}$$

$$(iv) \lim_{x \rightarrow 0} (1 + \sin x)^{\cot x}$$

$$(vii) \lim_{x \rightarrow 0} (\cos x)^{\cos e^2 x}$$

$$(viii) \lim_{x \rightarrow \frac{\pi}{4}} (\tan x)^{\tan 2x}$$

$$(ix) \lim_{x \rightarrow \infty} \left( \frac{ax+1}{ax-1} \right)^x$$

$$(x) \lim_{x \rightarrow 0} \left( \frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}}$$

Evaluate the following limits.

$$(i) \lim_{x \rightarrow 0} (\cos ax)^{\frac{b}{x^2}}$$

$$(ii) \lim_{x \rightarrow 0} \left( \frac{1 + \cos x}{2} \right)^{\frac{1}{x^2}}$$

$$(iii) \lim_{x \rightarrow 1} (1 - x^2)^{\frac{1}{\log(1-x)}}$$

$$(iv) \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^{\frac{1}{x^2}}$$

$$(v) \lim_{x \rightarrow 0} (\sin x)^{\tan x}$$

$$(iv) \lim_{x \rightarrow 0} (1 + \sin x)^{\cot x}$$

$$(vii) \lim_{x \rightarrow 0} (\cos x)^{\cos e^2 x}$$

$$(viii) \lim_{x \rightarrow \frac{\pi}{4}} (\tan x)^{\tan 2x}$$

$$(ix) \lim_{x \rightarrow \infty} \left( \frac{ax+1}{ax-1} \right)^x$$

$$(x) \lim_{x \rightarrow 0} \left( \frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}}$$

## Polar Curves

If we traverse in a hill section where the road is not straight, we often see caution boards hairpin bend ahead, sharp bend ahead etc. This gives an indication of the difference in the amount of bending of a road at various points which is the curvature at various points. In this chapter we discuss about the curvature, radius of curvature etc.

Consider a point P in the xy-Plane.

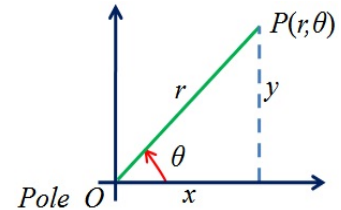
$r$  = length of OP = radial distance

$\theta$  = Polar angle

$(r, \theta) \rightarrow$  Polar co-ordinates

Let  $r = f(\theta)$  be the polar curve

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{-----(1)}$$



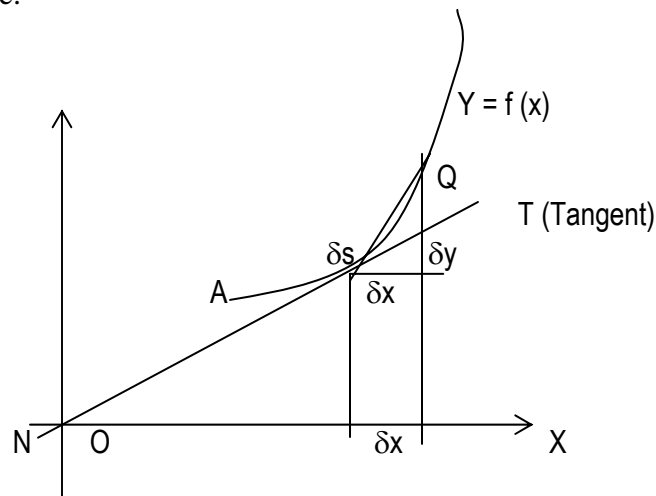
$$x = r \cos \theta \quad y = r \sin \theta$$

Relation (1) enables us to find the polar co-ordinates

$(r, \theta)$  when the Cartesian co-ordinates  $(x, y)$  are known.

### Expression for arc length in Cartesian form.

**Proof:** Let P  $(x, y)$  and Q  $(x + \delta x, Y + \delta y)$  be two neighboring points on the graph of the function  $y = f(x)$ . So that they are at length S and  $S + \delta s$  measured from a fixed point A on the curve.



From figure,  $\widehat{PQ} = \delta S,$

$$\widehat{AP} = S$$

$$\angle TPR = \psi \quad \text{and} \quad PR = \delta x, RQ = \delta y$$

$$\therefore \text{Arc PQ} = \delta S$$

From  $\triangle^{\text{le}}$  PQR, we have

$$[\text{Chord PQ}]^2 = PR^2 + QR^2$$

$$[\text{Chord PQ}]^2 = (\delta x)^2 + (\delta y)^2 \quad (\because \text{from figure})$$

When Q is very close to point P, the length of arc PQ is equal to the length of Chord PQ.

$$\text{i.e arc PQ} = \text{Chord PQ} = \delta s$$

$$\therefore (\delta s)^2 = (\delta x)^2 + (\delta y)^2 \text{ ----- (1)}$$

$\div (\delta x)^2$ , we get

$$\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

When  $Q \rightarrow P$  along the curve,  $\delta x \rightarrow 0$ ,  $\delta s \rightarrow 0$

$$\therefore \lim_{\delta x \rightarrow 0} \left(\frac{\delta s}{\delta x}\right)^2 = 1 + \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x}\right)^2$$

$$\text{i.e., } \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \text{ -----(2)}$$

Similarly, dividing (1) by  $\delta y$  and taking the limit as  $\delta y \rightarrow 0$ , we get

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \text{ -----(3)}$$

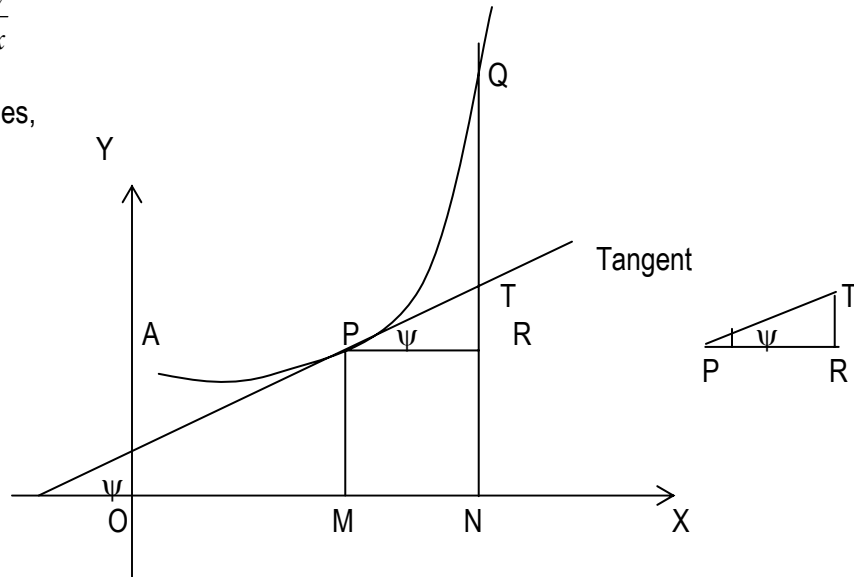


### Expressions for $\frac{ds}{dx}$ & $\frac{ds}{dy}$

Trace a tangent to the curve at the point P, it makes an angle  $\psi$  with the x – axis. From  $\triangle^{\text{le}}$  PRT, we have

$$\tan \psi = \frac{dy}{dx}$$

Equation (2) becomes,



$$\frac{ds}{dx} = \sqrt{1 + \tan^2 \psi}$$

$$= \sqrt{\sec^2 \psi} = \sec \psi$$

$$\therefore \frac{ds}{dx} = \sec \psi \text{ ( Or ) } \frac{dx}{ds} = \cos \psi$$

and equation (3) becomes

$$\frac{ds}{dy} = \sqrt{1 + \cot^2 \psi} = \sqrt{\operatorname{cosec}^2 \psi} = \operatorname{cosec} \psi$$

$$\therefore \frac{ds}{dy} = \operatorname{cosec} \psi \text{ or } \frac{dy}{ds} = \sin \psi$$

**Derive an expression for arc length in parametric form.**

**Solution:** Let the equation of the curve in Parametric form be  $x = f(t)$  and  $y = g(t)$ .

We have,

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

÷ by  $(\delta t)^2$ , we get

$$\left(\frac{\delta s}{\delta t}\right)^2 = \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$$

Taking the limit as  $\delta t \rightarrow 0$  on both sides, we get

$$\lim_{\delta t \rightarrow 0} \left(\frac{\delta s}{\delta t}\right)^2 = \lim_{\delta t \rightarrow 0} \left(\frac{\delta x}{\delta t}\right)^2 + \lim_{\delta t \rightarrow 0} \left(\frac{\delta y}{\delta t}\right)^2$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$(Or) \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \text{ ----- (4)}$$

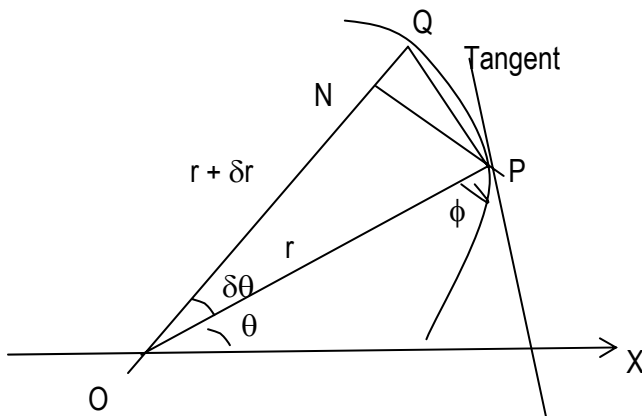
**Derive an expression for arc length in Cartesian form.**

**Solution:** Let P  $(r, \theta)$  and Q  $(r + \delta r, \theta + \delta \theta)$  be two neighboring points on the graph of the function  $r = f(\theta)$ . So that they are at lengths S and  $s + \delta s$  from a fixed point A on the curve.

$$\therefore PQ = (S + \delta s) - s = \delta s$$

Draw  $PN \perp OQ$

From  $\triangle OPN$ ,



$$\frac{PN}{OP} = \sin \delta\theta$$

$$\text{i.e., } \frac{PN}{r} = \sin \delta\theta \text{ (or) } PN = r \sin \delta\theta$$

$$\text{and } \frac{ON}{OP} = \cos \delta\theta$$

$$\text{i.e. } \frac{ON}{r} = \cos \delta\theta \text{ (or) } ON = r \cos \delta\theta$$

When Q is very close to P, the length of arc PQ is equal to  $\delta s$ , where  $\delta s$  is the length of chord PQ.

In  $\triangle PQN$ ,

$$(PQ)^2 = (PN)^2 + (QN)^2$$

$$\text{but } PN = r \sin \delta\theta \text{ (} \because \sin \delta\theta \approx \delta\theta \text{)}$$

$$\text{And } QN = OQ - ON$$

$$= (r + \delta r) - r \cos \delta\theta$$

$$= r + \delta r - r \text{ (} \because \cos \delta\theta \approx 1 \text{)}$$

$$\therefore QN = \delta r$$

$$\text{And } (PQ)^2 = (PN)^2 + (QN)^2$$

$$(\delta s)^2 + (r \delta\theta)^2 + (\delta r)^2 \text{ ----- (5)}$$

$\div$  by  $(\delta\theta)^2$  we get

$$\left( \frac{\delta s}{\delta\theta} \right)^2 = r^2 + \left( \frac{\delta r}{\delta\theta} \right)^2$$

When  $Q \rightarrow P$  along the curve  $\delta\theta \rightarrow 0$  as  $\delta s \rightarrow 0$  and  $\delta r \rightarrow 0$

$$\lim_{\delta\theta \rightarrow 0} \left( \frac{\delta S}{\delta\theta} \right)^2 = r^2 + \lim_{\delta\theta \rightarrow 0} \left( \frac{\delta r}{\delta\theta} \right)^2$$

$$\text{i.e.} \left( \frac{ds}{d\theta} \right)^2 = r^2 + \left( \frac{dr}{d\theta} \right)^2$$

$$\text{i.e., } \frac{ds}{d\theta} = \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \quad \text{----- (6)}$$

Similarly equation (5) ÷ by  $(\delta r)^2$  and taking the limits as  $\delta r \rightarrow 0$  we get

$$\lim_{\delta r \rightarrow 0} \left( \frac{\delta S}{\delta r} \right)^2 = \lim_{\delta r \rightarrow 0} r^2 \left( \frac{\delta\theta}{\delta r} \right)^2 + 1$$

$$\text{i.e.} \left( \frac{ds}{dr} \right)^2 = r^2 \left( \frac{d\theta}{dr} \right)^2 + 1$$

$$\therefore \frac{ds}{dr} = \sqrt{r^2 \left( \frac{d\theta}{dr} \right)^2 + 1} \quad \text{----- (7)}$$

**Note:** Angle between Tangent and Radius Vector:-

We have,

$$\tan \phi = r \frac{d\theta}{dr}$$

$$\text{i.e., } \frac{\sin \phi}{\cos \phi} = r \frac{d\theta}{dr}$$

$$= r \frac{d\theta}{ds} \cdot \frac{ds}{dr}$$

$$\frac{\sin \phi}{\cos \phi} = \frac{r d\theta / ds}{dr / ds}$$

$$\therefore \sin \phi = r \frac{d\theta}{ds} \text{ and } \cos \phi = \frac{dr}{ds}$$

Find  $\frac{ds}{dx}$  and  $\frac{ds}{dy}$  for the following curves:-

1)  $y = C \cosh\left(\frac{x}{c}\right)$

**Solution:**  $y = C \cosh\left(\frac{x}{c}\right)$

Differentiating y w.r.t x. we get

$$\frac{dy}{dx} = \sinh\left(\frac{x}{c}\right)$$

$$\begin{aligned}\therefore \frac{ds}{dx} &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\ &= \sqrt{1 + \sinh^2\left(\frac{x}{c}\right)} = \sqrt{\cosh^2\left(\frac{x}{c}\right)}\end{aligned}$$

$$\frac{ds}{dx} = \cosh\left(\frac{x}{c}\right)$$

Again Differentiating y w.r.t y we get

$$1 = C \sinh\left(\frac{x}{c}\right) \cdot \frac{1}{C} \cdot \frac{dx}{dy}$$

$$\text{i.e. } \frac{dx}{dy} = \operatorname{Cosech}\left(\frac{x}{c}\right)$$

$$\therefore \frac{ds}{dy} = \sqrt{1 + \operatorname{Cosec}^2\left(\frac{x}{c}\right)}$$

2)  $x^3 = ay^2$

Solution :  $x^3 = ay^2$

Differentiating w.r.t y and x separately we get

$$3x^2 \frac{dx}{dy} = 2ay \text{ and } 3x^2 = 2ay \frac{dy}{dx}$$

$$\text{i.e., } \frac{dx}{dy} = \frac{2ay}{3x^2} \text{ and } \frac{dy}{dx} = \frac{3x^2}{2ay}$$

We know that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1 + \left(\frac{3x^2}{2ay}\right)^2}$$

$$= \sqrt{1 + \frac{9x^3x}{4a^2y^2}} = \sqrt{1 + \frac{9ay^2x}{4a^2y^2}}$$

$$\frac{ds}{dx} = \sqrt{1 + \frac{9x}{4a}} \text{ and}$$

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \left(\frac{2ay}{3x^2}\right)^2}$$

$$\frac{ds}{dy} = \left(1 + \frac{4a^2y^2}{9x^4}\right)^{\frac{1}{2}} = \left(1 + \frac{4a}{9x}\right)^{\frac{1}{2}} \quad (\because x^3 = ay^2)$$

### 3. $y = \log \cos x$

**Solution .**  $y = \log \cos x$

Differentiating w.r.t x and y separately, we get

$$\frac{dy}{dx} = \frac{1}{\cos x} (-\sin x) = -\tan x$$

$$\text{i.e., } \frac{dy}{dx} = -\tan x$$

$$\text{and } 1 = \frac{1}{\cos x} (-\sin x) \frac{dx}{dy}$$

$$\text{i.e., } 1 = -\tan x \frac{dx}{dy} \quad (\text{Or}) \quad \frac{dx}{dy} = -\cot x$$

We have

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \text{and} \quad \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \tan^2 x} \quad \& \quad \frac{ds}{dy} = \sqrt{1 + \cot^2 x}$$

$$= \sqrt{\sec^2 x}$$

$$\therefore \frac{ds}{dx} = \sec x \quad \text{and} \quad \frac{ds}{dy} = (1 + \cot^2 x)^{\frac{1}{2}}$$

Find  $\frac{ds}{dt}$  for the following Curves:-

1.  $x = a (\cos t + t \sin t)$ ,  $y = a (\sin t - t \cos t)$

2.  $x = a \sec t$ ,  $y = b \tan t$

3.  $x = a \left( \cos t + \log \tan \frac{t}{2} \right)$ ,  $y = a \sin t$

**Solution of 1**

Given  $x = a (\cos t + t \sin t)$ ,  $y = a (\sin t - t \cos t)$

Differentiating  $x$  &  $y$  W.r.t 't', we get

$$\frac{dx}{dt} = a (-\sin t + \sin t + t \cos t)$$

$$\frac{dx}{dt} = a t \cos t$$

$$\text{and } \frac{dy}{dt} = a (\cos t - \cos t + t \sin t)$$

$$\frac{dy}{dt} = a t \sin t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$


---

$$= \sqrt{a^2 t^2 \cos^2 t + a^2 t^2 \sin^2 t}$$

$$\frac{ds}{dt} = at$$

### Solution of 2

$$x = a \sec t, y = b \tan t$$

$$\therefore \frac{dx}{dt} = a \sec t \tan t, \frac{dy}{dt} = b \sec^2 t$$

We have

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= (a^2 \sec^2 t \tan^2 t + b^2 \sec^4 t)^{\frac{1}{2}}$$

$$= [a^2 \sec^2 t (\sec^2 t - 1) + b^2 \sec^4 t]^{\frac{1}{2}}$$

$$= [a^2 \sec^4 t - a^2 \sec^2 t + b^2 \sec^4 t]^{\frac{1}{2}}$$

$$\frac{ds}{dt} = [(a^2 + b^2) \sec^4 t - a^2 \sec^2 t]^{\frac{1}{2}}$$

### Solution of 3

$$x = a (\cos t + \log \tan \frac{t}{2}), y = a \sin t$$

Differentiating x and y w.r.t 't' we get

$$\frac{dx}{dt} = a \left( -\sin t + \frac{1}{\tan \frac{t}{2}} \sec^2 \frac{t}{2} \cdot \frac{1}{2} \right), \frac{dy}{dt} = a \cos t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \left[ a^2 \left[ -\sin t + \frac{\sec^2 \frac{t}{2}}{2 \tan \frac{t}{2}} \right]^2 + a^2 \cos^2 t \right]^{\frac{1}{2}}, = \left[ a^2 - \frac{a^2 \sin t \sec^2 \frac{t}{2}}{\tan \frac{t}{2}} + \frac{a^2 \sec^4 \frac{t}{2}}{4 \tan^2 \frac{t}{2}} \right]$$

$$= a \cot t$$



Find  $\frac{ds}{d\theta}$  and  $\frac{dr}{d\theta}$  for the following curves

1.  $r = a(1 - \cos \theta)$

2.  $r^2 = a^2 \cos 2\theta$

3.  $r = a e^{\theta \cot \alpha}$

**Solution of 1**

$$r = a(1 - \cos \theta)$$

Differentiating  $r$  w.r.t  $\theta$  we get

$$\frac{dr}{d\theta} = a \sin \theta$$

Hence

$$\begin{aligned} \frac{ds}{d\theta} &= \left\{ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}} \\ &= \{ a^2 (1 - \cos \theta)^2 + a^2 \sin^2 \theta \}^{\frac{1}{2}} \\ &= \{ a^2 (1 - 2 \cos \theta + \cos^2 \theta) + a^2 \sin^2 \theta \}^{\frac{1}{2}} \\ &= \{ a^2 - 2a^2 \cos \theta + a^2 \}^{\frac{1}{2}} \\ &= \{ 2a^2 - 2a^2 \cos \theta \}^{\frac{1}{2}} \\ &= a \{ 2(1 - \cos \theta) \}^{\frac{1}{2}} \\ &= a \{ 2(2 \sin^2 \frac{\theta}{2}) \}^{\frac{1}{2}} \end{aligned}$$

$$\frac{ds}{d\theta} = 2a \sin \frac{\theta}{2}$$

$$\begin{aligned} \text{And } \frac{ds}{dr} &= \left\{ 1 + r^2 \left( \frac{d\theta}{dr} \right)^2 \right\}^{\frac{1}{2}} \\ &= \left\{ 1 + \frac{a^2 (1 - \cos \theta)^2}{a^2 \sin^2 \theta} \right\}^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 &= \left\{ \frac{\sin^2 \theta + (1 - \cos \theta)^2}{\sin^2 \theta} \right\}^{\frac{1}{2}} \\
 &= \frac{\{2(1 - \cos \theta)\}^{\frac{1}{2}}}{\sin \theta} = \frac{2 \sin \left( \frac{\theta}{2} \right)}{\sin \theta} = \frac{2 \sin \theta / 2}{2 \sin \theta / 2 \cos \theta / 2} \\
 \frac{ds}{dr} &= \frac{1}{\cos \theta / 2}
 \end{aligned}$$

**Solution of 2**

$$r^2 = a^2 \cos 2\theta$$

Differentiating W.r.t 'θ' we get

$$2r \frac{dr}{d\theta} = -a^2 \sin 2\theta \cdot 2$$

$$r \frac{dr}{d\theta} = -a^2 \sin 2\theta$$

$$\frac{dr}{d\theta} = \frac{-a^2 \sin 2\theta}{r}$$

Hence

$$\begin{aligned}
 \frac{ds}{d\theta} &= \left\{ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}} \\
 &= \left\{ r^2 + \left( \frac{-a^2 \sin 2\theta}{r} \right)^2 \right\}^{\frac{1}{2}} \\
 &= \left\{ r^2 + \frac{a^4 \sin^2 2\theta}{r^2} \right\}^{\frac{1}{2}} \\
 &= \left\{ a^2 \cos 2\theta + \frac{a^4 \sin^2 \theta}{a^2 \cos 2\theta} \right\}^{\frac{1}{2}} \\
 &= \left\{ \frac{a^4 \cos^2 2\theta + a^4 \sin^2 2\theta}{a^2 \cos 2\theta} \right\}^{\frac{1}{2}}
 \end{aligned}$$

$$= \left\{ \frac{a^4}{a^2 \cos 2\theta} \right\}^{\frac{1}{2}} = \frac{a}{\sqrt{\cos 2\theta}}$$

$$\text{Or } \frac{ds}{d\theta} = \left\{ \frac{a^4}{r^2} \right\}^{\frac{1}{2}} = \frac{a^2}{r} \quad (\because r^2 = a^2 \cos 2\theta)$$

$$\begin{aligned} \text{And } \frac{ds}{dr} &= \left\{ 1 + r^2 \left( \frac{d\theta}{dr} \right)^2 \right\}^{\frac{1}{2}} \\ &= \left\{ 1 + r^2 \left( \frac{r}{-a^2 \sin 2\theta} \right)^2 \right\}^{\frac{1}{2}} \\ &= \left\{ 1 + \frac{r^4}{a^4 \sin^2 2\theta} \right\}^{\frac{1}{2}} \\ &= \left\{ \frac{a^4 \sin^2 2\theta + a^4 \cos^2 2\theta}{a^4 \sin^2 2\theta} \right\}^{\frac{1}{2}} \\ &= \left\{ \frac{a^4}{a^4 \sin^2 2\theta} \right\}^{\frac{1}{2}} = \frac{1}{\sin 2\theta} = \operatorname{Cosec} 2\theta \end{aligned}$$

### Solution 3

$$r = a e^{\theta \cot \alpha}, \text{ here } \alpha \text{ is constant}$$

Differentiating w.r.t 'θ' we get

$$\frac{dr}{d\theta} = a e^{\theta \cot \alpha} \cdot \cot \alpha$$

Hence

$$\begin{aligned} \frac{ds}{d\theta} &= \left\{ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}} \\ &= \{ a^2 e^{2\theta \cot \alpha} + a^2 e^{2\theta \cot \alpha} \cot^2 \alpha \}^{\frac{1}{2}} \\ &= a e^{\theta \cot \alpha} \{ 1 + \cot^2 \alpha \}^{\frac{1}{2}} \\ &= a e^{\theta \cot \alpha} \{ \operatorname{Cosec}^2 \alpha \}^{\frac{1}{2}} \end{aligned}$$

$$\frac{ds}{d\theta} = a e^{\theta \cot \alpha} \cos \alpha$$

$$\begin{aligned} \text{and } \frac{ds}{dr} &= \left\{ 1 + r^2 \left( \frac{d\theta}{dr} \right)^2 \right\}^{\frac{1}{2}} \\ &= \left\{ 1 + a^2 e^{2\theta \cot \alpha} \frac{1}{a^2 e^{2\theta \cot \alpha \cot^2 \alpha}} \right\}^{\frac{1}{2}} \\ &= \{1 + \tan^2 \alpha\}^{\frac{1}{2}} \\ &= \{\sec^2 \alpha\}^{\frac{1}{2}} = \sec \alpha \end{aligned}$$

Exercises:

Find  $\frac{ds}{dr}$  and  $\frac{ds}{d\theta}$  to the following curves.

1.  $r^n = a^n \cos n\theta$

2.  $r(1 + \cos \theta) = a$

3.  $r\theta = a$

**Note:**

We have  $\sin \phi = r \frac{d\theta}{ds}$  and

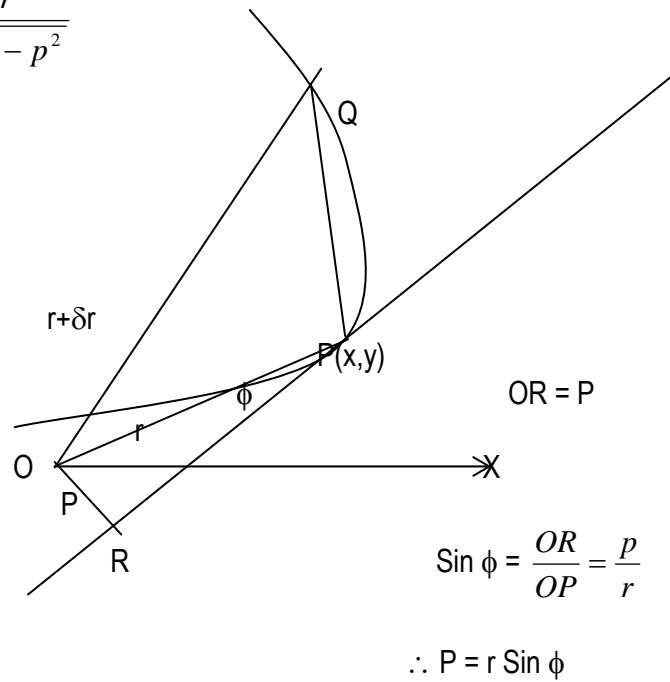
$$\cos \phi = \frac{dr}{ds}$$

$$\therefore \frac{dr}{ds} = \cos \phi = (1 - \sin^2 \phi)^{\frac{1}{2}}$$

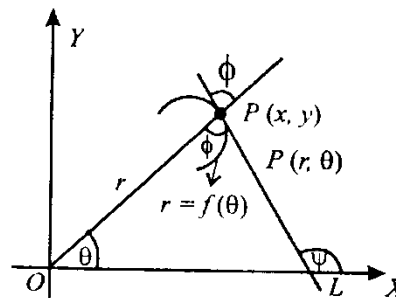
$$= \left[ 1 - \frac{p^2}{r^2} \right]^{\frac{1}{2}} \text{ Since } p = r \sin \phi.$$

$$\frac{dr}{ds} = \frac{\sqrt{r^2 - p^2}}{r}$$

$$\therefore \frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}}$$



Prove that with usual notations  $\tan \phi = r \left( \frac{d\theta}{dr} \right)$



Let  $P(r, \theta)$  be any point on the curve  $r = f(\theta)$

$\therefore \angle XOP = \theta$  and  $OP = r$

Let PL be the tangent to the curve at P subtending an angle  $\psi$  with the positive direction of the initial line (x - axis) and  $\phi$  be the angle between the radius vector OP and the tangent PL.

That is  $\angle OPL = \phi$

From the figure we have

$$\psi = \phi + \theta$$

(Recall from geometry that an exterior angle is equal to the sum of the interior opposite angles)

$$\Rightarrow \tan \psi = \tan (\phi + \theta)$$

$$\text{or } \tan \psi = \frac{\tan \phi + \tan \theta}{1 - \tan \phi \tan \theta} \quad \dots(1)$$

Let  $(x, y)$  be the Cartesian coordinates of P so that we have,

$$x = r \cos \theta, \quad y = r \sin \theta$$

Since  $r$  is a function of  $\theta$ , we can as well regard these as parametric equations in terms of  $\theta$ .

We also know from the geometrical meaning of the derivative that

$$\tan \psi = \frac{dy}{dx} = \text{slope of the tangent } PL$$

$$\text{ie., } \tan \psi = \frac{dy}{d\theta} \bigg/ \frac{dx}{d\theta} \quad \text{since } x \text{ and } y \text{ are function of } \theta$$

$$\text{ie., } \tan \psi = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)} = \frac{r \cos \theta + r' \sin \theta}{-r \sin \theta + r' \cos \theta} \quad \text{where } r' = \frac{dr}{d\theta}$$

Dividing both the numerator and denominator by  $r' \cos \theta$  we have,

$$\tan \psi = \frac{\frac{r \cos \theta}{r' \cos \theta} + \frac{r' \sin \theta}{r' \cos \theta}}{\frac{-r \sin \theta}{r' \cos \theta} + \frac{r' \cos \theta}{r' \cos \theta}}$$

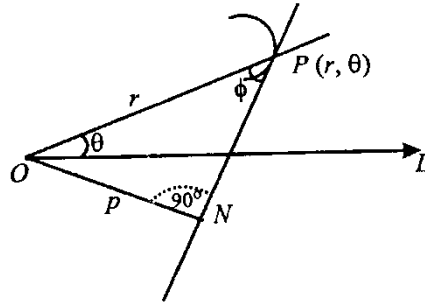
$$\text{Or } \tan \psi = \frac{\frac{r}{r'} + \tan \theta}{1 - \frac{r}{r'} \cdot \tan \theta} \quad \dots(2)$$

Comparing equations (1) and (2) we get

$$\tan \phi = \frac{r}{r'} = \frac{r}{\left(\frac{dr}{d\theta}\right)} \quad \text{or } \tan \phi = r \left(\frac{d\theta}{dr}\right)$$

**Prove with usual notations**  $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left( \frac{dr}{d\theta} \right)^2$  **or**

$$\frac{1}{p^2} = u^2 + \left( \frac{du}{d\theta} \right)^2 \text{ where } u = \frac{1}{r}$$



**Proof :**

Let O be the pole and OL be the initial line. Let P (r,  $\theta$ ) be any point on the curve and hence we have OP = r and  $\angle LOP = \theta$

Draw ON = p (say) perpendicular from the pole on the tangent at P and let  $\phi$  be the angle made by the radius vector with the tangent.

From the figure  $\angle ONP = 90^\circ$   $\angle LOP = \theta$

Now from the right angled triangle ONP

$$\sin \phi = \frac{ON}{OP}$$

$$\text{ie., } \sin \phi = \frac{p}{r} \quad \text{or} \quad p = r \sin \phi$$

$$\text{we have } p = r \sin \phi \quad \dots(1)$$

$$\text{and } \cot \phi = \frac{1}{r} \frac{dr}{d\theta} \quad \dots(2)$$

Squaring equation (1) and taking the reciprocal we get,

$$\frac{1}{p^2} = \frac{1}{r^2} \cdot \frac{1}{\sin^2 \phi} \quad \text{ie.,} \quad \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi$$

$$\text{Or} \quad \frac{1}{p^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

Now using (2) we get,

$$\frac{1}{p^2} = \frac{1}{r^2} \left[ 1 + \frac{1}{r^2} \left( \frac{dr}{d\theta} \right)^2 \right]$$

$$\text{Or} \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left( \frac{dr}{d\theta} \right)^2 \quad \dots(3)$$

Further, let  $\frac{1}{r} = u$

Differentiating w.r.t.  $\theta$  we get,

$$-\frac{1}{r^2} \left( \frac{dr}{d\theta} \right) = \frac{du}{d\theta} \Rightarrow \frac{1}{r^4} \left( \frac{dr}{d\theta} \right)^2 = \left( \frac{du}{d\theta} \right)^2, \text{ by squaring}$$

Thus (3) now becomes

$$\frac{1}{p^2} = u^2 + \left( \frac{du}{d\theta} \right)^2 \quad \dots(4)$$

### 1. Find the angle of intersection of the curves:

$$r = a(1 + \cos \theta) \text{ \& } r = b(1 - \cos \theta)$$

$$\text{Solution : } r = a(1 + \cos \theta) \quad : \quad r = b(1 - \cos \theta)$$

$$\Rightarrow \log r = \log a + \log(1 + \cos \theta) \quad : \quad \log r = \log b + \log(1 - \cos \theta)$$

Differentiating these w.r.t.  $\theta$  we get

$$\begin{aligned} \frac{1}{r} \frac{dr}{d\theta} &= 0 + \frac{-\sin \theta}{1 + \cos \theta} & : & \quad \frac{1}{r} \frac{dr}{d\theta} = 0 + \frac{\sin \theta}{1 - \cos \theta} \\ \cot \phi_1 &= \frac{-2 \sin(\theta/2) \cos(\theta/2)}{2 \cos^2(\theta/2)} & : & \quad \cot \phi_2 = \frac{2 \sin(\theta/2) \cos(\theta/2)}{2 \sin^2(\theta/2)} \end{aligned}$$

$$\text{ie., } \cot \phi_1 = -\tan(\theta/2) = \cot(\pi/2 + \theta/2) \quad : \quad \cot \phi_2 = \cot(\theta/2)$$

$$\Rightarrow \phi_1 = \pi/2 + \theta/2 \quad : \quad \phi_2 = \theta/2$$



$$\therefore \text{angle of intersection} = |\phi_1 - \phi_2| = |\pi/2 + \theta/2 - \theta/2| = \pi/2$$

Hence the curves intersect orthogonally.

**2. S.T. the curves**  $r = a(1 + \sin \theta)$  &  $r = a(1 - \sin \theta)$  **cut each other orthogonally**

**Solution :**  $\log r = \log a + \log (1 + \sin \theta)$  :  $\log r = \log a + \log (1 - \sin \theta)$

Differentiating these w.r.t  $\theta$  we get

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{\cos \theta}{1 + \sin \theta} \quad : \quad \frac{1}{r} \frac{dr}{d\theta} = \frac{-\cos \theta}{1 - \sin \theta}$$

$$\text{ie., } \cot \phi_1 = \frac{\cos \theta}{1 + \sin \theta} \quad : \quad \cot \phi_2 = \frac{-\cos \theta}{1 - \sin \theta}$$

$$\text{We have } \tan \phi_1 = \frac{1 + \sin \theta}{\cos \theta} \text{ and } \tan \phi_2 = \frac{1 - \sin \theta}{-\cos \theta}$$

$$\therefore \tan \phi_1 \cdot \tan \phi_2 = \frac{1 - \sin^2 \theta}{-\cos^2 \theta} = \frac{\cos^2 \theta}{-\cos^2 \theta} = -1$$

Hence the curves intersect orthogonally.

**3. Find the angle of intersection of the curves:**

$$r = \sin \theta + \cos \theta, r = 2 \sin \theta$$

**Solution :**  $\log r = \log (\sin \theta + \cos \theta)$  :  $r = 2 \sin \theta$

$$\Rightarrow \log r = \log (\sin \theta + \cos \theta) : \log r = \log 2 + \log (\sin \theta)$$

Differentiating these w.r.t  $\theta$  we get

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{\cos \theta - \sin \theta}{\sin \theta + \cos \theta} \quad : \quad \frac{1}{r} \frac{dr}{d\theta} = \frac{\cos \theta}{\sin \theta}$$

$$\text{ie., } \cot \phi_1 = \frac{\cos \theta (1 - \tan \theta)}{\cos \theta (1 + \tan \theta)} \quad : \quad \cot \phi_2 = \cot \theta \Rightarrow \phi_2 = \theta$$

$$\text{ie., } \cot \phi_1 = \cot (\pi/4 + \theta) \Rightarrow \phi_1 = \pi/4 + \theta$$

$$\therefore |\phi_1 - \phi_2| = |\pi/4 + \theta - \theta| = \pi/4$$

The angle of intersection is  $\pi/4$

**4. Find the angle of the curves:  $r = a \log \theta$  and  $r = a/\log \theta$**

**Solution :**  $r = a \log \theta$  :  $r = a/\log \theta$

$$\Rightarrow \log r = \log a + \log (\log \theta) \quad : \quad \log r = \log a - \log (\log \theta)$$

Differentiating these w.r.t  $\theta$ , we get,

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{1}{\log \theta \cdot \theta} \quad : \quad \frac{1}{r} \frac{dr}{d\theta} = -\frac{1}{\log \theta \cdot \theta}$$

$$\text{ie.,} \quad \cot \phi_1 = \frac{1}{\theta \log \theta} \quad : \quad \cot \phi_2 = -\frac{1}{\theta \log \theta}$$

Note : We can not find  $\phi_1$  and  $\phi_2$  explicitly.

$$\therefore \tan \phi_1 = \theta \log \theta \quad : \quad \tan \phi_2 = -\theta \log \theta$$

$$\text{Now consider, } \tan \phi_1 = \theta \log \theta \quad : \quad \tan \phi_2 = -\theta \log \theta$$

$$\text{Now consider, } \tan (\phi_1 - \phi_2) = \frac{2 \theta \log \theta}{1 + \tan \phi_1 \tan \phi_2} = \frac{2 \theta \log \theta}{1 - (\theta \log \theta)^2} \quad \dots\dots\dots(1)$$

We have to find  $\theta$  by solving the given pair of equations :

$$r = a \log \theta \quad \text{and} \quad r = a/\log \theta$$

$$\text{Equating the R.H.S we have } a \log \theta = \frac{a}{\log \theta}$$

$$\text{ie., } (\log \theta = 1)^2 = 1 \text{ or } \log \theta \Rightarrow \theta = e$$

Substituting  $\theta = e$  in (1) we get

$$\tan (\phi_1 - \phi_2) = \frac{2e}{1 - e^2} \quad (\because \log e = 1)$$

$$\therefore \text{ angle of intersection } \phi_1 - \phi_2 = \tan^{-1} \left( \frac{2e}{1 - e^2} \right) = 2 \tan^{-1} e$$

**5. Find the angle of intersection of the curves:**

$$r = a (1 - \cos \theta) \quad \text{and} \quad r = 2a \cos \theta$$

**Solution :**  $r = a (1 - \cos \theta)$  :  $r = 2a \cos \theta$

Taking logarithms we have,

$$\log r = \log a + \log (1 - \cos \theta) \quad : \quad \log r = \log 2a + \log (\cos \theta)$$

Differentiating these w.r.t  $\theta$ , we get,

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{\sin \theta}{1 - \cos \theta} \quad : \quad \frac{1}{r} \frac{dr}{d\theta} = \frac{-\sin \theta}{\cos \theta}$$

$$\text{ie., } \cot \phi_1 = \frac{2 \sin(\theta/2) \cos(\theta/2)}{2 \sin^2(\theta/2)} : \cot \phi_2 = -\tan \theta$$

$$\text{ie., } \cot \phi_1 = \cot(\theta/2) \quad : \quad \cot \phi_2 = \cot(\pi/2 + \theta)$$

$$\Rightarrow \phi_1 = \theta/2 \quad : \quad \phi_2 = \pi/2 + \theta$$

$$\therefore |\phi_1 - \phi_2| = |\theta/2 - \pi/2 - \theta| = |\pi/2 + \theta| \quad \dots(1)$$

$$\text{Now consider } r = a(1 - \cos \theta) = 2a \cos \theta$$

$$\text{Or } 3 \cos \theta = 1 \text{ or } \theta = \cos^{-1}(1/3)$$

Substituting this value in (1) we get,

$$\text{The angle of intersection } \pi/2 + 1/2 \cdot \cos^{-1}(1/3)$$

#### 6. Find the angle of intersection of the curves :

$$r = a\theta \text{ and } r = a/\theta$$

$$\text{Solution : } r = a\theta \quad : \quad r = a/\theta$$

$$\Rightarrow \log r = \log a + \log \theta \quad : \quad \log r = \log a - \log \theta$$

Differentiating these w.r.t  $\theta$ , we get,

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{1}{\theta} \quad : \quad \frac{1}{r} \frac{dr}{d\theta} = -\frac{1}{\theta}$$

$$\text{ie., } \cot \phi_1 = \frac{1}{\theta} \quad : \quad \cot \phi_2 = -\frac{1}{\theta}$$

$$\text{or } \tan \phi_1 = \theta \quad : \quad \tan \phi_2 = -\theta$$

Also by equating the R.H.S of the given equations we have

$$a\theta = a/\theta \text{ or } \theta^2 = 1 \Rightarrow \theta = \pm 1$$

$$\text{When } \theta = 1, \tan \phi_1 = 1, \tan \phi_2 = -1 \text{ and}$$

$$\text{When } \theta = -1, \tan \phi_1 = -1, \tan \phi_2 = 1.$$

$$\therefore \tan \phi_1 \cdot \tan \phi_2 = -1 \Rightarrow \phi_1 - \phi_2 = \pi/2$$

The curves intersect at right angles.

7. Find the pedal equation of the curve:  $r(1 - \cos \theta) = 2a$

**Solution :**  $r(1 - \cos \theta) = 2a$

$$\Rightarrow \log r + \log (1 - \cos \theta) = \log 2a$$

Differentiating w.r.t  $\theta$ , we get

$$\frac{1}{r} \frac{dr}{d\theta} + \frac{\sin \theta}{1 - \cos \theta} = 0 \quad \text{or} \quad \frac{1}{r} \frac{dr}{d\theta} = \frac{-\sin \theta}{1 - \cos \theta}$$

$$\therefore \cot \phi = \frac{-2 \sin(\theta/2) \cos(\theta/2)}{2 \sin^2(\theta/2)} = -\cot(\theta/2)$$

$$\text{ie., } \therefore \cot \phi = \cot(-\theta/2) \Rightarrow \phi = -(\theta/2)$$

Consider  $p = r \sin \phi$

$$\therefore p = r \sin(-\theta/2) \quad \text{or} \quad p = -r \sin(\theta/2)$$

$$\text{Now we have, } r(1 - \cos \theta) = 2a \quad \dots(1)$$

$$p = -r \sin(\theta/2) \quad \dots(2)$$

We have to eliminate  $\theta$  from (1) and (2)

$$(1) \text{ can be put in form } r \cdot 2 \sin^2(\theta/2) = 2a$$

$$\text{ie., } r \sin^2(\theta/2) = a$$

But  $p/r = \sin(\theta/2)$ , from (2)

$$\therefore r \left( \frac{p^2}{r^2} \right) = a \quad \text{or} \quad p^2 = ar$$

Thus  $p^2 = ar$  is the required pedal equation.

8. Find the pedal equation of the curve:  $r^2 = a^2 \sec 2\theta$

**Solution :**  $r^2 = a^2 \sec 2\theta$

$$\Rightarrow 2 \log r = 2 \log a + \log(\sec 2\theta)$$

Differentiating w.r.t  $\theta$ , we get,

$$\frac{2}{r} \frac{dr}{d\theta} = \frac{2 \sec 2\theta \tan 2\theta}{\sec 2\theta} \quad \text{ie., } \frac{1}{r} \frac{dr}{d\theta} = \tan 2\theta$$

$$\text{ie., } \cot \phi = \cot(\pi/2 - 2\theta) \Rightarrow \phi = \pi/2 - 2\theta$$

Consider  $p = r \sin \phi \therefore p = r \sin (\pi/2 - 2\theta)$  i.e.,  $p = r \cos 2\theta$

Now we have,  $r^2 = a^2 \sec 2\theta$  ... (1)

$$p = r \cos 2\theta \quad \dots (2)$$

From (2)  $p/r = \cos 2\theta$  or  $r/p = \sec 2\theta$

Substituting in (1) we get,  $r^2 = a^2 (r/p)$  or  $pr = a^2$

Thus  $pr = a^2$  is the required pedal equation.

### 9. Find the pedal equation of the curve: $r^n = a^n \cos n\theta$

**Solution :**  $r^n = a^n \cos n\theta$

$$\Rightarrow n \log r = n \log a + \log (\cos n\theta)$$

Differentiating w.r.t  $\theta$  we get

$$\frac{n}{r} \frac{dr}{d\theta} = \frac{-n \sin n\theta}{\cos n\theta} \text{ i.e., } \frac{1}{r} \frac{dr}{d\theta} = -\tan n\theta$$

$$\therefore \cot \phi = \cot (\pi/2 + n\theta) \Rightarrow \phi = \pi/2 + n\theta$$

Consider  $p = r \sin \phi$

$$\therefore p = r \sin (\pi/2 + n\theta) \text{ i.e., } p = r \cos n\theta$$

Now we have,  $r^n = a^n \cos n\theta$  ... (1)

$$p = r \cos n\theta \quad \dots (2)$$

$\therefore$  (1) as a consequence of (2) is  $r^n = a^n (p/r)$

Thus  $r^{n+1} = pa^n$  is the required pedal equation.

### 10. Find the pedal equation of the curve: $r^m = a^m (\cos m\theta + \sin m\theta)$

**Solution :**  $r^m = a^m (\cos m\theta + \sin m\theta)$

Differentiating w.r.t  $\theta$ , we get,

$$\frac{m}{r} \frac{dr}{d\theta} = \frac{-m \sin m\theta + m \cos m\theta}{\cos m\theta + \sin m\theta}$$

$$\text{i.e., } \frac{1}{r} \frac{dr}{d\theta} = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta} = \frac{\cos m\theta (1 - \tan m\theta)}{\cos m\theta (1 + \tan m\theta)}$$

$$\therefore \cot \phi = \cot (\pi/4 + m\theta) \Rightarrow \phi = \pi/4 + m\theta$$

Consider  $p = r \sin \phi$

$$\therefore p = r \sin (\pi/4 + m\theta)$$

$$\text{ie., } p = r [\sin (\pi/4) \cos m\theta + \cos (\pi/4) \sin m\theta]$$

$$\text{ie., } p = \frac{r}{\sqrt{2}} (\cos m\theta + \sin m\theta)$$

(we have used the formula of  $\sin (A + B)$  and also the values  $\sin (\pi/4) = \cos (\pi/4) = 1/\sqrt{2}$ )

$$\text{Now we have, } r^m = a^m (\cos m\theta + \sin m\theta) \quad \dots(1)$$

$$p = \frac{r}{\sqrt{2}} (\cos m\theta + \sin m\theta) \quad \dots(2)$$

Using (2) in (1) we get,

$$r^m = a^m \cdot \frac{p\sqrt{2}}{r} \text{ or } r^{m+1} = \sqrt{2} a^m p$$

Thus  $r^{m+1} = \sqrt{2} a^m p$  is the required pedal equation.

### 11. Establish the pedal equation of the curve:

$$r^n = a^n \sin n\theta + b^n \cos n\theta \text{ in the form } p^2 (a^{2n} + b^{2n}) = r^{2n+2}$$

**Solution :** We have  $r^n = a^n \sin n\theta + b^n \cos n\theta$

$$\Rightarrow n \log r = \log (a^n \sin n\theta + b^n \cos n\theta)$$

Differentiating w.r.t  $\theta$  we get

$$\frac{n}{r} \frac{dr}{d\theta} = \frac{na^n \cos n\theta - nb^n \sin n\theta}{a^n \sin n\theta + b^n \cos n\theta}$$

$$\text{Dividing by } n, \cot \phi = \frac{a^n \cos n\theta - b^n \sin n\theta}{a^n \sin n\theta + b^n \cos n\theta}$$

Consider  $p = r \sin \phi$

Since  $\phi$  cannot be found, squaring and taking the reciprocal we get,

$$\frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi \text{ or } \frac{1}{p^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} \left\{ 1 + \frac{(a^n \cos n\theta - b^n \sin n\theta)^2}{(a^n \sin n\theta + b^n \cos n\theta)^2} \right\}$$

$$\text{ie., } \frac{1}{p^2} = \frac{1}{r^2} \left\{ \frac{(a^n \sin n\theta + b^n \cos n\theta)^2 + (a^n \cos n\theta - b^n \sin n\theta)^2}{(a^n \sin n\theta + b^n \cos n\theta)^2} \right\}$$

$$\text{ie., } \frac{1}{p^2} = \frac{1}{r^2} \left\{ \frac{a^{2n}(\sin^2 n\theta + \cos^2 n\theta) + b^{2n}(\cos^2 n\theta + \sin^2 n\theta)}{(a^n \sin n\theta + b^n \cos n\theta)^2} \right\}$$

(product terms cancels out in the numerator)

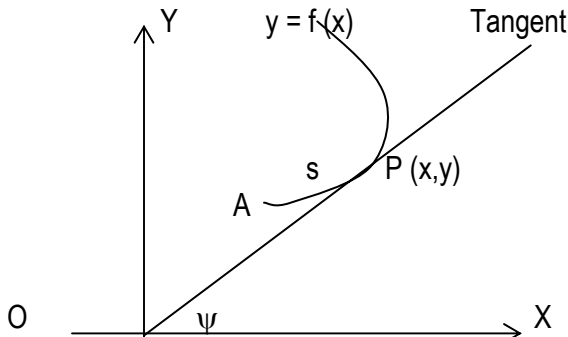
$$\text{ie., } \frac{1}{p^2} = \frac{1}{r^2} \cdot \frac{a^{2n} + b^{2n}}{(a^n \sin n\theta + b^n \cos n\theta)^2}$$

$$\text{or } \frac{1}{p^2} = \frac{1}{r^2} \cdot \frac{a^{2n} + b^{2n}}{(r^n)^2}, \text{ by using the given equation.}$$

$$\therefore p^2(a^{2n} + b^{2n}) = r^{2n+2} \text{ is the required pedal equation.}$$

## 12. Define Curvature and Radius of curvature

**Solution:** A Curve Cuts at every point on it. Which is determined by the tangent drawn.



Let P be a point on the curve  $y = f(x)$  at the length 's' from a fixed point A on it. Let the tangent at 'P' makes an angle  $\psi$  with positive direction of x – axis. As the point 'P' moves along curve, both s and  $\psi$  vary.

The rate of change  $\psi$  w.r.t s, i.e.,  $\frac{d\psi}{ds}$  as called the Curvature of the curve at 'P'.

The reciprocal of the Curvature at P is called the radius of curvature at P and is denoted by  $\rho$ .

$$\therefore \rho = \frac{1}{\frac{d\psi}{ds}} = \frac{ds}{d\psi}$$

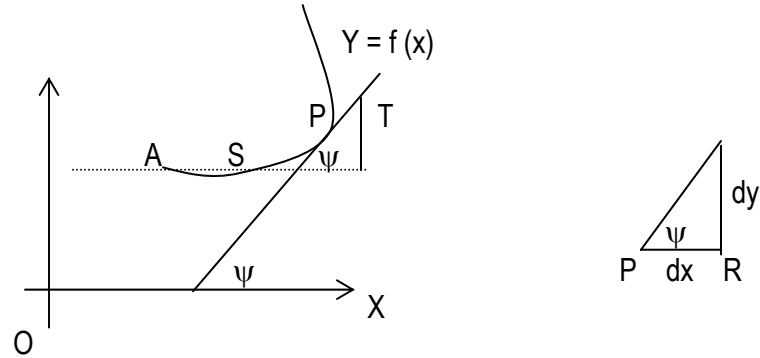
$$\therefore \rho = \frac{ds}{d\psi} \quad (\text{or}) \quad \frac{1}{\rho} = \frac{d\psi}{ds}$$

$$\text{Also denoted } \rho = \frac{1}{K} \quad \therefore K = \frac{d\psi}{ds}$$

K is read it as Kappa.

### 13. Derive an expression for radius of curvature in Cartesian form.

**Solution :** (1) Cartesian Form:



Let  $y = f(x)$  be the curve in Cartesian form.

We know that,  $\tan \psi = \frac{dy}{dx}$  (From Figure) ----- (1)

Where  $\psi$  is the angle made by the tangent at P with x – axis. Differentiating (1) W.r.t x, we get

$$\begin{aligned} \sec^2 \psi \cdot \frac{d\psi}{dx} &= \frac{d^2 y}{dx^2} \\ \text{i.e., } \frac{d^2 y}{dx^2} &= \sec^2 \psi \cdot \frac{d\psi}{dx} \\ &= (1 + \tan^2 \psi) \cdot \frac{d\psi}{dx} \\ &= (1 + \tan^2 \psi) \cdot \frac{d\psi}{dx} \cdot \frac{ds}{dx} \\ &= (1 + \tan^2 \psi) \cdot \frac{1}{\rho} \cdot \left\{ 1 + \left( \frac{dy}{dx} \right)^2 \right\}^{\frac{1}{2}} \\ &= \left\{ 1 + \left( \frac{dy}{dx} \right)^2 \right\} \cdot \frac{1}{\rho} \cdot \left\{ 1 + \left( \frac{dy}{dx} \right)^2 \right\}^{\frac{1}{2}} \quad \text{from eqn (1)} \end{aligned}$$



$$= \frac{1}{\rho} \left\{ 1 + \left( \frac{dy}{dx} \right)^2 \right\}^{\frac{3}{2}}$$

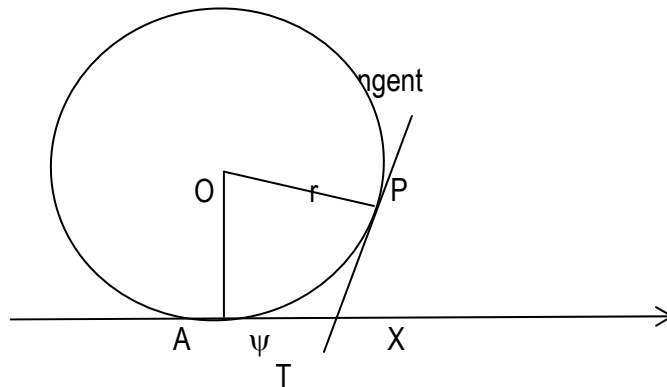
$$\therefore \rho = \frac{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}}{d^2 y / dx^2}$$

$$\therefore \rho = \frac{(1 + y_1^2)^{3/2}}{y_2} \dots \dots \dots (1)$$

Where  $y_1 = \frac{dy}{dx}$ ,  $y_2 = \frac{d^2 y}{dx^2}$

This is the formula for Radius of Curvature in Cartesian Form.

**14. Show that the Curvature of a Circle at any point on it, is a Constant**



**Solution:** Consider a Circle of radius  $r$ . Let  $A$  be a fixed point and 'P' be a given point on the circle such that arc  $AP = S$ .

Let the angle between the tangent to the Circle at  $A$  and  $P$  be  $\psi$ . Then clearly  $\angle AOP = \psi$ .

$$\therefore AP = r\psi$$

$$\text{i.e., } S = r\psi$$

This is the intrinsic equation of the circle.

Differentiating w.r.t  $S$  we get

$$1 = r \frac{d\psi}{ds} \text{ Or } K = \frac{d\psi}{ds} = \frac{1}{r}$$

$$\therefore K = \frac{1}{r} \text{ which is Constant}$$

Hence the Curvature of the Circle at any point on it is constant.

**15. Derive an expression for radius of curvature in parametric form.**

**Solution:** We have  $\rho = (1 + y_1^2)^{3/2}$

Let  $x = f(t)$ ,  $y = g(t)$  be the curve in Parametric Form.

$$\text{Then } y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{Y}}{\dot{X}}$$

$$Y_2 = \frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{\dot{Y}}{\dot{X}} \right) = \frac{d}{dt} \left( \frac{\dot{Y}}{\dot{X}} \right) \frac{dt}{dx}$$

$$= \frac{d}{dt} \left( \frac{\dot{Y}}{\dot{X}} \right) \cdot \left( \frac{1}{\frac{dx}{dt}} \right) = \frac{d}{dt} \left( \frac{\dot{Y}}{\dot{X}} \right) \cdot \frac{1}{\dot{X}}$$

$$= \frac{\ddot{X}\dot{Y} - \dot{X}\ddot{Y}}{(\dot{X})^2} \cdot \frac{1}{\dot{X}}$$

$$\therefore Y_2 = \frac{\ddot{X}\dot{Y} - \dot{X}\ddot{Y}}{(\dot{X})^3}$$

Substituting  $Y_1$  and  $Y_2$  in equation 1.

$$\rho = \frac{(1 + Y_1^2)^{3/2}}{Y_2}, \text{ we get}$$

$$\rho = \frac{\left[ \left( 1 + \left( \frac{\dot{Y}}{\dot{X}} \right)^2 \right)^{3/2} \right]}{\frac{\ddot{X}\dot{Y} - \dot{X}\ddot{Y}}{(\dot{X})^3}}$$

$$\rho = \frac{[(\dot{X})^2 + (\dot{Y})^2]^{3/2}}{\ddot{X}\dot{Y} - \dot{X}\ddot{Y}} \text{ ----- (2)}$$

Equation (2) is called the Radius of Curvature in Parametric Form.

**16. Derive an expression for radius of curvature in polar form.**

**Solution:** Let  $r = f(\theta)$  be the curve in the Polar Form. We know that, Angle between the tangent and radius vector,

$$\tan \phi = r \frac{d\theta}{dr}$$

$$= r \cdot \frac{1}{dr/d\theta}$$

$$\text{i.e., } \tan \phi = \frac{r}{\left(\frac{dr}{d\theta}\right)}$$

Differentiate w.r.t ' $\theta$ ' we get

$$\sec^2 \phi \cdot \frac{d\phi}{d\theta} = \frac{\frac{dr}{d\theta} \cdot \frac{dr}{d\theta} - r \frac{d^2 r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$\therefore \frac{d\phi}{d\theta} = \frac{1}{\sec^2 \phi} \left\{ \frac{\left(\frac{dr}{d\theta}\right)^2 r \frac{d^2 r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2} \right\}$$

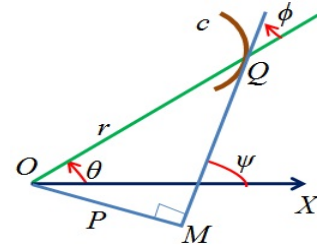
$$= \frac{1}{1 + \tan^2 \phi} \left\{ \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2} \right\}$$

$$= \frac{1}{1 + \frac{r^2}{\left(\frac{dr}{d\theta}\right)^2}} \left\{ \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2} \right\}$$

$$\frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

From figure  $\psi = \theta + \phi$

Differentiating w.r.t  $\theta$ , we get



$$\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

$$\therefore \frac{d\psi}{d\theta} = 1 + \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$\frac{d\psi}{d\theta} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

Also we know that  $\frac{ds}{d\theta} = \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}}$

$$\begin{aligned} \text{Now, } \rho &= \frac{ds}{d\psi} = \frac{ds}{d\theta} \cdot \frac{d\theta}{d\psi} \\ &= \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} \cdot \frac{\left(\frac{dr}{d\theta}\right)^2 + r^2}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}} \end{aligned}$$

$$\rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} \text{----- (3)}$$

Where  $r_1 = \frac{dr}{d\theta}$ ,  $r_2 = \frac{d^2r}{d\theta^2}$

Equation (3) as called the radius of curvature in Polar form.

### 17. Derive an expression for radius of curvature in pedal form.

**Solution:** Let  $p = r \sin \phi$  be the curve in Polar Form.

We have  $p = r \sin \phi$

Differentiating  $p$  W.r.t  $r$ , we get

$$\frac{dp}{dr} = \sin \phi + r \cos \phi \frac{d\phi}{dr}$$

$$\text{But } \sin \phi = r \frac{d\theta}{ds}, \cos \phi = \frac{dr}{ds}$$

$$\therefore \frac{dp}{dr} = r \frac{d\theta}{ds} + r \frac{dr}{ds} \frac{d\phi}{dr}$$

$$= r \frac{d\theta}{ds} + \frac{d\phi}{dr} \cdot \frac{dr}{ds}$$

$$= r \frac{d\theta}{ds} + r \frac{d\phi}{ds}$$

$$= r \frac{d}{ds} (\theta + \phi)$$

$$= r \frac{d\psi}{ds}$$

$$\therefore \frac{dp}{dr} = r \frac{d\psi}{ds}$$

$$\frac{dp}{dr} = r \cdot \frac{1}{\rho} \quad \therefore \frac{d\psi}{ds} = \frac{1}{\rho}$$

$$\therefore \rho = r \frac{dr}{dp} \quad \text{----- (4)}$$

Equation (4) is called Radius of Curvature of the Curve in Polar Form.

**18. Find the radius of curvature at (x, y) for the curve  $ay^2 = x^3$ .**

**Solution:** Given  $ay^2 = x^3$  ----- (1) is in Cartesian form.

We have, Radius of curvature in Cartesian form.

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} \quad \text{----- (2)}$$

Differentiating (1) w.r.t x we get,

$$a 2y \frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = y_1 = \frac{3x^2}{2ay} = \frac{3x^2}{2a \left( \frac{x^3}{a} \right)^{\frac{1}{2}}} = \frac{3\sqrt{x}}{2\sqrt{a}}$$

$$\therefore y_1 = \frac{3\sqrt{x}}{2\sqrt{a}}$$

Differentiating  $y_1$  w.r.t 'x' we get

$$y_2 = \frac{d^2y}{dx^2} = \frac{3}{2\sqrt{a}} \frac{1}{2\sqrt{x}} = \frac{3}{4\sqrt{a}\sqrt{x}}$$

Substitute  $Y_1$  and  $Y_2$  in (2), we get.

$$\rho = \frac{\sqrt{x}(4a+9x)^{\frac{3}{2}}}{6a}$$

**19. Find the radius of curvature at (x,y) for the curve  $y = c \log \sec \left( \frac{x}{c} \right)$**

**Solution:** Given  $y = c \log \sec \left( \frac{x}{c} \right)$  ----- (1)

Differentiating (1) w.r.t x, we get

$$y_1 = c \cdot \frac{1}{\sec \left( \frac{x}{c} \right)} \cdot \sec \left( \frac{x}{c} \right) \tan \left( \frac{x}{c} \right) = c \tan \left( \frac{x}{c} \right) \cdot \frac{1}{c} = \tan \left( \frac{x}{c} \right)$$

Differentiating  $y_1$  W.r.t x we get

$$y_2 = \sec^2 \left( \frac{x}{c} \right) \frac{1}{c} = \frac{1}{c} \sec^2 \left( \frac{x}{c} \right)$$

Substitute  $y_1$  and  $y_2$  in  $\delta = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$

$$\therefore \delta = \frac{\left[ 1 + \tan^2 \left( \frac{x}{c} \right) \right]^{\frac{3}{2}}}{\frac{1}{c} \sec^2 \left( \frac{x}{c} \right)} = \frac{\left[ \sec^2 \left( \frac{x}{c} \right) \right]^{\frac{3}{2}}}{\frac{1}{c} \sec^2 \left( \frac{x}{c} \right)}$$

$$\rho = c \sec \left( \frac{x}{c} \right)$$

20. Find the radius of curvature at the point 't' on the curve  $x = a(t + \sin t)$ ,  $y = a(1 - \cos t)$ .

**Solution:** Given Curves are in Parametric Form

$$\therefore \text{Radius of Curvature, } \rho = \frac{\left((\dot{x})^2 + (\dot{y})^2\right)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \quad \text{----- (1)}$$

Differentiating the given Curves W.r.t t, we get

$$\dot{x} = \frac{dx}{dt} = a(1 + \cos t) \quad \dot{y} = \frac{dy}{dt} = a \sin t$$

Differentiating W.r.t 't' we get

$$\ddot{x} = -a \sin t, \quad \ddot{y} = a \cos t$$

Substitute  $\dot{x}$ ,  $\ddot{y}$ ,  $\dot{y}$  and  $\ddot{x}$  in (1), we get

$$\begin{aligned} \rho &= \frac{\left(a^2(1 + \cos t)^2 + a^2 \sin^2 t\right)^{3/2}}{a(1 + \cos t)a \cos t - a \sin t(-a \sin t)} \\ &= \frac{a^3 \{1 + 2\cos t + \cos^2 t + \sin^2 t\}^{3/2}}{a^2 \{\cos t + \cos^2 t + \sin^2 t\}} \\ &= \frac{a \{2(1 + \cos t)\}^{3/2}}{(1 + \cos t)} = \frac{a \{2 \cdot 2 \cos^2 t/2\}^{3/2}}{2 \cos^2 t/2} = \frac{8a \cos^3 t/2}{2 \cos^2 t/2} \\ \rho &= 4a \cos t/2 \end{aligned}$$

21. Find the Radius of Curvature to  $x = a\{\cos t + \log \tan(t/2)\}$ ,  $y = a \sin t$  at t.

**Solution:** Here  $x = a\{\cos t + \log \tan(t/2)\}$ ,  $y = a \sin t$

$$\begin{aligned} \frac{dx}{dt} &= a \left\{ -\sin t + \frac{1}{\tan t/2} \cdot \sec^2 t/2 \cdot \frac{1}{2} \right\} \\ &= a \left\{ -\sin t + \frac{1}{2 \sin t/2 \cos t/2} \right\} \\ &= a \left\{ -\sin t + \frac{1}{2 \sin t/2} \right\} \\ &= a \{ (1 - \sin^2 t) / \sin t \} \end{aligned}$$

$$\frac{dx}{dt} = a \cos^2 t / \sin t$$

$$\text{and } \frac{dy}{dx} = a \cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \cos t}{a \cos t \cos t / \sin t} = \tan t$$

$$\therefore \frac{dy}{dx} = \tan t$$

Differentiating W.r.t 'x' we get

$$\frac{d^2 y}{dx^2} = \sec^2 t \frac{dt}{dx} = \sec^2 t \frac{1}{\frac{dx}{dt}} = \sec^2 t \cdot \frac{1}{a \cos^2 t / \sin t}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{a} = \sec^4 t \sin t$$

Substitute  $\frac{dy}{dx}$  &  $\frac{d^2 y}{dx^2}$  in  $\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$ , we get

$$\text{i.e., } \rho = \frac{(1 + \tan^2 t)^{3/2}}{\frac{1}{a} \sec^4 t \sin t} = \frac{a \sec^3 t}{\sec^4 t \sin t} = a \frac{\cos t}{\sin t} = a \cot t$$

$$\therefore \rho = a \cot t.$$

**22. Find the Radius of Curvature to  $\sqrt{x} + \sqrt{y} = \sqrt{a}$  at  $\left(\frac{a}{4}, \frac{a}{4}\right)$**

**Solution:** Given  $\sqrt{x} + \sqrt{y} = \sqrt{a}$  ----- (1)

Differentiating (1) w.r.t to 'x' we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\text{i.e., } y_1 = \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = -\frac{(\sqrt{a} - \sqrt{x})}{\sqrt{x}} \text{ (From (1))}$$

$$y_1 = 1 - \frac{\sqrt{a}}{\sqrt{x}} \text{ ----- (2)}$$



Also  $y_2 = \frac{d^2 y}{dx^2} = -\sqrt{a} \left( -\frac{1}{2} \right) x^{-3/2} = \frac{\sqrt{a}}{2x^{3/2}} \quad \text{----- (3)}$

At the given point  $\left( \frac{a}{4}, \frac{a}{4} \right)$

Then  $y_1 = 1 - \frac{\sqrt{a}}{\sqrt{a/2}} = -1$  &  $y_2 = 1 - \frac{\sqrt{a}}{2(a/4)^{3/2}} = \frac{4}{a}$

$\therefore$  Substitute  $y_1$  and  $y_2$  in  $\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$

$$\rho = \frac{(1 + (-1)^2)^{3/2}}{4/a} = \frac{2^{3/2} a}{4} = \frac{2\sqrt{2}a}{4} = \frac{a}{\sqrt{2}}$$

$$\rho = \frac{a}{\sqrt{2}}$$

**23. Show that for the Cardioids  $r = a(1 + \cos\theta)$ ,  $\rho^2 / r^2$  is a constant**

**Solution:**  $r = a(1 + \cos\theta)$

$$\frac{dr}{d\theta} = -a \sin\theta$$

We have,

$$\frac{1}{P^2} = \frac{1}{r^2} + \frac{1}{r^4} \left( \frac{dr}{d\theta} \right)^2, \text{ is Pedal Equation.}$$

$$= \frac{1}{r^2} + \frac{1}{r^4} a^2 \sin^2\theta$$

$$= \frac{r^2 + a^2 \sin^2\theta}{r^4} = \frac{a^2(1 + \cos\theta)^2 + a^2 \sin^2\theta}{r^4}$$

$$= \frac{2a^2(1 + \cos\theta)}{r^4}$$

$$\therefore \frac{1}{P^2} = \frac{2a^2}{r^4} \left( \frac{r}{a} \right) \quad (\because r = a(1 + \cos\theta))$$

$$\therefore \frac{r}{a} = 1 + \cos\theta$$

$$\frac{1}{P^2} = \frac{2a}{r^3}$$

$$\therefore P^2 = \frac{r^3}{2a}$$

Differentiating w.r.t 'P' we get

$$2P = \frac{1}{2a} 3r^2 \frac{dr}{dp}$$

$$\frac{dr}{dp} = \frac{4ap}{3r^2}$$

Now,  $\rho = r \frac{dr}{dp} = \frac{4ap}{3r}$

And  $\frac{\rho^2}{r^2} = \frac{1}{r} \left[ \frac{16a^2 p^2}{9r^2} \right] = \frac{16a^2}{9r^3} \cdot \frac{r^3}{2a} = \frac{8a}{9}$

**24. Find the Radius of Curvature of the Curve**  $\frac{1}{P^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}$

**Solution:** Given  $\frac{1}{P^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}$

Differentiating w.r.t to P, we get

$$\frac{-2}{P^3} = \frac{-1}{a^2 b^2} 2r \frac{dr}{dp}$$

$$\therefore \frac{dr}{dp} = \frac{a^2 b^2}{p^3 r}$$

$$\therefore \rho = r \cdot \frac{dr}{dp} = r \cdot \frac{a^2 b^2}{p^3 r} = \frac{a^2 b^2}{p^3}$$

25. Find the Radius of Curvature at  $(r, \theta)$  on  $r = \frac{a}{\theta}$

**Solution:** Given  $r = \frac{a}{\theta}$

Differentiating w.r.t to ' $\theta$ ' we get

$$\frac{dr}{d\theta} = \frac{-a}{\theta^2} = \frac{-a}{\theta} \cdot \frac{1}{\theta} = \frac{-r}{\theta}$$

$$\frac{dr}{d\theta} = \frac{-r}{\theta}$$

$$\begin{aligned} \text{We have } \frac{1}{P^2} &= \frac{1}{r^2} + \frac{1}{r^4} \left( \frac{dr}{d\theta} \right)^2 \\ &= \frac{1}{r^2} + \frac{1}{r^4} \frac{r^2}{\theta^2} = \frac{1}{r^2} + \frac{1}{r^2 \theta^2} \end{aligned}$$

$$\frac{1}{P^2} = \frac{1}{r^2} \left( 1 + \frac{1}{\theta^2} \right) = \frac{1}{r^2} \left( \frac{\theta^2 + 1}{\theta^2} \right)$$

$$\therefore P = \frac{r\theta}{\sqrt{\theta^2 + 1}}$$

$$P = \frac{r\theta}{\sqrt{\theta^2 + 1}} = \frac{r \cdot \frac{a}{r}}{\sqrt{\frac{a^2}{r^2} + 1}} = \frac{a \cdot r}{\sqrt{a^2 + r^2}}$$

Differentiating above result w.r.t to 'P' we get

$$1 = \frac{\sqrt{a^2 + r^2} \cdot a \frac{dr}{dp} - r \cdot a \frac{1}{2\sqrt{a^2 + r^2}} 2r \frac{dr}{dp}}{(a^2 + r^2)}$$

$$a^2 + r^2 = \left( \sqrt{a^2 + r^2} \cdot a - \frac{r^2 a}{\sqrt{a^2 + r^2}} \right) \frac{dr}{dp}$$

$$a^2 + r^2 = \frac{(a^2 + r^2) \cdot a - r^2 a}{\sqrt{a^2 + r^2}} \frac{dr}{dp}$$

$$(a^2 + r^2)\sqrt{a^2 + r^2} = a^3 \frac{dr}{dp}$$

$$\frac{(a^2 + r^2)^{3/2}}{a^3} = \frac{dr}{dp}$$

$$\text{Thus, } \rho = r \cdot \frac{dr}{dp} = \frac{r \cdot (a^2 + r^2)^{3/2}}{a^3}$$

$$\therefore \rho = \frac{r}{a^3} (a^2 + r^2)^{3/2}$$

### Exercises:

- (1) Find the Radius of the Curvature at the point  $(s, \psi)$  on  $S = a \log \tan \left( \frac{\pi}{4} + \frac{\psi}{2} \right)$
- (2) Find the Radius of the Curvature of  $xy = C^2$  at  $(x, y)$
- (3) Find the Radius of the Curvature of  $xy^3 = a^4$  at  $(a, a)$
- (4) Find the Radius of Curvature at the point  $\theta$  on  $x = C \sin 2\theta (1 + \cos 2\theta)$ ,  
 $y = C \cos 2\theta (1 - \cos 2\theta)$
- (5) If  $\rho_1$  and  $\rho_2$  are the radii of curvature at the extremities of any chord of the cardioid  
 $r = a (1 + \cos \theta)$  which passes through the Pole prove that  $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$

## DIFFERENTIAL CALCULUS – III

### PARTIAL DIFFERENTIATION

#### Introduction :-

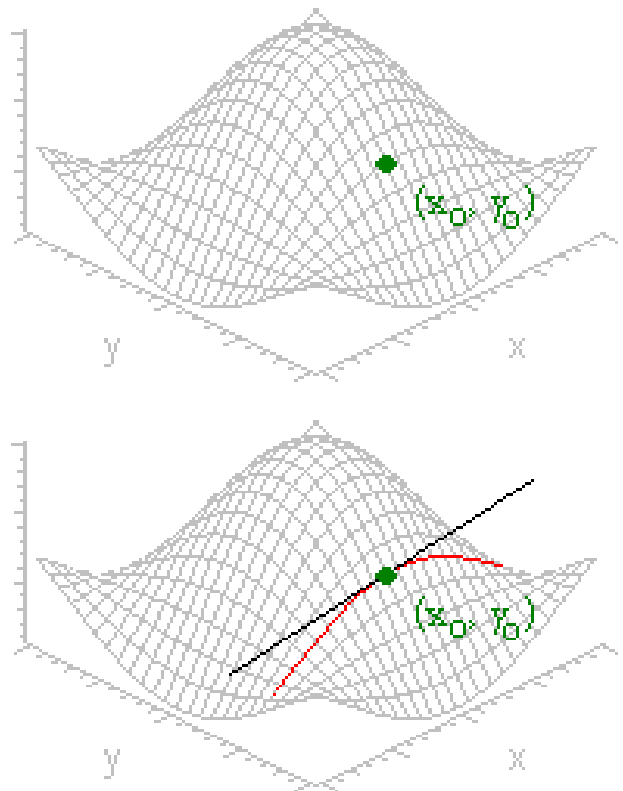
Partial differential equations abound in all branches of science and engineering and many areas of business. The number of applications is endless.

Partial derivatives have many important uses in math and science. We shall see that a partial derivative is not much more or less than a particular sort of directional derivative. The only trick is to have a reliable way of specifying directions ... so most of this note is concerned with formalizing the idea of direction

So far, we had been dealing with functions of a single independent variable. We will now consider functions which depend on more than one independent variable; Such functions are called functions of several variables.

#### Geometrical Meaning

Suppose the graph of  $z = f(x, y)$  is the surface shown. Consider the partial derivative of  $f$  with respect to  $x$  at a point  $(x_0, y_0)$ . Holding  $y$  constant and varying  $x$ , we trace out a curve that is the intersection of the surface with the vertical plane  $y = y_0$ . The partial derivative  $f_x(x_0, y_0)$  measures the change in  $z$  per unit increase in  $x$  along this curve. That is,  $f_x(x_0, y_0)$  is just the slope of the curve at  $(x_0, y_0)$ . The geometrical interpretation of  $f_y(x_0, y_0)$  is analogous.



## Real-World Applications:

### Rates of Change:

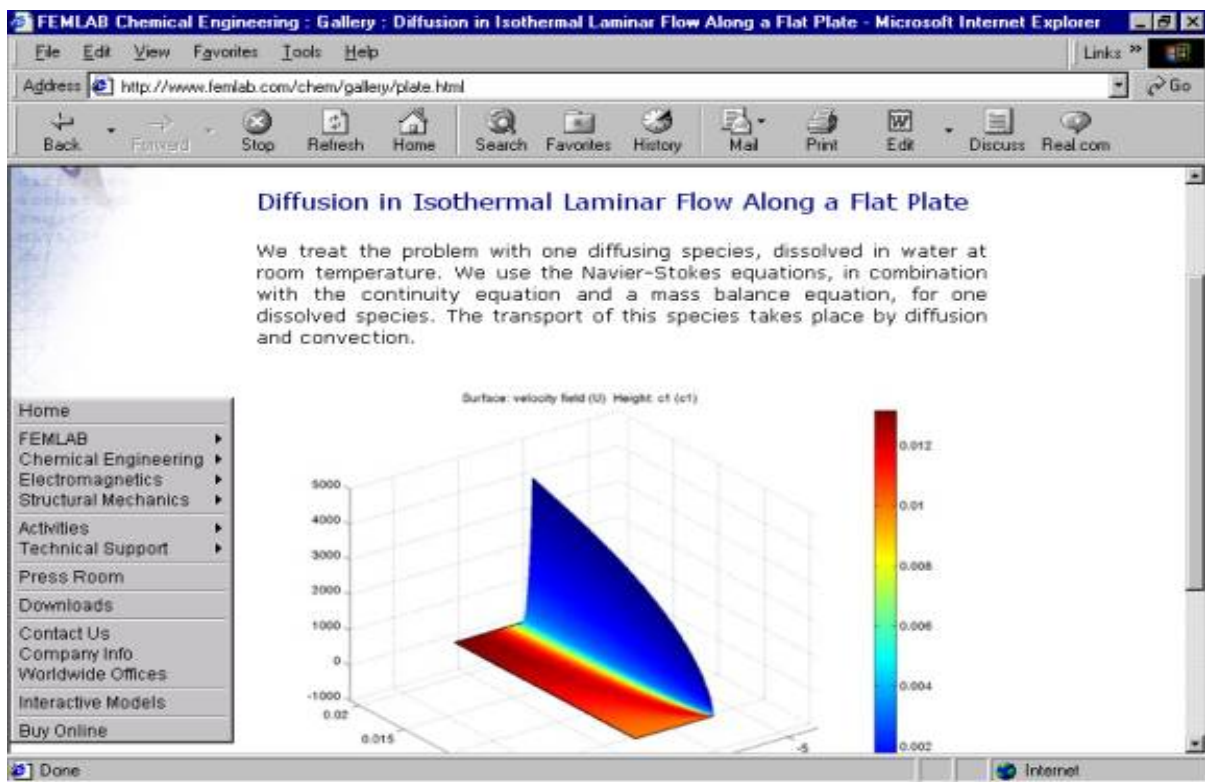
In the Java applet we saw how the concept of partial derivative could be applied geometrically to find the slope of the surface in the  $x$  and  $y$  directions. In the following two examples we present partial derivatives as rates of change. Specifically we explore an application to a temperature function (this example does have a geometric aspect in terms of the physical model itself) and a second application to electrical circuits, where no geometry is involved.

### I. Temperature on a Metal Plate

The screen capture below shows a current website illustrating thermal flow for chemical engineering. Our first application will deal with a similar flat plate where temperature varies with position.

\* The example following the picture below is taken from the current text in SM221,223:

[Multivariable Calculus](#) by James Stewart.



Suppose we have a flat metal plate where the temperature at a point  $(x,y)$  varies according to position. In particular, let the temperature at a point  $(x,y)$  be given by,

$$T(x, y) = 60 / (1 + x^2 + y^2)$$

where  $T$  is measured in  $^{\circ}\text{C}$  and  $x$  and  $y$  in meters.

**Question:** what is the rate of change of temperature with respect to distance at the point  $(2,1)$  in (a) the  $x$ -direction? and (b) in the  $y$ -direction ?

Let's take (a) first.

What is the rate of change of temperature with respect to distance at the point  $(2,1)$  in (a) the  $x$ -direction?

What observations and translations can we make here?

Rate of change of temperature indicates that we will be computing a type of derivative.

Since the temperature function is defined on two variables we will be computing a partial derivative. Since the question asks for the rate of change in the  $x$ -direction, we will be holding  $y$  constant. Thus, our question now becomes:

What is  $\frac{dT}{dx}$  at the point  $(2,1)$ ?

$$T(x, y) = 60 / (1 + x^2 + y^2) = 60(1 + x^2 + y^2)^{-1}$$

$$\frac{\partial T}{\partial x} = -60(2x)(1 + x^2 + y^2)^{-2}$$

$$\frac{\partial T}{\partial x}(2,1) = -60(4)(1 + 4 + 1)^{-2} = -\frac{20}{3}$$

Conclusion :

The rate of change of temperature in the  $x$ -direction at  $(2,1)$  is  $-\frac{20}{3}$  degrees per meter;

note this means that the temperature is decreasing !

**Part (b):**

The rate of change of temperature in the y-direction at (2,1) is computed in a similar manner.

$$T(x, y) = 60 / (1 + x^2 + y^2) = 60(1 + x^2 + y^2)^{-1}$$

$$\frac{\partial T}{\partial x} = -60(2y)(1 + x^2 + y^2)^{-2}$$

$$\frac{\partial T}{\partial x}(2,1) = -60(2)(1 + 4 + 1)^{-2} = -10/3$$

Conclusion :

The rate of change of temperature in the y-direction at (2,1) is  $-10/3$  degrees per meter;

note this means that the temperature is decreasing !

**II. Electrical Circuits: Changes in Current**

The following is adapted from an example in a former text for SM221,223 [Multivariable Calculus](#) by Bradley and Smith.

\* In an electrical circuit with electromotive force (EMF) of E volts and resistance R ohms, the current, I, is

$I = E/R$  amperes.

**Question: (a)** At the instant when  $E=120$  and  $R=15$  , what is the rate of change of current with respect to voltage.

(b) What is the rate of change of current with respect to resistance?

**(a)** Even though no geometry is involved in this example, the rate of change questions can be answered with **partial derivatives**.

we first note that I is a function of E and R, namely,

$$I(E,R) = ER^{-1}$$

---



The rate of change of current with respect to voltage =

the partial derivative of  $I$  with respect to voltage, holding resistance constant is

$$\frac{\partial I}{\partial E} = R^{-1}$$

when  $E=120$  and  $R=15$ , we have  $\frac{\partial I}{\partial E} = 15^{-1} \approx 0.0667$

verbal conclusion : If the resistance is fixed at 15 ohms, the current is increasing with respect to voltage at the rate of 0.0667 amperes per volt when the EMF is 120 volts.

### Part (b):

What is the rate of change of current with respect to resistance?

Using similar observations to part (a) we conclude:

The partial derivative of  $I$  with respect to resistance, holding voltage constant =  $\frac{\partial I}{\partial R} = ER^{-1}$

when  $E=120$  and  $R=15$ , we have  $\frac{\partial I}{\partial R}(120,15) = -120(15)^{-1} \approx -0.5333$

**Conclusion** : If the EMF is fixed at 120 volts, the current is decreasing with respect to resistance at the rate of 0.5333 amperes per ohm when the resistance is 15 ohms.

### Key Words :-

Then the partial derivative of  $z$  w.r.t  $x$  is given by

$$z_x = \frac{\partial z}{\partial x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

The partial derivative of  $z$  w.r.t  $y$  is given by

$$z_y = \frac{\partial z}{\partial y} = \lim_{\delta y \rightarrow 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

1. If  $u = e^{ax-by} \sin(ax+by)$  show that  $b \frac{\partial u}{\partial x} - a \frac{\partial u}{\partial y} = 2abu$

**Solution :**  $u = e^{ax-by} \sin(ax+by)$

$$\therefore \frac{\partial u}{\partial x} = e^{ax-by} \cos(ax+by) \cdot a + a \cdot e^{ax-by} \sin(ax+by)$$

$$\text{ie., } \frac{\partial u}{\partial x} = a e^{ax-by} \cos(ax+by) + au \quad \dots(1)$$

$$\text{Also } \frac{\partial u}{\partial y} = e^{ax-by} \cos(ax+by) \cdot (-b) + (-b) e^{ax-by} \sin(ax+by)$$

$$\text{ie., } \frac{\partial u}{\partial y} = -b e^{ax-by} \cos(ax+by) - bu \quad \dots(2)$$

$$\text{Now } b \frac{\partial u}{\partial x} - a \frac{\partial u}{\partial y} \text{ by using (1) and (2) becomes}$$

$$\begin{aligned} &= a b e^{ax-by} \cos(ax+by) + abu - a(-b e^{ax-by} \cos(ax+by) - bu) \\ &= 2abu \end{aligned}$$

$$\text{Thus } b \frac{\partial u}{\partial x} - a \frac{\partial u}{\partial y} = 2abu$$

2. If  $u = e^{ax+by} f(ax-by)$ , prove that

$$b \frac{\partial u}{\partial x} = a \frac{\partial u}{\partial y} = 2abu$$

**Solution :**  $u = e^{ax+by} f(ax-by)$ , by data

$$\frac{\partial u}{\partial x} = e^{ax+by} \cdot f'(ax-by) \cdot a + a e^{ax+by} f(ax-by)$$

$$\text{Or } \frac{\partial u}{\partial x} = a e^{ax+by} \cdot f'(ax-by) + a u \quad \dots(1)$$

$$\text{Next, } \frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by) \cdot b + b e^{ax+by} f(ax-by)$$

$$\text{Or } \frac{\partial u}{\partial y} = b e^{ax+by} f'(ax-by) + bu \quad \dots(2)$$

$$\text{Now consider L.H.S} = b \frac{\partial u}{\partial x} + a \frac{\partial u}{\partial y}$$

$$= b \left\{ a e^{ax+by} f'(ax-by) + au \right\} + a \left\{ b e^{ax+by} f'(ax-by) + bu \right\}$$

$$= ab e^{ax+by} f'(ax-by) + abu - ab e^{ax+by} f'(ax-by) + abu$$

$$= 2abu = \text{R.H.S}$$

$$\text{Thus } b \frac{\partial u}{\partial x} + a \frac{\partial u}{\partial y} = 2 a b u$$

3. If  $u = \log \sqrt{x^2 + y^2 + z^2}$ , show that  $(x^2 + y^2 + z^2) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 1$

**Solution :** By data  $u = \log \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} \log (x^2 + y^2 + z^2)$

The given  $u$  is a symmetric function of  $x, y, z$ ,

(It is enough if we compute only one of the required partial derivative)

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2 + z^2} \cdot 2x = \frac{x}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2 + z^2} \right)$$

$$\text{ie., } \frac{(x^2 + y^2 + z^2)1 - x \cdot 2x}{(x^2 + y^2 + z^2)^2} = \frac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^2} \quad \dots(1)$$

$$\text{Similarly } \therefore \frac{\partial^2 u}{\partial y^2} = \frac{z^2 + x^2 - y^2}{(x^2 + y^2 + z^2)^2} \quad \dots(2)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2} \quad \dots(3)$$

Adding (1), (2) and (3) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{x^2 + y^2 + z^2}$$

$$\text{Thus } (x^2 + y^2 + z^2) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 1$$

4. If  $u = \log (\tan x + \tan y + \tan z)$ , show that,

$$\sin 2x u_x + \sin 2y u_y + \sin 2z u_z = 2$$

**Solution :**  $u = \log (\tan x + \tan y + \tan z)$  is a symmetric function.

$$u_x = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\sin 2x u_x = \frac{(2 \sin x \cos x) \sec^2 x}{\tan x + \tan y + \tan z}$$

$$\text{Or } \sin 2x u_x = \frac{2 \tan x}{\tan x + \tan y + \tan z} \quad \dots(1)$$

$$\text{Similarly } \sin 2y u_y = \frac{2 \tan y}{\tan x + \tan y + \tan z} \quad \dots(2)$$

$$\sin 2z u_z = \frac{2 \tan z}{\tan x + \tan y + \tan z} \quad \dots(3)$$

Adding (1), (2) and (3) we get,

$$\sin 2x u_x + \sin 2y u_y + \sin 2z u_z = \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2$$

$$\text{Thus } \sin 2x u_x + \sin 2y u_y + \sin 2z u_z = 2$$

5. If  $u = \log (x^3 + y^3 + z^3 - 3xyz)$  then prove that  $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$  and hence show

$$\text{that } \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \frac{-9}{(x + y + z)^2}$$

**Solution :**  $u = \log (x^3 + y^3 + z^3 - 3xyz)$  is a symmetric function

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \quad \dots(1)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz} \quad \dots(2)$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \quad \dots(3)$$

Adding (1), (2) and (3) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x^3 + y^3 + z^3 - 3xyz)}$$

Recalling a standard elementary result,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

We have,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$\text{Thus } \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$$

$$\begin{aligned} \text{Further } \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u &= \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) u \\ &= \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) \\ &= \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left( \frac{3}{x + y + z} \right), \text{ by using the earlier result.} \\ &= \frac{\partial}{\partial x} \left( \frac{3}{x + y + z} \right) + \frac{\partial}{\partial y} \left( \frac{3}{x + y + z} \right) + \frac{\partial}{\partial z} \left( \frac{3}{x + y + z} \right) \\ &= \frac{-3}{(x + y + z)^2} + \frac{-3}{(x + y + z)^2} + \frac{-3}{(x + y + z)^2} = \frac{-9}{(x + y + z)^2} \end{aligned}$$

$$\text{Thus } \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \frac{-9}{(x + y + z)^2}$$

6. If  $u = f(r)$  and  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,

prove that  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$

**Solution :** Observing the required partial derivative we conclude that  $u$  must be a function of  $x, y$ . But

$u = f(r)$  by data and hence we need to have  $r$  as a function of  $x, y$ . Since  $x = r \cos \theta$ ,  $y = r \sin \theta$  we have  $x^2 + y^2 = r^2$ .

$\therefore$  we have  $u = f(r)$  where  $r = \sqrt{x^2 + y^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r^3} (r^2 - x^2) + \frac{f''(r)}{r^2} \cdot x^2 \text{ and}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3} (r^2 - y^2) + \frac{f''(r)}{r^2} \cdot y^2$$

Adding these results we get,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{f'(r)}{r^3} \{2x^2 - (x^2 + y^2)\} + \frac{f''(r)}{r^2} (x^2 + y^2) \\ &= \frac{f'(r)}{r^3} \cdot r^2 + \frac{f''(r)}{r^2} \cdot r^2 = \frac{1}{r} f'(r) + f''(r) \end{aligned}$$

Thus  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$

7. Prove that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$

**Proof :** Since  $u = f(x, y)$  is a homogeneous function of degree  $n$  we have by the definition,

$$u = x^n g(y/x) \quad \dots(1)$$

Let us differentiate this w.r.t  $x$  and also w.r.t  $y$

$$\therefore \frac{\partial u}{\partial x} = x^n \cdot g'(y/x) \cdot \left(-\frac{y}{x^2}\right) + nx^{n-1} g(y/x)$$

$$\text{ie., } \frac{\partial u}{\partial x} = x^{n-2} y g'(y/x) + nx^{n-1} g(y/x) \quad \dots(2)$$

$$\text{Also } \frac{\partial u}{\partial y} = x^n \cdot g'(y/x) \cdot \left(\frac{1}{x}\right)$$

$$\text{ie., } \frac{\partial u}{\partial y} = x^{n-1} \cdot g'(y/x) \quad \dots(3)$$

Now consider  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$  as a consequence of (2) and (3)

$$= x \left[ -x^{n-2} y g'(y/x) + n x^{n-1} g(y/x) \right] + y \left[ x^{n-1} g'(y/x) \right]$$

$$= -x^{n-1} y g'(y/x) + n x^n g(y/x) + x^{n-1} y g'(y/x)$$

$$= n \cdot x^n g(y/x)$$

$$= n u, \text{ by using (1)}$$

Thus we have proved Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u ; x u_x + y u_y = n u$$

8. **Prove that**  $x^2 \frac{\partial^2 u}{\partial x^2} + 2 x y \frac{\partial^2 u}{\partial x \partial y} = n(n-1)u$

**Proof :** Since  $u = f(x, y)$  is a homogeneous function of degree  $n$ , we have Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u \quad \dots(1)$$

Differentiating (1) partially w.r.t.  $x$  and also w.r.t.  $y$  we get,

$$\left( x \frac{\partial^2 u}{\partial x^2} + 1 \cdot \frac{\partial u}{\partial x} \right) + y \frac{\partial^2 u}{\partial x \partial y} = n \frac{\partial u}{\partial x} \quad \dots(2)$$

$$\text{Also, } x \frac{\partial^2 u}{\partial y \partial x} + \left( y \frac{\partial^2 u}{\partial y^2} + 1 \cdot \frac{\partial u}{\partial y} \right) = n \frac{\partial u}{\partial y} \quad \dots(3)$$

We shall now multiply (2) by  $x$  and (3) by  $y$ .

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + x y \frac{\partial^2 u}{\partial x \partial y} = n x \frac{\partial u}{\partial x} \text{ and}$$

$$x y \frac{\partial^2 u}{\partial y \partial x} + y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} = n y \frac{\partial u}{\partial y}$$

Adding these using the fact that  $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$  we get,

$$\left( x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \right) + \left( x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = n \left( x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

$$\text{ie., } x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + n u = n (n u), \text{ by using (1)}$$

$$\text{or } x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + n (nu) - nu = n (n - 1) u$$

$$\text{Thus } x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + n (n - 1) u$$

$$\text{ie., } x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = n (n - 1) u$$

9. If  $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$  show that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$

**Solution :** (Observe that the degree is 0 in every term)

$$u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$$

We shall divide both numerator and denominator of every term by x.

$$u = \frac{1}{y/x + z/x} + \frac{y/x}{z/x + 1} + \frac{z}{1 + y/x} = x^0 \{g(y/x, z/x)\}$$

$$\Rightarrow u \text{ is homogeneous of degree } 0. \therefore n = 0$$

$$\text{We have Euler's theorem, } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = n u$$

$$\text{Putting } n = 0 \text{ we get, } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

10. If  $u = \log \left( \frac{x^4 + y^4}{x + y} \right)$  show that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$

**Solution :** we cannot put the given u in the form  $x^n g(y/x)$

$$\therefore e^u = \frac{x^4 + y^4}{x + y} = \frac{x^4 (1 + y^4 / x^4)}{x (1 + y/x)} = x^3 \left\{ \frac{1 + (y/x)^4}{1 + (y/x)} \right\}$$

$$\text{ie., } e^u = x^3 g(y/x) \Rightarrow e^u \text{ is homogeneous of degree } 3 \therefore n = 3$$

Now applying Euler's theorem for the homogeneous function  $e^u$



We have  $x \frac{\partial(e^u)}{\partial x} + y \frac{\partial(e^u)}{\partial y} = n e^u$

ie.,  $x e^u \frac{\partial u}{\partial x} + y e^u \frac{\partial u}{\partial y} = 3 e^u$

Dividing by  $e^u$  we get  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$

11. If  $u = \tan^{-1} \left( \frac{x^3 + y^3}{x - y} \right)$  show that

(i)  $x u_x + y u_y = \sin 2 u$

(ii)  $x^2 u_{xx} + 2 x y u_{xy} + y^2 u_{yy} = \sin 4 u - \sin 2 u$

**Solution :** (i)  $u = \tan^{-1} \left( \frac{x^3 + y^3}{x - y} \right)$  by data

$$\Rightarrow \tan u = \frac{x^3 + y^3}{x - y} = \frac{x^3(1 + y^3/x^3)}{x(1 - y/x)} = x^2 \left\{ \frac{1 + (y/x)^3}{1 - (y/x)} \right\}$$

ie.,  $\tan u = x^2/g(y/x) \Rightarrow \tan u$  is homogeneous of degree 2.

Applying Euler's theorem for the function  $\tan u$  we have,

$$x \frac{\partial(\tan u)}{\partial x} + y \frac{\partial(\tan u)}{\partial y} = n \cdot \tan u ; n = 2$$

ie.,  $x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$

or  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u} = 2 \cos^2 u \frac{\sin u}{\cos u} = 2 \cos u \sin u = \sin 2 u$

$\therefore x u_x + y u_y = \sin 2u$

(ii) We have  $x u_x + y u_y = \sin 2 u$  ... (1)

Differentiating (1) w.r.t  $x$  and also w.r.t  $y$  partially we get

$$x u_{xx} + 1 \cdot u_x + y u_{xy} = 2 \cos 2u \cdot u_x$$
 ... (2)

And  $x u_{yx} + y u_{yy} + 1 \cdot u_y = 2 \cos 2u \cdot u_y$  ... (3)

Multiplying (2) by  $x$  and (3) by  $y$  we get,

$$x^2 u_{xx} + x u_x + xy u_{xy} = 2 \cos 2u \cdot x u_x$$

$$xy u_{yx} + y^2 u_{yy} + y u_y = 2 \cos 2u \cdot y u_y$$

Adding these by using the fact that  $u_{yx} = u_{xy}$ , we get

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} + (xy_x + yu_y) = 2 \cos 2u (xu_x + y u_y)$$

By using (1) we have,

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 2 \cos 2u \sin 2u - \sin 2u$$

(since  $\sin 2\theta = 2 \cos \theta \sin \theta$ , first term in the R.H.S becomes  $\sin 4u$ )

$$\text{Thus } x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = \sin 4u - \sin 2u$$

12. If  $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$  Prove that  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$

>> here we need to convert the given function  $u$  into a composite function.

$$\text{Let } u = f(p, q, r) \text{ where } p = \frac{x}{y}, q = \frac{y}{z}, r = \frac{z}{x}$$

$$\text{ie., } \{u \rightarrow (p, q, r) \rightarrow (x, y, z)\} \Rightarrow u \rightarrow x, y, z$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial x}$$

$$\text{ie., } \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{1}{y} + \frac{\partial u}{\partial q} \cdot 0 + \frac{\partial u}{\partial r} \cdot \left(-\frac{z}{x^2}\right)$$

$$\therefore x \frac{\partial u}{\partial x} = \frac{x}{y} \frac{\partial u}{\partial p} - \frac{z}{x} \frac{\partial u}{\partial r} \quad \dots(1)$$

Similarly by symmetry we can write,

$$y \frac{\partial u}{\partial y} = \frac{y}{z} \frac{\partial u}{\partial q} - \frac{x}{y} \frac{\partial u}{\partial p} \quad \dots(2)$$

$$z \frac{\partial u}{\partial z} = \frac{z}{x} \frac{\partial u}{\partial r} - \frac{y}{z} \frac{\partial u}{\partial q} \quad \dots(3)$$

$$\text{Adding (1), (2) and (3) we get } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

13. If  $u = f(x - y, y - z, z - x)$  show that  $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

>> Let  $u = f(p, q, r)$  where  $p = x - y, q = y - z, r = z - x$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial x}$$

$$\text{ie., } \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot 1 + \frac{\partial u}{\partial q} \cdot 0 + \frac{\partial u}{\partial r} (-1)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \quad \dots(1)$$

Similarly we have by symmetry

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \quad \dots(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \quad \dots(3)$$

Adding (1), (2) and (3) we get,  $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

14. If  $z = f(x, y)$  where  $x = r \cos \theta$  and  $y = r \sin \theta$

$$\text{Show that } \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 = \left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2$$

**Solution :**  $\{z \rightarrow (x, y) \rightarrow (r, \theta)\} \Rightarrow z \rightarrow (r, \theta)$

$$\therefore \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}; \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta}$$

$$\text{ie., } \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta \quad \dots(1)$$

$$\text{and } \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} (-r \sin \theta) + \frac{\partial z}{\partial y} (r \cos \theta) = r \left[ -\frac{\partial z}{\partial x} \sin \theta + \frac{\partial z}{\partial y} \cos \theta \right]$$

$$\text{or } \frac{1}{r} \frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} \sin \theta + \frac{\partial z}{\partial y} \cos \theta$$

squaring and adding (1), (2) and collecting suitable terms have,

$$\begin{aligned} \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x}\right)^2 [\cos^2 \theta + \sin^2 \theta] \\ &+ \left(\frac{\partial z}{\partial y}\right)^2 [\sin^2 \theta + \cos^2 \theta] + 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \cos \theta \sin \theta - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \sin \theta \cos \theta \\ \therefore \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \text{ ie., R.H.S = L.H.S} \end{aligned}$$

15. If  $z = f(x, y)$  where  $x = e^u + e^{-v}$ ,  $y = e^{-u} - e^v$

Prove that  $x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$

**Solution :**  $\{z \rightarrow (x, y) \rightarrow (u, v)\} \Rightarrow z \rightarrow (u, v)$

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}; \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

$$\text{ie., } \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot e^u + \frac{\partial z}{\partial y} (-e^{-u}) \quad \dots(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (-e^{-v}) + \frac{\partial z}{\partial y} (-e^v) \quad \dots(2)$$

Consider R.H.S =  $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$  and (1) - (2) yields

$$\frac{\partial z}{\partial x} (e^u + e^{-v}) - \frac{\partial z}{\partial y} (e^{-u} - e^v) = \frac{\partial z}{\partial x} \cdot x - \frac{\partial z}{\partial y} \cdot y$$

$$\text{Thus } \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} \quad \text{ie., R.H.S = L.H.S}$$

16. Find  $\frac{\partial(u, v, w)}{\partial(x, y, z)}$  where  $u = x^2 + y^2 + z^2$ ,  $v = xy + yz + zx$ ,  $w = x + y + z$

**Solution :** The definition of  $J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

But  $u = x^2 + y^2 + z^2$ ,  $v = xy + yz + zx$ ,  $w = x + y + z$

Substituting for the partial derivatives we get

$$J = \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

Expanding by the first row,

$$\begin{aligned} J &= 2x \{(x+z) - (y+x)\} - 2y \{(y+z) - (y+x)\} \\ &\quad + 2z \{(y+z) - (x+z)\} \\ &= 2x(z-y) - 2y(z-x) + 2z(y-x) \\ &= 2(xz - xy - yz + xy + yz - xz) = 0 \quad \text{Thus } J = 0 \end{aligned}$$

17. If  $u = \frac{yz}{x}$ ,  $v = \frac{zx}{y}$ ,  $w = \frac{xy}{z}$ , show that  $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

**Solution :** by data  $u = \frac{yz}{x}$ ,  $v = \frac{zx}{y}$ ,  $w = \frac{xy}{z}$

$$\begin{aligned} \frac{\partial(u, v, w)}{\partial(x, y, z)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix} \\ &= \frac{-yz}{x^2} \left\{ \left( \frac{-zx}{y^2} \right) \left( \frac{-xy}{z^2} \right) - \left( \frac{x}{z} \right) \left( \frac{x}{y} \right) \right\} \\ &\quad - \frac{z}{x} \left\{ \frac{z}{y} \left( \frac{-zx}{y^2} \right) - \frac{y}{z} \cdot \frac{x}{y} \right\} + \frac{y}{x} \left\{ \frac{z}{y} \cdot \frac{x}{z} - \frac{y}{z} \left( \frac{-zx}{y^2} \right) \right\} \\ &= \frac{-yz}{x^2} \left\{ \frac{x^2}{yz} - \frac{x^2}{yz} \right\} - \frac{z}{x} \left\{ \frac{-x}{z} - \frac{x}{z} \right\} + \frac{y}{x} \left\{ \frac{x}{y} + \frac{x}{y} \right\} \\ &= 0+1+1+1+1=4 \end{aligned}$$

Thus  $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

18. If  $u + v = e^x \cos y$  and  $u - v = e^x \sin y$  find the jacobian of the functions  $u$  and  $v$  w.r.t  $x$  and  $y$ .

**Solution :** we have to find  $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

Using the given data we have to solve for  $u$  and  $v$  in terms of  $x$  and  $y$ .

By data  $u + v = e^x \cos y$  .....(1)

$u - v = e^x \sin y$  .....(2)

(1) + (2) gives :  $2u = e^x (\cos y + \sin y)$

(2) - (2) gives :  $2v = e^x (\cos y - \sin y)$

ie.,  $u = \frac{e^x}{2} (\cos y + \sin y)$  ;  $v = \frac{e^x}{2} (\cos y - \sin y)$

$\therefore \frac{\partial u}{\partial x} = \frac{e^x}{2} (\cos y + \sin y), \frac{\partial v}{\partial x} = \frac{e^x}{2} (-\sin y - \cos y)$

Now  $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{e^x}{2} (\cos y + \sin y) & \frac{e^x}{2} (-\sin y + \cos y) \\ \frac{e^x}{2} (\cos y - \sin y) & \frac{-e^x}{2} (\sin y + \cos y) \end{vmatrix}$

$= \frac{e^x}{2} \cdot \frac{e^x}{2} \{ -(\cos y + \sin y)^2 - (\cos y - \sin y)^2 \}$

$= \frac{-e^{2x}}{4} \{ 1 + \sin 2y + (1 - \sin 2y) \} = \frac{-e^{2x}}{2}$

**Thus**  $\frac{\partial(u, v)}{\partial(x, y)} = \frac{-e^{2x}}{2}$

19. (a) If  $x = r \cos \theta$ ,  $y = r \sin \theta$  find the value of  $\frac{\partial(r, \theta)}{\partial(x, y)}$

(b) Further verify that  $\frac{\partial(x, y)}{\partial(r, \theta)} \cdot \frac{\partial(r, \theta)}{\partial(x, y)} = 1$

(a) **Solution :** We shall first express  $r, \theta$  in terms of  $x$  and  $y$ .

We have  $x = r \cos \theta$ ,  $y = r \sin \theta$  by data.

$$\therefore x^2 + y^2 = r^2 \text{ and } \frac{y}{x} = \tan \theta \text{ or } \theta = \tan^{-1}(y/x)$$

$$\text{Consider } r^2 = x^2 + y^2$$

Differentiating partially w.r.t  $x$  and also w.r.t  $y$  we get,

$$2r \frac{\partial r}{\partial x} = 2x \quad \text{and} \quad 2r \frac{\partial r}{\partial y} = 2y$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\text{Also consider } \theta = \tan^{-1}(y/x)$$

$$\therefore \frac{\partial \theta}{\partial x} = \frac{1}{1 + (y/x)^2} \cdot \frac{-y}{x^2} \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x}$$

$$\text{i.e.,} \quad \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\text{Now } \frac{\partial(r, \theta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix}$$

$$\text{i.e.,} \quad = \frac{x^2}{r(x^2 + y^2)} + \frac{y^2}{r(x^2 + y^2)} = \frac{(x^2 + y^2)}{r(x^2 + y^2)} = \frac{1}{r}$$

$$\therefore \frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{r}$$

**Solution (b):** Consider  $x = r \cos \theta$ ,  $y = r \sin \theta$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r$$

$$\therefore \frac{\partial(x, y)}{\partial(r, \theta)} = r$$

$$\text{From (1) and (2): } \frac{\partial(x, y)}{\partial(x, \theta)} \cdot \frac{\partial(r, \theta)}{\partial(x, y)} = r \cdot \frac{1}{r} = 1$$

**20. If  $x = u(1 - v)$ ,  $y = uv$  then show that  $JJ' = 1$**

**Solution :**  $x = u(1 - v)$ ;  $y = uv$

$$\frac{\partial x}{\partial u} = (1 - v), \quad \frac{\partial y}{\partial u} = v \quad \frac{\partial x}{\partial v} = -u, \quad \frac{\partial y}{\partial v} = u$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} (1 - v) & -u \\ v & u \end{vmatrix}$$

$$= (1 - v)u + uv = u \quad \therefore J = u$$

Next we shall express  $u$  and  $v$  in terms of  $x$  and  $y$ .

By data  $x = u - uv$  and  $y = uv$

$$\text{Hence } x + y = u. \text{ Also } v = \frac{y}{u} = \frac{y}{x + y}$$

$$\text{Now we have, } u = x + y; v = \frac{y}{x + y} \quad \therefore \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 1,$$



$$\frac{\partial v}{\partial x} = \frac{(x+y) \cdot 0 - y \cdot 1}{(x+y)^2} = -\frac{y}{(x+y)^2}$$

$$\begin{aligned} \therefore J' &= \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -y & x \end{vmatrix} \\ &= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u} \end{aligned}$$

$$\text{Thus } J' = \frac{1}{u} \text{ Hence } J \cdot J' = u \cdot \frac{1}{u} \text{ Thus } JJ' = 1$$

## 21. State Taylor's Theorem for Functions of Two Variables.

**Statement:** Considering  $f(x+h, y+k)$  as a function of a single variable  $x$ , we have by Taylor's Theorem

$$f(x+h, y+k) = f(x, y+k) + h \frac{\partial f(x, y+k)}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 f(x, y+k)}{\partial x^2} + \dots \quad (1)$$

Now expanding  $f(x, y+k)$  as function of  $y$  only,

$$f(x, y+k) = f(x, y) + k \frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!} \frac{\partial^2 f(x, y)}{\partial y^2} + \dots$$

$$\therefore \text{(i) takes the form } f(x+h, y+k) = f(x, y) + h \frac{\partial f(x, y)}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 f(x, y)}{\partial x^2} + \dots +$$

$$+ h \frac{\partial}{\partial x} \left\{ f(x, y) + k \frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!} \frac{\partial^2 f(x, y)}{\partial y^2} + \dots \right\}$$

$$+ \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} \left\{ f(x, y) + k \frac{\partial f(x, y)}{\partial y} + \dots \right\}$$

$$\text{Hence } f(x+h, y+k) = f(x, y) + h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} + \frac{1}{2!} \left( h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \right) + \dots \quad (1)$$

In symbol we write it as

$$F(x+h, y+k) = f(x,y) + \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f + \frac{1}{2!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f + \dots$$

Taking  $x = a$  and  $y = b$ , (1) becomes

$$f(a+h, b+k) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2!} [h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)] + \dots$$

Putting  $a+h = x$  and  $b+k = y$  so that  $h = x-a$ ,  $k = y-b$ , we get

$$\begin{aligned} F(x,y) &= f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] \\ &= \frac{1}{2!} [(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \dots \quad (2) \end{aligned}$$

This is Taylor's expansion of  $f(x,y)$  in powers of  $(x-a)$  and  $(y-b)$ . It is used to expand  $f(x,y)$  in the neighborhood of  $(a,b)$

corollary, putting  $a = 0$ ,  $b = 0$  in (2), we get

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!} [x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \dots \quad (3)$$

This is Maclaurin's Expansion of  $f(x,y)$

**22. Expand  $e^x \log(1+y)$  in powers of  $x$  and  $y$  up to terms of third degree.**

**Solution:** Here

$$f(x,y) = e^x \log(1+y) \quad \therefore f(0,0) = 0$$

$$f_x(x,y) = e^x \log(1+y) \quad \therefore f_x(0,0) = 0$$

$$f_y(x,y) = e^x \frac{1}{1+y} \quad \therefore f_y(0,0) = 1$$

$$f_{xx}(x,y) = e^x \log(1+y) \quad \therefore f_{xx}(0,0) = 0$$

$$f_{xy}(x,y) = e^x \frac{1}{1+y} \quad \therefore f_{xy}(0,0) = 1$$

$$f_{yy}(x,y) = -e^x (1+y)^{-2} \quad \therefore f_{yy}(0,0) = -1$$

$$f_{xxx}(x,y) = e^x \log(1+y) \quad \therefore f_{xxx}(0,0) = 0$$

$$f_{xyy}(x,y) = e^x \frac{1}{1+y} \quad \therefore f_{xyy}(0,0) = 1$$

$$f_{xyy}(x,y) = -e^x (1+y)^{-2} \quad \therefore f_{xyy}(0,0) = -1$$

$$f_{yyy}(x,y) = 2e^x (1+y)^{-3} \quad \therefore f_{yyy}(0,0) = 2$$

Now, Maclaurin's expansion of  $f(x,y)$  gives

$$f(x,y) = f(0,0) + x(f_x(0,0) + y f_y(0,0)) + \frac{1}{2!} \{x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\} + \frac{1}{3!} \{x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\} + \dots$$

$$\begin{aligned} \therefore e^x \log(1+y) &= 0 + x.0 + y(1) + \frac{1}{2!} \{x^2.0 + 2xy(1) + y^2(-1)\} \\ &\quad + \frac{1}{3!} \{x^3.0 + 3x^2y(1) + 3xy^2(-1) + y^3(2)\} + \dots \\ &= y + xy - \frac{1}{2} y^2 + \frac{1}{2} (x^2y - xy^2) + \frac{1}{2} y^3 + \dots \end{aligned}$$

**23. Expand  $f(x,y) = e^x \cos y$  by Taylor's Theorem about the point  $\left(1, \frac{\pi}{4}\right)$  up to the Second degree terms.**

**Solution:**  $f(x,y) = e^x \cos y$  and  $a = 1, b = \frac{\pi}{4} \quad \therefore f = \left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$

$$f_x(x,y) = e^x \cos y \quad \therefore f_x\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$f_y(x,y) = -e^x \sin y \quad \therefore f_y\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

$$f_{xx}(x,y) = e^x \cos y \quad \therefore f_{xx}\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$f_{xy}(x,y) = -e^x \sin y \quad \therefore f_{xy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

$$f_{yy}(x,y) = -e^x \cos y \quad \therefore f_{yy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

Hence by Taylor's Theorem, we obtain

$$\begin{aligned}
 f(x,y) &= f\left(1, \frac{\pi}{4}\right) + \left[(x-1)f_x + \left(y - \frac{\pi}{4}\right)f_y\right] + \\
 &\quad \frac{1}{2!} \left[(x-1)^2 f_{xx} + 2(x-1)\left(y - \frac{\pi}{4}\right) f_{xy}\right] + \dots \\
 \text{i.e., } e^x \cos y &= \frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right)\right] + \frac{1}{2!} \\
 &\quad \left[(x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)\left(y - \frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right) + \left(y - \frac{\pi}{4}\right)^2 \left(\frac{-e}{\sqrt{2}}\right)\right] + \dots \\
 e^x \cos y &= \frac{e}{\sqrt{2}} \left[1 + (x-1) - \left(y - \frac{\pi}{4}\right)\right] + \frac{1}{2!} \left[(x-1)^2 - 2(x-1)\left(y - \frac{\pi}{4}\right) - \left(y - \frac{\pi}{4}\right)^2\right] + \dots
 \end{aligned}$$

**Exercise:**

- 1) Expand  $e^{xy}$  up to Second degree terms by using Maclaurin's theorem
- 2) Expand  $\log(1 - x - y)$  up to Third degree terms by using Maclaurin's theorem
- 3) Expand  $x^2y$  about the point  $(1, -2)$  by Taylor's expansion
- 4) Obtain the Taylor's expansion of  $e^x \sin y$  about the point  $\left(0, \frac{\pi}{2}\right)$  up to Second degree terms
- 5) Expand  $e^{\sin x}$  up to the term containing  $x^4$

## Maxima and Minima:-

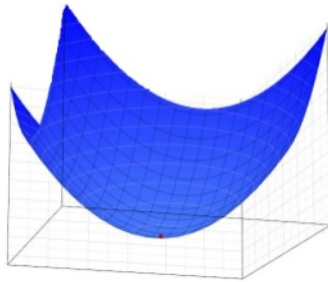
In mathematics, the maximum and minimum (plural: maxima and minima) of a function, known collectively as extrema (singular: extremum), are the largest and smallest value that the function takes at a point within a given neighborhood.

A function  $f(x, y)$  is said to have a Maximum value at  $(a, b)$  if there exists a neighborhood point of  $(a, b)$  (say  $(a+h, b+k)$ ) such that  $f(a, b) > f(a+h, b+k)$ .

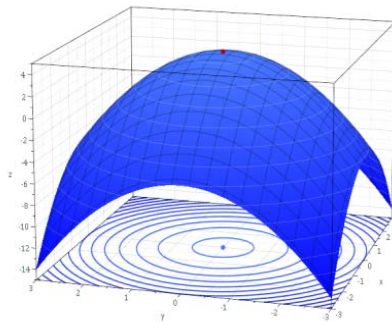
Similarly,

Minimum value at  $(a, b)$  if there exists a neighborhood point of  $(a, b)$  (say  $(a+h, b+k)$ ) such that  $f(a, b) < f(a+h, b+k)$ .

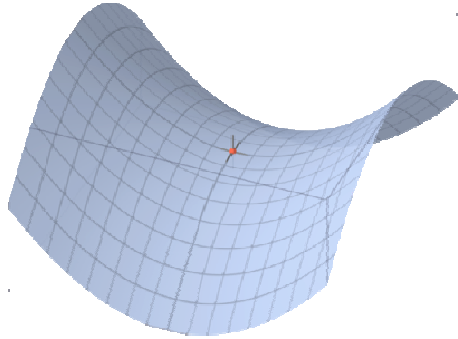
A Minimum point on the graph (in red)  $f(x, y) = x^2 + y^2(1-x)^3$



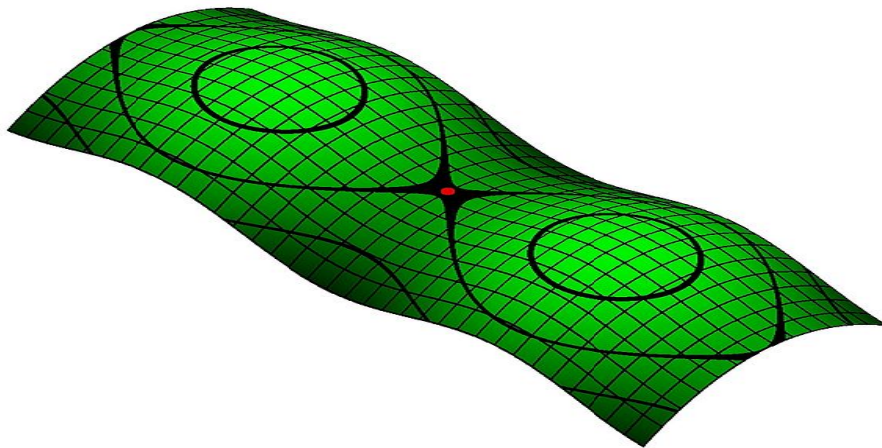
A Maximum point on the graph is at the top (in red)



A saddle point on the graph of  $z=x^2-y^2$  (in red)



Saddle point between two hills.



### Necessary and Sufficient Condition:-

- If  $f_x = 0$  and  $f_y = 0$  (Necessary Condition)
  - Function will be minimum if  $AC - B^2 > 0$  and  $A > 0$
  - Function will be maximum if  $AC - B^2 > 0$  and  $A < 0$
  - Function will be neither maxima nor minima if  $AC - B^2 < 0$
  - If  $AC - B^2 = 0$  we cannot make any conclusion without any further analysis
- where  $A = f_{xx}$ ,  $B = f_{xy}$ ,  $C = f_{yy}$

---

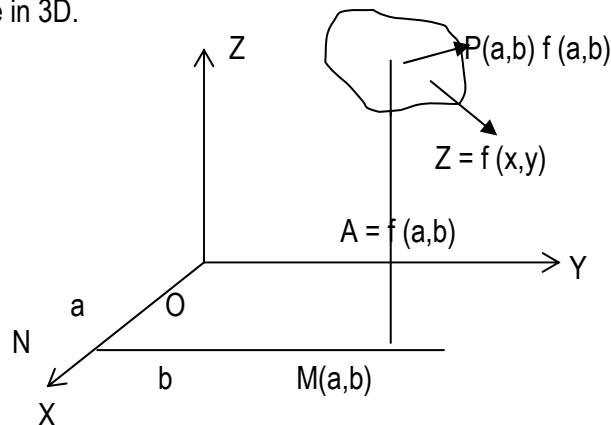
A

### Working Procedure:-

- First we find Stationary points by considering  $f_x = 0$  and  $f_y = 0$ .
- Function will be minimum if  $AC - B^2 > 0$  and  $A > 0$  at that stationary point
- Function will be maximum if  $AC - B^2 > 0$  and  $A < 0$  at that stationary point
- Function will be neither maximum nor minimum if  $AC - B^2 < 0$  at that stationary point and it is called as *SADDLE POINT*.

**25. Explain Maxima & Minima for Functions of Two Variables& hence obtain the Necessary Conditions for Maxima, Minima.**

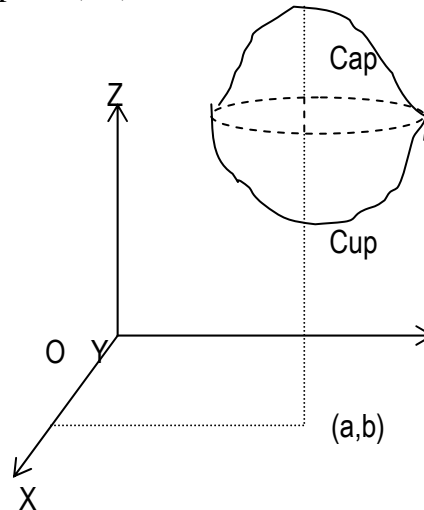
**Solution:** Let  $Z = f(x, y)$  be a given function of two independent variables  $x$  &  $y$ . The above equation represents a surface in 3D.



A given points  $(a, b)$  on the surface has Co-ordinates  $[a, b, f(a, b)]$

**Definition:**

The function  $Z = f(x, y)$  is said to be a maximum at the point  $(a, b)$  if  $f(x, y) < f(a, b)$  in the neighborhood of the point  $(a, b)$



**Definition:**

The function  $Z = f(x, y)$  is said to possess a minimum at the point  $(a, b)$  if  $f(x, y) > f(a, b)$  in the neighborhood of the point  $(a, b)$

**Necessary Condition for Maxima, Minima:**

If  $Z = f(x, y)$  has a max or min at  $(a, b)$  then  $f_x(a, b) = 0$ ,  $f_y(a, b) = 0$

Sufficient Conditions for Maxima, Minima:

Put  $R = f_{xx}(a, b)$ ,  $S = f_{xy}(a, b)$ ,  $T = f_{yy}(a, b)$

(1) Suppose  $S^2 - RT > 0$

There is no maxima or minima at  $(a, b)$

(2) Suppose  $S^2 - RT < 0$

Thus there is maxima or minima at  $(a, b)$  according as  $R < 0$  Or  $R > 0$

(3) Suppose  $S^2 - RT = 0$ , Then there is a saddle point at  $(a, b)$



26. Find the maxima and minima of the functions  $f(x,y) = x^3 + y^3 - 3axy$ ,  $a > 0$  is constant.

**Solution:** Given  $f(x,y) = x^3 + y^3 - 3axy$

$$f_x = 3x^2 - 3ay, f_y = 3y^2 - 3ax$$

$$f_{xx} = 6x, f_{yy} = 6y.$$

Put  $f_x = 0, f_y = 0$  and solve

$$\text{i.e., } 3x^2 - 3ay = 0 \text{ \& } 3y^2 - 3ax = 0$$

$$\text{i.e., } x^2 = ay \text{ \& } y^2 = ax$$

$$\Rightarrow y = \frac{x^2}{a} \quad \therefore \Rightarrow \left( \frac{x^2}{a} \right)^2 = ax \quad (\because x^2 = ay)$$

$$\therefore \frac{x^4}{a^2} = ax$$

$$\therefore x^4 = a^3x$$

$$\text{i.e., } x(x^3 - a^3) = 0$$

$$\therefore x = 0, x = a$$

$$\Rightarrow y = 0, y = \pm a$$

$\therefore$  The critical Or stationary points are  $(0,0)$ ,  $(a,a)$  and  $(a,-a)$

(1) At  $(0,0)$

$$R = f_{xx}(0,0) = 0$$

$$S = f_{xy}(0,0) = -3a$$

$$T = f_{yy}(0,0) = 0$$

$$\therefore S^2 - RT = 9a^2 - 0 = 9a^2 > 0$$

$\therefore$  There is neither a maximum or a minimum at  $(0,0)$

---

27. Examine the following functions for extreme values  $f = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

**Solution:**

$$f_x = 4x^3 - 4x + 4y$$

$$f_y = 4y^3 - 4x - 4y$$

$$f_{xy} = 4, \quad f_{xx} = 12x^2 - 4, \quad f_{yy} = 12y^2 - 4$$

Put  $f_x = 0, f_y = 0$  and solve

$$\text{i.e., } 4x^3 - 4x + 4y = 0 \quad \rightarrow (1)$$

$$4y^3 + 4x - 4y = 0 \quad \rightarrow (2)$$

Adding (1) & (2), we get

$$4(x^3 + y^3) = 0$$

$$\text{i.e., } x^3 + y^3 = 0$$

$$\text{i.e., } y = -x$$

Substitute  $y = -x$  in (1), we get

$$4x^3 - 4x - 4x = 0$$

$$\text{i.e., } 4x^3 - 8x = 0$$

$$\text{i.e., } x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$$

$$\text{i.e., } x = 0 \text{ \& } x^2 - 2 = 0$$

$$\text{i.e., } x = 0 \text{ \& } x = \pm \sqrt{2}$$

$$x = \sqrt{2}, -\sqrt{2}$$

$$\therefore x = 0, \sqrt{2}, -\sqrt{2} \text{ and corresponding values of } y \text{ are } y = 0, -\sqrt{2}, \sqrt{2}$$

$$\therefore \text{ The critical points are } (0,0), \left( \sqrt{2}, -\sqrt{2} \right), \left( -\sqrt{2}, \sqrt{2} \right)$$

(1) at (0,0)

$$R = f_{xx}(0,0) = -4$$


---

$$S = f_{xy}(0,0) = 4$$

$$T = f_{yy}(0,0) = -4$$

$$\therefore S^2 - RT = 16 - (-4)(-4) = 16 - 16 = 0$$

i.e.,  $S^2 - RT = 0$ , These is a saddle point at  $(0,0)$

(2) at  $(\sqrt{2}, -\sqrt{2})$

$$R = f_{xx}(\sqrt{2}, -\sqrt{2}) = 24 - 4 = 20$$

$$S = f_{xy}(\sqrt{2}, -\sqrt{2}) = 4$$

$$T = f_{yy}(\sqrt{2}, -\sqrt{2}) = 20$$

$$\therefore S^2 - RT = 16 - (20)(20) = 16 - 400 = -384 < 0$$

Thus these is neither maximum nor minimum according to  $R < 0$  or  $R > 0$  at  $(\sqrt{2}, -\sqrt{2})$

Hence  $R = 20 > 0$

$\therefore$  There is a minimum at  $(\sqrt{2}, -\sqrt{2})$

$$\begin{aligned} \therefore f_{\min} &= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2 \\ &= 4 + 4 - 4 - 8 - 4 \\ &= -8 \end{aligned}$$

(3) at  $(-\sqrt{2}, \sqrt{2})$

$$R = f_{xx}(-\sqrt{2}, \sqrt{2}) = 20 > 0$$

$$S = f_{xy}(-\sqrt{2}, \sqrt{2}) = 4$$

$$T = f_{yy}(-\sqrt{2}, \sqrt{2}) = 20$$

$$\therefore S^2 - RT = 16 - 400 = -384 < 0$$

Since  $R > 0$ ,  $\therefore$  There is minima at  $(-\sqrt{2}, \sqrt{2})$

$\therefore f_{\min} = -8$  at  $(-\sqrt{2}, \sqrt{2})$

$\therefore$  Extreme Value = -8 at  $(-\sqrt{2}, \sqrt{2})$  &  $(-\sqrt{2}, \sqrt{2})$

**Exercise:**

- 1) Find the extreme values of  $f = x^3 y^2 (1 - x - y)$
- 2) Determine the maxima or minima of the function  $\sin x + \sin y + \sin (x + y)$
- 3) Examine the function  $f(x, y) = 1 + \sin(x^2 + y^2)$  for extremum.

28. If  $PV^2 = K$  and if the relative errors in  $P$  is 0.05 and in  $V$  is 0.025 show that the error in  $K$  is 10%.

**Solution :**  $PV^2 = K$  by data. Also  $\frac{\delta P}{P} = 0.05$  and  $\frac{\delta V}{V} = 0.025$

$$\Rightarrow \log P + 2 \log V = \log K$$

$$\Rightarrow \delta(\log P) + 2\delta(\log V) = \delta(\log K)$$

$$\text{i.e., } \frac{1}{P} \delta P + 2 \cdot \frac{1}{V} \delta V = \frac{1}{K} \delta K$$

$$\text{i.e., } 0.05 + 2(0.025) = \frac{\delta K}{K} \text{ or } \frac{\delta K}{K} = 0.1$$

$$\therefore \frac{\delta K}{K} \times 100 = (0.1) \times 100 = 10$$

Thus the error in  $K$  is 10%.

29. The time  $T$  of a complete oscillation of a simple pendulum is given by the formula

$$T = 2\pi\sqrt{l/g}.$$

- (i) If  $g$  is a constant find the error in the calculated value of  $T$  due to an error of 3% in the value of  $l$ .
- (ii) Find the maximum error in  $T$  due to possible errors upto 1% in  $l$  and 3% in  $g$ .

**Solution :**

$$(i) \quad T = 2\pi\sqrt{l/g}, \quad g = \text{Constant}, \quad \frac{\delta l}{l} \times 100 = 3$$

$$\Rightarrow \log T = \log 2\pi + \frac{1}{2}(\log l - \log g)$$

$$\Rightarrow \delta(\log T) = \delta(\log 2\pi) + \frac{1}{2}\delta(\log g)$$

$$\text{i.e., } \frac{\delta T}{T} = 0 + \frac{1}{2} \frac{\delta l}{l} - 0$$

$$\text{or } \frac{\delta T}{T} \times 100 = \frac{1}{2} \left( \frac{\delta l}{l} \times 100 \right) = \frac{1}{2}(3) = 1.5$$

$\therefore$  the error in  $T = 1.5\%$ .

(ii) If  $g$  is not a constant we have,

$$\frac{\delta T}{T} \times 100 = \frac{1}{2} \left( \frac{\delta l}{l} \times 100 \right) - \frac{1}{2} \left( \frac{\delta g}{g} \times 100 \right)$$

The error in  $T$  will be maximum if the error in  $l$  is positive and the error in  $g$  is negative (or vice-versa) as the difference in errors converts in to a sum.

$$\therefore \max \left( \frac{\delta T}{T} \times 100 \right) = \frac{1}{2}(+1) - \frac{1}{2}(-3) = 2$$

$\therefore$  the maximum error in  $T$  is 2%.

30. The current measured by a tangent galvanometer is given by the relation  $c = k \tan \theta$  where  $\theta$  is the angle of deflection. Show that the relative error in  $c$  due to a given error in  $\theta$  is minimum when  $\theta = 45^\circ$ .

**Solution :** Consider  $c = k \tan \theta$ .  $K$  is taken as a constant.

$$\Rightarrow \log c = \log k + \log (\tan \theta)$$

$$\Rightarrow \delta(\log c) = \delta(\log k) + \delta \log (\tan \theta)$$

$$\text{i.e., } \frac{1}{c} \delta c = 0 + \frac{\sec^2 \theta}{\tan \theta} \delta \theta$$

$$\text{i.e., } \frac{\delta c}{c} = \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\cos^2 \theta} \delta \theta \quad \text{or} \quad \frac{\delta \theta}{c} = \frac{\delta \theta}{\sin \theta \cos \theta}$$

$$\text{i.e., } \frac{\delta c}{c} = \frac{2}{\sin 2\theta} \delta \theta$$

The relative error in  $c$  being  $\delta c / c$  minimum when the denominator of the R.H.S. is maximum and the maximum value of a sine function is 1.

$$\therefore \sin 2\theta = 1 \Rightarrow 2\theta = 90^\circ \text{ or } \theta = 45^\circ.$$

Thus the relative error in  $c$  is minimum when  $\theta = 45^\circ$

31. If  $T = \frac{1}{2} mv^2$  is the kinetic energy, find approximately the change in  $T$  as  $m$  changes

from 49 to 49.5 and  $v$  changes from 1600 to 1590. 6 Marks

**Solution :** We have by data  $T = \frac{1}{2} mv^2$  and

$$m = 49, m + \delta m = 49.5 \quad \therefore \delta m = 0.5$$

$$v = 1600, v + \delta v = 1590 \therefore \delta v = -10$$


---

We have to find  $\delta T$ . (logarithm is not required)

$$\therefore \delta T = \frac{1}{2} \delta (mv^2)$$

$$\frac{1}{2} \{m(2v\delta v) + \delta m \cdot v^2\}$$

$$i.e., = \frac{1}{2} \{(49) (2) (1600) (-10) + (0.5) (1600)^2\} = 1,44,000$$

Thus the change in  $T = \delta T = -1,44,000$

32. The pressure  $p$  and the volume  $v$  of a gas are concentrated by the relation  $pv^{1.4} = \text{constant}$ . Find the percentage increase in pressure corresponding to a diminution of  $\frac{1}{2}\%$  in volume.

**Solution :**

$$pv^{1.4} = \text{Constant} = c(\text{say}), \text{ by data.}$$

$$\Rightarrow \log p + 1.4 \log v = \log c$$

$$\Rightarrow \delta(\log p) + 1.4\delta(\log v) = \delta(\log c)$$

$$i.e., \frac{\delta p}{p} + 1.4 \left( \frac{\delta v}{v} \right) = 0; \text{ But } \frac{\delta v}{v} \times 100 = -\frac{1}{2}, \text{ by data.}$$

$$\therefore \frac{\delta p}{p} \times 100 + 1.4 \left( \frac{\delta v}{v} \times 100 \right) = 0 \text{ or } \frac{\delta p}{p} \times 100 = +0.7.$$

Thus the percentage increase in pressure = 0.7

33. Find the percentage error in the area of an ellipse when an error of +1% is made in measuring the major and minor axis.

**Solution :** For the ellipse  $x^2/a^2 + y^2/b^2 = 1$  the area (A) is given by  $\pi ab$  where  $2a$  and  $2b$  are the lengths of the major and minor axis.

Let  $2a = x$  and  $2b = y$ .

$$\text{By data } \frac{\delta x}{x} \times 100 = 1, \frac{\delta y}{y} \times 100 = 1.$$

$$A = \pi ab = \pi \cdot \frac{x}{2} \cdot \frac{y}{2} = \frac{\pi}{4} xy$$

$$\therefore \log A = \log (\pi / 4) + \log x + \log y$$

$$\Rightarrow \delta(\log A) = \delta \log (\pi / 4) + \delta(\log x) + \delta(\log y)$$

$$\text{i.e., } \frac{\delta A}{A} = 0 + \frac{\delta x}{x} + \frac{\delta y}{y} \text{ or } \frac{\delta A}{A} \times 100 = \frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100$$

$$\therefore \frac{\delta A}{A} \times 100 = 1 + 1 = 2$$

Thus error in the area = 2%

27. If the sides and angles of a triangle ABC vary in such way that the circum radius remains constant, prove that

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0$$

**Solution :** If the triangle ABC is inscribed in a circle of radius r and if a,b,c respectively denotes the sides opposite to the angles A,B,C we have the sine rule (formula) given by

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$$

$$\text{or } a = 2r \sin A, b = 2r \sin B, c = 2r \sin C$$

$$\Rightarrow \delta a = 2r \delta (\sin A), \delta b = 2r \delta (\sin B), \delta c = 2r \delta (\sin C)$$

$$\text{i.e., } \delta a = 2r \cos A \delta A, \delta b = 2r \cos B \delta B, \delta c = 2r \cos C \delta C$$

$$\text{or } \frac{\delta a}{\cos A} = 2r \delta A, \frac{\delta b}{\cos B} = 2r \delta B, \frac{\delta c}{\cos C} = 2r \delta C$$

Adding all these results we get,

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 2r(\delta A + \delta B + \delta C) = 2r \delta (A + B + C)$$

But  $A+B+C = 180 = \pi$  radians = constant.

$$\Rightarrow \delta (A + B + C) = \delta (\text{constant}) = 0$$

$$\text{Thus } \frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0$$





d)  $\rho = (1+(y')^2)^{3/2}$

10) The function for which Rolle's theorem is true is:

- a)  $f(x)=\log x$  in the interval  $[1/2,2]$
- b)  $f(x)=|x+1|$  in the interval  $[-2,2]$
- c)  $f(x)=|x|$  in the interval  $[-1,1]$
- d) Non of the above

11) The value of 'c' in Rolle's theorem, where  $-\pi/2 < c < \pi/2$  and  $f(x)=\cos x$  is equal to:

- a)  $\pi/4$
- b)  $\pi/3$
- c)  $\pi$
- d) 0

12). The expansion of  $\tan x$  in powers of  $x$  by Maclaurin's theorem is valid in the interval :

- a)  $(-\infty, \infty)$
- b)  $(-3\pi/2, 3\pi/2)$
- c)  $(-\pi, \pi)$
- d)  $(-\pi/2, \pi/2)$

13) The value of 'c' in Lagrange's mean value theorem, where  $[1,2]$  and  $f(x)=x(x-1)$  is :

- a)  $5/4$
- b)  $3/2$
- c)  $7/4$
- d)  $11/6$

14) The value of 'c' in Rolle's theorem, where  $[0, \pi]$  and  $f(x)=\sin x$  is equal to:

- a)  $\pi/6$
- b)  $\pi/3$
- c)  $\pi/2$
- d) Non of these

15) The maximum value of  $\log x/x$  is:

- a) 1
- b) e
- c)  $2/e$
- d)  $1/e$

16) The maximum value of  $(1/x)^x$  is equal to:

- a) e
- b) 1
- c)  $e^{1/e}$
- d)  $(1/e)^e$

17) The difference between the maximum and minimum values of the function  $a \sin x + b \cos x$  is:

- a)  $2\sqrt{a^2 + b^2}$
- b)  $2(a^2 + b^2)$
- c)  $a^2 + b^2$
- d)  $-\sqrt{a^2 + b^2}$

18) Which one of the following statement is correct for the function  $f(x)=x^3$

- a)  $f(x)$  has a maximum value at  $x=0$
- b)  $f(x)$  has a minimum value at  $x=0$
- c)  $f(x)$  has neither a maximum nor a minimum value at  $x=0$
- d)  $f(x)$  has no point of inflexion

19) which one of the following is not an indeterminate form

- a)  $\infty + \infty$
- b)  $\infty - \infty$
- c)  $\infty / \infty$
- d)  $0 \times \infty$

20) In Lagrange's mean value theorem,  $f'(c)=$

- a)  $f(a) - f(b) / b - a$
- b)  $f(b) - f(a) / a - b$
- c)  $f(b) - f(a) / b - a$
- d) none

21) In Cauchy's mean value theorem ,  $f'(c) / g'(c) =$

- a)  $f(a) - f(b) / g(b) - g(a)$       b)  $f(b) - f(a) / g(b) - g(a)$   
 c)  $f(b) + f(a) / g(b) + g(a)$       d) none

22)  $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$  is equal to :

- a) 0      b)  $\infty$       c)  $\log(a/b)$       d)  $\log(a-b)$

23) The value of  $\lim_{x \rightarrow 0} \frac{\log x}{x - 1}$  is equal to:

- a) -1      b)  $\infty$       c) 1      d) 0

24) The value of  $\lim_{x \rightarrow 0} (1 - \cos x / 3 x^2)$  is equal to:

- a) 3      b) 1/3      c) 1/6      d) 1/9

25)  $\lim_{x \rightarrow 0} (e^x + e^{-x} - 2) / x^2$  is equal to :

- a) 1      b) -1      c) 1/2      d) -1/2

26) The value of  $\lim_{x \rightarrow 0} (1 + x)^{1/x}$  is :

- a) 1      b) -1      c) 1/e      d) e

27) The value of  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \dots$

- a) 1      b) 2      c) 3      d) 0

28) The formulae for radius of curvature in polar form is

- a)  $\rho = r dr / dp$       b)  $\rho = [(r^2 + (r_1')^2)^{3/2}] / [r^2 + 2(r_1')^2 - r r_2']$   
 c)  $\rho = (r^2 + (r_1')^2)^{3/2}$       d) None

29) The value of  $\lim_{x \rightarrow 0} 1^{1/x} = \dots$

- a) 1      b)  $\infty$       c)  $-\infty$       d) 0
-

*KEY ANSWERS:*

1-a	2-a	3-a	4-a	5-a	6-a	7-a	8-b	9-b	10-a
11-d	12-d	13-b	14-c	15-d	16-c	17-a	18-c	19-a	20-c
21-b	22-c	23-c	24-c	25-a	26-c	27-a	28-b	29-b	--