An estimated multi-country DSGE model of endogenous growth for policy analysis.

Tom Holden[1]
Deutsche Bundesbank[2]
University of Surrey[3]

Shifu Jiang[4]
Hong Kong Monetary Authority[5]
University of Surrey[3]

08 March 2019

Abstract: *This paper builds and estimates a multi-country DSGE model with endogenous growth for policy analysis. The model features multiple countries, multiple sectors and multiple policy levers. The model features an international technological frontier, advanced by incremental process innovation, based on the work of Holden (2016; 2017). Product innovation absorbs long-run scale effects, while firm entry absorbs much of the impact of short run fluctuations in demand on process innovation. Slow international diffusion of productivity improvements is driven by complementarities between human capital and advanced technologies. Intuitively, new human capital is needed to understand new technologies. The model is estimated on data from 1870-2013 for six regions. The estimated model suggests policy can have powerful effects on the level of output over the medium term, and even some permanent effects. However, these long-run effects are swamped by the model's endogenous persistence even over forty-year horizons.*

Keywords: *medium frequency cycles, DSGE, open economy, endogenous growth, policy*
JEL Classification: *C51, E32, E62, F1, F2, F4, H2, H5, I2, J11, J24, J61, N1, O3, O4*

[1] Contact email: thomas.holden@gmail.com Web-site: http://www.tholden.org/
[2] Deutsche Bundesbank, Wilhelm-Epstein-Straße 14, 60431 Frankfurt am Main, Germany
[3] School of Economics, FASS, University of Surrey, Guildford, GU2 7XH, England
[4] Contact email: naive.j.1989@gmail.com Web-site: https://sites.google.com/view/shifujiang/home
[5] Hong Kong Monetary Authority, 55th Floor, Two International Finance Centre, 8 Finance Street, Central, Hong Kong.

The research leading to these results has received funding from The European Union's Horizon 2020 Framework Programme for Research and Innovation, under grant agreement "Modelling of the Research and Innovation Policies" (MONROE, grant no. 727114).

1. Introduction

All countries share the benefits of research. If a Chinese firm develops a new manufacturing process for solar panels, that same process can then be used simultaneously in all other countries. Even if a country does not immediately adopt the new technology, they may still benefit from cheaper solar panels. As a result, single country models are likely to overstate the home country importance of local R&D, while understating its wider benefits. These spill-overs also generates a role for policy makers in supra-national institutions such as the EU. While an increase in German R&D may not benefit Germany enough to warrant the increase, an increase in R&D from all EU countries is likely to benefit all EU countries.

There is a need then for a multi-country model of endogenous growth suitable for use in regular policy analysis. Usefulness for policy analysis imposes many restrictions on the model. It is misleading to argue for policies that may provide long-run benefits with a model that misses the channels by which they may cause short-run costs. Thus, the model must capture shorter and medium run dynamics as well as long-run effects. Furthermore, if the model struggles to explain the data, it is unlikely to provide reasonable policy prescriptions. Hence, the model must be rich enough to be estimated on all relevant data, with the model explaining differences in productivity both over time and across countries. The model must also capture the principal channels by which policy could affect outcomes. In this paper, we build a multi-country DSGE model with endogenous growth that answers these challenges.

Matching the data can be a struggle for models of endogenous growth. While US GDP per capita experiences large medium-frequency fluctuations (Comin & Gertler 2006), in the long-run, the logarithm of real US GDP per capita looks very close to a straight line. Indeed, formal statistical tests reject the null of a unit root in real US GDP per capita (Holden 2016). While this result is less clear elsewhere, this is to be expected as the US is the world's technological leader, while other countries are attempting to catch up to its trend. However, many regions do seem to share near parallel growth trends with that of the advanced developed world, as shown in Figure 1. There also appears to be low frequency correlation across countries, with, for example, the relatively high US productivity of the 1970s echoed in high output around the world. This is suggestive of a dominant role for a common component of productivity worldwide. With the US at the technological frontier, this common productivity must have similar lower frequency properties to US GDP per capita. Thus, the common component in world productivity ought to be near stationary but subject to large fluctuations at medium-frequencies. It turns out that this is particularly difficult to ensure in the presence of population growth and migration, making it crucially important that these features are allowed for in the model.[6]

Holden (2016) produces a model capable of reconciling large medium-frequency movements with long-run trend reversion, even in the presence of varying population growth. To preserve our ability to match these facts and to enable us to capture the global nature of technology, we embed the core of the Holden (2016) model as a multi-nationally owned sector within the model we present here. The Holden (2016) model removes the long-run scale effect via duplication of process innovation over industries, with the measure of industries being advanced by product innovation. Since the measure of industries cannot respond quickly to fluctuations in demand, a further margin is needed to ensure that there is not implausibly large variance at frequency zero. In the Holden (2016) model, this is entry of firms into each industry. The model generates

[6] See Holden (2016) for discussion of the effects of population growth.

medium-frequency dynamics in productivity as the returns to invention are higher in good times, and endogenously new industries end up with relatively high productivity. Thus an increase in demand leads to more relatively new industries and hence higher aggregate productivity.

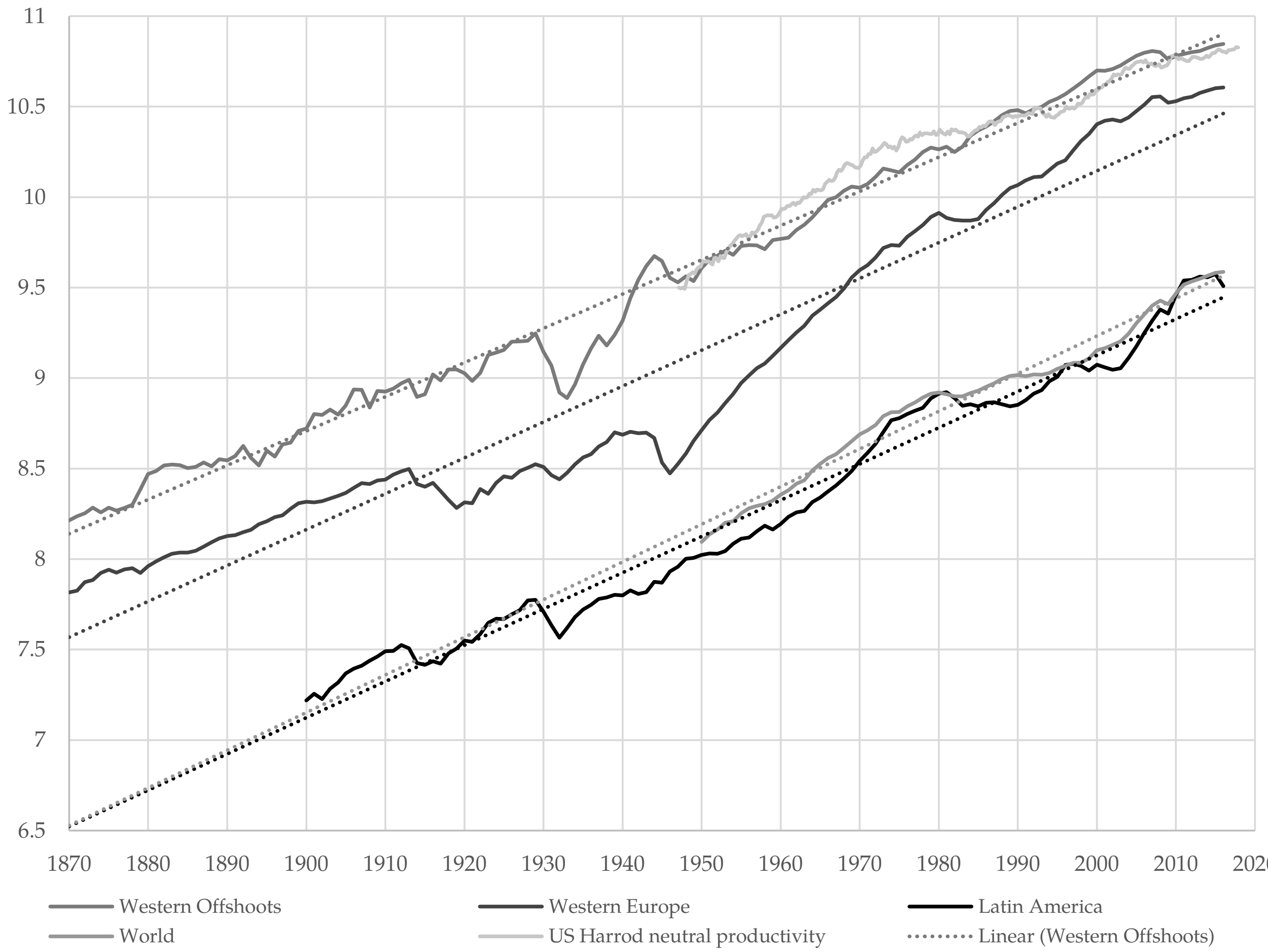

Figure 1: Log GDP per capita in Western Offshoots, Western Europe, Latin America and the World (data from the Maddison project (Inklaar et al. 2018)), Log US Harrod neutral productivity (derived from Fernald's utilisation adjusted series (Fernald 2012))

Growth within the aforementioned multi-nationally owned sector will drive long-run growth in all countries. This is preferable to modelling a different technology level in each country, with countries attempting to catch-up towards the maximum across countries, as that would introduce intractable non-differentiabilities (the maximum). Our approach also avoids the counter-factual implications of the alternative of making global technology some composite of national components: an improvement in scythe design from a third world nation should not have an impact on productivity in countries no longer using scythes in farming.

Despite technological growth coming from a multi-nationally owned sector, our model will still be able to generate substantial differences in growth in output per hour across countries, thanks to differing levels of human capital. While human capital accumulation cannot be the driver of long-run growth—a farmer with a PhD and a scythe is likely to be as productive as a scythe-bearing farmer without a PhD—new human capital does appear to be necessary to take advantage of frontier productivity growth. Indeed, broadly construed, human capital is precisely what is needed for adoption: to adopt a new technology, I must first understand it, i.e. I must have the relevant human capital. In our model, countries with below equilibrium levels of human capital can potentially experience faster growth than that of the technological leader, as they bring their human capital stock up to the necessary level to deal with advanced technologies. The gradual accumulation of human capital will be key to the model's ability to generate slow catch-up in output per hour across countries.

With trend reversion in productivity taking a decade or more, estimating on a long sample is essential to capture the dynamics of output across countries. However, over long horizons, there are permanent movements in many quantities which should be stationary according to standard models. For example, the labour share appears to be declining over time in many countries (see e.g. Elsby, Hobijn, & Sahin (2013) and Karabarbounis & Neiman (2014a; 2014b)). To enable our model to capture such dynamics, we allow for non-Cobb-Douglas production, so that movements in factor specific productivities can drive changes in expenditure shares. Further permanent movements in observables are mopped up by permanent shocks to policy instruments.

Permanent movements in policy instruments will also be key to explaining differing cross-country outcomes. A key question for any model of policy's effects on productivity is to explain the origins of the permanent differences in productivity across regions that we observe in Figure 1. The trivial explanation for this "preferences and production processes are different across countries" is both misguided and entirely useless for policy makers. Given the considerable mixing of populations we observe over long time spans, differences in the preferences of a country's people cannot persist indefinitely. Furthermore, if a factory in one country can produce a certain quantity of outputs with a certain quantity of inputs, then were that same factory replicated in another country, it should still be able to produce outputs in the same ratio to inputs. Of course, for this to hold true it is crucial that the inputs are really the same in both countries: for example, this must include the knowledge and skill of the workers and managers. While there are some fixed inputs which differ systematically across countries (climate, terrain, etc.) for modern economies these inputs are unlikely to be particularly significant. Instead, in our model differences across countries will be solely driven by permanent movements in policy instruments. This will allow an analysis of what policy makers could do to close long-run gaps in productivity.

In the next section, we outline the multi-nationally owned sector which drives growth in the model. The rest of the model is described in Section 3. Section 4 describes our estimation procedure and its results are given in Section 5. Then, in Section 6 we analyse the policy implications of our model through a number of scenarios. We conclude in Section 7.

2. The growth engine

We begin by describing the core of the model. This takes the form of a multi-national sector producing (perishable) "engineering" services, which will enter the production of other goods. The production of engineering will use a freely traded intermediate good we call "widget composites", to be defined later. It may help the reader to think of these widget composites as being labour while reading the following, however. Although our use of this engineering sector is chiefly motivated by expositional brevity and modelling tractability, the reader may like to think of engineering services as high-tech, short-lived intermediate goods such as computer chips. All of the model's growth will ultimately be driven by productivity improvements within this engineering sector. The presentation of the below will closely follow the earlier working papers, Holden (2016) and Holden (2017).

The sector producing "engineering" is multi-national in the sense that both its inputs ("widget composites") and its output (engineering) are freely tradeable across countries without transport costs. Additionally, the firms within the sector will all discount the future using the same stochastic discount factor. Thus, while the inputs to the sector will come from particular countries, and the sector's output will be used in particular countries, where these countries are will not matter for the behaviour of the firms in the sector. This multi-nationality of engineering ensures that the benefits of productivity improvements eventually spill over to all countries. Since ideas are not physical objects, and so are not subject to national borders, it is natural to model their production in a way which ensures that the technological frontier is truly global. Additionally, the multi-national setup of the engineering sector greatly enhances the model's tractability. However, we accept that national borders do limit the extent to which "engineering services", or other intermediate goods embodying technology may be transferred across countries. In section 3 we will show that transport costs in the inputs and outputs of the engineering sector may be effectively reintroduced without compromising tractability.

The engineering production sector comprises a continuum of narrow industries, each of which contains finitely many firms producing a unique product. The measure of industries is increased by the invention of new products, which start their life patent-protected. However, we assume that product inventors lack the necessary human capital to produce their product at scale themselves, and so they must licence out their patent to manufacturing firms. The duration of patent-protection is given by a geometric distribution, in line with Serrano's (2010) evidence on the large proportion of patents that are allowed to expire early, perhaps because they are challenged in court or perhaps because another new product is a close substitute. Allowing for a distribution of protection lengths also allows us to give a broader interpretation to protection within our model. Even in the absence of patent protection, the combination of contractual agreements such as NDAs, and difficulties in reverse engineering, is likely to enable the inventor of a new product to extract rents for a period.

Our model of endogenous competition within each industry is derived from Jaimovich (2007). One important departure from the Jaimovich model is that in our model, entry decisions take place one period in advance. This is natural as we wish to model research as taking place after entry but before production. Productivity within a firm is increased by performing research or appropriation. We regard process research as incremental, with regular small changes rather than the unpredictable jumps found in Schumpetarian models (Aghion & Howitt 1992; Phillips & Wrase 2006; Wälde 2005).

Throughout, we assume that only products are patentable,[7] and so by exerting effort firms are able to "appropriate" process innovations from other industries to aid in the production of their own product. This appropriation is costly since technologies for producing other products will not be directly applicable to producing a firm's own product. We assume that technology transfer within an industry is costless however, due to intra-industry labour flows and the fact that all firms in an industry are producing the same product. This is important for preserving the tractability of the model, as it means that without loss of generality we may think of all firms as just existing for two periods, in the first of which they enter and perform research, and in the second of which they produce.

The broad timing of our model is as follows. At the beginning of period t invention takes place, creating new industries. All holders of current patents (including these new inventors) then decide what level of licence fee to charge. Then, based on these licence fees and the level of overhead costs, firms choose whether to enter each industry. Next, firms perform appropriation, raising their next-period productivity towards that of the frontier, then research, further improving their productivity next period. In period $t+1$, they then produce using their newly improved production process. Meanwhile, a new batch of firms will be starting this cycle again.

We now give the detailed structure of the engineering production sector.

2.1. Aggregators

The aggregate "engineering" good is produced by a perfectly competitive industry from the aggregated output $X_t(i)$ of each industry $i \in \mathbb{I}_{t-1} \subset \mathbb{R}$, using the following Dixit-Stiglitz-Ethier (Dixit & Stiglitz 1977; Ethier 1982) style technology:

$$X_t = |\mathbb{I}_{t-1}| \left[\frac{1}{|\mathbb{I}_{t-1}|} \int_{i \in \mathbb{I}_{t-1}} X_t(i)^{\frac{1}{1+\lambda}} di\right]^{1+\lambda}$$

where $\frac{1+\lambda}{\lambda}$ is the elasticity of substitution between goods and where the exponent on the measure of industries ($|\mathbb{I}_{t-1}|$)[8] has been chosen to remove any preference for variety in consumption.[9] We normalize the price of the aggregate engineering good to 1.

Similarly, each industry aggregate good $X_t(i)$ is produced by a perfectly competitive industry from the intermediate goods $X_t(i,j)$ for $j \in \{1, \dots, J_{t-1}(i)\}$,[10] using the technology:

$$X_t(i) = J_{t-1}(i) \left[\frac{1}{J_{t-1}(i)} \sum_{j=1}^{J_{t-1}(i)} X_t(i,j)^{\frac{1}{1+\eta\lambda}}\right]^{1+\eta\lambda}$$

where $\eta \in (0,1)$ controls the degree of differentiation between firms, relative to that between industries.

2.2. Intermediate firms

2.2.1. Pricing

Firm j in industry i has access to the linear production technology $X_t(i,j) = A_t(i,j)\mathcal{W}_t^{\mathrm{P}}(i,j)$ for production in period t, where $A_t(i,j)$ is their productivity and $\mathcal{W}_t^{\mathrm{P}}(i,j)$ is their input of widget composites, which will be a composite of various capital, labour and intermediate good types. As in Jaimovich (2007),

[7] See Holden (2016) for further discussion.

[8] The $t-1$ subscript here reflects the fact that industries are invented one period before their product is available to consumers.

[9] Incorporating a preference for variety would not change the long-run stability of our model. However, it appears to be counter-factual. See Holden (2016).

[10] Again, the $t-1$ subscript reflects the fact that firms enter one period before production.

strategic profit maximisation then implies that in a symmetric equilibrium, the price of the good in industry i is given by $P_t(i) = \left(1+\mu_{t-1}(i)\right)\frac{P_t^{\mathrm{W}}}{A_t(i,j)} = \left(1+\mu_{t-1}(i)\right)\frac{P_t^{\mathrm{W}}}{A_t(i)}$, where $\mu_t(i) := \lambda\frac{\eta J_t(i)}{J_t(i)-(1-\eta)} \in (\eta\lambda,\lambda]$ is the industry i mark-up in period $t+1$, P_t^{W} is the wage in the engineering sector and $A_t(i) = A_t(i,j)$ is the productivity shared by all firms in industry i in symmetric equilibrium. From aggregating across industries, we then have that $P_t^{\mathrm{W}} = \frac{A_t}{1+\mu_{t-1}}$ where $\frac{1}{1+\mu_t} = \left[\frac{1}{|\mathbb{I}_t|}\int_{i\in\mathbb{I}_t}\left[\frac{1}{1+\mu_t(i)}\right]^{\frac{1}{\lambda}}di\right]^{\lambda}$ determines the aggregate mark-up μ_{t-1} and where:

$$A_t := \frac{\left[\frac{1}{|\mathbb{I}_{t-1}|}\int_{i\in\mathbb{I}_{t-1}}\left[\frac{A_t(i)}{1+\mu_{t-1}(i)}\right]^{\frac{1}{\lambda}}di\right]^{\lambda}}{\left[\frac{1}{|\mathbb{I}_{t-1}|}\int_{i\in\mathbb{I}_{t-1}}\left[\frac{1}{1+\mu_{t-1}(i)}\right]^{\frac{1}{\lambda}}di\right]^{\lambda}}$$

is a measure of the aggregate productivity level.[11]

2.2.2. Sunk costs: rents, appropriation and research

Following Jaimovich (2007), we assume that the number of firms in an industry is pinned down by the zero profit condition that equates pre-production costs to production period profits. Firms raise equity in order to cover these upfront costs, which come from four sources.

Firstly, firms must pay a fixed operating cost of $\mathcal{W}^{\mathrm{F}}$ units of widget composites that covers things such as bureaucracy, human resources, facility maintenance, training, advertising, shop set-up and capital installation/creation. Asymptotically, the level of fixed costs will not matter, but including it here will help in our explanation of the importance of patent protection for long run growth.

Secondly, if the product produced by industry i is currently patent-protected, then firms must pay a rent of $\mathcal{R}_t(i)$ engineering to the patent-holder for the right to produce in their industry. Since all other sunk costs are paid in units of widget composites, for convenience we define $\mathcal{W}_t^{\mathcal{R}}(i) := \frac{\mathcal{R}_t(i)}{P_t^{\mathcal{W}}}$, i.e. the widget composite amount equivalent in cost to the rent.

Thirdly, firms will expand effort on appropriating the previous process innovations of the leading industry. We define the level of the leading technology within industry i by $A_t^*(i) := \max_{j\in\{1,\dots,J_{t-1}(i)\}} A_t(i,j)$ and the level of the best technology anywhere by $A_t^* := \sup_{i\in\mathbb{I}_{t-1}} A_t^*(i)$. Due to free in-industry transfer, even without exerting any appropriation effort, firms in industry i may start their research from $A_t^*(i)$ in period t. By employing appropriation workers, a firm may raise this level towards A_t^*.

We write $A_t^{**}(i,j)$ for the base from which firm $j \in \{1,\dots,J_t(i)\}$ will start research in period t. This base is given by the output of the appropriation process, the returns of which we assume to take the form:

$$A_t^{**}(i,j) = \left[A_t^*(i)^\tau + \left(A_t^{*\tau} - A_t^*(i)^\tau\right)\frac{\mathcal{F}_t^{\mathrm{A}}(i,j)}{1+\mathcal{F}_t^{\mathrm{A}}(i,j)}\right]^{\frac{1}{\tau}}, \tag{1}$$

where $\tau > 0$ controls whether the catch-up amount is a proportion of the technology difference in levels ($\tau = 1$), log-levels ($\tau = 0$) or anything in between or beyond, and where $\mathcal{F}_t^{\mathrm{A}}(i,j)$ is the effective input to appropriation. This in turn is given by $\mathcal{F}_t^{\mathrm{A}}(i,j) := E_t^{\mathrm{A}}(i)\mathcal{W}_t^{\mathrm{A}}(i,j)$, where $\mathcal{W}_t^{\mathrm{A}}(i,j)$ is the quantity of widget

[11] Due to the non-linear aggregation, it will not generically be the case that aggregate output is aggregate labour input times A_t. However, the aggregation chosen here is the unique one under which aggregate mark-ups are known one period in advance, as industry mark-ups are.

composites that the firm devotes to appropriation in period t, and $E_t^{\mathrm{A}}(i) := A_t^*(i)^{-\zeta^{\mathrm{A1}}} A_t^{*\zeta^{\mathrm{A2}}} |\mathbb{I}_t|^{\phi^{\mathrm{A}}} \Psi^{\mathrm{A}}$, where Ψ^{A} is the productivity of widget composites used for appropriation, $\zeta^{\mathrm{A1}} > 0$ controls the extent to which appropriation is getting harder over time due to the increased complexity of later technologies, $\zeta^{\mathrm{A2}} \geq 0$ gives the strength of the spillover from process innovation to appropriation, and where $\phi^{\mathrm{A}} \geq 0$ gives the strength of the spillover from product innovation to process appropriation.

This specification captures the key idea that the further a firm is behind the frontier, the more productive will be appropriation. Allowing for spillovers from product and process innovation is essential to demonstrate that in our model endogenous growth is not a knife-edge result, unlike in that of Li (2000). Finally, allowing for appropriation (and research, and invention) to get harder over time is both realistic, and essential for the tractability of our model, since it will lead our model to have a finite dimensional state vector asymptotically, despite all the heterogeneity across industries. In order for this to be the case, we assume that ϕ^{A} is small enough that $A_t^{*-\zeta^{\mathrm{A}}} |\mathbb{I}_t|^{\phi^{\mathrm{A}}} \to 0$ as $t \to \infty$, where $\zeta^{\mathrm{A}} = \zeta^{\mathrm{A1}} - \zeta^{\mathrm{A2}} > 0$.

Fourthly, firms will perform research. If firm $j \in \{1, \dots, J_t(i)\}$ has an effective research input of $\mathcal{F}_t^{\mathrm{R}}(i,j)$, then we assume that its productivity level in period $t+1$ will be given by:

$$A_{t+1}(i,j) = A_t^{**}(i,j)\left(1 + \gamma Z_{t+1}(i,j)\mathcal{F}_t^{\mathrm{R}}(i,j)\right)^{\frac{1}{\gamma}}, \tag{2}$$

where $\gamma > 0$ controls the "parallelizability" of research and $Z_{t+1}(i,j) > 0$ is a stationary stochastic process representing the luck component of research, with $\mathbb{E}Z_t \equiv 1$.[12] If $\gamma = 1$, research may be perfectly parallelized, so arbitrarily large quantities may be performed within a given period without loss of productivity, but if γ is large, then, in line with the evidence of Siliverstovs and Kancs (2012), the returns to research decline as the firm attempts to pack more into one period. Much as with appropriation, the effective input to research is given by $\mathcal{F}_t^{\mathrm{R}}(i,j) := E_t^{\mathrm{R}}(i)\mathcal{W}_t^{\mathrm{R}}(i,j)$, where $\mathcal{W}_t^{\mathrm{R}}(i,j)$ is the quantity of widget composites the firm employs in research in period t, and $E_t^{\mathrm{R}}(i) := A_t^{**}(i)^{-\zeta^{\mathrm{R1}}} A_t^{*\zeta^{\mathrm{R2}}} |\mathbb{I}_t|^{\phi^{\mathrm{R}}} \Psi^{\mathrm{R}}$, where Ψ^{R} is the productivity of widget composites used in research, ζ^{R1} controls the extent to which research is getting harder over time, $\zeta^{\mathrm{R2}} \geq 0$ gives the spillover from frontier process innovation and $\phi^{\mathrm{R}} \geq 0$ gives the spillover from product innovation to process. We again assume that ϕ^{R} is small enough that $A_t^{*-\zeta^{\mathrm{R}}} |\mathbb{I}_t|^{\phi^{\mathrm{R}}} \to 0$ as $t \to \infty$, where $\zeta^{\mathrm{R}} = \zeta^{\mathrm{R1}} - \zeta^{\mathrm{R2}} > 0$. We also assume that $\zeta^{\mathrm{R1}} > \zeta^{\mathrm{A1}}$, $\zeta^{\mathrm{R2}} \leq \zeta^{\mathrm{A2}}$ and $\phi^{\mathrm{R}} \leq \phi^{\mathrm{A}}$ implying that the difficulty of research is increasing over time faster than the difficulty of appropriation. This is made because research is very much specific to the industry in which it is being conducted, whereas appropriation is a similar task across all industries attempting to appropriate the same technology, and hence is more likely to have been standardised, or to benefit from other positive spillovers.

In the following, for tractability, we will assume that $Z_t(i,j) := Z_t$ so that all firms receive the same "idea" shock. We make this assumption chiefly for simplicity, but it may be justified by appeal to common inputs to private research, such as university research output or the availability of new tools, or by appeal to in-period labour market movements carrying ideas with them. Holden (2016) shows that allowing for industry-specific shocks has minimal impact on our results, providing shocks are bounded above.

[12] Peretto (1999) also looks at research that drives incremental improvements in productivity, and chooses a similar specification. The particular one used here is inspired by Groth, Koch, and Steger (2009).

2.2.3. Research and appropriation effort decisions

Firms are owned by households worldwide and so they choose research and appropriation to maximize:

$$\mathbb{E}_t\left[\Xi_{t,t+1}\left(P_t(i,j)-\frac{P_{t+1}^{\mathcal{W}}}{A_{t+1}(i,j)}\right)X_t(i,j)\right]-\left[\mathcal{W}_t^R(i,j)+\mathcal{W}_t^A(i,j)+\mathcal{W}_t^{\mathcal{R}}(i)+\mathcal{W}^F\right]P_t^{\mathcal{W}},$$

where $\Xi_{t,t+1}$ is the international stochastic discount factor from period t to $t+1$. (A common stochastic discount factor across countries exists due to complete international markets.) It may be shown that, for firms in frontier industries (those for which $A_t^*(i)=A_t^*$), if an equilibrium exists, then it is unique and symmetric within an industry; but we cannot rule out the possibility of asymmetric equilibria more generally.[13] However, since the coordination requirements of asymmetric equilibria render them somewhat implausible, we restrict ourselves to the unique equilibrium in which all firms within an industry choose the same levels of research and appropriation.

Then, providing $\frac{1}{\mu_t(i)}<\min\{\gamma,\tau\}$ and $\gamma>\zeta^{\mathrm{R}1}$ (for the second order conditions[14] and for uniqueness), combining the first order and free entry conditions then gives us that, in the limit as $\operatorname{var} Z_{t+1}\to 0$:[15]

$$\mathcal{F}_t^{\mathrm{R}}(i)=\max\left\{0,\frac{d_t(i)E_t^{\mathrm{R}}(i)\left(\mathcal{W}_t^{\mathrm{A}}(i,j)+\mathcal{W}_t^{\mathcal{R}}(i)+\mathcal{W}_t^{\mathrm{F}}\right)-\mu_t(i)}{\gamma\mu_t(i)-d_t(i)}\right\} \tag{3}$$

and:

$$\mathcal{F}_t^{\mathrm{A}}(i)=\max\left\{0,-f_t(i)+\sqrt{\max\left\{0,f_t(i)^2+g_t(i)\right\}}\right\}, \tag{4}$$

where $d_t(i)\in(0,1)$[16] is small when firm behaviour is highly distorted by firms' incentives to deviate from choosing the same price as the other firms in their industry, off the equilibrium path (so $d_t(i)\to 1$ as $J_t(i)\to\infty$), and $f_t(i)$ and $g_t(i)$ are increasing in an industry's distance from the frontier,[17] as the further behind a firm is, the greater are the returns to appropriation.

Equations (3) and (4) mean that research and appropriation levels are increasing in the other sunk costs a firm must pay prior to production, but decreasing in mark-ups. They also mean that the strategic distortions caused by there being a small number of firms within an industry tend to reduce research and appropriation levels. Other sunk costs matter for research levels because when other sunk costs are high, entry into the industry is lower, meaning that each firm receives a greater slice of production-period profits, and so has correspondingly amplified research incentives.

Why mark-up increases decrease research incentives is clearest when those mark-up increases are driven by exogenous decreases in the elasticity of substitution. When products are close substitutes, then by performing research (and cutting its price) a firm may significantly expand its market-share, something that will not happen when the firm's good is a poor substitute for its rivals. When $d_t(i)\approx 1$ (i.e. there are a lot of firms in the industry) firms act as if they faced an exogenous elasticity of substitution $\frac{1+\mu_t(i)}{\mu_t(i)}$, and so when mark-ups are high they will want to perform little research. When $d_t(i)$ is small (i.e. there are only a few

---

[13] See Holden (2016) for further discussion.

[14] See Holden (2016) for further details and discussion.

[15] The first order and zero profit conditions are the same as those in Holden (2016), and the reader is referred to that paper for details. We do not assume $\sigma_Z=0$ when simulating, but it leads here to expressions that are easier to interpret. The full set of FOCs is also contained in the appendix.

[16] $d_t(i):=1-\frac{\omega_t(i)}{1+\omega_t(i)}\frac{(\lambda-\mu_t(i))(\mu_t(i)-\eta\lambda)}{\lambda(1-\eta)\mu_t(i)}$, where $\omega_t(i):=\frac{J_t(i)(1-\eta)}{(J_t(i)-(1-\eta))^2(1+\mu_t(i))}$.

[17] $f_t(i):=\frac{1}{2}\left[1+\frac{d_t(i)}{\tau\mu_t(i)}\frac{1+(\gamma-\zeta^{\mathrm{R}})\mathcal{L}_t^{\mathrm{R}}(i)}{1+\gamma\mathcal{L}_t^{\mathrm{R}}(i)}\right]\left[1-\left(\frac{A_t^*(i)}{A_t^*}\right)^{\tau}\right]-1$, $g_t(i):=\frac{d_t(i)}{\tau\mu_t(i)}\frac{1+(\gamma-\zeta^{\mathrm{R}})\mathcal{L}_t^{\mathrm{R}}(i)}{1+\gamma\mathcal{L}_t^{\mathrm{R}}(i)}E_t^{\mathrm{A}}(i)\left[L_t^{\mathrm{R}}(i)+L_t^{\mathcal{R}}(i)+L^{\mathrm{F}}\right]\left[1-\left(\frac{A_t^*(i)}{A_t^*}\right)^{\tau}\right]-\left(\frac{A_t^*(i)}{A_t^*}\right)^{\tau}$.

firms[18]) then firms' behaviour is distorted by strategic considerations. Each firm realises that if they perform extra research today then their competitors will accept lower mark-ups the next period. This reduces the extent to which research allows market-share expansion, depressing research incentives.

The key thing to note about (3) and (4) is that research and appropriation are independent of the level of demand, except insomuch as demand affects mark-ups or the level of strategic distortion. This is because when demand is high there is greater entry, so each firm still faces roughly the same demand. This is essential for removing the short-run scale effect.

In industries that are no longer patent-protected, rents will be zero (i.e. $\mathcal{W}_t^{\mathcal{R}}(i) \equiv 0$). Since research is getting harder at a faster rate than appropriation ($\zeta^{\mathrm{R1}} > \zeta^{\mathrm{A1}}$, $\zeta^{\mathrm{R2}} \le \zeta^{\mathrm{A2}}$, $\phi^{\mathrm{R}} \le \phi^{\mathrm{A}}$), at least asymptotically, no research will be performed in these industries. This is because $d_t(i) E_t^{\mathrm{R}}(i) [\mathcal{W}_t^{\mathrm{A}}(i) + \mathcal{W}^{\mathrm{F}}] - \mu_t(i)$ is asymptotically negative as $\mu_t(i) \in (\eta\lambda, \lambda]$. For growth to continue forever in the absence of patent protection, we would require that the overhead cost ($\mathcal{W}^{\mathrm{F}}$) was growing over time at exactly the right rate to offset the increasing difficulty of research. This does not seem particularly plausible. However, it will turn out that optimal patent rents grow at exactly this rate, so with patent protection we will be able to sustain long run growth even when overhead costs are asymptotically dominated by the costs of research.

2.3. Inventors and patent protection

Each new industry is controlled by an inventor who owns the international patent rights to the product the industry produces. Until the inventor's product goes on sale, they can successfully protect their revenue stream through contractual arrangements, such as non-disclosure agreements. This means that even in the absence of patent-protection, an inventor will receive one period of revenues. They then have a probability of $1 - q$ of being granted a patent to enable them to extract rents for a second period. After this, if they have a patent at t, then they face a constant probability of $1 - q$ of having a patent at $t + 1$.

We suppose, however, that inventors lack the necessary human capital to produce their product at scale themselves. Instead, in the period the product is invented, and in each subsequent period in which they have a patent, the inventor optimally chooses the rent $\mathcal{R}_t(i)$ (or equivalently $\mathcal{W}_t^{\mathcal{R}}(i)$) to charge all the firms that wish to produce their product.

The reader should have a firm such as Apple in mind when thinking about these inventors. Apple has no manufacturing plants and instead maintains its profits by product innovation and tough bargaining with suppliers.

[18] The minimum value of $d_t(i)$ occurs when there is more than one firm in the industry. If there is a single firm in an industry, then, as you would expect, very little research will be performed (because the firm's only incentive to cut prices comes from competition from other industries, competition which is very weak, since those industries are producing poor substitutes to its own good). However, this drop in research incentives works entirely through the mark-up channel, and $d_t(i) \to 1$ as $J_t(i) \to 1$. One intuition for this is that there can be no strategic behaviour when there is only a single firm.

2.3.1. Optimal rent decisions

Inventors' businesses are also owned by households worldwide; hence, an inventor's problem is to choose $\mathcal{W}_{t+s}^{\mathcal{R}}(i)$ for $s \in \mathbb{N}$ to maximise their expected present value, subject to an enforceability constraint on rents. Their period t present value is given by:

$$V_t^{\mathbb{I}}(i) := \mathbb{E}_t \sum_{s=0}^{\infty} (1-q)^s \left[\prod_{k=1}^{s} \Xi_{t+k-1t+k} (1-\delta_{\mathbb{I},t+k}) \right] \mathcal{W}_{t+s}^{\mathcal{R}}(i) P_{t+s}^{\mathcal{W}} J_{t+s}(i),$$

where $\delta_{\mathbb{I},t}$ is the rate at which products drop out of the engineering aggregator, due to changing tastes, or the invention of replacement products.[19] In line with this intuition, we assume that $\delta_{\mathbb{I},t} = \delta_{\mathbb{I}} \left(\frac{|\mathbb{I}_t|}{G_{\mathbb{I}} |\mathbb{I}_{t-1}|} \right)^{\psi} \tilde{\delta}_{\mathbb{I},t}$, where $\psi > 0$ gives the elasticity of product depreciation with respect to product growth, $G_{\mathbb{I}}$ is the value of $\lim_{t\to\infty} \frac{|\mathbb{I}_t|}{|\mathbb{I}_{t-1}|}$ in the non-stochastic limit of the model, and $\tilde{\delta}_{\mathbb{I},t}$ is a stationary stochastic process, with $\mathbb{E}\tilde{\delta}_{\mathbb{I},t} \equiv 1$. We also assume that the random event of losing patent protection is independent of that of product depreciation, so some (but not all) products will drop out of the aggregator without ever going out of patent protection. We define $\mathbb{P}_t$ to be the set of patent protected industries. The above logic implies that:

$$|\mathbb{P}_t| - (1-\delta_{\mathbb{I},t})(1-q)|\mathbb{P}_{t-1}| = |\mathbb{I}_t| - (1-\delta_{\mathbb{I},t})|\mathbb{I}_{t-1}|.$$

If the rents charged by a patent-holder go too high, a firm is likely to ignore them completely in the hope that either they will be lucky, and escape having their profits confiscated from them by the courts (since proving patent infringement is often difficult), or that the courts will award damages less than the licence fee. See Holden (2016) for a full discussion of the plausibility of this. As shown by that paper, this leads patent holders and firms to bargain over rents, leading patent-holders to set:

$$\mathcal{W}_t^{\mathcal{R}}(i) = \frac{1-\rho}{\rho} \left[\mathcal{W}_t^{\mathrm{R}}(i) + \mathcal{W}_t^{\mathrm{A}}(i) + \mathcal{W}^{\mathrm{F}} \right], \tag{5}$$

at least for sufficiently large t, where $\rho \in (0,1)$ is the bargaining power of the firm, in the sense of the generalized Nash bargaining solution. The simple form of this expression comes from the fact that a firm's production period revenues (which is what is being bargained over) are precisely equal to the costs they face prior to production, thanks to the free entry condition. A full description of the legally motivated bargaining process is contained in Holden (2016), along with a discussion of some technical complications pertaining to off equilibrium play.

From combining (3) and (5) then, at least for sufficiently large t, in the limit as $\operatorname{var} Z_{t+1} \to 0$, we have that:

$$\mathcal{F}_t^{\mathrm{R}}(i) = \frac{\rho \mu_t(i) - d_t(i) E_t^{\mathrm{R}}(i) \left(\mathcal{W}_t^{\mathrm{A}}(i) + \mathcal{W}^{\mathrm{F}} \right)}{d_t(i) - \gamma \rho \mu_t(i)}. \tag{6}$$

For there to be growth in the long run then, we now just require that $d_t(i) > \gamma \rho \mu_t(i)$, which together with the second order and appropriation uniqueness conditions implies $\gamma\rho < \frac{1}{\mu_t(i)} < \min\{\gamma, \tau\}$.[20] We see that, once optimal rents are allowed for, research is no longer decreasing in mark-ups within an industry, at least for firms at the frontier. Mathematically, this is because the patent-holder sets rents as such a steeply increasing function of research levels. More intuitively, you may think of the patent-holder as effectively controlling

[19] This means that were it not for the invention of new products, there would be a measure $(1-\delta_{I,t})|I_{t-1}|$ of industries producing in period $t+1$.

[20] If the number of firms in protected industries is growing over time then $d_t(i) \to 1$, so asymptotically these conditions are equivalent.

how much research is performed by firms in their industry, and as taking most of the rewards from this research. It is then unsurprising that we reach these Schumpeterian conclusions.[21]

2.3.2. Invention

We consider invention as a costly process undertaken by inventors until the expected returns from inventing a new product fall to zero. New products appear at the end of the product spectrum, and are contained in the set of products aggregated into the engineering good until they are hit with the product depreciation shock. Therefore, the product index i will always refer to the same product, once it has been invented.

There is, however, no reason to think that newly invented products will start with a competitive production process. A newly invented product may be thought of as akin to a prototype: yes, identical prototypes could be produced by the same method, but doing this is highly unlikely to be commercially viable. Instead, there will be rapid investment in improving the product's production process until it may be produced as efficiently as its rivals can be. In our model, this investment in the production process is performed not by the inventor but by the manufacturers. Prototyping technology has certainly improved over time;[22] in light of this, we assume that a new product i is invented with a production process of level $A_t^*(i) = S_t A_t^*$, where $S_t \in (0,1)$ is an exogenous process controlling initial relative productivity. We assume that either S_t is stationary, so prototyping technology is keeping up with frontier productivity, or that S_t has a negative growth rate asymptotically, so firms start progressively further behind over time.

Just as we expect process research to be getting harder over time, as all the obvious process innovations have already been discovered, so too we may expect product invention to be getting harder over time, as all the obvious products have already been invented. In addition, the necessity of actually finding a way to produce a prototype will result in the cost of product invention increasing in $A_t^*(i)$, the initial productivity level of the process for producing the new product. Finally, following Li (2000), we allow for spillovers from process innovation to product innovation. As a result of these considerations, we assume that the widget composite cost is given by $\frac{\mathcal{F}_t^{\mathrm{I}}}{E_t^{\mathrm{I}}}$, where $\mathcal{F}_t^{\mathrm{I}}$ is a stationary stochastic process determining the difficulty of invention, with $\Pr\left(\mathcal{F}_t^{\mathrm{I}} > \underline{\mathcal{F}^{\mathrm{I}}}\right) \equiv 1$ and $\mathbb{E}\mathcal{F}_t^{\mathrm{I}} \equiv \mathcal{F}^{\mathrm{I}} > \underline{\mathcal{F}^{\mathrm{I}}}$, and where $E_t^{\mathrm{I}} := (S_t A_t^*)^{-\zeta^{\mathrm{I1}}} A_t^{*\zeta^{\mathrm{I2}}} |\mathbb{I}_{t-1}|^{-\phi^{\mathrm{I}}}$, with $\zeta^{\mathrm{I2}} \geq 0$ giving the strength of spillovers from process innovation and $\phi^{\mathrm{I}} \geq 0$ and $\zeta^{\mathrm{I1}} > \zeta^{\mathrm{I2}}$ controlling the rate at which inventing a new product gets more difficult because of, respectively, an increased number of existing products or an increased level of initial productivity.

The entry and exit of products is limited by two factors. Firstly, products cannot be uninvented, and secondly, thanks to free entry, there cannot be positive returns to entering. Combining these with incentive compatibility gives the following complementary slackness type condition:

$$\min\left\{|\mathbb{P}_t| - (1-q)(1-\delta_{\mathbb{I},t})|\mathbb{P}_{t-1}|, \frac{\mathcal{F}_t^{\mathrm{I}}}{E_t^{\mathrm{I}}} P_t^{(\omega)} - V_t^{\mathbb{I}}(\sup \mathbb{I}_t)\right\} = 0.$$

[21] The empirical evidence (Scott 1984; Levin, Cohen & Mowery 1985; Aghion et al. 2005; Tingvall & Poldahl 2006) suggests that the cross-industry relationship between competition and research takes the form of an inverted-U. See Holden (2016) for a discussion of how this might be generated in our model.

[22] Examples of recent technologies that have raised the efficiency of prototype production include 3D printing and computer scripting languages such as Python.

When the $|\mathbb{P}_t| \geq (1-q)(1-\delta_{\mathbb{I},t})|\mathbb{P}_{t-1}|$ constraint binds, then the measure of firms will have to adjust instead, meaning there could be an asymmetry in the response of mark-ups to certain shocks.

2.3.3. The life cycle of an industry

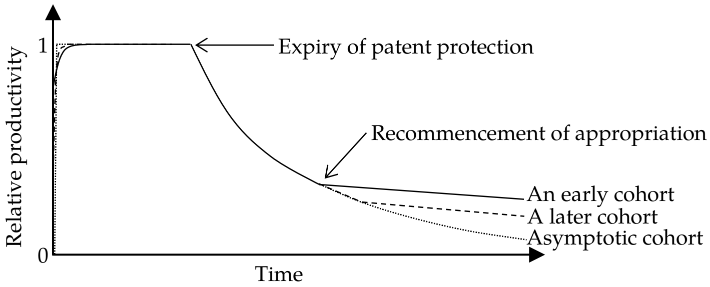

Figure 2: The stylized life cycle of an industry

We are now close to being able to describe the life cycle of an industry in our model. Industries start out with productivity behind that of the frontier, but thanks to the high cost of entry to patent-protected industries, they have strong incentives to catch up to it via appropriation. Once they reach it, thanks to the rents charged by patent holders they will perform research pushing the frontier forward over time. Now, some industries will be hit with the "product-depreciation" shock prior to going out of patent protection, but others will fall out of patent protection first. We have already shown that these industries will not perform research asymptotically, and in Holden (2016) we show that their appropriation choice implies that the relative productivity of non-protected industries is tending to zero asymptotically. This is empirically plausible since productivity differences across industries have been steadily increasing over time,[23] and it is important for the tractability of our model since it enables us to focus on the asymptotic case in which non-protected firms never perform appropriation. It is also in line with the long delays in the diffusion of technology found by Mansfield (1993) amongst others.

Armed with this knowledge we can depict the full lifecycle of industries from different cohorts. We do this in Figure 2.

2.4. Market clearing

The engineering sector is closed with the widget composite market clearing condition:

$$\mathcal{W}_t = \frac{\mathcal{F}_t^{\mathrm{I}}}{E_t^{\mathrm{I}}}\left[|\mathbb{P}_t| - (1-q)(1-\delta_{\mathbb{I},t})|\mathbb{P}_{t-1}|\right] + \int_{i\in\mathbb{I}_t} \left(\mathcal{W}_t^{\mathrm{R}}(i) + \mathcal{W}_t^{\mathrm{A}}(i) + \mathcal{W}^{\mathrm{F}}\right) J_t(i)\, di$$

$$+ P_t^{\mathcal{W}^{-\frac{1+\lambda}{\lambda}}} \frac{X_t}{|\mathbb{I}_{t-1}|}\int_{i\in\mathbb{I}_{t-1}} \frac{1}{A_t(i)}\left(\frac{A_t(i)}{1+\mu_{t-1}(i)}\right)^{\frac{1+\lambda}{\lambda}} di.$$

As in Holden (2016), as $t \to \infty$ the model will converge to stationarity. Under these asymptotics the previous equation implies:

$$\mathcal{W}_t = \frac{\mathcal{F}_t^{\mathrm{I}}}{E_t^{\mathrm{I}}}\left[|\mathbb{P}_t| - (1-q)(1-\delta_{\mathbb{I},t})|\mathbb{P}_{t-1}|\right] + |\mathbb{P}_t|\mathcal{W}_t^{\mathrm{RP}} J_t^{\mathrm{P}} + X_t P_t^{\mathcal{W}^{-\frac{1+\lambda}{\lambda}}} \frac{1}{|\mathbb{I}_{t-1}|}\int_{i\in\mathbb{I}_{t-1}} \frac{1}{A_t(i)}\left(\frac{A_t(i)}{1+\mu_{t-1}(i)}\right)^{\frac{1+\lambda}{\lambda}} di,$$

[23] Some indirect evidence for this is provided by the increase in wage inequality, documented in e.g. Autor, Katz, and Kearney (2008). Further evidence is provided by the much higher productivity growth rates experienced in manufacturing, compared to those in services (mostly unpatented and unpatentable), documented in e.g. Duarte and Restuccia (2009).

where variables with superscript P refer to the value in patent protected industries. Now, if we define the aggregate productivity of non-protected industries for period t production by:

$$A^{\mathrm{N}}_{t-1} := \left[\frac{1}{|\mathbb{I}_{t-1}|-|\mathbb{P}_{t-1}|}\int_{i\in\mathbb{I}_{t-1}\setminus\mathbb{P}_{t-1}} A_t(i)^{\frac{1}{\lambda}}\,d\mathrm{i}\right]^{\lambda} = \left[\frac{1}{|\mathbb{I}_{t-1}|-|\mathbb{P}_{t-1}|}\int_{i\in\mathbb{I}_{t-1}\setminus\mathbb{P}_{t-1}} A_{t-1}(i)^{\frac{1}{\lambda}}\,d\mathrm{i}\right]^{\lambda},$$

(as non-protected industries have constant productivity), then:

$$\frac{1}{|\mathbb{I}_{t-1}|}\int_{i\in\mathbb{I}_{t-1}}\frac{1}{A_t(i)}\left(\frac{A_t(i)}{1+\mu_{t-1}(i)}\right)^{\frac{1+\lambda}{\lambda}}di = \frac{1}{|\mathbb{I}_{t-1}|}\left[\left(\frac{1}{1+\eta\lambda}\right)^{\frac{1+\lambda}{\lambda}}\int_{i\in\mathbb{I}_{t-1}\setminus\mathbb{P}_{t-1}} A_t(i)^{\frac{1}{\lambda}}\,di + |\mathbb{P}_{t-1}|\frac{1}{A^*_t}\left(\frac{A^*_t}{1+\mu^{\mathrm{P}}_{t-1}}\right)^{\frac{1+\lambda}{\lambda}}\right]$$

$$= \frac{|\mathbb{I}_{t-1}|-|\mathbb{P}_{t-1}|}{|\mathbb{I}_{t-1}|}{A^{\mathrm{N}}_{t-1}}^{\frac{1}{\lambda}}\left(\frac{1}{1+\eta\lambda}\right)^{\frac{1+\lambda}{\lambda}} + \frac{|\mathbb{P}_{t-1}|}{|\mathbb{I}_{t-1}|}{A^*_t}^{\frac{1}{\lambda}}\left(\frac{1}{1+\mu^{\mathrm{P}}_{t-1}}\right)^{\frac{1+\lambda}{\lambda}},$$

so the widget composite market clearing condition implies:

$$\begin{aligned}\mathcal{W}_t = {}& \frac{\mathcal{F}^{\mathrm{I}}_t}{E^{\mathrm{I}}_t}\left[|\mathbb{P}_t| - (1-q)(1-\delta_{\mathbb{I},t})|\mathbb{P}_{t-1}|\right] + |\mathbb{P}_t|\mathcal{W}^{\mathrm{RP}}_t J^{\mathrm{P}}_t \\ & + X_t {P^{\mathcal{W}}_t}^{-\frac{1+\lambda}{\lambda}}\left[\frac{|\mathbb{I}_{t-1}|-|\mathbb{P}_{t-1}|}{|\mathbb{I}_{t-1}|}{A^{\mathrm{N}}_{t-1}}^{\frac{1}{\lambda}}\left(\frac{1}{1+\eta\lambda}\right)^{\frac{1+\lambda}{\lambda}} + \frac{|\mathbb{P}_{t-1}|}{|\mathbb{I}_{t-1}|}{A^*_t}^{\frac{1}{\lambda}}\left(\frac{1}{1+\mu^{\mathrm{P}}_{t-1}}\right)^{\frac{1+\lambda}{\lambda}}\right].\end{aligned}$$

Furthermore, since non-protected industries have constant productivity:

$$\begin{aligned}A^{\mathrm{N}}_t &= \left[\frac{1}{|\mathbb{I}_t|-|\mathbb{P}_t|}\left[\int_{i\in(\mathbb{I}_t\setminus\mathbb{P}_t)\setminus(\mathbb{I}_{t-1}\setminus\mathbb{P}_{t-1})} {A^*_t}^{\frac{1}{\lambda}}\,d\mathrm{i} + (1-\delta_{\mathbb{I},t})\int_{i\in\mathbb{I}_{t-1}\setminus\mathbb{P}_{t-1}} A_{t-1}(i)^{\frac{1}{\lambda}}\,d\mathrm{i}\right]\right]^{\lambda} \\ &= \left[\frac{(|\mathbb{I}_t|-|\mathbb{P}_t|)-(1-\delta_{\mathbb{I},t})(|\mathbb{I}_{t-1}|-|\mathbb{P}_{t-1}|)}{|\mathbb{I}_t|-|\mathbb{P}_t|}{A^*_t}^{\frac{1}{\lambda}} + \frac{(1-\delta_{\mathbb{I},t})(|\mathbb{I}_{t-1}|-|\mathbb{P}_{t-1}|)}{|\mathbb{I}_t|-|\mathbb{P}_t|}{A^{\mathrm{N}}_{t-1}}^{\frac{1}{\lambda}}\right]^{\lambda}.\end{aligned}$$

Using A^{N}_t, we can also simplify the productivity aggregation condition giving:

$$P^{\mathcal{W}}_t = \left[\frac{1}{|\mathbb{I}_{t-1}|}\int_{\mathrm{i}\in\mathbb{I}_{t-1}}\left[\frac{A_t(i)}{1+\mu_{t-1}(\mathrm{i})}\right]^{\frac{1}{\lambda}}d\mathrm{i}\right]^{\lambda} = \left[\frac{|\mathbb{I}_{t-1}|-|\mathbb{P}_{t-1}|}{|\mathbb{I}_{t-1}|}\left(\frac{A^{\mathrm{N}}_{t-1}}{1+\eta\lambda}\right)^{\frac{1}{\lambda}} + \frac{|\mathbb{P}_{t-1}|}{|\mathbb{I}_{t-1}|}\left(\frac{A^*_t}{1+\mu^{\mathrm{P}}_{t-1}}\right)^{\frac{1}{\lambda}}\right]^{\lambda}.$$

Along similar lines, we have that:

$$\frac{1}{1+\mu_t} = \left[\frac{|\mathbb{I}_t|-|\mathbb{P}_t|}{|\mathbb{I}_t|}\left(\frac{1}{1+\eta\lambda}\right)^{\frac{1}{\lambda}} + \frac{|\mathbb{P}_t|}{|\mathbb{I}_t|}\left(\frac{1}{1+\mu^{\mathrm{P}}_t}\right)^{\frac{1}{\lambda}}\right]^{\lambda},$$

which gives A_t as $P^{\mathcal{W}}_t = \frac{A_t}{1+\mu_{t-1}}$.

3. The rest of the model

3.1. Preliminary

Before presenting the rest of the model in detail, we provide an overview of its broad structure. It is also helpful to introduce some conventions we follow throughout. The world consists of $\mathcal{N}+1$ countries where country 0 represents the rest of world (ROW) and is captured in a stylised fashion. All countries will produce at least some of the input to the multinational engineering sector ("widget composites") and will use some of its output ("engineering") in the production of other goods. The intermediate engineering good will embody technology in the model: all growth will ultimately be driven by growth in the engineering sector. For the engineering sector to be multinational as previously discussed, the stochastic discount factor must be constant across countries. This implies that we must have complete international financial markets. However, we are able to fix the counter-factual implications of international complete markets in simpler models by: 1) assuming that only some agents within each country have access to such markets, and 2) ensuring the stochastic discount factor responds to unobserved states via the use of Epstein-Zin preferences (Epstein & Zin 1989).

With growth coming from a common source across countries, a naïve model might struggle to explain the variance in productivity across countries. To remedy this, we assume that advanced human capital is required to use advanced technologies. This is a way of capturing the adoption process, which is emphasised by e.g. Comin's recent work (Comin & Gertler 2006; Comin & Hobijn 2010). In each country $n \in \{1, \dots \mathcal{N}\}$, engineering, together with (skilled and unskilled) labour and (Human, physical, and public) capital, is used to produce 3 goods, namely widgets (used to produce widget composites), tradable goods, and non-tradable goods. We use $S_0 = \{T, NT, W\}$ to denote the set of sectors producing these goods (respectively, tradeables, non-tradeables and widgets). Widgets and tradable goods are differentiated across countries and assembled to form widget composites and tradable composites, respectively. Then tradable composites and non-tradable goods are combined to produce an "almost final" good.

There are 7 final goods, including a durable good, a non-durable good, physical capital, human capital and the government versions of the latter three. The government version of human capital is referred to as R&D capital, which is needed to produce widget composites. It may be interpreted as government-funded fundamental research conducted in Universities. However, human capital (public or private) alone cannot drive growth. Public knowledge is used to help in the improvement of productivity within the engineering sector, and private knowledge is essential to take advantage of technological improvements coming from that sector. The set of private final goods is $SP_1 = \{KP, HP, D, ND\}$ (respectively, physical capital, human capital, durables, and non-durables) and the set of government final goods is $SG_1 = \{KG, HG, CG\}$ (respectively, government physical capital, government R&D capital, and government consumption). Each final good is produced by almost final goods and a certain type of labour. The labour input reflects hours of distribution, packaging, sales, marketing, or teaching, which are likely to differ substantially across sectors. Government final goods require the same types of labour as their corresponding private goods, so it is useful to define $S_1 = \{K, H, D, NDCG\}$.

The multisector structure above is important for at least two reasons. First, it allows us to capture movements and possible trends in relative prices. For example, the price of investment goods has been declining relative to consumption goods, suggesting higher growth in the productivity of investment goods.

Second, as suggested by Perli and Sakellaris (1998), a slow transfer of labour from the human capital sector to other sectors is key to the propagation of shocks. For simplicity, all sectors other than the engineering sector features perfectly competitive markets. Distortions to these markets may nonetheless be captured in the model via the model's shock processes, much like the wedges of Chari, Kehoe, McGrattan (2007).

On the demand side, there are two types of households as per Iacoviello (2005): patient and impatient households differentiated by their discount factors. In addition, the former has access to both incomplete domestic financial markets and complete international financial markets while the latter can only borrow from the former up to a collateral constraint. We use subscript l (lender) to denote patient households and subscript b (borrower) to denote impatient households.

Throughout, we make heavy use of an aggregator that is CES in the short-run but Cobb-Douglas in the long-run, first introduced in Holden (2017). This enables us to bring non-unitary elasticities to an environment in which factors are growing at different rates, while preserving the existence of a balanced growth path. For future reference, this hybrid aggregator $z_t = \mathcal{H}(a_t, x_t, y_t; \alpha, e, \varrho)$ has the form

$$z_t = a_t \bar{x}_{t-1}{}^{\alpha} \left[\alpha \left(\frac{x_t}{\bar{x}_{t-1}} \right)^{\frac{e-1}{e}} + (1-\alpha) y_t^{\frac{e-1}{e}} \right]^{\frac{e}{e-1}},$$

where:

$$\bar{x}_t = \left(\frac{x_t}{y_t} \right)^{1-\varrho} \bar{x}_{t-1}{}^{\varrho},$$

z_t is output, x_t and y_t are input, a_t is a shock, $\alpha \in (0,1)$ is the long-run share of x_t, $e > 0$ is the short-run elasticity of substitution, and ϱ controls the degree of persistence of the "habit" stock term $\bar{x}_t$. The aggregator collapse to a CES one in the limit as $\varrho \to 1$. Intuitively, $\bar{x}_t$ captures the fact that firms that have become accustomed to high ratios of x_t to y_t will experience substantial drops in productivity should they attempt to produce with a lower ratio.

In what follows, we describe each agent in country $n \in \{1, \dots \mathcal{N}\}$ and then we describe country 0, the rest of the world. We keep engineering as the numeraire (i.e. all prices are in units of engineering). To keep notations concise, we use ss to denote the steady state[24] or the trend of a bracketed expression when this causes no confusion, where the trend of a variable is understood as the path to which it would converge in the absence of shocks. I.e. $(f(a_t, b_t) - ss)$ means $f(a_t, b_t)$ relative to its trend. Variables without time subscripts (e.g. X) denote the steady states or deterministic trends (always evaluated in period t) of the corresponding variables (e.g. X_t). Many first order conditions are standard and hence are left to appendix 9.

Note that in the following many parameters are given n (country) subscripts which are later specialised to be constant across countries, in line with the arguments given in the introduction. The additional generality here ensures that the model could be readily generalized to one in which there were systematic differences across countries, should policy makers be interested in such a model.

[24] Since the model will contain unit root shock processes, strictly there will be a continuum of steady states. In the following, by "the steady state", we mean the point that would be the steady state were all shock processes in fact stationary.

3.2. Households

The population of patient and impatient households in country n is governed respectively by:

$$N_{nl,t} = N_t \left\{ \widetilde{N}_{n,t}(1-\varpi_{n,t}) + \psi_l \left[\log\left(\frac{V_{nl,t}}{N_{nl,t}}\right) - \frac{1}{\mathcal{N}} \sum_{m=1}^{\mathcal{N}} \log\left(\frac{V_{ml,t}}{N_{nl,t}}\right) - ss \right] \right\},$$

$$N_{nb,t} = N_t \left\{ \widetilde{N}_{n,t}\varpi_{n,t} + \psi_b \left[\log\left(\frac{V_{nb,t}}{N_{nb,t}}\right) - \frac{1}{\mathcal{N}} \sum_{m=1}^{\mathcal{N}} \log\left(\frac{V_{mb,t}}{N_{nb,t}}\right) - ss \right] \right\},$$

where N_t is global population with a growth rate of $\log\left(\frac{N_t}{N_{t-1}}\right)$ following a stationary $AR(1)$ process. Allowing for a common stochastic trend in population is important if there is to be movement in population across countries.

The terms in the braces allow two channels of migration. First, $\widetilde{N}_{n,t}$ and $\varpi_{n,t}$ represent, when $\psi_l = \psi_b = 0$, the shares of country n's population and the shares of impatient households. Second, households move towards countries in which people are happier than average. $V_{nl,t}$ and $V_{nb,t}$ are the value functions of each type of households. ψ_l and ψ_b are the welfare elasticities of population. Hence, people "die" more in low value function countries and are "born" more in high value function countries. This is a relatively tractable approach to capturing international migration driven by differences in quality of life across countries.

Country n's population is $N_{n,t} = N_{nl,t} + N_{nb,t}$. The population of the ROW is $N_{0,t} = N_t - \left(\sum_{n=1}^{\mathcal{N}} N_{n,t}\right)$.

3.2.1. Patient households

The representative patient household makes decisions on consumption, investment, and labour supply. The household consumes three goods, namely durable goods $D_{nl,t}$, nondurable goods $C^{ND}_{nl,t}$, and public goods $CG_{n,t}$. To derive utility from durable goods, the household must conduct home production using the technology:

$$C^D_{nl,t} = \Omega^h_{n,t}\left(v^D_{nl,t} D_{nl,t}\right)^{\alpha^h_n} {L^h_{nl,t}}^{1-\alpha^h_n}, \tag{7}$$

where $C^D_{nl,t}$ is the consumption of durable goods, $v^D_{nl,t}$ is a utilisation rate, $L^h_{nl,t}$ is the labour input for home production, and $\Omega^h_{n,t}$ is a stationary $AR(1)$ process of productivity. Home production here should be interpreted in the loosest possible sense: sitting and watching TV is home production as it is combining a durable good with household time. The consumption of the three goods enters utility function by the hybrid aggregators:

$$\frac{CP_{nl,t}}{N_{nl,t}} = \mathcal{H}\left(\Omega^{CP}_{n,t}, \frac{C^D_{nl,t}}{N_{nl,t}}, \frac{C^{ND}_{nl,t}}{N_{nl,t}}, \alpha^{CD}_n, e^{CP}_n, \varrho^{CD}_n\right), \tag{8}$$

$$\frac{C_{nl,t}}{N_{nl,t}} = \mathcal{H}\left(\Omega^{C}_{n,t}, \frac{C^{CP}_{nl,t}}{N_{nl,t}}, \frac{C^{CG}_{nl,t}}{N_{n,t}}, \alpha^{CP}_n, e^{CC}_n, \varrho^{CP}_n\right), \tag{9}$$

where the public good is divided by domestic population $N_{n,t}$ because we assume public goods are rival at the national level. For instance, governments need to build more public hospitals as population grows. Using the hybrid aggregator here enables us capture non-unitary short to medium-run elasticities of substitution between durables and non-durables and between private and public consumption.

The household accumulates respectively stocks of durable goods, private physical capital $KP^s_{n,t}$, and private human capital $HP^s_{n,t}$, for s $\in$ S$_0$, according to:

$$D_{nl,t} = \left(1 - \delta^D_{n,t} - \frac{\theta^{UD0}_n}{1+\theta^{UD1}_n} {v^D_{nl,t}}^{1+\theta^{UD1}_n}\right) D_{nl,t-1} + I^D_{nl,t} \exp\left[-\frac{\theta^{GD}_n}{2}\left(\log\frac{G^D_{nl,t}}{G^D_{nl,t-1}}\right)^2 - \frac{\theta^{DP}_n}{2}\left(\log\frac{D_{nl,t}}{D^P_{nl,t-1}}\right)^2\right], \tag{10}$$

$$KP^s_{n,t} = \left(1 - \delta^{KP,s}_{n,t} - \frac{\theta^{UK0}_n}{1+\theta^{UK1}_n} {v^{K,s}_{n,t}}^{1+\theta^{UK1}_n}\right) KP^s_{n,t-1} + I^{KP,s}_{n,t} \exp\left[-\frac{\theta^{GKP}_n}{2}\left(\log \frac{G^{KP,s}_{n,t}}{G^{KP,s}_{n,t-1}}\right)^2 - \frac{\theta^{KPP}_n}{2}\left(\log \frac{KP^s_{n,t}}{KP^{P,s}_{n,t-1}}\right)^2\right], \quad (11)$$

$$HP^s_{n,t} = (1 - \delta^{HP,s}_{n,t}) HP^s_{n,t-1} + I^{HP,s}_{n,t} \exp\left[-\frac{\theta^{GHP}_n}{2}\left(\log \frac{G^{HP,s}_{n,t}}{G^{HP,s}_{n,t-1}}\right)^2 - \frac{\theta^{HPP}_n}{2}\left(\log \frac{HP^s_{n,t}}{HP^{P,s}_{n,t-1}}\right)^2\right], \quad (12)$$

where $G^D_{nl,t}$, $G^{KP,s}_{n,t}$, $G^{HP,s}_{n,t}$ are growth rates of the respective stocks; $I^D_{nl,t}$, $I^{KP,s}_{n,t}$, $I^{HP,s}_{n,t}$ are investment levels in each stock; $D^P_{nl,t}$, $KP^{P,s}_{n,t}$, $HP^{P,s}_{n,t}$ are planned stocks; and, $v^D_{nl,t}$, $v^{K,s}_{n,t}$ are utilization rates. Depreciation takes place both exogenously due to $\delta^D_{n,t}$, $\delta^{KP,s}_{n,t}$, $\delta^{HP,s}_{n,t}$, and endogenously due to variable utilization as in Baxter and Farr (2005). Further details on the exogenous depreciation process are deferred to section 4.2.

Investment is subject to two adjustment costs. First, there is a stock growth adjustment cost which plays a similar role to the standard investment adjustment cost favoured by e.g. Christiano, Eichenbaum & Evans (2005). One advantage of our approach is that it ensures that the stock variables are always increasing in investment, something which is not true under standard specifications of investment adjustment costs. However, since fluctuations in growth rates are driven by fluctuations in investment, the implications will be very similar. In our context, it is especially helpful as ratios of growth rates to lagged growth rates have unit steady-state whereas ratios of investment to lagged investment are growing, complicating the steady-state.

The second adjustment cost depends on deviations from an investment plan set one period before. This specification nests the standard one-period time-to-build frictions as $\theta^{DP}_n \to \infty$ etc. When θ^{DP}_n, θ^{KPP}_n and θ^{HPP}_n are finite, this specification allows for some investment to be productive in the period in which it is performed. This is particularly important in our model in which periods are years not quarters: while a one quarter time to build seems reasonable, a one year one is a stretch.

The household has Epstein-Zin preferences, with value function:

$$V_{nl,t} = \left\{(1-\beta_{nl,t}) {N_{nl,t}}^{\xi_n} \left(\frac{U_{nl,t}}{N_{nl,t}}\right)^{1-\sigma_n} + \beta_{nl,t}\left[\mathbb{E}_t\left(V^{1-\gamma^V_n}_{nl,t+1}\right)\right]^{\frac{1-\sigma_n}{1-\gamma^V_n}}\right\}^{\frac{1}{1-\sigma_n}}. \quad (13)$$

These preferences allow us to distinguish between risk aversion, controlled by $\gamma^V_n \geq 0$, and the intertemporal elasticity of substitution $\frac{1}{\sigma_n} \geq 0$. The household prefers an early resolution of uncertainty if $\gamma^V_n > \sigma_n$, and a later resolution if $\gamma^V_n < \sigma_n$. Given complete international financial markets, Kollmann (2016) shows that this feature is useful to explain the correlation between real exchange rates and relative consumption, even when the model is solved at first order. [25] Intuitively, this is because unobserved states have an impact on future value and thus on the stochastic discount factor, lessening the degree of consumption growth synchronization required across countries. The parameter $\xi \in [0,1]$ controls whether the household is an average utilitarian ($\xi = 0$) or a total utilitarian ($\xi = 1$), or anything in between, with respect to its members. I.e. does the household value consumption per head higher in states of the world in which total population is higher? While total utilitarian is more standard in DSGE models, there is a large philosophical literature

[25] In terms of the accuracy when solving DSGE models with recursive preferences, Caldara et al. (2012) find second and third order perturbation methods competitive with Chebyshev polynomials and value function iteration. Employing a first order approximation inevitably means some sacrifice in accuracy. However, Epstein-Zin preferences still helps match consumption dynamics in the presence of complete markets, even with a first order approximation. This is because whereas in a closed economy model, only the lead of the stochastic discount factor enters, and it enters multiplying other terms, within an open economy complete markets model, the stochastic discount factor enters directly via the equations equating stochastic discount factors across countries.

on the appropriateness of average vs total utilitarianism. In our context, this parameter picks up the extent to which the representative household cares about the utility of both its descendants and new immigrants.

To ensure the stationarity of consumption in the model, the discount factor is decreasing in the ratio between non-durable goods and output. To help tractability, we assume this discount factor is treated as exogenous, so the representative household does not internalise its effects on the discount factor. As observed by e.g. Kollmann (2016), this has a minimal effect on dynamics. The law of motion for the discount factor is given by:

$$\operatorname{logit}\beta_{nl,t} = \rho_n^\beta \operatorname{logit}\beta_{nl,t-1} + \left(1-\rho_n^\beta\right)\operatorname{logit}\left[\bar{\beta}_{nl} - b_n^\beta \log\frac{\left(C_{nb,t}^{ND}+C_{nl,t}^{ND}\right)P_n^{ND}}{Y_n P_n}\right] + \sigma_n^\beta \varepsilon_{n,t}^\beta, \tag{14}$$

where $\bar{\beta}_{nl}$ is a parameter with $\bar{\beta}_{nl} > \bar{\beta}_{nb}$. The contemporary utility $U_{nl,t}$ depends on final consumption composites $C_{nl,t}$ and various types of labour supply $L_{nl,t}^s$, for:

$$s \in S_2 \equiv \{T, NT, W, ST, SNT, SW, D, NDCG, K, H\}.$$

From left to right, elements in S_2 indicate unskilled labour working in tradable, non-tradable, and widget sectors (T, NT, W), skilled labour working in these sectors (ST, SNT, SW), and labour used to produce durable goods, nondurable (or public) goods, capital, and human capital $(D, NDCG, K, H)$.

Broadly following Jaimovich and Rebelo (2009), the contemporary utility (felicity) is given by:

$$\frac{U_{nl,t}}{N_{nl,t}} = \frac{C_{nl,t}}{N_{nl,t}} - h_n\frac{C_{nl,t-1}}{N_{nl,t-1}} - \frac{\bar{C}_{nl,t-1}}{N_{nl,t-1}}\left[\kappa_{n,t}^0 + \sum_{s\in S_2}\frac{\kappa_{n,t}^s}{1+\nu_n^s}\left(\frac{L_{nl,t}^s}{N_{nl,t}}\right)^{1+\nu_n^s} + \frac{\kappa_{n,t}^h}{1+\nu_n^h}\left(\frac{L_{nl,t}^h}{N_{nl,t}}\right)^{1+\nu_n^h}\right], \tag{15}$$

where h_n controls the degree of habit,

$$\frac{\bar{C}_{nl,t}}{N_{nl,t}} = \left(\frac{C_{nl,t}}{N_{nl,t}}\right)^{1-\varrho_n^C}\left(\frac{\bar{C}_{nl,t-1}}{N_{nl,t-1}}\right)^{\varrho_n^C} \tag{16}$$

is a smoothed consumption habit stock, making preferences non–time-separable in consumption and labour, the $\kappa_{n,t}^{\cdot}$ variables are preference shocks, and $\varrho_n^C \in [0,1]$ controls the strength of short-run wealth effects on labour supply[26]. This preference nests two special cases: when $\varrho_n^C = 1$ (Greenwood–Hercowitz–Huffman (1988), GHH preferences) and when $\varrho_n^C = 0$ (King–Plosser–Rebelo (1988), KPR preferences).

Note that each type of labour enters the utility function separately with its own elasticity of labour supply parameter. This ensures that there is not too much substitutability in labour supply across sectors. This is consistent with recent evidence suggesting that labour moves across sectors slowly after a shock (Acemoglu et al. 2016; Autor, Dorn & Hanson 2016). In particular, this specification avoids quick swings to the engineering sector which might otherwise result in overly strong productivity movements.

The household maximizes its life-time utility (13) subject to (7)-(12), (14)-(16), and the following budget constraint:

$$\begin{aligned}
&\sum_{s\in S_2} L_{nl,t}^s W_{n,t}^s\left(1-\tau_{n,t}^{L,s}\right) + \sum_{s\in S_0} R_{n,t}^{KP,s} v_{n,t}^{K,s} KP_{n,t}^s\left(1-\tau_{n,t}^{K,s}\right) + \sum_{s\in S_0} R_{n,t}^{HP,s} HP_{n,t}^s\left(1-\tau_{n,t}^{H,s}\right)\\
&\geq P_{n,t}^{HP} I_{n,t}^{HP} + P_{n,t}^{KP} I_{n,t}^{KP} + P_{n,t}^{D} I_{nl,t}^{D}\left(1+\tau_{n,t}^{D}\right) + P_{n,t}^{ND} C_{nl,t}^{ND}\left(1+\tau_{n,t}^{ND}\right)\\
&-B_{n,t-1} + Q_{n,t}B_{n,t} + \mathcal{A}_{n,t}\\
&+\tau_{nl,t} Y_{n,t} P_{n,t} + \Pi_{n,t},
\end{aligned}$$

where the first line is after-tax labour income and capital income; the second line is the expenditure on new capital and goods; the third line contains holdings of a one-period risk-free domestic bond $B_{n,t}$ with price

[26] Unlike Jaimovich and Rebelo (2009), we lag $\bar{C}_{nl,t}$ in the utility function, to ensure that there are no contemporaneous wealth effects for any ϱ_n^C.

$Q_{n,t}$, as well as state contingent net payments to the rest of the world $\mathcal{A}_{n,t}$, coming from the complete set of state-contingent bonds available to the household; the fourth line contains a lump-sum tax $\tau_{nl,t}Y_{n,t}P_{n,t}$ and dividend income $\Pi_{n,t}$. Note that the lump-sum tax $\tau_{nl,t}$ is normalized as a proportion of output.

3.2.2. Impatient households

Impatient households have the same preferences as patient households except that they discount future utility more heavily. Notably, the same preference shock realisations hit both types of households. Impatient households differ from patient households in several further ways. First, they only have access to a one-period risk-free domestic bond. Second, they do not own firms nor any form of capital, including human capital. This is a natural consequence of their impatience: they have no desire to save in any form, including in physical or human capital accumulation. Since in production human capital will always be combined with skilled labour, it is natural to additionally assume that impatient households cannot supply skilled labour. As a result, they provide only unskilled labour. In order to keep the preferences of the impatient households otherwise identical to those of patient households, we keep the terms $L^{ST}_{nb,t}$, $L^{SNT}_{nb,t}$, and $L^{SW}_{nb,t}$ in their utility function though, and instead assume that they are additional unskilled labour supply to the corresponding sector. I.e. it is as if the household members who have the capability to acquire human capital and thus provide skilled labour are forced to provide unskilled labour as they have no human capital. Consequently, the total supply of unskilled labour to S_0 sectors is $L^{ST}_{nb,t}+L^{T}_{nb,t}$, $L^{SNT}_{nb,t}+L^{NT}_{nb,t}$, and $L^{SW}_{nb,t}+L^{W}_{nb,t}$ respectively.

The representative impatient household maximises the impatient equivalent of (13) subject to the impatient equivalent of (7)-(10), (14)-(16), and the following two constraints:

$$
\begin{aligned}
&\sum_{s\in\{S_0,S_1\}} L^{s}_{nb,t}W^{s}_{n,t}(1-\tau^{L,s}_{n,t}) + \sum_{s\in\{T,NT,W\}} L^{Ss}_{nb,t}W^{s}_{n,t}(1-\tau^{L,s}_{n,t})\\
&\geq P^{D}_{n,t}I^{D}_{nb,t}(1+\tau^{D}_{n,t}) + P^{ND}_{n,t}C^{ND}_{nb,t}(1+\tau^{ND}_{n,t})\\
&+B_{n,t-1}-Q_{n,t}B_{n,t}\\
&+\tau_{nb,t}Y_{n,t}P_{n,t},
\end{aligned}
$$

$$
B_{n,t}-\rho^{B}_{n}B_{n,t-1} \leq (1-\rho^{B}_{n})m_{n,t}D_{nb,t}P^{D}_{n,t} \tag{17}
$$

where (17) limits borrowing by the value of durable good stocks. As suggested by Guerrieri and Iacoviello (2017), the inertia parameter $\rho^{B}_{n}\in(0,1)$ may reflect several features that help capture data. For example, with multi-period debt contracts, the borrowing constraints are reset only for households refinancing their debts. When we take the model to the data, we treat housing as durable goods rather than capital, reflecting the fact that a lot of borrowing is secured against housing.

3.2.3. Household aggregation

Before proceeding, we define a few aggregate variables that are used in the rest of the model: aggregate supply of skilled labour is $L^{Ss}_{n,t}=L^{Ss}_{nl,t}, s\in S_0$; aggregate supply of unskilled labour is $L^{s}_{n,t}=L^{s}_{nl,t}+L^{s}_{nb,t}+L^{Ss}_{nb,t}, s\in S_0$; aggregate supply of labour in final goods sectors is $L^{s}_{n,t}=L^{s}_{nl,t}+L^{s}_{nb,t}, s\in S_1$; aggregate demand of nondurable goods, new durable goods, new physical capital, and new human capital is respectively:

$$
\begin{aligned}
C^{ND}_{n,t} &= C^{ND}_{nb,t}+C^{ND}_{nl,t},\\
I^{D}_{n,t} &= I^{D}_{nb,t}+I^{D}_{nl,t},\\
I^{KP}_{n,t} &= \sum_{s\in S_0} I^{KP,s}_{n,t},\\
I^{HP}_{n,t} &= \sum_{s\in S_0} I^{HP,s}_{n,t}.
\end{aligned}
$$

3.3. Production

We describe the supply side of the economy from the final output to lower layers. All goods are produced by firms in perfectly competitive industries, though firms face taxes on some inputs. All first order conditions are deferred to the appendix.

3.3.1. Final goods

Each of the 7 final goods is produced by a Cobb-Douglas technology. Firms hire the corresponding type of labour and buy almost final goods to produce with the production function:

$$I^s_{n,t} = \Omega^s_{n,t} {Y^s_{n,t}}^{\alpha^s_n} {L^s_{n,t}}^{1-\alpha^s_n}, \qquad s \in \{SP_1, SG_1\}$$

where $\{SP_1, SG_1\} = \{KP, HP, D, ND, KG, HG, CG\}$, $I^{ND}_{n,t} \equiv C^{ND}_{n,t}$ is non-durable goods; $I^{CG}_{n,t} \equiv CG_{n,t}$ is public consumption goods; $I^{KG}_{n,t}$ and $I^{HG}_{n,t}$ are investment in public physical and R&D capital respectively; $Y_{n,t} = \sum_{s\in\{SP_1,SG_1\}} Y^s_{n,t}$ is the total demand for almost final goods; $\Omega^s_{n,t}$ is an exogenous good specific productivity shock. Government goods $CG_{n,t}$, $I^{KG}_{n,t}$, and $I^{HG}_{n,t}$ require the same type of labour as, respectively, private goods $C^{ND}_{n,t}$, $I^{KP}_{n,t}$, and $I^{HP}_{n,t}$. Their total demand for labour is:

$$L^K_{n,t} = L^{KP}_{n,t} + L^{KG}_{n,t},$$
$$L^H_{n,t} = L^H_{n,t} + L^H_{n,t},$$
$$L^{NDCG}_{n,t} = L^{ND}_{n,t} + L^{CG}_{n,t}.$$

Furthermore, we assume $\alpha^{CG}_n = \alpha^{ND}_n$, $\alpha^{KG}_n = \alpha^{KP}_n$, and $\alpha^{HG}_n = \alpha^{HP}_n$, so that public goods are produced by a similar technology to their private equivalents. Allowing for α^s_n to differ across $s \in \{SP_1, SG_1\}$ is important as it permits the model to generate different growth rates for different final goods, even though the input of almost final goods has the same growth rate across good types. Thus, for example, the model can generate a decline in the relative price of investment goods over time.

3.3.2. Almost final goods and trade

Almost final goods are a CES aggregate of tradable composites $Y^{TC}_{n,t}$ and non-tradable goods $Y^{NT}_{n,t}$:

$$Y_{n,t} = \Omega^Y_{n,t} \left[\alpha^{NC}_n \left(\frac{Y^{NT}_{n,t}}{Y^{NT}_n} \right)^{\frac{e^{NC}_n - 1}{e^{NC}_n}} + \left(1 - \alpha^{NC}_n\right) \left(\frac{Y^{TC}_{n,t}}{Y^{TC}_n} \right)^{\frac{e^{NC}_n - 1}{e^{NC}_n}} \right]^{\frac{e^{NC}_n}{e^{NC}_n - 1}}, \tag{18}$$

where $\Omega^Y_{n,t}$ is an exogenous almost final good specific productivity shock, Y^{NT}_n is the steady-state $Y^{NT}_{n,t}$, Y^{TC}_n is the steady-state $Y^{TC}_{n,t}$, and hence Ω^Y_n is the steady-state of $Y_{n,t}$.

The tradable composites are assembled by goods importers using a CES technology:

$$Y^{TC}_{n,t} = \Omega^{TC}_{n,t} \left[\sum_{m=0}^{\mathcal{N}} \left(\frac{\widetilde{N}_m}{\sum_{m=0}^{\mathcal{N}} \widetilde{N}_m} \right)^{\frac{1}{e^T_n}} {Y^T_{n,m,t}}^{\frac{e^T_n - 1}{e^T_n}} \right]^{\frac{e^T_n}{e^T_n - 1}}, \tag{19}$$

where the subscript (n, m) means country n imports and country m exports, the differentiated tradable goods are weighted by steady-state population shares, which will help keep the model symmetric in per capita terms. Intuitively, the goods produced by larger countries are more varied than those produced by smaller countries, and so more highly in demand. (The range of movies produced in the US is far broader than the range produced in Luxembourg.) We could have modelled this with a continuum of varieties in each country, and free-entry conditions, then endogenously the measure of varieties per-country would have ended up proportional to population shares in steady state. The overall effect would have been practically identical to what we have here though, only with additional complication. Much as ever, $\Omega_{n,t}^{TC}$ is an exogenous tradeable composite good specific productivity shock.

Likewise, there are widget importers producing widget composites in two steps. The first step is to use a CES technology combining widgets from each country:

$$Y_{n,t}^{WP} = \Omega_{n,t}^{WP}\left[\sum_{m=1}^{\mathcal{N}}\left(\frac{\widetilde{N}_m}{\sum_{m=1}^{\mathcal{N}}\widetilde{N}_m}\right)^{\frac{1}{e_n^W}} Y_{n,m,t}^{W}{}^{\frac{e_n^W-1}{e_n^W}}\right]^{\frac{e_n^W}{e_n^W-1}}. \tag{20}$$

In the second step, government R&D capital $HG_{n,t-1}$ is added in to produce widget composites $Y_{n,t}^{WC}$:

$$Y_{n,t}^{X} = Y_{n,t}^{WC} = \Omega_{n,t}^{WC} HG_{n,t-1}{}^{\alpha_n^{HW}} Y_{n,t}^{WP}{}^{1-\alpha_n^{HW}} - F_{n,t}^{WC}, \tag{21}$$

where $\Omega_{n,t}^{WC}$ is an exogenous widget composite specific productivity shocks, $F_{n,t}^{WC}$ is a fixed cost growing at the same rate as $Y_{n,t}^{WC}$; $Y_{n,t}^{X}$ is a production by-product we refer to as "claims on engineering", which the country needs to claim for the output of the engineering sector. We will detail the role of these claims shortly. To be clear: this is one production process which simultaneously generates two output goods, widget composites and claims on engineering. The competitive firms producing widget composites are able to cover the fixed cost as they make positive profits due to the decreasing returns to scale implied by the free use of government R&D capital.

Importers are subject to iceberg trade and transport costs depending on n and $m \neq n$: $\iota_{n,m,t}^{T} = \iota_{n,t}^{MT} + \iota_{m,t}^{XT}$ for tradable goods and $\iota_{n,m,t}^{W} = \iota_{n,t}^{MW} + \iota_{m,t}^{XW}$ for widgets. These costs may reflect tariff and non-tariff barriers to trade.

Aggregate global widget supply is given by $\sum_{m=1}^{\mathcal{N}} Y_{n,t}^{WC} = \mathcal{W}_t$, so $P_{n,t}^{WC} = P_t^{\mathcal{W}}$ for all n.

3.3.3. Core goods production

Core goods, namely tradable goods, non-tradable goods, and widgets, are produced according to the map shown in Figure 3. However, each country's use of engineering is limited by their ownership of claims on engineering. In particular, effective engineering $\mathcal{X}_{n,t}$ is produced from engineering and claims on engineering by a perfectly competitive industry with access to the production process:

$$\mathcal{X}_{n,t} = X_{n,t}\exp\left[-o_n\left(\log\left(\frac{X_{n,t}}{X_t}\right) - \log\left(\frac{Y_{n,t}^X}{Y_t^X}\right)\right)^2\right], \tag{22}$$

where $X_t = \sum_{n=1}^{\mathcal{N}} X_{n,t}$ is the total output of engineering, $Y_t^X = \sum_{n=1}^{\mathcal{N}} Y_{n,t}^X$ is the total claims on engineering and $o_n \geq 0$ is a parameter. The adjustment costs in the brackets captures the idea that a country is most productive if its use of engineering corresponds to its input into that sector. Otherwise, the country will be importing either engineering or widgets whose use it may not fully understand. As a country's use of engineering increases, so too will its demand for claims on engineering. This will push up the price of claims on engineering, encouraging manufacturers of widget composites in that country to produce more, even though the price of widget composites is the same across countries. The net effect is very similar to that of transport costs, but this approach preserves the multi-nationality of the widget sector, which ensures tractability.

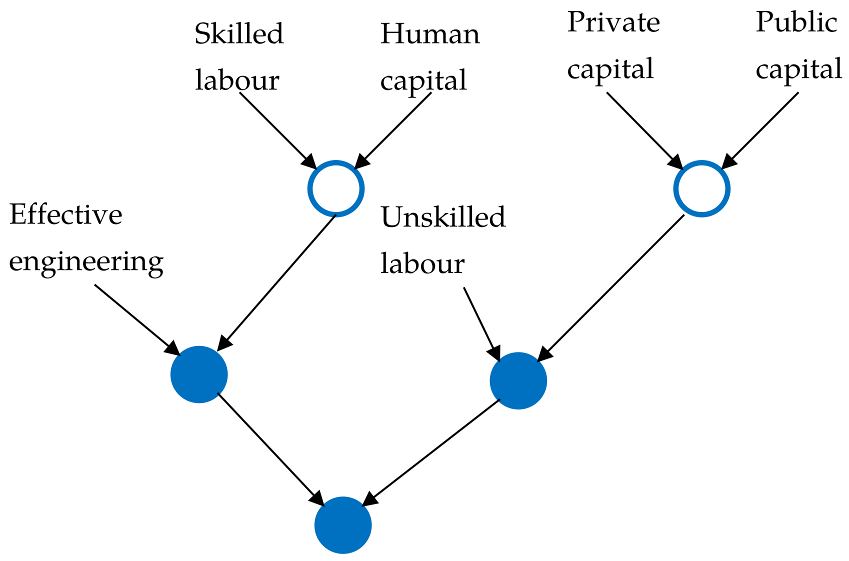

Figure 3: Map of the production of core goods

Note: Empty circles represent Cobb-Douglas production functions. Filled circles represent the hybrid function $\mathcal{H}$.

For s $\in$ $S_0 = \{T, NT, W\}$, the technology in each step of production is shown below:

$$Y_{n,t}^{HL,s} = \Omega_{n,t}^{HL,s} {HP_{n,t}^{s}}^{\alpha_n^{HL}} {L_{n,t}^{Ss}}^{1-\alpha_n^{HL}}$$

$$K_{n,t}^{s} = \Omega_{n,t}^{K,s} {KG_{n,t-1}}^{\alpha_n^{K}} \left(v_{n,t}^{K,s} KP_{n,t}^{s}\right)^{1-\alpha_n^{K}} - F_{n,t}^{K}$$

$$Y_{n,t}^{KL,s} = \mathcal{H}(\Omega_{n,t}^{KL,s}, K_{n,t}^{s}, L_{n,t}^{s}; \alpha_n^{KL}, e_n^{KL,s}, \varrho_n^{KL})$$

$$Y_{n,t}^{HLX,s} = \mathcal{H}(\Omega_{n,t}^{HLX,s}, Y_{n,t}^{HL,s}, \mathcal{X}_{n,t}; \alpha_n^{HLX}, e_n^{HLX,s}, \varrho_n^{HLX})$$

$$Y_{n,t}^{s} = \mathcal{H}(\Omega_{n,t}^{s}, Y_{n,t}^{HLX,s}, Y_{n,t}^{KL,s}; \alpha_n^{Y}, e_n^{Y,s}, \varrho_n^{Y})$$

where $F_{n,t}^{K}$ is a fixed cost growing at the same rate as $K_{n,t}^{s}$. As ever, each good is produced by a perfectly competitive industry, and $\Omega_{n,t}^{\cdot}$ is a good specific exogenous productivity shock. Note that we again assume a constant return to scale production function for the capital aggregate, even in the presence of a public capital input to production. Intuitively, if private factors are doubled but e.g. no new public roads are built, less than twice as much would be produced due to congestion on public roads. Thanks to the constant return to scale technology, firms can make positive profits even with perfectly competitive markets. We add the fixed cost to obtain an extra margin to match the data, such as the shares of labour income in total output. Note also that the share parameters do not differ across core goods. This helps identification as we will not observe expenditure shares for particular core goods.

We employ the hybrid aggregator $\mathcal{H}$ where a non-unity elasticity of substitution could be important. For example, a small $e_n^{HLX,s}$ allows complementarity between human capital and engineering, which captures the necessity of investment and training before new technologies may be adopted and prevents too fast technology diffusion (Rotemberg 2003). These elasticities can also differ across sectors, so factors tend to stay in the sector where they are more complementary.

3.3.4. The architecture of productivity shocks

There are 29 productivity and preference shocks "Ω" in each country. They are $AR(1)$ processes in logarithms with cross sectional correlations, which are constructed using common shocks as below:

$$\begin{array}{llllllll}
\Omega_{n,t}^{h} & = & \Omega_{n}^{h} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{hi} & & & \\
\Omega_{n,t}^{CP} & = & \Omega_{n}^{CP} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{CALL} & \Omega_{n,t}^{CPi} & & \\
\Omega_{n,t}^{C} & = & \Omega_{n}^{C} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{CALL} & \Omega_{n,t}^{Ci} & & \\
\Omega_{n,t}^{KP} & = & \Omega_{n}^{KP} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S1ALL} & \Omega_{n,t}^{PALL} & \Omega_{n,t}^{KALL} & \\
\Omega_{n,t}^{HP} & = & \Omega_{n}^{HP} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S1ALL} & \Omega_{n,t}^{PALL} & \Omega_{n,t}^{HALL} & \\
\Omega_{n,t}^{D} & = & \Omega_{n}^{D} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S1ALL} & \Omega_{n,t}^{PALL} & \Omega_{n,t}^{Di} & \\
\Omega_{n,t}^{ND} & = & \Omega_{n}^{ND} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S1ALL} & \Omega_{n,t}^{PALL} & \Omega_{n,t}^{NDALL} & \\
\Omega_{n,t}^{KG} & = & \Omega_{n}^{KG} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S1ALL} & \Omega_{n,t}^{GALL} & \Omega_{n,t}^{KALL} & \\
\Omega_{n,t}^{HG} & = & \Omega_{n}^{HG} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S1ALL} & \Omega_{n,t}^{GALL} & \Omega_{n,t}^{HALL} & \\
\Omega_{n,t}^{CG} & = & \Omega_{n}^{CG} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S1ALL} & \Omega_{n,t}^{GALL} & \Omega_{n,t}^{NDALL} & \\
\Omega_{n,t}^{Y} & = & \Omega_{n}^{Y} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{Yi} & & & \\
\Omega_{n,t}^{TC} & = & \Omega_{n}^{TC} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{TradeALL} & \Omega_{n,t}^{TCi} & & \\
\Omega_{n,t}^{WP} & = & \Omega_{n}^{WP} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{TradeALL} & \Omega_{n,t}^{WPi} & & \\
\Omega_{n,t}^{WC} & = & \Omega_{n}^{WC} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{WCi} & & & \\
\Omega_{n,t}^{KT} & = & \Omega_{n}^{KT} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{TALL} & \Omega_{n,t}^{KKALL} & , \\
\Omega_{n,t}^{HLT} & = & \Omega_{n}^{HLT} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{TALL} & \Omega_{n,t}^{HLALL} & \\
\Omega_{n,t}^{KLT} & = & \Omega_{n}^{KLT} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{TALL} & \Omega_{n,t}^{KLALL} & \\
\Omega_{n,t}^{HLXT} & = & \Omega_{n}^{HLXT} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{TALL} & \Omega_{n,t}^{HLXALL} & \\
\Omega_{n,t}^{T} & = & \Omega_{n}^{T} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{TALL} & \Omega_{n,t}^{HLXKLALL} & \\
\Omega_{n,t}^{KNT} & = & \Omega_{n}^{KNT} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{NTALL} & \Omega_{n,t}^{KKALL} & \\
\Omega_{n,t}^{HLNT} & = & \Omega_{n}^{HLNT} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{NTALL} & \Omega_{n,t}^{HLALL} & \\
\Omega_{n,t}^{KLNT} & = & \Omega_{n}^{KLNT} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{NTALL} & \Omega_{n,t}^{KLALL} & \\
\Omega_{n,t}^{HLXNT} & = & \Omega_{n}^{HLXNT} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{NTALL} & \Omega_{n,t}^{HLXALL} & \\
\Omega_{n,t}^{NT} & = & \Omega_{n}^{NT} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{NTALL} & \Omega_{n,t}^{HLXKLALL} & \\
\Omega_{n,t}^{KW} & = & \Omega_{n}^{KW} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{WALL} & \Omega_{n,t}^{KKALL} & \\
\Omega_{n,t}^{HLW} & = & \Omega_{n}^{HLW} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{WALL} & \Omega_{n,t}^{HLALL} & \\
\Omega_{n,t}^{KLW} & = & \Omega_{n}^{KLW} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{WALL} & \Omega_{n,t}^{KLALL} & \\
\Omega_{n,t}^{HLXW} & = & \Omega_{n}^{HLXW} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{WALL} & \Omega_{n,t}^{HLXALL} & \\
\Omega_{n,t}^{W} & = & \Omega_{n}^{W} & \Omega_{n,t}^{ALL} & \Omega_{n,t}^{S0ALL} & \Omega_{n,t}^{WALL} & \Omega_{n,t}^{HLXKLALL} &
\end{array} \tag{23}$$

where all terms on the RHS are multiplicative, the first column on the RHS is steady states and the remaining columns are $AR(1)$ series in logarithms with zero steady states.

3.4. Government

Given the relatively high tax burden in the Euro Area (40% of GDP), we allow governments to collect taxes from several sources. Importantly, governments can tax (or subsidise) the employment of engineering $\tau_{n,t}^{X,s}, s \in S_0$. Households pay capital income taxes $\tau_{n,t}^{H,s}, s \in S_0$ and $\tau_{n,t}^{K,s}, s \in S_0$, labour income taxes $\tau_{n,t}^{L,s}, s \in S_2$, and lump-sum taxes $\tau_{nl,t}$ or $\tau_{nb,t}$. Firms pay sales taxes $\tau_{n,t}^{D}$, $\tau_{n,t}^{ND}$, $\tau_{n,t}^{NT}$, $\tau_{n,t}^{TC}$ on durable, nondurable, non-tradable, and tradable composite goods. Governments can distort trade by imposing tariffs $\iota^{MT}_{n,t}$, $\iota^{XT}_{n,t}$, $\iota^{MW}_{n,t}$, $\iota^{XW}_{n,t}$ on the import and export of tradable goods and widgets. Governments can subsidise households and firms by levying negative taxes. For example, governments may foster growth by subsidising human capital and engineering. Of course, real world policy changes usually act as simultaneous changes to multiple taxes, but it is convenient to keep taxes separate for modelling.

On the expenditure side, governments provide a public good $CG_{n,t}$ and invest in public physical capital $KG_{n,t}$ and public R&D capital $HG_{n,t}$. The two types of capital evolve respectively according to

$$KG_{n,t} = (1-\delta^{KG}_{n,t})KG_{n,t-1} + I^{KG}_{n,t}\exp\left[-\frac{\theta^{GKP}_n}{2}\left(\log\frac{G^{KG}_{n,t}}{G^{KG}_{n,t-1}}\right)^2\right],$$
$$HG_{n,t} = (1-\delta^{HG}_{n,t})HG_{n,t-1} + I^{HG}_{n,t}\exp\left[-\frac{\theta^{GHP}_n}{2}\left(\log\frac{G^{HG}_{n,t}}{G^{HG}_{n,t-1}}\right)^2\right],$$

where we assume standard one-period time-to-build frictions. This is more plausible for government capital due to the long lags in government decision making.

The government's budget is balanced by a government bond, which however, is abstracted from the model, i.e. we do not track the evolution of government debt implied by the rest of the model. In a model tracking government debt, fiscal stability would require that the fiscal deficit respond negatively to government debt. The parametric restrictions implied by this make estimation much harder. Our approach here is simpler, and does not lose much information as long as government expenditure and revenue are observable. Our approach here is equivalent to assuming that the government budget is balanced period by period by a lump-sum tax on patient households, but we do not advocate this interpretation.

Government expenditure and revenue are given respectively by:

$$\text{GovExp}_{n,t} = \sum_{s\in SG_1} \tau^s_{n,t} Y_{n,t} P_{n,t},$$

$$\begin{aligned}
\text{GovRev}_{n,t} = {} & \phi^{MT}_n \iota^{MT}_{n,t} \sum_{m=0,m\neq n}^{\mathcal{N}} P^T_{m,t} Y^T_{n,m,t} + \phi^{XT}_n \iota^{XT}_{n,t} \sum_{m=0,m\neq n}^{\mathcal{N}} P^T_{n,t} Y^T_{m,n,t} \\
& + \phi^{MW}_n \iota^{MW}_{n,t} \sum_{m=1,m\neq n}^{\mathcal{N}} P^W_{m,t} Y^W_{n,m,t} + \phi^{XW}_n \iota^{XW}_{n,t} \sum_{m=1,m\neq n}^{\mathcal{N}} P^W_{n,t} Y^W_{m,n,t} \\
& + P^{NT}_{n,t} Y^{NT}_{n,t} \tau^{NT}_{n,t} \phi^{NT}_n + P^{TC}_{n,t} Y^{TC}_{n,t} \tau^{TC}_{n,t} \phi^{TC}_n + P^D_{n,t} I^D_{n,t} \tau^D_{n,t} \phi^D_n + P^{ND}_{n,t} I^{ND}_{n,t} \tau^{ND}_{n,t} \phi^{ND}_n \\
& + \sum_{s\in\{S_0,S_1\}} L^s_{nb,t} W^s_{n,t} \tau^{L,s}_{n,t} \phi^L_n + \sum_{s\in S_0} L^{Ss}_{nb,t} W^s_{n,t} \tau^{L,s}_{n,t} \phi^L_n + \sum_{s\in S_2} L^s_{nl,t} W^s_{n,t} \tau^{L,s}_{n,t} \phi^L_n \\
& + \sum_{s\in S_0} R^{KP,s}_{n,t} v^{K,s}_{n,t} KP^s_{n,t} \tau^{K,s}_{n,t} \phi^{K,s}_n + \sum_{s\in S_0} R^{HP,s}_{n,t} HP^s_{n,t} \tau^{H,s}_{n,t} \phi^{H,s}_n \\
& + \sum_{s\in S_0} P^{\mathcal{X}}_{n,t} \mathcal{X}^S_{n,t} \tau^{\mathcal{X},s}_{n,t} \phi^{\mathcal{X},s}_n \\
& + (\tau_{nl,t} + \tau_{nb,t}) Y_{n,t} P_{n,t},
\end{aligned}$$

where each distortionary tax enters by a fraction of $\phi^{\cdot}_n$ and lump-sum taxes and government expenditure are normalised by $Y_{n,t}P_{n,t}$. This setup implies that a proportion $1-\phi^{\cdot}_n$ of each tax is purely an exogenous wedge as in Chari, Kehoe, and McGrattan (2007) (possibly capturing non-monetary obstacles such as bureaucracy). For simplicity, all taxes are assumed to be contained in observed government revenue, regardless of the sign of the corresponding $\tau^{\cdot}_{n,t}$ variable. This may be justified if we assume that negative taxes are actually capturing tax rebates which offset against positive taxes.

Since:

$$I_{n,t}^{s} P_{n,t}^{s} = \tau_{n,t}^{s} Y_{n,t} P_{n,t}, \qquad s \in SG_1,$$

the Cobb-Douglas production functions of final goods imply:

$$\tau_{n,t}^{s} \alpha_{n}^{s} = \frac{Y_{n,t}^{s}}{Y_{n,t}}, \qquad s \in SG_1.$$

So, $\tau_{n,t}^{S}, S \in SG_1$ determines how many almost final goods are used to produce public final goods.

Fiscal instruments are governed by the following rules. Government expenditures responds to their lags, the growth rate of output, and hours worked per capita. The latter two capture the response of spending to the business cycle. For $i \in SG_1$:

$$\log \tau_{n,t}^{i} = \log \tau_{n,t-1}^{i} + \phi_{n}^{Y,i}\left(\frac{Y_{n,t}}{Y_{n,t-1}} - ss\right) + \phi_{n}^{L,i}\left(\log \frac{L_{n.t}}{N_{n,t}} - ss\right) + \phi_{n}^{u,i}\left(u_{n,t}^{ALL} + u_{n,t}^{SG_1}\right) + u_{n,t}^{i},$$

where $u_{n,t}^{ALL}$ is a common shock on all fiscal instruments, $u_{n,t}^{SG_1}$ is a common shock on public spending, and $u_{n,t}^{i}$ is idiosyncratic shocks. Common shocks add correlation across fiscal instruments, which may pick up e.g. the dynamics of debt stabilisation.

Similarly, for $i \in \{b, l\}$, lump-sum taxes are governed by:

$$\tau_{ni,t} = \tau_{ni,t-1} + \phi_{n}^{Y,i}\left(\frac{Y_{n,t}}{Y_{n,t-1}} - ss\right) + \phi_{n}^{L,i}\left(\log \frac{L_{n.t}}{N_{n,t}} - ss\right) + \phi_{n}^{u,i}\left(u_{n,t}^{ALL} + u_{n,t}^{bl}\right) + u_{n,t}^{i}.$$

Distortionary taxes follow:

$$\tau_{n,t}^{i} = \tau_{n,t-1}^{i} + \phi_{n}^{u,i}\left(u_{n,t}^{ALL} + u_{n,t}^{CALL}\right) + u_{n,t}^{i}, \qquad i = \{D, ND\},$$

$$\tau_{n,t}^{i} = \tau_{n,t-1}^{i} + \phi_{n}^{u,i} u_{n,t}^{ALL} + u_{n,t}^{i}, \qquad i = \{NT, TC\},$$

$$\iota_{n,t}^{i} = \iota_{n,t-1}^{i} + \phi_{n}^{u,i}\left(u_{n,t}^{ALL} + u_{n,t}^{\iota ALL}\right) + u_{n,t}^{i}, \qquad i = \{TX, TM, WX, WM\},$$

$$\tau_{n,t}^{i,s} = \tau_{n,t-1}^{i,s} + \phi_{n}^{u,i,s}\left(u_{n,t}^{ALL} + u_{n,t}^{iALL} + u_{n,t}^{sALL}\right), \qquad i = \{H, K, \mathcal{X}\}, s \in S_0 .$$

Borrowing an idea from Mattesini and Rossi (2010), we capture the progressiveness of labour taxes by setting:

$$\tau_{n,t}^{i} = \tau_{n,t}^{La} + \tau_{n,t}^{Lb} \log \widehat{w}_{n,t}^{i}, \qquad i \in S_2$$

where $\widehat{w}_{n,t}^{i}$ is detrended wage rates, and:

$$\tau_{n,t}^{La} = \tau_{n,t-1}^{La} + \phi_{n}^{u,\tau,La}\left(u_{n,t}^{ALL} + u_{n,t}^{LALL}\right) + u_{n,t}^{La},$$

$$\log \tau_{n,t}^{Lb} = \log \tau_{n,t-1}^{Lb} + \phi_{n}^{u,\tau,Lb}\left(u_{n,t}^{ALL} + u_{n,t}^{LALL}\right) + u_{n,t}^{Lb}.$$

Notably, all fiscal instruments contain a unit root because the observed time series of taxes are very smooth, without clear mean-reversion. The levels of these unit root tax processes will drive permanent differences in outcomes across countries. These unit root processes thus save us from estimating separate steady-states for taxes in each country. In addition, by assuming unit roots we save on the estimation of assorted persistence parameters.

3.5. The rest of the world

The ROW does not contribute to R&D, or to the production of widgets, widget composites or engineering. It receives an endowment of tradable goods, $Y_{0,t}^{T}$, which can be sold to country $m \in \{1, \dots, \mathcal{N}\}$ at the price $P_{0,t}^{T}$. The ROW also consumes tradable composites $Y_{0,t}^{TC}$. However, we assume implicitly that the ROW enjoys freely the outcome of R&D thanks to unmodelled technology diffusion. Hence, in the long run, the ROW variables grow at the same rates as their counterparts in countries $1-n$. After detrending and taking logarithms, $P^T_{0,t}$ and $Y^{TC}_{0,t}$ are governed by a $VAR(1)$:

$$\begin{bmatrix}\log \hat{P}^T_{0,t}\\ \log \hat{Y}^{TC}_{0,t}\end{bmatrix} = \begin{bmatrix}(1-\rho_0^{PP})\log \hat{P}^T_0 - \rho_0^{PY}\log \hat{Y}^{TC}_0\\ -\rho_0^{YP}\log \hat{P}^T_0 + (1-\rho_0^{YY})\log \hat{Y}^{TC}_0\end{bmatrix} + \begin{bmatrix}\rho_0^{PP} & \rho_0^{PY}\\ \rho_0^{YP} & \rho_0^{YY}\end{bmatrix}\begin{bmatrix}\log \hat{P}^T_{0,t-1}\\ \log \hat{Y}^{TC}_{0,t-1}\end{bmatrix} + \begin{bmatrix}\sigma_0^P\epsilon^P_{0,t}\\ \sigma_0^Y\left(\epsilon^P_{0,t}\rho_0 + \epsilon^Y_{0,t}\sqrt{1-{\rho_0}^2}\right)\end{bmatrix},$$

where variables with a hat are detrended variables. The steady state of the VAR is given by:

$$Y^{TC}_0 = \frac{\widetilde{N}_0}{\widetilde{N}_1} Y^{TC}_1 y^{TC}_0,$$
$$P^T_0 = P^T_1 p^T_0,$$

where p^T_0 and y^{TC}_0 are parameters to be estimated. Scaling the demand of the ROW by its relative size helps ensure that parameters remain stable as the number of countries is varied.

We assume that ROW's demand for country m's tradable goods is:

$$Y^T_{0,m,t} = \left(\frac{P^T_{0,t}}{(1+\iota^T_{0,m,t})P^T_{m,t}}\right)^{e^T_{0,m}} Y^{TC}_{0,t}\frac{\widetilde{N}_m}{\sum_{m=0}^{\mathcal{N}}\widetilde{N}_m},$$

where the elasticity $e^T_{0,m}$ potentially differs across m.

3.6. Detrending

We solve the model around its asymptotic long-run trend. We solve for the trend of each variable in this section. A detrended variable equals the variable divided by its trend and is denoted with a hat "$\widehat{\ }$". We use "$\overset{Tr}{\Leftrightarrow}$" to indicate that the trend of terms on the LHS equals to the trend of terms on the RHS.

First, the production of core goods in sector $s \in S_0$ implies that the terms below share the same trend, which we assume to be ${A^*_t}^{b+1}N_t$:

$$\begin{array}{ccccccccc}
Y^s_{n,t}P^s_{n,t} & \overset{Tr}{\Leftrightarrow} & Y^{KL,s}_{n,t}P^{KL,s}_{n,t} & \overset{Tr}{\Leftrightarrow} & L^s_{n,t}W^s_{n,t} & & \\
 & & & \overset{Tr}{\Leftrightarrow} & K^s_{n,t}R^{K,s}_{n,t} & \overset{Tr}{\Leftrightarrow} & KP^s_{n,t}R^{KP,s}_{n,t} \\
 & \overset{Tr}{\Leftrightarrow} & Y^{HLX,s}_{n,t}P^{HLX,s}_{n,t} & \overset{Tr}{\Leftrightarrow} & \mathcal{X}^s_{n,t}P^{\mathcal{X}}_{n,t} & \overset{Tr}{\Leftrightarrow} & X_{n,t} & . \\
 & & & \overset{Tr}{\Leftrightarrow} & Y^{HL,s}_{n,t}P^{HL,s}_{n,t} & \overset{Tr}{\Leftrightarrow} & L^{Ss}_{n,t}W^{Ss}_{n,t} \\
 & & & & & \overset{Tr}{\Leftrightarrow} & HP^s_{n,t}R^{HP,s}_{n,t}
\end{array}$$

Since labour must be proportional to population in the long run, it follows that $\widehat{W}^s_{n,t} = \frac{W^s_{n,t}}{{A^*_t}^{b+1}}$ and $\widehat{W}^{Ss}_{n,t} = \frac{W^{Ss}_{n,t}}{{A^*_t}^{b+1}}$. It is also clear that $P^{\mathcal{X}}_{n,t}$ must be stationary according to (22).

Next, focusing on widgets, we have $P^{WC}_{n,t}Y^{WC}_{n,t} \overset{Tr}{\Leftrightarrow} P^{WP}_{n,t}Y^{WP}_{n,t} \overset{Tr}{\Leftrightarrow} P^W_{n,t}Y^W_{n,t}$. From the engineering sector we know $\hat{P}^{WC}_{n,t} = \frac{P^{WC}_{n,t}}{A^*_t}$ and hence $\hat{Y}^{WC}_{n,t} = \frac{Y^{WC}_{n,t}}{{A^*_t}^b N_t}$. Assuming $\widehat{HG}_{n,t} = \frac{HG_{n,t}}{{A^*_t}^c N_t}$ gives $\hat{Y}^{WP}_{n,t} = \frac{Y^{WP}_{n,t}}{{A^*_t}^{\frac{(b-c\alpha^{HW}_n)}{1-\alpha^{HW}_n}} N_t}$ and $\hat{P}^{WP}_{n,t} = \frac{P^{WP}_{n,t}}{{A^*_t}^{1+b-\frac{(b-c\alpha^{HW}_n)}{1-\alpha^{HW}_n}}}$. The CES production function (20) implies $P^W_{n,t}$ and $Y^W_{n,t}$ for all n have the same trends as $P^{WP}_{n,t}$ and $Y^{WP}_{n,t}$, which is why b and c don't have the subscript n. Note that because "α"s are fixed across $s \in S_0$, the trends of $P^W_{n,t}$ and $Y^W_{n,t}$ must also be the trends of $P^T_{n,t}$, $Y^T_{n,t}$, $P^{NT}_{n,t}$, and $Y^{NT}_{n,t}$. Using the CES productions functions of almost final goods, the trends of $P^{TC}_{n,t}$, $Y^{TC}_{n,t}$, $P_{n,t}$, and $Y_{n,t}$ are then also known.

Now, consider the production of final goods:

$$\begin{aligned} I_{n,t}^{s} P_{n,t}^{s}, s \in \{SP_1, SG_1\} &\overset{Tr}{\Leftrightarrow} W_{n,t}^{s} L_{n,t}^{s}, s \in S_1 \\ &\overset{Tr}{\Leftrightarrow} Y_{n,t} P_{n,t} \end{aligned}.$$

Denoting $d = \frac{(b - c\alpha_n^{HW})}{1-\alpha_n^{HW}}$ and using the Cobb-Douglas production functions, we can solve:

$$\hat{P}_{n,t}^{s} = \frac{P_{n,t}^{s}}{A_t^{*1+b-d\alpha_n^{s}}}, \hat{I}_{n,t}^{s} = \frac{I_{n,t}^{s}}{N_t A_t^{*d\alpha_n^{s}}}, s \in \{SP_1, SG_1\}.$$

It follows the capital accumulation functions that:

$$R_{n,t}^{KP,s} = \frac{R_{n,t}^{KP,s}}{A_t^{*1+b-d\alpha_n^{KP}}}, \widehat{KP}_{n,t}^{s} = \frac{KP_{n,t}^{s}}{N_t A_t^{*d\alpha_n^{KP}}}, R_{n,t}^{HP,s} = \frac{R_{n,t}^{HP,s}}{A_t^{*1+b-d\alpha_n^{HP}}}, \widehat{HP}_{n,t}^{S} = \frac{HP_{n,t}^{s}}{N_t A_t^{*d\alpha_n^{HP}}}, s \in S_0,$$

and similarly $HG_{n,t}$ and $KG_{n,t}$. Then, we can solve c from:

$$\frac{(b - c\alpha_n^{HW})}{1-\alpha_n^{HW}} \alpha_n^{HG} \equiv d\alpha_n^{HG} = c.$$

We have thus solved the trends of all production factors. The last step is to solve for b. Using production functions in core goods sectors, we have, for $s \in S_0$,

$$\hat{Y}_{n,t}^{HL,s} = \frac{Y_{n,t}^{HL,s}}{N_t A_t^{*d\alpha_n^{HP}\alpha_n^{HL}}}, \hat{P}_{n,t}^{HL,s} = \frac{P_{n,t}^{HL,s}}{A_t^{*1+b-d\alpha_n^{HP}\alpha_n^{HL}}},$$

$$\hat{K}_{n,t}^{s} = \frac{K_{n,t}^{s}}{N_t A_t^{*d\alpha_n^{KG}\alpha_n^{K}+d\alpha_n^{KP}(1-\alpha_n^{K})}}, \hat{R}_{n,t}^{K,s} = \frac{R_{n,t}^{K,s}}{A_t^{*1+b-d\alpha_n^{KG}\alpha_n^{K}-d\alpha_n^{KP}(1-\alpha_n^{K})}},$$

$$Y_{n,t}^{KL,s} = \frac{Y_{n,t}^{KL,s}}{N_t A_t^{*\left(d\alpha_n^{KG}\alpha_n^{K}+d\alpha_n^{KP}(1-\alpha_n^{K})\right)\alpha_n^{KL}}}, P_{n,t}^{KL,s} = \frac{P_{n,t}^{KL,s}}{A_t^{*1+b-\left(d\alpha_n^{KG}\alpha_n^{K}+d\alpha_n^{KP}(1-\alpha_n^{K})\right)\alpha_n^{KL}}},$$

$$\hat{Y}_{n,t}^{HLX,s} = \frac{Y_{n,t}^{HLX,s}}{N_t A_t^{*d\alpha_n^{HP}\alpha_n^{HL}\alpha_n^{HLX}+(1+b)(1-\alpha_n^{HLX})}}, \hat{P}_{n,t}^{HLX,s} = \frac{P_{n,t}^{HLX,s}}{A_t^{*-(d\alpha_n^{HP}\alpha_n^{HL}-1-b)\alpha_n^{HLX}}},$$

$$Y_{n,t}^{s} = \frac{Y_{n,t}^{s}}{N_t A_t^{*d\alpha_n^{HP}\alpha_n^{HL}\alpha_n^{HLX}+(1+b)(1-\alpha_n^{HLX})\alpha_n^{Y}+\left(d\alpha_n^{KG}\alpha_n^{K}+d\alpha_n^{KP}(1-\alpha_n^{K})\right)\alpha_n^{KL}(1-\alpha_n^{Y})}}, P_{n,t}^{s} = \frac{P_{n,t}^{s}}{A_t^{*1+b-d}}.$$

Using the previous result $\hat{Y}_{n,t}^{W} = \frac{Y_{n,t}^{W}}{A_t^{*d} N_t}$, b can be solved from:

$$d\alpha_n^{HP}\alpha_n^{HL}\alpha_n^{HLX} + (1+b)(1-\alpha_n^{HLX})\alpha_n^{Y} + \left(d\alpha_n^{KG}\alpha_n^{K} + d\alpha_n^{KP}(1-\alpha_n^{K})\right)\alpha_n^{KL}(1-\alpha_n^{Y}) = d.$$

The trend of the remaining variables can be found easily. For further reference, we list the followings:

$$\hat{C}_{nl,t} = \frac{C_{nl,t}}{N_t A_t^{*\alpha_n^{CP}(\alpha_n^{CD}(d\alpha_n^{D}\alpha_n^{h}-d\alpha_n^{ND})+d\alpha_n^{ND}-d\alpha_n^{CG})+d\alpha_n^{CG}}},$$

$$\hat{V}_{nl,t} = \frac{V_{nl,t}}{N_t^{\frac{\xi_n}{1-\sigma_n}} A_t^{*\alpha_n^{CP}(\alpha_n^{CD}(d\alpha_n^{D}\alpha_n^{h}-d\alpha_n^{ND})+d\alpha_n^{ND}-d\alpha_n^{CG})+d\alpha_n^{CG}}},$$

$$\hat{\lambda}_{nl,t}^{B} = \frac{\lambda_{nl,t}}{N_t^{\frac{\xi_n}{1-\sigma_n}-1} A_t^{*\alpha_n^{CD}(d\alpha_n^{D}\alpha_n^{h}-d\alpha_n^{ND})-1-b+d\alpha_n^{ND}+(\alpha_n^{CP}-1)(\alpha_n^{CD}(d\alpha_n^{D}\alpha_n^{h}-d\alpha_n^{ND})+d\alpha_n^{ND}-d\alpha_n^{CG})}}.$$

Note that the growth rates of $V_{nl,t}$ and $\lambda_{nl,t}$ determine the interest rates on international financial markets. So, the trends of $V_{nl,t}$ and $\lambda_{nl,t}$ must be constant across countries. Or, as a slightly stronger restriction, all parameters underlying their trends must be constant across countries.

4. Estimation

We will use "world", "model area", and "rest of the world" to respectively refer to earth, the whole of country 1 to $\mathcal{N}$, and country 0. We estimate the (log-)linearized[27] model on annual data for the ROW and six other countries/regions: the US, the UK, Germany and France, (most of) the rest of EU (REU, including Belgium, Denmark, Finland, Italy, Netherlands, Portugal, Spain, Sweden), and the rest of model area (RMA, including Australia, Canada, Japan, Norway, Switzerland). They are indexed as regions one to six, respectively. Our choice of these 17 countries is based on data availability. Of course, Norway and Switzerland are more closely connected to the REU than to the RMA. However, in order to study EU policy, it is important that EU and non-EU countries are kept separate.

4.1. Preparing the model

The model presented in the last section is highly parameterised. As such, it is difficult to estimate, especially when $\mathcal{N}$ is large. However, as argued in the introduction, plausibly all parameters in production functions and preference should be identical across countries. A factory from one country can be moved to another, and, providing inputs are truly kept constant, should produce the same output. Likewise, the presence of migration means that all populations will mix in the long-run, so preference differences across countries should not be permanent.

In light of these arguments, and to help reduce dimensionality, we make essentially all parameters (including those of shocks) constant across countries. Constraining the parameters governing production and preference shock processes to be constant across countries can be justified by the fact that these are just as much parameters of production functions or preferences as are elasticities. Constraining the steady states, standard deviations and responsiveness of fiscal instruments to be constant across countries seems much more palatable once one remembers that these processes all have unit roots. Given that the processes have infinite variance, the same process can potentially explain both relatively high taxes in France and relatively low taxes in the US. Furthermore, as a result of the infinite variance, the "steady state" will not have the conventional interpretation as any kind of long-run or cross-country mean. Instead, it is just the point of linearization, which will be pinned down by its impact on the responses of all variables to all shocks. As this differential impact only comes from the model's non-linearities, there is little information in the data to separately pin down each country's steady-states of these variables. However, by using the same parameters across countries, the data at least allows us to obtain reasonable average effects across countries.

Aside from the unit rooms already introduced in fiscal instruments, we also introduce unit roots into further policy relevant variables for additional help in capturing permanent differences across countries. Specifically, population shares $\widetilde{N}_{n,t}$ are random walks with steady states calibrated for each country, but their standard deviations are fixed across countries. Trade elasticities with the ROW $e_{0,n}^{T}$ (potentially affected by foreign policy), the fractions of impatient households $\varpi_{n,t}$ (potentially affected by financial literacy policy

[27] Linearization is perhaps of particular concern due to the presence of shocks with a unit root. Thus, the model has no tendency to return to the steady-state about which the model is approximated. I.e. the model may end up far from the region in which the first order approximation is reasonable. This concern is justified, but in fact it is not specific to the unit root case. With highly persistent shocks, or strong endogenous persistence (as in this model), the model may take longer to return to steady-state than the length of the data, so accuracy is likely to be just as compromised. To deal with these accuracy concerns, we will be careful in interpreting results.

interventions), and the loan-to-value ratio $m_{n,t}$ (potentially affected by macroprudential policy) are random walks with all of their parameters, including steady-states, fixed across countries.

With most parameters fixed across countries, it is possible to make the model symmetric in per capital terms in the steady state. To do so, the steady-state fixed costs F_n^K and F_n^{WC}, and the steady-state level of almost final goods Y_n,[28] must be proportional across countries with the proportion given by the steady-state relative population $\frac{\widetilde{N}_n}{\sum_{m=0}^{N}\widetilde{N}_m}$. Furthermore, symmetry of the steady state requires that tariffs have zero steady states as they affect only the demand for foreign goods but not that for domestic goods.

The symmetry across countries also helps in finding the model's steady state, and hence greatly aids estimation. The symmetry also ensures that parameters remain stable as new countries are added. In econometric terms, the estimates of the model with two countries should be consistent for the model with four countries, and the estimates of the model with four countries should be consistent for the model with six countries.[29] Since, in practice, we estimate the model by gradually adding more countries, starting with the US and Germany, then adding France and the UK, this is helpful.

Finally, from estimates with four countries we found that the estimated parameters governing the exogenous physical capital depreciation rates, $\delta_{n,t}^{KP,s}$ for s $\in \mathrm{S}_0$ and δ_t^{KG}, are very close to each other. Hence, we fix these parameters across sectors. For the same reason, we do the same to Human capital depreciation rates $\delta_{n,t}^{HP,s}$ for s $\in \mathrm{S}_0$ and δ_t^{HG}. We also turn off the $\tau_{n,t}^{TC}$ shock, which is redundant given the model's other shocks.

We end up with a total of 316 parameters to estimate. While this is still large relative to the prior literature (for example, Smets & Wouters (2003) estimate 34 parameters), relative to the number of observables, this is quite sparsely parameterised. In particular, with six countries, we have 155 observables, giving an average of about 2.0 parameters per series, essentially equivalent to estimating 155 independent AR(1) models. Compare this to estimating a $VAR(1)$ on 155 observables which would result in 36115 parameters!

4.2. Steady-state computation

One of the greatest challenges with a model of this scale is in finding the steady-state. We take a nested approach. In particular, it turns out that given the level of certain aggregates, we can find the steady-state number of firms per protected industry by solving a univariate non-linear problem. This univariate problem has at most three solutions, but we know that the largest one is the correct one.[30] Furthermore, a good initial guess for this univariate problem is available from a constant mark-up approximation. Having this univariate problem nested within the overall steady-state finding routine isolates the problems caused by multiplicity, and the high degree of non-linearity in the steady-state of the endogenous growth part of the model.

The overall steady-state finding routine uses a Newton-type method. This involves repeatedly calculating the residual of the model's static equations given guesses of certain aggregates (calculating these residuals requires solving the aforementioned univariate problem). Extensive algebraic simplifications reduces the number of unknowns and residuals to only 28. To facilitate finding a solution to this problem, we maintain a cache of inputs and outputs, and map new inputs to the closest previously observed input.

[28] By construction, this is equal to Ω_n^Y.

[29] More precisely, any parameter that is identified in the two-country model (as the sample length goes to infinity) will be a consistent estimator for the same parameter in the four or six country model.

[30] See Holden (2016).

When difficulties are encountered, we take homotopy steps from a known point towards the desired one. For the sake of speed, the code calculating the residuals of the model's equations is compiled to MEX using MATLAB Coder.

4.3. Data and measurement equations

Matching medium-frequency dynamics requires using long span data, so we can distinguish between permanent and highly persistent movements. Our primary data source is the Jordà-Schularick-Taylor Macrohistory Database (Jordà, Schularick & Taylor 2016), which contains macro series since 1870 for all 17 countries we consider. This is augmented with data from the Penn World Tables (Feenstra, Inklaar & Timmer 2015) and the Maddison project (Inklaar et al. 2018) and data from each country's central bank and statistic agency. UK data is collected from "A millennium of macroeconomic data" maintained by the Bank of England[31] (Thomas, Hills & Dimsdale 2010). World population data is from the United Nations[32] and the Maddison project. Our sample is annual data from 1870 to 2017, though not all variables are available for the full sample. In particular, some series are only available until the very recent. These series can still help smooth and forecast key variables, even if they don't have much impact on the likelihood. Appendix 10 describes the data in detail.

In choosing observables, we follow the principle that there should be one variable in growth rates for each of the model's stochastic trends. Other variables should be in ratios to these, to avoid throwing away information about steady-states. We thus include the world population and the model area real GDP growth rate, which correspond to the two stochastic trends A_t^* and N_t in the model. All other series are expressed as ratios. We observe in each region the regional population as shares of the world population; the regional GDP as shares of the model area GDP[33]; the private consumption to GDP ratio; the (private or gross) investment to GDP ratio; the total R&D expenditure to GDP ratio; the advanced secondary and tertiary education expenditure to GDP ratio;[34] the government expenditure (or separately government consumption and government investment) to GDP ratio; the export to GDP ratio; deflator inflation of private consumption, gross investment, government consumption, government investment, export, and import relative to that of GDP; the logarithm of hours worked per capita; the compensation of employees to GDP ratio; the consumption of fixed capital to GDP ratio; the ex post real interest rate[35]; the spread between households' borrowing cost and the risk-free interest rate; the government revenue to GDP ratio. The availability of regional data varies slightly across regions. The data is neither demeaned nor detrended.

In order to facilitate examining the EC's objectives for R&D spending across countries, we match the total R&D expenditure share in the model to the R&D expenditure share from the Eurostat data.[36] However, given that R&D in the model may not directly map to observed R&D, and that the data may be of limited

[31] See https://www.bankofengland.co.uk/statistics/research-datasets .

[32] See https://www.un.org/esa/population/publications/sixbillion/sixbilpart1.pdf
and United Nations, Department of Economic and Social Affairs, Population Division (2017). World Population Prospects: The 2017 Revision, DVD Edition.

[33] The avoid perfect multicollinearity, the GDP share of region 6 is dropped.

[34] This is levels 3 to 8 of the 2011 International Standard Classification of Education. In many countries, this corresponds to the education levels which are optional, or at least subject to extensive choice. Thus, it is a natural match to the human capital expenditure in our model which is strictly optional.

[35] We treat the ex post real interest rate in Germany during the periods of hyperinflation as unobservable.

[36] Additionally, we assume that R&D is part of measured investment, in line with modern national accounting practices, which provides further information on R&D.

reliability, we allow observed R&D shares to be some parameter times the R&D shares from the model, where the parameter ($\exp rdy$) is constant both across time and across countries.

The need for the $\exp rdy$ parameter stems from assorted differences between R&D in the model and R&D in the data. Firstly, process R&D in our model captures small incremental improvements, many of which may be made by ordinary employees in the course of their usual work, and which will usually not result in a patent or other tangible intellectual property product. As such, the bulk of the time spent on these improvements is very unlikely to be captured in the national accounts. Secondly, the national accounts "intellectual property products" measure includes some R&D that is aimed at creating differentiation rather than improving productivity (better mapping to fixed costs), as well as some activities not traditionally considered as R&D. For example, in the US NIPA, measured investment in intellectual property products includes expenditure on software copies/originals as well as investment in producing movies/TV/books/music. Finally, measured public R&D is likely to capture some activities that are so "basic" as to have no impact on aggregate productivity for the foreseeable future, while in the model it captures quite practical knowledge. Our additional parameter gives some flexibility in allowing for such discrepancies between the model and the data.

Because we are using historical data which may be unreliable at times, we allow for a measurement error for each observable. However, the standard deviations of the same observables are the same across countries. To specify the measurement equations, we define the corresponding concepts in the model. GDP is given by

$$\begin{aligned}
GDP_{n,t} &= I_{n,t}^{KP}P_{n,t}^{KP} + I_{n,t}^{HP}P_{n,t}^{HP} + I_{n,t}^{D}P_{n,t}^{D} + I_{n,t}^{ND}P_{n,t}^{ND} + I_{n,t}^{KG}P_{n,t}^{KG} + I_{n,t}^{HG}P_{n,t}^{HG} + I_{n,t}^{CG}P_{n,t}^{CG} \\
&+ \sum_{m=0,m\neq n}^{\mathcal{N}} \left(1 + (1-\phi_n^{XT})\iota_{n,t}^{XT} + (1-\phi_m^{MT})\iota_{m,t}^{MT}\right)P_{n,t}^{T}Y_{m,n,t}^{T} \\
&+ \sum_{m=1,m\neq n}^{\mathcal{N}} \left(1 + (1-\phi_n^{XW})\iota_{n,t}^{XW} + (1-\phi_m^{MW})\iota_{m,t}^{MW}\right)P_{n,t}^{W}Y_{m,n,t}^{W} \\
&- \sum_{m=0,m\neq n}^{\mathcal{N}} \left(1 + (1-\phi_m^{XT})\iota_{m,t}^{XT} + (1-\phi_n^{MT})\iota_{n,t}^{MT}\right)P_{m,t}^{T}Y_{n,m,t}^{T} \\
&- \sum_{m=1,m\neq n}^{\mathcal{N}} \left(1 + (1-\phi_m^{XW})\iota_{m,t}^{XW} + (1-\phi_n^{MW})\iota_{n,t}^{MW}\right)P_{m,t}^{W}Y_{n,m,t}^{W} \\
&+ Y_{n,t}^{WC}P_{n,t}^{WC}\,\frac{\frac{\mathcal{F}_t^I}{E_t^I}\left[|\mathbb{P}_t| - (1-q)(1-\delta_{\mathbb{I},t})|\mathbb{P}_{t-1}|\right] + |\mathbb{P}_t|\mathcal{W}_t^{RP}J_t^P}{Y_{n,t}^{WC}},
\end{aligned}$$

where the last term is country n's contribution to R&D in the engineering sector, which is considered as non-residential fixed investment by the recent System of National Accounts (SNA).

GDP can be decomposed into personal consumption expenditure (PCE), government consumption expenditure (GCE), Private domestic investment (PDI), Government gross investment (GGI), and net exports. Note that new housing is considered as residential fixed investment by the SNA but as durable goods in the model. Also, investment in education is treated as a (public or private) service by the SNA. Our definitions reflect these differences:

$$PCE_{n,t} + GCE_{n,t}\alpha_{n,t}^{pub\_edu} = I_{n,t}^{D}P_{n,t}^{D}(1-\alpha_n^{resid}) + I_{n,t}^{ND}P_{n,t}^{ND} + I_{n,t}^{HP}P_{n,t}^{HP},$$

$$PDI_{n,t} = I_{n,t}^{KP}P_{n,t}^{KP} + I_{n,t}^{D}P_{n,t}^{D}\alpha_{n,t}^{resid} + Y_{n,t}^{WC}P_{n,t}^{WC}\,\frac{\frac{\mathcal{F}_t^I}{E_t^I}\left[|\mathbb{P}_t| - (1-q)(1-\delta_{\mathbb{I},t})|\mathbb{P}_{t-1}|\right] + |\mathbb{P}_t|\mathcal{W}_t^{RP}J_t^P}{Y_t^{WC}},$$

$$GCE_{n,t}(1-\alpha_{pub\_edu}) = I_{n,t}^{CG} P_{n,t}^{CG},$$
$$GGI_{n,t} = I_{n,t}^{KG} P_{n,t}^{KG} + I_{n,t}^{HG} P_{n,t}^{HG},$$

where $\alpha_{n,t}^{pub\_edu}$ is the share of public education expenditure in GCE, $\alpha_{n,t}^{resid}$ is the shares of housing in durable goods, both are random walks. The real terms and deflators of national account variables are calculated by chain-type fisher indexes, following the same treatment in the data.

Statistics agencies construct consumption of fixed capital (CFC) using the perpetual inventory method. That is, a fixed depreciation rate is attached to each type of asset. Consequently, the movements of CFC data reflect compositional changes in investment goods (i.e. in some periods shorter life goods are purchased than in others) and cannot possibly track the stochastic depreciation due to variable capacity utilisation. To be consistent with data, we approximate the perpetual inventory method as follows. For a given type of asset, let I_t be net-of-adjustment-cost investment, K_t be the stock and δ_t^* be the exogenous depreciation rate of assets purchased in period t, which will follow an $AR(1)$ process. Then according to the perpetual inventory method:

$$K_t = \sum_{s=-\infty}^{t} (1-\delta_s^*)^{t-s} I_s.$$

The effective depreciation rate from period t to period $t+1$, δ_t, should satisfy:

$$K_{t+1} = (1-\delta_t)K_t + I_{t+1},$$

thus:

$$1-\delta_t = \frac{K_{t+1}-I_{t+1}}{K_t} = \frac{\sum_{s=-\infty}^{t+1}(1-\delta_s^*)^{t+1-s}I_s - I_{t+1}}{\sum_{s=-\infty}^{t}(1-\delta_s^*)^{t-s}I_s} = \frac{\sum_{s=-\infty}^{t}(1-\delta_s^*)^{t+1-s}I_s}{\sum_{s=-\infty}^{t}(1-\delta_s^*)^{t-s}I_s}.$$

In order to produce an expression which does not require us to track the entire history of δ_t^* and I_t^*, we log-linearize this expression around $\delta_t^* = \delta^*$, where δ^* is the steady-state of δ_t^*. This gives:

$$\log(1-\delta_t) \approx \frac{\sum_{s=-\infty}^{t}(1-\delta^*)^{t-s}I_s \log(1-\delta_s^*)}{\sum_{s=-\infty}^{t}(1-\delta^*)^{t-s}I_s},$$

which has a recursive representation as required. The contribution to CFC of this type of asset is δ_t times the stock value determined according to its law of motion in the model. We apply this method to physical capital, residential durable goods, and public R&D capital. The sum of them is linked to the data.

The rest of the model's observables are defined as follows. The credit spread is the difference between the shadow price of domestic bonds relative to the risk-free interest rate:

$$spread_t = \frac{1}{\mathbb{E}_t\left[\frac{\partial V_{nb,t}}{\partial V_{nb,t+1}} \frac{\lambda_{B,nb,t+1}}{\lambda_{B,nb,t}}\right]} - \frac{1}{Q_t}.$$

Compensation of employees includes both wages and human capital return:

$$Compensation_t = \sum_{s\in S_0} R_{n,t}^{HP,s} HP_{n,t}^s + \sum_{s\in S_2} L_{nl,t}^s W_{n,t}^s + \sum_{s\in\{S_0,S_1\}} L_{nb,t}^s W_{n,t}^s + \sum_{s\in\{T,NT,W\}} L_{nb,t}^{Ss} W_{n,t}^s.$$

Hours worked is linked to data via a parameter *hpop* that maps the units of the data to the units of the model:

$$\log\frac{Hours_t}{Population_t} = \left[\log\frac{\left(\sum_{s\in S_2} L_{nl,t}^s + L_{nb,t}^s\right)}{N_{n,t}} - ss\right] + hpop.$$

4.4. Estimation details

The linearized model is estimated by Bayesian methods, which combine the parameters' prior distributions with the model's likelihood function. Since evaluating the likelihood is so slow (over one core minute per likelihood evaluation even with a compiled steady state routine[37]), we do not attempt to integrate over the posterior density with Markov Chain Monte Carlo. Instead, we maximise the posterior density to find the posterior mode, and then use a Laplace approximation to obtain credible sets.

In the likelihood, we allow for a fairly diffuse initial distribution of the state, as the presence of unit roots means that the model has no stationary distribution. This would be desirable even without unit roots in the model though, as the diffuse initial distribution of the state captures the fact that many state variables were away from their long-run levels at the start of the sample. It also enables us to capture some structural change, via gradual convergence of shock processes towards their long-run levels. This helps as many of the "great ratios" are trending in the data. In particular, we construct the initial distribution of the state as follows. We first produce a scaled version of the model's transition matrix with spectral radius equal to the minimum of the original transition matrix's spectral radius, and an estimated parameter $\mathfrak{a} \in [0,1]$ in the unit interval. We then calculate the covariance of the stationary distribution of the distorted model with the scaled transition matrix. Suppose this matrix is $\mathcal{C}$ and just its diagonal is contained in the matrix $\mathcal{D}$. Then the final initial covariance matrix we use for the state is $\mathfrak{d}\left(\mathcal{D}^{\frac{1-\mathfrak{b}}{2}}\mathcal{C}^{\mathfrak{b}}\mathcal{D}^{\frac{1-\mathfrak{b}}{2}}\right)^{\mathfrak{c}}$ for estimated parameters $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$.

Our initial search for the posterior mode used the CMA-ES algorithm (Hansen 2006) with a large population (tracking 32 points), which has good global search performance. We adapted both the algorithm and Dynare (Adjemian et al. 2011) to permit this maximisation to be run over multiple cores. From a region of reasonably high density, we then used a Newton type method (again adapted to support parallelisation) and a novel parallel compass search algorithm to obtain the actual maximum.[38] Total running time was several months.[39]

4.5. Priors

The broad structure of priors for the model's parameters are given in Table 1 below. Where possible, we follow the literature in assigning priors. For parameters that are typically not estimated in the literature, such as factor shares, we employ fairly diffuse priors to cover reasonable values. All parameters other than the steady-state population shares are fixed across countries and hence have no subscript n. Full details on the origins of these priors are given after the table.

---

[37] Only around two seconds per likelihood evaluation were spent in computing the steady state when using the compiled code.
[38] These are implemented in the custom version of Dynare available from: https://github.com/tholden/dynare/releases .
[39] Precise run time figures are difficult as the code has been started and stopped many times, with small changes made and then the estimation resumed from the same point.

Table 1: Treatment of parameters

Note: N, LiN, G, IG, B, U denote respectively normal, logit-normal, gamma, inverse gamma, beta, and uniform distributions.

Variable	**Description**	**Priors (mean, std.)**
Stationary distribution		
$\mathfrak{a}$	Eigenvalue trunc. for stationary dist. calculation	$LiN(0.83, 0.31)$[40]
$\mathfrak{b}$	Strength of correlation in stationary dist.	$U(0,1)$
$\log \mathfrak{c}$	Exponentiation of stationary dist. covariance	$N(0,5)$
$\log \mathfrak{d}$	Scale of stationary dist. covariance	$N(0,10)$
Engineering sector		
λ	The degree of difference between industries	$G(1, 0.1)$
η	The degree of difference between firms, relative to that between industries	$B(0.1, 0.01)$
ϕ^R	The spillover from product innovation to research	$G(0.0015, 0.00075)$
ζ^R	The increasing difficulty in research relative to the spillover from process innovation	$G(0.008, 0.004)$
γ	Parallelizability of research	$G(5, 5)$
ρ	The bargaining power of engineering producers	$B(0.1, 0.05)$
q	Probability of losing a patent	$B(0.05, 0.01)$
ψ	The elasticity of product depreciation	$G(0.1, 0.05)$
Z	Research idea (steady state)	Normalised
$\tilde{\delta}_{\mathbb{I}}$	The exogenous product depreciation rate (steady state)	$G(0.01, 0.005)$
$\mathcal{F}^I$	Difficulty of product invention (steady state)	$G(5, 5)$
Population		
ψ^b	The elasticity of borrower migration	$G(10^{-5}, 5 * 10^{-6})$
ψ^l	The elasticity of lender migration	$G(10^{-5}, 5 * 10^{-6})$
GN	The growth rate of world population (steady state)	$G(1.013, 0.01)$
ϖ	The ratio of impatient households (steady state)	$B(0.5, 0.22)$
$\tilde{N}_n$	Population shares (steady state)	Calibrated
Preference		
σ	The inversed intertemporal elasticity of substitution	$G(1, 0.5)$
γ^V	Risk aversion	$G(10, 5)$
ξ	The degree of total utilitarianism	$LiN(0.5, 0.050)$ [41]
h	Habit	$B(0.5, 0.22)$
$\bar{\beta}_b$	The discount factor of impatient households	$B(0.89, 0.005)$
$\bar{\beta}_l$	The discount factor of patient households	$B(0.96, 0.005)$
b^β	The elasticity of discount factor	$G(0.005, 0.001)$
"ν"s	The inversed elasticities of labour supply	$G(2, 0.75)$
κ^0	Disutility shock (steady state)	$N(0, 0.1)$
"κ"s	Weights on labour disutility (steady state)	$G(1, 0.1)$

[40] A logit transformation of $\mathfrak{a}$ has a Normal prior with mean 5 and standard deviation 5.

[41] A logit transformation of ξ has a Normal prior with mean 0 and standard deviation 1.1.

Variable	Description	Priors (mean, std.)
The borrowing constraint		
ρ^B	Borrowing constraint inertia	$U\left(0.5, \sqrt{1/12}\right)$
m	The loan-to-value ratio (steady state)	$U\left(0.5, \sqrt{1/12}\right)$
Capital and durable goods accumulation		
$\theta^{GKP}, \theta^{GHP}, \theta^{GD}$	Capital adjustment cost	$G(10, 5)$
$\theta^{PKP}, \theta^{PHP}, \theta^{PD}$	Time-to-build frictions	$G(10, 5)$
$\theta^{KU0}, \theta^{DU0}$	Slops of utilization depreciation	$G(0.2, 0.1)$
$\theta^{KU1}, \theta^{DU1}$	Elasticities of utilization depreciation	$G(1, 0.5)$
δ^D	The depreciation rate of durable goods (steady state)	$B(0.05, 0.025)$
"δ^K"s	The depreciation rate of physical capital (steady state)	$B(0.1, 0.05)$
"δ^H"s	The depreciation rate of human capital (steady state)	$B(0.023, 0.013)$
Production		
"Ω"s	Productivity shocks (steady state)	Normalised
α^{NC}	Non-tradeable share of final goods	$B(0.5,0.22)$
"α"s	Factor or goods shares in production or preference	$LiN(0.5,0.1181)$ [42]
"ϱ"s	The speed towards Cobb-Douglas or GHH	$B(0.5, 0.22)$
"e"s	The short-run elasticity of substitution between goods or production factors	$G(1, 0.5)$
F^{WC}, F^K	Normalised fixed cost of country 1	$G(1,1)$
o	Frictions in engineering adoption	$G(100, 70)$
Fiscal policy		
$\tau^{KG}, \tau^{HG}, \tau^{CG}$	Normalised government expenditure (steady state)	$B(0.2, 0.1)$
τ^{Lb}	Elasticity of progressive labour taxes (steady state)	$B(0.1, 0.05)$
The rest of the "τ"s	Taxes (steady state)	$N(0, 0.1)$
"ι"s	Tariffs (steady state)	Fixed at 0
"ϕ^Y"s	Responsiveness to output growth	$N(0, 0.5)$
"ϕ^L"s	Responsiveness to labour	$N(0, 0.5)$
"ϕ^u"s	Responsiveness to common shocks	$N(0, 0.5)$
"ϕ"s	One minus the extent to which taxes are pure wedges	$U\left(0.5, \sqrt{1/12}\right)$
ROW		
p_0^T	The relative price of ROW tradable goods	$G(1, 0.5)$
y_0^{TC}	The ROW's relative demand for tradable goods	$G(1, 0.5)$
e_0^T	The ROW's price elasticity of tradable goods demand	$G(1, 0.5)$
Measurement equations		
$hpop$	Hour unit transformer	$G(6.5, 0.1)$
rdy	The R&D spending matcher	$N(0, 0.2)$
α^{pub_edu}	The shares of public education expenditure in GCE	$B(0.25, 0.05)$
α^{resid}	The shares of housing in durable goods	$B(0.33, 0.05)$

[42] A logit transform of the "α"s is normally distributed with mean 0 and standard deviation 0.5.

Variable	Description	Priors (mean, std.)
Standard deviations and persistence		
"σ"s	Standard deviations of measurement errors	$IG(0.001, 0.001)$
	Standard deviations of policy shocks	$IG(0.001, 0.02)$
	Standard deviations of other exogenous processes	$IG(0.005, 0.1)$
"ρ"s	First order persistence of exogenous processes	$LiN(0.5, 0.2227)$ [43]

There is not much prior information regarding the parameters in the engineering sector. Their means are chosen based on the calibration exercise of Holden (2016) and their standard deviations are chosen to be relatively large. In Holden (2016), γ is chosen to match growth rate, $\mathcal{F}^I$ is used to vary the number of firms in protected industries. ϕ^R, ζ^R, p, $\tilde{\delta}_{\mathbb{I}}$ are adjusted to match mark ups of 20% in frontier industries and 5% in non-frontier ones. The mark up domain is $(\eta\lambda, \lambda]$. q is set to make the mean of patent durations of 20 years.

With regard to parameters governing population, the mean of the population growth rate is set to its average in data. The elasticities of migration have priors such that doubling welfare in a country would attract immigration of about 0.002% of global population. The prior on impatient household shares is centred at 50% with a rather flat distribution.

Priors of preference parameters are similar to those in the literature. The literature that employs Epstein–Zin types of preference has found the intertemporal elasticity of substitution (IES) on both sides of unity, depending on the focus of the paper. Work on long-run risks and bond premium (e.g. Rudebusch & Swanson 2012) typically find a small IES of around 0.5. Work on exchange rates and international capital flow (e.g. Colacito et al. 2018; Kollmann 2017) uses a larger IES around 1.5. We set the prior mean of σ to 1 and the prior standard deviation large enough to cover those values used in the literature. The risk-aversion parameter γ^V has been found to be relatively large in the prior literature. We take 10, an intermediate value in the literature, as the prior mean. We let the data to speak for the degree of total utilitarianism, so its standard deviation is relatively large. The rest of preference parameters have priors that are similar to Smets and Wouters (2003; 2007). The inverse labor elasticities, "ν"s, have a mean of 2 and standard deviation of 0.5. The habit parameter h is smaller than the literature because the Jaimovich and Rebelo preference has added extra persistence. The mean of patient households' discount factor would imply an 4% annual interest rate if there is no growth in the model. Its standard deviation is large enough to allow the model matching data in this endogenous growth model. The mean of impatient households' discount factor implies an 8% credit spread. We use uniform priors on the loan-to-value ratio m and the inertia parameter ρ^B, so they are free to adjust to make the borrowing constraint bind in the steady state. The disutility weights of each type of labour have a standard deviation of 0.1, which is about enough to account for the cross-sector preference.

Parameters governing capital and durable goods movements are also standard. The priors allow $\theta^{GKP}, \theta^{GHP}, \theta^{PKP}, \theta^{PHP}$ to imply a similar size of adjustment cost as Smets and Wouters. Steady-state depreciation rates are set to 0.1 for physical capital, 0.0228 for human capital (implying 30 years half-life), and 0.05 for durable goods (standard in models with housing). The priors on variable utilization are consistent with Baxter and Farr (2005).

[43] A logit transform of the "ρ"s is normally distributed with mean 0 and stand deviation 1.1.

On the production side, "ϱ"s control the speed at which the hybrid production function converges to a Cobb-Douglas function in the long run. They have a beta distribution with the mean of 0.5 and a standard deviation of 0.22. We use a logit-normal distribution on factor shares to reflect the belief that values closer to 1 or 0 are very unlikely. We let data decide if two production factors are complementary or substitutive, so the short-run elasticities of substitution are centred at unity. We use a very loose priors on fixed costs and the parameter of engineering adoption frictions.

Concerning fiscal policy, the elasticity of progressive labour taxes has a mean of 0.1, which is roughly consistent with income tax bands in Europe. The mean of government expenditure implies that the spending on each types of public goods is about 20% of output. Distortionary taxes partially capture pure wedges, so they are set to be symmetric around 0 with relatively large standard deviations. The coefficients of policy's reactions are also normal with zero mean. The prior of the extent to which taxes are wedges is uniform over [0,1].

The ROW parameters are all centred at 1 with a standard deviation of 0.5 to cover reasonable values. Parameters in measurement equations are set to values that are consistent with data. Persistences of all exogenous processes follows a logit normal distribution to prevent values nearly hitting the boundary. Seemingly random walk or trend behaviours in the data should be captured by factors that are designed to contain unit roots, e.g. fiscal instruments. Finally, priors of standard deviations are standard. Policy shocks are assumed to be smaller than non-policy shocks. For productivity shocks, the standard deviations are chosen such that the sum of those shocks according to (23) has a standard deviation of 0.005. We ensure measurement errors not to play a significant role by assigning a small standard deviation with a relative tight prior.

4.5.1. Hierarchical priors

To exploit our prior information in full, we amend some of the priors in Table 1 where we believe one parameter is informative on another. This is the case when, for example, the similar parameters appear in multiple sectors. To allow prior correlation, we employ a hierarchical structure. In what follows, the RHS variables are all normally distributed with means and standard deviations chosen to make the marginal priors of original variables on the LHS equal to those listed in Table 1.

The inverse elasticities of labour supply are correlated across sectors:

$$
\begin{array}{rcccccc}
\log \nu^{W} & = & \nu^{ALL} & + & \nu^{S0ALL} & + & \nu^{WALL} \\
\log \nu^{T} & = & \nu^{ALL} & + & \nu^{S0ALL} & + & \nu^{TALL} \\
\log \nu^{NT} & = & \nu^{ALL} & + & \nu^{S0ALL} & + & \nu^{NTALL} \\
\log \nu^{SW} & = & \nu^{ALL} & + & \nu^{SS0ALL} & + & \nu^{WALL} \\
\log \nu^{ST} & = & \nu^{ALL} & + & \nu^{SS0ALL} & + & \nu^{TALL} \\
\log \nu^{SNT} & = & \nu^{ALL} & + & \nu^{SS0ALL} & + & \nu^{NTALL}. \\
\log \nu^{D} & = & \nu^{ALL} & + & \nu^{S1ALL} & + & \nu^{Di} \\
\log \nu^{NDCG} & = & \nu^{ALL} & + & \nu^{S1ALL} & + & \nu^{NDCGi} \\
\log \nu^{K} & = & \nu^{ALL} & + & \nu^{S1ALL} & + & \nu^{Ki} \\
\log \nu^{H} & = & \nu^{ALL} & + & \nu^{S1ALL} & + & \nu^{Hi} \\
\log \nu^{h} & = & \nu^{ALL} & + & & & \nu^{hi}
\end{array}
$$

Elasticities of substitution between production factors can vary across S_0 and their hierarchical priors are given by:

$$
\begin{aligned}
\log e^{KLT} &= e^{KLALL} + e^{TALL} \\
\log e^{HLXT} &= e^{HLXALL} + e^{TALL} \\
\log e^{YT} &= e^{YALL} + e^{TALL} \\
\log e^{KLNT} &= e^{KLALL} + e^{NTALL} \\
\log e^{HLXNT} &= e^{HLXALL} + e^{NTALL}. \\
\log e^{YNT} &= e^{YALL} + e^{NTALL} \\
\log e^{KLW} &= e^{KLALL} + e^{WALL} \\
\log e^{HLXW} &= e^{HLXALL} + e^{WALL} \\
\log e^{YW} &= e^{YALL} + e^{WALL}
\end{aligned}
$$

Persistence (and similarly standard deviations) of productivity shocks has the hierarchical structure below:

$$
\begin{aligned}
\operatorname{logit} \rho^{\Omega ALL} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriS0ALL}} + \rho^{\Omega ALLi} \\
\operatorname{logit} \rho^{\Omega S0ALL} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriS0ALL}} + \rho^{\Omega S0ALLi} \\
\operatorname{logit} \rho^{\Omega TALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriS0ALL} + \rho^{\Omega TALLi} \\
\operatorname{logit} \rho^{\Omega NTALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriS0ALL} + \rho^{\Omega NTALLi} \\
\operatorname{logit} \rho^{\Omega WALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriS0ALL} + \rho^{\Omega WALLi} \\
\operatorname{logit} \rho^{\Omega KKALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriIntALL} + \rho^{\Omega KKALLi} \\
\operatorname{logit} \rho^{\Omega HLALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriIntALL} + \rho^{\Omega HLALLi} \\
\operatorname{logit} \rho^{\Omega KLALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriIntALL} + \rho^{\Omega KLALLi} \\
\operatorname{logit} \rho^{\Omega HLXALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriIntALL} + \rho^{\Omega HLXALLi} \\
\operatorname{logit} \rho^{\Omega HLXKLALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriIntALL} + \rho^{\Omega HLXKLALLi} \\
\operatorname{logit} \rho^{\Omega S1ALL} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriS1ALL}} + \rho^{\Omega S1ALLi} \\
\operatorname{logit} \rho^{\Omega KALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriS1ALL} + \rho^{\Omega KALLi} \\
\operatorname{logit} \rho^{\Omega HALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriS1ALL} + \rho^{\Omega HALLi} \\
\operatorname{logit} \rho^{\Omega NDALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriS1ALL} + \rho^{\Omega NDALLi} \quad . \\
\operatorname{logit} \rho^{\Omega Di} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriS1ALL}} + \rho^{\Omega Dii} \\
\operatorname{logit} \rho^{\Omega PALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriPGALL} + \rho^{\Omega PALLi} \\
\operatorname{logit} \rho^{\Omega GALL} &= \rho^{\Omega PriALL} + \rho^{\Omega PriPGALL} + \rho^{\Omega GALLi} \\
\operatorname{logit} \rho^{\Omega CALL} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriPGALL}} + \rho^{\Omega CALLi} \\
\operatorname{logit} \rho^{\Omega hi} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriPGALL}} + \rho^{\Omega hii} \\
\operatorname{logit} \rho^{\Omega CPi} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriPGALL}} + \rho^{\Omega CPii} \\
\operatorname{logit} \rho^{\Omega Ci} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriPGALL}} + \rho^{\Omega Cii} \\
\operatorname{logit} \rho^{\Omega TrALL} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriPGALL}} + \rho^{\Omega TrALLi} \\
\operatorname{logit} \rho^{\Omega Yi} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriPGALL}} + \rho^{\Omega Yii} \\
\operatorname{logit} \rho^{\Omega TCi} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriPGALL}} + \rho^{\Omega TCii} \\
\operatorname{logit} \rho^{\Omega WPi} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriPGALL}} + \rho^{\Omega WPii} \\
\operatorname{logit} \rho^{\Omega WCi} &= \rho^{\Omega PriALL} + \phantom{\rho^{\Omega PriPGALL}} + \rho^{\Omega WCii}
\end{aligned}
$$

Finally, the hierarchical structure of steady-state fiscal instruments (and similarly of "ϕ^Y"s, "ϕ^L"s, "ϕ^u"s, "ϕ"s and the standard deviations of policy shocks) is:

$$\begin{array}{rcccccc}
\operatorname{logit} \tau^{KG} & = & \tau^{SG1ALL} & + & \tau^{KGi} & & \\
\operatorname{logit} \tau^{HG} & = & \tau^{SG1ALL} & + & \tau^{HGi} & & \\
\operatorname{logit} \tau^{CG} & = & \tau^{SG1ALL} & + & \tau^{CGi} & & \\
\tau^{b} & = & \tau^{ALL} & + & \tau^{lbALL} & + & \tau^{bi} \\
\tau^{l} & = & \tau^{ALL} & + & \tau^{lbALL} & + & \tau^{li} \\
\tau^{D} & = & \tau^{ALL} & + & \tau^{CALL} & + & \tau^{Di} \\
\tau^{ND} & = & \tau^{ALL} & + & \tau^{CALL} & + & \tau^{NDi} \\
\tau^{NT} & = & \tau^{ALL} & + & & & \tau^{NTi} \\
\tau^{La} & = & \tau^{ALL} & + & \tau^{LALL} & + & \tau^{Lai} \\
\operatorname{logit} \tau^{Lb} & = & \tau^{ALL} & + & \tau^{LALL} & + & \tau^{Lbi} \\
\tau^{HW} & = & \tau^{ALL} & + & \tau^{HALL} & + & \tau^{WALL} \\
\tau^{HT} & = & \tau^{ALL} & + & \tau^{HALL} & + & \tau^{TALL} \\
\tau^{HNT} & = & \tau^{ALL} & + & \tau^{HALL} & + & \tau^{NTALL} \\
\tau^{KW} & = & \tau^{ALL} & + & \tau^{KALL} & + & \tau^{WALL} \\
\tau^{KT} & = & \tau^{ALL} & + & \tau^{KALL} & + & \tau^{TALL} \\
\tau^{KNT} & = & \tau^{ALL} & + & \tau^{KALL} & + & \tau^{NTALL} \\
\tau^{XW} & = & \tau^{ALL} & + & \tau^{XALL} & + & \tau^{WALL} \\
\tau^{XT} & = & \tau^{ALL} & + & \tau^{XALL} & + & \tau^{TALL} \\
\tau^{XNT} & = & \tau^{ALL} & + & \tau^{XALL} & + & \tau^{NTALL}
\end{array}.$$

The hierarchical structure given above make clearest how correlation in priors arises. In practice, however, we can estimate less parameters than those contained in the above equations. To see how to do this, consider some hierarchical priors:

$$a_1 \sim N(m, \sigma_a^2),$$
$$a_2 \sim N(m, \sigma_b^2),$$
$$m \sim N(\mu, \sigma_m^2).$$

To reduce the dimension from 3 to 2, a_1 and a_2 can be written as:

$$a_1 = \nu + k_{11}e_1 + k_{12}e_2,$$
$$a_2 = \nu + k_{21}e_1 + k_{22}e_2,$$

where $e_1 \sim N(0,1)$ and $e_2 \sim N(0,1)$, for some k_{11}, k_{12}, k_{21} and k_{22}. More generally, if we have a vector of parameters $a \sim N(\mu, \Sigma)$ where Σ may not be full rank, we can reduce the dimensionality by taking the Schur decomposition $UDU' = \Sigma$ and then selecting the columns of U corresponding to the non-zero elements of the diagonal of D, say $U_{\cdot 1}$ and D_{11} respectively. Then for $z \sim N(0, I)$, $\mu + U_{\cdot 1}D_{11}z$ is a draw from the same distribution as a.

4.5.2. Likelihood penalties

In addition to the penalization of the likelihood coming from the prior, we further penalize the likelihood to ensure that there is an interior maximum, given the hard constraints coming from e.g. the second order conditions. We also penalize the likelihood to ensure that parameters remain in the region in which there are no issues coming from the limited machine precision. (With such a large model, and unit root shock processes, errors caused by limited machine precision can become a significant issue.) These penalties are equivalent to having a richer prior with further sources of correlation in the prior.[44]

The penalties designed to ensure an interior maximum are as follows:

- $\max\{0, -\log(SOC)\}$, where SOC is a quantity which is positive if the second order conditions for engineering sector firms' research are satisfied in steady-state.[45]
- $\max\{0, -\log\left(\mu^{\mathrm{P}} - \frac{1}{\gamma}\right)\}$, where μ^{P} is the steady-state mark-up level in protected industries in the engineering sector. This ensures that $\mu^{\mathrm{P}} > \frac{1}{\gamma}$ which is required for the second order conditions and for growth to continue.
- $\max\{0, -\log\left(1 - \eta\lambda\omega^{\mathrm{P}}\right)\}$, where $\omega^{\mathrm{P}} = \frac{J^P(1-\eta)}{\left(J^P-(1-\eta)\right)^2(1+\mu^{\mathrm{P}})}$ and μ^{P} and J^{P} are respectively the steady-state mark-up level and number of firms in protected industries in the engineering sector. This is also required for the second order conditions.
- $\max\{0, -\log\left(J^{\mathrm{P}} - (1-\eta)\right)\}$, where J^P is the steady-state number of firms per protected industry in the engineering sector. This penalty ensures that mark-ups are finite and positive.
- $\max\{0, -\log(MaxResids)\}^2$, where *MaxResids* is the unique local maximum of the function whose zeros determine the steady-state of the number of firms per protected industry in the engineering sector.[46] If this quantity is positive, there are three solutions for the number of firms per protected industry, and it is correct to pick the one with the most firms (see Holden (2016)). If it is negative, only the solution with few firms exists, which is the wrong solution to pick (again, see Holden (2016)). Without this penalty constraining *MaxResids* to be positive, there is a jump in the posterior density as the parameters move from the region with three solutions to the region with only one. This makes it hard to maximise the posterior density, and may interfere with the calculation of the posterior's Hessian if the optimum is on the edge of this region.
- $2 \times (10000 \max\{0, \bar{e} - 1\})^4$ where $\bar{e}$ is the largest Eigenvalue included in the "stable" part of the solution to the model. This ensures that the stable part is truly stable without imposing a hard limit on the maximum acceptable Eigenvalue. This is important as the presence of unit roots in the model mean that we should expect "stable" Eigenvalues of one, which, due to numerical inaccuracies, will end up larger than one.

The penalties included purely to avoid numerical inaccuracies are as follows:

- $0.00001 \sum_{t=1}^{T} \log\left(\kappa_{F_t}\right)^4$, where κ_{F_t} is the two-norm condition number of the covariance matrix of period t's measurement conditional on information up to period t. This ensures that the measurement covariance matrix is well behaved with respect to inversion (which is necessary to evaluate the likelihood).
- $0.0001 \log(\kappa_A)^4$ where κ_A is the two-norm condition number of the matrix which must be inverted while solving the model in order to obtain the impact effect of shocks on state variables. This ensures that small changes in parameters do not lead to gigantic changes in the impact of shocks.

---

[44] In order to impose the custom penalties described here, changes to Dynare (Adjemian et al. 2011) were required. These are implemented in the custom version of Dynare available from: https://github.com/tholden/dynare/releases .

[45] *SOC* is defined in section 9.3.2 within the appendix.

[46] We refer the reader to the file "`GetJPSlow.m`" from the model's source code for exact details of this residual function.

5. Results

5.1. Parameter estimates

Table 2 gives the estimates we obtain for the model's main parameters, using data from our six regions (US, UK, France, Germany, REU, RMA). To reduce the barrage of numbers, we omit any persistences of shock processes, shock standard deviations, and all of the parameters governing fiscal instruments or other random walk processes. (Since the steady-states of random-walk processes do not have the standard interpretation, reporting them risks confusion.) To ease the computational burden of calculating posterior standard deviations, we report posterior standard deviations conditional on the values of all other parameters, i.e. based solely upon the diagonal of the Hessian. For parameters given hierarchical priors, we omit the posterior standard deviation, as the correlation in the prior means that even calculating conditional standard deviations would require the full Hessian. For parameters with logit normal priors, we use the delta method to construct standard errors. To further speed up the computation of posterior standard deviations, we use an asymptotic outer product gradient approximation to the Hessian of the posterior density.[47]

Table 2: Main parameter estimates

Variable	Priors (mean, std.)	Posterior mode	Conditional posterior std. dev.
$\mathfrak{a}$	$LiN(0.83, 0.31)$	0.9922	$9.2237 * 10^{-4}$
$\mathfrak{b}$	$U(0,1)$	1.0000	0.0038
$\log \mathfrak{c}$	$N(0,5)$	-0.0059	0.0283
$\log \mathfrak{d}$	$N(0,10)$	2.9874	0.1018
λ	$G(1, 0.1)$	1.0864	0.0125
η	$B(0.1, 0.01)$	0.1129	0.0014
ϕ^R	$G(0.0015, 0.00075)$	0.0011	0.0353
ζ^R	$G(0.008, 0.004)$	0.0060	0.0604
γ	$G(5,5)$	10.3095	1.4655
$\mathfrak{p}$	$B(0.1, 0.05)$	0.0890	0.0015
$\mathfrak{q}$	$B(0.05, 0.01)$	0.0504	0.0026
ψ	$G(0.1, 0.05)$	0.0760	0.0563
$\tilde{\delta}_{\mathbb{I}}$	$G(0.01, 0.005)$	0.0238	0.0038
$\mathfrak{F}^I$	$G(5,5)$	10.5361	0.4278
ψ^b	$G(10^{-5}, 5 * 10^{-6})$	$7.2116 * 10^{-6}$	$6.9672 * 10^{-6}$
ψ^l	$G(10^{-5}, 5 * 10^{-6})$	$8.4662 * 10^{-6}$	$1.6158 * 10^{-4}$
GN	$G(1.013, 0.01)$	1.0155	$2.4362 * 10^{-4}$
σ	$G(1, 0.5)$	0.5569	0.0158
γ^V	$G(10, 5)$	2.3399	0.0719
ξ	$LiN(0.5, 0.050)$	0.6984	0.0095
h	$B(0.5, 0.22)$	0.6717	$4.5916 * 10^{-4}$
$\bar{\beta}_b$	$B(0.89, 0.005)$	0.8951	0.0029
$\bar{\beta}_l$	$B(0.96, 0.005)$	0.9744	$1.8925 * 10^{-4}$
b^β	$G(0.005, 0.001)$	0.0082	$1.2312 * 10^{-4}$
ν^W	$G(2, 0.75)$	1.0249	
ν^T	$G(2, 0.75)$	0.6306	
ν^{NT}	$G(2, 0.75)$	0.7903	

[47] The contributions of the prior and penalties are distributed equally over all individual observations (some variable, in some country, at some period). Again, this feature is only present in the custom version of Dynare available from: https://github.com/tholden/dynare/releases .

Variable	Priors (mean, std.)	Posterior mode	Conditional posterior std. dev.
ν^{SW}	$G(2, 0.75)$	0.9627	
ν^{ST}	$G(2, 0.75)$	0.5923	
ν^{SNT}	$G(2, 0.75)$	0.7424	
ν^{D}	$G(2, 0.75)$	0.52222	
ν^{NDCG}	$G(2, 0.75)$	0.5972	
ν^{K}	$G(2, 0.75)$	0.4935	
ν^{H}	$G(2, 0.75)$	0.6166	
ν^{h}	$G(2, 0.75)$	0.8258	
κ^{0}	$N(0, 0.1)$	0.1508	0.0020
κ^{h}	$G(1, 0.1)$	0.9861	0.1870
κ^{T}	$G(1, 0.1)$	1.1131	0.1013
κ^{NT}	$G(1, 0.1)$	0.9951	0.0732
κ^{W}	$G(1, 0.1)$	0.9040	0.0560
κ^{ST}	$G(1, 0.1)$	1.0061	0.0379
κ^{SNT}	$G(1, 0.1)$	1.0225	0.0571
κ^{SW}	$G(1, 0.1)$	0.9754	0.0562
κ^{D}	$G(1, 0.1)$	0.9705	0.1172
κ^{NDCG}	$G(1, 0.1)$	0.9356	0.0518
κ^{K}	$G(1, 0.1)$	1.0168	0.0689
κ^{H}	$G(1, 0.1)$	0.9773	0.3370
ρ^{B}	$U\left(0.5, \sqrt{1/12}\right)$	0.7146	0.1029
θ^{GKP}	$G(10, 5)$	6.7977	1.0604
θ^{GHP}	$G(10, 5)$	7.2444	2.8008
θ^{GD}	$G(10, 5)$	10.3711	0.9740
θ^{PKP}	$G(10, 5)$	7.9131	1.2622
θ^{PHP}	$G(10, 5)$	10.0738	2.8041
θ^{PD}	$G(10, 5)$	4.3639	0.6375
θ^{KU0}	$G(0.2, 0.1)$	0.1502	0.0314
θ^{DU0}	$G(0.2, 0.1)$	0.1837	0.2781
θ^{KU1}	$G(1, 0.5)$	1.9140	0.1703
θ^{DU1}	$G(1, 0.5)$	0.5333	0.0102
δ^{D}	$B(0.05, 0.025)$	0.0297	$6.0922 * 10^{-4}$
δ^{K}	$B(0.1, 0.05)$	0.0166	$6.6685 * 10^{-4}$
δ^{H}	$B(0.023, 0.013)$	0.0332	0.0052
α^{CP}	$LiN(0.5,0.1181)$	0.7969	0.0021
α^{CD}	$LiN(0.5,0.1181)$	0.7029	0.0025
α^{h}	$LiN(0.5,0.1181)$	0.6045	0.0052
α^{KP}	$LiN(0.5,0.1181)$	0.7726	0.0067
α^{D}	$LiN(0.5,0.1181)$	0.5615	0.0020
α^{ND}	$LiN(0.5,0.1181)$	0.4496	0.0028
α^{HP}	$LiN(0.5,0.1181)$	0.4659	0.0081
α^{HW}	$LiN(0.5,0.1181)$	0.4041	0.0067
α^{HL}	$LiN(0.5,0.1181)$	0.2643	0.0154
α^{K}	$LiN(0.5,0.1181)$	0.8316	0.0036
α^{KL}	$LiN(0.5,0.1181)$	0.2492	0.0041
α^{HLX}	$LiN(0.5,0.1181)$	0.0978	0.0030
α^{Y}	$LiN(0.5,0.1181)$	0.3432	0.0020
α^{NC}	$B(0.5,0.22)$	0.8144	0.0020

Variable	Priors (mean, std.)	Posterior mode	Conditional posterior std. dev.
ϱ^{C}	$B(0.5, 0.22)$	0.7422	0.0227
ϱ^{CP}	$B(0.5, 0.22)$	0.9823	0.0080
ϱ^{CD}	$B(0.5, 0.22)$	0.6059	0.1643
ϱ^{KL}	$B(0.5, 0.22)$	0.8627	0.0247
ϱ^{HLX}	$B(0.5, 0.22)$	0.9932	$1.2121 * 10^{-4}$
ϱ^{Y}	$B(0.5, 0.22)$	0.9916	$3.2069 * 10^{-4}$
e^{CC}	$G(1, 0.5)$	0.7732	0.0401
e^{CP}	$G(1, 0.5)$	0.2179	0.0568
e^{NC}	$G(1, 0.5)$	0.0951	0.0264
e^{T}	$G(1, 0.5)$	0.3467	0.0015
e^{W}	$G(1, 0.5)$	8.0749	0.3866
e^{KLT}	$G(1, 0.5)$	0.4072	
e^{KLNT}	$G(1, 0.5)$	0.3785	
e^{KLW}	$G(1, 0.5)$	0.4517	
e^{HLXT}	$G(1, 0.5)$	1.4107	
e^{HLXNT}	$G(1, 0.5)$	0.7580	
e^{HLXW}	$G(1, 0.5)$	0.5550	
e^{YT}	$G(1, 0.5)$	1.2806	
e^{YNT}	$G(1, 0.5)$	1.2079	
e^{YW}	$G(1, 0.5)$	0.9267	
F^{WC}	$G(1,1)$	0.0184	0.0110
F^{K}	$G(1,1)$	0.5981	0.0310
o	$G(100, 70)$	54.9386	149.4562
p_0^{T}	$G(1, 0.5)$	1.6597	0.0053
y_0^{TC}	$G(1, 0.5)$	1.0099	0.0166
$hpop$	$G(6.5, 0.1)$	6.4676	0.0447
rdy	$N(0, 0.2)$	0.0288	0.0371

The following parameters are particularly notable. γ has a posterior mode far above its prior mean. This means there are strong decreasing returns to firm process innovation. Effectively, there are tight constraints on the amount of research that a firm can pack into a period. p is estimated to be moderately smaller than its prior mean, implying that the data favours patent holders having substantial bargaining power. q is close to 0.05, implying patents last for roughly 20 years, as in reality.

Despite a prior placing high weight on high degrees of risk aversion (prior mean of 10), we estimate that risk aversion is in fact barely above 2. This is gratifying as there are many reasons for scepticism about extreme values of risk aversion (see e.g. Dillenberger, Gottlieb & Ortoleva (2018)). Furthermore, the elasticity of intertemporal substitution is around one half, in line with the meta-analysis of Havranek et al. (2015). However, the household is estimated as having much stronger habits than in the prior mean. These are perhaps partially substituting for high degrees of risk aversion. ξ is estimated to be above one half, implying that households do value consumption per head higher when population is higher.

Labour supply elasticities are all higher than in the prior (i.e. the "ν" parameters are smaller). An elastic labour supply is often helpful in matching aggregate time series, particularly when labour market frictions are not modelled. These low elasticities also amplify labour flows between sectors. Unsurprisingly though, standard labour for producing widgets is supplied most inelastically, with a coefficient above one. Skilled labour used in widget production is the next most inelastically supplied variety. This effectively dampens the response of engineering to shocks, ensuring that there are not excessive movements in engineering productivity via the model's endogenous growth mechanism.

While capital adjustment costs are still substantial, they are generally lower than the prior mean. Thus, the model needs fewer frictions to explain the persistence of the data than expected (i.e. than in other models). This is perhaps unsurprising in hindsight given the strong endogenous persistence coming from the model's endogenous growth mechanism.

Taken literally, the share parameters ("α"s) would imply a rather small private capital share of output. (The steady-state labour share is around 80% in the model compared to 60% in the data, and the steady-state investment share is about 9%, compared to 18% in the data.) However, the presence of unit root taxes mean that the model can readily explain the observed data, even with a steady-state which is quite far from the data. It may also mean that parameters governing steady-states, such as these, may be weakly identified. In future work, we may impose a prior on the steady-state to help with this.

The "ϱ" parameters imply a fairly high degree of persistence of the production habit stock which governs how long it takes for the long-run unitary elasticity of substitution to dominate the short-run non-unitary one. Thus, the model is likely to behave like one with CES production, and so growth may appear "unbalanced" even over quite long horizons. Interestingly, the persistence is greatest for the aggregator combining engineering with the skilled labour and human capital bundle. This aggregator exhibits strong complementarities in the short-run within the widget and non-tradeable sector, matching our motivating intuition that human capital and engineering should be gross complements. The higher elasticity of substitution for this aggregator in the tradeable sector may perhaps reflect knowledge spill-overs coming from the international goods trade.

There is also a high degree of estimated complementarity between tradeables and non-tradeables, and across tradeables from different countries. This complementarity is helpful in generating comovement both within and across countries. We also find strong complementarity between durable services and non-durable goods in household preferences. This is natural, as, for example, a car is often necessary to enjoy trips to restaurants and cinemas.

5.2. Smoothed series

This section reports selected smoothed series obtained by applying the Kalman smoother to the six-country model. While the smoother tracks well the levels of the series in our observation set, including total R&D expenditure shares, and total human capital investment shares, it struggles to accurately track decompositions of these quantities (e.g. public vs private or across sectors). This is due to the presence of unit roots in the model which lead to high uncertainty about levels. Thus, where appropriate we report the series relative to their mean over the sample, to focus attention on the estimated dynamics.[48]

We start by looking at the model implied total R&D shares of GDP, in Figure 4. Since our data for this variable only begins in 1981, the paths before this date are entirely inferred from the other observables and the model's dynamics, via the Kalman smoother. The broad picture is one of a gradual increase in R&D expenditure over time. Unsurprisingly, the great depression coincided with a particularly stark fall in R&D

[48] We also omit the first two observations (1870 and 1871) from the smoothed results, which features a large jump in several variables as the model smooths between the diffuse initial distribution and the initial observations.

around the world, helping to explain the low productivity in the following years. The 1950s were a great global boom time for R&D though, according to the model's view of the data.

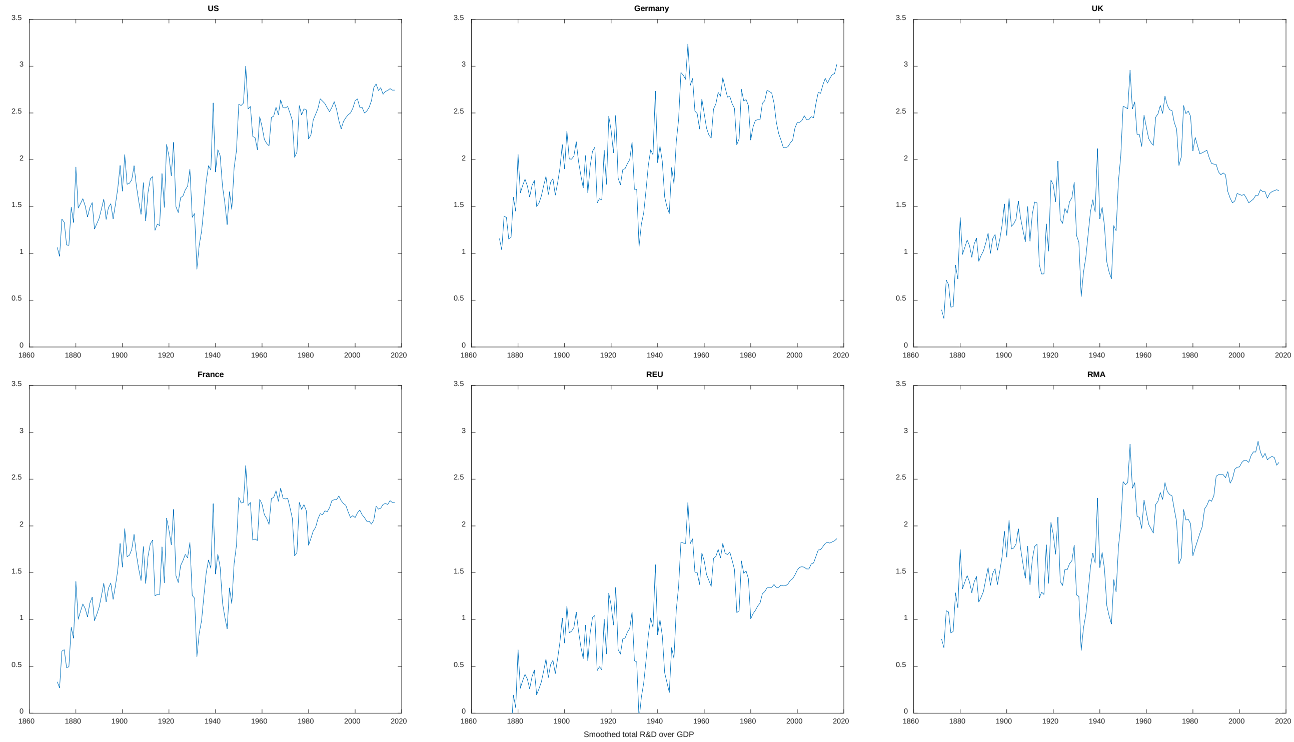

Figure 4: Smoothed total R&D over GDP

In Figure 5, we show the decomposition of this total quantity into public and private contributions. This is a non-trivial exercise as neither public nor private R&D expenditure are contained in our dataset, other than indirectly via their contribution to investment. According to the model's view of the data, most countries' public R&D share of GDP has been steadily falling. With the era of public large scale nuclear and space research behind us, this seems plausible. However, many countries experienced a mid-century surge in public R&D, no doubt associated with military research, the post WW2 arms race and reconstruction work. This picture of declining public R&D should be concerning to European policy-makers: it suggests that the "golden era" of European public R&D is behind us. In light of this, it is unsurprising that the EC is encouraging European countries to increase their R&D spending.

The picture of the decline of European public R&D is even clearer in Figure 6 which plots the level of public R&D capital stocks across countries, relative to the trend of this variable. Of course, the public R&D capital stocks are growing over time in all countries, but this figure implies they are growing less quickly than their long-run trend. While the US's public R&D capital stock (relative to trend) has fallen by only about 40 log points[49] over the sample, the German public R&D capital stock has fallen by over 100 log points, and the UK and French quantities have fallen by around 150 log points. Part of this may be the effects of the high levels of public R&D capital that European powers had built up before and during WW2, for military purposes, but even with this in mind, this represents a shocking fall.

[49] I.e. the logarithm of the public R&D capital stock divided by its trend has decreased by 0.4 over the sample.

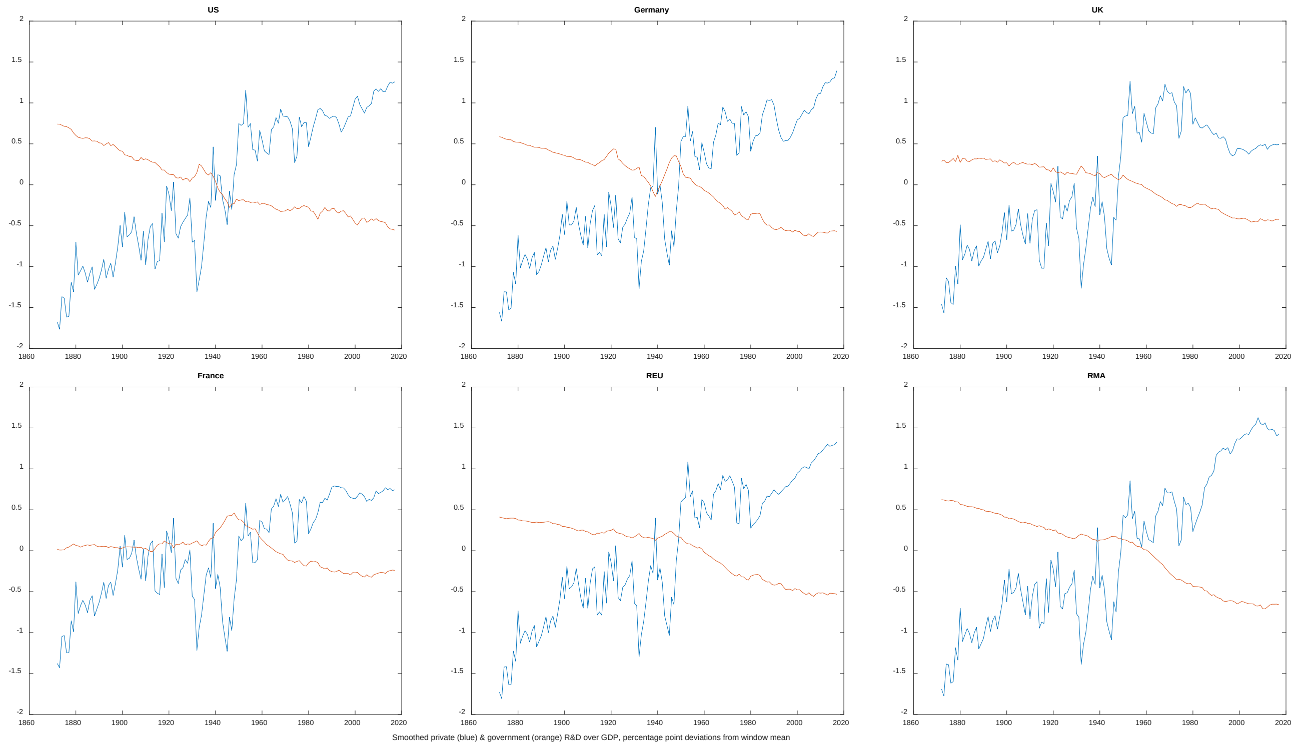

Figure 5: Smoothed private (blue) & government (orange) R&D over GDP, percentage point deviations from window mean

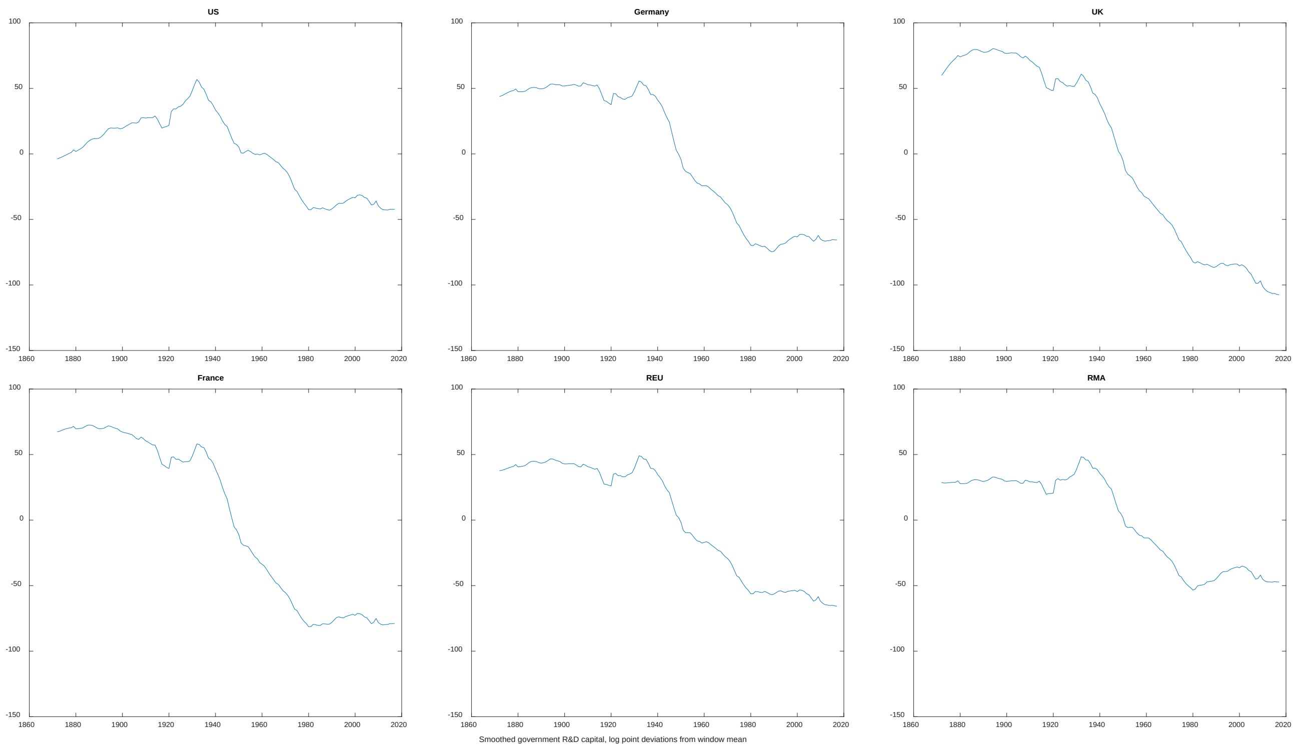

Figure 6: Smoothed government R&D stocks, relative to trend, log point deviations from window mean

For all countries, the model views private R&D expenditure shares as having increased over the sample. All countries have experienced substantial volatility in private R&D though, according to the model's view of the data. Particularly noticeable are the sharp drops in R&D expenditure during the great depression and WW2. Clearly then, these smoothed estimates of private R&D shares are pro-cyclical, implying that the dynamics of the estimated model favour pro-cyclical private R&D. This stems from the mechanisms of the Holden (2016) model. In good times, the returns to invention are high as the flow of profits obtained by sellers of the new good will be higher due to greater demand. Thus, product innovation is pro-cyclical. The extra demand also leads to firm entry within each industry, meaning more duplication of process research and hence higher total process innovation as well. While mark-ups fall, slightly depressing process incentives, this effect is dominated by the increase in the number of firms performing process research.

Since the gradual process of human capital accumulation is one of the key determinants of output per hour in our model, it is also instructive to look at changes in the levels of human capital stocks across countries. Figure 7 plots the smoothed human capital investment shares of GDP. The data on this variable starts in 2005, but has many missing values, with an average of between 5 and 6 observations per region. The Kalman smoother views human capital as reasonably stable in the US, moderately increasing in the UK, and declining everywhere else. WW2 was seen as a time of major human capital investment, in line with many (including women) being retrained to aid wartime production.

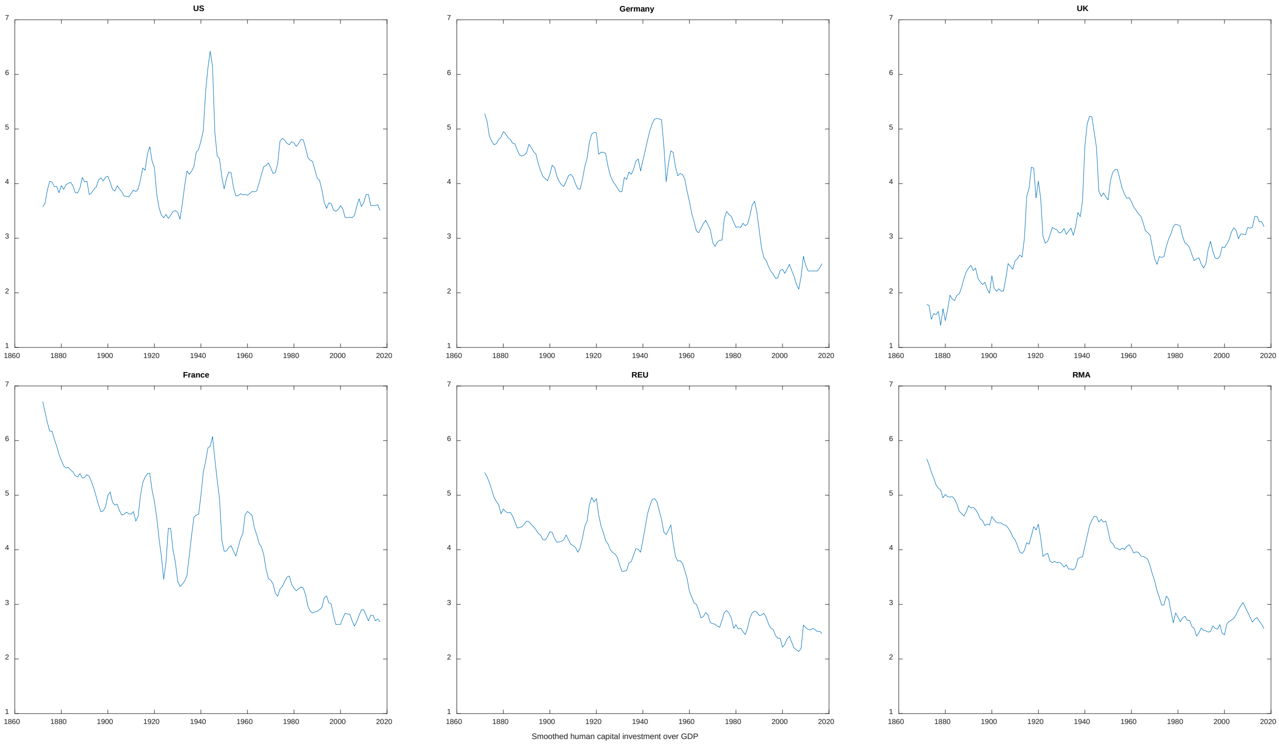

Figure 7: Smoothed human capital investment over GDP

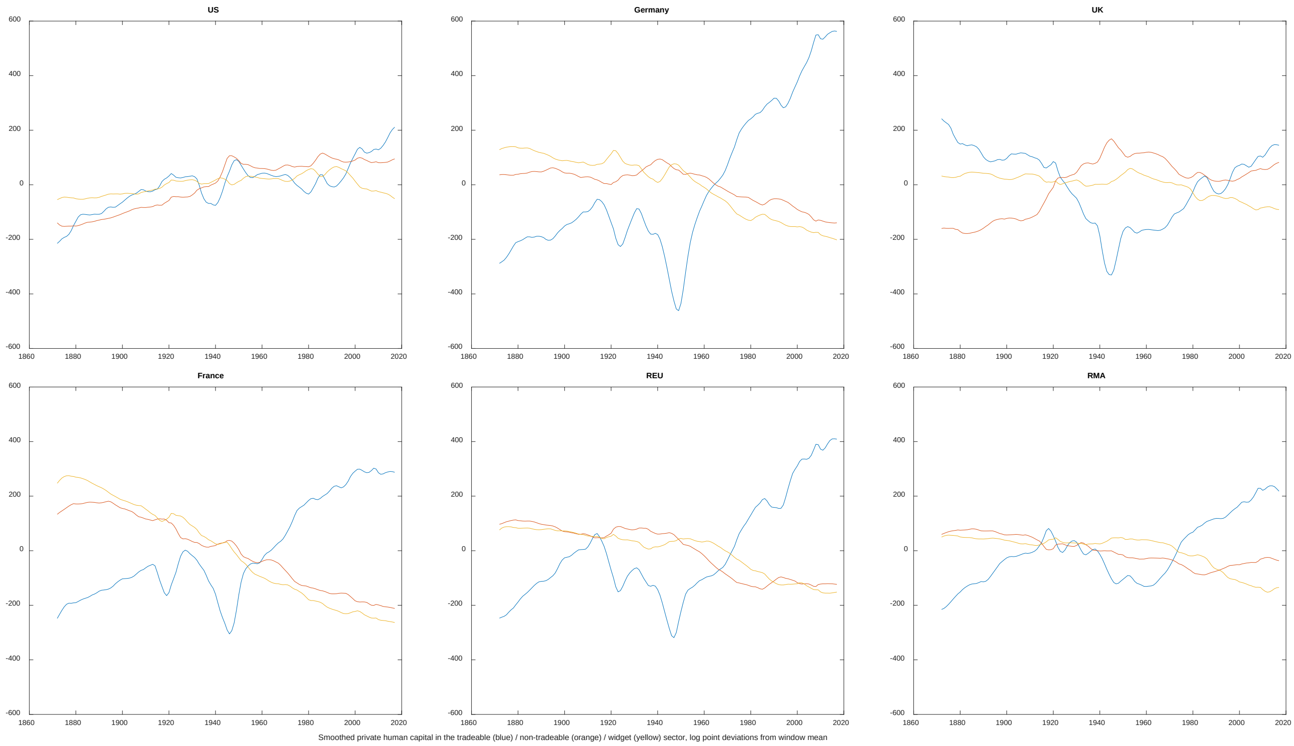

Figure 8: Smoothed private human capital in the tradeable sector (blue) / non-tradeable (orange) / widget (yellow), relative to trend, log point deviations from window mean

Human capital is likely to be particularly important in the widget and non-tradeable sectors, where it is a gross complement to engineering. Figure 8 shows the human capital stocks in each sector, relative to trend. Again, the picture is not universally wonderful for Europe. The US has achieved huge (over 200 log points) increases in human capital in the non-tradeable sector (relative to trend) and has at least prevented a decline in human capital in the widget sector over the sample. However, in all other regions, human capital in these two sectors has been falling (relative to trend) with only the exception of the UK's non-tradeable sector. For example, there has been a fall of around 500 log points in human capital (relative to trend) in the widget sector in France. The US's continuing high level of human capital in the widget sector means that its output per hour there will be particularly high. This in turn means that the US will contribute most to the advancement of the global technological frontier, and that the US will benefit from a low effective price of engineering, increasing output in all other sectors. In reality, this is manifested in the US's dominance of the tech industry. However, Europe can at least take some comfort in the gigantic surge in human capital in the tradeable sector it has experienced since WW2. This is particularly apparent in Germany, perhaps helping to explain the strength of the German automotive and industrial export industry.

5.3. Impulse response functions

We now examine the model's dynamic behaviour in more detail by looking at impulse responses. Due to the symmetry in parameters across countries, and our use of (log-)linearization, the percentage effect of a certain shock to French variables on France is identical to the percentage effect of a shock to the equivalent German variable on Germany (and likewise for other countries). Additionally, the shock to French variables will have the same percentage impact on Germany as the shock to German variables would have on France (and again likewise for other countries). For example, if an exogenous 1% shock to US almost final good productivity raises US GDP by 0.5% and UK GDP by 0.1%, then an exogenous 1% shock to UK almost final good productivity will raise UK GDP by 0.5% and US GDP by 0.1%. This would not be true in a higher order approximation to the model, as these impulse responses would then be a function of the level of state variables, which differ substantially across countries.

Given this, we do not need to show IRFs for every pair of countries. Instead it suffices to show IRFs for a generic "home" country (the country experiencing the shock) and a generic "foreign" country. These responses capture the average effect across countries of such shocks. This is ideal for a supra-national policy maker who is interested in the average effect their policy change might have over their region of interest. (Recall that policy changes are modelled as shocks to the policy instrument in question.) While we would have liked to have presented IRFs to a higher order approximation to the model (which would have differed across countries), without also estimating the model at higher order, these would have been misleading. Unfortunately, estimating a model as large as this at higher order is essentially impossible. That said, presenting average effects is entirely standard in many fields of applied economics. For example, in micro-econometric studies, it is standard to look at the average effect of treatment on the treated, despite potential heterogeneity on the effects of the treatment across individuals. The average effect is still informative of what might be expected in normal circumstances. Furthermore, given that we have a structural model, we can readily go on to describe how these responses might change depending on country-specific characteristics (their state variables).

Since the model we have presented has so many shocks, it is not practical to present IRFs to all of them. Instead, here we focus on those non-policy shocks for which a 1 standard deviation impulse results in a maximum change in home GDP over the first 40 years following the shock of at least 0.01%, relative to what it would be had no shock arrived. I.e. we focus on the most practically relevant non-policy shocks. In total we show six sets of impulse responses in this section. Four to global shocks, and two to country specific shocks. We look properly at policy shocks in the next section.

All plots show variables in log point deviation (1 meaning 1 log point, i.e. approximately 1%) from the level they would have attained had no shock hit, with the exception of variables given as a ratio, which are in percentage points (1 meaning 1 percentage point). All variables not explicitly given as a ratio are in per capita terms. Where the foreign country is affected differently to the home country, we show plots for both countries. In the plot labels, "Private H Investment" means "private investment in human capital", and the "government deficit" is the primary deficit i.e. government spending minus tax. Throughout this paper, when we refer to the "government deficit" we will mean the primary deficit.

5.3.1. Global population shock

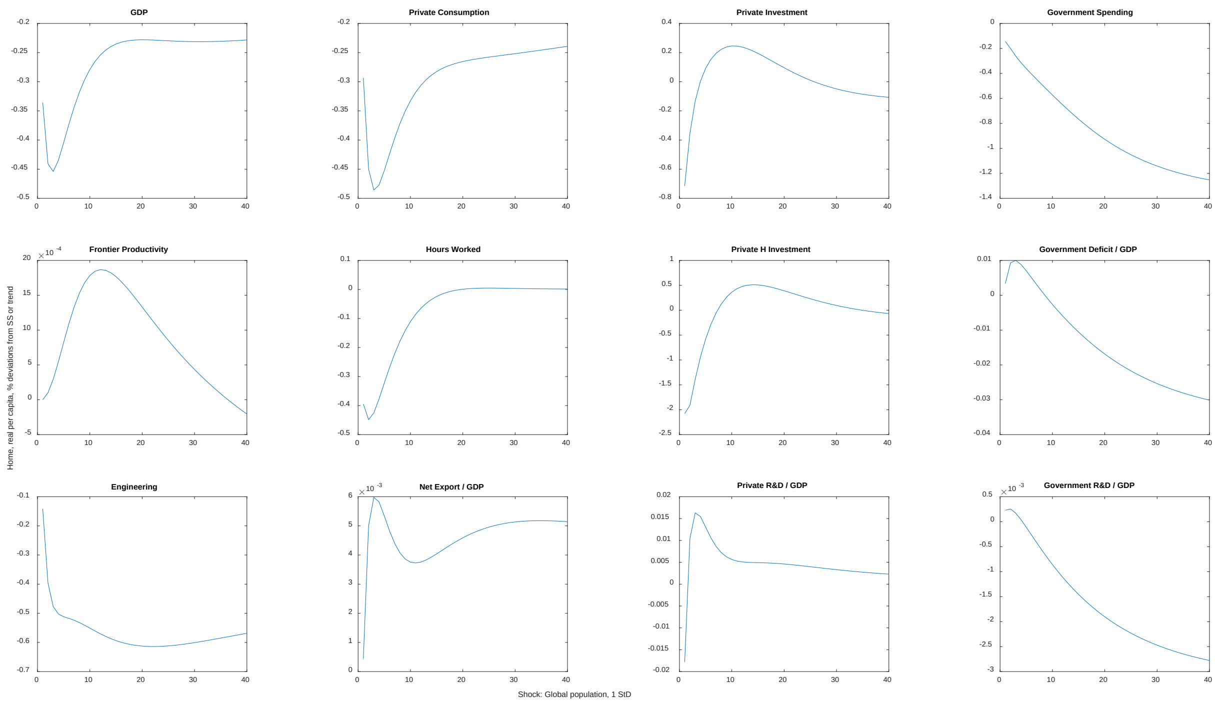

Figure 9: Impulse response in the home country to a 1 standard deviation increase in global population. All variables other than ratios are in per capita terms.

A global population shock simultaneously increases the growth rate of population in all countries, including the rest of the world. With higher population, but capital stocks fixed in the short-run, consumption per head must fall. Additionally, since there must be additional investment to bring capital per head up to its old value, there is additional need to substitute away from consumption over the medium term. In fact, the model's ability to generate comovement is strong enough that investment even falls on impact. This is thanks to the Jaimovich-Rebello (2009) preferences: with lower consumption per head, the household is closer to its subsistence level of felicity, so has to cut hours in order to keep felicity above the subsistence level. Viewed another way, the fall in consumption per head reduces demand, depressing wages, which also depresses hours due to the absence of wealth effects with these preferences. Since aggregate demand falls in per capita, so too does demand for engineering goods (per capita), reducing profits there. Additionally, the supply of widgets (per capita) will contract due to the drop in hours, further reducing engineering profits. This dampens the rate of invention of new products, meaning that fewer products are relatively new, and hence that aggregate productivity in the engineering sector is relatively lower, as endogenously new industries have higher productivity. (See Holden (2016).)

In the long-run, the magnitude (and sign) of the permanent effect on GDP is determined by the magnitude (and sign) of the permanent movement in frontier productivity. While frontier productivity initially rises as the slow adjustment of the measure of industries means there are fewer firms per industry and hence higher mark-ups, it does eventually fall below trend, suggesting small permanent negative effects of this shock. This subsequent fall is driven by the rebound in mark-ups as new industries enter to take advantage of the high growth during the recovery period.

5.3.2. Global engineering research idea shock

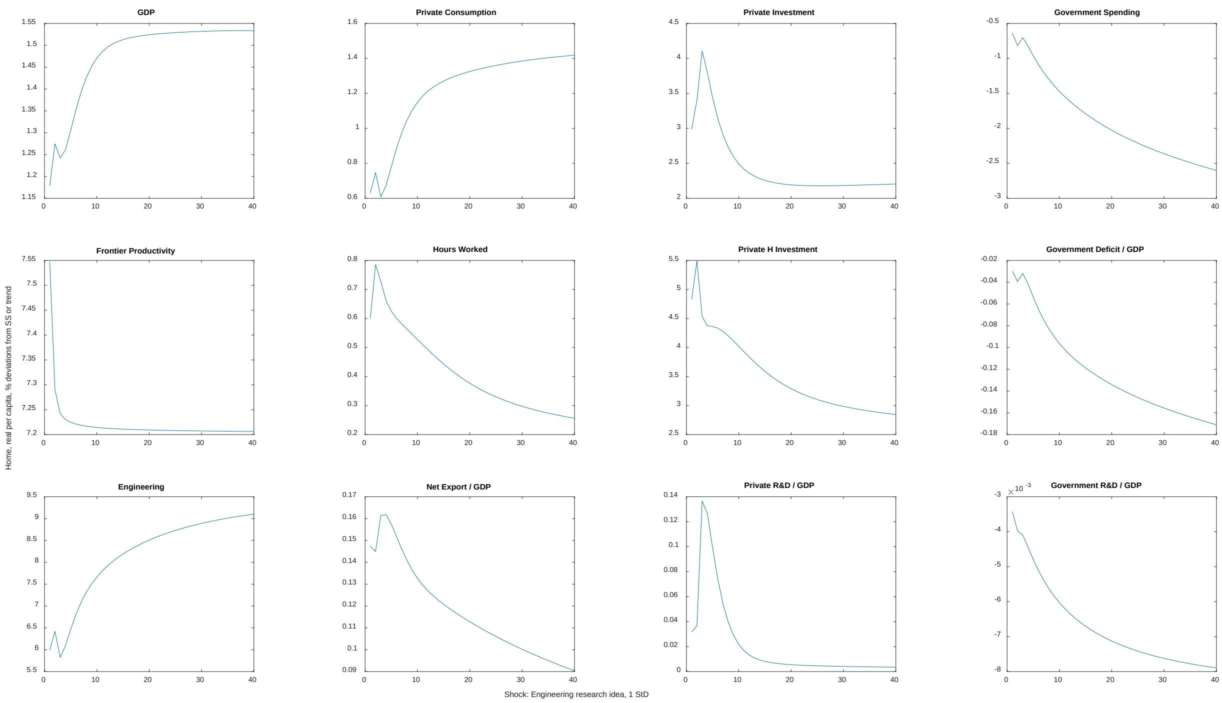

Figure 10: Impulse response in the home country to a 1 standard deviation increase in research ideas.

Somewhat contrary to the results of Holden (2016), the research idea shocks emerge as a major driving shock in our estimated model. A one standard deviation research idea shock permanently increases frontier productivity in the engineering sector by about 7.2%, and output worldwide by about 1.5%. As such, these represent major breakthroughs. Such shocks will lead to a common $I(1)$ trend in productivity across countries. About two thirds of the permanent impact on output of this shock materialises on impact, with GDP then increasing rapidly over the next ten years to make-up the other one third of the permanent impact. The shock leads to a boom in hours worked, consumption and investment, thanks in part to the comovement generated by our preference specification. The response of human capital investment is particularly large, as increased human capital is necessary to take advantage of the permanently higher level of engineering. Even private R&D experiences a boom after the initial shock, as the returns to inventing new products has increased.

5.3.3. Global engineering invention cost shock

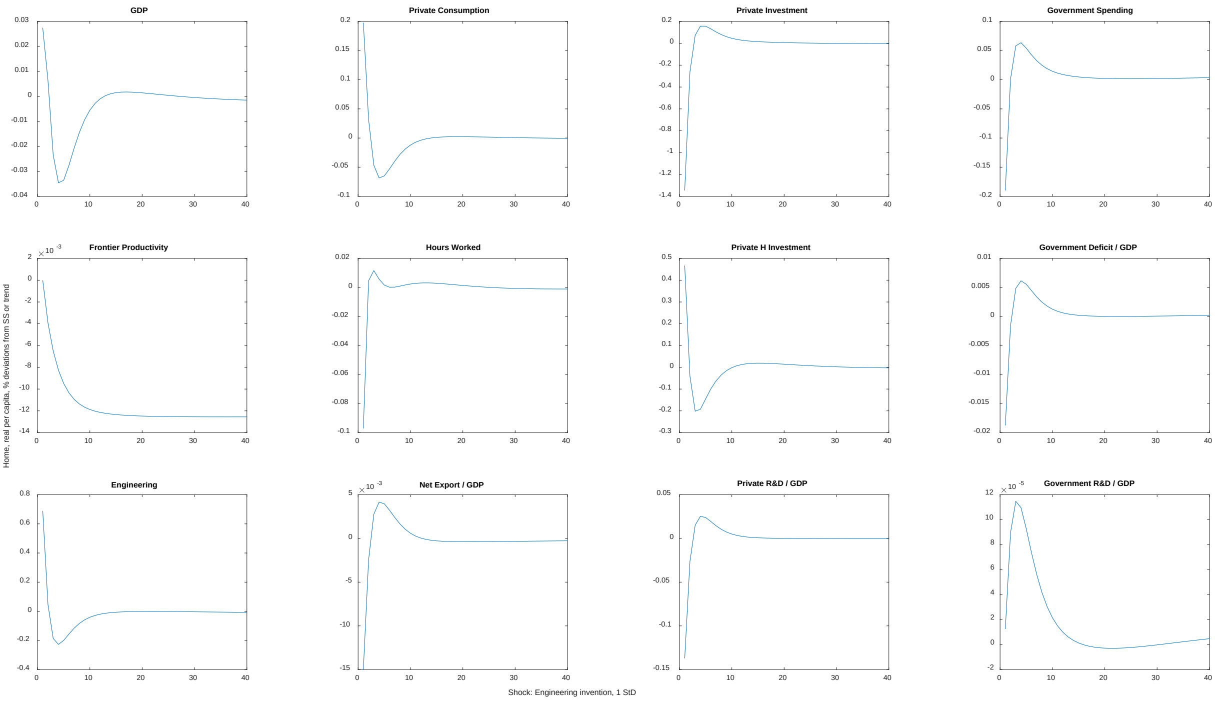

Figure 11: Impulse response in the home country to a 1 standard deviation increase in the cost of invention.

This shock temporarily increases the cost of inventing a new product. As a result, there is a pause in the growth of the measure of industries. With the measure of industries fixed, but population still growing, the number of firms per industry must increase, pushing down mark-ups and hence decreasing research incentives. This produces the permanent fall in frontier productivity we observe. Additionally, with fewer new products being invented, a greater proportion of industries are producing products which are no longer patent protected. Since such industries endogenously have lower productivity, this results in a fall in engineering output after the first year. The initial rise in engineering output is driven by “widgets” being reallocated from invention to production. This reallocation effect is strong enough that even output increases on input.

5.3.4. ROW price shock

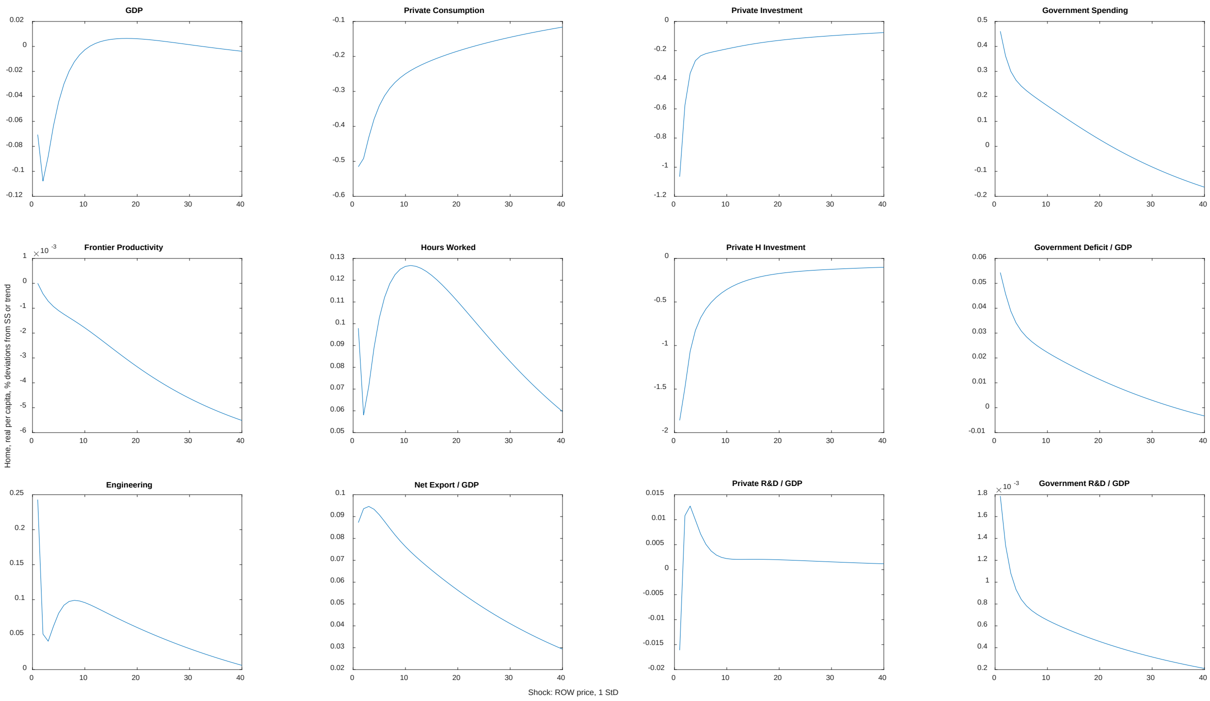

Figure 12: Impulse response in the home country to a 1 standard deviation increase in ROW prices

The last important global shock is a shock which permanently increases the price that the ROW charges to other countries for ROW goods. This shock also implies permanently higher demand from the ROW for home country goods, as home country goods become relatively cheap. The former effect leads the home country to import less. The latter effect leads the home country to export more. As a result, net exports rise, crowding out private consumption. The effect is particularly strong due to the complementarity between tradeables and non-tradeables, which produces lower demand for home non-tradeables given the drop in foreign tradeables.

5.3.5. Skilled labour productivity shock

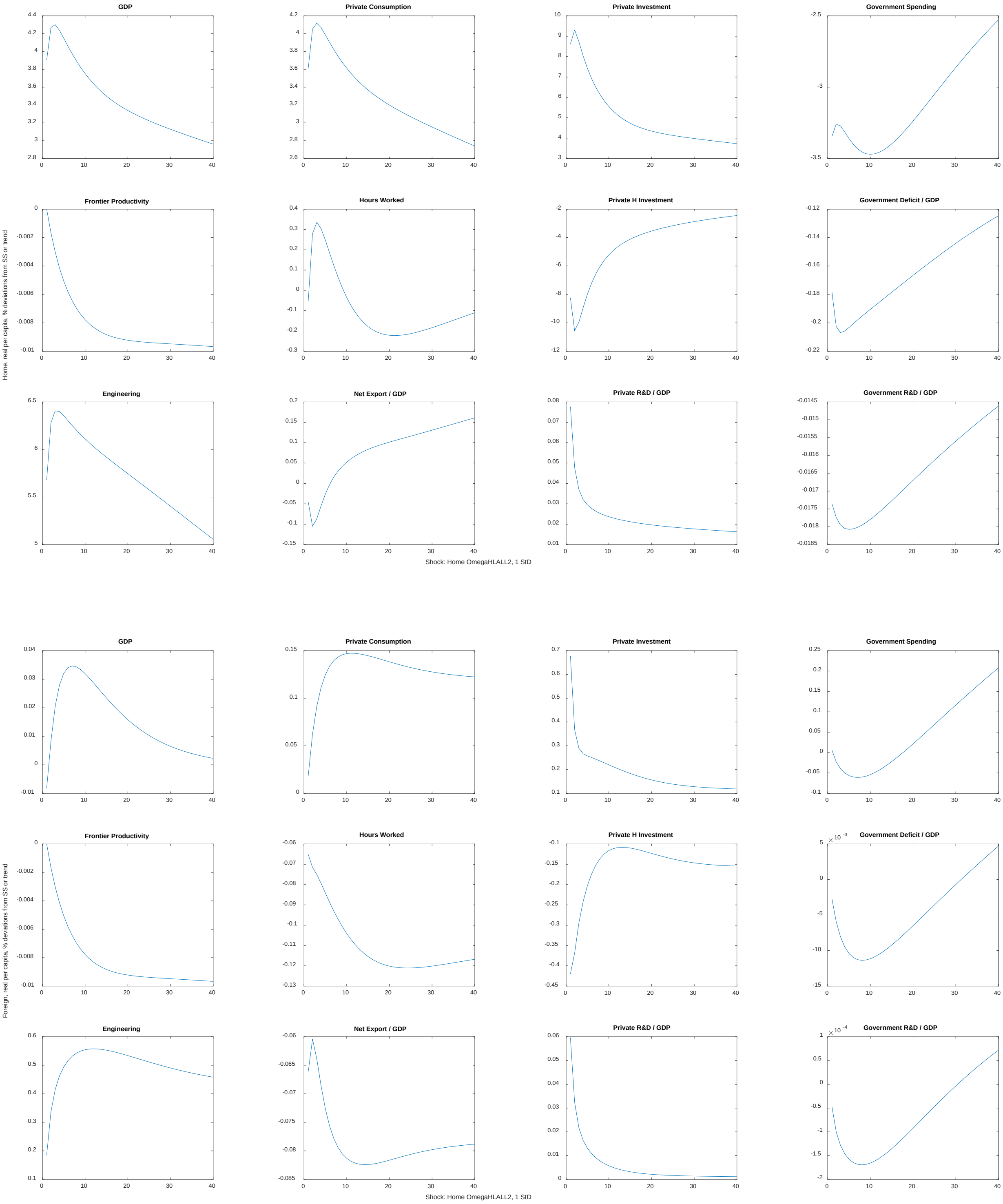

Figure 13: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 standard deviation increase in home skilled labour productivity.

A skilled labour productivity shock is a shock to the productivity of the intermediate produced from skilled labour and human capital. However, the original exogenous productivity shock is amplified and made more persistent via the engineering mechanism (higher engineering demand leading to higher engineering productivity via inventor entry). This also leads to permanently lower frontier productivity (firm entry means lower mark-ups so lower process improvement incentives). Note that this shock leads to a big drop in private human capital investment, as the increased productivity substitutes for human capital, and the returns to investment are higher elsewhere. The result is that there is a particularly large expansion in R&D and engineering.

5.3.6. Traded composite (widgets composite and tradeable composite) productivity shock

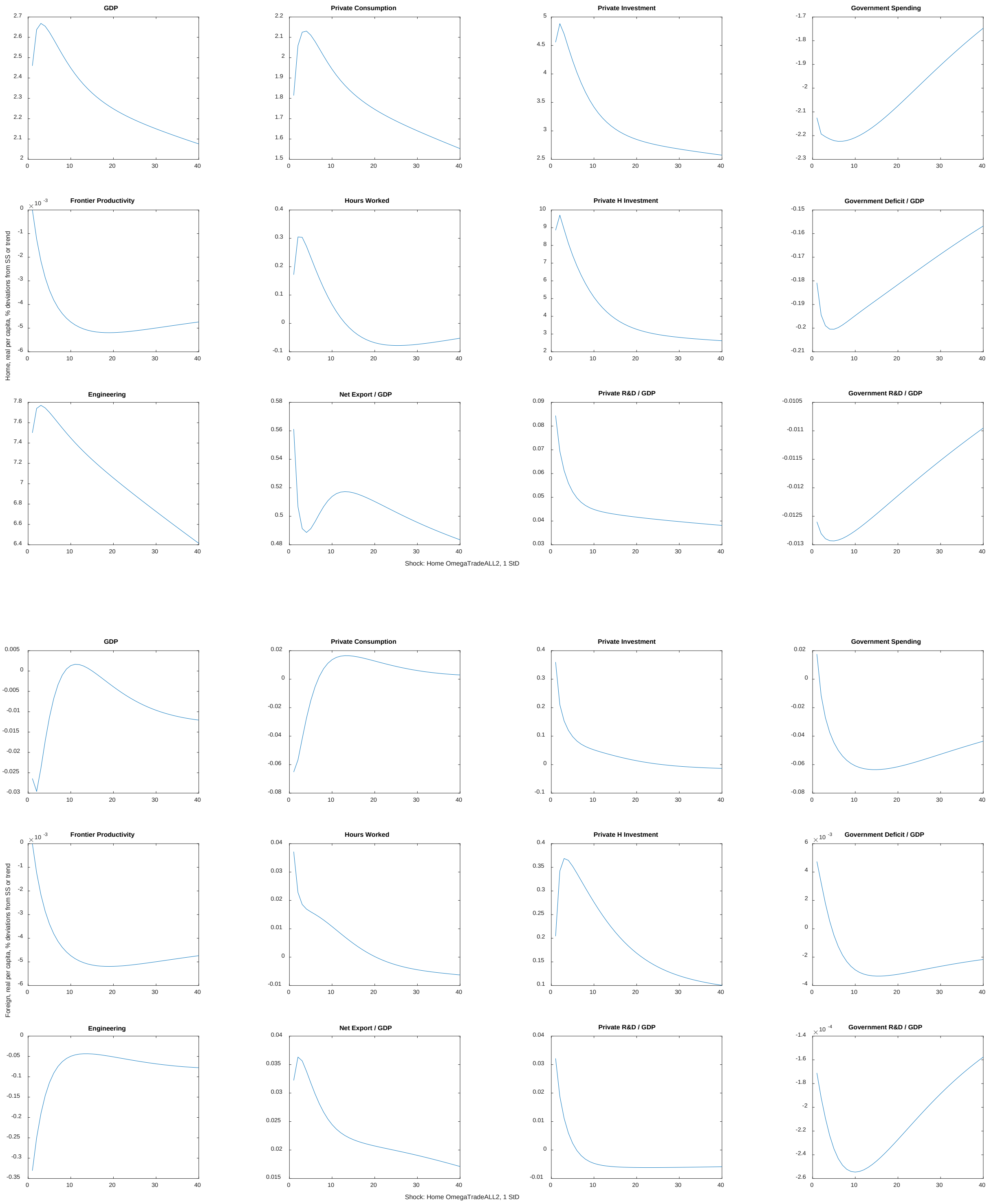

Figure 14: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 standard deviation increase in home traded composite productivity.

A traded composite productivity shock increases the productivity with which both widget composites and tradeable composites are produced in the home country. This might reflect an increase in productivity in shipping. With more widget composites being produced, there is an expansion in global engineering production, from which the home country particularly benefits as it is producing more widget composites. With an expansion in the engineering sector, more new products are invented, pushing up productivity in the sector. This produces a sizeable boom in output in the home country. To fully benefit from the increased engineering, both countries invest in human capital, however, the foreign country is still hurt by the shock, as increased exports, and reduced imports, crowd out consumption.

6. Scenarios for policy analysis

There are three components to our policy analysis scenarios. Firstly, we examine the impulse responses to permanent policy shocks in the estimated model. These impulse responses give a guide to policy makers on what policy levers they could pull in order to accomplish their goals. Secondly, we present forecasts of the model's key variables for the next 100 years. These show how the model predicts the global economy with evolve given "business as usual". This will help policy makers in assessing where intervention may be needed. Note though that this "business as usual" scenario incorporates policy's estimated responses to other endogenous variables, thus just holding policy instruments constant may not be sufficient to achieve even this baseline scenario. Finally, we present conditional forecasts assuming that policy makers adjust some of their instruments to produce a given increase in R&D and human capital investment shares.

The detailed numerical results of these scenarios are all contained either in this model's GitHub repository:

https://github.com/tholden/MONROE-DSGE

or in a supplemental results file available from:

https://is.gd/MONROE_DSGE

6.1. Impulse responses to policy shocks

We begin by presenting impulse responses to the majority of the model's policy shocks. Since policy makers are not constrained to make changes in policy instruments of similar magnitudes to past changes in those instruments, looking at the responses to one standard deviation shocks is not particularly relevant. Instead, we show the response to a one percentage point change in the policy instrument, e.g. increasing taxes or expenditure shares by one percentage point. The only policy shocks we exclude in the below are those whose impact is a linear combination of other policy shocks, i.e. common shocks, as well as the lump-sum tax on patient households, which only has an effect on government deficits.

Exactly as before, all plots show variables in log point deviation (1 meaning 1 log point, i.e. approximately 1%) from the level they would have attained had no shock hit, with the exception of variables given as a ratio, which are in percentage points (1 meaning 1 percentage point). Again as before, we show plots for both the home country (the one experiencing the policy change) and a representative foreign country. Finally recall that in the plot labels, "Private H Investment" means "private investment in human capital".

To help the reader, we mark with (*) in the subsection title any shock for which GDP and the deficit move in opposite directions over the entire forty-year period. If an increase in GDP can be brought about along with a reduction in the deficit, then such a policy change is almost a "free lunch", at least according to the model. Recall though that the increase in GDP is usually accompanied by an increase in hours, so the impact on welfare is still ambiguous. Additionally, one more substantial caveat is in order: the instruments corresponding to "free lunches" tend to be those that are estimated as barely entering the government budget constraint at all, i.e. they represent pure efficiency wedges. As such, there is no loss in revenue from somehow cutting such an efficiency wedge. To avoid confusion, we also report the ϕ parameters which govern the extent to which the tax enters the government budget constraint in the section title.

We begin with plots showing tax changes, then we show changes in tariffs, before finally showing changes in spending.

6.1.1. Basic rate labour "tax" (*) ($\phi^L = 0.00$)

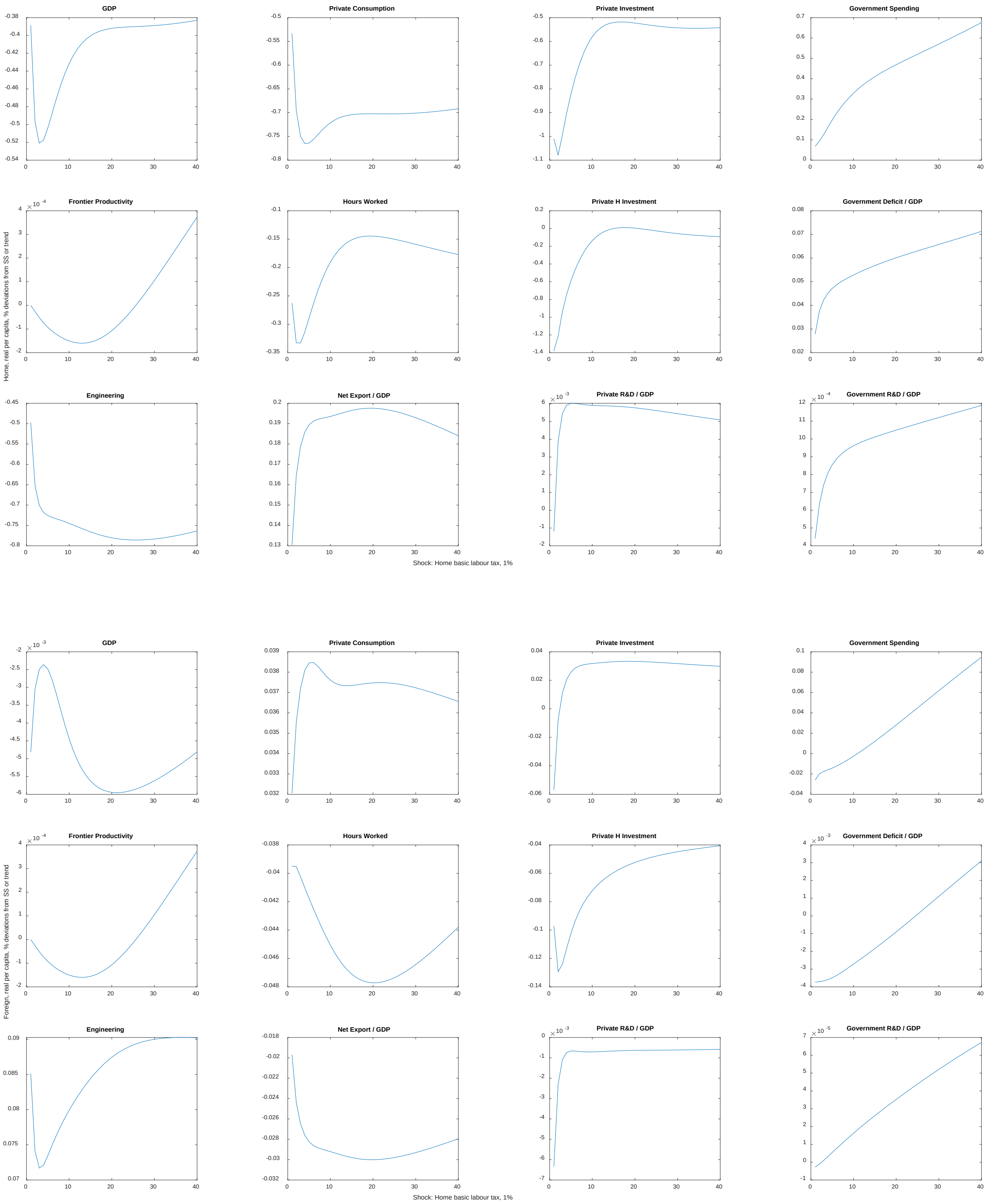

Figure 15: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home basic labour taxes.

This is a shock to $\tau^{La}_{n,t}$, nominally the basic rate of labour tax, but in fact estimated to be a pure labour efficiency wedge. Increasing the wedge by one percentage point results in a drop in home hours worked of around 0.3%, with a corresponding 0.5% drop in GDP. As a pure wedge, increasing the tax does not help the deficit. Thus, cutting labour efficiency wedges is a powerful stimulus tool in the model.

6.1.2. Labour "tax" progressivity shock (*) ($\phi^L = 0.00$)

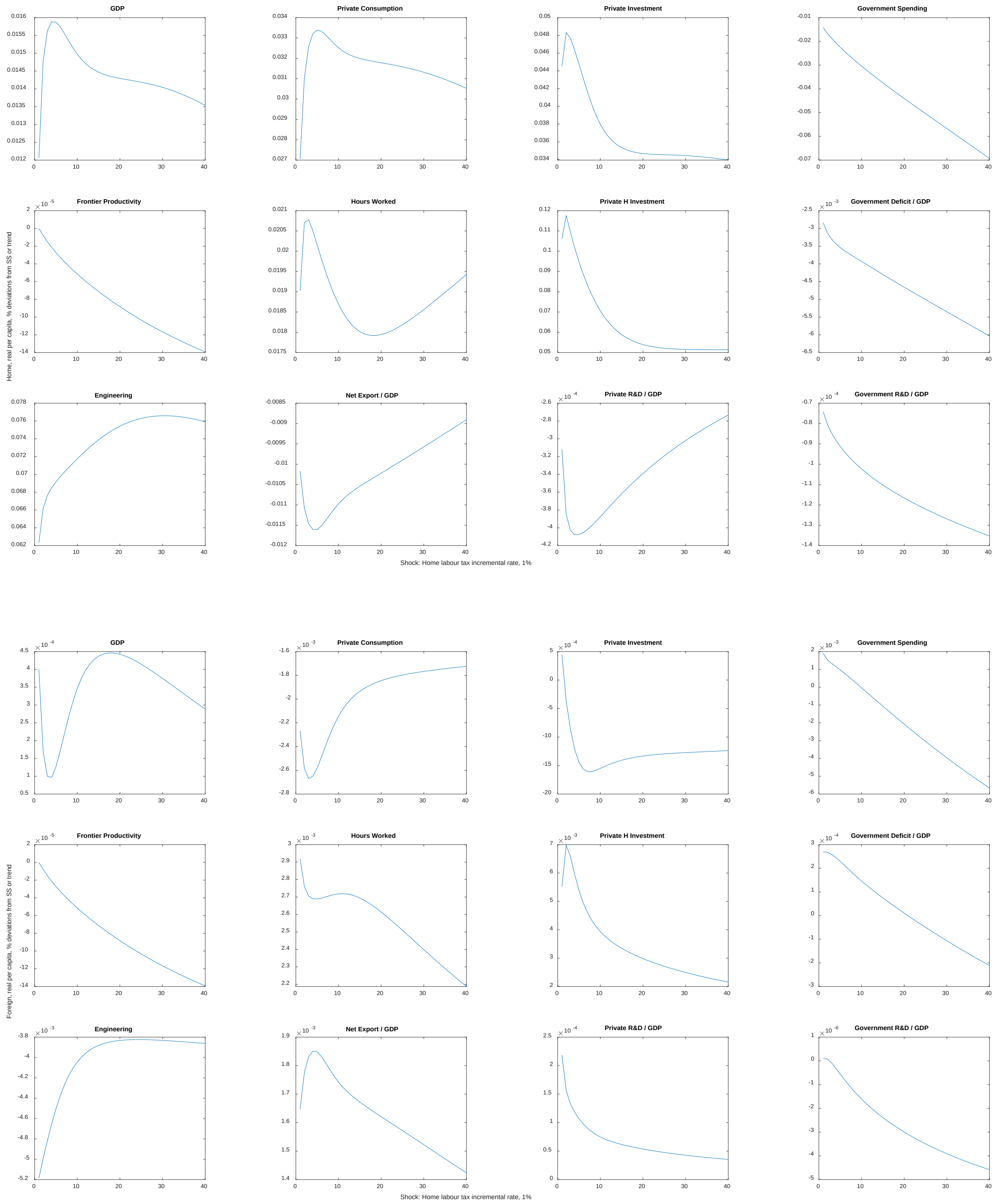

Figure 16: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home labour tax progressivity.

This is a shock to $\tau^{Lb}_{n,t}$. Since all steady-state wages are below one, an increase in $\tau^{Lb}_{n,t}$ means all workers receive a tax cut, with the least paid getting the biggest cut. Thus, hours worked increase, leading to a boom. As before, this "tax" is estimated to not appear in the government budget constraint, and the deficit would no doubt be worse if it did.

6.1.3. Human capital tax shock ($\phi^{HW} = 0.70, \phi^{HT} = 0.52, \phi^{HNT} = 0.66$)

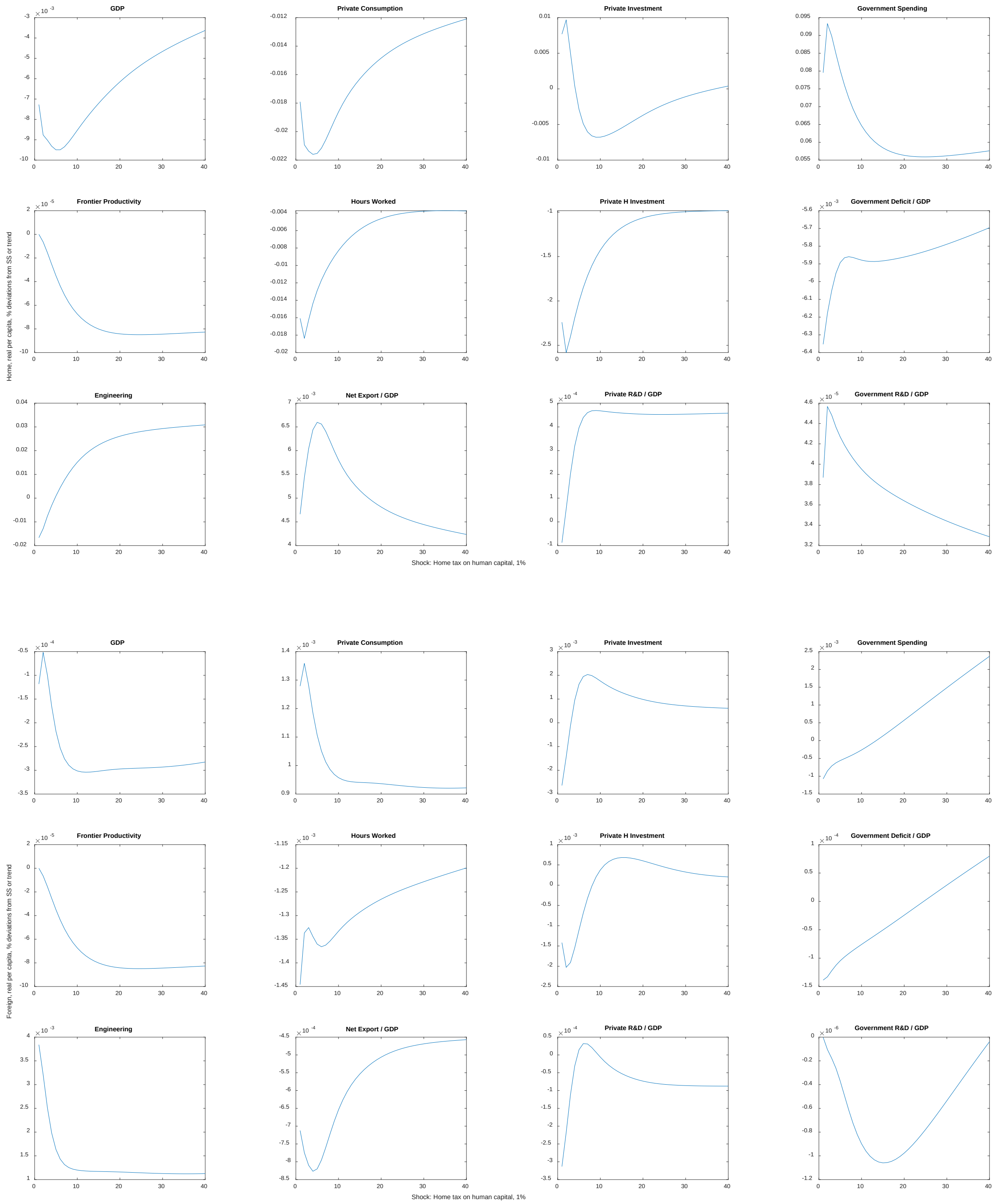

Figure 17: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in human capital taxes (1 percentage point in the non-tradeable sector, slightly smaller in other sectors (0.92 and 0.88 percentage points respectively)).

A tax on the returns to human capital has quite a small effect on GDP, in part due to a substitution into physical capital, and in part due to the estimated small human capital share of production. Unsurprisingly, it results in a drop in human capital investment and hours worked (as supplying skilled labour is required to benefit from human capital). The initial response is larger to bring the stock to its new level.

6.1.4. Physical capital tax shock ($\phi^{KW} = 0.85, \phi^{KT} = 0.73, \phi^{KNT} = 0.83$)

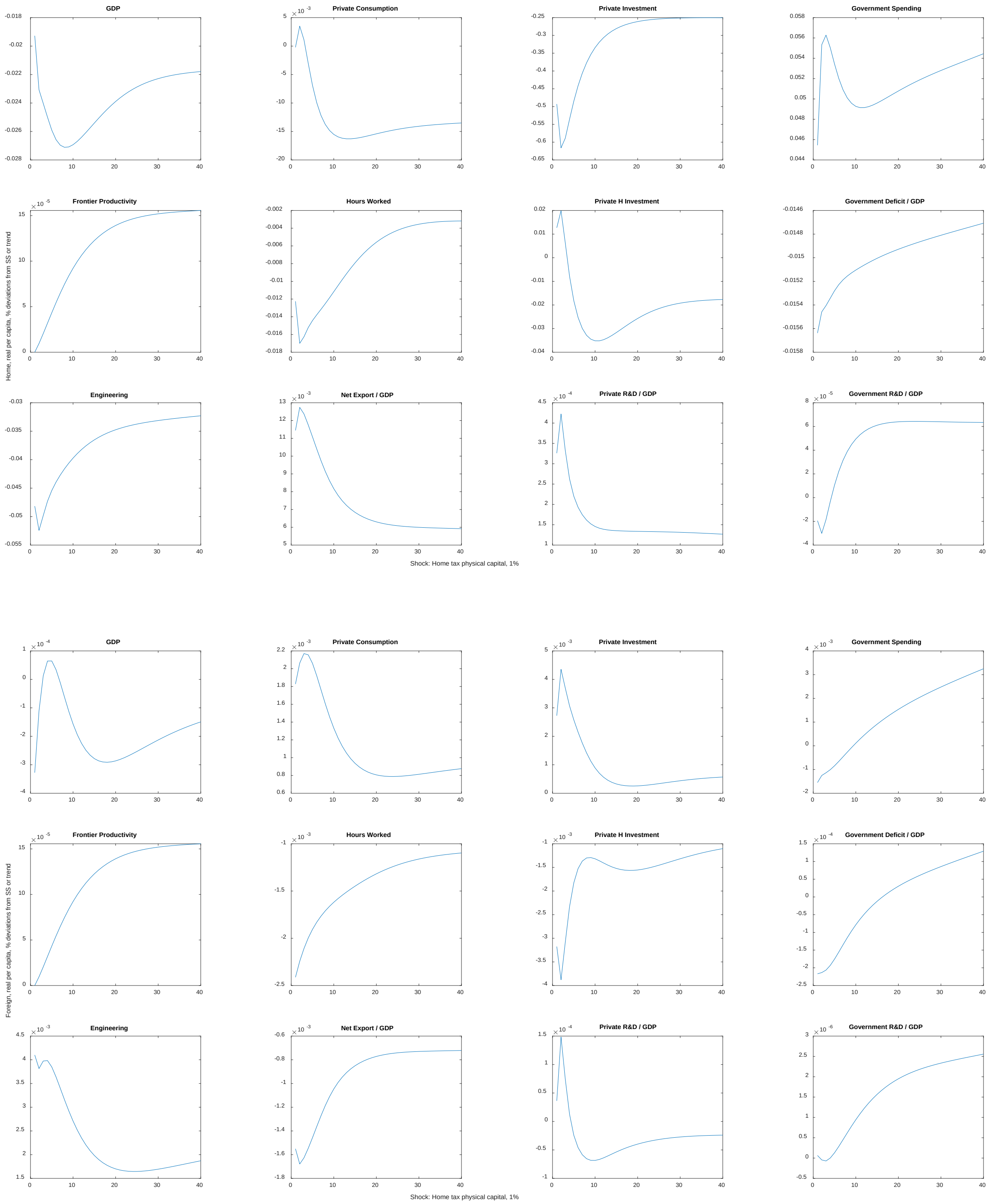

Figure 18: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in physical capital taxes (1 percentage point in the tradeable sector, slightly smaller in other sectors (0.99 and 0.97 percentage points respectively)).

A tax on the returns to physical capital has a larger, but still quite small, effect on GDP, again due to the relatively small estimated physical capital share of production. (As previously mentioned, this may in part be due to weak identification.)

6.1.5. Engineering tax shock ($\phi^{\mathcal{X}W} = 0.67, \phi^{\mathcal{X}T} = 0.49, \phi^{\mathcal{X}NT} = 0.63$)

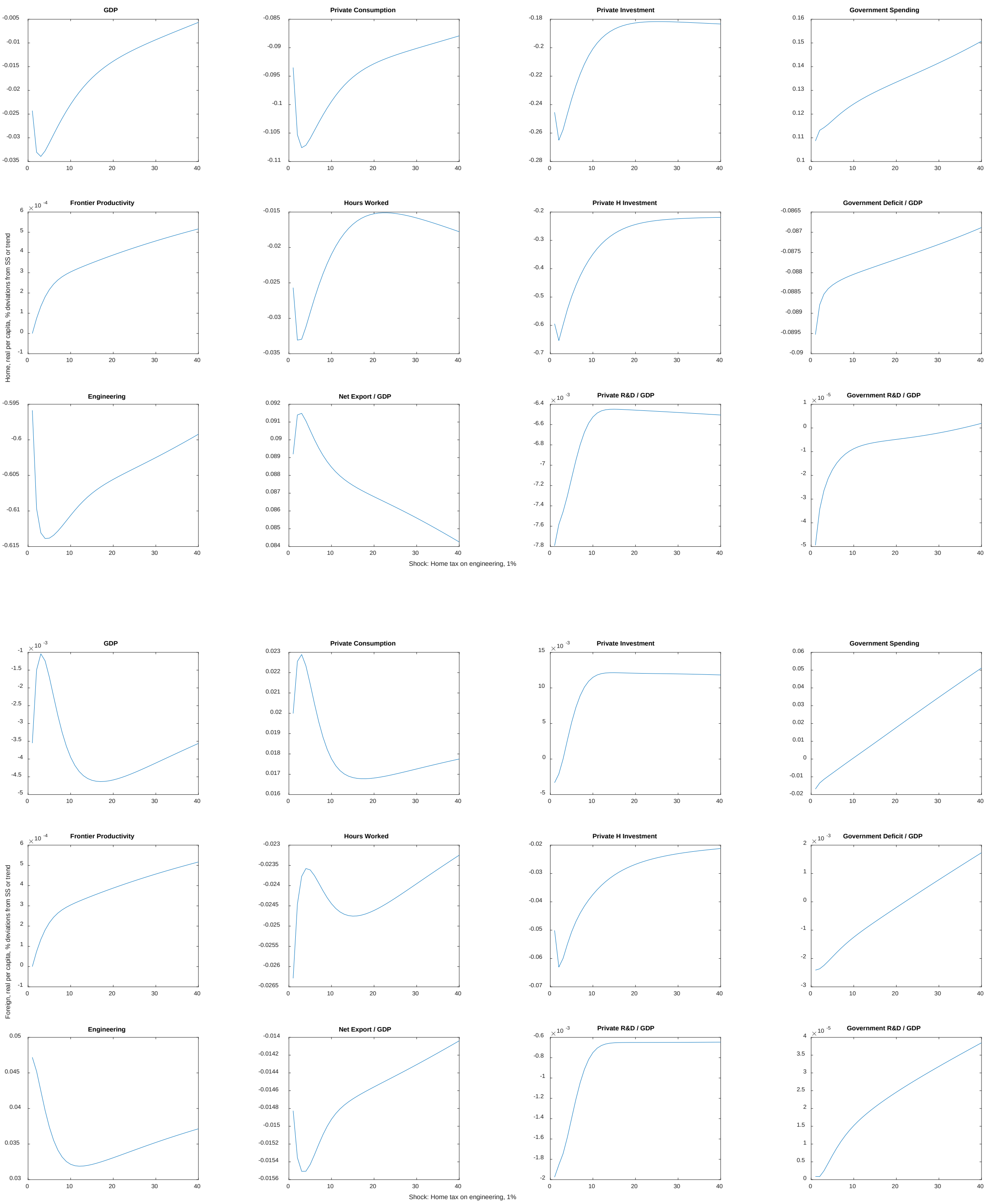

Figure 19: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home taxes on engineering goods (1 percentage point in the non-tradeable sector, smaller in other sectors (0.81 and 0.72 points respectively)).

This tax is a permanent increase in the tax imposed on purchases of engineering services. As a result there is a large drop in engineering use, which via our model's amplification mechanism leads to a drop in global productivity. With engineering and human capital investment being complements, there is a particularly large drop in human capital investment. Subsidising engineering may be a powerful policy tool.

6.1.6. Tradeable sector tax shock ($\phi^{HT} = 0.52, \phi^{KT} = 0.73, \phi^{\chi T} = 0.49$)

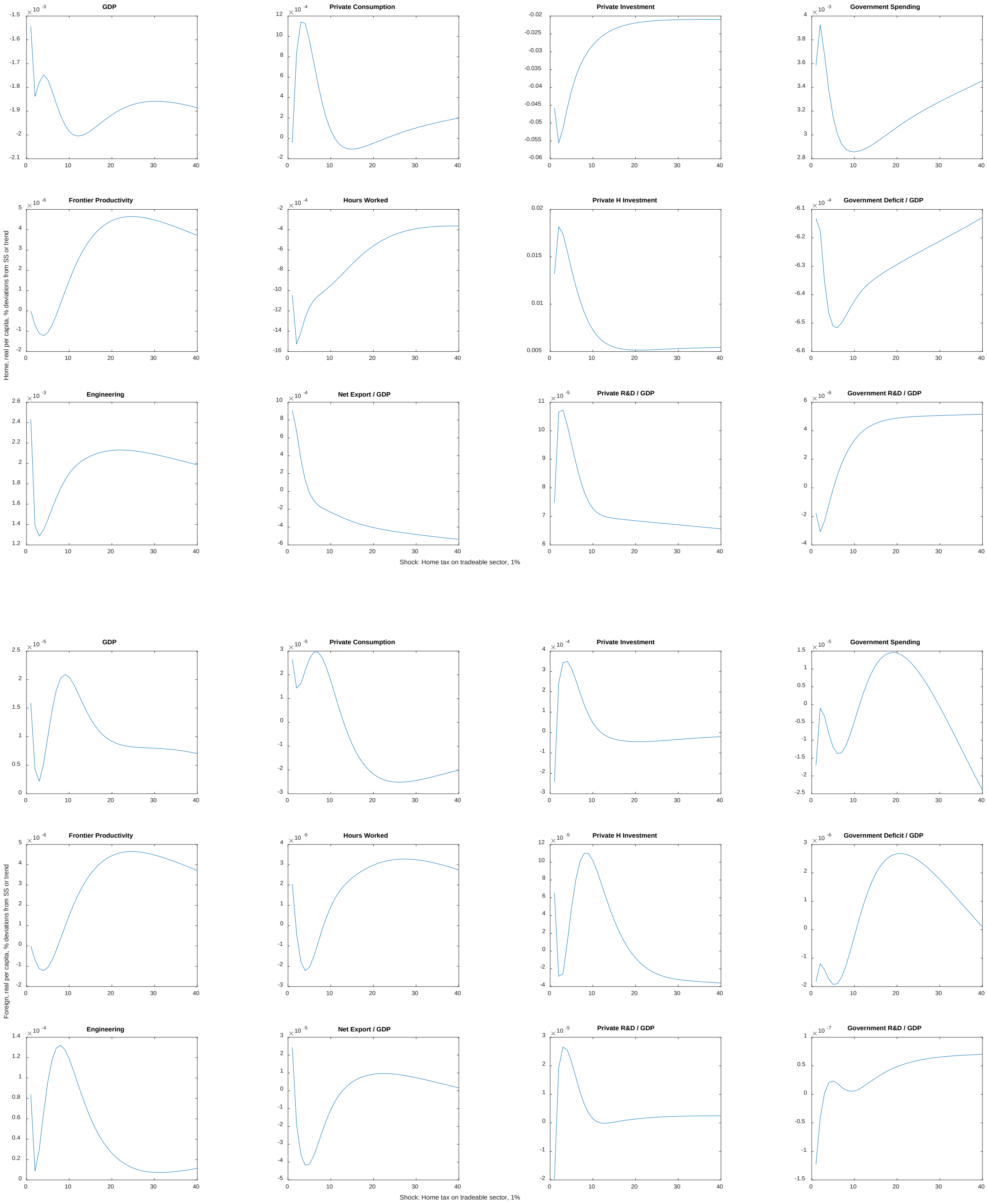

Figure 20: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home taxes on the tradeable goods sector. More precisely: a 1 percentage point increase in capital taxes in the tradeable sector, combined with a 0.19 and 0.06 percentage point fall in human capital and engineering taxes in that sector, respectively.

This tax is a permanent increase in the tax imposed on capital in the tradeable goods sector, with some partially offsetting decreases of other taxes in that sector. (These magnitudes are freely estimated, and were not constrained to have identical signs.) The net impact is muted, with investment falling most.

6.1.7. Non-tradeable sector tax shock ($\phi^{HNT} = 0.66, \phi^{KNT} = 0.83, \phi^{\mathcal{X}NT} = 0.63$)

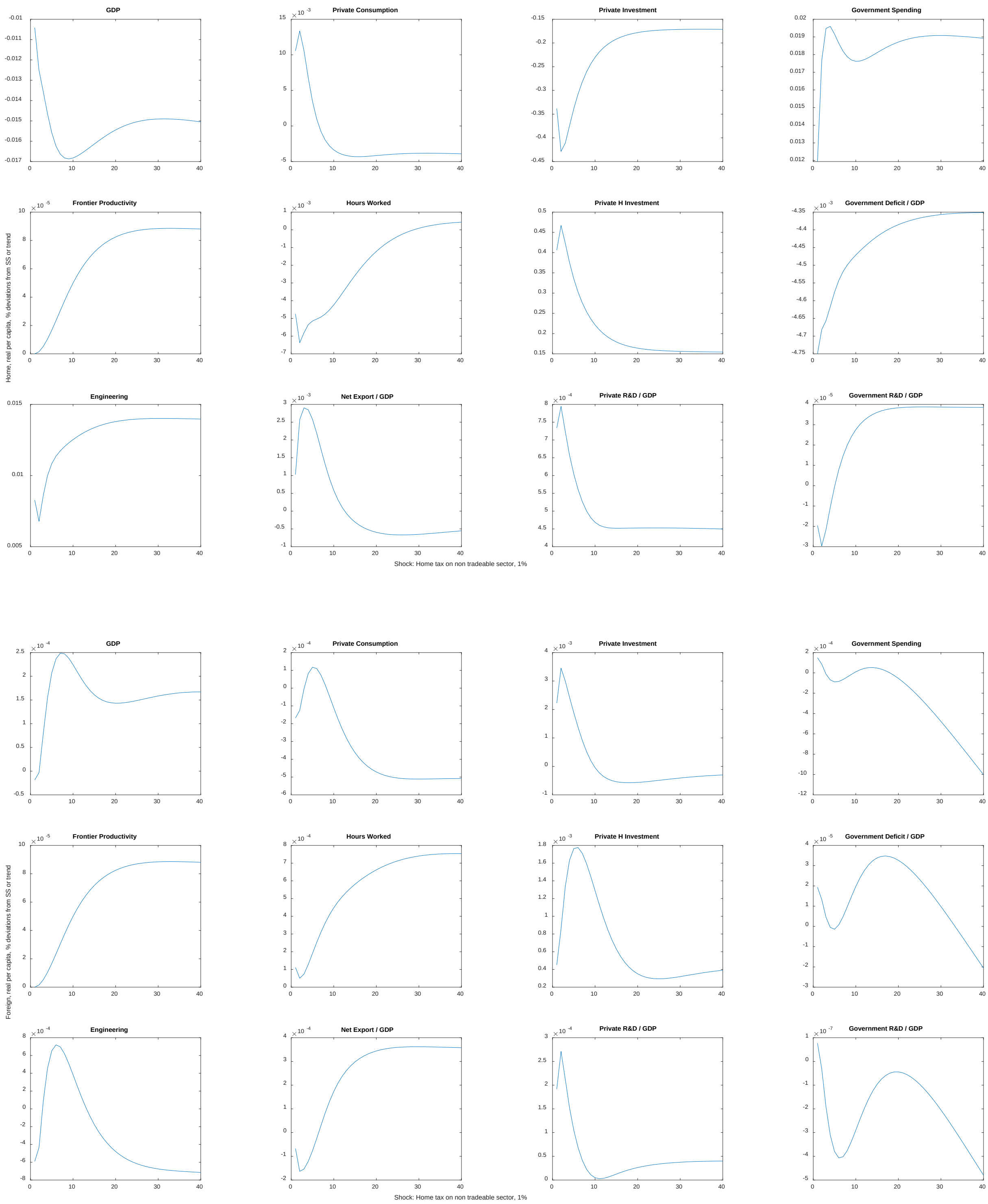

Figure 21: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home taxes on the non-tradeable goods sector. More precisely: a 1 percentage point increase in capital taxes in the tradeable sector, combined with a 0.22 and 0.09 percentage point fall in human capital and engineering taxes in that sector, respectively.

This tax is a permanent increase in the tax imposed on capital in the tradeable goods sector, with some partially offsetting decreases of other taxes in that sector. The response is broadly similar to before, but with a larger impact due to the lack of good substitutes for non-tradeable goods.

6.1.8. Widget sector tax shock ($\phi^{HW} = 0.70, \phi^{KW} = 0.85, \phi^{\mathcal{X}W} = 0.67$)

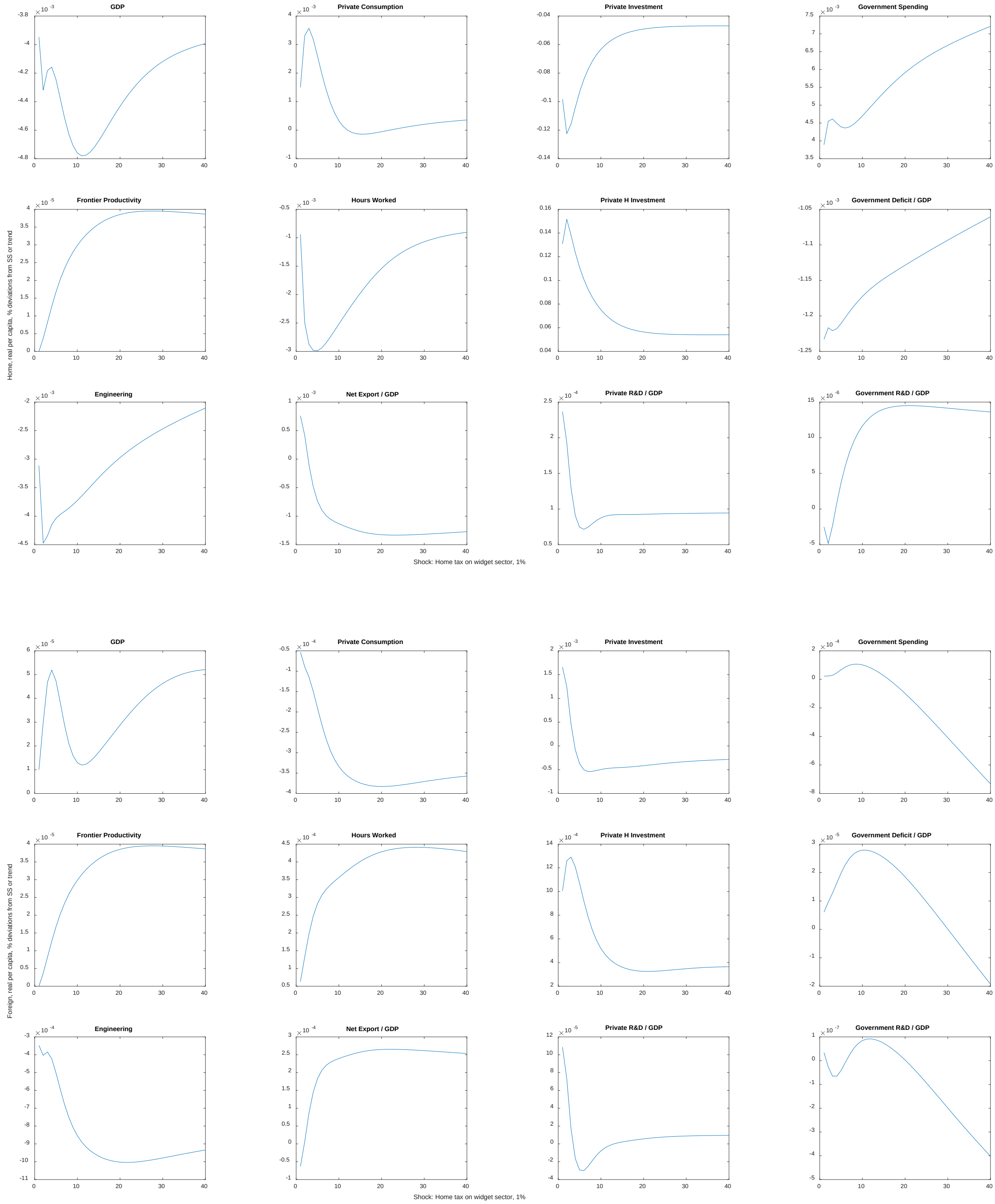

Figure 22: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home taxes on the widget sector. More precisely: a 1 percentage point increase in capital taxes in the tradeable sector, combined with a 0.20 and 0.07 percentage point fall in human capital and engineering taxes in that sector, respectively.

This tax is a permanent increase in the tax imposed on capital in the widgets sector, with some partially offsetting decreases of other taxes in that sector. Again the impact is small, though here the engineering sector is hit negatively, due to the need for widgets to produce engineering.

6.1.9. Non-durable goods “tax” shock ($\phi^{ND} = 0.09$)

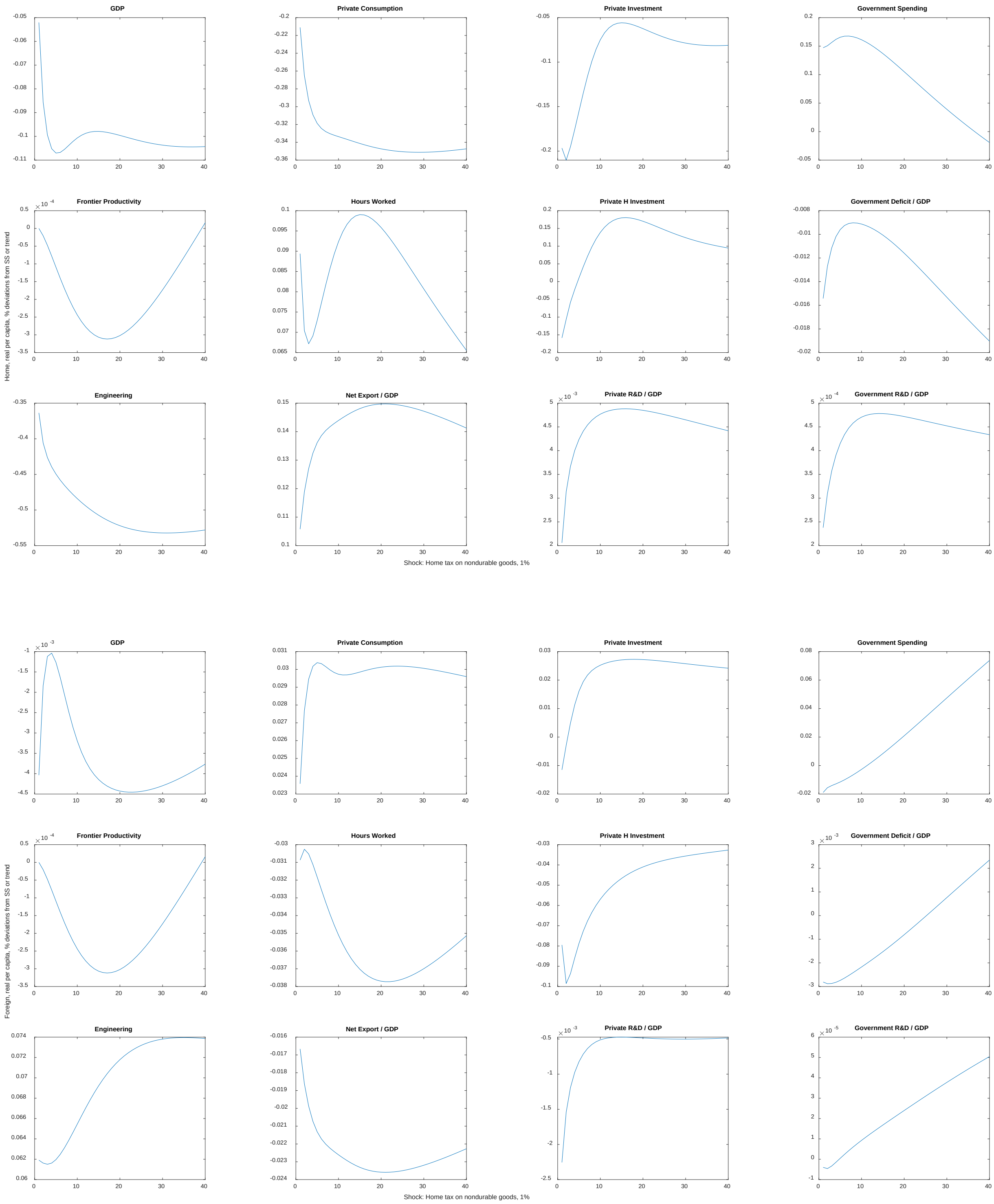

Figure 23: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home taxes on non-durable goods.

A non-durable goods “tax” shock is a permanent increase on the tax on purchasing non-durable goods (e.g. VAT). Unsurprisingly, this produces a permanent decline in consumption and GDP, and reduces the deficit (despite being almost entirely a pure efficiency wedge). It has a positive effect on hours though, as impatient households increase hours worked to limit the fall in their consumption.

6.1.10. Durable goods “tax” shock ($\phi^D = 0.05$)

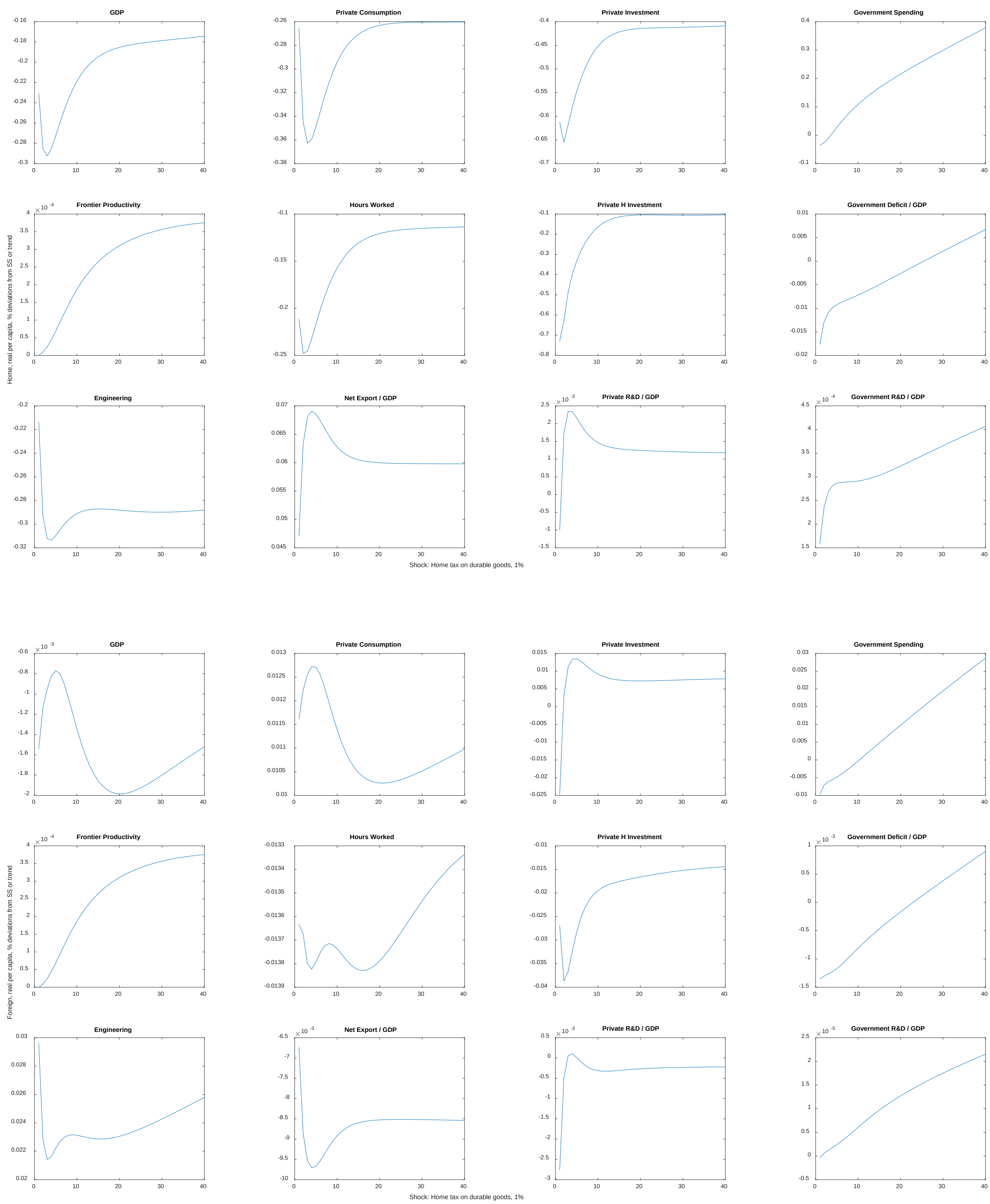

Figure 24: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 standard deviation increase in home taxes on durable goods.

A durable goods tax shock is a permanent increase on the tax on durable goods. The effects are very similar to the effects of the previous shock, except here hours also fall. This is because the benefits for impatient households to preserving the level of durable good consumption is significantly weaker than that for non-durable good consumption, due to their high rate of discounting.

6.1.11. Non-tradeable goods “tax” shock (*) ($\phi^{NT} = 0.00$)

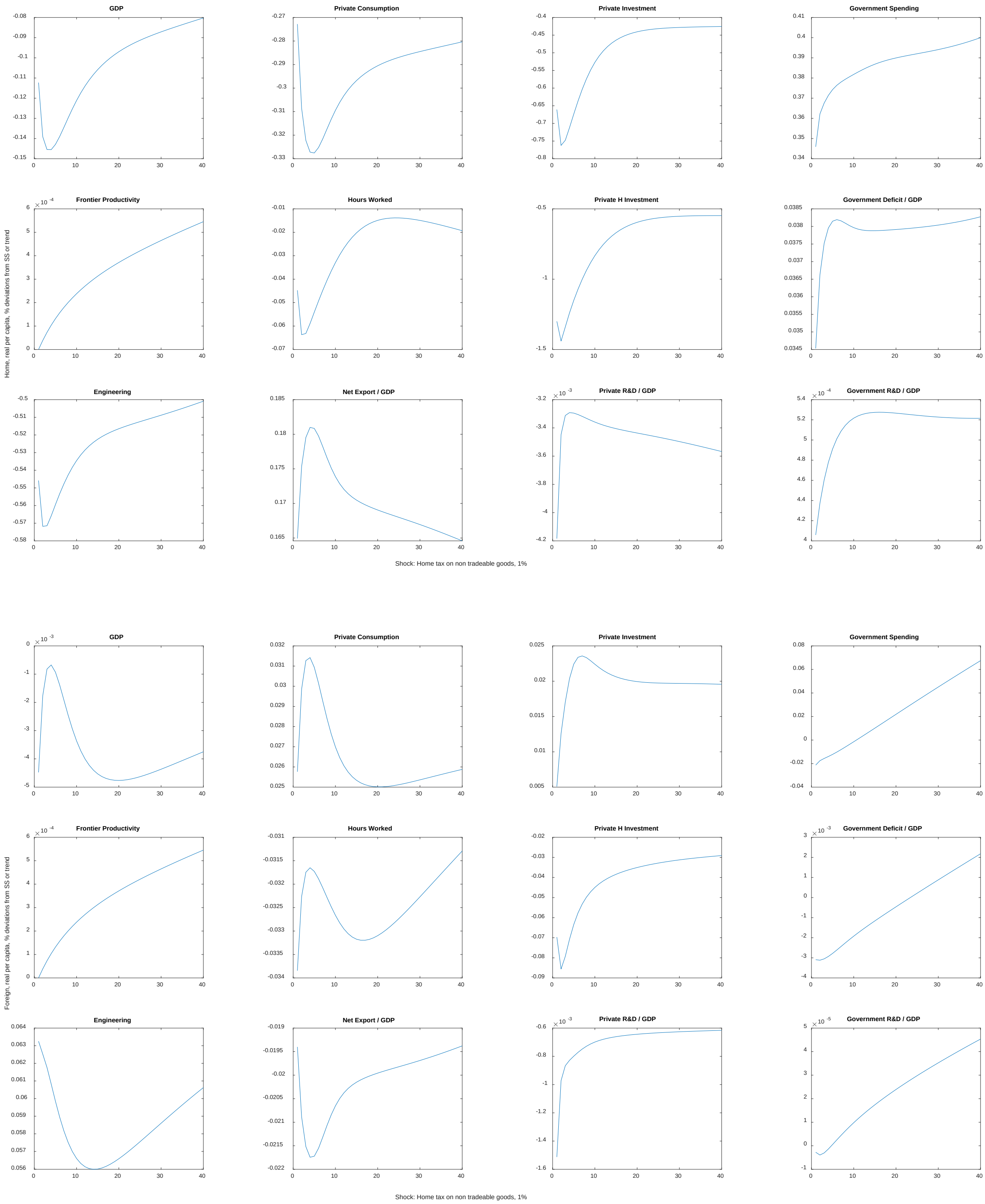

Figure 25: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home taxes on non-tradeable goods.

A non-tradeable goods tax shock is a permanent increase in the tax imposed on non-tradeable goods (e.g. services), though it is estimated to be a pure efficiency wedge. Given this, the worsening of the deficit is unsurprising. Furthermore, due to the strong complementarity between non-tradeables and tradeables, tradeable demand also falls, pushing up net exports.

6.1.12. Lump-sum tax on impatient households shock (*) (pure tax)

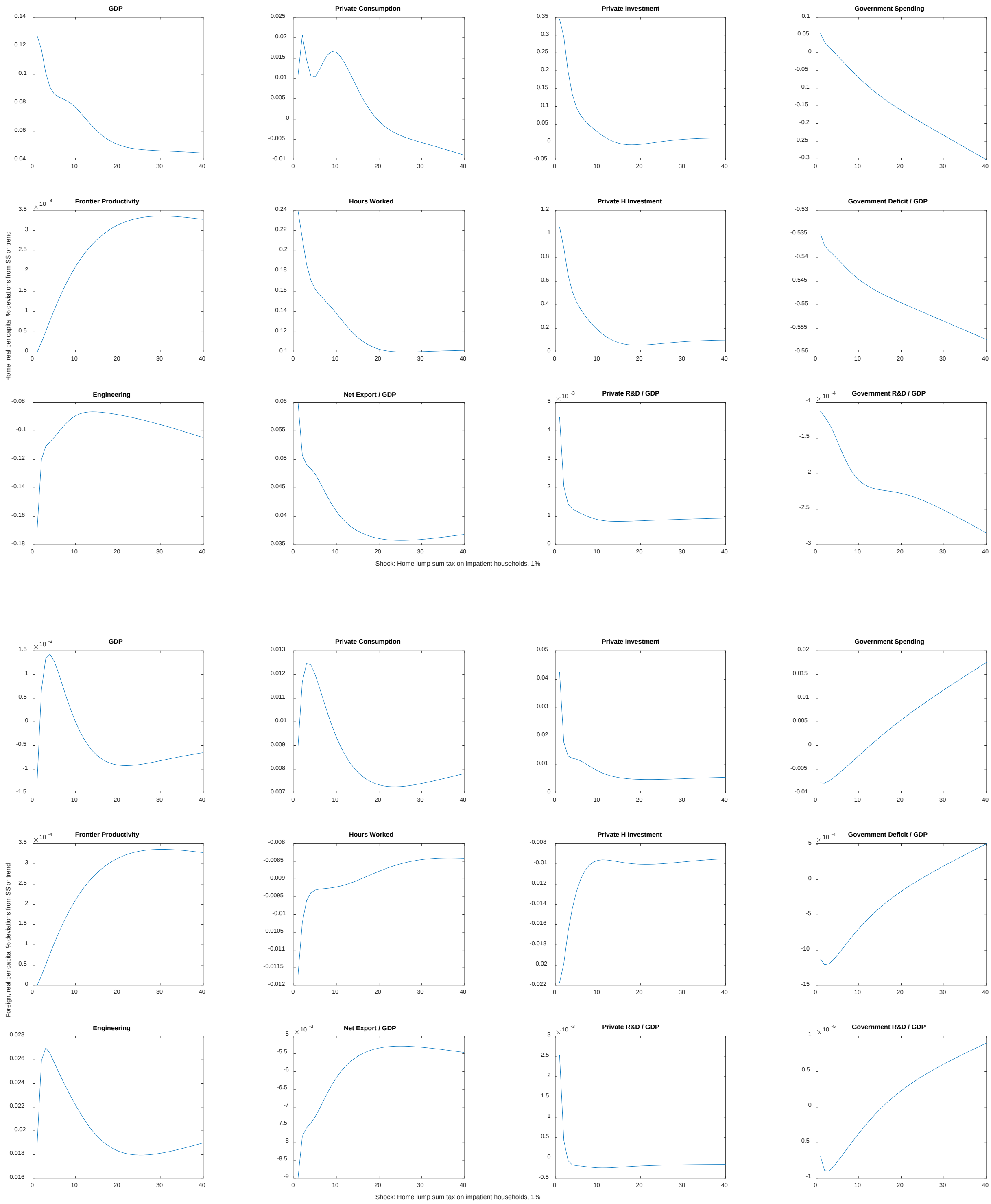

Figure 26: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point of GDP increase in the lump sum tax on impatient households.

An increase in the lump-sum tax on impatient households leads them to work more, to avoid a painful large drop in consumption. With the increase in unskilled hours, output rises. However, the resulting reduction in wages pushes down skilled hours, and so engineering falls in the home country, dampening the output rise. This is the only true tax shock in the model which raises GDP while lowering the deficit.

6.1.13. Tradeable export tariff shock ($\phi^{XT} = 0.52$)

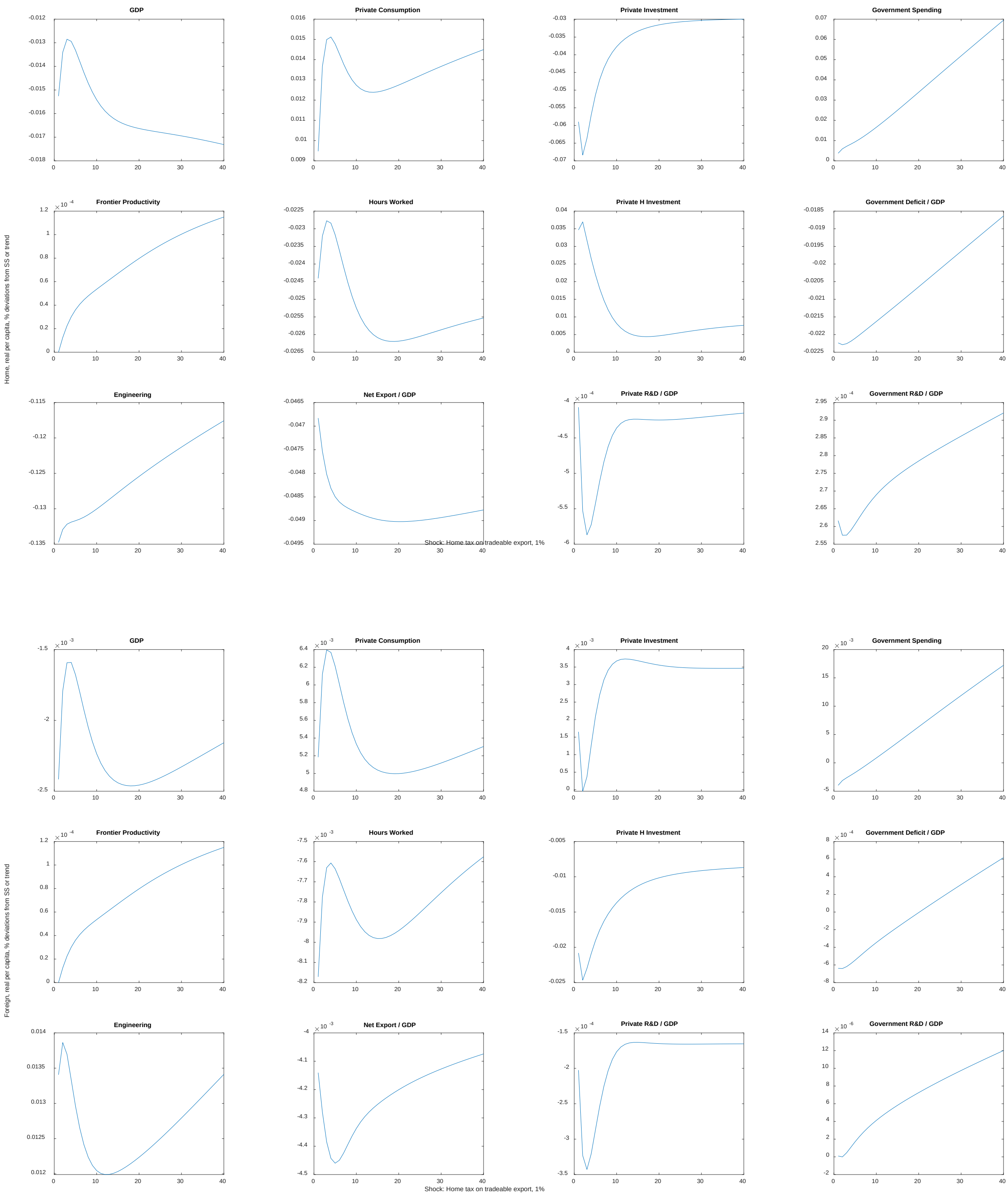

Figure 27: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home tradeable export tariffs.

These graphs show a permanent increase in export tariffs on tradeable goods. Unsurprisingly it leads to a decline in net exports, which crowds in consumption. GDP falls though as the drop in net exports dominates. There is a relatively large negative effect on engineering due to the drop in demand for home good. This leads to a productivity effect which amplifies the drop in GDP.

6.1.14. Widget export tariff shock ($\phi^{XW} = 0.52$)

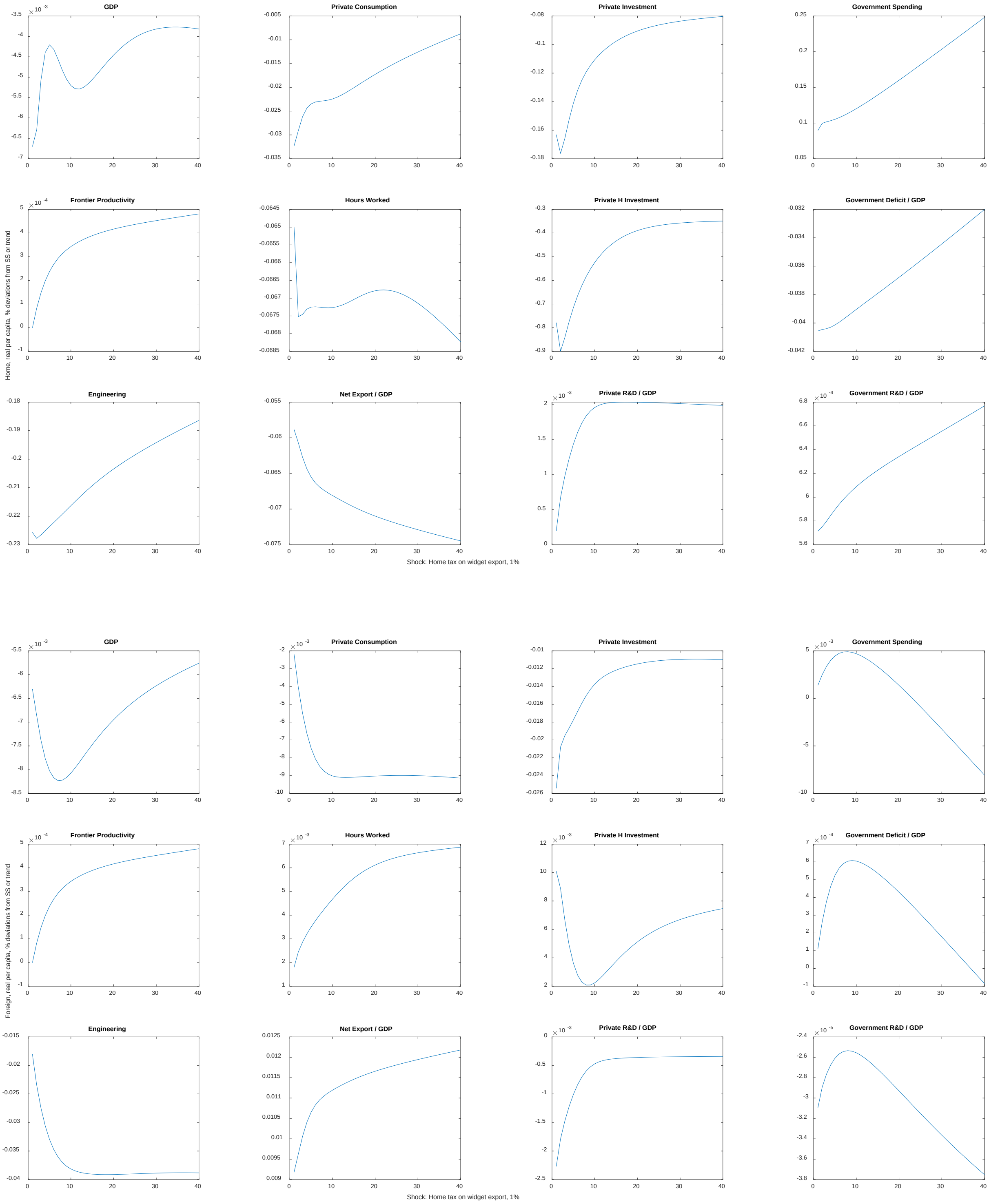

Figure 28: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home widget export tariffs.

These graphs show a permanent increase in export tariffs on widgets. Much as before, net exports fall. Here though, the amplification via the engineering channel is large enough that both GDP and consumption fall. There is also a big drop in human capital investment, due to the complementarity between engineering and human capital.

6.1.15. Tradeable import tariff shock ($\phi^{MT} = 0.51$)

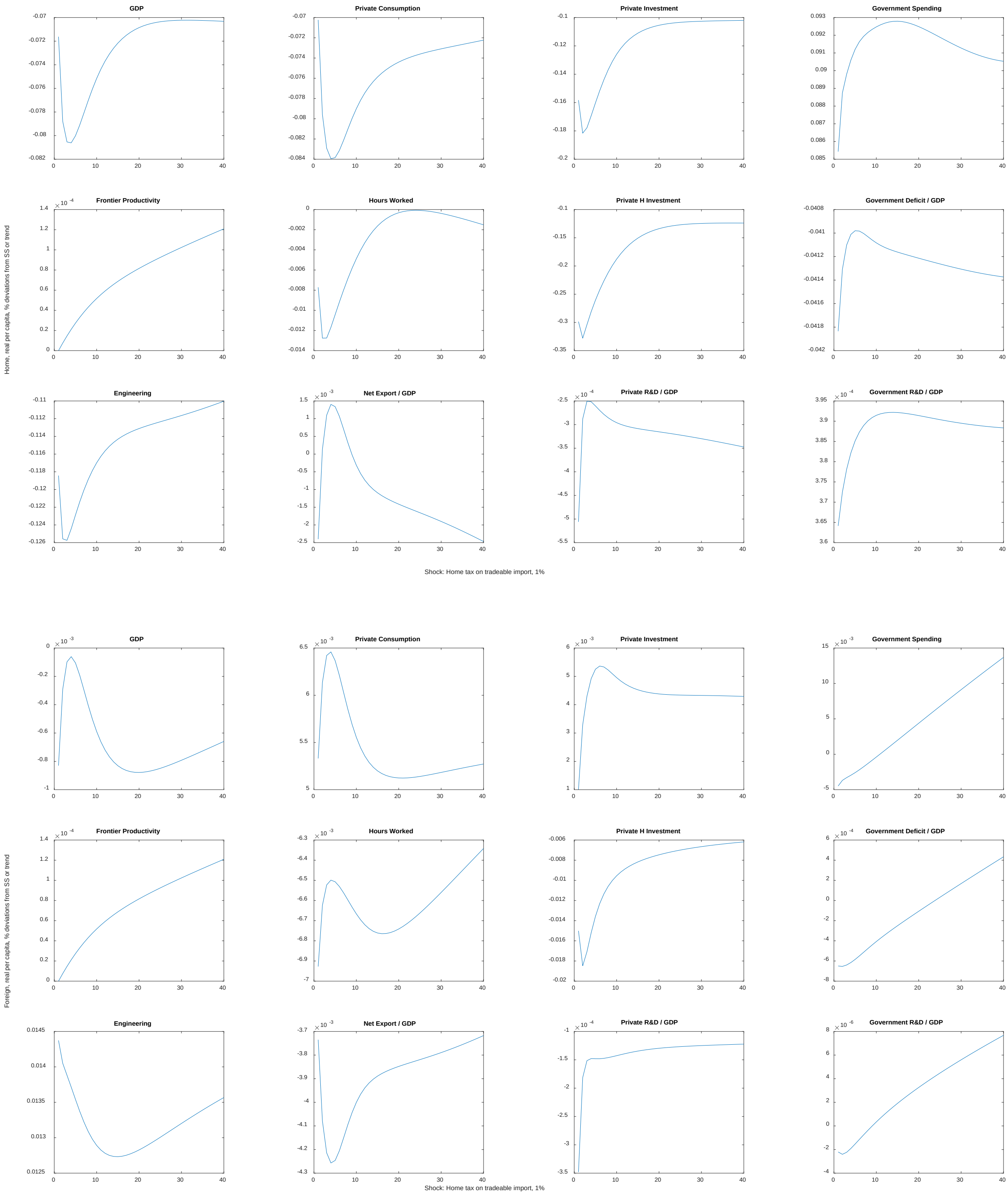

Figure 29: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home tradeable import tariffs.

These graphs show a permanent increase in import tariffs on tradeable goods. Unsurprisingly it leads to a rise in net exports. GDP and consumption both fall though due to the complementarity between home and foreign tradeable goods, as well as the complementarity with home non-tradeable. Again the effect is amplified via engineering's drop.

6.1.16. Widget import tariff shock ($\phi^{MW} = 0.52$)

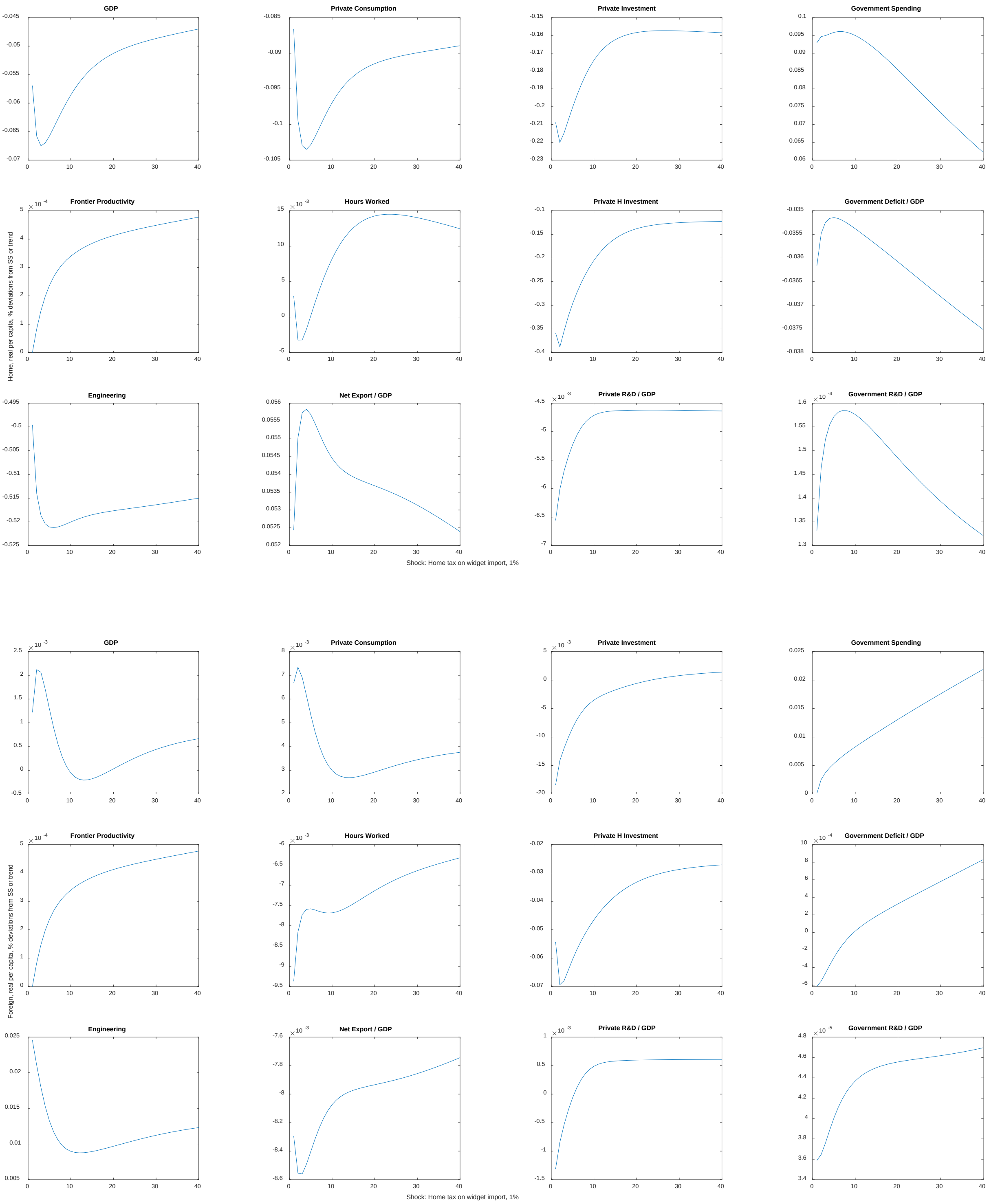

Figure 30: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point increase in home widget import tariffs.

Our final tariff shock is a permanent increase in import tariffs on widgets. The response is much as before, but with additional movement in engineering due to the need for widgets in engineering.

6.1.17. Government consumption share shock

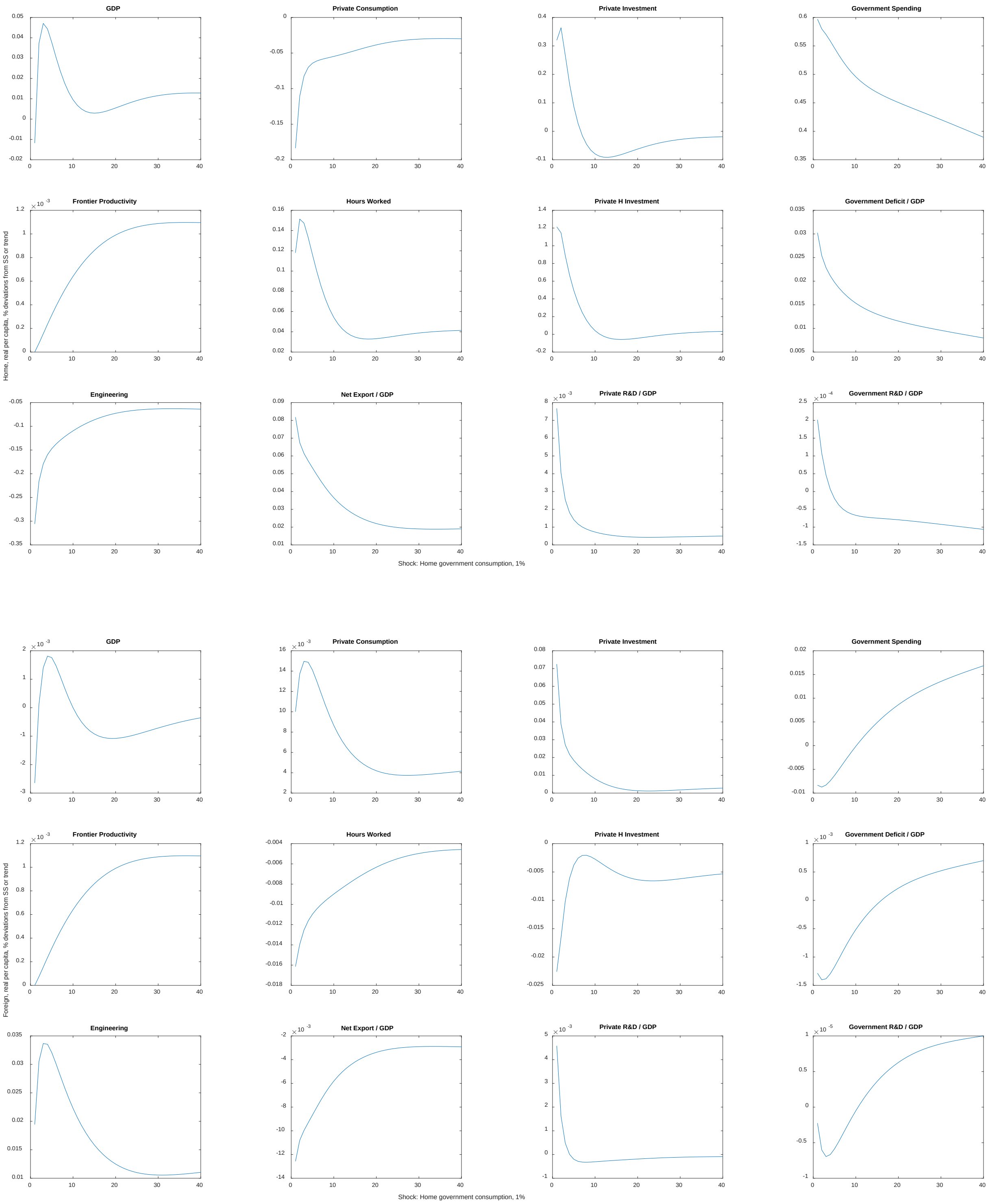

Figure 31: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point of GDP increase in government consumption.

A permanent increase in government spending reduces GDP on impact (due to crowding out of consumption), but slightly increases GDP from then on, though the multiplier is clearly less than one. With resources taken away from widget production, engineering falls, further limiting the rise in GDP.

6.1.18. Government physical capital investment share shock

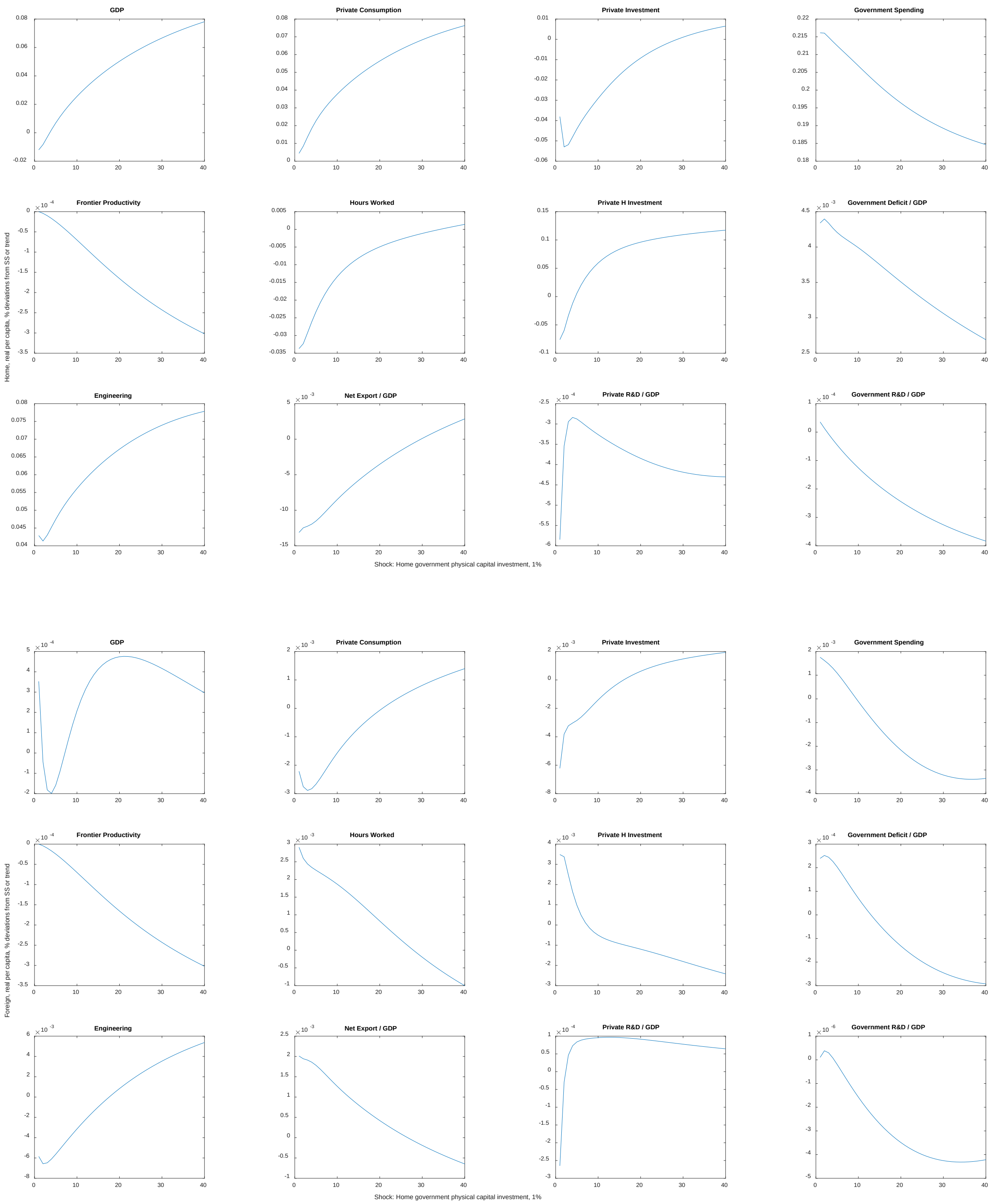

Figure 32: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point of GDP increase in government physical capital investment.

A permanent increase in government physical capital investment partially crowds out private investment, but still leads to a substantial increase in GDP and consumption. The increased government capital pushes up output per hour in all sectors, increasing engineering output which further amplifies the rise in GDP. After a few years, human capital investment rises to take advantage of the higher engineering.

6.1.19. Government R&D capital investment share shock

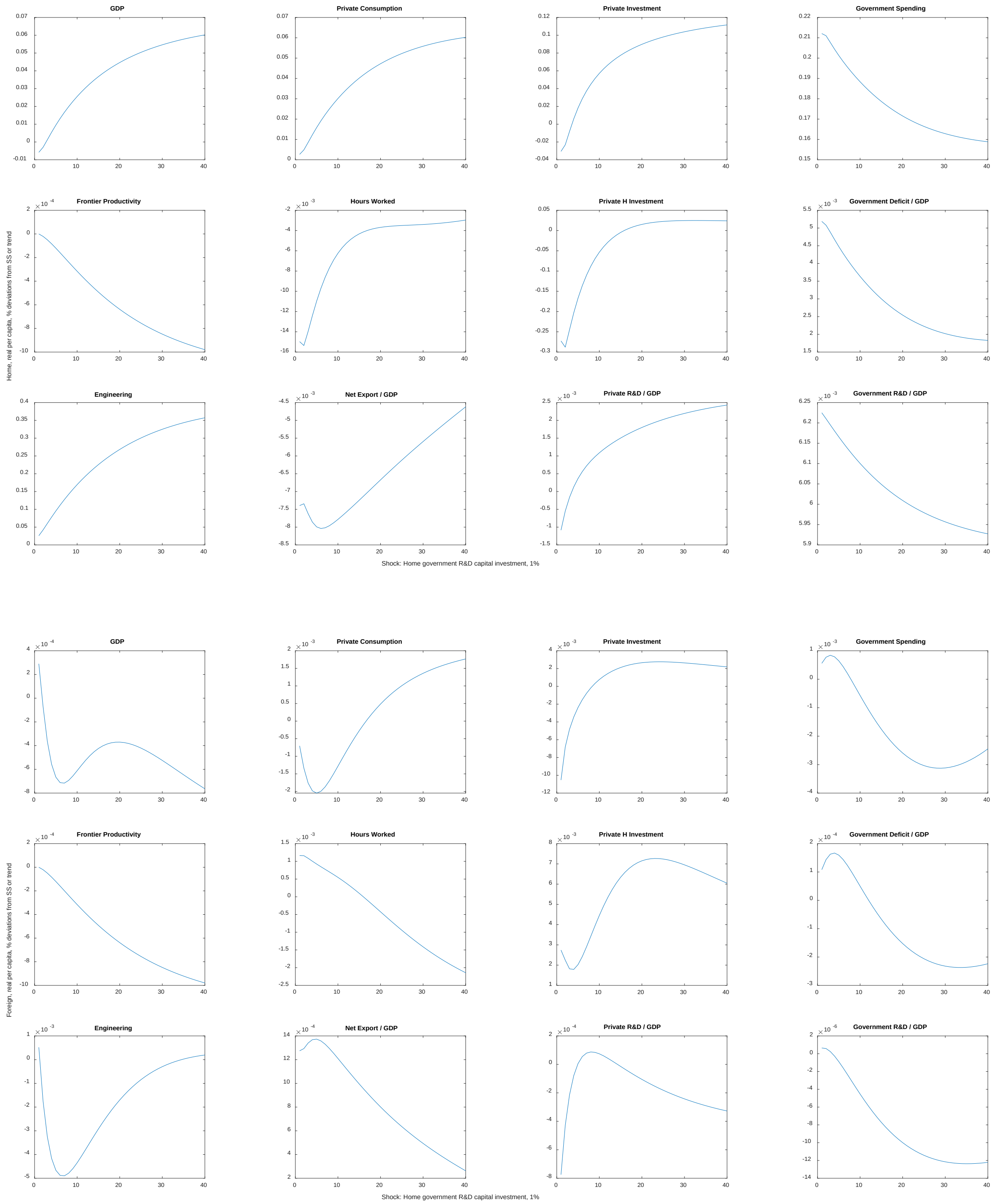

Figure 33: Impulse response in the home country (top panels) and foreign country (bottom panels) to a 1 percentage point of GDP increase in government R&D capital investment.

A permanent increase in government R&D capital investment leads to higher engineering output, thus increasing GDP. Private investment rises to keep up with the increase in engineering. However, human capital investment falls initially, as labour substitutes between producing government R&D capital and producing human capital (e.g. university professors reduce their teaching to pursue grants).

6.2. Unconditional forecasts (the "baseline" scenario)

We now present our model's forecasts for the next one hundred years. These are produced by first running the Kalman smoother at the estimated posterior mode to find the distribution of the model's state variables in the final observation period (2017). We then use this as a base from which to run forward with repeated Kalman predict steps. We chiefly show paths for observable variables, as these are the most precisely identified. In the plots below, the displayed confidence bands are at the 90% level, and only capture uncertainty coming from the shocks, not the uncertainty coming from the parameters. We omit confidence bands for the years up to 2017 so that the start of the forecast period is clear. (Given that there are many missing observations, even prior to 2017 the Kalman smoother may be having to infer values, so there could be non-trivial uncertainty pre-2017 as well.) The rest of this section proceeds region by region.

6.2.1. United States

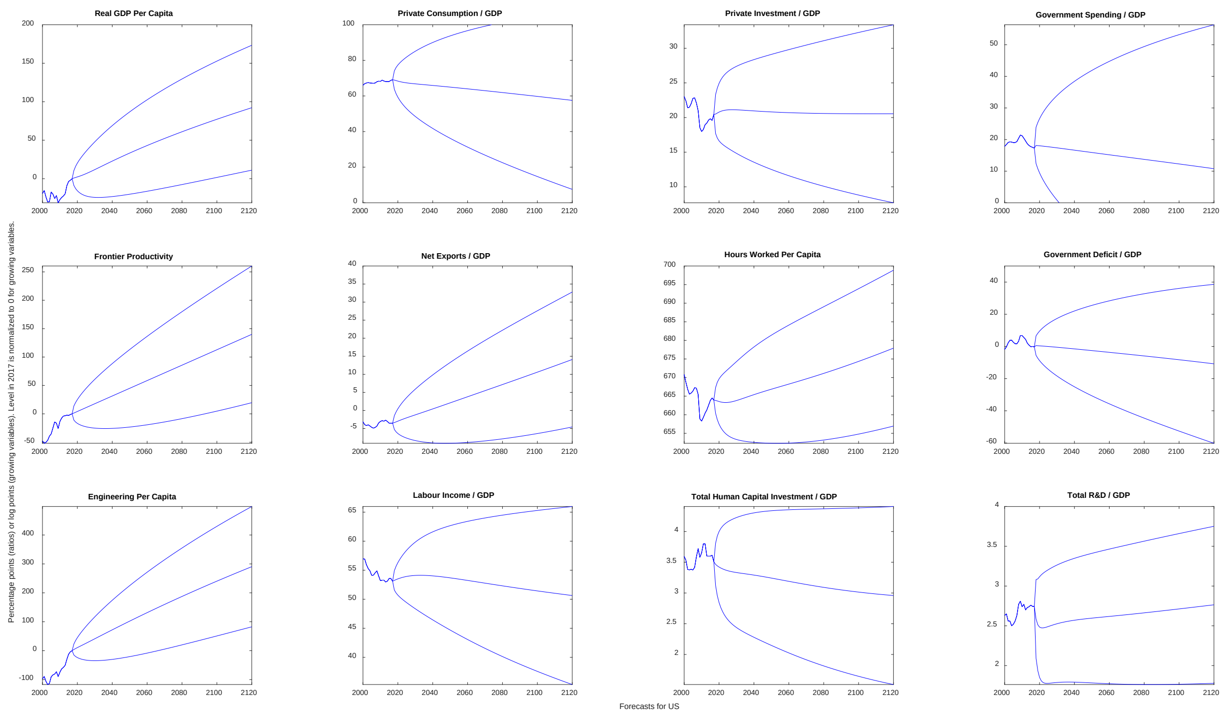

Figure 34: Forecasts for the United States. Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables. The confidence bands are at a 90% level.

According to the model, US per capita real growth in the next 100 years is likely to be slower than in the last 100 years, averaging only 1% despite an anticipated rise in hours worked per capita. At the same time, the model predicts continued structural change in the US economy. The labour share will continue to fall, reaching almost 50% by 2120, and the consumption share will also fall. Meanwhile, the US will begin exporting much more heavily, becoming a net exporter by the middle of the century. Human capital investment is expected to decline from its current level (no doubt contributing to the modest growth), while R&D investment is expected to initially decline before gradually recovering. The model is anticipating further cuts in government spending, leading to a substantial government surplus by the end of the sample. However, it should be noted that the confidence bands are all incredibly wide on these predictions. While this may be in part due to the well-known tendency of Bayesian estimated DSGE models to overpredict standard deviations (see e.g. Christiano, Trabandt & Walentin (2011)), it also reflects the extreme difficulty of making predictions at such long horizons.

6.2.2. Germany

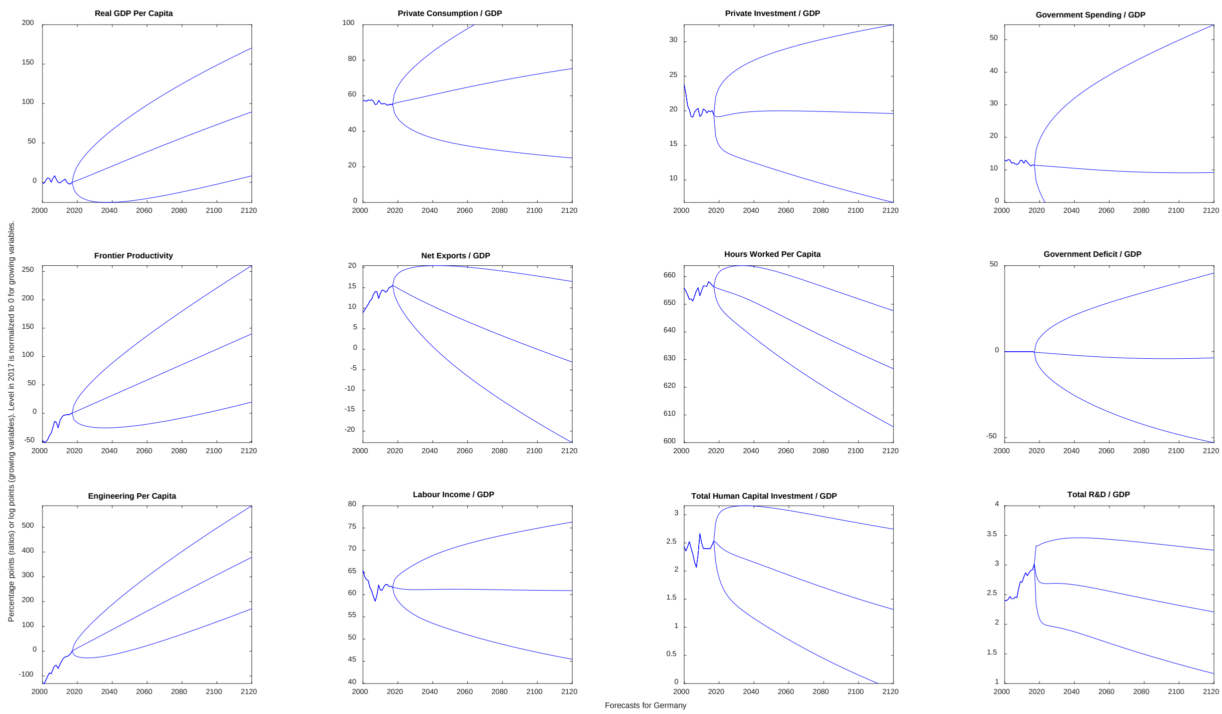

Figure 35: Forecasts for Germany. Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables. The confidence bands are at a 90% level.

German GDP growth over the sample is anticipated to be slightly worse than that in the US, in part as the model predicts a substantial decline in hours per capita and human capital investment. Meanwhile, Germany's role as a net exporter is predicted to end around the end of the century. Such a reduction in German net exports is plausible given the rise of the skilled Chinese middle class, and the rise of electric cars. The German private consumption share is predicted to rise to make up for the fall in net exports. German R&D is also predicted to fall substantially, so while it is currently at the EU's 3% target, it is not predicted to stay there. Interestingly, Germany is predicted to experience more growth in "engineering" use per capita than the US, perhaps due to exporting a lower share of its "widgets".

6.2.3. The United Kingdom

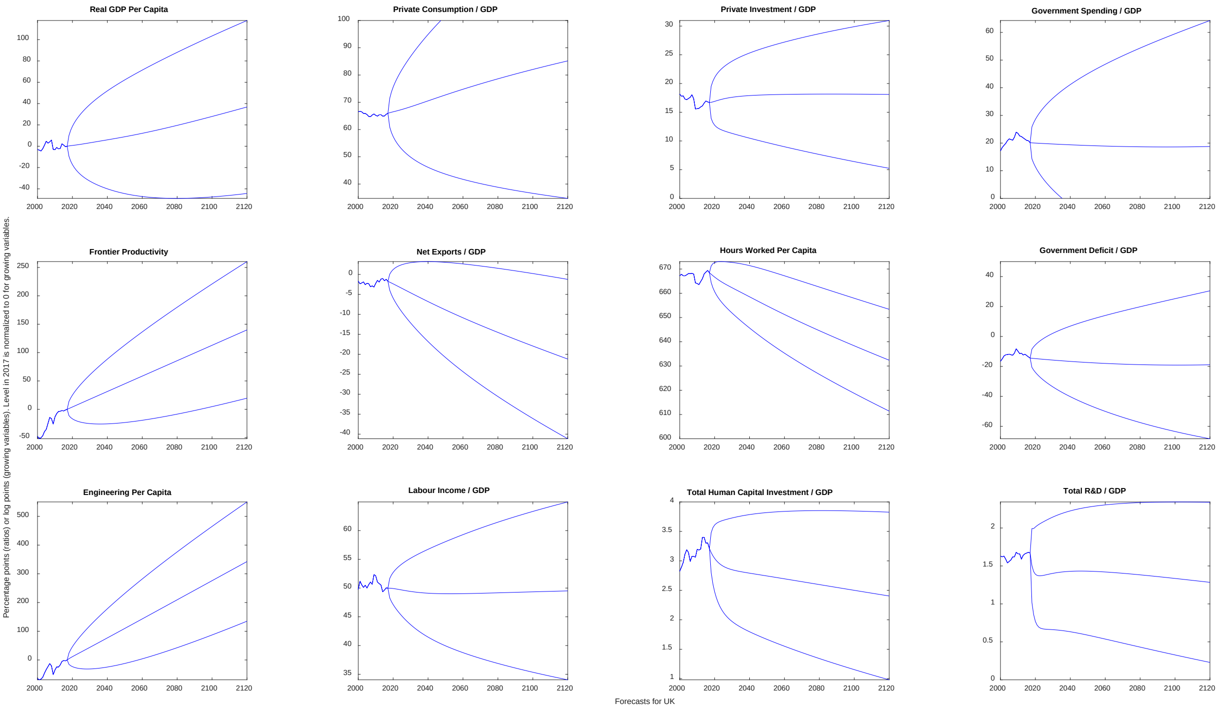

Figure 36: Forecasts for the United Kingdom. Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables. The confidence bands are at a 90% level.

The UK's outlook is particularly bleak. By 2120, GDP per capita will not even have grown 40%, meaning the annual growth rate per capita is below 0.4%. This partly reflects an expected decline in hours worked, human capital investment and R&D investment. Meanwhile, net exports are expected to fall substantially.

6.2.4. France

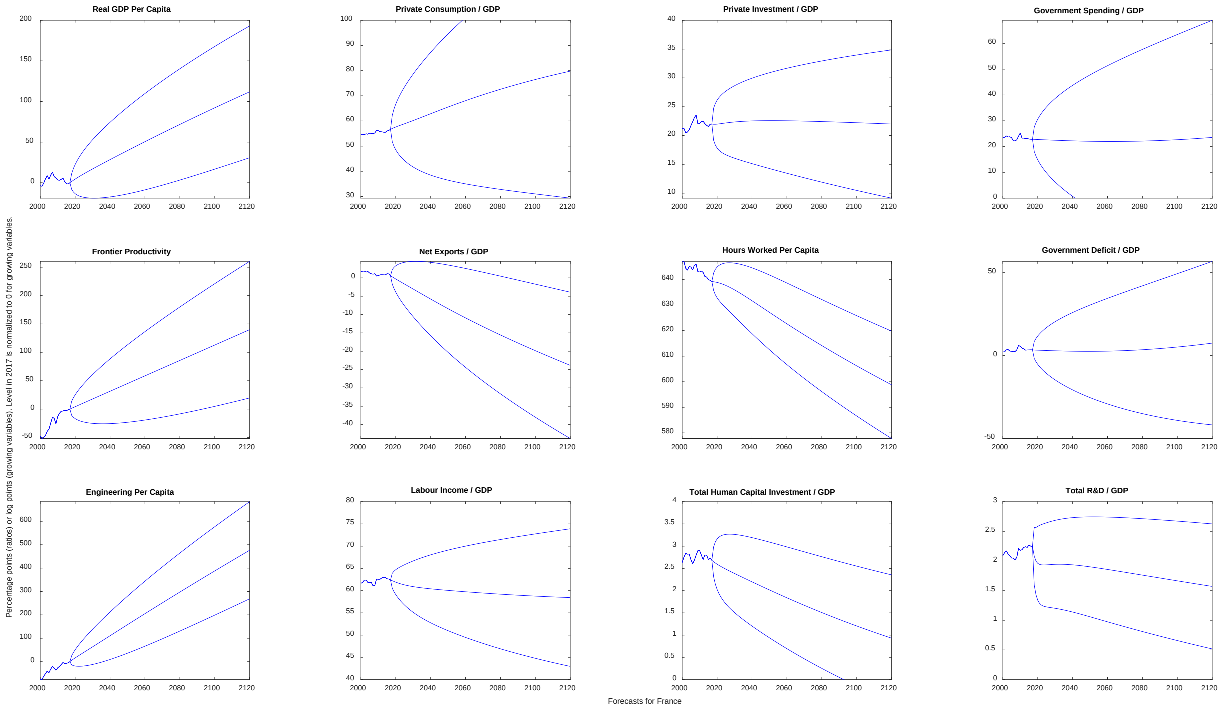

Figure 37: Forecasts for France. Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables. The confidence bands are at a 90% level.

France's GDP growth is expected to be particularly strong over the next 100 years. Higher even than the US or Germany's. This is accomplished despite anticipated declines in hours worked and human capital investment. The most obvious explanation for this is that French capital stocks were relatively low at the start of the forecast horizon, so France can benefit from some catch-up growth.

6.2.5. “REU” (Belgium, Denmark, Finland, Italy, Netherlands, Portugal, Spain, Sweden)

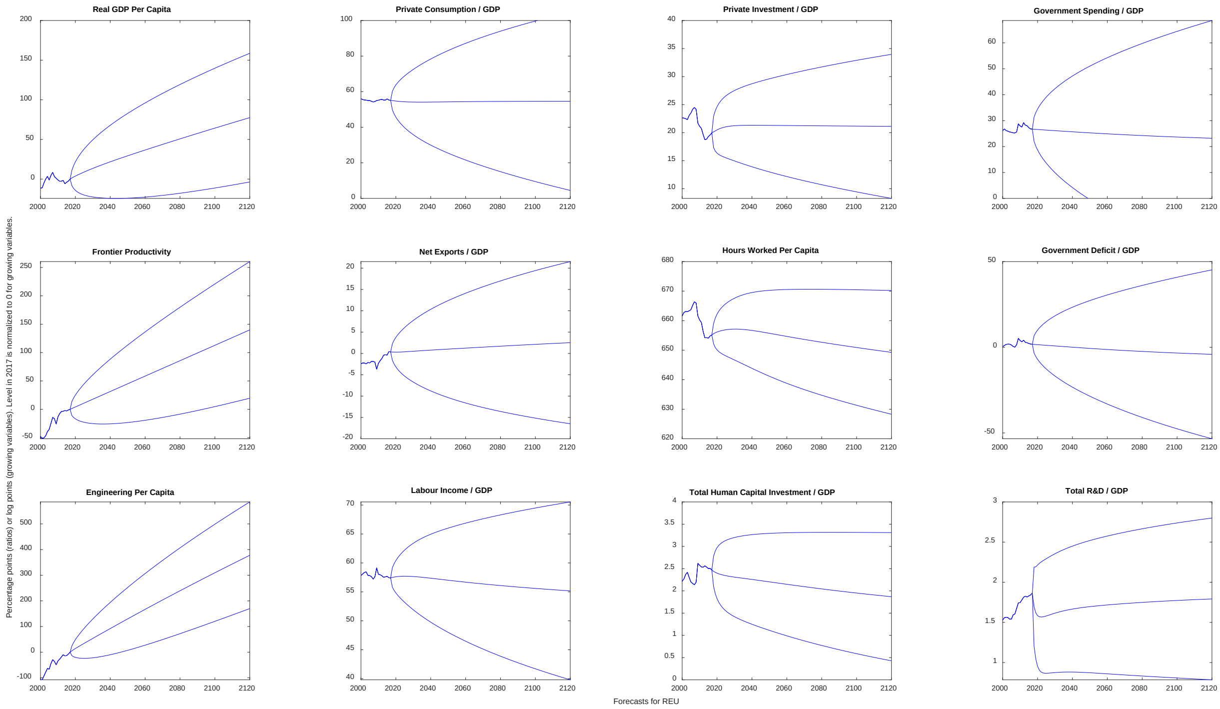

Figure 38: Forecasts for “REU” (Belgium, Denmark, Finland, Italy, Netherlands, Portugal, Spain, Sweden). Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables. The confidence bands are at a 90% level.

The REU region is expected to experience moderately weaker growth than the US, Germany or France, but still stronger growth than the UK. There does not seem to be evidence of further catch-up from the REU zone. The REU zone is expected to slightly strengthen its net exports, however, it is predicted to experience a drop in labour income share, driven by reductions in hours per capita and human capital investment. REU governments are predicted to be running a surplus by the end of the sample.

6.2.6. "RMA" (Australia, Canada, Japan, Norway, Switzerland)

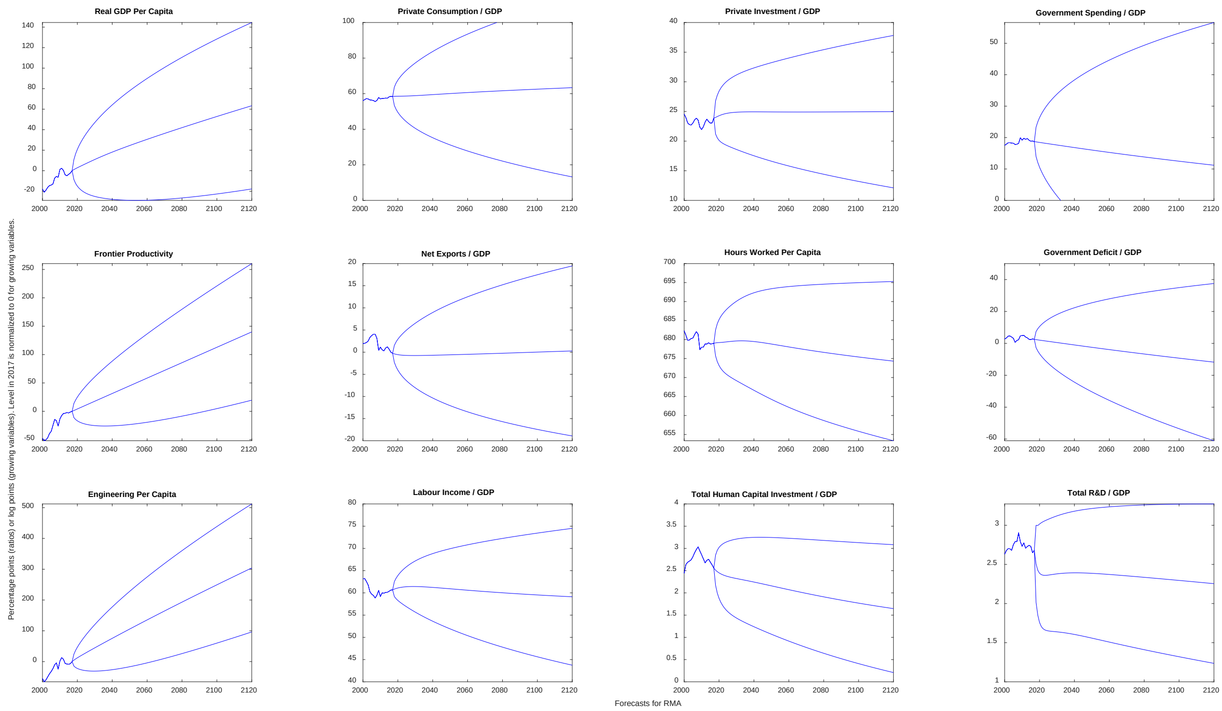

Figure 39: Forecasts for "RMA" (Australia, Canada, Japan, Norway, Switzerland). Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables. The confidence bands are at a 90% level.

The RMA region is also predicted to struggle over the next 100 years, achieving only about 60% growth by 2120. This is the second worse after the UK. This may in part be driven by the drop in the R&D and human capital investment shares that the model predicts for the RMA region. The model also predicts a large drop in the government expenditure share in the region, which may be exacerbating the weak growth, though it does ensure a reduction in the deficit.

6.3. Forecasts with additional R&D and human capital stimulus

We now look at an alternative scenario in which a sequence of unexpected changes to human capital taxes and government R&D capital investment are made in EU countries, in order to increase the human capital investment and R&D shares. The precise targets we use are guided by the common scenario produced by the MONROE project, to which this paper is a contribution.

In particular, the MONROE common scenario stipulates that the share of highly educated people should increase by 0.1 percentage points per year, and that R&D shares should reach their EU targets by 2030, and then stay there. Unfortunately, it is hard to transfer these targets literally to the model of this paper.

Firstly, we do not have a share of highly educated people in the model. The closest we have to it is the model's human capital stock. However, given the model's adjustment costs, achieving a rapid increase in human capital stocks is likely to be costly. Furthermore, it is unclear what magnitude of increase would be appropriate. Instead then, we assume that EU policy makers attempt to ensure that the human capital investment share of GDP in EU countries is $0.01 \times t$ percentage points higher than it is in our unconditional forecast, where t is the number of years since 2018. Using a 0.01 percentage point per year increase has a back of the envelope justification from the fact that human capital investment shares are around 3%, while population shares of degree holders are around 30%. Thus, scaling by 10% should get us something broadly reasonable.

Secondly, whereas in the other model's being produced in the MONROE project, R&D shares are constant in the unconditional "no change" or "baseline" scenario, our unconditional forecasts feature large movements in R&D shares over the next 100 years. Thus, the additional R&D we would need to get R&D levels up to the EU's targets will not agree with the additional R&D that the other models would need. This would result in substantially different stimulus being required in our model compared to those of our project partners. Instead then, we ensure that R&D shares in EU countries are higher than our unconditional forecasts by the difference between the R&D shares in the baseline used by our MONROE partners, and those shares in their intervention scenario. To be precise, R&D shares are higher than our unconditional forecasts by the values in percentage points given in the following table:

	GERMANY	UK	FRANCE	REU
2015	0	0	0	0
2016	0.020667	0.000667	0.048667	0.038095
2017	0.041333	0.001333	0.097333	0.076189
2018	0.062	0.002	0.146	0.114284
2019	0.082667	0.002667	0.194667	0.152379
2020	0.103333	0.003333	0.243333	0.190473
2021	0.124	0.004	0.292	0.228568
2022	0.144667	0.004667	0.340667	0.266662
2023	0.165333	0.005333	0.389333	0.304757
2024	0.186	0.006	0.438	0.342852
2025	0.206667	0.006667	0.486667	0.380946
2026	0.227333	0.007333	0.535333	0.419041
2027	0.248	0.008	0.584	0.457136
2028	0.268667	0.008667	0.632667	0.49523
2029	0.289333	0.009333	0.681333	0.533325
2030+	0.31	0.01	0.73	0.571419

Table 3: Increments to R&D shares in percentage points over the unconditional forecasts

Since we have two policy targets per EU country, we need two policy instruments per EU country. Given that we are looking to manipulate human capital and the R&D share, the two natural instruments are taxes (or subsidies) on human capital, and government R&D expenditure. We assume that EU governments coordinate to jointly hit the eight objectives using their eight instruments each year. We assume that other agents in the model are surprised by the EU governments' policy intervention each period.

6.3.1. United States

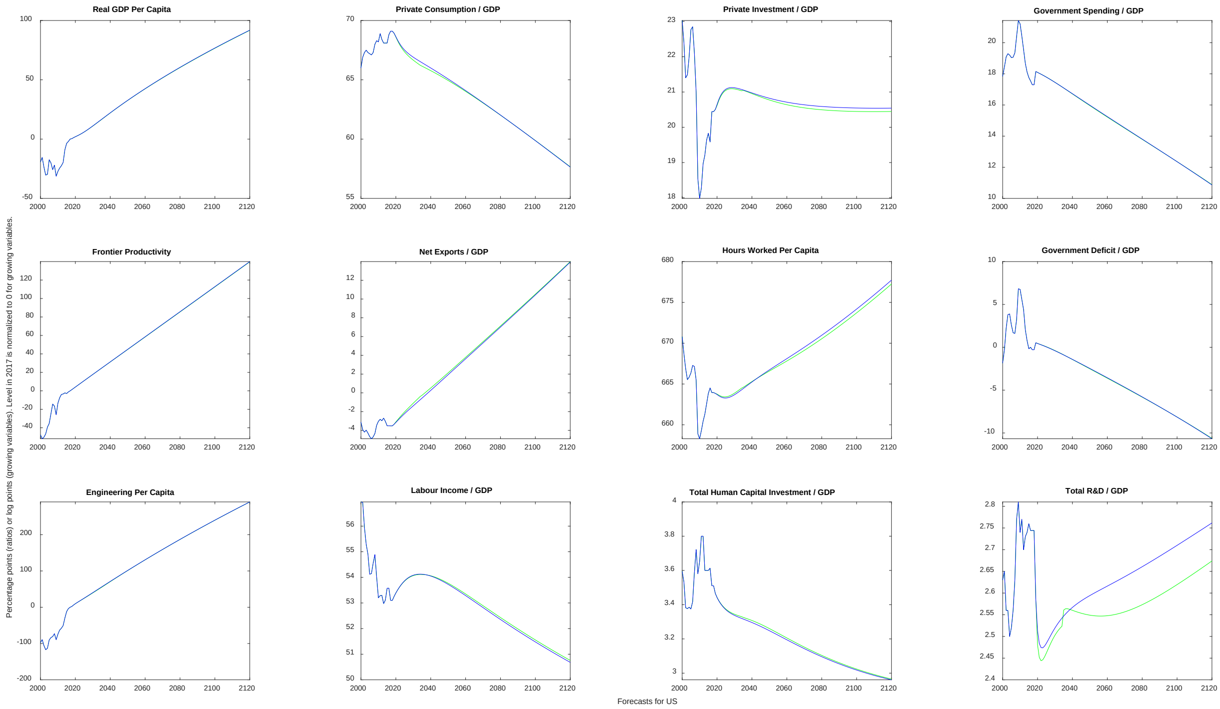

Figure 40: Forecasts given additional EU R&D and human capital expenditure for the United States, in green. Blue lines are the baseline (as in the previous section). Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables.

This policy intervention has a fairly marginal effect on the US. The only major impact is that the US R&D share ends up being lower, as more frontier R&D is being done in the EU instead. However, with the estimated parameters, this barely has any impact on output growth in the US, and only slightly depresses US consumption and investment. Frontier productivity in the engineering sector does not move, so this intervention has no permanent effects on engineering productivity.

6.3.2. Germany

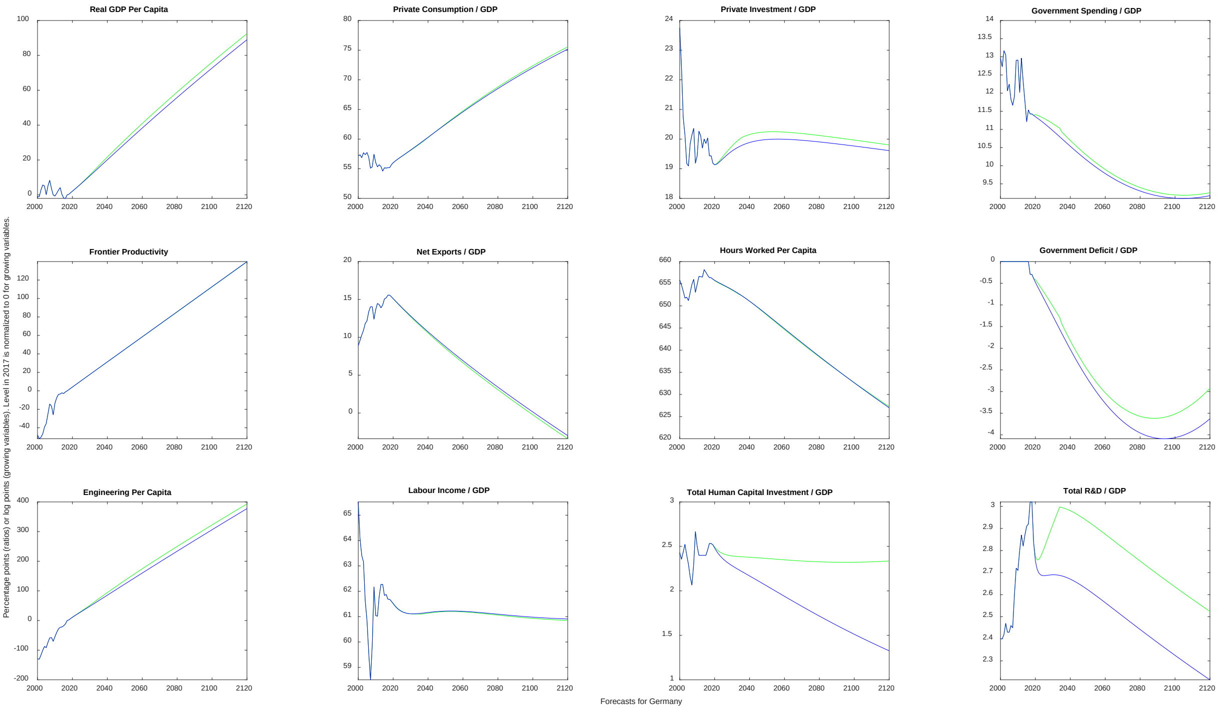

Figure 41: Forecasts given additional EU R&D and human capital expenditure for Germany, in green. Blue lines are the baseline (as in the previous section). Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables.

The policy intervention does lead to an increase in German GDP, in part thanks to higher private investment. However, it is relatively small: only a few percentage points by 2120. With more government R&D being performed, German engineering use increases as engineering becomes relatively cheaper. The policy intervention turns out to be of a perfect magnitude to prevent the decline in the human capital investment share of German GDP. However, while it manages to increase German R&D expenditure during the period of increasing interventions up to 2030, beyond that point the R&D share again resumes its decline. As expected, the government deficit worsens, but only by about 1% of GDP at the peak.

6.3.3. The United Kingdom

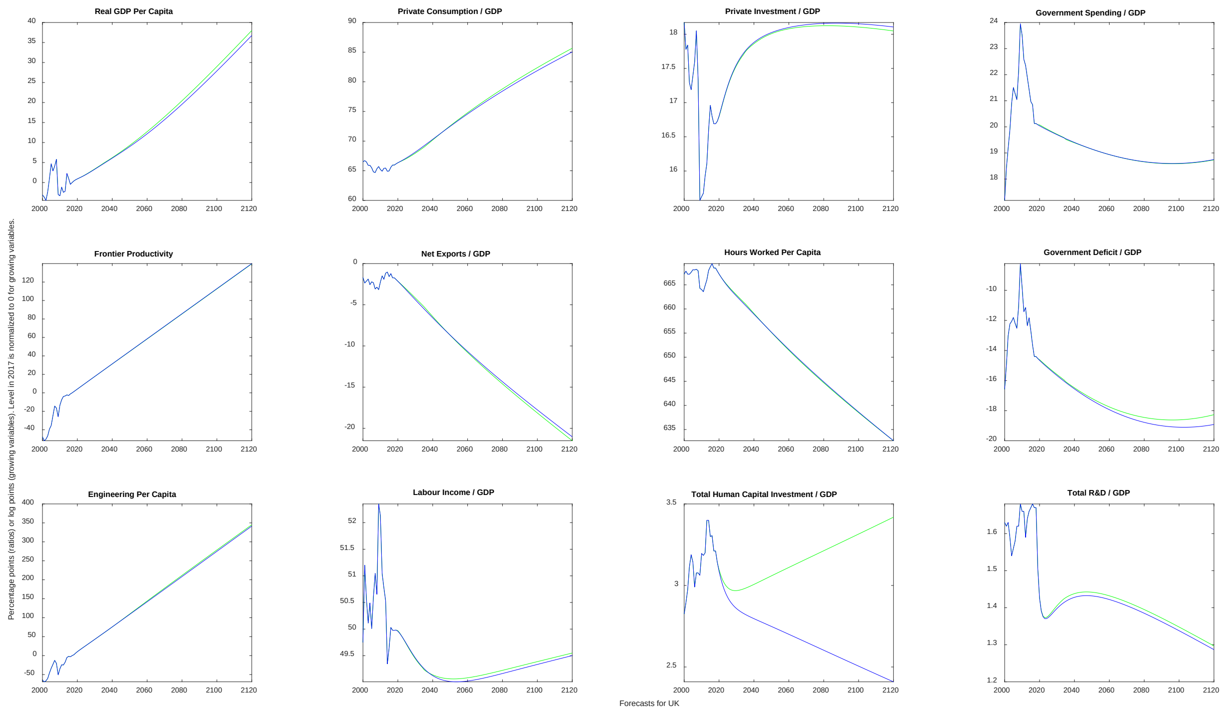

Figure 42: Forecasts given additional EU R&D and human capital expenditure for the United Kingdom, in green. Blue lines are the baseline (as in the previous section). Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables.

The effect on the UK is broadly similar, though the R&D expenditure share moves less, due to the very modest target given to the UK. Human capital investment grows substantially here though, leading to an increase in the labour income share.

6.3.4. France

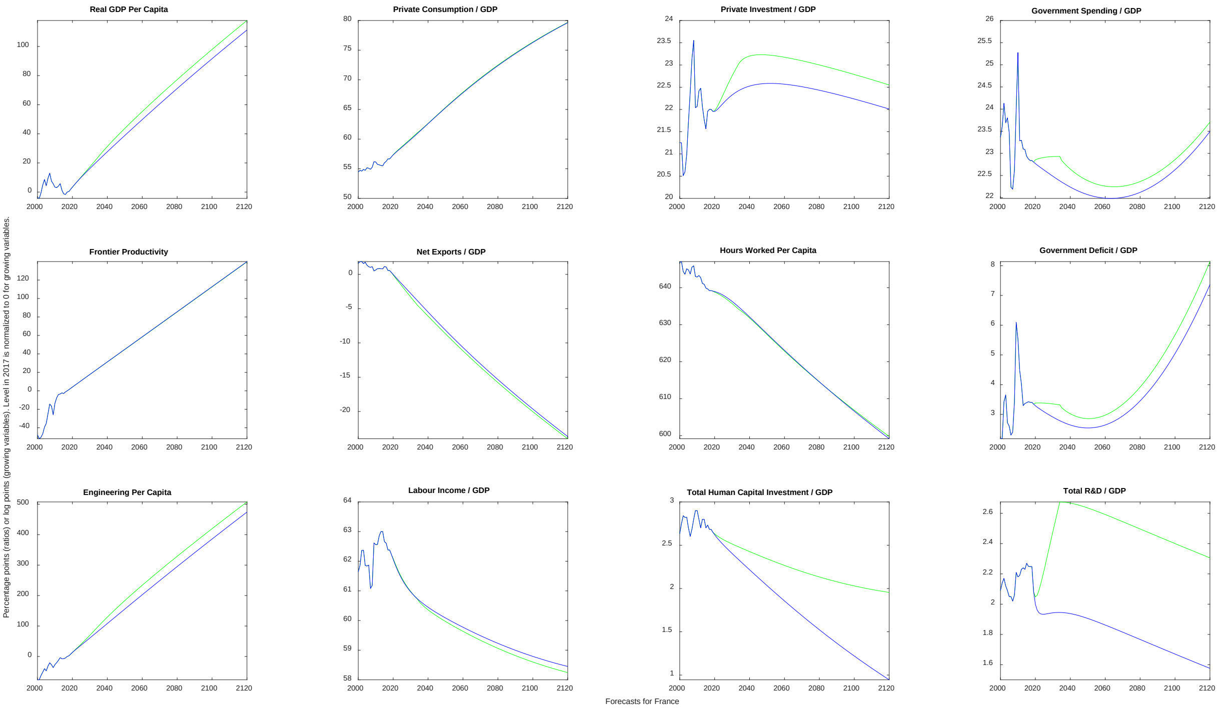

Figure 43: Forecasts given additional EU R&D and human capital expenditure for France, in green. Blue lines are the baseline (as in the previous section). Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables.

France's outcomes are also quite like Germany's. However, France experiences a larger increase in private investment, leading to slightly higher GDP by 2120. Despite this, the magnitude of the human capital investment intervention is not sufficient to completely reverse the expected decline in the French human capital investment share.

6.3.5. “REU” (Belgium, Denmark, Finland, Italy, Netherlands, Portugal, Spain, Sweden)

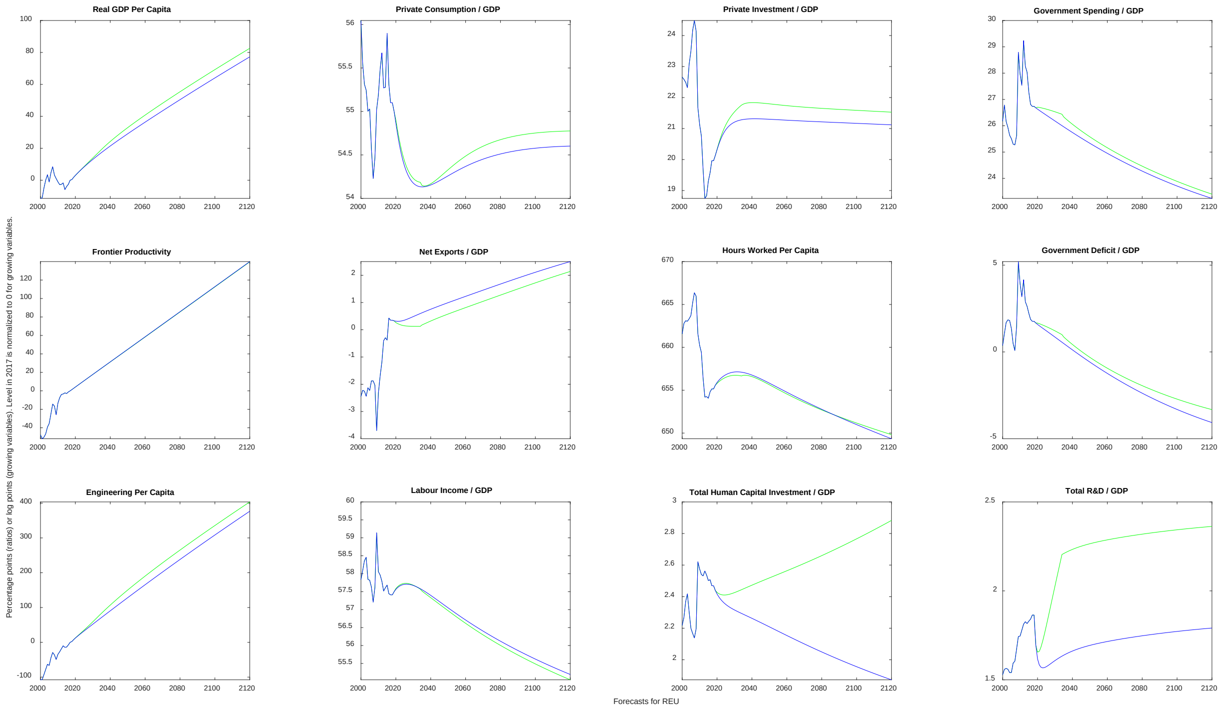

Figure 44: Forecasts given additional EU R&D and human capital expenditure for “REU” (Belgium, Denmark, Finland, Italy, Netherlands, Portugal, Spain, Sweden), in green. Blue lines are the baseline (as in the previous section). Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables.

The story for the REU region is again similar, though with the REU’s relatively stable consumption and investment, these increases are clearer in the plots than they were for other countries.

6.3.6. “RMA” (Australia, Canada, Japan, Norway, Switzerland)

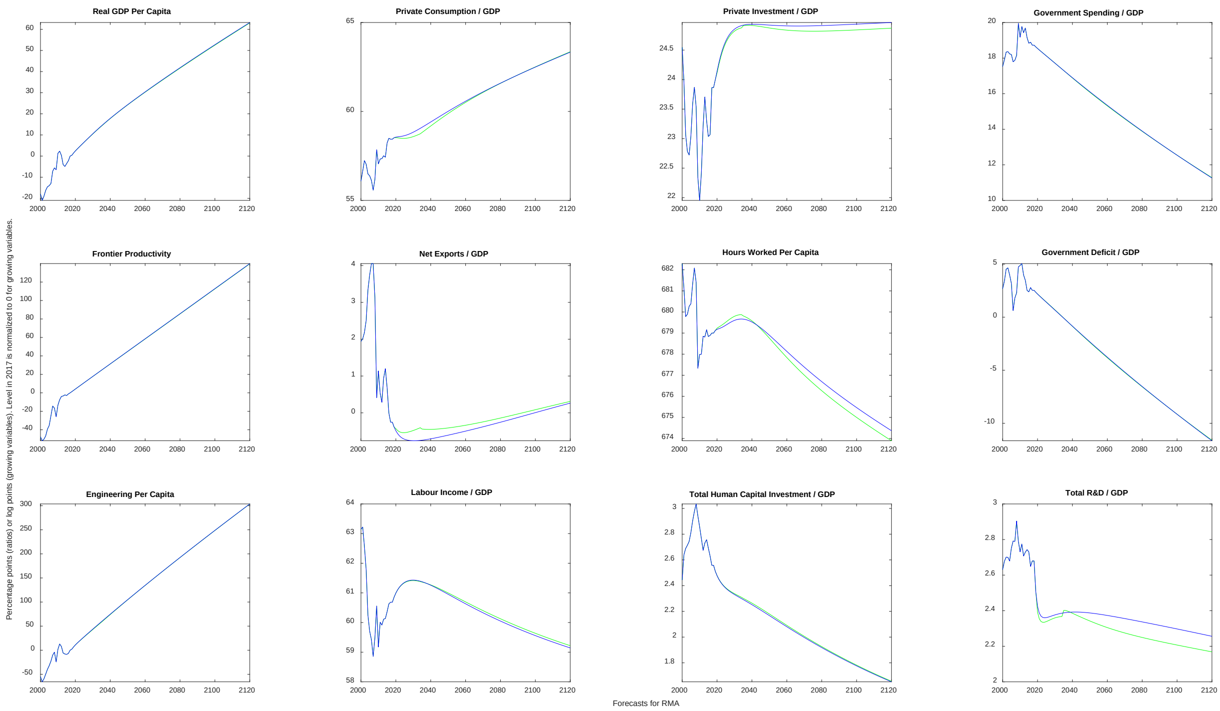

Figure 45: Forecasts given additional EU R&D and human capital expenditure for “RMA” (Australia, Canada, Japan, Norway, Switzerland), in green. Blue lines are the baseline (as in the previous section). Percentage points (ratios) or log points (growing variables and hours per capita). The level in 2017 is normalized to 0 for growing variables.

Finally, the effects on the RMA region are broadly similar to those on the US. Net exports increase so that the RMA can supply the additional demand from the EU, leading to an initial increase in hours worked. RMA R&D falls as a greater share of world R&D is now being done in the EU.

7. Conclusion

This paper has presented a multi-country, multi-sector DSGE model with endogenous growth. The model was designed for policy analysis, and features many channels by which policy may have a short-run, medium-run or long-run impact on output and R&D.

The model was estimated on a long span of data across six regions (the US, the UK, Germany, France, (most of) the rest of the EU, other advanced nations). Our estimated parameters support our prior intuition that complementarity between human capital and advanced technologies is a key channel for generating gradual diffusion of technological advances.

From running the Kalman smoother on the model, we showed that the model inferred level of European public R&D spending has been declining over the last 30 years, as a proportion of GDP. The model's best interpretation of the data also features declining levels of European human capital relative to trend in all sectors except tradeables.

From examining the impulse responses to the model's most important shocks we have exhibited the strength of the model's mechanisms for generating persistence, amplification and comovement. Every shock we examined impacted engineering demand and through that also impacted productivity in the engineering sector. While these impacts were concentrated over medium horizons, we also saw that all shocks had at least some permanent effect on the level of output through their effects on the process improvement decisions of frontier firms.

One potential challenge that this model raises for policy makers is that the permanent effects on frontier productivity and the medium-term effects on engineering usually have opposite signs. An increase in engineering demand leads both to more product entry—implying high productivity in the medium term as new industries are more productive, and more firm entry within each industry—implying lower productivity in the long run as higher competition lowers mark-ups and thus process improvement incentives. However, since the magnitude of the permanent effects on engineering is swamped by the direct effect of each of the permanent policy changes we consider, policy makers can perhaps safely ignore these permanent effects on frontier productivity.

From examining the impulse responses to policy shocks, we found that raising the lump sum tax on impatient households could increase output while reducing the deficit, making it a potential powerful tool for stimulating output over the medium-term. However, such a tax may not be particularly politically feasible, since it hits some of the worse off members of society.

We also found that the policy variables directly targeting human capital and R&D were effective in increasing output. For example, increasing human capital accumulation subsidies does indeed increase human capital and output, and increasing government R&D capital increases both output and private sector R&D over the medium term. As such, it would certainly be possible for European governments to bring about a rise in human capital and R&D expenditure.

In our conditional forecast exercise, we show the consequences of the EU doing just that. The effects of our considered policy intervention were quite moderate, with only a small level effect on output. While policy makers may have hoped such an intervention would permanently increase EU growth rates, in an interconnected world, this was always unlikely, particularly given scale effect considerations. However, the intervention does succeed in ensuring that a greater proportion of world R&D occurs in the EU. There may be political or security reasons why policy makers view this as desirable, independent of the effect on GDP.

8. References

Acemoglu, Daron, David Autor, David Dorn, Gordon H Hanson & Brendan Price. 2016. 'Import Competition and the Great US Employment Sag of the 2000s'. *Journal of Labor Economics* 34 (S1): S141–S198.

Adjemian, Stéphane, Houtan Bastani, Fréderic Karamé, Michel Juillard, Junior Maih, Ferhat Mihoubi, George Perendia, Johannes Pfeifer, Marco Ratto & Sébastien Villemot. 2011. *Dynare: Reference Manual Version 4*. CEPREMAP.

Aghion, Philippe, Nick Bloom, Richard Blundell, Rachel Griffith & Peter Howitt. 2005. 'Competition and Innovation: An Inverted-U Relationship.' *Quarterly Journal of Economics* 120 (2) (May): 701–728.

Aghion, Philippe & Peter Howitt. 1992. 'A Model of Growth Through Creative Destruction'. *Econometrica* 60 (2) (March): 323–351.

Autor, David H, David Dorn & Gordon H Hanson. 2016. 'The China Shock: Learning from Labor-Market Adjustment to Large Changes in Trade'. *Annual Review of Economics* 8: 205–240.

Autor, David H, Lawrence F Katz & Melissa S Kearney. 2008. 'Trends in U.S. Wage Inequality: Revising the Revisionists'. *Review of Economics and Statistics* 90 (2): 300–323.

Baxter, Marianne & Dorsey D Farr. 2005. 'Variable Capital Utilization and International Business Cycles'. *Journal of International Economics* 65 (2): 335–347.

Caldara, Dario, Jesus Fernandez-Villaverde, Juan F Rubio-Ramirez & Wen Yao. 2012. 'Computing DSGE Models with Recursive Preferences and Stochastic Volatility'. *Review of Economic Dynamics* 15 (2): 188–206.

Chari, Varadarajan V, Patrick J Kehoe & Ellen R McGrattan. 2007. 'Business Cycle Accounting'. *Econometrica* 75 (3): 781–836.

Christiano, Lawrence J., Martin Eichenbaum & Charles L. Evans. 2005. 'Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy'. *Journal of Political Economy* 113 (1): 1–45.

Christiano, Lawrence J., Mathias Trabandt & Karl Walentin. 2011. 'Introducing Financial Frictions and Unemployment into a Small Open Economy Model'. *Journal of Economic Dynamics and Control* 35 (12): 1999–2041.

Colacito, Ric, Mariano Max Massimiliano Croce, Steven Wei Ho & Philip Howard. 2018. 'BKK the EZ Way. International Long-Run Growth News and Capital Flows'.

Comin, Diego & Mark Gertler. 2006. 'Medium-Term Business Cycles'. *American Economic Review* 96 (3) (June): 523–551.

Comin, Diego & Bart Hobijn. 2010. 'An Exploration of Technology Diffusion'. *American Economic Review* 100 (5) (December): 2031–2059.

Dillenberger, David, Daniel Gottlieb & Pietro Ortoleva. 2018. *Stochastic Impatience and the Separation of Time and Risk Preferences*. Penn Institute for Economic Research, Department of Economics, University of Pennsylvania.

Dixit, Avinash K. & Joseph E. Stiglitz. 1977. 'Monopolistic Competition and Optimum Product Diversity'. *The American Economic Review* 67 (3) (June): 297–308.

Duarte, Margarida & Diego Restuccia. 2009. 'The Role of the Structural Transformation in Aggregate Productivity'. Working Papers tecipa-348. University of Toronto, Department of Economics.

Elsby, Michael, Bart Hobijn & Ayseful Sahin. 2013. 'The Decline of the U.S. Labor Share'. *Brookings Papers on Economic Activity* 44 (2 (Fall)). Brookings Papers on Economic Activity: 1–63.

Epstein, Larry G. & Stanley E. Zin. 1989. 'Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework'. *Econometrica* 57 (4) (July 1): 937–969.

Ethier, Wilfred J. 1982. 'National and International Returns to Scale in the Modern Theory of International Trade'. *The American Economic Review* 72 (3) (June): 389–405.

Feenstra, Robert C., Robert Inklaar & Marcel P. Timmer. 2015. 'The Next Generation of the Penn World Table'. *American Economic Review* 105 (10) (October): 3150–82.

Fernald, John G. 2012. *A Quarterly, Utilization-Adjusted Series on Total Factor Productivity*. Working Paper Series. Federal Reserve Bank of San Francisco.

Greenwood, Jeremy, Zvi Hercowitz & Gregory W. Huffman. 1988. 'Investment, Capacity Utilization, and the Real Business Cycle'. *The American Economic Review* 78 (3): 402–417.

Groth, Christian, Karl-Josef Koch & Thomas Steger. 2009. 'When Economic Growth Is Less than Exponential'. *Economic Theory* (July 4).

Guerrieri, Luca & Matteo Iacoviello. 2017. 'Collateral Constraints and Macroeconomic Asymmetries'. *Journal of Monetary Economics* 90: 28–49.

Hansen, Nikolaus. 2006. 'The CMA Evolution Strategy: A Comparing Review'. In *Towards a New Evolutionary Computation*, 75–102. Springer.

Havranek, Tomas, Roman Horvath, Zuzana Irsova & Marek Rusnak. 2015. 'Cross-Country Heterogeneity in Intertemporal Substitution'. *Journal of International Economics* 96 (1) (May 1): 100–118.

Holden, Tom. 2016. 'Medium Frequency Cycles in a Stationary World.'

———. 2017. *Medium Frequency Cycles, Global Imbalances, and the Puzzles of Open Economy Macroeconomics.* SSRN Scholarly Paper. Rochester, NY: Social Science Research Network.

Iacoviello, Matteo. 2005. 'House Prices, Borrowing Constraints, and Monetary Policy in the Business Cycle'. *American Economic Review* 95 (3) (June): 739–764.

Inklaar, Robert, Harmen de Jong, Jutta Bolt & Jan van Zanden. 2018. *Rebasing 'Maddison': New Income Comparisons and the Shape of Long-Run Economic Development*. GGDC Research Memorandum. Groningen Growth and Development Centre, University of Groningen.

Jaimovich, Nir. 2007. 'Firm Dynamics and Markup Variations: Implications for Sunspot Equilibria and Endogenous Economic Fluctuations'. *Journal of Economic Theory* 137 (1) (November): 300–325.

Jaimovich, Nir & Sergio Rebelo. 2009. 'Can News about the Future Drive the Business Cycle?' *American Economic Review* 99 (4) (September): 1097–1118.

Jordà, Òscar, Moritz Schularick & Alan M. Taylor. 2016. 'Macrofinancial History and the New Business Cycle Facts'. In *NBER Macroeconomics Annual 2016, Volume 31*, 213–263. University of Chicago Press.

Karabarbounis, Loukas & Brent Neiman. 2014a. 'Capital Depreciation and Labor Shares Around the World: Measurement and Implications'. *National Bureau of Economic Research Working Paper Series* No. 20606.

———. 2014b. 'The Global Decline of the Labor Share*'. *The Quarterly Journal of Economics* 129 (1) (February 1): 61–103.

King, Robert G., Charles I. Plosser & Sergio T. Rebelo. 1988. 'Production, Growth and Business Cycles: I. The Basic Neoclassical Model'. *Journal of Monetary Economics* 21 (2) (March 1): 195–232.

Kollmann, Robert. 2016. 'International Business Cycles and Risk Sharing with Uncertainty Shocks and Recursive Preferences'. *Trade and Macroeconomics: New Approaches to Enduring Puzzles* 72 (November 1): 115–124.

———. 2017. 'Explaining International Business Cycle Synchronization: Recursive Preferences and the Terms of Trade Channel'.

Levin, Richard C., Wesley M. Cohen & David C. Mowery. 1985. 'R & D Appropriability, Opportunity, and Market Structure: New Evidence on Some Schumpeterian Hypotheses'. *The American Economic Review* 75 (2) (May): 20–24.

Li, Chol-Won. 2000. 'Endogenous vs. Semi-Endogenous Growth in a Two-R&D-Sector Model'. *The Economic Journal* 110 (462) (March 1): C109–C122.

Mansfield, Edwin. 1993. 'The Diffusion of Industrial Robots in Japan and the United States'. *Research Policy* 22 (2) (April): 105.

Mattesini, Fabrizio & Lorenza Rossi. 2010. 'Progressive Taxation of Labor Income, Taylor Principle and Monetary Policy'.

Peretto, Pietro F. 1999. 'Cost Reduction, Entry, and the Interdependence of Market Structure and Economic Growth'. *Journal of Monetary Economics* 43 (1) (February 19): 173–195.

Perli, Roberto & Plutarchos Sakellaris. 1998. 'Human Capital Formation and Business Cycle Persistence'. *Journal of Monetary Economics* 42 (1): 67–92.

Phillips, Kerk L. & Jeff Wrase. 2006. 'Is Schumpeterian `creative Destruction' a Plausible Source of Endogenous Real Business Cycle Shocks?' *Journal of Economic Dynamics and Control* 30 (11) (November): 1885–1913.

Rotemberg, Julio J. 2003. 'Stochastic Technical Progress, Smooth Trends, and Nearly Distinct Business Cycles'. *American Economic Review* 93 (5): 1543–1559.

Rudebusch, Glenn D. & Eric T. Swanson. 2012. 'The Bond Premium in a DSGE Model with Long-Run Real and Nominal Risks'. *American Economic Journal: Macroeconomics* 4 (1): 105–43.

Scott, John. 1984. 'Firm versus Industry Variability in R&D Intensity'. In *R & D, Patents, and Productivity*. NBER.

Serrano, Carlos J. 2010. 'The Dynamics of the Transfer and Renewal of Patents'. *The RAND Journal of Economics* 41 (4) (December 1): 686–708.

Siliverstovs, Boriss & D'Artis Kancs. 2012. *R&D and Non-Linear Productivity Growth of Heterogeneous Firms*. SSRN Scholarly Paper. Rochester, NY: Social Science Research Network.

Smets, Frank & Rafael Wouters. 2003. 'An Estimated Dynamic Stochastic General Equilibrium Model of the Euro Area'. *Journal of the European Economic Association* 1 (5): 1123–1175.

———. 2007. 'Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach'. *American Economic Review* 97 (3) (June): 586–606.

Thomas, Ryland, Sally Hills & Nicholas Dimsdale. 2010. *The UK Recession in Context — What Do Three Centuries of Data Tell Us?* SSRN Scholarly Paper. Rochester, NY: Social Science Research Network.

Tingvall, Patrik Gustavsson & Andreas Poldahl. 2006. 'Is There Really an Inverted U-Shaped Relation between Competition and R&D?' *Economics of Innovation and New Technology* 15 (3): 101.

Wälde, Klaus. 2005. 'Endogenous Growth Cycles'. *International Economic Review* 46 (3): 867–894.

9. Appendix: Equilibrium conditions

In this appendix, we list the first order conditions (FOCs) of each agent's optimisation problem in country $n \in \{1, \ldots \mathcal{N}\}$.

9.1. Households

The representative patient household maximize its life-time utility (13) subject to (7)-(12), (14)-(16), and the budget constraint. The representative impatient household maximises the impatient equivalent of (13) subject to the financial constraint (17), the impatient equivalent of (7)-(9), (14)-(16), and the budget constraint. For $m \in \{b, l\}$, $s \in S_0$, let the pairs of Lagrange multipliers and constraints be as follows: $\lambda^{CD}_{nm,t}$ is with (1); $\lambda^{CP}_{nm,t}N_{nm,t}$ and $\lambda^{\overline{CD}}_{nm,t}$ are with (8); $\lambda^{C}_{nm,t}N_{nm,t}$ and $\lambda^{\overline{CP}}_{nm,t}$ are with (9); $\lambda^{D}_{nm,t}$ is with (10); $\lambda^{KP,s}_{n,t}$ is with (11); $\lambda^{HP,s}_{n,t}$ is with (12); $\lambda^{\bar{C}}_{nm,t}N_{nm,t}$ is with (16); $\lambda^{B}_{nm,t}$ is with the budget constraint; $\lambda^{F}_{n,t}$ is with (11), $\lambda^{GD}_{nm,t}$ is with $G^{D}_{nm,t}D_{nl,t-1} = D_{nl,t}$; $\lambda^{GKP,s}_{n,t}$ is with $G^{KP,s}_{n,t}KP^{s}_{n,t-1} = KP^{s}_{n,t}$; $\lambda^{GHP,s}_{n,t}$ is with $G^{HP,s}_{n,t}HP^{s}_{n,t-1} = HP^{s}_{n,t}$. Further let $EV_{nm,t} \equiv \left[\mathbb{E}_t\left(V_{nm,t+1}^{1-\gamma_n^V}\right)\right]^{\frac{1}{1-\gamma_n^V}}$ and $\frac{\partial V_{nm,t}}{\partial V_{nm,t+1}} \equiv \beta_{nm,t}{V_{nm,t}}^{\sigma_n}{EV_{nm,t}}^{\gamma_n^V-\sigma_n}{V_{nm,t+1}}^{-\gamma_n^V}$. First order conditions are

$B_{n,t}$:	$\begin{aligned} &\mathbb{E}_t \frac{\partial V_{nl,t}}{\partial V_{nl,t+1}} \frac{\lambda^B_{nl,t+1}}{\lambda^B_{nl,t}} = Q_{n,t}; \\ &Q_{n,t} = \frac{\lambda^F_{n,t}}{\lambda^B_{nb,t}} + \mathbb{E}_t \frac{\partial V_{nb,t}}{\partial V_{nb,t+1}} \left(\frac{\lambda^B_{nb,t+1}}{\lambda^B_{nb,t}} - \frac{\lambda^F_{n,t+1}}{\lambda^B_{nb,t}} \rho^B_n \right); \end{aligned}$
$A_{n,t}$:	$\frac{\partial V_{nl,t-1}}{\partial V_{nl,t}} \frac{\lambda^B_{nl,t}}{\lambda^B_{nl,t-1}} = \Xi_{t-1,t};$
$C_{nm,t}$:	$\begin{aligned} &{V_{nm,t}}^{\sigma_n}(1-\beta_{nm,t}){N_{nm,t}}^{\xi_n-1+\sigma_n}{U_{nm,t}}^{-\sigma_n} \\ &-\mathbb{E}_t \frac{\partial V_{nm,t}}{\partial V_{nm,t+1}} {V_{nm,t+1}}^{\sigma_n}(1-\beta_{nm,t+1}){N_{nm,t+1}}^{\xi_n-1+\sigma_n}{U_{nm,t+1}}^{-\sigma_n} \frac{N_{nm,t+1}}{N_{nm,t}} h_n \\ &+\lambda^{\bar{C}}_{nm,t}(1-\varrho^C_n)\left(\frac{C_{nm,t}}{N_{nm,t}}\right)^{-\varrho^C_n}\left(\frac{\bar{C}_{nm,t-1}}{N_{nm,t-1}}\right)^{\varrho^C_n} \\ &+\lambda^C_{nm,t} = 0; \end{aligned}$
$\bar{C}_{nm,t}$:	$\begin{aligned} &-\mathbb{E}_t \frac{\partial V_{nm,t}}{\partial V_{nm,t+1}} {V_{nm,t+1}}^{\sigma_n}(1-\beta_{nm,t+1}){N_{nm,t+1}}^{\xi_n-1+\sigma_n}{U_{nm,t+1}}^{-\sigma_n} \frac{N_{nb,t+1}}{N_{nb,t}} \\ &\left[\kappa^0_{n,t+1} + \sum_{s\in S_2} \frac{\kappa^s_{n,t+1}}{1+\nu^s_n}\left(\frac{L^s_{nm,t+1}}{N_{nm,t+1}}\right)^{1+\nu^s_n} + \frac{\kappa^h_{n,t+1}}{1+\nu^h_n}\left(\frac{L^h_{nm,t+1}}{N_{nm,t+1}}\right)^{1+\nu^h_n}\right] \\ &-\lambda^{\bar{C}}_{nm,t} \\ &+\mathbb{E}_t \frac{\partial V_{nm,t}}{\partial V_{nm,t+1}} \lambda^{\bar{C}}_{nm,t+1} \left\{ \left(\frac{C_{nm,t+1}}{N_{nm,t+1}}\right)^{1-\varrho^C_n} \frac{N_{nm,t+1}}{N_{nm,t}} \varrho^C_n \left(\frac{\bar{C}_{nm,t}}{N_{nm,t}}\right)^{\varrho^C_n-1} \right\} = 0; \end{aligned}$
$CP_{nm,t}$:	$\begin{aligned} &-\lambda^C_{nm,t}\Omega^C_{n,t}{\overline{CP}_{nm,t-1}}^{(\alpha^{CP}_n-1)\frac{e^C_n-1}{e^C_n}}\left(\frac{C_{nm,t}}{\Omega^C_{n,t}CP_{nm,t}}\right)^{\frac{1}{e^C_n}}\alpha^{CP}_n \\ &-\lambda^{\overline{CP}}_{nm,t}\left(\frac{1}{CG_{n,t}}\frac{N_{n,t}}{N_{nm,t}}\right)^{1-\varrho^{CP}_n}(1-\varrho^{CP}_n){CP_{nm,t}}^{-\varrho^{CP}_n}{\overline{CP}_{nm,t-1}}^{\varrho^{CP}_n} + \lambda^{CP}_{nm,t} = 0; \end{aligned}$

$\overline{CP}_{nm,t}$:	$-\mathbb{E}_t \frac{\partial V_{nm,t}}{\partial V_{nm,t+1}} \lambda^{C}_{nm,t+1} \left\{ \alpha^{CP}_n \frac{C_{nm,t+1}}{\overline{CP}_{nm,t}} \right.$ $\left. - \Omega^{C}_{n,t+1} \overline{CP}_{nm,t}{}^{(\alpha^{CP}_n - 1)\left(1-\frac{1}{e^C_n}\right)-1} \left[\frac{C_{nm,t+1}}{\Omega^{C}_{n,t+1}}\right]^{\frac{1}{e^C_n}} \alpha^{CP}_n CP_{nm,t+1}{}^{\frac{e^C_n - 1}{e^C_n}} \right\}$ $-\mathbb{E}_t \frac{\partial V_{nm,t}}{\partial V_{nm,t+1}} \lambda^{\overline{CP}}_{nm,t+1} \left(\frac{CP_{nm,t+1}}{CG_{n,t+1}} \frac{N_{n,t+1}}{N_{nb,t+1}} \right)^{1-\varrho^{CP}_n} \varrho^{CP}_n \overline{CP}_{nm,t}{}^{\varrho^{CP}_n - 1}$ $+\lambda^{\overline{CP}}_{nm,t} = 0;$
$C^{ND}_{nm,t}$:	$-\lambda^{CP}_{nm,t} \Omega^{CP}_{n,t} \overline{C}^{D}_{nm,t-1}{}^{\alpha^{CD}_n \frac{e^{CP}_n - 1}{e^{CP}_n}} (1 - \alpha^{CD}_n) \left(\frac{CP_{nm,t}}{\Omega^{CP}_{n,t} C^{ND}_{nm,t}} \right)^{\frac{1}{e^{CP}_n}}$ $-\lambda^{\overline{CD}}_{nm,t} C^{D}_{nm,t}{}^{1-\varrho^{CD}_n} \overline{C}^{D}_{nm,t-1}{}^{\varrho^{CD}_n} (\varrho^{CD}_n - 1) C^{ND}_{nm,t}{}^{\varrho^{CD}_n - 2} - \lambda_{B,nm,t} P^{ND}_{n,t} (1 + \tau^{ND}_{n,t}) = 0;$
$C^{D}_{nm,\mathrm{t}}$:	$-\lambda^{CP}_{nm,t} \Omega^{CP}_{n,t} \bar{C}^{D}_{nm,t-1}{}^{(\alpha^{CD}_n - 1)\frac{e^{CP}_n - 1}{e^{CP}_n}} \left(\frac{CP_{nm,t}}{\Omega^{CP}_{n,t} C^{D}_{nm,t}} \right)^{\frac{1}{e^{CP}_n}} \alpha^{CD}_n$ $-\lambda^{\overline{CD}}_{nm,t} C^{ND}_{nm,t}{}^{\varrho^{CD}_n - 1} \bar{C}^{D}_{nm,t-1}{}^{\varrho^{CD}_n} (1 - \varrho^{CD}_n) C^{D}_{nm,t}{}^{-\varrho^{CD}_n} + \lambda^{CD}_{nm,t} = 0;$
$\bar{C}^{D}_{nm,t}$:	$-\mathbb{E}_t \frac{\partial V_{nm,t}}{\partial V_{nm,t+1}} \lambda^{CP}_{nm,t+1} \left\{ \alpha^{CD}_n \frac{CP_{nm,t+1}}{\bar{C}^{D}_{nm,t}} \right.$ $\left. - \Omega^{CP}_{n,t+1} \bar{C}^{D}_{nm,t}{}^{(\alpha^{CD}_n - 1)\left(1-\frac{1}{e^{CP}_n}\right)-1} \left(\frac{CP_{nm,t+1}}{\Omega^{CP}_{n,t+1}} \right)^{\frac{1}{e^{CP}_n}} \alpha^{CD}_n C^{D}_{nm,t+1}{}^{\frac{e^{CP}_n - 1}{e^{CP}_n}} \right\}$ $-\mathbb{E}_t \frac{\partial V_{nm,t}}{\partial V_{nm,t+1}} \lambda^{\overline{CD}}_{nm,t+1} \left(\frac{C^{D}_{nm,t+1}}{C^{ND}_{nm,t+1}} \right)^{1-\varrho^{CD}_n} \varrho^{CD}_n \bar{C}^{D}_{nm,t}{}^{\varrho^{CD}_n - 1}$ $+\lambda^{\overline{CD}}_{nm,t} = 0;$
$D_{nm,t}$:	$-\lambda^{CD}_{nm,t} \alpha^{h}_n \frac{C^{D}_{nm,t}}{D_{nm,t}}$ $+\lambda^{D}_{nm,t} \left(-I^{D}_{nm,t} \exp\left[-\frac{\theta^{GD}_n}{2} \left(\log \frac{G^{D}_{nm,t}}{G^{D}_{nm,t-1}} \right)^2 - \frac{\theta^{DP}_n}{2} \left(\log \frac{D_{nm,t}}{D^{P}_{nm,t-1}} \right)^2 \right] \theta^{DP}_n \left(\log \frac{D_{nm,t}}{D^{P}_{nm,t-1}} \right) \frac{1}{D_{nm,t}} \right.$ $\left. - 1 \right)$ $+\mathbb{E}_t \frac{\partial V_{nm,t}}{\partial V_{nm,t+1}} \lambda^{D}_{nm,t+1} \left(1 - \delta^{D}_{n,t} - \frac{\theta^{UD0}_n}{1 + \theta^{UD1}_n} \upsilon^{D}_{nm,t}{}^{1+\theta^{UD1}_n} \right)$ $-\lambda^{GD}_{nm,t}$ $+\mathbb{E}_t \frac{\partial V_{nm,t}}{\partial V_{nm,t+1}} \lambda^{GD}_{nm,t+1} G^{D}_{nm,t+1}$ $+\mathcal{I}(m = b) \lambda^{F}_{n,t} (1 - \rho^{B}_n) m_{n,t} P^{D}_{n,t} = 0,$ where $\mathcal{I}$ is a binary indicator function;
$I^{D}_{nm,t}$:	$-\lambda^{B}_{nm,t} P^{D}_{n,t} (1 + \tau^{D}_{n,t}) + \lambda^{D}_{nm,t} \exp\left[-\frac{\theta^{GD}_n}{2} \left(\log \frac{G^{D}_{nm,t}}{G^{D}_{nm,t-1}} \right)^2 - \frac{\theta^{DP}_n}{2} \left(\log \frac{D_{nm,t}}{D^{P}_{nm,t-1}} \right)^2 \right] = 0;$

$G^D_{nm,t}$:	$\begin{aligned} &-\lambda^D_{nm,t} I^D_{nm,t} \exp\left[-\frac{\theta^{GD}_n}{2}\left(\log\frac{G^D_{nm,t}}{G^D_{nm,t-1}}\right)^2 - \frac{\theta^{DP}_n}{2}\left(\log\frac{D_{nm,t}}{D^P_{nm,t-1}}\right)^2\right]\theta^{GD}_n\left(\log\frac{G^D_{nm,t}}{G^D_{nm,t-1}}\right)\frac{1}{G^D_{nm,t}} \\ &+\mathbb{E}_t\frac{\partial V_{nm,t}}{\partial V_{nm,t+1}}\lambda^D_{nm,t+1}I^D_{nm,t+1}\exp\left[-\frac{\theta^{GD}_n}{2}\left(\log\frac{G^D_{nm,t+1}}{G^D_{nm,t}}\right)^2 - \frac{\theta^{DP}_n}{2}\left(\log\frac{D_{nm,t+1}}{D^P_{nm,t}}\right)^2\right]\theta^{GD}_n\left(\log\frac{G^D_{nm,t+1}}{G^D_{nm,t}}\right)\frac{1}{G^D_{nm,t}} \\ &+\lambda^{GD}_{nm,t}D_{nm,t-1}=0; \end{aligned}$
$D^P_{nm,t}$:	$\mathbb{E}_t\frac{\partial V_{nm,t}}{\partial V_{nm,t+1}}\lambda^D_{nm,t+1}I^D_{nm,t+1}\exp\left[-\frac{\theta^{GD}_n}{2}\left(\log\frac{G^D_{nm,t+1}}{G^D_{nm,t}}\right)^2 - \frac{\theta^{DP}_n}{2}\left(\log\frac{D_{nm,t+1}}{D^P_{nm,t}}\right)^2\right]\theta^{DP}_n\left(\log\frac{D_{nm,t+1}}{D^P_{nm,t}}\right)\frac{1}{D^P_{nm,t}}=0;$
$v^D_{nm,t}$:	$-\lambda^{CD}_{nm,t}\alpha^h_n\frac{C^D_{nm,t}}{v^D_{nm,t}} - \lambda^D_{nm,t}\theta^{UD0}_n {v^D_{nm,t}}^{\theta^{UD1}_n} D_{nl,t-1} = 0;$
$KP^s_{n,t}, s \in S_0$;	$\begin{aligned} &\lambda^B_{nl,t}R^{KP,s}_{n,t}v^{K,s}_{n,t}(1-\tau^{K,s}_{n,t}) \\ &+\lambda^{KP,s}_{n,t}\left(-I^{KP,s}_{n,t}\exp\left[-\frac{\theta^{GKP}_n}{2}\left(\log\frac{G^{KP,s}_{n,t}}{G^{KP,s}_{n,t-1}}\right)^2 - \frac{\theta^{KPP}_n}{2}\left(\log\frac{KP^s_{n,t}}{KP^{P,s}_{n,t-1}}\right)^2\right]\theta^{KPP}_n\left(\log\frac{KP^s_{n,t}}{KP^{P,s}_{n,t-1}}\right)\frac{1}{KP^s_{n,t}} - 1\right) \\ &+\mathbb{E}_t\frac{\partial V_{nl,t}}{\partial V_{nl,t+1}}\lambda^{KP,s}_{n,t+1}\left(1-\delta^{KP,s}_{n,t+1}-\frac{\theta^{UK0}_n}{1+\theta^{UK1}_n}{v^{K,s}_{n,t+1}}^{1+\theta^{UK1}_n}\right) \\ &-\lambda^{GKP,s}_{n,t} \\ &+\mathbb{E}_t\frac{\partial V_{nl,t}}{\partial V_{nl,t+1}}\lambda^{GKP,s}_{n,t+1}G^{KP,s}_{n,t+1}=0; \end{aligned}$
$I^{KP,s}_{n,t}, s \in S_0$;	$\frac{P^{KP}_{n,t}}{\exp\left[-\frac{\theta^{GKP}_n}{2}\left(\log\frac{G^{KP,s}_{n,t}}{G^{KP,s}_{n,t-1}}\right)^2 - \frac{\theta^{KPP}_n}{2}\left(\log\frac{KP^s_{n,t}}{KP^{P,s}_{n,t-1}}\right)^2\right]} = \frac{\lambda^{KP,s}_{n,t}}{\lambda^B_{nl,t}};$
$G^{KP,s}_{n,t}, s \in S_0$;	$\begin{aligned} &-\lambda^{KP,s}_{n,t}I^{KP,s}_{n,t}\exp\left[-\frac{\theta^{GKP}_n}{2}\left(\log\frac{G^{KP,s}_{n,t}}{G^{KP,s}_{n,t-1}}\right)^2 - \frac{\theta^{KPP}_n}{2}\left(\log\frac{KP^s_{n,t}}{KP^{P,s}_{n,t-1}}\right)^2\right]\theta^{GKP}_n\left(\log\frac{G^{KP,s}_{n,t}}{G^{KP,s}_{n,t-1}}\right)\frac{1}{G^{KP,s}_{n,t}} \\ &+\mathbb{E}_t\frac{\partial V_{nl,t}}{\partial V_{nl,t+1}}\lambda^{KP,s}_{n,t+1}I^{KP,s}_{n,t+1}\exp\left[-\frac{\theta^{GKP}_n}{2}\left(\log\frac{G^{KP,s}_{n,t+1}}{G^{KP,s}_{n,t}}\right)^2 - \frac{\theta^{KPP}_n}{2}\left(\log\frac{KP^s_{n,t+1}}{KP^{P,s}_{n,t}}\right)^2\right]\theta^{GKP}_n\left(\log\frac{G^{KP,s}_{n,t+1}}{G^{KP,s}_{n,t}}\right)\frac{1}{G^{KP,s}_{n,t}} \\ &+\lambda^{GKP,s}_{n,t}KP^s_{n,t-1}=0; \end{aligned}$
$KP^{P,s}_{n,t}, s \in S_0$;	$\mathbb{E}_t\frac{\partial V_{nl,t}}{\partial V_{nl,t+1}}\lambda^{KP,s}_{n,t+1}I^{KP,s}_{n,t+1}\exp\left[-\frac{\theta^{GKP}_n}{2}\left(\log\frac{G^{KP,s}_{n,t+1}}{G^{KP,s}_{n,t}}\right)^2 - \frac{\theta^{KPP}_n}{2}\left(\log\frac{KP^s_{n,t+1}}{KP^{P,s}_{n,t}}\right)^2\right]\theta^{KPP}_n\left(\log\frac{KP^s_{n,t+1}}{KP^{P,s}_{n,t}}\right)\frac{1}{KP^{P,s}_{n,t}}=0;$
$v^{K,s}_{n,t}, s \in S_0$;	$\lambda^B_{nl,t}R^{KP,s}_{n,t}KP^s_{n,t}(1-\tau^{K,s}_{n,t}) - \lambda^{KP,s}_{n,t}\theta^{UK0}_n {v^{K,s}_{n,t}}^{\theta^{UK1}_n} KP^s_{n,t-1} = 0;$

$HP^s_{n,t}, s \in S_0$;	$\lambda_{B,nl,t} R^{HP,s}_{n,t}(1-\tau^{H,s}_{n,t})$ $+\lambda^{HP,s}_{n,t}\left(-I^{HP,s}_{n,t}\exp\left[-\frac{\theta^{GHP}_n}{2}\left(\log\frac{G^{HP,s}_{n,t}}{G^{HP,s}_{n,t-1}}\right)^2-\frac{\theta^{HPP}_n}{2}\left(\log\frac{HP^s_{n,t}}{HP^{P,s}_{n,t-1}}\right)^2\right]\theta^{HPP}_n\left(\log\frac{HP^s_{n,t}}{HP^{P,s}_{n,t-1}}\right)\frac{1}{HP^s_{n,t}} - 1\right)$ $+\mathbb{E}_t\frac{\partial V_{nl,t}}{\partial V_{nl,t+1}}\lambda^{HP,s}_{n,t+1}(1-\delta^{HP,s}_{n,t+1})$ $-\lambda^{GHP,s}_{n,t}$ $+\mathbb{E}_t\frac{\partial V_{nl,t}}{\partial V_{nl,t+1}}\lambda^{GHP,s}_{n,t}G^{HP,s}_{n,t+1}=0;$
$I^{HP,s}_{n,t}, s \in S_0$;	$-\lambda_{B,nl,t}P^{HP}_{n,t}+\lambda^{HP,s}_{n,t}\exp\left[-\frac{\theta^{GHP}_n}{2}\left(\log\frac{G^{HP,s}_{n,t}}{G^{HP,s}_{n,t-1}}\right)^2-\frac{\theta^{HPP}_n}{2}\left(\log\frac{HP^s_{n,t}}{HP^{P,s}_{n,t-1}}\right)^2\right]=0;$
$G^{HP,s}_{n,t}, s \in S_0$;	$-\lambda^{HP,s}_{n,t}I^{HP,s}_{n,t}\exp\left[-\frac{\theta^{GHP}_n}{2}\left(\log\frac{G^{HP,s}_{n,t}}{G^{HP,s}_{n,t-1}}\right)^2-\frac{\theta^{HPP}_n}{2}\left(\log\frac{HP^s_{n,t}}{HP^{P,s}_{n,t-1}}\right)^2\right]\theta^{GHP}_n\left(\log\frac{G^{HP,s}_{n,t}}{G^{HP,s}_{n,t-1}}\right)\frac{1}{G^{HP,s}_{n,t}}$ $+\mathbb{E}_t\frac{\partial V_{nl,t}}{\partial V_{nl,t+1}}\lambda^{HP,s}_{n,t+1}I^{HP,s}_{n,t+1}\exp\left[-\frac{\theta^{GHP}_n}{2}\left(\log\frac{G^{HP,s}_{n,t+1}}{G^{HP,s}_{n,t}}\right)^2-\frac{\theta^{HPP}_n}{2}\left(\log\frac{HP^s_{n,t+1}}{HP^{P,s}_{n,t}}\right)^2\right]\theta^{GHP}_n\left(\log\frac{G^{HP,s}_{n,t+1}}{G^{HP,s}_{n,t}}\right)\frac{1}{G^{HP,s}_{n,t}}$ $+\lambda^{GHP,s}_{n,t}HP^s_{n,t-1}=0;$
$HP^{P,s}_{n,t}, s \in S_0$;	$\mathbb{E}_t\frac{\partial V_{nl,t}}{\partial V_{nl,t+1}}\lambda^{HP,s}_{n,t+1}I^{HP,s}_{n,t+1}\exp\left[-\frac{\theta^{GHP}_n}{2}\left(\log\frac{G^{HP,s}_{n,t+1}}{G^{HP,s}_{n,t}}\right)^2-\frac{\theta^{HPP}_n}{2}\left(\log\frac{HP^s_{n,t+1}}{HP^{P,s}_{n,t}}\right)^2\right]\theta^{HPP}_n\left(\log\frac{HP^s_{n,t+1}}{HP^{P,s}_{n,t}}\right)\frac{1}{HP^{P,s}_{n,t}}=0;$
$L^s_{nl,t}, s \in \{S_2\}$:	$-V_{nl,t}{}^{\sigma_n}(1-\beta_{nl,t})N_{nl,t}{}^{\xi_n-1+\sigma_n}U_{nl,t}{}^{-\sigma_n}\frac{\bar{C}_{nl,t-1}}{N_{nl,t-1}}\kappa^s_{n,t}\left(\frac{L^s_{nl,t}}{N_{nl,t}}\right)^{\nu^s_n}$ $+\lambda^B_{nl,t}W^s_{n,t}(1-\tau^{L,s}_{n,t})=0;$
$L^s_{nb,t}, s \in \{S_0, S_1\}$:	$-V_{nb,t}{}^{\sigma_n}(1-\beta_{nb,t})N_{nb,t}{}^{\xi_n-1+\sigma_n}U_{nb,t}{}^{-\sigma_n}\frac{\bar{C}_{nb,t-1}}{N_{nb,t-1}}\kappa^s_{n,t}\left(\frac{L^s_{nb,t}}{N_{nb,t}}\right)^{\nu^s_n}$ $+\lambda^B_{nb,t}W^s_{n,t}(1-\tau^{L,s}_{n,t})=0;$
$L^{Ss}_{nb,t}, s \in \{S_0\}$:	$-V_{nb,t}{}^{\sigma_n}(1-\beta_{nb,t})N_{nb,t}{}^{\xi_n-1+\sigma_n}U_{nb,t}{}^{-\sigma_n}\frac{\bar{C}_{nb,t-1}}{N_{nb,t-1}}\kappa^{Ss}_{n,t}\left(\frac{L^{Ss}_{nb,t}}{N_{nb,t}}\right)^{\nu^{Ss}_n}$ $+\lambda^B_{nb,t}W^s_{n,t}(1-\tau^{L,s}_{n,t})=0;$
$L^h_{nm,t}$:	$-V_{nm,t}{}^{\sigma_n}(1-\beta_{nm,t})N_{nm,t}{}^{\xi_n-1+\sigma_n}U_{nm,t}{}^{-\sigma_n}\frac{\bar{C}_{nm,t-1}}{N_{nm,t-1}}\kappa^h_{n,t}\left(\frac{L^h_{nm,t}}{N_{nm,t}}\right)^{\nu^h_n}$ $-\lambda^{CD}_{nm,t}(1-\alpha^h_n)\frac{C^D_{nm,t}}{L^h_{nm,t}}=0;$

9.2. Production

Firms use labour, capital, and engineering to produce final goods via multiple steps. In the process, all markets are perfectly competitive. So, we have a series FOCs linking marginal product to relative prices. Let

Final goods: for $s \in \{SP_1, SG_1\}$

$$P^s_{n,t}\alpha^s_n \frac{I^s_{n,t}}{Y^{\{SP_1\}}_{n,t}} = P_{n,t},$$

$$P^s_{n,t}(1-\alpha^s_n)\frac{I^s_{n,t}}{L^s_{n,t}} = W^s_{n,t},$$

where $P^s_{n,t}$ is the price of final goods, $P_{n,t}$ is the price of almost final goods.

Almost final goods:

$$Y^{NT}_{n,t} = \left(\frac{P_{n,t}\Omega^Y_{n,t}\alpha^{NC}_n}{P^{NT}_{n,t}(1+\tau^{NT}_{n,t})Y^{NT}_n}\right)^{e^{NC}_n} \frac{Y_{n,t}}{\Omega^Y_{n,t}} Y^{NT}_n,$$

$$Y^{TC}_{n,t} = \left(\frac{P_{n,t}\Omega^Y_{n,t}(1-\alpha^{NC}_n)}{P^{TC}_{n,t}(1+\tau^{TC}_{n,t})Y^{TC}_n}\right)^{e^{NC}_n} \frac{Y_{n,t}}{\Omega^Y_{n,t}} Y^{TC}_n,$$

where $P^{NT}_{n,t}$ is the price of non-tradable goods, $P^{TC}_{n,t}$ is the price of tradable composites.

Tradable and widget composites:

$$Y^W_{n,m,t} = \left(\frac{P^{WP}_{n,t}\Omega^{WP}_{n,t}}{(1+\iota^W_{n,m,t})P^W_{m,t}}\right)^{e^W_n} \frac{Y^{WP}_{n,t}}{\Omega^{WP}_{n,t}} \frac{\widetilde{N}_m}{\sum_1^{\mathcal{N}} \widetilde{N}_m},$$

$$Y^T_{n,m,t} = \left(\frac{P^{TC}_{n,t}\Omega^{TC}_{n,t}}{(1+\iota^T_{n,m,t})P^T_{m,t}}\right)^{e^T_n} \frac{Y^{TC}_{n,t}}{\Omega^{TC}_{n,t}} \frac{\widetilde{N}_m}{\sum_0^{\mathcal{N}} \widetilde{N}_m},$$

$$(P^{WC}_{n,t} + P^X_{n,t})(1-\alpha^{HW}_n)\frac{Y^{WC}_{n,t} + F^{WC}_{n,t}}{Y^{WP}_{n,t}} = P^{WP}_{n,t},$$

where $P^T_{n,t}$ is the price of tradable goods, $P^W_{m,t}$ is the price of widgets, $P^{WP}_{n,t}$ is the price of an intermediate product $Y^{WP}_{n,t}$, $P^{WC}_{n,t}$ is the price of widget composites, $P^X_{n,t}$ is the price of claims on engineering.

Core goods firms: for $s \in S_0$

$$P^{HL,s}_{n,t}\alpha^{HL}_n \frac{Y^{HL,s}_{n,t}}{HP^s_{n,t}} = R^{HP\{S_0\}}_{n,t},$$

$$P^{HL,s}_{n,t}(1-\alpha^{HL}_n)\frac{Y^{HL,s}_{n,t}}{L^{Ss}_{n,t}} = W^{Ss}_{n,t},$$

$$R^{K,s}_{n,t}(1-\alpha^K_n)\frac{K^s_{n,t} + F^K_{n,t}}{v^{K,s}_{n,t}KP^s_{n,t}} = R^{KP,s}_{n,t},$$

$$P^{\mathcal{X}}_{n,t}\left[\exp\left[-o_n\left(\log\left(\frac{X_{n,t}}{X_t}\right) - \log\left(\frac{Y^X_{n,t}}{Y^X_t}\right)\right)^2\right] - \mathcal{X}_{n,t}2o_n\left(\log\left(\frac{X_{n,t}}{X_t}\right) - \log\left(\frac{Y^X_{n,t}}{Y^X_t}\right)\right)\frac{X_t - X_{n,t}}{X_{n,t}X}\right] = 1,$$

$$P^{\mathcal{X}}_{n,t}\mathcal{X}_{n,t}2o_n\left(\log\left(\frac{X_{n,t}}{X_t}\right) - \log\left(\frac{Y^X_{n,t}}{Y^X_t}\right)\right)\frac{Y^X_t - Y^X_{n,t}}{Y^X_{n,t}Y^X_t} = P^X_{n,t},$$

$$\frac{P_{n,t}^{KL,s}\Omega_{n,t}^{KL}\bar{K}_{n,t-1}^{s}{}^{(\alpha_n^{KL}-1)\left(1-\frac{1}{e_n^{KL,s}}\right)}\left(\frac{Y_{n,t}^{KL,s}}{\Omega_{n,t}^{KL}K_{n,t}^{s}}\right)^{\frac{1}{e_n^{KL,s}}}\alpha_n^{KL}-R_{n,t}^{K,s}}{W_{n,t}^{s}-P_{n,t}^{KL,s}\Omega_{n,t}^{KL}\bar{K}_{n,t-1}^{s}{}^{\alpha_n^{KL}\left(1-\frac{1}{e_n^{KL,s}}\right)}\left(\frac{Y_{n,t}^{KL,s}}{\Omega_{n,t}^{KL}L_{n,t}^{s}}\right)^{\frac{1}{e_n^{KL,s}}}(1-\alpha_n^{KL})}K_{n,t}^{s}=L_{n,t}^{s},$$

$$\begin{aligned}&\left[P_{n,t}^{KL,s}\Omega_{n,t}^{KL}\bar{K}_{n,t-1}^{s}{}^{(\alpha_n^{KL}-1)\left(1-\frac{1}{e_n^{KL,s}}\right)}\left(\frac{Y_{n,t}^{KL,s}}{\Omega_{n,t}^{KL}K_{n,t}^{s}}\right)^{\frac{1}{e_n^{KL,s}}}\alpha_n^{KL}-R_{n,t}^{K,s}\right]K_{n,t}^{s}\\&\quad=\mathbb{E}_t\Xi_{t+1}\Bigg[P_{n,t+1}^{KL,s}\Omega_{n,t+1}^{KL}\bar{K}_{n,t}^{s}{}^{(\alpha_n^{KL}-1)\left(1-\frac{1}{e_n^{KL,s}}\right)}\left(\frac{Y_{n,t+1}^{KL,s}}{\Omega_{n,t+1}^{KL}K_{n,t+1}^{s}}\right)^{\frac{1}{e_n^{KL,s}}}\alpha_n^{KL}K_{n,t+1}^{s}\\&\quad-\alpha_n^{KL}P_{n,t+1}^{KL,s}Y_{n,t+1}^{KL,s}(1-\varrho_n^{KL})-R_{n,t+1}^{K,s}K_{n,t+1}^{s}\varrho_n^{KL}\Bigg],\end{aligned}$$

$$\frac{P_{n,t}^{HLX,s}\Omega_{n,t}^{HLX}\bar{Y}_{n,t-1}^{HL,s}{}^{(\alpha_n^{HLX}-1)\left(1-\frac{1}{e_n^{HLX,s}}\right)}\left(\frac{Y_{n,t}^{HLX,s}}{\Omega_{n,t}^{HLX}Y_{n,t}^{HL,s}}\right)^{\frac{1}{e_n^{HLX,s}}}\alpha_n^{HLX}-P_{n,t}^{HL,s}}{\left(1+\tau_{n,t}^{\mathcal{X},s}\right)P_{n,t}^{\mathcal{X}}-P_{n,t}^{HLX,s}\Omega_{n,t}^{HLX}\bar{Y}_{n,t-1}^{HL,s}{}^{\alpha_n^{HLX}\left(1-\frac{1}{e_n^{HLX,s}}\right)}\left(\frac{Y_{n,t}^{HLX,s}}{\Omega_{n,t}^{HLX}\mathcal{X}_{n,t}^{s}}\right)^{\frac{1}{e_n^{HLX,s}}}(1-\alpha_n^{HLX})}Y_{n,t}^{HL,s}=\mathcal{X}_{n,t}^{s},$$

$$\begin{aligned}&\left[P_{n,t}^{HLX,s}\Omega_{n,t}^{HLX}\bar{Y}_{n,t-1}^{HL,s}{}^{(\alpha_n^{HLX}-1)\left(1-\frac{1}{e_n^{HLX,s}}\right)}\left(\frac{Y_{n,t}^{HLX,s}}{\Omega_{n,t}^{HLX}Y_{n,t}^{HL,s}}\right)^{\frac{1}{e_n^{HLX,s}}}\alpha_n^{HLX}-P_{n,t}^{HL,s}\right]Y_{n,t}^{HL,s}\\&\quad=\mathbb{E}_t\Xi_{t+1}\Bigg[P_{n,t+1}^{HLX,s}\Omega_{n,t+1}^{HLX}\bar{Y}_{n,t}^{HL,s}{}^{(\alpha_n^{HLX}-1)\left(1-\frac{1}{e_n^{HLX,s}}\right)}\left(\frac{Y_{n,t+1}^{HLX,s}}{\Omega_{n,t+1}^{HLX}Y_{n,t+1}^{HL,s}}\right)^{\frac{1}{e_n^{HLX,s}}}\alpha_n^{HLX}Y_{n,t+1}^{HL,s}\\&\quad-\alpha_n^{HLX}P_{n,t+1}^{HLX,s}Y_{n,t+1}^{HLX,s}(1-\varrho_n^{HLX})-P_{n,t+1}^{HL,s}Y_{n,t+1}^{HL,s}\varrho_n^{HLX}\Bigg],\end{aligned}$$

$$\frac{P_{n,t}^{s}\Omega_{n,t}^{s}\bar{Y}_{n,t-1}^{HLX,s}{}^{(\alpha_n^{Y}-1)\left(1-\frac{1}{e_n^{Y,s}}\right)}\left(\frac{Y_{n,t}^{s}}{\Omega_{n,t}^{s}Y_{n,t}^{HLX,s}}\right)^{\frac{1}{e_n^{Y,s}}}\alpha_n^{Y}-P_{n,t}^{HLX,s}}{P_{n,t+1}^{KL,s}-P_{n,t}^{s}\Omega_{n,t}^{s}\bar{Y}_{n,t-1}^{HLX,s}{}^{\alpha_n^{Y}\left(1-\frac{1}{e_n^{Y,s}}\right)}\left(\frac{Y_{n,t}^{s}}{\Omega_{n,t}^{s}Y_{n,t}^{KL,s}}\right)^{\frac{1}{e_n^{Y,s}}}(1-\alpha_n^{Y})}Y_{n,t}^{HLX,s}=Y_{n,t}^{KL,s},$$

$$\begin{aligned}&\left[P_{n,t}^{s}\Omega_{n,t}^{s}\bar{Y}_{n,t-1}^{HLX,s}{}^{(\alpha_n^{Y}-1)\left(1-\frac{1}{e_n^{Y,s}}\right)}\left(\frac{Y_{n,t}^{s}}{\Omega_{n,t}^{s}Y_{n,t}^{HLX,s}}\right)^{\frac{1}{e_n^{Y,s}}}\alpha_n^{Y}-P_{n,t}^{HLX,s}\right]Y_{n,t}^{HLX,s}\\&\quad=\mathbb{E}_t\Xi_{t+1}\Bigg[P_{n,t+1}^{s}\Omega_{n,t+1}^{s}\bar{Y}_{n,t}^{HLX,s}{}^{(\alpha_n^{Y}-1)\left(1-\frac{1}{e_n^{Y,s}}\right)}\left(\frac{Y_{n,t+1}^{s}}{\Omega_{n,t+1}^{s}Y_{n,t+1}^{HLX,s}}\right)^{\frac{1}{e_n^{Y,s}}}\alpha_n^{Y}Y_{n,t+1}^{HLX,s}\\&\quad-\alpha_n^{Y}P_{n,t+1}^{s}Y_{n,t+1}^{s}(1-\varrho_n^{Y})-P_{n,t+1}^{HLX,s}Y_{n,t+1}^{HLX,s}\varrho_n^{Y}\Bigg].\end{aligned}$$

9.3. Engineering sector

9.3.1. Aggregate relationships in the engineering sector

$$P_{n,t}^{WC}=P_t^{\mathcal{W}}=\frac{A_t}{1+\mu_{t-1}},$$

$$\left(\frac{1}{1+\mu_t}\right)^{\frac{1}{\lambda}} = (1-\mathscr{s}_t)\left(\frac{1}{1+\eta\lambda}\right)^{\frac{1}{\lambda}} + \mathscr{s}_t\left(\frac{1}{1+\mu_t^{\mathrm{P}}}\right)^{\frac{1}{\lambda}},$$

$$\left(\frac{A_t}{1+\mu_{t-1}}\right)^{\frac{1}{\lambda}} = (1-\mathscr{s}_{t-1})\left(\frac{A_{t-1}^{\mathrm{N}}}{1+\eta\lambda}\right)^{\frac{1}{\lambda}} + \mathscr{s}_{t-1}\left(\frac{A_t^{*}}{1+\mu_{t-1}^{\mathrm{P}}}\right)^{\frac{1}{\lambda}},$$

$$A_t^{\mathrm{N}\frac{1}{\lambda}} = \left(1-\frac{1-\delta_{\mathbb{I},t}}{G_{|\mathbb{I}|,t}}\frac{1-\mathscr{s}_{t-1}}{1-\mathscr{s}_t}\right)A_t^{*\frac{1}{\lambda}} + \frac{1-\delta_{\mathbb{I},t}}{G_{|\mathbb{I}|,t}}\frac{1-\mathscr{s}_{t-1}}{1-\mathscr{s}_t}A_{t-1}^{\mathrm{N}\ \frac{1}{\lambda}},$$

$$\mathscr{s}_t = \left(1-\frac{1-\delta_{\mathbb{I},t}}{G_{|\mathbb{I}|,t}}\right) + \frac{1-\delta_{\mathbb{I},t}}{G_{|\mathbb{I}|,t}}(1-q)\mathscr{s}_{t-1}.$$

9.3.2. Firm decisions in the engineering sector

$$\mu_t^{\mathrm{P}} = \lambda\frac{\eta J_t^{\mathrm{P}}}{J_t^{\mathrm{P}}-(1-\eta)},$$

$$\omega_t^{\mathrm{P}} = \frac{J_t^{\mathrm{P}}(1-\eta)}{\left(J_t^{\mathrm{P}}-(1-\eta)\right)^2(1+\mu_t^{\mathrm{P}})},$$

$$d_t^{\mathrm{P}} := 1-\frac{\omega_t^{\mathrm{P}}}{1+\omega_t^{\mathrm{P}}}\frac{(\lambda-\mu_t^{\mathrm{P}})(\mu_t^{\mathrm{P}}-\eta\lambda)}{\lambda(1-\eta)\mu_t^{\mathrm{P}}},$$

$$G_{A^*,t} = \left(1+\gamma Z_t\mathcal{F}_{t-1}^{\mathrm{RP}}\right)^{\frac{1}{\gamma}},$$

$$\frac{1}{|\mathbb{I}_t|J_t^{\mathrm{P}}}\frac{\mu_t^{\mathrm{P}}}{1+\mu_t^{\mathrm{P}}}\left(\frac{1+\mu_t}{1+\mu_t^{\mathrm{P}}}\right)^{\frac{1}{\lambda}}\mathbb{E}_t\Xi_{t,t+1}X_{t+1}\left(\frac{A_{t+1}^{*}}{A_{t+1}}\right)^{\frac{1}{\lambda}}\frac{d_t^{\mathrm{P}}}{\mu_t^{\mathrm{P}}}\frac{Z_{t+1}E_t^{\mathrm{RP}}}{1+\gamma Z_{t+1}\mathcal{F}_t^{\mathrm{RP}}} = P_t^{\mathscr{w}},$$

$$\frac{1}{|\mathbb{I}_t|J_t^{\mathrm{P}}}\frac{\mu_t^{\mathrm{P}}}{1+\mu_t^{\mathrm{P}}}\left(\frac{1+\mu_t}{1+\mu_t^{\mathrm{P}}}\right)^{\frac{1}{\lambda}}\mathbb{E}_t\Xi_{t,t+1}X_{t+1}\left(\frac{A_{t+1}^{*}}{A_{t+1}}\right)^{\frac{1}{\lambda}} = \frac{1}{\mathscr{p}}\frac{\mathcal{F}_t^{\mathrm{RP}}}{E_t^{\mathrm{RP}}}P_t^{\mathscr{w}}.$$

$$\begin{aligned}
SOC = {} & J^{\mathrm{P}^3}\gamma\eta^3\left(J^{\mathrm{P}}-1+\eta\right)^3\lambda^3 \\
& + 2\,\eta^2\left(J^{\mathrm{P}}-1+\eta\right)\Bigg(\bigg(J^{\mathrm{P}^3}+(2\,\eta-2)J^{\mathrm{P}^2}+3\,(-1+\eta)^2J^{\mathrm{P}}\,\left(\frac{1}{2}\right)+\left(\eta-\frac{1}{2}\right)(-1+\eta)^2\bigg)\gamma \\
& - \left(\frac{1}{2}\right)\left(J^{\mathrm{P}}-1+\eta\right)\left(J^{\mathrm{P}^2}+(3\,\eta-3)J^{\mathrm{P}}+\eta^2-3\,\eta+2\right)\Bigg)J^{\mathrm{P}^2}\lambda^2 \\
& + \eta\Bigg(\left(J^{\mathrm{P}^2}+(-1+\eta)J^{\mathrm{P}}+(-1+\eta)^2\right)\left(J^{\mathrm{P}^3}+(2\,\eta-2)J^{\mathrm{P}^2}+(-1+\eta)^2J^{\mathrm{P}}+\eta(-1+\eta)^2\right)\gamma \\
& - \bigg(2\left(J^{\mathrm{P}^4}+(4\,\eta-4)J^{\mathrm{P}^3}+(4\,\eta^2-9\,\eta+5)J^{\mathrm{P}^2}+(3\,\eta^3-8\,\eta^2+7\,\eta-2)J^{\mathrm{P}}\right. \\
& \left. + \eta(\eta-2)(-1+\eta)^2\right)\bigg)\left(J^{\mathrm{P}}-1+\eta\right)\Bigg)J^{\mathrm{P}}\,\lambda - J^{\mathrm{P}^6}+(-5\,\eta+5)J^{\mathrm{P}^5}+\left(-8\,\eta^2+17\,\eta-9\right)J^{\mathrm{P}^4} \\
& + \left(-8\,\eta^3+22\,\eta^2-20\,\eta+6\right)J^{\mathrm{P}^3}+\left(-7\,\eta^4+22\,\eta^3-22\,\eta^2+6\,\eta+1\right)J^{\mathrm{P}^2} \\
& - \left(3\,\left(\eta^2-\left(\frac{4}{3}\right)\eta-1\right)\right)(-1+\eta)^3J^{\mathrm{P}}-\left(\eta^2-\eta-1\right)(-1+\eta)^4.
\end{aligned}$$

9.3.3. Inventor decisions

$$V_t^{\mathbb{I}} = \frac{1-\mathscr{p}}{\mathscr{p}}\frac{\mathcal{F}_t^{\mathrm{RP}}}{E_t^{\mathrm{RP}}}P_t^{\mathscr{w}}J_t^{\mathrm{P}} + (1-q)\mathbb{E}_t(1-\delta_{\mathbb{I},t+1})\Xi_{t,t+1}V_{t+1}^{\mathbb{I}},$$

$$\min\left\{\mathscr{s}_t-\frac{(1-q)(1-\delta_{\mathbb{I},t})\mathscr{s}_{t-1}}{G_{|\mathbb{I}|,t}},\frac{\mathcal{F}_t^{\mathrm{I}}}{E_t^{\mathrm{I}}}P_t^{\mathscr{w}}-V_t^{\mathbb{I}}\right\} = 0,$$

$$\delta_{\mathbb{I},t} = \delta_{\mathbb{I}}\left(\frac{G_{|\mathbb{I}|,t}}{G_{\mathbb{I}}}\right)^{\psi}\tilde{\delta}_{\mathbb{I},t}.$$

9.3.4. Market clearing

$$
\begin{aligned}
\sum_{m=1}^{\mathcal{N}} Y_{n,t}^{WC} = \mathcal{W}_t = \frac{\mathcal{F}_t^{\mathrm{I}}}{E_t^{\mathrm{I}}} |\mathbb{I}_t| \left[\mathcal{s}_t - \frac{(1-q)(1-\delta_{\mathbb{I},t}) \mathcal{s}_{t-1}}{G_{|\mathbb{I}|,t}} \right] + |\mathbb{I}_t| \mathcal{s}_t \frac{\mathcal{F}_t^{\mathrm{RP}}}{E_t^{\mathrm{RP}}} J_t^{\mathrm{P}} \\
+ X_t P_t^{\mathcal{W}^{-\frac{1+\lambda}{\lambda}}} \left[(1-\mathcal{s}_{t-1}) {A_{t-1}^{\mathrm{N}}}^{\frac{1}{\lambda}} \left(\frac{1}{1+\eta\lambda} \right)^{\frac{1+\lambda}{\lambda}} + \mathcal{s}_{t-1} {A_t^{*}}^{\frac{1}{\lambda}} \left(\frac{1}{1+\mu_{t-1}^{\mathrm{P}}} \right)^{\frac{1+\lambda}{\lambda}} \right].
\end{aligned}
$$

10. Appendix: Data sources

Observables described in section 4.2 are constructed using data collected from the following sources. We sometimes face trade-offs between the number of observations and quality. For example, consumption deflator series are shorter than CPI series, but the latter are a rough approximation of the true consumption deflator. We typically use data with better quality where possible.

1. The US:
   a. Jordà-Schularick-Taylor Macrohistory Database (1870 – 2016):
      i. GDP
      ii. Real GDP per capita
      iii. Gross domestic capital formation to GDP ratio
      iv. Government revenues
      v. Export
      vi. Import
      vii. Short-term nominal interest rate
   b. Bureau of Economic Analysis (1929 – 2017):
      i. Personal consumption expenditure to GDP ratio (table 1.1.10)
      ii. Government consumption expenditure (table 3.9.5)
      iii. Government gross investment (table 3.9.5)
      iv. Deflators of Private consumption, Private investment, Government consumption, Government investment, Export, Import (table 1.1.9)
      v. Hours Worked by Full-Time and Part-Time Employees, domestic industries (1948 – 2013, table 6.9)
      vi. Compensation of employees (table 2.1)
      vii. Consumption of fixed capital (table 5.1)
   c. Federal Reserve Economic Data:
      i. Finance rate on personal loans at commercial banks, 24-month loan (1973 – 2017)
   d. The Maddison project:
      i. Population (1870 – 2016)
   e. OECD:
      i. Gross domestic expenditure on research and development (GERD from Main Science and Technology Indicators, 1981 – 2016)
      ii. ISCED2011 levels 3 to 8 expenditure at all public and private educational institutions (Educational finance indicators, 1995, 2000, 2005, 2008 – 2016)
2. The UK (Great Britain and Northern Ireland)
   a. "A millennium of macroeconomic data" (1870 – 2016):
      i. GBP/USD exchange rate (table A33)
      ii. Population (table A18)
      iii. GDP (table A11)
      iv. Household and NPISH consumption (table A11)
      v. Gross fixed capital formation (table A11)

vi. General government consumption of goods and services (table A11)
vii. Central government Gross fixed capital formation (1946 – 2013, table A27)
viii. Central government income (table A27)
ix. Value of goods and services exports (table A36)
x. Value of goods and services imports (table A36)
xi. Deflators of GDP, Household and NPISH consumption, Gross fixed capital formation, Government consumption, Export, Import (table A47)
xii. Number of person employed (table A50)
xiii. Average hours worked per year per head for all workers (table A54)
xiv. Compensation of employees (table A17)
xv. Capital Consumption (table A11)
xvi. BOE policy rate (table A31)
xvii. Household Personal Loan Rate (1970 – 2013, table M12)

b. OECD:
i. Gross domestic expenditure on research and development (GERD from Main Science and Technology Indicators, 1981 – 2016)
ii. ISCED2011 levels 3 to 8 expenditure at all public and private educational institutions (Educational finance indicators, 1995, 2000, 2005, 2008 – 2016)

3. All the other countries[50]
a. Jordà-Schularick-Taylor Macrohistory Database (1870 – 2016):
i. Local currency/USD exchange rate[51]
ii. GDP
iii. Real GDP per capita
iv. Gross domestic capital formation to GDP ratio
v. Government expenditure
vi. Government revenue
vii. Export
viii. Import
ix. Short-term nominal interest rate
- For the REU, EURO area money market rate is collected from IMF's IFS database
- For the RMA, its nominal interest rate is the US rate.

b. Penn World Tables (1950 – 2014)
i. Personal consumption expenditure to GDP ratio
ii. Deflators of private consumption, gross domestic investment, government consumption, export, important
iii. Number of person engaged

[50] For each country in REU and RMA, we have the value of government expenditure but the price of government consumption. It can be shown that the regional real government spending is constructed as the Fisher index of these two variables if 1) the proportion of national government expenditure quantity to regional expenditure quantity equals to the same proportion for government consumption; and 2) the proportion of the national government expenditure price to regional expenditure price equals to the same proportion for government consumption.
[51] For EMU countries, local currency is the currency before the Euro was introduced.

iv. Average annual hours worked by persons engaged
v. Share of labour compensation in GDP
vi. Average depreciation rate of the capital stock
c. National central bank dataset
i. Finance rate on personal consumption loans (starting year varies)
d. The Maddison project
i. Population (1870 – 2016)
e. OECD:
i. Gross domestic expenditure on research and development (GERD from Main Science and Technology Indicators and Eurostat table 2020, 1981 – 2016)[52]
ii. ISCED2011 levels 3 to 8 expenditure at all public and private educational institutions (Educational finance indicators, 1995, 2000, 2005, 2008 – 2016)

4. World population: United Nations and the Maddison project (1870 – 2016)[53]

[52] We combine the two data sources by interpolation in which there is an I(2) component common to OECD and Eurostata data, I(1) OECD/ Eurostata specific components, and an measurement error. The results essentially agree with the EC data when that exists.

[53] Maddison project data is used from 1950 onwards. Before 1950, Maddison project data is only available in certain years. Data from the United Nations is used as a high frequency indicator to interpolate between these points.