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Three Generations from Triality

One internal block, three consistent readings

Key Insight. The octonionic model does not postulate three generations by hand. Instead, it starts from a single eight-dimensional internal block with Spin(8) triality. The three triality-related irreducible representations — vector, left-handed spinor, right-handed spinor — are read as three coherent ways of organizing the same internal data. When this structure is embedded into the Albert algebra, it naturally unfolds into three fermion generations with correlated mass and mixing patterns.

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The Standard Model contains three generations of quarks and leptons. They have identical gauge quantum numbers but very different masses and mixings. From the perspective of the usual gauge-group story this is a mystery: why not one generation, or five, or an arbitrary number?

Most approaches simply accept three generations as an experimental fact and add family indices. The octonionic model takes a different route. It asks whether the number three could arise from the internal representation theory of the exceptional structures that underlie the model.

The key player is the triality of Spin(8) and its embedding into the Albert algebra $H_3(\mathbb{O})$. In the triality picture, the number three is not an adjustable model choice but a built-in feature of Spin(8): there are exactly three triality-related 8-dimensional representations, no more and no fewer.

Spin(8) and triality

The group Spin(8), the double cover of SO(8), has an exceptional property: it possesses three inequivalent eight-dimensional irreducible representations,

$$V_8, \qquad S_8^+, \qquad S_8^-,$$

usually called the vector, left-handed spinor, and right-handed spinor representations. An outer automorphism of Spin(8) permutes these three representations. This is the triality symmetry.

All three irreducible 8-dimensional representations of Spin(8) can be realised on copies of the octonions $\mathbb{O}$; what differs between V_8, S_8^+ and S_8^- is not the underlying vector space, but the way Spin(8) acts on it. This is precisely what makes the picture of a single internal eight-dimensional block with three distinct readings natural.

In abstract group theory, triality is often presented as a curiosity of Dynkin type D_4. In the present model it is taken seriously as structural input:

- We start from *one* internal eight-dimensional block, not three unrelated ones.
- We insist that this block can be read in three coherent ways: as V_8, as S_8^+, and as S_8^-.
- The consistency of these three readings constrains how internal operators can act.

This "one block, three readings" principle will later be matched to the three observed fermion generations.

From octonions to the Albert algebra

The internal degrees of freedom of the model are organised using the Albert algebra $H_3(\mathbb{O})$, the Jordan algebra of 3×3 hermitian octonionic matrices. Its automorphism group is the exceptional Lie group F_4:

$$F_4 \;=\; \mathrm{Aut}\big(H_3(\mathbb{O})\big).$$

A convenient picture is to view each diagonal entry as an octonionic "slot". A generic element of $H_3(\mathbb{O})$ can be written as

$$H \;=\; \begin{pmatrix} h_1 & x_{12} & x_{13} \\ \bar{x}_{12} & h_2 & x_{23} \\ \bar{x}_{13} & \bar{x}_{23} & h_3 \end{pmatrix},$$

with $h_i \in \mathbb{O}$ on the diagonal and $x_{ij} \in \mathbb{O}$ off diagonal. In the triality-based reading, the three h_i play asymmetric but tightly related roles: one is assigned a vector structure, the others carry left- and right-handed spinor structures. The three fermion generations will ultimately be read off from these three correlated diagonal directions.

Within $H_3(\mathbb{O})$, this picture can be summarised more schematically:

- One diagonal octonion slot is associated with the vector representation.
- The second diagonal slot carries a left-handed spinor structure.
- The third diagonal slot carries a right-handed spinor structure.

Triality then manifests itself as a structured permutation of these roles, implemented by elements of F_4 that reshuffle the internal directions in a controlled way.

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Reading generations from triality sectors

When we couple the internal $H_3(\mathbb{O})$ structure to the spacetime Dirac operator and the mass map $\Pi(H)$, each triality sector produces a family of fermionic modes. Schematically:

- The *vector-like* reading of the internal block organises one set of states with a characteristic mass pattern.
- The *left-handed spinor* reading gives rise to a second set of states, with masses related but not identical to the first set.
- The *right-handed spinor* reading yields a third set, again correlated but distinct.

These three correlated sets are identified with the three fermion generations. In this way, the three triality-related sectors can be associated with the observed triplets (e, μ, τ), (u, c, t) and (d, s, b) (and their neutrino partners), not as three unrelated copies, but as three correlated readings of the same internal block. The crucial point is not only that there are exactly three representations, but that they are related by an *outer* automorphism: they are three faces of one internal object, not three arbitrary copies.

Constraints on mass and mixing patterns

Because the three generations arise from a single internal block with triality symmetry, their mass and mixing parameters cannot be chosen independently. Several qualitative features follow:

1. **Hierarchy:** The eigenvalues of the mass map $\Pi(H)$, when restricted to the three triality-related sectors, typically split into bands with a built-in hierarchy. This echoes the observed pattern of light, medium, and heavy generations.
2. **Mixing structure:** The allowed off-diagonal couplings between triality sectors are constrained by the Jordan structure of $H_3(\mathbb{O})$ and the embedding of the Standard Model gauge group. This shapes the form of the CKM and PMNS matrices.
3. **Stability:** Because the three generations share a common internal origin, small deformations of the vacuum configuration $\langle H \rangle$ tend to move all three in a correlated way, rather than producing arbitrary new families.

The goal is not to "explain every digit" of the mass spectrum, but to show that three generations with a hierarchical and mixing-rich structure are the *natural* outcome of the exceptional geometry.

Comparison with ad hoc family replication

In more conventional settings, one starts from a gauge group and then adds three copies of the fermion content:

$$\psi \longrightarrow (\psi^{(1)}, \psi^{(2)}, \psi^{(3)}),$$

with a family index labelling generations. This move works phenomenologically, but it does not tell us why there are three copies, nor why their masses are ordered the way we see them.

The octonionic approach reverses the logic:

- Start from a single octonionic/Spin(8) block with triality.
- Embed it into an exceptional algebra ($H_3(\mathbb{O})$ with F_4 symmetry).
- Let the internal mass map and vacuum pick out three correlated triality sectors.

Family replication is not assumed; it is read off from the structure of the internal algebra.

References

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Three generations appear as three triality-related faces of a single exceptional internal block, not as three arbitrary copies.

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