P1: Solve the differential equation

$$dY_t = \mu dt + \sigma Y_t dW_t$$

Hint: Use the integrating factor

$$F_t = exp(-\sigma W_t + \frac{1}{2}\sigma^2 t)$$

to prove that

$$Y_t = \frac{Y_0 + \mu \int_0^t F_s ds}{F_t}$$

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To prove that the Y_t is the solution, we only need to show Y_t satisfies the SDE.
First, we look at F_t

$$\begin{aligned} F_t &= exp(-\sigma W_t + \frac{1}{2}\sigma^2 t) \\ \Rightarrow dF_t &= \frac{1}{2}\sigma^2 F_t dt - \sigma F_t dW_t + \frac{1}{2}\sigma^2 F_t (dW_t)^2 \\ &= \sigma^2 F_t dt - \sigma F_t dW_t \end{aligned}$$

Then, starting from F_t, we can get dY_t

$$\begin{aligned} dY_t &= \frac{\partial Y_t}{\partial t} dt + \frac{\partial Y_t}{\partial F_t} dF_t + \frac{1}{2}\frac{\partial^2 Y_t}{\partial F_t^2}(dF_t)^2 \\ &= \frac{\mu}{F_t} F_t dt - \frac{1}{F_t} Y_t(\sigma^2 F_t dt - \sigma F_t dW_t) + \frac{1}{F_t^2} Y_t \sigma^2 F_t^2 dt \\ &= \mu dt + \sigma Y_t dW_t - \sigma^2 Y_t dt + \sigma^2 Y_t dt \\ &= \mu dt + \sigma Y_t dW_t \end{aligned}$$

Proof Completed

P2: Let X_t denote geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

where μ, σ are constants.
Prove the following: if $\mu - \frac{\sigma^2}{2} < 0$ then the probability p that X_t starting from $x < R$ ever hits R is given by

$$p = (\frac{x}{R})^{\gamma_1}$$

where $\gamma_1 = 1 - \frac{2\mu}{\sigma^2}$(Hint: Compute the infinitesimal operator of $X_t^{\gamma_1}$ and use the Dynkin formula)
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Remark: Some technicalities about stopping times and optional sampling theorem to be clarified later.
Define: Let τ be the time that X_t hits R.
Define: $\tau_k = min\{k, \tau\}$, then τ_k is a bounded stopping time.
Define: $f(X_t) = \{ \begin{array}{ll} x_t^{\gamma_1}, & t \leq \tau_k \\ 0, & t > \tau_k \end{array}$
Then, for $f(X_t)$, we would get

$$\begin{aligned} df(X_t) &= \frac{\partial f}{\partial X_t} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2 \\ &= \gamma_1 X_t^{\gamma_1 - 1}(\mu X_t dt + \sigma X_t dW_t) + \frac{1}{2}\gamma_1(\gamma_1 - 1)X_t^{\gamma_1 - 2}\sigma^2 X_t^2 st \\ &= \sigma\gamma_1 {X_t}^{\gamma_1} dW_t \end{aligned}$$

Taking integrals from 0 to t, we would get

$$f(X_{\tau_k}) - f(X_0) = \int_0^{\tau_k} \sigma\gamma_1 X_t^{\gamma_1} dW_t$$

Taking expectation on both sides, we would have

$$\mathbb{E}[f(X_{\tau_k})] = f(X_0)$$

Write out the probabilities

$$P(\text{X hits R before } \tau_k)R^{\gamma_1} + P(\text{X did not hit R}) \times 0 = x^{\gamma_1}$$

Then, taking $k \to \infty$, we can conclude

$$P(\text{X hits R before K}) = (\frac{x}{R})^{\gamma_1}$$

P3: Consider the solution to the PDE

$$\begin{aligned} \frac{\partial F(t,x)}{\partial t} + \mathcal{A}F(t,x) - r(x)F(t,x) &= -h(x) \\ F(T,x) &= \Phi(x) \end{aligned}$$

Where

$$\mathcal{A}F(t,x) = \frac{\partial F}{\partial x}\mu(t,x) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(t,x)$$

Letting

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

Prove that

$$F(t,x) = \mathbb{E}_t[\int_t^T e^{-\int_t^s r(X_u)du} h(X_s)ds + e^{-\int_t^T r(X_u)du}\Phi(X_t)]$$

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We build a stochastic process, write out its intergal form and cancel out dW_s terms when taking expectation.

Define

$$Z_s = e^{-\int_t^s r(X_u)du} F(s, X_s)$$

Using Ito's lemma:

$$dZ_s = -e^{-\int_t^s r(x_u)du} F(s, X_s)ds + e^{-\int_t^s r(x_u)du} dF(s, X_s)$$

Write out $dF(s, X_s)$ using Ito's lemma

$$\begin{aligned} dF(s, X_s) &= \frac{\partial F}{\partial s}ds + \frac{\partial F}{\partial X_s}dX_s + \frac{1}{2}\frac{\partial^2 F}{\partial X_s^2}(dX_s)^2 \\ &= [\frac{\partial F}{\partial S} + \mathcal{A}F]ds + \frac{\partial F}{\partial X_s}\sigma dW_s \end{aligned}$$

According to the BVP,

$$\frac{\partial F}{\partial s} + \mathcal{A}F = r(s)F(s, X_s) - h(X_s)$$

Threfore,

$$dF(s, X_s) = [r(s)F(s, X_s) - h(X_s)]ds + \frac{\partial F}{\partial X_s}\sigma dW_s$$

Plug this back into Z_t,

$$dZ_s = e^{-\int_t^s r(X_u)du}(-h(X_s))ds + e^{-\int_t^s r(X_u)du}\frac{\partial F}{\partial X_s}\sigma dW_s$$

For simplicity, we compile all dW_s terms into a function C. Then, integrate from t to T

$$Z_T - Z_t = -\int_t^T e^{-\int_t^s r(X_u)du} h(X_s)ds + \int_t^T c(X_s, s)dW_s$$

Plug in the expression for Z, then

$$Z_T = e^{-\int_t^T r(X_u)du} F(T, X_T) = e^{-\int_t^T r(X_u)du} Z_t \qquad = e^0 F(t, X_t) = F(t, X_t)$$

Therefore, we could write out expressions

$$F(t, X_t) = e^{-\int_t^T r(x_u)du} \Phi(X_T) + \int_t^T e^{-\int_t^s r(X_u)du} h(X_s)ds + \int_t^T c(s, X_s)dW_s$$

Taking expectation on both sides conditioning on $\mathcal{F}_{\sqcup}^{\mathcal{X}}$, then the dW_t term would cancel out

$$F(t, X_t) = \mathbb{E}_t[e^{-\int_t^T r(X_u)du} \Phi(X_T) + \int_t^T e^{-\int_t^s r(X_u)} h(X_s)ds]$$