Abstract Linear Algebra **Midterm exam 3** Nov. 20th, 2019

First name:________________ **Last name:**________________

Problem 1 (12 points). For the matrix

$$A = \begin{bmatrix} 3 & -3 & -1 \\ 2 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

(a) compute the eigenvalues of matrix A and their corresponding algebraic multiplicities. **(5 points)**

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 3-\lambda & -3 & -1 \\ 2 & -2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)\det\left(\begin{bmatrix} 3-\lambda & -3 \\ 2 & -2-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)\left((3-\lambda)(-2-\lambda)+6\right)$$

$$= -(1-\lambda)^2\lambda$$

$\lambda=0$ is an eigenvalue of A with multiplicity 1

$\lambda=1$ is an eigenvalue of A with multiplicity 2

(b) compute the corresponding eigenspaces of matrix A **(6 points)** and decide if A is diagonalizable **(1 point)**.

E_1 : $A - I = \begin{bmatrix} 2 & -3 & -1 \\ 2 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ reduced row echelon form

is $\begin{bmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $E_1 = N(A-I) = \text{span}\left\{\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\right\}$

$\dim(E_1) = 2$ = algebraic multiplicity of eigenvalue 1

E_0 = $A - 0 \cdot I = \begin{bmatrix} 3 & -3 & -1 \\ 2 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ reduced row echelon form is

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $E_0 = N(A) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}$

$\dim(E_0) = 1$ = algebraic multiplicity of eigenvalue 0

A is diagonalizable

Problem 2 (4 points). Let $A = S\Lambda S^{-1}$ where

$$S = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Compute A^n (n is an arbitrary positive integer).

$$S^{-1} = \frac{1}{\det(S)} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$A^n = (S\Lambda S^{-1})^{n} = S\Lambda^n S^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2^n & \\ & 3^n \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2^n & 2\cdot 3^n \\ 2\cdot 2^n & 3\cdot 3^n \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -3\cdot 2^n + 4\cdot 3^n & 2\cdot 2^n - 2\cdot 3^n \\ -6\cdot 2^n + 6\cdot 3^n & 4\cdot 2^n - 3\cdot 3^n \end{bmatrix}$$

Problem 3 (12 points). Consider the inner product space $\mathbb{R}^4$ associated with the standard inner product $\langle x, y\rangle = y^t x$. Let W be a subspace of $\mathbb{R}^4$ spanned by column vectors $[\,1 \;\; 1 \;\; 0 \;\; 0\,]^t$ and $[\,0 \;\; 1 \;\; 1 \;\; 0\,]^t$.

(a) Find a generating set for $W^\perp$. (4 points)

$$W^\perp = \{ x \in R^4 : \langle x, y\rangle = 0 \quad \forall\, y \in W \}$$

$$x \in W^\perp \iff \begin{cases} \left\langle x, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle = 0 \\ \left\langle x, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\rangle = 0 \end{cases} \iff \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{} \overset{x}{} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = -x_2$
$x_3 = -x_2$
x_4 arbitrary

$$\Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ -x_2 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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$$W^\perp = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) Find orthonormal bases for W and $W^{\perp}$ respectively by Gram-Schmidt process. **(4 points)**

① For W let $v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}$ $\hat{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$

let $v_2 = \frac{\hat{v}_2}{\|\hat{v}_2\|} = \frac{1}{\sqrt{\frac{3}{2}}} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ $\{v_1, v_2\}$ is an orthonormal basis for W

②. For $W^{\perp}$ we find $\begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \perp \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ hence only need to normalize these two vectors and we have $\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis for $W^{\perp}$

(c) Find the vector in W that is closest to the vector $[\, 1 \;\; 1 \;\; 1 \;\; 1 \,]^t$. **(4 points)**

only need to project $[1 \; 1 \; 1 \; 1]^t$ onto W

the vector by Thm 6.6 is $\langle [1 \; 1 \; 1 \; 1]^t, [\frac{1}{\sqrt{2}} \; \frac{1}{\sqrt{2}} \; 0 \; 0] \rangle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}$

$+ \langle [1 \; 1 \; 1 \; 1]^t, \frac{\sqrt{2}}{\sqrt{3}} [-\frac{1}{2} \; \frac{1}{2} \; 1 \; 0] \rangle \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \cdot \sqrt{\frac{2}{3}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix}$

Problem 4 (6 points). Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be **distinct** eigenvalues of a linear operator T and $v_1, v_2, \cdots, v_k$ be corresponding eigenvectors. By Theorem 5.5, we know $v_1, v_2, \cdots, v_k$ are linearly independent. Use this fact to show that $v_1 + v_2 + \cdots + v_k$ is **not** an eigenvector of T.

If $v_1 + \cdots + v_k$ is an eigenvector of T

then $T(v_1 + \cdots + v_k) = \lambda (v_1 + \cdots + v_k)$ for some λ

but $T(v_1 + \cdots + v_k) = Tv_1 + Tv_2 + \cdots + Tv_k = \sum_{i=1}^{k} \lambda_i v_i$

we then have $\sum_{i=1}^{k} (\lambda_i - \lambda) v_i = 0$ by linear independency of $\{v_1, \cdots, v_k\}$ then $\lambda = \lambda_1 = \cdots \lambda_k$ $\rightarrow\leftarrow$ with the assumption that $\lambda_1 \cdots \lambda_k$ are distinct

Problem 5 (12 points). Consider the inner product space $V = M_{2\times 2}(\mathbb{R})$ associated with the Frobenius inner product $\langle A, B\rangle = trace(B^t A)$. We have verified that the standard ordered basis $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is also an orthonormal basis for V.

(a) Let $f : V \to \mathbb{R}$ be the map $f\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{12} + a_{21}$. Show that f is linear, hence a linear functional **(2 point)**. Find the matrix M such that $f(\cdot) = \langle \cdot, M\rangle$ **(5 points)**.

by Remark of Thm 6.8 $M = \sum_{i=1}^{4} \overline{f(E_i)}\, E_i$

hence $M = f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + f\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$+ f\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + f\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) For $A \in V$, define $T_A : V \to V$ by $T_A(X) = AX$. Verify that $(T_A)^* = T_{A^t}$. **(5 points)**

We need to verify $\langle T_A(X), Y\rangle = \langle X, T_{A^t}(Y)\rangle$ for all $X, Y \in M_{2\times 2}(\mathbb{R})$ but $\langle T_A(X), Y\rangle = \mathrm{trace}(Y^t A X)$

$= \mathrm{trace}(Y^t (A^t)^t X) = \mathrm{trace}((A^t Y)^t X) = \langle X, A^t Y\rangle$

$= \langle X, T_{A^t}(Y)\rangle$

Problem 6 (4 points). Let $A \in M_{n\times n}(R)$. Circle answers for following questions. **(1 point each)**

(a) A is invertible if and only if zero is not an eigenvalue of A. **True** ✓ **False**

(b) If the characteristic polynomial of A splits, then A is diagonalizable. **True** **False** ✓

(c) Let λ be an eigenvalue of A. There could be no eigenvector corresponding to λ. **True** **False** ✓

(d) If A is diagonalizable, then A is invertible. **True** **False** ✓

Extra credit problem (10 points). Let V be a finite dimensional inner product space and T be a linear transformation on V. Let N be the null space of T. Prove that the range of T^* is equal to the orthogonal complement of N. That is $\mathscr{R}(T^*) = N^{\perp}$.
(Hint: first prove part (a) then either (b1) or (b2))

(a) Show that $\mathscr{R}(T^*) \subset N^{\perp}$. **(5 points)**

Let $y \in \mathcal{R}(T^*)$ then $\exists\ x \in V$ $y = T^* x$

Now $\forall\ z \in N$ we have $\langle y, z \rangle = \langle T^* x, z \rangle = \overline{\langle z, T^* x \rangle}$

$= \overline{\langle Tz, x \rangle} = \overline{\langle 0, x \rangle} = 0$ hence $y \in N^{\perp}$

(b1) Show that $dim(\mathscr{R}(T^*)) = dim(N^{\perp})$. **(5 points)**

i) $V = N \oplus N^{\perp} \Rightarrow \dim V = \dim(N) + \dim(N^{\perp})$

ii) By dimension thm $\dim V = \dim(N) + \dim(\mathcal{R}(T))$

iii) $\dim(\mathcal{R}(T)) = \text{rank}(T) = \text{rank}\, T^* = \dim(\mathcal{R}(T^*))$

By i) ii) iii) $\dim(\mathcal{R}(T^*)) = \dim(N^{\perp})$

(b2) Show that $N^{\perp} \subset \mathscr{R}(T^*)$ by proving that $\mathscr{R}(T^*)^{\perp} \subset N$. **(5 points)**

let $y \in \mathcal{R}(T^*)^{\perp}$ then $\langle y, T^*(x) \rangle = 0$ $\forall\ x \in V$

$\Rightarrow \langle T(y), x \rangle = 0$ $\forall\ x \in V$ in particular, take $x = T(y)$.

then $\|T(y)\| = 0 \Rightarrow T(y) = 0 \Rightarrow y \in N$