P1: Suppose that $\Pi(t, S_t)$ is the arbitrage-free price of a derivative security on S_t with some terminal payoff $\Pi(t, S_T) = \Phi(S_T)$ and r is the interest rate. Prove that under the probability measure Q we have that

$$d\Pi(t, S_t) = r\Pi(t, S_t)dt + \sigma_\Pi \Pi(t, S_t)dW_t$$

for an appropriate function $\sigma_\Pi(t, S_t)$
****
Under the measure Q, $S(t)$ would follow the dynamics:

$$S(t) = rS(t)dt + \sigma S(t)dW_t$$

Then, for the derivative security and stock, the value process would be

$$dV(t) = V[\frac{\mu_\Pi d\Pi(t)}{\Pi} + \frac{\mu_S dS(t)}{S}]$$

We follow the procedure for deriving the PDE for Black-Scholes model in the lecture notes:

$$\begin{aligned}
\Pi(t) &= F(t, S(t)) \\
d\Pi(t) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}ds + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2 \\
&= [\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S}rS(t) + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2]dt + \frac{\partial F}{\partial S}\sigma S dW_t \\
&= \alpha_\Pi F dt + \sigma_\Pi F dW
\end{aligned}$$

The value process can be re-written as

$$dV(t) = V[(\mu_\Pi \alpha_\Pi + \mu_s \alpha_s)dt + (\mu_\Pi \sigma_\Pi + \mu_S \sigma_S)dW]$$

For the arbitrage free condition. We have deterministic value process with an interest rate of r. Therefore, for the value process, we have the conditions:

- $\mu_\Pi \sigma_\Pi + \mu_S \sigma_S = 0$
- $\mu_\Pi + \mu_S = 1$
- $\mu_\Pi \alpha_\Pi + \mu_S \alpha_S = r$

Plug in expressions for F, we can arrive at a PDE for F

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial S}rS(t) + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2 = rF$$

since $\Pi(t) = F(t, S(t))$, we can conclude that Π indeed follows dynamics.

P2: Suppose that the stock market follows the dynamics

$$dS_t = \mu S_t dt + \sigma S_t d\bar{W}_t$$

and the itnerest rate is constant and equal to r. Consider a "digital claim"

$$\Phi(S_T) = \begin{cases} 1 & \text{if } S_T > K \\ 0 & \text{if } S_T < K \end{cases}$$

where $K > 0$. Give a formula for the arbitrage-free price $\Pi(t, S_t)$ of the digital claim at time t when the stock price is S_t
****
For the given dynamics, we assume S_t follows the dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Then, from Black-Scholes pricing

$$\begin{aligned}\Pi(t, S_t) &= e^{-r(T-t)}\mathbb{E}^Q[\Phi(S_t)|S_t] \\ &= e^{-r(T-t)}\{P^Q[S_T > K|S_t] \times 1 + P^Q[S_T < K|S_t] \times 0\} \\ &= e^{-r(T-t)}P^Q[S_T > K|S_t]\end{aligned}$$

Since S_T is a geometric brownian motion. Then S_t itself is a log-normal random variable. The above probability can be solved by doing a proper integration with respect to the density function of log-normal random variables.

$$P^Q(S_T > K|S_t) = 1 - F_N(\frac{1}{\sigma\sqrt{T-t}}[(r - \frac{\sigma^2}{2})(T-t) + \ln\frac{K}{S_t}])$$

P3: Keep the same dynamics for the stock prize and the same assumption on the interest rate as above. Give a formula for the arbitrage free price $\Pi(t, S_t)$ of a put option with the payoff

$$\Phi(S_T) = max\{0, K - S_T\}$$

****
Similar to problem 2, this can be solved by finding the expectation of a function of a log-normal random variable.

$$S_T = s_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(W_T - W_t)}$$
$$\Pi(t, S_t) = e^{-r(T-t)}\mathbb{E}^Q[\Phi(S_T)|S_t = s]$$

Therefore, we find the proper limit on $\frac{1}{\sqrt{T-t}(W_T - W_t)}$, which is a standard normal random variable.

$$S_T < K \Leftrightarrow \frac{1}{\sqrt{T-t}}(W_T - W_t) < \frac{1}{\sigma\sqrt{T-t}}[\ln \frac{K}{S_t} - (r - \frac{\sigma^2}{2})(T-t)]$$

The above gave the bound for the integral. Now, we express Φ using the standard normal variable in this interval.

$$\Phi = K - S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}z}$$

Then the prizing can be written as

$$\Pi(t, S_t) = e^{-r(T-t)} \int\limits_{z\in A} \Phi(t, z) f(z) dz$$

Solution completed.