SF2955 Homework 2

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1 Introduction

In this report a basic epidemic model whose unknown model parameters will be estimated by Bayesian inference. The posterior will be a complex probability distribution, which will be sampled from using a Markov chain Monte Carlo (MCMC) algorithm.

2 Model description

The epidemic is assumed to be spreading in a population of P individuals. Using a SIR model, a discrete-time stochastic process $(S_t, I_t, R_t)_{t \in N}$, where S_t is individuals who were not infected before day t, I_t is the number of individuals that are infected at the beginning of day t, R_t is the number of individuals that are removed at the beginning of day t, either recovered, become immune, or died before day t. The population is assumed to be isolated, each day t, $P = S_t + I_t + R_t$, so that the process can be described by two states only.

The outbreak starts with an initial number of I_0 cases, so the initial number that are not infected $S_0 = P - I_0$. During day t, an additional number ΔI of individuals become infected, which means the number of S_t decreases the same amount. An additional number ΔR either recover or die. So the number of infected individuals increase by ΔI and decrease with ΔR.

$$\begin{cases} S_{t+1} = S_t - \Delta_t^I, \\ I_{t+1} = I_t + \Delta_t^I - \Delta_t^R, \quad t \in N \\ R_{t+1} = R_t + \Delta_t^R, \end{cases} \tag{1}$$

With a constant probability of p, the distribution of the flow of R_t is:

$$\Delta_t^R \sim Bin(I_t, p^{i \to r}) \tag{2}$$

The probability that someone becomes infected on day t is:

$$p_t^{s \to i} = 1 - exp(-\lambda(t)\frac{I_t}{P}) \tag{3}$$

The variable $\lambda(t) > 0$ is dependent on the average number of interactions per individual on day t.

$$\lambda_t^I \sim NegBin(\kappa, \varphi) \tag{4}$$

With parameters $\varphi \in (0, 1)$ and;

$$\kappa = (\frac{1}{\varphi} - 1)S_t p_t^{s \to i} \tag{5}$$

Throughout this assignment, $\varphi = 0.995$. The time frame is divided into d intervals, where $\lambda(t)$ is constant.

$$0 = t_0 < t_1.. < t_d = T \tag{6}$$

$$\lambda(t) = \sum_{i=1}^{d-1} \lambda_i 1_{t_{i-1},t_i}(t) + \lambda_d 1_{t_{d-1},d_i}(t) \tag{7}$$

3 Problem 1

Given a parameter vector $\theta = (\lambda, t, p^{i \to r})$, the Markov chain $(S_t, I_t)_{t \in N}$ transition probabilities are determined:

$$q_0(s_t, i_t; s_{t+1}, i_{t+1}) = P_0(S_{t+1} = s_{t+1}, I_{t+1} = i_{t+1} \mid S_t = s_t, I_t = i_t) \tag{8}$$

Using equation (1) it can be rewritten as follows:

$$P_0(s_t - \Delta_t^I = s_{t+1}, i_t + \Delta_t^I - \Delta_t^R = i_{t+1}) \tag{9}$$

$$P_0(\Delta_t^I = s_t - s_{t+1})P(\Delta_t^I - \Delta_t^R = i_{t+1} - i_t \mid \Delta_t^I = s_t - s_{t+1}) \tag{10}$$

$$P_0(\Delta_t^I = s_t - s_{t+1})P(\Delta_t^R = i_t - i_{t+1} + s_t - s_{t+1}) \tag{11}$$

4 Prior distributions

In order to provide a Bayesian model, a prior distribution of $\pi(\theta)$ for θ needs to be specified. The parameters are assumed to be priori independent, $\pi(\theta) = \pi(\lambda)\pi(t)\pi(p^{i\to r})$. First t is assigned a flat prior over the set of breakpoints, the components of λ are priori independent with $\Gamma(\alpha_i, \beta_i)$,

$$\pi(\lambda) = \prod_{i=1}^{d} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} e^{-\beta_i \lambda_i} \tag{12}$$

Throughout the assignment, $\alpha_i = 2$. Finally, Beta is assigned prior:

$$\pi(p^{i\to r}) = \frac{1}{B(a.b)} (p^{i\to r})^{a-1} (1 - p^{i\to r})^{b-1} \tag{13}$$

5 Data

In order to calibrate the model, COVID-19 data from Germany and Iran will be used. The parameters θ will be estimated on the basis of two realizations, $(i_t, r_t)_{t=0}^{T}$ and $(I_t, R_t)_{t=0}^{T}$

6 Problem 2

Determining the likelihood, where $P(S_0 = s_0, I_0 = i_0 \mid \theta) = 1$:

$$f(y \mid \theta) = P(S_0 = s_0, I_0 = i_0, ..., S_T = s_T, I_T = i_T \mid \theta) = \tag{14}$$

Using chain rule:

$$P(S_0, I_0)P(S_1, I_1 \mid S_0, I_0)(S_2, I_2 \mid S_0, I_0, S_1, I_1)..P(S_{n+1}, I_{n+1} \mid S_n, I_n, S_{n-1}, I_{n-1}, ..S_0, I_0) = \tag{15}$$

Simplifying the equation:

$$\prod_{i=0}^{n} P(S_{n+1}, I_{n+1} \mid S_n, I_n) \tag{16}$$

From problem 1, the expression can be written as:

$$\prod_{i=0}^{n} P_0(\Delta_t^I = s_t - s_{t+1}) P(\Delta_t^R = i_t - i_{t+1} + s_t - s_{t+1}) \tag{17}$$

7 Problem 3

Now computing the normalising constant, the full conditionals $\pi(\lambda \mid y, t, p^{i \to r}$, $\pi(t \mid y, p^{i \to r}, \lambda)$, $\pi(p^{i \to r} \mid y, \lambda, t)$. Assuming they are priori independent:

$$\pi(\theta \mid y) \propto (y \mid \theta)\pi(\lambda)\pi(t)\pi(p^{i-r}) \tag{18}$$

Starting with $\pi(\lambda \mid y, t, p^{i-r})$ and its distributions:

$$\pi(\lambda \mid y, t, p^{i-r}) \propto f(y \mid \theta)\pi(\lambda) \propto \tag{19}$$

$$\prod_{i=1}^{d} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} e^{-\beta_i \lambda_i} \prod_{i=0}^{T} P(\Delta_t^I = s_t - s_{t+1}) P(\Delta_t^R = i_t - i_{t+1} + s_t - s_{t+1}) \tag{20}$$

Now moving to $\pi(t \mid y, p^{i \to r}$

$$\propto \prod_{i=0}^{T} P(\Delta_t^I = s_t - s_{t+1}) P(\Delta_t^R = i_t - i_{t+1} + s_t - s_{t+1}) 1_{0 < t_1 .. < t_{d-1} < T}(t) \tag{21}$$

Finally, $\pi(p^{i \to r} \mid y, \lambda, t) \propto$

$$\frac{1}{B(a.b)} (p^{i \to r})^{a-1} (1 - p^{i \to r})^{b-1} \prod_{i=0}^{T} P(\Delta_t^I = s_t - s_{t+1}) P(\Delta_t^R = i_t - i_{t+1} + s_t - s_{t+1}) \tag{22}$$

$$\propto (p^{i \to r})^{a-1} (1 - p^{i \to r})^{b-1} \prod_{i=0}^{T} P(\Delta_t^R = i_t - i_{t+1} + s_t - s_{t+1}) \tag{23}$$

8 Problem 4

Implementing a hybrid sampler simulating from the joint posterior $\pi(\theta \mid y)$ for the available data sets, using a standard Gibbs step for $p^{i \to r}$, and the components λ and t are updated using local Metropolis-Hastings moves. Using a Gaussian random walk proposal for each λ_i, a candidate is generated according to:

$$\lambda_i^* = \lambda_i + \sigma\epsilon \tag{24}$$

Where ϵ is normally distributed and $\sigma > 0$ is an algorithmic parameter. For each breakpoint, a candidate is generated using a random walk proposal:

$$t_i^* = t_i + \epsilon \tag{25}$$

Where ϵ has a discrete uniform distribution, $U(-M, M)$.

9 Problem 5

Investigating the sensitivity of the posterios and mixing with regard to the hyperparameters $(\beta_i), a, b$ showed that they did not have any significant impact on the results, only the beta distribution. The algorithmic parameters σ, M, causes higher variance to λ, t with an lower values.

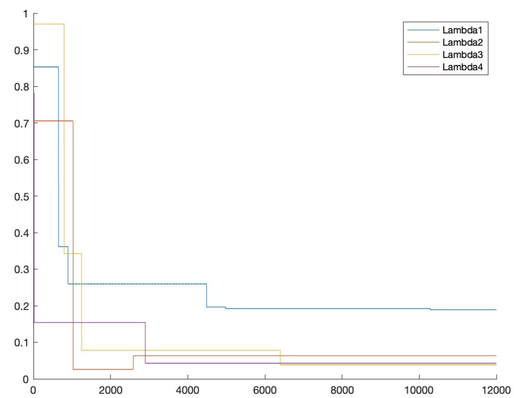


Figure 1: Proposed lambdas

In figure 1, $\sigma = 100$. Increasing β_i seems to decrease the variance of the proposed lambdas, especially the first one.

10 Problem 6

Starting with the German data, with $\sigma = 0.05$, the breakpoint converges to $t = 34$, which is the turning point for the infected in Germany.

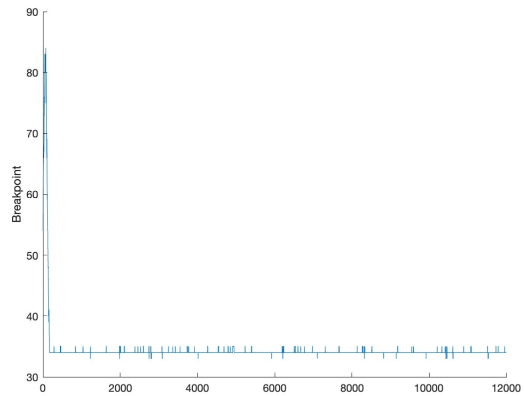


Figure 2: Breakpoint for German data

Having instead 3 breakpoints, they converge to 20, 28, and 34. Which has about the same time between them as the implemented COVID-19 containment laws. The mean of $p^{i \to r}$ was computed to be 0.065. The lambdas showed an increase in variance the more they were, but their values were as expected.

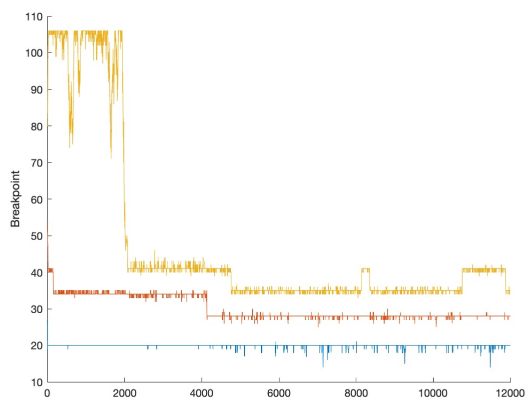


Figure 3: 3 breakpoints for German data

Looking at Iran's data of infected, two breakpoints should be made out easily, at around 35 and 65.

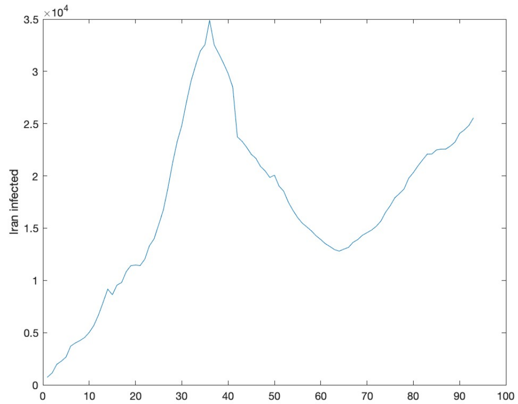
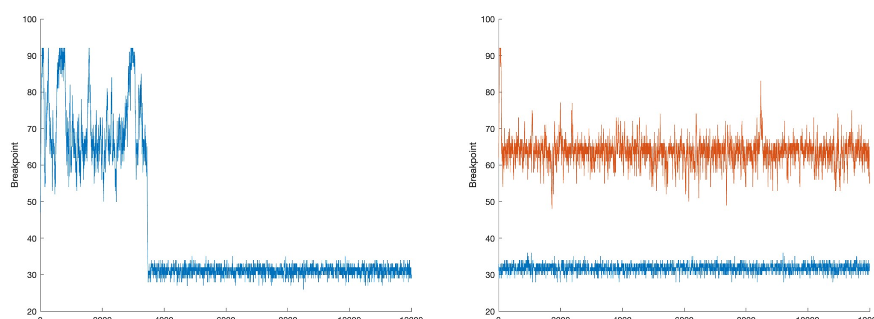


Figure 4: Iran infected

With one breakpoint the logarithm seems indecisive between the two, while two breakpoints converges much more nicely.

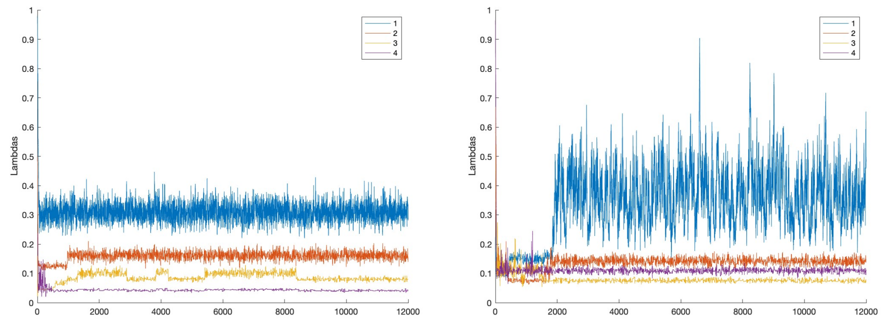


(a) 1 breakpoint

(b) 2 breakpoints

Figure 5: Iran breakpoints

Looking at the countries' lambdas, the stages of COVID-19 containment is clearer in Germany's lambdas, while both countries have a bigger lambda with an increase in infection.



(a) Germany lambdas

(b) Iran lambdas

Figure 6: 4 lambdas

11 Problem 7

Since $p^{i \to r}$ does not require λ or t, an MCM algorithm is not required. Looking the data given one can instead decide on parameters to use with a Beta distribution.