Question

A box has 100 pens of which 10 are defective . What is the probability that out of a sample of 5 pens drawn one by one with replacement at most one is defective?

(A)$\left(\frac{9}{10}\right)^5$
(B)$\frac{1}{2}\left(\frac{9}{5}\right)^4$
(C)$\frac{1}{2}\left(\frac{9}{10}\right)^5$
(D)$\frac{1}{2}\left(\frac{9}{5}\right)^4+\left(\frac{9}{10}\right)^5$

Solution:

Parameter	Values	Description
n	5	Number of defective pens
p	0.1	probability of drawing a defective pen
μ	0.5	np
σ	0.671	$\sqrt{np(1-p)}$
X		Defective pens

using Gaussian

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{1}$$

The CDF of Y:

$$F_Y(y) = 1 - \Pr\left(Y > y\right) \tag{2}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{y-\mu}{\sigma}\right) \tag{3}$$

But,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}\left(0,1\right) \tag{4}$$

$$\tag{5}$$

the Q-function is defined as:

$$Q\left(x\right) = \Pr\left(Y > x\right)\ \forall x \in Y \sim \mathcal{N}(0.5, 0.45) \tag{6}$$

therefore the cdf will be:

$$F_Y\left(y\right) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y > \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y < \mu \end{cases} \tag{7}$$

Probability that out of a sample of 5 pens drawn one by one with replacement at most is defective is

$$\Pr\left(0.5 \le X \le 1.5\right) = Q\left(\frac{0.5-\mu}{\sigma} \le X \le \frac{1.5-\mu}{\sigma}\right) \tag{8}$$

$$= Q\left(\frac{0.5-\mu}{\sigma} \le X \le \frac{1.5-\mu}{\sigma}\right) \tag{9}$$

The gaussian distribution function is defined as:

$$p_Y\left(x\right) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{10}$$

Probability that out of a sample of 5 pens drawn one by one with replacement at most is defective is

$$Y = 5 \tag{11}$$
$$p_Y(2) = \frac{1}{\sqrt{2\pi(0.9)}} e^{-\frac{(5-1)^2}{2(0.9)}} \tag{12}$$
$$= \frac{1}{\sqrt{2\pi(0.9)}} e^{-8.88} \tag{13}$$
$$= 5.85 \times 10^{-5} \tag{14}$$

Q function

Solving using Q function is defined

$$Q(x) = \int_x^{\infty} f(x)\, dx \tag{15}$$

then CDF of Y is:

$$\Pr(Y < x) = \int_{-\infty}^{x} f(x)\, dx \tag{16}$$
$$= 1 - \int_x^{\infty} f(x)\, dx \tag{17}$$
$$= 1 - Q(x) \tag{18}$$

and for finding $\Pr\left(Z = \frac{X-\mu}{\sigma}\right)$ Using approximation,

$$\Pr\left(Z = \frac{Y-\mu}{\sigma}\right) \approx \Pr\left(\frac{Y+0.5-\mu}{\sigma} < Z < \frac{Y-0.5-\mu}{\sigma}\right) \tag{19}$$
$$\approx \Pr\left(Z < \frac{Y+0.5-\mu}{\sigma}\right) - \Pr\left(Z < \frac{Y-0.5-\mu}{\sigma}\right) \tag{20}$$
$$\approx Q\left(\frac{Y-0.5-\mu}{\sigma}\right) - Q\left(\frac{Y+0.5-\mu}{\sigma}\right) \tag{21}$$

$$Y = 5 \tag{22}$$
$$\Pr(Z = 4.219) \approx Q(3.69) - Q(4.74) \tag{23}$$
$$\approx 0.000112127 - 1.06859 \times 10^{-}6 \tag{24}$$
$$\approx 1.11 \times 10^{-4} \tag{25}$$

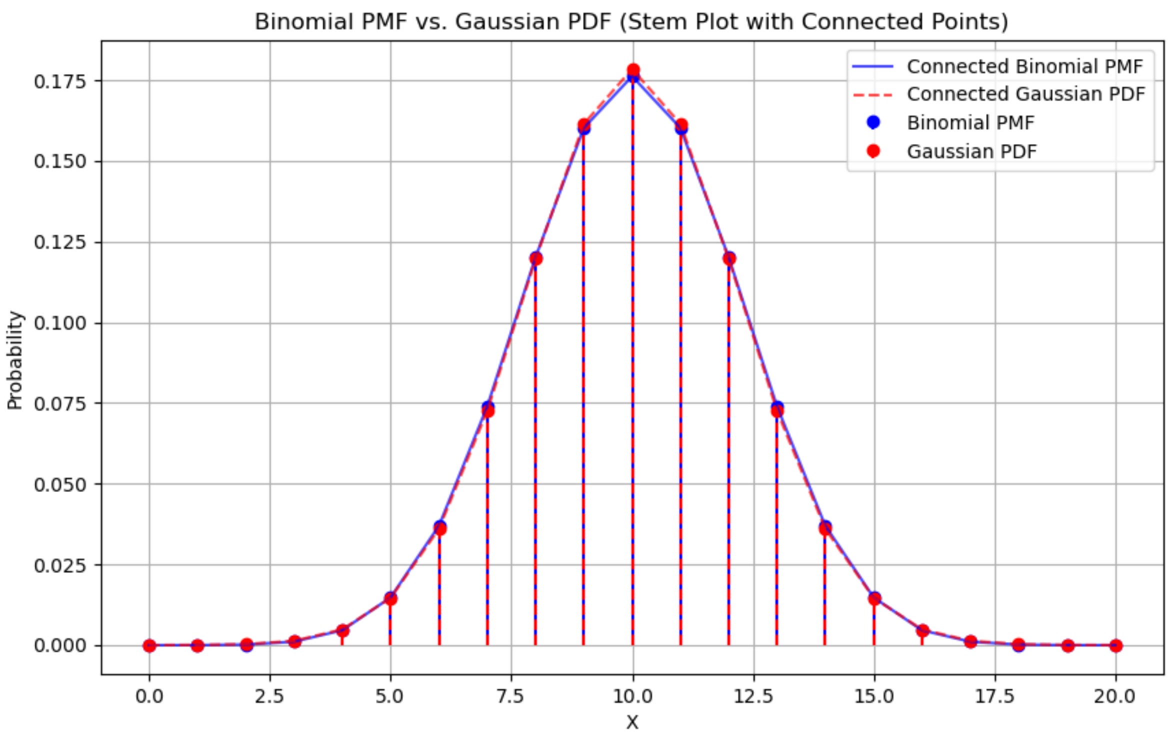

Fig. 0. pmf of binomial and pdf of Gaussian of X and Y marked balls