**P1:** Use Ito's lemma to prove that

$$\int_0^t W_s^2 dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s ds$$

Using Ito's lemma to write our difference form of $W_t^3$

$$\begin{aligned} d(\frac{1}{3}W_t^3) &= 0dt + W_t^2 dW_t + W_t(dW_t)^2 \\ &= W_t^2 dW_t + W_t dt \end{aligned}$$

Integrate on both sides, we get

$$\frac{1}{3}W_t^3 = \int_0^t W_s^2 dW_s + \int_0^t W_s ds$$

Proof Completed

**P2:** Check whether the process $X(t) = W_1(t) \times W_2(t)$ is a martingale, where $W_1(t)$,$W_2(t)$ are independent brownian motions.

A little hand-waivy, we use $\mathcal{F}_s^{W_1}, \mathcal{F}_s^{W_2}$ to denote the $\sigma$-algebra induced by $W_1$ and $W_2$. We want to show that

$$\mathbb{E}[X(t)|F_s^X] = X(s), \forall s \le t$$

Since $X(t) = W_1 \times W_2$, we have $\mathcal{F}_t^X = \mathcal{F}_t^{W_1} \otimes \mathcal{F}_t^{W_2}$ from the independence of $W_1$ and $W_2$. Then,

$$\begin{aligned} &\mathbb{E}[W_x(t) \times W_2(t)|\mathcal{F}_s^{W_1} \otimes \mathcal{F}_s^{W_2}] \\ =&\mathbb{E}[W_1(t)|F_s^{W_1}] \times \mathbb{E}[W_2(t)|F_s^{W_2}] \\ =&W_1(s) \times W_2(s) \\ =&X(s) \end{aligned}$$

Since $W_t$, the brownian motion is a martingle.
Proof Completed

**P3:** Let $g(s)$ be a bounded, deterministic function of time. Show that

$$X_t = \int_0^t g(s) dW_s$$

is normally distributed with mean zero and standard deviation $\sqrt{\int\limits_0^t g^2(s)ds}$. Hint: a) use Ito's lemma to show that $Z_t = e^{-\frac{\eta^2}{2}\int\limits_0^t g^2(s)ds + \eta\int\limits_0^t g(s)dW_s}$ is a Martingale. b) Use this observation to conclude that

$$\mathbb{E}(e^{\eta X_t}) = e^{\frac{\eta^2}{2}\int\limits_0^t g^2(s)ds}$$

Which is the moment generating function of a normally distributed variable with zero mean and standard deviation equal to $\sqrt{\int\limits_0^t g^2(s)ds}$

Denote

$$Y_t = -\frac{\eta^2}{2}\int\limits_0^t g^2(s)ds + \eta\int\limits_0^t g(s)dW_s$$

Then, by Ito's lemma, we would have

$$dY_t = -\frac{\eta^2}{2}g^2(t) + \eta g(t)dW_t$$

Follwing hint (1), we try to show $Z_t = e^{Y_t}$ is a martingale.

$$\begin{aligned} dZ_t &= \frac{\partial Z_t}{\partial t}dt + \frac{\partial Z_t}{\partial Y_t}dY_t + \frac{1}{2}\frac{\partial^2 Z_t}{\partial Y_t^2}dY_t^2 \\ &= odt + e^{Y_t}(-\frac{\eta^2}{2}g^2(t) + \eta g(t)) + \frac{1}{2}e^{Y_t}(-\frac{\eta^2}{2}g^2(t) + \eta g(t)dW_t)^2 \\ &= \eta g(t)e^{Y_t}dW_t \end{aligned}$$

There is no $dt$ term in the above expression, therefore $Z_t$ is indeed an Martingale.

$$\mathbb{E}[Z_t] = Z_0 = 1$$

Notice, $g(s)$ is a deterministic function, therefore

$$\mathbb{E}[Z_t] = \mathbb{E}[e^{\eta X_t}]e^{-\frac{\eta^2}{2}\int\limits_0^t g^2(s)ds}$$

Combining the above two results, we can conclude

$$\mathbb{E}[e^{\eta X_t}] = e^{\frac{\eta^2}{2}\int\limits_0^t g^2(s)ds}$$

Following hint (2), this concludes that

$$X_t \sim \mathcal{N}(0, \sqrt{\int\limits_0^t g^2(s)ds})$$

**P4**: Let

$$\beta_k(t) = \mathbb{E}[W_t^k], \text{ for } k = 0, 1, 2...$$

Use Ito's formula to prove that

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds$$

Define $Z_{k,t} = W_t^k$, then we are intested in

$$\beta_k(t) = \mathbb{E}[Z_{k,t}]$$

By Ito's lemma

$$\begin{aligned} dZ_{k,t} &= \frac{\partial Z_{k,t}}{\partial t}dt + \frac{\partial Z_{k,t}}{\partial W_t}dW_t + \frac{1}{2}\frac{\partial^2 Z_{k,t}}{\partial W_t^2}(dW_t)^2 \\ &= 0 + kW_t^{k-1}dW_t + \frac{1}{2}k(k-1)w_t^{k-2}dt \\ \Rightarrow Z_{k,t} &= \int_0^t kW_s^{k-1}dW_s + \frac{1}{2}k(k-1)\int_0^t W_s^{k-2}ds \end{aligned}$$

Take expectation on both sies:

$$\mathbb{E}[Z_{k,t}] = \frac{1}{2}k(k-1)\mathbb{E}[\int_0^t W_s^{k-2}ds]$$

If we allow the exchange between integration and expectation, we naturally get the result

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds$$

Proof Completed