Problem Statement

The sum of the squares of the first ten natural numbers is,

$$1^2 + 2^2 + ... + 10 = 385$$

The square of the sum of the first ten natural numbers is,

$$(1 + 2 + ... + 10)^2 = 55^2 = 3025$$

Hence the difference between the sum of the squares of the first ten natural numbers and the square of the sum is $3025 - 385 = 2640$.

Find the difference between the sum of the squares of the first one hundred natural numbers and the square of the sum.

Solution

Much like with the first problem, this problem is trivial to solve with a computer without using any fancy formulas. However, it can be broken down through sum formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

So:

$$\left(\sum_{i=1}^{n} i\right)^2 - \sum_{i=1}^{n} i^2 = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6}$$

A proof of the first formula can be found my first Project Euler solution. A derivation of the second formula can be found below. Simplifying:

$$\begin{aligned}\left(\sum_{i=1}^{n} i\right)^2 - \sum_{i=1}^{n} i^2 &= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \\ &= n(n+1)\left(\frac{3n(n+1) - 2(2n+1)}{12}\right) \\ &= n(n+1)\left(\frac{3n^2 - n - 2}{12}\right)\end{aligned}$$

Checking:

$$n = \implies 10(11)(288)/12 = 2640$$

Plugging in:

$$n = 100 \implies 100(101)(29898)/12 = 25,164,150$$

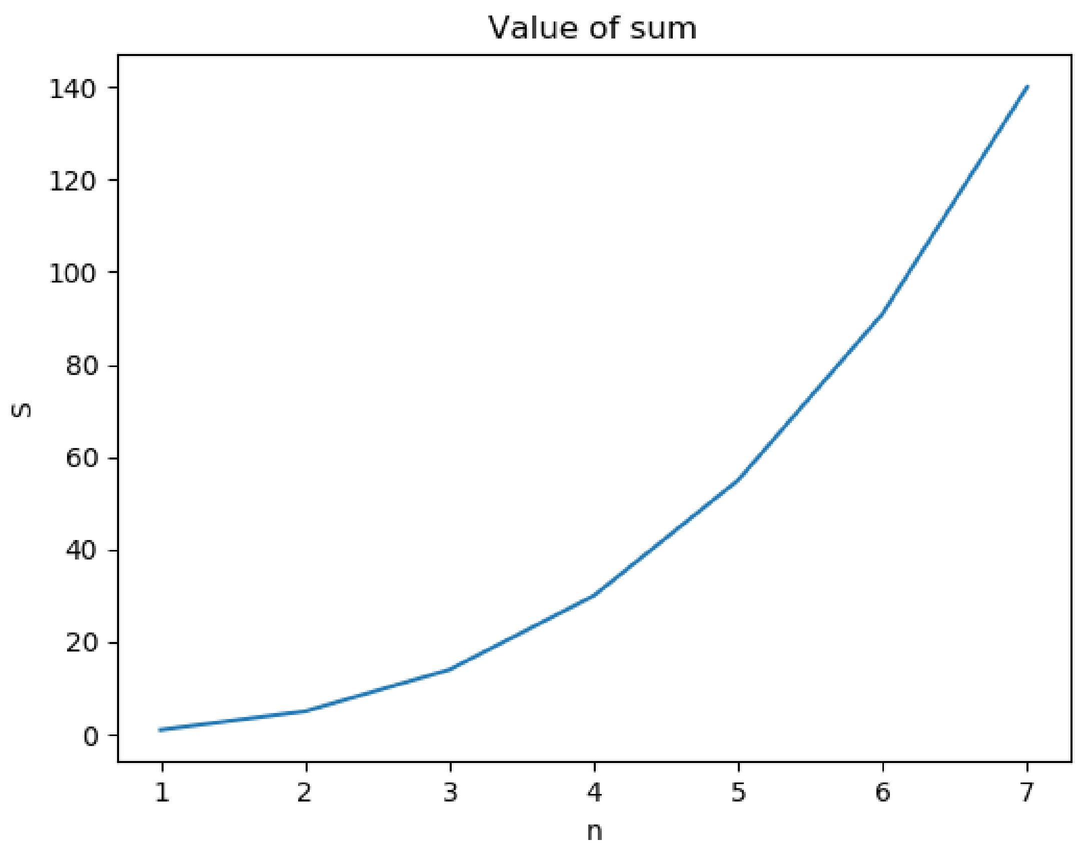
which is confirmed by the algorithm.

Deriving Sum Formulas

Based on the proof of the sum formula for i, once a formula has been found for the sum, proving that it is the sum is a matter of properly apply proof by induction. However, the real work of the problem is coming up with a proper formula. Thus consider the values of the sum:

n	$\sum_{i=1}^{n} i^2$
1	1
2	5
3	14
4	30
5	55
6	91
7	140

plotted below.



It would be nice if, like the first sum formula, this formula was also a polynomial in terms of n. Based on the values, I will first attempt to fit it to a parabola:

$$S = an^2 + bn + c$$

which gives the following linear system:

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 14 \end{bmatrix}$$

Putting in lower echelon form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 14 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

So $c = 2$ and:

$$-2b - 6 = 1 \implies b = -7/2$$
$$a - 7/2 + 2 = 1 \implies a = 5/2$$

Verifying the solution:

$$(5/2) - 7/2 + 2 = 1$$
$$(5/2)(4) - (7/2)(2) + 2 = 5$$
$$(5/2)(9) - (7/2)(3) + 2 = 14$$
$$(5/2)(16) - (7/2)(4) + 2 = 28$$

which is not correct for $n = 4$. Thus, a parabola is an insufficient polynomial. Let's try then a cubic polynomial, repeating the process:

$$S = an^3 + bn^2 + cn + d$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 14 \\ 30 \end{bmatrix}$$

Solving, by once again:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 14 \\ 30 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 0 & -8 & -6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 14 \\ -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 0 & -6 & -6 & -5 \\ 0 & -8 & -6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -4 \\ -5 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & -6 & -7 \\ 0 & -6 & -6 & -5 \\ 0 & -8 & -6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -4 \\ -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & -6 & -7 \\ 0 & 0 & 3 & 11/2 \\ 0 & 0 & 6 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1/2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & -6 & -7 \\ 0 & 0 & 3 & 11/2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1/2 \\ 0 \end{bmatrix}$$

So:

$$d = 0$$
$$3c + (11/2)d = 1/2 \implies c = 1/6$$
$$-4b - 6c - 2d = -3 \implies b = 1/2$$
$$a + b + c + d = 1 \implies a = 1/3$$

Or:

$$S(n) = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

Plugging in:

$$S(1) = 1$$
$$S(2) = 5$$
$$S(3) = 14$$
$$S(4) = 30$$
$$S(5) = 55$$
$$S(6) = 91$$

Finally, performing the proof by induction:

$$\sum_{i=1}^{n+1} i^2 = (n+1)^2 + \sum_{i=1}^{n} i^2 = \frac{(n+1)}{6}(6(n+1) + n(2n+1)) = \frac{(n+1)}{6}(2n^2 + 7n + 6)$$
$$= \frac{(+1)(n+2)(2n+3)}{6} = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

Completing the induction proof. Based on this pattern, I expect that summing up i^3 will require a 4th order polynomial, which could be similarly derived.