Bayesian Hierarchical Dynamic Factor Models

Anthony M. Thomas, Jr.
anthony.thomas3@mavs.uta.edu

Department of Mathematics
The University of Texas at Arlington

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Section 1

Factor Analysis

1. **Factor Analysis** is a method that uses the covariances between a set of observed variables to described them in terms of a smaller set of unobservable variables called factors.

Bayesian Inference

Bayesian Inference can be described by three parts:

1. Build a model based on data $\boldsymbol{X}$ and parameters $\boldsymbol{\Theta}$
   - Likelihood: $p\left(\boldsymbol{X}|\boldsymbol{\Theta}\right)$
   - Prior: $p\left(\boldsymbol{\Theta}\right)$
2. Compute the posterior pdf
   - Posterior:

$$p\left(\boldsymbol{\Theta}|\boldsymbol{X}\right) = \frac{p\left(\boldsymbol{X}|\boldsymbol{\Theta}\right)p\left(\boldsymbol{\Theta}\right)}{p\left(\boldsymbol{X}\right)}$$

3. Report summaries
   - Posterior mean: $\mathrm{E}\left[\boldsymbol{\Theta}|\boldsymbol{X}\right]$
   - Posterior (co)variances: $\mathrm{Var}\left[\boldsymbol{\Theta}|\boldsymbol{X}\right]$

Section 2

Why Variational Bayesian Inference?

Consider a model with data $\boldsymbol{X}$ and unknowns $\boldsymbol{Z}$. The goal is to compute the posterior

$$p\left(\boldsymbol{Z}|\boldsymbol{X}\right) = \frac{p\left(\boldsymbol{X}|\boldsymbol{Z}\right)p\left(\boldsymbol{Z}\right)}{p\left(\boldsymbol{X}\right)} \tag{1}$$

and report posterior means and covariances.

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Why Variational Bayesian Inference?

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2. The gold standard is to use MCMC

Why Variational Bayes?

MCMC methods allow sampling from intractable distributions, but takes too long to run. VB methods are faster.

Variational Bayes

VB turns the inference problem into an

Section 3

Classical Factor Analysis Model

The Classical (orthogonal) FA model assumes assumes the form

$$\boldsymbol{X} = \boldsymbol{\Lambda} \boldsymbol{F} + \boldsymbol{e}$$

where

1. $\boldsymbol{X} = (X_1, \ldots, X_N)^\top$ denotes the vector of observations
2. $\boldsymbol{\Lambda} = [\lambda_{nk}]_{N \times K}$ denotes the matrix of factor loadings
3. $\boldsymbol{F} = (F_1, \ldots, F_K)^\top$ denotes the vector of latent factors
4. $\boldsymbol{e} = (e_1, \ldots, e_N)^\top$ denotes the vector of latent error terms

Normal Theory Assumptions

The Normal Theory Classical FA model assumes the form

$$\boldsymbol{X} = \boldsymbol{\Lambda}\boldsymbol{F} + \boldsymbol{e}$$

and adds the assumptions that

1. $\boldsymbol{F} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_K)$
2. $\boldsymbol{e} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ where $\boldsymbol{\Sigma} = \mathrm{diag}\left(\sigma_1^2, \ldots, \sigma_N^2\right)$
3. F_k and e_n are independent for every pair k, n

Section 4

Hierarchical Dynamic Factor Model

$$\mathbf{X}_{bst} = \mathbf{\Lambda}_{H.bs}(L)\mathbf{H}_{bst} + \mathbf{e}_{Xbst} \quad (3)$$

$$\mathbf{H}_{bst} = \mathbf{\Lambda}_{G.bs}(L)\mathbf{G}_{bt} + \mathbf{e}_{Hbst} \quad (4)$$

$$\mathbf{G}_{bt} = \mathbf{\Lambda}_{F.b}(L)\mathbf{F}_{t} + \mathbf{e}_{Gbt} \quad (5)$$

$$\mathbf{\Psi}_{F}(L)\mathbf{F}_{t} = \boldsymbol{\epsilon}_{Ft}, \quad (6)$$