Programmation 1

TD n°7

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$$\frac{}{\rho \vdash x := e \Rightarrow \rho[x \mapsto [\![e]\!]\rho]}\ (:=) \qquad \frac{}{\rho \vdash \texttt{skip} \Rightarrow \rho}\ (\texttt{Skip})$$

$$\frac{\rho \vdash c_1 \Rightarrow \rho' \quad \rho' \vdash c_2 \Rightarrow \rho''}{\rho \vdash c_1; c_2 \Rightarrow \rho''}\ (\texttt{Seq})$$

$$\frac{\rho \vdash c_1 \Rightarrow \rho'}{\rho \vdash \texttt{if}\ e\ \texttt{then}\ c_1\ \texttt{else}\ c_2 \Rightarrow \rho'}\ (\texttt{if}_1)\ \text{si } [\![e]\!]\rho \neq 0 \qquad \frac{\rho \vdash c_2 \Rightarrow \rho''}{\rho \vdash \texttt{if}\ e\ \texttt{then}\ c_1\ \texttt{else}\ c_2 \Rightarrow \rho'}\ (\texttt{if}_2)\ \text{si } [\![e]\!]\rho = 0$$

$$\frac{\rho \vdash c \Rightarrow \rho' \quad \rho' \vdash \texttt{while}\ e\ \texttt{do}\ c \Rightarrow \rho''}{\rho \vdash \texttt{while}\ e\ \texttt{do}\ c \Rightarrow \rho''}\ (\texttt{while})\ \text{si } [\![e]\!]\rho \neq 0 \qquad \frac{}{\rho \vdash \texttt{while}\ e\ \texttt{do}\ c \Rightarrow \rho}\ (\texttt{while}_{\text{fin}})\ \text{si } [\![e]\!]\rho = 0$$

FIGURE 1 – La sémantique opérationnelle à grands pas de IMP.

$$\begin{aligned}
(x := e \cdot C, \rho) &\to (C, \rho[x \mapsto [\![e]\!]\rho]) \\
(\texttt{skip} \cdot C, \rho) &\to (C, \rho) \\
(c_1; c_2 \cdot C, \rho) &\to (c_1 \cdot c_2 \cdot C, \rho) \\
(\texttt{if}\ e\ \texttt{then}\ c_1\ \texttt{else}\ c_2 \cdot C, \rho) &\to (c_1 \cdot C, \rho) \quad \text{si } [\![e]\!]\rho \neq 0 \\
(\texttt{if}\ e\ \texttt{then}\ c_1\ \texttt{else}\ c_2 \cdot C, \rho) &\to (c_2 \cdot C, \rho) \quad \text{si } [\![e]\!]\rho = 0 \\
(\texttt{while}\ e\ \texttt{do}\ c \cdot C, \rho) &\to (c \cdot \texttt{while}\ e\ \texttt{do}\ c \cdot C, \rho) \quad \text{si } [\![e]\!]\rho \neq 0 \\
(\texttt{while}\ e\ \texttt{do}\ c \cdot C, \rho) &\to (C, \rho) \quad \text{si } [\![e]\!]\rho = 0
\end{aligned}$$

FIGURE 2 – La sémantique opérationnelle à petits pas de IMP.

Théorèmes petit pas

Déterminisme la réduction est déterministe

Progrès les seules configurations ne possédant pas de successeur sont de la forme (ε, ρ)

$$\frac{}{(x,\rho) \to_{pp} (\widehat{\dot{\rho(x)}},\rho)}\ (\texttt{Var}) \qquad \frac{(e_1,\rho) \to_{pp} (e_1',\rho)}{(e_1 \dot{+} e_2,\rho) \to_{pp} (e_1' \dot{+} e_2,\rho)}\ (+_\ell)$$

$$\frac{(e_2,\rho) \to_{pp} (e_2',\rho)}{(\dot{n} \dot{+} e_2,\rho) \to_{pp} (\dot{n} \dot{+} e_2',\rho)}\ (+_r) \qquad \frac{}{(\dot{n} \dot{+} \dot{m},\rho) \to_{pp} (\widehat{\dot{n+m}},\rho)}\ (+_{\text{fin}})$$

$$\frac{(e,\rho) \to_{pp} (e',\rho)}{(\dot{-}e,\rho) \to_{pp} (\dot{-}e',\rho)}\ (-) \qquad \frac{}{(\dot{-}\dot{n},\rho) \to_{pp} (\widehat{\dot{-n}},\rho)}\ (-_{\text{fin}})$$

FIGURE 3 – Sémantique opérationnelle à petits pas des expressions arithmétiques.

Théorèmes grand pas

Déterminisme l'arbre de dérivation d'un jugement est guidé par la syntaxe et donc unique.

Correction s'il existe une dérivation $\rho \vdash c \Downarrow \rho_\infty$ alors il existe une dérivation $(c \cdot \varepsilon, \rho) \to^* (\varepsilon, \rho_\infty)$

Adéquation s'il existe une dérivation $(c \cdot \varepsilon, \rho) \to^* (\varepsilon, \rho_\infty)$ alors il existe une dérivation $\rho \vdash c \Downarrow \rho_\infty$

1 Operational semantics

Exercise 1 : Operational semantics

Let c be a program and ρ an environment. Show the equivalence between :

1. There exists an infinite derivation of $(c \cdot \varepsilon, \rho)$
2. There exists no ρ_∞ such that $\rho \vdash c \Downarrow \rho_\infty$.

Exercise 2 :

The operational semantics defined in exercise 1 may appear artificial. Indeed, it does not describe how the expressions are calculated.

We are interested in the small step operational semantics of expressions, like in figure 3.

1. Give a proof of

$$((x \dot{+} (\dot{-}y)) \dot{+} \dot{2}, \rho[x \mapsto 3, y \mapsto 2]) \to_{pp}^* (\dot{3}, \rho[x \mapsto 3, y \mapsto 2])$$

2. State then prove the progress theorem (théorème de progrès)
3. State then prove the determinism theorem (théorème de déterminisme)
4. Show the correctness of denotational semantics
5. Show the adequacy of denotational semantics

2 Lattices and orderings

Inf-demi-treillis complet

Un *inf-demi-treillis complet* est un ensemble ordonné $(X, \leq)$ non vide tel que toute famille $F \subseteq X$ a une borne inférieure $\bigwedge F$.

Exercise 3 : Complete lattices

1. Show that a complete inf-semi-lattice (called a complete meet-semilattice) is in fact a complete lattice.
2. Show that the set of all subsets of any set A (its powerset) is a complete lattice.
3. Justify that the set of open sets $\mathcal{O}$ of a topological space $(X, \mathcal{O})$ is a complete lattice. What is the sup of a family F of open sets ? What is its inf ?

Knaster-Tarski

Soit $(X, \leq)$ un treillis complet et $f : X \to X$ une fonction monotone. Alors l'ensemble des points fixes de f est un treillis complet non vide.

Exercise 4 : A proof of Knaster-Tarski

Let f be a monotonic function from X to X where X is a complete lattice.

1. Show that f has a greatest and least fixed point.
2. Deduce that the set of fixed points is a complete lattice.

Exercise 5 : Using Knaster-Tarski

Prove the Cantor-Schröder-Bernstein theorem : if A and B are two sets such that there exist two injective functions f and g respectively from A to B and from B to A, then A is in bijection with B.

Hint (preserved in French for full effect) : faire un dessin avec deux patates, tout serait si beau si on pouvait trouver X tel que $f(X)^c$....

3 DCPOs

Rappel sur les familles dirigées

Une famille D non vide d'un ensemble $(X, \leq)$ est dirigée si et seulement si

$$\forall (x, y) \in D, \exists z \in D, z \geq x \wedge z \geq y$$

Rappels sur les DCPOs

Un DCPO est un ensemble partiellement ordonné $(X, \leq)$ tel que toute famille dirigée possède un sup. Un DCPO est *pointé* s'il existe un élément minimal.

Exercise 6 : Which is which ?

Draw the following sets and indicate which are DCPOs, which are complete lattices, which are pointed, and justify.

1. $\mathbf{1} = \{\perp\}$.
2. $\mathbf{Bool}_\perp = \{0, 1, \perp\}$ with $x < y$ if and only if $x = \perp$ and $y \neq \perp$.
3. $\mathbb{N}$ with the usual ordering.
4. $\omega + 1$ with the usual ordering.
5. $\mathbb{N}^2$ with the product ordering.
6. $\{[x, y] \mid x, y \in I, x \leq y\}$ with the ordering $\supseteq$ where $I = [0, 1]$.
7. $\{[x, y] \mid x, y \in I \cap \mathbb{Q}, x \leq y\}$ with the ordering $\supseteq$ where $I = [0, 1]$.

Exercise 7 : Knaster-Tarski VS Scott

We endow $[0,1]$ with the usual ordering which makes it DCPO and complete as a lattice.

1. Show that a monotonic function $f : [0,1] \to [0,1]$ admits a fixed point.
2. Show that if $f : [0,1] \to [0,1]$ is a Scott-continuous function then it has a fixed point. Moreover, this fixed point is the limit of the sequence $x_i \triangleq f^i(0)$.
3. Show the equivalence between the two following propositions for a monotonic function $f : [0,1] \to [0,1]$.
   — f preserves the sup
   — f is left-continuous over $[0,1]$
4. Deduce that $\sup f^i(0)$ is not always a fixed point of f by giving a counter-example.