First name: ____________________ **Last name:** ____________________

Problem 1 (15 points).

(a) Let $v_1, v_2, \cdots, v_n$ be n vectors in a vector space V over F. Write down the definition of $v_1, v_2, \cdots, v_n$ being linear independent (**2 points**) and the definition of $v_1, v_2, \cdots, v_n$ being linear dependent (**2 points**).

Vectors $v_1, v_2, \cdots, v_n$ are called linear independent if the equation $\sum_{i=1}^n a_i v_i$ has only trivial solution $a_1 = a_2 = \cdots = a_n = 0$ and they are called linear dependent if the equation has a nontrivial solution, that is at least one coefficient $a_i \neq 0$.

(b) Prove that $v_1, \cdots, v_n$ are linear dependent if (**3 points**) and only if (**3 points**) at least one vector from $v_1, \cdots, v_n$ can be represented as a linear combination of the other vectors.

If $v_1, \cdots, v_n$ are linear dependent, then equation $\sum_{i=1}^n a_i v_i = 0$ has a nontrivial solution. Assume $a_i \neq 0$ then $a_i v_i = -\sum_{j\neq i} a_j v_j$ and hence $v_i = -\sum_{j\neq i} \frac{a_j}{a_i} v_j$ that is we write v_i as a linear combination of other vectors. Conversely, if $v_i = -\sum_{j\neq i} \frac{a_j}{a_i} v_j$, then there is a nontrivial solution $(a_1, a_2, \cdots, a_{i-1}, -1, a_{i+1}, \cdots, a_n)$ to the equation $sum_{i=1}^n a_i v_i = 0$.

(c) Circle answers for following questions. (**1 point** each)

i. $\{\emptyset\}$ is the smallest vector space. **True** **False**
This is false because every vector space must have the zero vector by definition.

ii. Zero vector $\{0\}$ is linear independent. **True** **False**
False

iii. Zero vector space has no basis. **True** **False**
$\{\emptyset\}$ is the basis for $\{0\}$.

iv. Considering $\mathbb{C}$ as a vector space over $\mathbb{R}$, are vectors 1 and i linear dependent? **Yes** **No**
No

v. Considering vector space R^3, are vectors $[\pi, 1, 7]$, $[1, \sqrt{2}, 3]$, $[-1, e, 2]$, and $[-\log(1), 1, 1]$ linear dependent? **Yes** **No**
Yes, we have four vectors in a three dimensional space. They must be linear dependent.

Problem 2 (6 points). Let $a \in R$ be a constant and consider following subsets of $\mathscr{F}(R, R)$ parameterized by a

$$S_a = \left\{ f \in \mathscr{F}(R,R) : f(0) = a \right\} \subset \mathscr{F}(R,R).$$

(a) Show S_0 is a subspace of $\mathscr{F}(R, R)$. (**5 points**)

S_0 contains functions from R to R that satisfy $f(0) = 0$. The zero vector in $\mathscr{F}(R,R)$ is the zero function Z. That is $Z(x) = 0$ for all $x \in R$. In particular, $Z(0) = 0$, hence $Z \in S_0$. Let $f, g \in S_0$, then $(f+g)(0) = f(0) + g(0) = 0$ and hence $f + g \in S_0$. Similarly, $(af)(0) = af(0) = 0$, $af \in S_0$. Therefore S_0 is a subspace of $\mathscr{F}(R,R)$.

(b) Give a one-line proof that S_a is **NOT** a subspace of $\mathscr{F}(R,R)$ for $a \neq 0$. (**1 point**)

The zero function Z is not in S_a for $a \neq 0$.

Problem 3 (10 points). Consider **consistent** linear system $Ax = b$ with augmented matrix

$$[A|b] = \begin{bmatrix} -7 & -35 & 9 & -13 & c \\ 1 & 5 & -1 & 1 & 6 \\ 3 & 15 & -2 & 0 & 19 \end{bmatrix}$$

where c is an unknown number.

(a) Find a matrix B that is row equivalent to $[A|b]$ and is in reduced row echelon form. Label any row operations you preform. Please note that at some stage the unknown c can be decided by the **consistency** assumption. (**5 points**)

$[A|b]$ is row equivalent to

$$\begin{bmatrix} 1 & 5 & 0 & -2 & 7 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & c+40 \end{bmatrix}$$

By consistency assumption $c = -40$. Hence

$$B = \begin{bmatrix} 1 & 5 & 0 & -2 & 7 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) The value of the unknown c is -40 . (**2 points**)

(c) Parameterize the linear system and write down the solution space. (**3 points**)
The linear system corresponding to B reads

$$x_1 = -5x_2 + 2x_4 + 7, \quad x_3 = 3x_4 + 1.$$

The solution space is then $[7\ 0\ 1\ 0]^t + span\{[-5\ 1\ 0\ 0]^t, [2\ 0\ 3\ 1]^t\}$.

Problem 4 (13 points). Let V denote the set of all solutions to the system of linear equations

$$x_1 - x_2 + 2x_4 - 3x_5 + x_6 = 0$$
$$2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 = 0$$

(a) Show that $S = \{[0,-1,0,1,1,0], [1,0,1,1,1,0]\}$ is a linearly independent subset of V. (**2 points** for linear independency and **2 points** for "subset")

To prove linear independence, solve linear equation

$$[0,-1,0,1,1,0]a + [1,0,1,1,1,0]b = [0,0,0,0,0,0].$$

and we find this yields $a = b = 0$. To prove subset, need to plug in equation and see if the two vectors solve both of the equations.

(b) Extend S to a basis for V. To do this, you may first find a basis for V by reducing the system to row echelon form (**3 points**) and then consider using replacement theorem to replace two vectors from this basis with the two vectors in S (**3 points**).

The row reduce echelon form is

$$\begin{bmatrix} 1 & 0 & -1 & 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & -1 & 2 & 2 & 0 \end{bmatrix}$$

Hence V is spanned by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and we can extend S to

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

which also form a basis for V.

(c) Fill in the blanks. (**1 point each**)

i. The number of dependent variable(s) of the linear system is 2 .

ii. The number of free variable(s) of the linear system is 4 .

iii. The dimension of V is 4 .

Problem 5 (6 points). Let $\{u, v, w\}$ be a basis for V, show that

(a) Show that $\{u+w, v, u-v+w\}$ is **NOT** a basis for V. (**3 points**)
We find $u+w-v = u-v+w$, that is $u+w, v, u-v+w$ are linear dependent hence not a basis.

(b) Show that $\{u+w, v, u-v+2w\}$ is a basis for V. (**3 points**)
$u+w, v, u-v+2w \in V$ hence $span(\{u+w, v, u-v+2w\}) \subset V$. also we find $\{u+w, v, u-v+2w\}$ is linear independent. Because $dim(V) = 3$, $\{u+w, v, u-v+2w\}$ is then a basis for V.

Extra credit problem (10 points). Let V be a real vector space of all real infinite sequences

$$V = \left\{(a_1, a_2, \cdots) : a_i \in R\right\}.$$

Consider the subset U of V consisting of all sequences satisfy the linear recurrence relation

$$a_{k+2} - 5a_{k+1} + 3a_k = 0, \quad \text{for } k = 1, 2, \cdots.$$

(a) Show that U is a subspace of V. (**5 points**)

i. $(0, 0, 0, \cdots) \in U$.

ii. suppose $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ are in U, then their sum is $(a_1+b_1, a_2+b_2, \cdots)$ for any $k \geq 1$

$$(a_{k+2}+b_{k+2})-5(a_{k+1}+b_{k+1})+3(a_k+b_k) = a_{k+2}-5a_{k+1}+3a_k+b_{k+2}-5b_{k+1}+3b_k = 0$$

hence their sum is also in U.

iii. Similarly, we can verify $a(a_1, a_2, \cdots) \in U$.

(b) Show that U is finite-dimensional and find a basis for U. (**5 points**)

Idea: For $k \geq 3$, a_k is determined by $a_k = 5a_{k-1} - 3a_{k-2}$. That is the whole sequence is settled down after one chooses (a_1, a_2). Indeed, U is isomorphic to F^2 hence finite dimensional and let $(a_1 = 1, a_2 = 0)$ and $(b_1 = 0, b_2 = 1)$ we have a basis

$$\{(1, 0, -3, 15, \cdots), (0, 1, 5, 12, \cdots)\}$$

for U.