Problem Statement

Each new term in the Fibonacci sequence is generated by adding the previous two terms. By starting with 1 and 2, the first 10 terms will be:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$$

By considering the terms in the Fibonacci sequence whose values do not exceed four million, find the sum of the even-valued terms.

Solution

Note Fibonacci numbers are defined by the following recursive formula:

$$a_n = a_{n-1} + a_{n-2}$$
$$a_0 = 0$$
$$a_1 = 1$$

where the problem statement starts with what I am calling a_2.

From below, an explicit formula for the Fibonacci numbers is:

$$a_n = \frac{\sqrt{5}}{5} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{\sqrt{5}}{5} \left(\frac{1 - \sqrt{5}}{2} \right)^n < \frac{\sqrt{5}}{5} \left(\frac{1 + \sqrt{5}}{2} \right)^n$$

Thus, I first find n such that:

$$\frac{\sqrt{5}}{5} \left(\frac{1 + \sqrt{5}}{2} \right)^n < 4 \times 10^6$$

$$n \ln \left(\frac{1 + \sqrt{5}}{2} \right) < \ln \left(4\sqrt{5} \times 10^6 \right)$$

$$n < \frac{\ln \left(4\sqrt{5} \times 10^6 \right)}{\ln \left(\frac{1+\sqrt{5}}{2} \right)} \approx 33.26$$

So, putting $n = 33$ and $n = 34$ into the equation:

$$a_{33} = 3,524,578$$
$$a_{34} = 5,702,887$$

So $n = 33$ is the largest of the Fibonacci numbers to consider. Finally, due to the parity of even and odd numbers, we see that every third Fibonacci number is even. Even number plus even number is even, odd number plus odd number is even, but other combinations are odd. The sequence starts with an even and an odd number, so $a_2 = a_1 + a_0$ is odd, $a_3 = a_2 + a_1$ is even, $a_4 = a_3 + a_2$ is odd,

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and so on. Thus summing only the even Fibonacci numbers below 4 million is equivalent to:

$$S = \sum_{k=0}^{11} a_{3k} = \frac{\sqrt{5}}{5} \sum_{k=0}^{11} \left[\left(\left(\frac{1+\sqrt{5}}{2} \right)^3 \right)^k - \left(\left(\frac{1-\sqrt{5}}{2} \right)^3 \right)^k \right]$$

Finally, using:

$$\sum_{k=0}^{n} c^k = \frac{1 - c^{n+1}}{1 - c}$$

In this case, $n = 11$ and $c = (1 \pm \sqrt{5})^3/8$:

$$S = \frac{\sqrt{5}}{5} \left(\frac{1 - \left(\frac{1+\sqrt{5}}{2} \right)^{36}}{1 - \left(\frac{1+\sqrt{5}}{2} \right)^3} - \frac{1 - \left(\frac{1-\sqrt{5}}{2} \right)^{36}}{1 - \left(\frac{1-\sqrt{5}}{2} \right)^3} \right)$$

Solving Linear Homogeneous Recursive Systems

The Fibonacci numbers fall under a more general class of problems called linear homogeneous recursive systems. In these problems, the goal is to convert a specific recursive formula into an explicit formula. The solutions work much like linear homogeneous ordinary differential equations. Once a solution or collection of solutions is found, we can take linear combinations of those solutions to solve the system. To see this, let me first present the system of interest: consider, a sequence of numbers a_n such that:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$$

where c_i are not all uniformly zero and some initial condition a_i for $0 \leq i < k$ provided. Such a system is called a linear homogeneous recursive system with k degrees of freedom. Observe that if we have one sequence of numbers a_i that satisfy the equation and a second sequence of numbers b_i that satisfy the system then:

$$d_i = \alpha_a a_i + \alpha_b b_i$$

also satisfies the system. In other words, linear combinations of sequences also satisfy the recursive relationship. To see this, just plug in the solution and expand as follows:

$$d_n = \alpha_a a_n + \alpha_b b_n$$
$$= \alpha_a \left(c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} \right) + \alpha_b \left(c_1 b_{n-1} + c_2 b_{n-2} + ... + c_k b_{n-k} \right)$$
$$= c_1 \left(\alpha_a a_{n-1} + \alpha_b b_{n-1} \right) + ... + c_k \left(\alpha_a a_{n-k} + \alpha_b b_{n-k} \right)$$
$$= c_1 d_{n-1} + ... + c_k d_{n-k}$$

Thus the d_n satisfy the relationship. Since the recursive system has k degrees of freedom from the k initial conditions, I expect that if we find k linearly independent sequences s_{ij} (here I have the ith term in the jth such sequence), that I can solve for a_n based on the a_k by solving the linear system:

$$a_i = \sum_{j=0}^{k-1} \alpha_j s_{ij}$$

for the $0 \leq i < k$. If these s_{ij} are actually linearly independent, this will create a solve-able system of linear equations for the α_j and then:

$$a_n = \sum_{j=0}^{k-1} \alpha_j s_{nj}$$

In order for this to actually be useful, we then also want the n term of each of our sequences to be readily computable. Note, this is not a formal proof of this being the solution, like for ODEs, but it does work at least in some cases.

Fortunately, such systems have a so-called characteristic polynomial that solves the system for a general term a_n with a readily computable formula. This polynomial is:

$$P(r) = r^k - c_1 r^{k-1} - c_2 r_{k-2} - ... - c_k$$

The roots of this polynomial then make up the explicit formula for our solution. Observe:

$$P(r) = 0 \implies r^k - c_1 r^{k-1} - c_2 r_{k-2} - ... - c_k = 0$$
$$r^{n-k}\left(r^k - c_1 r^{k-1} - c_2 r^{k-2} - ... - c_k\right) = 0$$
$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - ... - c_k r^{n-k} = 0$$
$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + ... + c_k r^{n-k}$$

so if $a_n = r^n$ then:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$$

which is exactly the recursive formula. By taking linear combinations of these root solutions we then can get the particular solution for the initial condition specified by the a_i, $0 \leq i < k$.

As an example, consider the Fibonacci sequence. Note that if I call:

$$a_0 = 0$$
$$a_1 = 1$$

Then my a_2 is the first term in the sequence specified in the problem statement. Recall that the Fibonacci numbers are defined as:

$$a_n = a_{n-1} + a_{n-2}$$

Thus the characteristic polynomial is:

$$r^2 - r - 1 = 0$$

which by quadratic formula is:

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1 + \sqrt{5}}{2}$$

$$r_2 = \frac{1 - \sqrt{5}}{2}$$

Using a linear combination of these roots:

$$a_n = \alpha_1 r^n + \alpha_2 r^n$$

$$a_0 = \alpha_1 + \alpha_2$$

$$a_1 = \alpha_1 r_1 + \alpha_2 r_2$$

Since, $a_0 = 0$ and $a_1 = 1$

$$\alpha_1 + \alpha_2 = 0 \implies \alpha_2 = -\alpha_1$$

$$\left(\frac{1 + \sqrt{5}}{2} \right) \alpha_1 + \left(\frac{1 - \sqrt{5}}{2} \right) \alpha_2 = 1$$

$$\left(\left(\frac{1 + \sqrt{5}}{2} \right) - \left(\frac{1 - \sqrt{5}}{2} \right) \right) \alpha_1 = 1$$

$$\sqrt{5} \alpha_1 = 1$$

$$\alpha_1 = \frac{\sqrt{5}}{5}$$

$$\alpha_2 = -\frac{\sqrt{5}}{5}$$

So:

$$a_n = \frac{\sqrt{5}}{5} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{\sqrt{5}}{5} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

One thing not mentioned in the above is recursive formulas with characteristic polynomials with roots of higher multiplicity. Like with ODEs, it turns out that:

$$a_n = nr^n, n^2 r^n, ..., n^m r^n$$

where m is the multiplicity are also solutions. Observe:

$$a_n = nr^n \iff nr^n = c_1(n-1)r^{n-1} + c_2(n-2)r^{n-2} + ... + c_k(n-k)r^{n-k}$$

$$a_n = nr^n \iff r^{n-k}\left(nr^k - c_1(n-1)r^{k-1} + c_2(n-2)r^{k-2} + ... + c_k(n-k)\right) = 0$$

$$a_n = nr^n \iff r^{n-k}\left((n-k)\left(r^k - c_1 r^{k-1} + c_2 r^{k-2} + ... + c_k\right)\right.$$
$$\left. + r\left(kr^{k-1} - c_1(k-1)r^{k-2} + c_2(k-2)r^{k-3} + ... + c_{k-1}\right)\right) = 0$$

where the above splitting and factorization gives the characteristic polynomial and it's derivative. Observe:

$$P(r) = r^k - c_1 r^{k-1} - c_2 r^{k-2} - ... - c_k$$
$$P'(r) = kr^{k-1} - c_1(k-1)r^{k-2} - c_2(k-2)r^{k-3} - ... - c_{k-1}$$

So the above becomes:

$$nr^{n-k}((n-k)P(r) + rP'(r)) = 0$$

So if both $P(r) = 0$ and $P'(r) = 0$, then the above is true. However, we've specifically stated that r is a root of at least multiplicity 2, so:

$$P(r) = 0$$
$$P(x) = (x-r)^m \hat{P}(x)$$

where $\hat{P}(x)$ is some other polynomial and $m \geq 2$, but then:

$$P'(x) = m(x-r)^{m-1}\hat{P}(x) + (x-r)^m \hat{P}'(x)$$

and we see that $P'(r) = 0$. Thus, we conclude that nr^n is also a solution of recursive formula if the root has a multiplicity greater than or equal to 2. Note, for the higher order terms are similar, except instead of stopping at P', the formula requires higher order derivatives. The steps for the proof are as follows:

- Plug in the proposed solution to the recursive formula.

- Show that the formula holds if r is actually a root of the polynomial and the appropriate derivatives.

- Show that because the root has the appropriate multiplicity it is actually a root of the characteristic polynomial and it's derivatives.

- Conclude that $n^m r^n$ is a sequence that obeys the recursive relationship.

Fibonacci Numbers and the Golden Ratio

Two numbers are said to be in the golden ratio if $a > b > 0$ and:

$$\frac{a+b}{a} = \frac{a}{b}$$

Or rearranging:

$$a^2 - ab - b^2 = 0$$

Thus, choosing $b = 1$, we have:

$$a^2 - a - 1 = 0$$

This is the same as the characteristic polynomial for the Fibonacci numbers. And since $a > b$, we have:

$$a = \frac{1 + \sqrt{5}}{2}$$

And since $b = 1$, we see:

$$a = \frac{a}{b} = \frac{1 + \sqrt{5}}{2}$$

which is called the golden ratio. Recall that:

$$a_n = \frac{\sqrt{5}}{5} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{\sqrt{5}}{5} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

for the Fibonacci numbers and observe, that:

$$\frac{1 - \sqrt{5}}{2} < 1$$

So in the limit that $n \to \infty$:

$$\lim_{n \to \infty} \frac{\sqrt{5}}{5} \left(\frac{1 - \sqrt{5}}{2} \right)^n = 0$$

So the ratio:

$$\lim_{n \to \infty} \frac{a_n}{a_{n-1}} = \lim_{n \to \infty} \frac{\frac{\sqrt{5}}{5} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{\sqrt{5}}{5} \left(\frac{1-\sqrt{5}}{2} \right)^n}{\frac{\sqrt{5}}{5} \left(\frac{1+\sqrt{5}}{2} \right)^{n-1} - \frac{\sqrt{5}}{5} \left(\frac{1-\sqrt{5}}{2} \right)^{n-1}} = \frac{\left(\frac{1+\sqrt{5}}{2} \right)^n}{\left(\frac{1+\sqrt{5}}{2} \right)^{n-1}} = \frac{1 + \sqrt{5}}{2}$$

Or exactly the golden ratio.

An Alternative Formula for nth Fibonacci Number

In this section, I derive an alternative, but not necessarily more useful formula for the nth Fibonacci number using binomial expansion. In the above, we derived a formula for the nth Fibonacci number:

$$a_n = \frac{\sqrt{5}}{5} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{\sqrt{5}}{5} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

$$a_n = \frac{\sqrt{5}}{5} \frac{1}{2^n} \left(\left(1 + \sqrt{5} \right)^n - \left(1 - \sqrt{5} \right)^n \right)$$

The goal here is to simplify this formula using the binomial expansion:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(x - y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (-y)^k$$

Note, these formula's are simply a matter of counting all the combinations, which is exactly what n choose k tells us: choose k different y's out n different bins. So:

$$(x + y)^n - (x - y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \left(y^k - (-y)^k \right) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \left(1 + (-1)^{k+1} \right)$$

which only keeps the odd values of k:

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \left(1 + (-1)^{k+1} \right) = 2 \left(\binom{n}{1} x^{n-1} y^1 + \binom{n}{3} x^{n-3} y^3 + \ldots + \binom{n}{n-1} x^1 y^{n-1} \right)$$

(if n is even otherwise the last term will be $k = n$ not $k = n - 1$). Since $x = 1$ and $y = \sqrt{5}$ in this case:

$$\left(1 + \sqrt{5} \right)^n - \left(1 - \sqrt{5} \right)^n = \sum_{k=0}^{n} \binom{n}{k} \sqrt{5}^k \left(1 + (-1)^{k+1} \right) = 2\sqrt{5} \sum_{k=1}^{n} \binom{n}{k} \sqrt{5}^{k-1} \left(\frac{1 + (-1)^{k+1}}{2} \right)$$

And:

$$a_n = \frac{1}{2^{n-1}} \sum_{k=1}^{n} \binom{n}{k} \sqrt{5}^{k-1} \left(\frac{1 + (-1)^{k+1}}{2} \right)$$

where:

$$\frac{1 + (-1)^{k+1}}{2} = \begin{cases} 0, & k \text{ even} \\ 1, & k \text{ odd} \end{cases}$$

Finally, keeping only the odd values for k, we see that:

$$k - 1 = 2m \tag{1}$$

where m is an integer, so:

$$\sqrt{5}^{k-1} = 5^m$$

and

$$a_n = \begin{cases} \dfrac{1}{2^{n-1}} \displaystyle\sum_{m=0}^{n/2-1} \binom{n}{2m+1} 5^m, & n \text{ even} \\[3em] \dfrac{1}{2^{n-1}} \displaystyle\sum_{m=0}^{(n-1)/2} \binom{n}{2m+1} 5^m, & n \text{ odd} \end{cases}$$

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Geometric Sums

Another result in this solution is the geometric sum:

$$\sum_{k=0}^{n} c^k = \frac{1 - c^{n+1}}{1 - c}$$

For proof consider:

$$S_n = \sum_{k=0}^{n} c^k$$

$$S_{n+1} = \sum_{k=0}^{n+1} c^k$$

Then:

$$S_{n+1} - S_n = c^{n+1}$$

$$S_{n+1} = 1 + cS_n \implies c^{n+1} = 1 + cS_n - S_n$$

$$S_n = \frac{c^{n+1} - 1}{c - 1}$$

$$S_n = \frac{1 - c^{n+1}}{1 - c}$$