First name: ________________ **Last name:** ________________

Problem 1 (10 points). Consider a linear transformation $T : P_2(R) \to R^3$ defined by

$$T(p(x)) = \begin{bmatrix} p(1) \\ p'(0) \\ \int_0^1 p(x)dx \end{bmatrix}.$$

(a) Compute $[T]_\beta^\gamma$ where $\beta = \{1, x, x^2\}$ and $\gamma = \{e_1, e_2, e_3\}$ are the standard ordered bases of $P_2(R)$ and R^3 respectively. **(5 points)**

$$T(1) = \begin{bmatrix} 1 \\ 0 \\ \int_0^1 1\,dx \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad T(x) = \begin{bmatrix} 1 \\ 1 \\ \int_0^1 x\,dx \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$$T(x^2) = \begin{bmatrix} 1 \\ 0 \\ \int_0^1 x^2 dx \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{3} \end{bmatrix} \qquad [T]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

(b) Compute $[T(1 + 2x + x^2)]_\gamma$. **(2 points)**

$$[T(1+2x+x^2)]_\gamma = [T]_\beta^\gamma [(1+2x+x^2)]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ \frac{7}{3} \end{bmatrix}$$

(c) Compute $\det([T]_\beta^\gamma)$ **(2 points)**. Decide if T an isomorphism **(1 point)**.

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}\right) = (-1)^{2+2} \cdot 1 \cdot \det\left(\begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{3} \end{bmatrix}\right)$$

$$= -\frac{2}{3}$$

Because $\det([T]_\beta^\gamma) \neq 0$ T is invertible hence an isomorphis

Problem 2 (6 points). Decide the value of k in following equation (show procedure of your work)

$$\det\left(\begin{bmatrix} b_1 & b_2 & b_3 \\ 2a_1 & 2a_2 & 2a_3 \\ 3c_1+b_1 & 3c_2+b_2 & 3c_3+b_3 \end{bmatrix}\right) = k\det\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right).$$

$$\det\left(\begin{bmatrix} b \\ 2a \\ 3c+b \end{bmatrix}\right) \overset{R_3 \to R_3 - R_1}{=} \det\left(\begin{bmatrix} b \\ 2a \\ 3c \end{bmatrix}\right) = 2\det\begin{bmatrix} b \\ a \\ 3c \end{bmatrix}$$

$$= 6\det\left(\begin{bmatrix} b \\ a \\ c \end{bmatrix}\right) \overset{R_1 \leftrightarrow R_2}{=} -6\det\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Problem 3 (10 points). Consider vector spaces V and W over a field F and a **one to one** linear transformation T from V to W.

(a) show that T maps a linear independent subset of V to a linear independent subset of W. Namely, show if S is a linear independent subset of V and $\{v_1, \cdots, v_n\} \subset S$, then $\{T(v_1), \cdots, T(v_n)\}$ are linear independent. **(5 points)**

let $a_1, \cdots, a_n \in F$ and $\sum a_i T(v_i) = 0$ by linearity of T · $T(\sum a_i v_i) = \sum a_i T(v_i) = 0$. Because T is one to one $N(T) = \{0\}$ hence $\sum a_i v_i = 0$ $(\sum a_i v_i \in N(T))$

By linear independency of $\{v_1, \cdots, v_n\}$ $a_i = 0$ for all i.
There for $T(v_1), T(v_2), \cdots, T(v_n)$ are linearly independent.

(b) Suppose V is finite dimensional, state the dimension theorem **(3 points)** and show that $dim(V) \leq dim(W)$ **(2 points)**.

Dimension Thm : $\dim(V) = \text{Nullity}(T) + \text{rank}(T)$

T is one to one $N(T) = \{0\} \Rightarrow \dim(N(T)) = 0$

$\text{Nullity}(T) = \dim(N(T)) = 0$ $\quad \dim(V) = \text{rank}(T)$

$R(T)$ is a subspace of $W \Rightarrow \dim(R(T)) \leq \dim W$

$\Rightarrow \dim(V) = \text{rank}(T) = \dim(R(T)) \leq \dim W$

Problem 4 (6 points). Find the inverse matrix for the following matrix A:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

$[A \mid Id]$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & +1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Problem 5 (18 points). For $A \in M_{2\times 2}(R)$, define $T_A : M_{2\times 2}(R) \to M_{2\times 2}(R)$ by

$$T_A(X) = AX, \quad X \in M_{2\times 2}(R).$$

(a) Show T_A is a linear transformation on $M_{2\times 2}(R)$. **(3 points)**

$\forall X, Y \in M_{2\times 2}(R)$ $\quad T_A(X+Y) = A(X+Y) = AX + AY = T_A(X) + T_A(Y)$

$\forall c \in R \quad X \in M_{2\times 2}(R) \quad T_A(cX) = A(cX) = cAX = c(AX) = c\,T_A(X)$

(b) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, compute $[T_A]_\beta$ where $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is the standard ordered basis of $M_{2\times 2}(R)$. **(4 points)**

$\cancel{T_A\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a \end{bmatrix}}$

$T_A\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$ $\quad [T_A]_\beta = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$

$T_A\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$

$T_A\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}$

$T_A\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$

(c) Show that T_A is invertible transformation if and only if A is invertible matrix. **(6 points)**

Way 1: $[T_A]_\beta$ invertible $\Leftrightarrow$ T_A invertible (chapter 2)

A invertible $\Leftrightarrow \det(A) \neq 0$, $[T_A]_\beta$ invertible $\Leftrightarrow$ $\det([T_A]_\beta) \neq 0$ (chapter 4)

$\det([T_A])_\beta = (\det A)^2$ (please check) hence T_A invertible $\Leftrightarrow$ A inver is

Way 2: "$\Rightarrow$" T_A invertible $\Rightarrow$ T_A is onto $\Rightarrow \exists B, T_A(B) = Id \Rightarrow \exists B \; AB = Id$

"$\Leftarrow$" A is invertible $T_{A^{-1}} \circ T_A(X) = A^{-1}AX = X$ $T_A \circ T_{A^{-1}}(X) = AA^{-1}X = X$

$\Rightarrow T_{A^{-1}}$ is inverse of T_A

(d) By part (a), $U := \{T_A : A \in M_{2\times 2}(R)\}$ is a subset of $\mathscr{L}(M_{2\times 2}(R))$. Show that U is a subspace of $\mathscr{L}(M_{2\times 2}(R))$. **(4 points)**

①. zero function is in U

$0_{2\times 2} \in M_{2\times 2}(R)$ $T_{0_{2\times 2}}(X) = 0 \cdot X = 0$ hence

$T_{0_{2\times 2}}$ is zero function and it is in U

②. let $A, B \in M_{2\times 2}(R)$ $(T_A + T_B)(X) = T_A(X) + T_B(X) = AX + BX$

$= (A+B)X = T_{A+B}(X)$ $\forall X \in M_{2\times 2}(R) \Rightarrow T_A + T_B = T_{A+B} \in U$

③. let $A \in M_{2\times 2}(R)$ $c \in R$ $(cT_A)(X) = cT_A(X) = cAX = (cA)X = T_{cA}(X)$

$\forall X \in M_{2\times 2}(R)$

$\Rightarrow cT_A = T_{cA} \in U$

(e) The dimension of the subspace $U = \{T_A : A \in M_{2\times 2}(R)\}$ is $\underline{4}$. **(1 point)**

Extra credit problem (10 points). Let $P \in M_{n\times n}(R)$ be a projection. Recall a square matrix P is called a projection if $P^2 = P$.

(a) Show that restricted on range of P, L_P is identity map. That is if $y \in \mathscr{R}(P)$, then $L_P(y) = y$. **(3 points)**

$y \in \mathscr{R}(P) \Rightarrow \exists x \in R^n$ $y = Px$

$L_P(y) = Py = PPx = P^2x = Px = y$

(b) Show that $I - P$ is also a projection. **(2 points)**

$$(I-P)^2 = I - 2P + P^2 = I - 2P + P = I - P$$

(c) Show that $R^n = \mathscr{R}(L_P) \oplus \mathscr{R}(L_{I-P})$. **(3 points)**

$\forall\ v \in R^n$ $\quad v = Pv + (I-P)v$ where $Pv \in \mathcal{R}(P)$

$(I-P)v \in \mathcal{R}(I-P)$

so $R^n = \mathcal{R}(L_P) + \mathcal{R}(L_{I-P})$

let $y \in \mathcal{R}(L_P) \cap \mathcal{R}(L_{I-P})$ $\quad y = Px_1 = (I-P)x_2$ for some x_1, x_2

$y \overset{(a)}{=} Py = P(I-P)x_2$

$= Px_2 - P^2 x_2$

$= 0$

so $\mathcal{R}(L_P) \cap \mathcal{R}(L_{I-P}) = \{0\}$

$\therefore R^n = \mathcal{R}(L_P) \oplus \mathcal{R}(L_{I-}$

(d) Show that I_n is the only invertible projection in $M_{n\times n}(R)$. **(2 points)**

$I_n^2 = I_n \Rightarrow I_n$ is a projection.

now assume P is a invertible projection

$P^2 = P \Rightarrow P^{-1}P^2 = P^{-1}P = Id \Rightarrow P = Id$

so Id is the only invertible projection