Flight Control
Full Technical Report

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1 Introduction

In our investigation, we investigated the application of different controllers for different aircraft models.

For the purpose of this technical report, we will be focusing on the intricacies of designing effective controllers considering the impact of feedback, proportional gain and delay as well as the Zeigler-Nichols tuning method for PID controllers.

2 Discussion

2.1 Closed Loop Sinusoidal Disturbance

The aircraft model that we considered for this section was approximately the following:

$$G(s) = \frac{304.2s + 251.4}{s(s^3 + 15.7s^2 + 43.6s + 269.1)}$$

The controller is simply a proportional controller with delay, $K(s) = ke^{-Ds}$.

From the Bode plot in our investigation (Lab Report Section 2.2), where we consider $k = 1$ and $D = 0.6$, we have the following:
$|G(j\omega)K(j\omega)|_{\omega=2\pi\cdot 0.66} = 8.99\ dB = 10^{\frac{8.99}{20}} \approx 2.82$
$\angle G(j\omega)K(j\omega)|_{\omega=2\pi\cdot 0.66} = -225\,\mathrm{deg}$

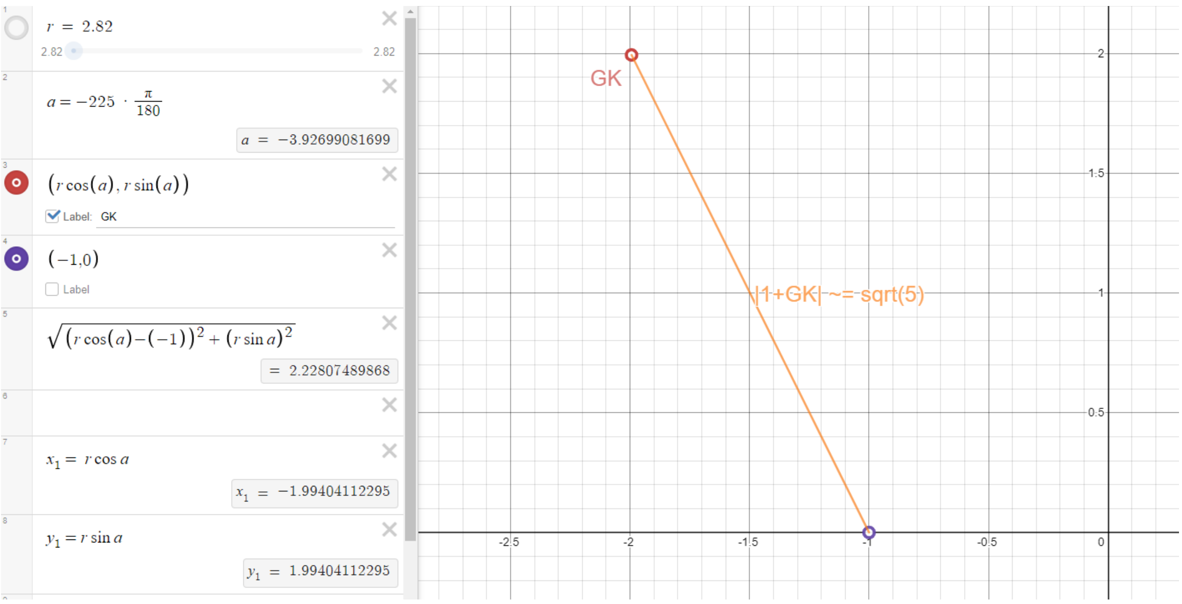

Figure 1: Argand diagram of $G(j\omega_1)K(j\omega_1)$ plotted with $|(1 + K(j\omega_1)G(j\omega_1))|$ in Desmos

From the plotted argand diagram, we find that $|(1 + K(j\omega_1)G(j\omega_1))| \approx \sqrt{5} \approx 2.23 \approx 6.99\ dB$.

Does the theory predict that your feedback will help attenuate the sinusoidal disturbance for stabilizing gains?

For the closed-loop system to be insensitive to noise we require the noise sensitivity function $|\frac{G}{1+L}| = |\frac{G}{1+GK}| << 1$, meaning that $|(1 + K(j\omega_1)G(j\omega_1))| >> |G(j\omega_1)|$. This is not robust since the proportional gain of K is set to 1, therefore we can consider the approximation $\frac{|G|}{|1+G|}$ and since this is not much smaller than 1, the feedback will not attenuate the sinusoidal disturbance for stabilising gains.

2.2 Delay Controller for Unstable Open-Loop

In this section we consider the proportional controller with delay $K(s) = ke^{-Ds}$ and the unstable aircraft model $G(s) = \dfrac{-2}{1 - sT}$ (Lab Report Section 2.3). For the closed loop to be stable, we require the roots of $1 + G(s)K(s) = 0$ to be in the LHP.

Note: We could also consider this as the number of anti-clockwise encirclements of -1 of $ke^{-Ds}G(s)$ must equal the number of unstable poles of the open loop GK (in this case, there is 1 unstable pole).

Below we analytically verify that if $D > T$ then no proportional gain exists to stabilise the system.

Calculating the roots is non-trivial as this is actually a state-space model, however as a formulation, we get the following:

$$1 + ke^{-Ds} \cdot \frac{-2}{1 - sT} = 0$$

$$(1 - sT) = 2ke^{-Ds} \tag{1}$$

The solution to this equation can be related to the Lambert (product log) function $W(s)$:

$$s = \frac{T\ W\Big(-\dfrac{2kDe^{-D/T}}{T}\Big) + D}{DT}$$

To satisfy stability, the solution must be less than 0:

$$\begin{aligned} T\ W\Big(-\frac{2kDe^{-D/T}}{T}\Big) + D < 0 \\ W(2k\ ze^{z}) < z \qquad (\text{where } z = -\frac{D}{T}) \end{aligned}$$

Note that we can take the inverse Lambert of both sides to simplify further since the inverse Lambert function is $W^{-1}(x) = xe^{x}$ which is positive for all values of x and is a one-to-one function for real numbers which in the case of $z = -\dfrac{D}{T}$ is true. Also since z here is a negative, by applying the inverse to both sides, we are making the RHS positive so the comparison sign must be flipped.

This simplifies the equation to the following: $2kze^{z} > ze^{z}$ Therefore we require $k > \dfrac{1}{2}$ for stability.

We can also find the restrictions related to D and T by using the Padé approximation [3] for the delay:

$$e^{-Ds} \approx \frac{1 - \frac{Ds}{2}}{1 + \frac{Ds}{2}}$$

From Eq.1, we get the following roots:

$$\begin{aligned} (1 - sT) &= 2k\frac{1 - \frac{Ds}{2}}{1 + \frac{Ds}{2}} \\ (1 - sT)(2 + Ds) &= 2k(2 - Ds) \\ s^2 - \frac{((2k+1)D - 2T)s}{DT} + \frac{4k-2}{DT} &= 0 \end{aligned}$$

Now consider marginal stability $k = \frac{1}{2}$ for simplification.

$$s^2 - \frac{2D - 2T}{DT} s = 0$$

Therefore the poles of the closed loop system are the following at $k = 1/2$:

$$s = 0, \frac{2D - 2T}{DT}$$

For stability, we require the poles to be < 0:

$s = 0$ is the pole due to marginal stability from k, this would be a stable pole for $k > \frac{1}{2}$. Now considering $\frac{2D - 2T}{DT} < 0$, we get $D < T$ for stability. Therefore for $D > T$, the system is unstable.

By plotting the Nyquist diagram, it is possible to see that for $D < T$ there is exactly one AC encirclement (or rather intersection) whereas for $D > T$, the encirclement/intersection is clockwise. Therefore we now know that for $D > T$, there is no proportional gain that can stabilise the system since we were checking for marginal stability.

To confirm the above results, the parametrised Nyquist plot is available at https://www.desmos.com/calculator/wmmyg9cqgl.

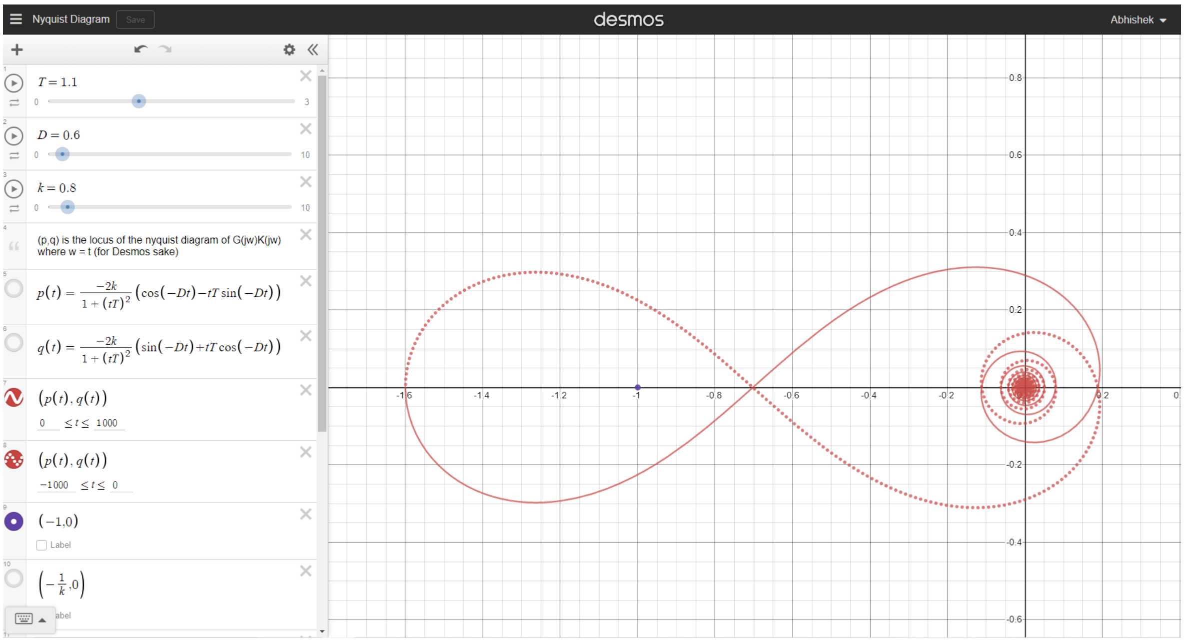

Figure 2: Parametrised Nyquist diagram plotted in Desmos

These equations are derived from:

$$\begin{aligned}
K(j\omega)G(j\omega) &= ke^{-jD\omega} \cdot \frac{-2}{1 - j\omega T} \\
&= ke^{-jD\omega} \cdot \frac{-2(1 + j\omega T)}{1 + \omega^2 T^2} \\
&= k(\cos(-D\omega) + j\sin(-D\omega)) \cdot \frac{-2(1 + j\omega T)}{1 + \omega^2 T^2} \\
&= \frac{-2k}{1 + \omega^2 T^2} \left(\cos(-D\omega) + j\sin(-D\omega)\right) \cdot (1 + j\omega T) \\
&= \frac{-2k}{1 + \omega^2 T^2} \Big(\big(\cos(-D\omega) - \omega T \sin(-D\omega)\big) + j\big(\sin(-D\omega) + \omega T \cos(-D\omega)\big) \Big) \\
K(j\omega)G(j\omega) &= \frac{-2k}{1 + \omega^2 T^2} \Big(\big(\cos(D\omega) + \omega T \sin(D\omega)\big) + j\big(-\sin(D\omega) + \omega T \cos(D\omega)\big) \Big)
\end{aligned}$$

2.3 Broom Balancing

The problem of stabilizing an unstable aircraft is similar in many respects to the problem of balancing a broom.

Consider the question: "What is the shortest upside down broom I can balance on my hand?" The linearised equations of motion (assuming negligible handle weight, length L, horizontal position of your hand x, angle θ to vertical, and considering one dimension) are:

$$\ddot{x} + L\ddot{\theta} = g\theta$$

Suppose we measure the horizontal position of the top of the broom, $y = x + L\theta$ and then use the feedback signal, $z = y + T\dot{y}$ where $T^2 = L/g$. Using this we can calculate the transfer function from x to z under this arrangement.

$$\ddot{y} = \ddot{x} + L\ddot{\theta} = g\theta$$

Therefore $y = x + \frac{L}{g}\ddot{y} = x + T^2\ddot{y}$. By taking the Laplace transform of this we can find the transfer function from x to y:

$$\overline{y}(s) = \frac{1}{1 - T^2 s^2}\ \overline{x}(s)$$

Now taking the Laplace transform of $z = y + T\dot{y}$ to find the transfer function from y to z:

$$\overline{z}(s) = (1 + sT)\ \overline{y}(s)$$

Combining the transfer functions, we get the overall transfer function from x to z:

$$\begin{aligned}
\overline{z}(s) &= \frac{1 + sT}{1 - s^2T^2}\ \overline{x}(s) \\
&= \frac{(1 + sT)}{(1 - sT)(1 + sT)}\ \overline{x}(s) \\
\overline{z}(s) &= \frac{1}{1 - sT}\ \overline{x}(s)
\end{aligned}$$

This assumes the particular proportional-derivative action controller given, but this is a reasonable choice.

The above transfer function for broom balancing is very similar to that of the unstable aircraft model that has the transfer function $G(s) = \frac{-2}{1 - sT}$.

For the unstable aircraft model (Lab Report Section 2.3), we determined by manual testing that $T_{\text{min}} = 0.5s$ is the smallest T that we can successfully stabilise. Therefore using $T^2 = \frac{L}{g}$, we find that $L_{\text{min}} = gT^2_{\text{min}} = 2.45\ m$. Comparing this to a real broomstick, a real broomstick is relatively smaller $(< 1.5m)$ and hence we cannot balance the broom.

2.4 Ziegler-Nichols

In this section, we consider the tuning of the PID controller using the Ziegler-Nichols tuning method to stabilise the given aircraft model (Lab Report Section 3.1). Ziegler-Nichols aim for a quarter-amplitude damping response and although this type of tuning provides very fast rejection of disturbances, it makes the loop very oscillatory with high overshoots [4].
There are two principal sets of Ziegler-Nichols rules for tuning PID controllers namely the open-loop method and the closed-loop method. A brief description of the two sets of rules are given below.

2.4.1 Process Reaction Curve (Open-Loop Method)

This method involves considering the step input response of the plant. From the characteristics of the step response (the reaction curve), we can determine the empirical measurements of the time period till steady-state gain T and the dead-time L which are then used to derive the PID coefficients from Table 1 .

Figure 3: Example step response curve in Desmos

Controller	K_P	T_i	T_d
P	$\frac{T}{L}$	∞	0
PI	$0.9\frac{T}{L}$	$\frac{L}{0.3}$	0
PID	$1.2\frac{T}{L}$	2L	0.5L

Table 1: Open-Loop Method Tuning Parameters

Advantages
A big advantage of this method is that only a single test is required and can be readily performed on real devices. The device is not pushed to stability limits and hence damage is unlikely to occur.

Disadvantages
There are several disadvantages specific to this method. Firstly, it requires the dead-time L to be small enough relative to T so that the response is not poor. The rules therefore depend on an accurate measurement of dead time, which is difficult to obtain delay-based processes with short dead times [2]. Secondly, this method is not applicable to unstable open-loop processes. Also the step-response of non-linear processes is highly sensitive to noise and hence this method does not perform so well for non-linear processes.

2.4.2 Continuous Cycling (Closed-Loop Method)

Consider the closed-loop system with only a proportional controller. Starting with a low value of K_P, increase K_P until a steady-state oscillation occurs. At this point, we record the magnitude (the ultimate gain) K_u and the period T_u of the oscillation [1].

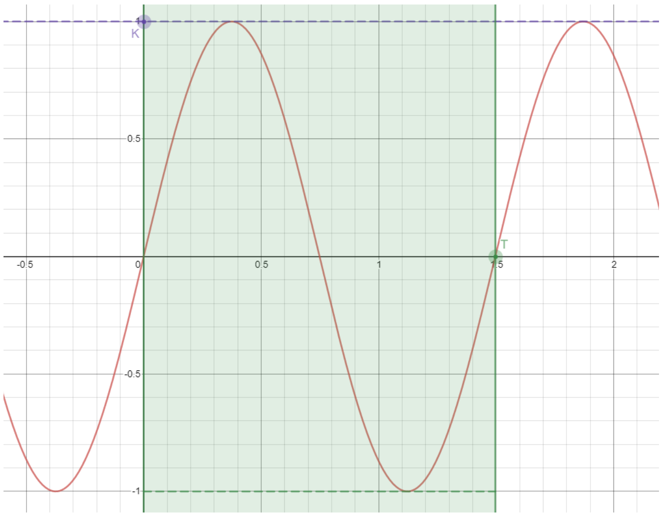

Figure 4: Example continuous cycling response curve in Desmos

From this we can get an estimate for the values of the PID coefficients from Table 2.

Controller	K_P	T_i	T_d
P	$0.5K_u$	∞	0
PI	$0.45K_u$	$\frac{T_u}{1.2}$	0
PID	$0.6K_u$	$0.5T_u$	$0.125T_u$

Table 2: Closed-Loop Method Tuning Parameters

Advantages

A major reason for the usage of this method is that all types of processes can be tested with this method and directly testing the closed-loop allows for direct tuning of the system for closed-loop stability.

Disadvantages

This method is time-consuming, expensive and hard to perform due to the inherent trial-and-error procedure to achieve continuous cycling. For the testing of a real-life plant, continuous cycling can damage the plant (as this is the stability limit) and hence is a major disadvantage of this method.

Summary

In summary, the Ziegler-Nichols tuning method is used to determine an initial/estimated set of working PID parameters for a given system and can also be applied to unknown systems. From either of the two methods, we can obtain the required parameters as needed for our controller. Both methods have their own drawbacks and depending on the real-life system, the open-loop method can be easier to test than the closed-loop. However, continuous cycling testing with real input can be very difficult to implement especially in the case of an aircraft.

2.5 Discrete Domain

The derivative of the PID controller (Lab Report Section 3.1) is approximated by using the backward difference for differentiation. Therefore the equivalent transfer function of the discretised PID controller is the following:

$$K(z) = K_p \overline{y}(z) + K_i \frac{T_s}{1 - z^{-1}} \overline{y}(s) + K_d \frac{1 - z^{-1}}{T_s} \overline{y}(s)$$

where T_s is the sampling time, $K_i = \dfrac{K_p}{T_i}$ and $K_d = K_p T_d$.

Here we consider the discretisation of the blocks with time-delays. Considering a DAC as a simple zero order hold, the continuous time plant $G(s) = e^{-\frac{D1s}{s}}$ placed in the arrangement illustrated in Fig.5 and assuming that the ADC and DAC operate synchronously with sampling period T, we find the discrete-time transfer function from $u(k)$ to $y(k)$.

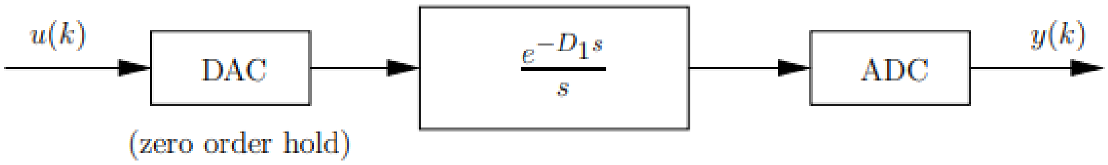

Figure 5: ADC/DAC arrangement for discretisation

This can be more precisely defined as the following:

$$u(k) \longrightarrow \boxed{\frac{(1 - e^{-sT})}{s} \mathcal{L}\{ \cdot \Big|_{k=\frac{t}{T}} \}} \longrightarrow \boxed{\frac{e^{-D_1 s}}{s}} \longrightarrow \boxed{\mathcal{L}^{-1}\{\cdot\}|_{t=kT}} \longrightarrow y(k)$$

Simplifying this in the Laplace domain, we get the following: $\overline{y}(s) = \dfrac{e^{-D_1 s}(1 - e^{-sT})}{s^2} \overline{u}(s)$

We can apply the inverse Laplace transfer function to get the time domain transfer function and then convert this into a discrete time transfer function g_k from u_k to y_k by setting $t = kT$, where $y_k = g_k * x_k$ and $H(t)$ is the step function centred at $t = 0$ in the time domain.

$$\begin{aligned}
g(k) &= \mathcal{L}^{-1}\{\frac{e^{-D_1 s}(1 - e^{-sT})}{s^2}\}\Big|_{t=kT} \\
&= (t - D_1)\ H(t - D_1) - (t - (T + D_1))\ H(t - (T + D_1))\Big|_{t=kT} \\
&= (kT - D_1)\ H(kT - D_1) - (kT - (T + D_1))\ H(kT - (T + D_1))
\end{aligned}$$

Let $D_1 = nT + D_0$ where n is an integer and $0 \leq D_0 < T$

$$= ((k - n)T - D_0)\ H\Big((k - n)T - D_0\Big) - ((k - (n + 1))T - D_0)\ H\Big((k - (n + 1))T - D_0\Big)$$

$$g(k) = \begin{cases} 0 & 0 \leq k \leq n \\ T - D_0 & k = n + 1 \\ T & k \geq n + 2 \end{cases} \tag{2}$$

The routine bodedisp.m obtains a time-delayed discrete-time frequency response by first calculating the discrete-time transfer function of the loop excluding the time-delay, then approximating the time-delay by an integer product of the sampling time.

Calculating the discrete-time transfer function without the time-delay and then approximating the time-delay by an integer product of the sampling time gives the following overall discrete-time transfer function:

$$g(k) = \mathcal{L}^{-1}\{\frac{1-e^{-sT}}{s}\frac{1}{s}\}\Big|_{t=k'T} = t-(t-T)H\Big(t-T\Big)\Big|_{t=k'T} = k'T-(k'-1)T\ H\Big((k'-1)T\Big)$$

Note however that we only need to consider $k' \geq 0$ and hence consider $g(k') = 0$ for $k' \leq 0$. Now let $k' = k-n$ due to the shift of nT.

$$g(k) = \begin{cases} (k-n)T-(k-(n+1))T\ H\Big((k-(n+1))T\Big) & k \geq n \\ 0 \qquad k < n \end{cases}$$

$$g(k) = \begin{cases} 0 & 0 \leq k \leq n \\ T & k \geq n+1 \end{cases} \tag{3}$$

Discuss the differences between these two schemes, considering accuracy under different time-delays and sampling times and ease of calculation.

For the case when $D_0 = 0$ meaning that the delay is an exact multiple of the sampling period T, both schemes (Eq.2 and Eq.3) are equivalent irrespective of T. However when $D_0 \neq 0$, there is an error of D_0 at $k = n+1$ for the second scheme (Eq.3). This is a very minor error as it is a single sample and for a large sampling frequency meaning a small sampling period T, this would have a negligible effect. Nonetheless, the second scheme (Eq.3) is easier to calculate as it is essentially a step function of magnitude T centred at $k = n+1$.

Note however that the error D_0 is only negligible when considering the delta function as the input. For any general input, this is not the case as discussed below.

There are a number of options available to analyse the delayed loop including the PID controller. One is to find the product of the plant and controller transfer functions, and then apply bodedisp.m. Is this a reasonable approach? Can you suggest some alternatives?

To analyse the delayed loop including the PID controller, finding the product of the plant and the controller transfer function and then applying bodedisp may not be reasonable since the differences in discrete time could be very large. In the case of exponentials in the time domain for example the plant transfer function $\frac{-1}{1-sT}$, the difference due to delay error D_0 from a proportional-delay controller is very large when the delay is approximated as an integer multiple of the sampling period $D_1 = nT$ instead of the actual delay $D_1 = nT + D_0$ for $0 \leq D_0 < T$. Therefore in the case of a PID controller, this method of approximation is most certainly unsuitable.

A more reasonable approach is to convert the equations to state space and use the state-space representation for analysis. This is because adding delay to a system shifts us into a different class of differential equations, specifically from ODEs to functional differential equations (more specifically delay differential equations) and this can have chaotic behaviour. In general analysing closed-loop stability of time-delay systems is non-trivial. We need to consider the delay-independent stability criteria as well as the delay-dependent criteria. A possible method to do this is to use model transformations and then apply mathematical techniques such as delay partitioning and consider Lyapunov stability [5].

3 Conclusion

In this report, we investigated the rejection of sinusoidal noise by comparing the open-loop system with the closed-loop system, the stability criteria for a specific model in closed-loop with a proportional-delay controller, the derivation of an broom balancing model analogous to that of an aircraft, evaluated the different Ziegler-Nichols tuning methods for tuning a PID controller and evaluated the effects of DAC and ADC in discrete-time as well as considering the impact on a model system. In essence, we have undergone the process of designing and evaluating real controllers for various aircraft models and understanding the effects of discretisation on the stability of the closed-loop implementation.

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4 Appendix