Take-home Final Exam 808V, due May 18, 9pm.

The Matlab code is available in **math.umd.edu/~tvp/808/thf**

1. We want to find the solution $x(t,\omega)$ of the SDE

$$dx = 4(-x^3 + x)dt + b\,dB(t), \qquad x(0) = X_0$$

with $b = .15$, $X_0 = .03$.

(a) Complete the provided code `sde_EM` for the Euler-Maruyama method.
Also compute $P(x(1) > 0.5)$.

(b) Find the forward Kolmogorov equation. Here it has the form $q_t + F_0(x)q + F_1(x)q_x - F_2(x)q_{xx} = 0$.
Complete the provided code `sde_FKE`. Also compute $P(x(1) > 0.5)$.

2. Let $B_1(t,\omega)$ denote the Brownian bridge process for $t \in [0,1]$ (standard Brownian motion constrained by the condition $B(1) = 0$). Recall that we have the KL expansion with $\int_0^1 v^{(k)}(t)v^{(\ell)}(t)\,dt = \delta_{kl}$, $\mathrm{E}\left[Y^{(k)}\right] = 0$, $\mathrm{E}[Y^{(k)}Y^{(\ell)}] = \delta_{k\ell}$

$$B_1(t) = \sum_{k=1}^{\infty} \lambda_k^{1/2} Y^{(k)}(\omega) v^{(k)}(t) \quad \text{with } \lambda_k = \frac{1}{k^2\pi^2}, \quad v^{(k)}(t) = \sqrt{2}\sin(k\pi t), \quad Y^{(k)} \sim N(0,1) \text{ independent}$$

We want to approximate $B_1(t)$ by a process of the form $X_N(t,\omega) = \sum_{k=1}^{N-1} Z^{(k)}(\omega)v^{(k)}(t)$.
Approximation 1: Use the **truncated KL series** $X_N := \sum_{k=1}^{N-1} \lambda_k^{1/2} Y^{(k)}(\omega)v^{(k)}(t)$
Approximation 2: Use the **piecewise linear interpolating function** $X_N(t)$ with values $X_N\left(\frac{k}{N}\right) = B\left(\frac{k}{N}\right)$ for $k = 1,\ldots,N-1$: here $Z^{(k)}(\omega) := B\left(\frac{k}{N},\omega\right)$ and $v^{(k)}(t)$ are the "hat functions" for the points $\frac{k}{N}$, $k = 1,\ldots,N-1$ (If N is a power of 2 this is exactly our original construction of the Brownian motion).
For each approximation **find a sharp error bound**

$$\mathrm{E}\left[\|B_1 - X_N\|^2_{L^2[0,1]}\right] \le CN^{-\alpha}$$

and **give values for C and α.**
Hint: First find $C_1 := \mathrm{E}\left[\|B_1\|^2_{L^2[0,1]}\right]$ using the KL expansion.
Let $B_h(t)$ denote the Brownian bridge on $[0,h]$ (standard Brownian motion constrained by the condition $B(h) = 0$) and find $C_h := \mathrm{E}\left[\|B_h\|^2_{L^2[0,h]}\right]$. Hint: $h^{1/2}B(t/h)$ is a standard Brownian motion, hence $B_h(t) = h^{1/2}B_1(t/h)$.
For Approximation 2 note that $B_1(t) - X_N(t)$ is a Brownian bridge on each subinterval $\left[\frac{k-1}{N}, \frac{k}{N}\right]$ for $k = 1,\ldots,N$.

3. We use the stochastic Galerkin method. Assume that we have $N_h = 2$ degrees of freedom for the space variable (for nodes x_1, x_2), $a(x,y) = a_0(x) + \sum_{k=1}^{n} a_k(x)y_k$ with $n = 2$, and we use degree $p = 1$ for each y_j. We use normalized Legendre polynomials $\psi_j(y_k)$, $k = 0,\ldots,p$ as a basis for $\mathcal{P}_p$.
We solve the linear system and get an array U_{i,j_1,j_2} corresponding to the basis function $\phi_i(x)\psi_{j_1-1}(y_1)\psi_{j_2-1}(y_2)$ where $\phi_j(x)$ are the nodal basis functions. Explain how to compute $\mathrm{E}\left[u_h(x_1)\right]$, $\mathrm{Var}\left[u_h(x_1)\right]$, $\mathrm{Cov}\left[u_h(x_1),\, u_h(x_2)\right]$ from the values U_{i,j_1,j_2}. Explain how the code in `stochgal.m` for the computation of the expectation and covariance works.

4. Consider the stochastic boundary value problem on the interval $D = (0,1)$

$$-\left(a(x)u'\right)' = f(x), \qquad u(0) = 0, \quad u(1) = 0$$

$$a(x) = a_0 + \sum_{k=0}^{\infty} \frac{1}{k^2}\cos(k\pi x)Y_k, \qquad Y_k \sim \mathrm{Unif}[-1,1], \text{ independent}$$

with $a_0(x) = 1.65$, $f(x) = 1$. We want to find $\mu(x) := \mathrm{E}\left[u(x)\right]$ and $C(x,\tilde{x}) := \mathrm{Cov}[u(x), u(\tilde{x})]$.

(a) Use MC-FEM: Use the provided code `mcfem`. Add code to print out values for $\mu(0.3)$ with a 95%-confidence interval, for $C(0.3, 0.8)$ and for $P(u(.8) > .05)$.

(b) Use SG-FEM: Use the provided code `stochgal`. Add code to print out values for $\mu(0.3)$ with a 95%-confidence interval, for $C(0.3, 0.8)$.
After you computed the Galerkin solution find $P(u(.8) > .05)$ by using Monte-Carlo (simulate $Y_k \sim \text{Unif}[-1, 1]$). *Hint:* Compute $\psi_k(x)$ by `legendreP(k,x)/sqrt(2/(2*k+1))` .Compute the vector $v = [\psi_0(0.8), \ldots, \psi_p(0.8)]$, then use `V=tenspraII(v,n); V=V(:);`

(c) If you use e.g. $N = 10$ how does the actual computation time grow with n for MC-FEM and for SG-FEM (e.g. with $p = 3$)?
Experiment with different values n, N, p, M. Using your computer, can you achieve higher accuracy with MC-FEM or with SG-FEM for this problem? Justify your answer (note that we don't know the exact solution).

Legendre polynomials: Let $\psi_k(y)$ denote the Legendre polynomial of degree $k \in \{0, 1, 2, \ldots\}$, normalized such that $\|\psi_k\|_{L^2[-1,1]} = 1$. Then we obtain for the matrices M and $\tilde{M}$

$$r_k := \int_{-1}^{1} y \cdot \psi_{k-1}(y) \cdot \psi_k(y)\, dy = (4 - k^{-2})^{-1/2}, \qquad k = 1, 2, 3, \ldots$$

$$M = I, \qquad \tilde{M} = \begin{bmatrix} 0 & r_1 & & \\ r_1 & \ddots & \ddots & \\ & \ddots & \ddots & r_p \\ & & r_p & 0 \end{bmatrix}$$