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Spectral geometry: bridge to gravity

When a Dirac spectrum encodes both spacetime and octonions

Key Insight. The central object of this sheet is a single operator, the Dirac operator D. In spectral geometry one defines a "spectral action"

$$S_{\text{spec}} = \operatorname{Tr} f(D^2/\Lambda^2),$$

and shows that its expansion produces terms of Einstein–Hilbert type, cosmological-constant type and gauge-kinetic type. This offers a concrete way to connect an internal Dirac matrix D_F from the octonionic construction with an effective spacetime geometry, without adding a separate gravitational action by hand.

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This sheet shifts the focus from the detailed octonionic construction back to a single unifying object: the Dirac operator. We briefly recall how a geometric Dirac operator encodes the metric of spacetime, how the spectral action principle turns its spectrum into an effective action, and how a finite internal matrix can be coupled to it. In this way, spectral geometry provides a precise framework in which internal structures can be tested against gravitational and gauge dynamics.

Dirac operators in ordinary geometry

In the standard (continuum) description of gravity one starts from a metric $g_{\mu\nu}$ on a four-dimensional manifold. From $g_{\mu\nu}$ one builds a Levi-Civita connection, curvature tensors and finally the Einstein–Hilbert action. Fermions enter through a *Dirac operator* D_M, constructed from $g_{\mu\nu}$ and the spin connection.

Concretely, D_M is a first-order differential operator of the schematic form

$$D_M = i\,\gamma^\mu \nabla_\mu,$$

where ∇_μ is the covariant derivative determined by the metric and γ^μ are gamma matrices adapted to $g_{\mu\nu}$. The metric enters D_M twice: through the choice of γ^μ and through the connection in ∇_μ. Thus, D_M can be viewed as a compact encoding of the metric data.

Spectral geometry turns this observation around: instead of viewing D_M as derived from $g_{\mu\nu}$, it asks how much of $g_{\mu\nu}$ and its curvature can be *recovered* from the spectrum of D_M. The answer is: a lot. Certain standard coefficients $a_n(D_M^2)$, the Seeley–DeWitt coefficients, carry integrated information about volume and curvature.

For our purposes we only need three facts:

- $a_0(D_M^2)$ is proportional to the total volume,
- $a_2(D_M^2)$ contains the scalar curvature R,
- $a_4(D_M^2)$ contains combinations of R^2 and, if gauge fields are included, terms like $F_{\mu\nu}F^{\mu\nu}$.

The spectral action and its structure

The spectral action principle takes the Dirac operator D as the primary input and defines

$$S_{\text{spec}} = \operatorname{Tr} f(D^2/\Lambda^2),$$

where Λ is a large mass scale and f is a fixed, positive test function. The trace is taken over the Hilbert space on which D acts.

If D is of the usual geometric form built from $g_{\mu\nu}$, one can express S_{spec} as an integral over spacetime. In four dimensions this yields schematically

$$S_{\text{spec}} \sim \int d^4x\,\sqrt{|g|}\,\Big(\underbrace{\Lambda^4\,a_0(D^2)}_{\text{vacuum / cosmological}} + \underbrace{\Lambda^2\,a_2(D^2)}_{\text{Einstein–Hilbert type}} + \underbrace{a_4(D^2)}_{\substack{\text{curvature}^2\\ \text{and gauge}}} + \cdots \Big).$$

Several points are important:

- the *same* operator D controls all three sectors: cosmological term, Einstein–Hilbert term and gauge kinetic terms,
- the organisation into these sectors is automatic once D is given,
- no separate gravitational Lagrangian is added by hand; it appears as part of the expansion of S_{spec}.

The precise numerical coefficients depend on f and on the detailed form of D, but the qualitative structure is robust.

Adding a finite internal space

So far D only knew about spacetime. To include internal quantum numbers one enlarges the Hilbert space from pure spinors to

$$\mathcal{H} = L^2(\text{spinors on } M) \otimes \mathcal{H}_F,$$

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where $\mathcal{H}_F$ is a finite-dimensional Hilbert space. All internal labels (gauge charges, generation indices, etc.) live in $\mathcal{H}_F$.

The Dirac operator is then modified to

$$D = D_M \otimes \mathbf{1} + \gamma^5 \otimes D_F.$$

Here:

- D_M is the geometric Dirac operator built from $g_{\mu\nu}$,
- D_F is a finite matrix acting on $\mathcal{H}_F$,
- γ^5 is the chirality matrix: it distinguishes left-handed from right-handed spinor components.

The factor γ^5 appears for a concrete reason. In realistic models the internal symmetries act differently on left- and right-handed fermions. Placing D_F next to γ^5 ensures that this chiral pattern is correctly reflected when D is squared and inserted into S_{spec}.

For the Standard Model there is a specific choice of $(\mathcal{H}_F, D_F)$ for which the leading terms in the expansion of S_{spec} reproduce the known gauge group and fermion content. This shows that a *finite* internal matrix D_F can already carry realistic particle-physics information once it is coupled to D_M in this way.

Octonionic data as input for an internal Dirac operator

In the octonionic part of this calendar we have already built a rich internal structure:

- an exceptional internal space based on $H_3(\mathbb{O})$,
- rotor and compressor operators acting on the states of a generation,
- mass maps such as $\Pi(\langle H \rangle)$.

Later sheets introduce additional invariants (for example universe parameters (p_S, p_G, p_T)) that classify sectors of this internal algebra.

From the viewpoint of spectral geometry, these objects are natural candidates for the blocks of a finite internal Dirac operator D_F:

- combinations of rotors and compressors suggest gauge-covariant parts of D_F,
- mass maps provide off-diagonal blocks that mix left- and right-handed components in D_F,
- sector invariants indicate which combinations of internal operators could appear in the coefficients $a_n(D^2)$ and thus influence effective coupling strengths in S_{spec}.

This sheet does not attempt to write down a final D_F. Its aim is more modest and more precise:

- given any explicit choice of D_F built from the octonionic operator toolbox, spectral geometry provides a clear recipe for constructing a combined operator

  $$D = D_M \otimes \mathbf{1} + \gamma^5 \otimes D_F$$

  and for computing the corresponding spectral action,
- the resulting S_{spec} automatically organises terms into cosmological, Einstein–Hilbert and gauge-kinetic parts.

The octonionic model supplies the internal matrix; the spectral framework supplies the rule "turn D into an action".

Summary of the bridge

The bridge built here has three steps:

1. A Dirac operator D_M compactly encodes a spacetime metric $g_{\mu\nu}$ and its curvature.
2. Enlarging D_M to $D = D_M \otimes \mathbf{1} + \gamma^5 \otimes D_F$ allows a finite internal matrix D_F to carry gauge and mass information.
3. The spectral action $S_{\text{spec}} = \operatorname{Tr} f(D^2/\Lambda^2)$ turns the spectrum of this combined Dirac operator into a unified expression whose expansion contains both gravitational and gauge-kinetic terms.

In this sense, the spectral-geometry framework gives a precise target: if the octonionic construction can produce a concrete internal Dirac matrix D_F, then there is a mathematically well-defined way to test how its spectrum feeds into an effective action that already knows about gravity.

References

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[2] A. H. Chamseddine, A. Connes and M. Marcolli, "Gravity and the standard model with neutrino mixing," *Adv. Theor. Math. Phys.* **11**, 991–1089 (2007).

The Dirac operator links spacetime and internal physics. Spectral geometry uses a combined Dirac operator to automatically generate an action that includes both gravity and gauge forces. Our octonionic model offers the internal part; spectral action shows how its properties shape the universe.

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