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Radius operator: scales as geometric invariants

From (α, β, γ) to (a_0, b_0, c_0) and energy hierarchies

Key Insight. The radius operator R extracts three characteristic length scales (a_0, b_0, c_0) from the heptagon structure. Powers of this matrix act as algebraic scale operators whose eigenvalues reproduce the observed energy hierarchies. For the first time, fundamental energy hierarchies emerge as geometric invariants of an internal operator, not as external input parameters.

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Yesterday's heptagon operator compressed the seven imaginary directions of the octonions into three eigenvalues (α, β, γ), capturing angular information about how these directions are oriented relative to each other. Today we take the next step: we pass from angular information to radial information. The *radius operator* R measures distance in the internal symmetry atlas defined by the heptagon and G_2.

We remain in a strict *matrix-first* viewpoint: all relevant internal quantities are realised as finite 8×8 matrices on the octonionic space. Numbers such as (α, β, γ) or (a_0, b_0, c_0) appear only as eigenvalues of these matrices, not as independent scalar fields on top of them.

The radius operator is constructed from the heptagon geometry and the octonionic structure, constrained by G_2-invariance and compatibility with the heptagon. Concretely, one can regard R as a matrix polynomial in the heptagon operator,

$$R = c_0\,\mathbf{1} + c_1\,H_7 + c_2\,H_7^2,$$

with real coefficients c_i fixed by these symmetry and compatibility conditions. Since R is a polynomial in H_7, the two operators commute and can be diagonalised simultaneously.

Its spectrum is again remarkably simple:

$$\mathrm{spec}(R) = \{a_0, b_0, c_0\},$$

three characteristic dimensionless radii with multiplicities adding up to 8. The triple (a_0, b_0, c_0) is the *radius spectrum*: three internal radii associated with the geometry encoded by the heptagon.

From radii to energy scales

In quantum field theory, hierarchies of energy scales usually arise from many small multiplicative steps. Analytically, such behaviour is often described using exponential functions. In our octonionic, matrix-based setting we achieve the same effect *algebraically*, without introducing a transcendental matrix exponential.

Instead of forming $\exp(R)$, we consider integer powers of the radius operator,

$$\Sigma_N := R^N, \qquad N \in \mathbb{N}_{>0},$$

as *scale operators*. Because R is a finite 8×8 matrix, all powers R^N are well-defined. By the Cayley–Hamilton theorem, R satisfies its own characteristic polynomial of degree three. As a result, every high power R^N can be reduced to a quadratic polynomial in R:

$$R^N \;=\; p_0(N)\,\mathbf{1} \;+\; p_1(N)\,R \;+\; p_2(N)\,R^2,$$

with coefficients $p_i(N)$ that depend algebraically on the eigenvalues a_0, b_0, c_0.

In a basis where R is diagonal, it has diagonal entries a_0, b_0, c_0 (with their respective multiplicities). The powers R^N then automatically have diagonal entries

$$a_0^N, \qquad b_0^N, \qquad c_0^N.$$

Thus the action of the scale operator Σ_N on the internal space is entirely determined by the *powers* of the three radii.

We now use these pure matrix powers to model physical energy scales. Introduce a single reference unit μ_0 (for instance in GeV) and define three characteristic energy scales by

$$\Lambda_{\mathrm{P}} = \mu_0\,a_0^N, \qquad \Lambda_{\mathrm{EW}} = \mu_0\,b_0^N, \qquad \Lambda_{\mathrm{QCD}} = \mu_0\,c_0^N,$$

for some fixed integer N. Only ratios of scales are physically meaningful at this level, and these ratios are now given by

$$\frac{\Lambda_{\mathrm{P}}}{\Lambda_{\mathrm{EW}}} = \left(\frac{a_0}{b_0}\right)^N, \quad \frac{\Lambda_{\mathrm{EW}}}{\Lambda_{\mathrm{QCD}}} = \left(\frac{b_0}{c_0}\right)^N, \quad \frac{\Lambda_{\mathrm{P}}}{\Lambda_{\mathrm{QCD}}} = \left(\frac{a_0}{c_0}\right)^N.$$

Already for moderate integers N (for example $N \approx$ 20–40), small algebraic differences between a_0, b_0, c_0 can give rise to very large multiplicative separations between the associated scales. This is the discrete, purely algebraic analogue of an exponential hierarchy – implemented entirely inside the finite matrix algebra of the octonionic model.

The key message is not the exact numerical fit—that requires detailed numerics and will be addressed in the main text—but the *structural fact*:

> There exist three distinguished internal radii (a_0, b_0, c_0) and their powers under repeated application of the radius operator generate three physically relevant energy scales.

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Why three scales?

Empirically, particle physics is organised around three strikingly different characteristic scales:

1. The Planck scale, where gravity becomes comparable to other interactions.
2. The electroweak scale, where $SU(2)_L \times U(1)_Y$ symmetry is broken.
3. The QCD scale, where confinement and chiral symmetry breaking dominate.

In the model, this triad is mirrored by the triad (a_0, b_0, c_0):

- The number of qualitatively distinct scales is fixed by the structure of R and its spectrum, not by phenomenological needs.
- The relative ordering and separation of these scales can be traced back to ratios such as a_0/b_0 and b_0/c_0 and to the integer N that counts how often R is applied.

This turns a long-standing "why these three?" question into a statement about the eigenstructure of an internal operator and its algebraic powers.

Radius operator within the attractor picture

On the second Advent Sunday (7 December), the calendar will present the idea of an *attractor* for scales. In that picture:

- The triple (a_0, b_0, c_0) defines three preferred radii in the internal space.
- Renormalisation-group (RG) flow in the physical theory is naturally modelled by repeated application of scale operators built from R.
- The observed scales appear as stable regimes (or fixed points) of this discrete flow rather than arbitrary initial conditions.

The radius operator R is thus the internal, geometric backbone of this attractor story. Without R and its discrete spectrum, the attractor mechanism would have nothing to lock onto.

Conceptual gain from $\operatorname{spec}(R) = (a_0, b_0, c_0)$

Introducing R and its spectrum brings several conceptual benefits:

1. **Geometric origin of hierarchies:** large ratios between energy scales (Planck vs. electroweak vs. QCD) are no longer mere accidents but are linked to differences between eigenvalues of a symmetry-constrained operator and their algebraic powers.
2. **Minimality:** three radii suffice—no large list of independent scale parameters is needed at the fundamental level.
3. **Spectral language:** scale information is encoded spectrally, aligning with the later use of spectral geometry and spectral actions.

This is why we list the radius operator as the first explicit bridge from abstract octonionic geometry to physically observed hierarchies. It turns the question "Why are there three widely separated scales?" into "What is the spectrum of the radius operator, and how do its powers act on the internal space?"—a question that can, in principle, be answered within the matrix-first, octonionic framework.

References

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