Question : Ten playing cards numbered 1, 2, 3, .., 10 are placed face down on a table. One card is drawn at random, its number recorded, and then replaced face down. A card is drawn again at random. The probability that the number on the second draw is greater than the number on the first draw (rounded off to two decimal places) is

Solution:

Let, X be the random variable representing number on the first draw.
Y be the random variable representing the number on the second card drawn.

Parameter	Values	Description
X	i, $1 \leq i \leq 10$	card with number i
Y	i, $1 \leq i \leq 10$	card with number i

$$p_X(X = i) = \frac{1}{10}, 1 \leq i \leq 10 \tag{1}$$

$$p_Y(Y = i) = \frac{1}{10}, 1 \leq i \leq 10 \tag{2}$$

Let $Z = X - Y$,

$$p_X(Z = k) = \begin{cases} \frac{11-x-k}{10}, & \text{if} 1 \leq k \leq 10 - x \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

$$p_X(k) = \Pr(X - Y = k) = \Pr(X = k + Y) \tag{4}$$

$$= \sum_{i=1}^{10} \Pr(X = k + i \mid Y = i)\, p_Y(i) \tag{5}$$

X and Y are independent

$$\Pr(X = k + i) = p_X(k + i) \tag{6}$$

$$\therefore\ p_X(k) = \sum_{i=1}^{10-x} p_X(k + i)\, p_Y(i) \tag{7}$$

$$= p_X(i) * p_Y(i) \tag{8}$$

Now consider,

$$p_X(k) = \frac{1}{10} \sum_{i=1}^{10-x} p_X(k + i) = \frac{1}{10} \sum_{i=1}^{10-x} p_X(i) \tag{9}$$

The probability that the second card's number is greater than the first card's number is given by:

$$\Pr(Y > X) = \sum_{x=1}^{10} \sum_{k=1}^{10-x} \Pr(Z = k \mid X = x) \cdot \Pr(X = x) \tag{10}$$

$$\Pr(Y > X) = \sum_{x=1}^{10} \sum_{k=1}^{10-x} \frac{1}{10} \cdot \frac{11 - x - k}{10} \tag{11}$$

$$\Pr(Y > X) = 0.45 \tag{12}$$

Therefore, the probability that the number on the second draw is greater than the number on the first draw is approximately 0.45 or 45%.