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Heptagon operator: seven directions, one spectrum

From seven imaginary units to three eigenvalues (α, β, γ)

Key Insight. The seven imaginary octonion units can be arranged on a heptagon that encodes their multiplication rules. From this data one constructs the *heptagon operator* H_7, a single linear operator whose eigenvalues (α, β, γ) summarise the internal geometry. These three numbers are not free parameters; they are geometric invariants that will later feed into coupling constants and mixing angles. Today we introduce H_7 and its spectrum as a compact fingerprint of the sevenfold structure of $\mathbb{O}$.

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Seven imaginary units on a heptagon

Octonions $\mathbb{O}$ have seven imaginary units $e_1, \ldots, e_7$. Their multiplication can be depicted on an oriented heptagon: each directed edge (and certain chords) carries a triple (e_i, e_j, e_k) with

$$e_i e_j = e_k, \qquad e_j e_k = e_i, \qquad e_k e_i = e_j,$$

and reversed order introduces a minus sign. This Fano-type diagram is more than a mnemonic; it packages the nonassociative multiplication table into a single geometric picture.

Constructing the heptagon operator

From the seven units e_i and the heptagon geometry, one defines a linear operator H_7 on the underlying real 8-dimensional space. Schematic example:

$$H_7 = \sum_{i=1}^{7} c_i \, L_{e_i},$$

where $L_{e_i} : x \mapsto e_i x$ denotes left multiplication by e_i, and the coefficients c_i are chosen to respect the heptagon symmetry (for instance, equal on edges belonging to the same orbit under G_2).

The important facts are:

- H_7 encodes all seven directions in a single operator.
- H_7 is constrained by G_2 symmetry, so its spectrum is highly structured.

Details of the precise coefficients are not needed on this page; they are spelled out in the technical notes.

Spectrum: three eigenvalues (α, β, γ)

A key feature of H_7 is that its eigenspectrum is much simpler than an arbitrary 8×8 matrix would allow. In favourable constructions,

$$\mathrm{spec}(H_7) = \{\alpha, \beta, \gamma\},$$

with appropriate multiplicities that add up to 8. The numbers

$$(\alpha, \beta, \gamma)$$

are then *heptagon eigenvalues*: geometric invariants that capture how the seven imaginary directions are arranged relative to each other.

Qualitatively:

- α may correspond to an eigen-subspace aligned with a particular family of edges.
- β and γ correspond to other symmetry-related subspaces.

The crucial point: instead of seven arbitrary numbers, we get only three characteristic eigenvalues.

Why (α, β, γ) matter for physics

Later in the calendar, (α, β, γ) will reappear in several contexts:

- In the construction of the *radius operator* R and its spectrum (a_0, b_0, c_0) (5 December).
- In geometric expressions for coupling constants, such as the fine-structure constant α and the strong coupling α_s.
- In defining angles between rotor directions that enter the Weinberg angle θ_W.

In other words, the heptagon eigenvalues are a compact *seed* from which many later observables can be grown. They serve as a small set of internal numbers that eventually manifest as physical constants.

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Heptagon operator within the operator toolbox

On the second Advent Sunday (7 December), the calendar will present a full *operator toolbox*:

- heptagon operator H_7 with eigenvalues (α, β, γ),
- radius operator R with radii (a_0, b_0, c_0),
- sign/signature operators,
- rotors (antisymmetric generators of forces),
- compressors (symmetric mass and mixing operators).

The heptagon operator is the first piece of this toolbox to appear explicitly. Its role is to turn the combinatorial data "seven imaginary units on a heptagon" into spectral data that can be plugged into operators, potentials and eventually quantitative formulas.

Conceptual gain from H_7

Compared to working directly with seven basis elements e_i, the heptagon-operator viewpoint offers:

1. **Compression:** seven directions are summarised by three eigenvalues.
2. **Invariants:** (α, β, γ) are invariant under G_2-compatible re-labellings of the heptagon—they are not artefacts of a particular basis choice.
3. **Spectral language:** we move from basis-dependent multiplication tables to basis-independent spectral data, which is the natural language for later spectral geometry.

Thus, H_7 is the first place where the octonionic multiplication table starts to look like something a physicist would recognise as "spectral parameters".

References

[1] J. C. Baez, "The octonions," *Bull. Amer. Math. Soc.* **39**, 145–205 (2002).

[2] C. Furey, "Charge quantization from a number operator," *Phys. Lett. B* **742**, 195–199 (2015).

[3] [Internal notes on the heptagon operator and its spectrum: `chap03_neu.tex; appK_neu.tex; oktonionen-basen.tex`.]

The heptagon operator H_7 compresses the seven imaginary directions of $\mathbb{O}$ into three eigenvalues (α, β, γ). These invariants are the first internal numbers that will later reappear in couplings, scales and mixing angles.

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