December 7, 2025 (Second Advent Sunday)

Operator toolbox

The rotor/compressor toolbox for all observables

Key Insight. In the octonionic model, all observable quantities — couplings, masses, mixings and scales — are traced back to invariants of two operator families on the internal space: antisymmetric *rotors* (forces) and symmetric *compressors* (masses and mixings). The heptagon operator encodes the seven imaginary octonion directions; the radius operator R with spectrum (a_0, b_0, c_0) defines an attractor mechanism for fundamental scales. Together they form a minimal but sufficient operator toolbox: nothing else is added by hand.

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We've spent the first week building the stage: octonions, their automorphism group G_2, triality, and the idea that internal degrees of freedom live in an eight-dimensional exceptional space. But a stage is not a play. To make physical predictions — to connect this abstract geometry to the numbers we measure in experiments — we need actors. In the octonionic model, those actors are operators.

Today marks the Second Advent Sunday, and with it a shift in focus. We move from "what is the internal space?" to "what do we do with it?" The answer is surprisingly compact: all observable quantities — coupling constants, particle masses, mixing angles, energy scales — are traced back to the eigenvalues and norms of just two families of operators acting on the internal space.

The first family consists of **rotors**: antisymmetric operators that generate internal rotations. Think of them as the mathematical embodiment of forces. When two rotors don't commute — when rotating in one internal direction and then another gives a different result than doing it in the opposite order — the size of that mismatch (measured by the norm of the commutator) becomes a coupling constant. A large mismatch means a strong force; a small mismatch means a weak one.

The second family consists of **compressors**: symmetric operators with real eigenvalues. These encode masses and mixing patterns. The eigenvalues of a compressor give you particle masses; the eigenvectors tell you which combinations of fields have definite mass. When two compressors act on overlapping but slightly different subspaces — say, one for down-type quarks and one for up-type quarks — the misalignment between their eigenbases produces a mixing matrix like the CKM matrix.

Two specific operators play starring roles in this toolbox:

1. The **heptagon operator** H_7, which packages the seven imaginary octonion directions into a single object. Despite being built from seven directions, it has only three independent eigenvalues (α, β, γ), which are geometric invariants fixed by the G_2 symmetry. These three numbers will reappear as seeds for gauge couplings and flavor hierarchies.

2. The **radius operator** R, which measures how far different sectors of the internal space sit from preferred axes. Its spectrum (a_0, b_0, c_0) defines three characteristic radii. When one considers powers of R, the eigenvalues are raised to corresponding powers, and even moderate gaps between a_0, b_0, c_0 are amplified into huge hierarchies between the associated scales. A difference of order 40 in these radii can translate into many orders of magnitude between the corresponding physical energies.

Why is this toolbox attractive? Because it's both minimal and sufficient. It's *minimal* in the sense that we're not adding independent Yukawa matrices, unmotivated extra symmetries, or long lists of unrelated constants. Everything comes from the same two operator families. And it's *sufficient* in the sense that, in principle, all the quantities that enter phenomenology can be expressed in terms of the invariants of these operators.

The remaining Advent days will flesh out this claim with concrete examples: how the fine-structure constant $\alpha \approx 1/137$ emerges from rotor norms, how fermion masses arise from compressor spectra, how mixing matrices reflect misaligned eigenbases. But the conceptual foundation is laid today: physics is not a list of parameters — it's the spectrum of a small set of operators on an exceptional internal space.

Let us see how rotors and compressors turn geometry into numbers.

From algebra to operators

Octonions and their automorphism group G_2 give us a rigid internal stage. But physical predictions are not read directly from the multiplication table; they arise from *operators* acting on the internal space. In this model, two operator families play the central role:

- **Rotors** — antisymmetric operators G_a generating internal rotations: they encode forces and couplings.

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- **Compressors** — symmetric operators C with real spectra: they encode masses and mixing patterns.

Once this toolbox is in place, every later sheet becomes a story about eigenvalues, eigenvectors, commutators and norms of these operators.

The heptagon operator: seven directions, three invariants

The seven imaginary octonion units are arranged on the Fano-plane heptagon. Instead of handling them one by one, the model packages them into a single *heptagon operator* H_7 acting on the internal space:

$$H_7 \;=\; \sum_{i=1}^{7} c_i \, E_i,$$

where the E_i encode the seven imaginary directions and the coefficients c_i are fixed by the G_2–symmetric vacuum configuration and encode the orientation of the Fano plane within the algebra. In other words, H_7 is not chosen by hand; it is induced by H_{vac} and the octonionic geometry. Geometrically, H_7 summarizes how non-associativity is distributed in the vacuum: it acts as a "non-associativity profile" of H_{vac} along the seven imaginary directions of the internal space.

Despite being built from seven directions, H_7 has only three independent eigenvalues,

$$\mathrm{Spec}(H_7) = (\alpha, \beta, \gamma),$$

which are invariants of the G_2–orbit of H_7. These three numbers will reappear as seeds for:

- gauge couplings (fine-structure, strong coupling, weak mixing),
- relative positions of flavour sectors,
- and parts of the hierarchy structure.

The radius operator and three fundamental scales

From the heptagon structure one constructs a *radius operator* R, whose eigenvalues are algebraic functions of (α, β, γ). It measures how far internal directions sit from preferred axes. Its spectrum is

$$\mathrm{Spec}(R) = (a_0, b_0, c_0), \quad a_0 > b_0 > c_0.$$

Iterated applications of R — or, equivalently, integer powers R^N with moderate N — yield eigenvalues

$$\mathrm{Spec}(R^N) = (a_0^N, b_0^N, c_0^N).$$

For suitable choices of N these powers can be associated with three characteristic energy scales,

$$E_{\text{Planck}} \sim a_0^N, \qquad E_{\text{EW}} \sim b_0^N, \qquad E_{\text{QCD}} \sim c_0^N.$$

Thus, the familiar hierarchy of Planck, electroweak and QCD scales is encoded in a few geometric invariants of R rather than inserted as three independent inputs; large separations between these scales come from raising a small set of radii to suitable powers, not from introducing transcendental functions as new primitives.

Rotors: forces from commutator norms

Rotors are antisymmetric operators G_a generating continuous internal symmetries. These rotors G_a are precisely the generators that populate the connection A_μ in yesterday's transport equation $D\Psi = 0$.

Their commutators measure how two internal directions fail to commute. The squared norm of a commutator,

$$\big\| [G_a, G_b] \big\|^2,$$

acts as the prototype for a coupling constant. Symbolically, one can write

$$\alpha \sim \big\| [G_{\text{em}}, G_{\text{ref}}] \big\|^2, \qquad \alpha_s \sim \big\| [G_{\text{color}}, G_{\text{ref}}] \big\|^2,$$

and

$$\sin^2 \theta_W \sim \big\| [G_{\text{weak}}, G_{\text{ref}}] \big\|^2.$$

Choosing different rotor pairs recovers different interactions. In this view, a "strong" force is literally a large commutator norm in the internal algebra; a "weak" one corresponds to nearly commuting rotors. In this sense, the rotor algebra reads off interaction strengths from the same non-associative profile that is encoded in H_7.

Compressors: masses and mixings as spectra

Compressors are symmetric operators with real eigenvalues. The most important example is the mass map $\Pi(H_{\text{vac}})$ constructed from a vacuum configuration $H_{\text{vac}} \in H_3(\mathbb{O})$:

$$\Pi(H_{\text{vac}})\Psi = m \, \Psi.$$

Its eigenvalues m provide prototype fermion masses; its eigenvectors define the associated mass eigenstates. Additional compressors act in flavour subspaces; misalignment between their eigenbases produces mixing matrices:

- down- vs. up-quark compressors $\Rightarrow$ CKM matrix,
- charged-lepton vs. neutrino compressors $\Rightarrow$ PMNS matrix.

Masses and mixings thus share a common origin: they are different ways of reading the same symmetric operators.

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Attractor behaviour of scales and hierarchies

The combination of radius operator and compressors suggests an *attractor* picture:

- The spectrum (a_0, b_0, c_0) singles out preferred scales.
- Renormalisation-group (RG) flow tends to pull effective parameters towards these scales.
- Small deformations of H_{vac} move eigenvalues, but certain patterns (like the large gap between Planck and EW) remain stable.

A useful picture is that of a landscape with three deep valleys at E_{Planck}, E_{EW} and E_{QCD}. Fields and couplings start their renormalisation journey at some point in this landscape and then "roll down" along the RG flow. Even if they are kicked around by quantum corrections, they tend to end up in the same valleys defined by (a_0, b_0, c_0).

In that sense, hierarchies are not fragile coincidences but *attractor positions*. The renormalisation flow behaves less like a fine-tuned balancing act and more like a persistent cat: no matter how often you try to move it, it will eventually curl up again in its favourite spot in the operator landscape.

Why this toolbox is minimal and sufficient

What makes the toolbox attractive is its balance between simplicity and power:

- It is *minimal*: no independent Yukawa matrices, no unmotivated extra symmetries, no long list of unrelated constants.
- It is *sufficient*: in principle, all quantities that enter phenomenology—masses, mixing angles, couplings, scales—can be expressed in terms of the invariants of these operators.

The remaining Advent days flesh out this claim with concrete examples: from specific couplings (fine-structure, strong, Weinberg angle) to numerical mass prototypes and flavour structures.

References

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One toolbox, two operator families: rotors and compressors turn octonionic geometry into concrete numbers for couplings, masses and scales.

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