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G$_2$ as the minimal exceptional symmetry
The 14-dimensional gatekeeper of the octonions

Key Insight. G$_2$ is the smallest exceptional Lie group and the full automorphism group of the octonions. Every map in G$_2$ preserves octonionic multiplication and the norm. In the model, this makes G$_2$ the *gatekeeper* of the internal structure: any internal operator, symmetry or interaction must respect G$_2$ invariance. Today we meet G$_2$ as the minimal exceptional symmetry from which the larger exceptional group F$_4$ will later emerge.

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What is G$_2$?

THE group G_2 can be defined in many equivalent ways. For the octonionic story, the most natural is:

$$G_2 = \text{Aut}(\mathbb{O}),$$

the group of all linear transformations of $\mathbb{O}$ that preserve the octonionic product and the norm. It is a 14-dimensional, compact, connected, simply-connected Lie group and the smallest of the five exceptional Lie groups.

Concretely:

- G_2 preserves the multiplication table of the seven imaginary units $e_1, \ldots, e_7$.

- It preserves the standard norm $|x|^2 = x\bar{x}$.

- It acts transitively on the unit sphere of imaginary octonions, with stabiliser isomorphic to $SU(3)$.

In other words, G_2 is the full continuous symmetry group of the octonionic number system itself.

Why "minimal exceptional" matters

As a Lie group, G_2 is:

- too small to host all Standard Model symmetries directly,

- but large enough to control the essential nonassociative structure of $\mathbb{O}$,

- and exceptional—meaning it does not fit into the infinite A_n, B_n, C_n, D_n series.

This makes G_2 an ideal starting point:

- It is restrictive enough to strongly constrain internal operators.

- It is flexible enough to embed subgroups that resemble $SU(3)_C \times SU(2)_L \times U(1)_Y$ in appropriate ways.

- It naturally sits inside the larger exceptional group F_4, the automorphism group of the Albert algebra $H_3(\mathbb{O})$.

G$_2$ as a gatekeeper of allowed operators

In the model, internal operators (heptagon operator, radius operator, rotors, compressors) are not arbitrary matrices; they must be compatible with the G_2-structure. Informally:

> If an operator would break G_2 in an uncontrolled way, it is not part of the fundamental toolbox.

This has two important consequences:

1. **Restricted parameter space:** many couplings and mass terms that are allowed in a generic field-theory Lagrangian are simply forbidden by G_2.

2. **Natural subgroups:** gauge groups that actually appear (or approximate) in low-energy physics are precisely those that can be embedded in G_2 (and later in F_4) in a structurally compatible way.

G$_2$ thus serves as a first filter between "any algebraic construction on $\mathbb{R}^8$" and "constructions that respect the octonionic number system".

From G$_2$ to F$_4$

Later in the calendar, the Albert algebra $H_3(\mathbb{O})$ will appear, and with it the larger exceptional group F_4:

$$F_4 = \text{Aut}\big(H_3(\mathbb{O})\big).$$

The relationship is hierarchical:

- G_2 controls the algebra of $\mathbb{O}$ itself.

- $H_3(\mathbb{O})$ builds 3×3 Hermitian matrices over $\mathbb{O}$.

- F_4 controls the automorphisms of this larger Jordan algebra.

From the perspective of the calendar:

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- early days: G_2 and triality structure the octonionic stage,

- middle days: F_4 organises the symmetry atlas on the Albert algebra,

- later days: potentials and equilibria on this atlas fix physical scales and constants.

G_2 is the first rung on this exceptional ladder.

Conceptual gain from G_2

Putting G_2 at the base of the internal symmetry story has clear advantages:

1. **Uniqueness:** There is only one real division algebra with 7 imaginary units, and only one connected Lie group that preserves it: G_2. This is about as far from "model building by choice" as one can get.

2. **Rigidity:** Once octonions are chosen, G_2 is fixed. Internal symmetries are no longer free groups to be dialled but are inherited from this starting point.

3. **Roadmap:** The inclusion $G_2 \subset F_4$ provides a clear path from basic number system to full symmetry atlas.

In this sense, G_2 is not just an exotic group in a classification table; it is the minimal exceptional guardian of the octonionic world.

References

[1] F. Engel, "Ein neues, dem linearen Komplexe analoges Gebilde," *Ber. Verh. Königl. Sächs. Ges. Wiss. Leipzig* **52**, 63–74 (1900).

[2] R. L. Bryant, "Metrics with exceptional holonomy," *Ann. Math.* **126**, 525–576 (1987).

[3] J. C. Baez, "The octonions," *Bull. Amer. Math. Soc.* **39**, 145–205 (2002).

[4] [Internal notes on G_2 and its role in the symmetry atlas: `chap02_neu.tex`; `appB_neu.tex`; `oktonionen-basen.tex`.]

G_2 is the 14-dimensional automorphism group of the octonions and the minimal exceptional symmetry. In the model it acts as a gatekeeper: only operators and symmetries compatible with G_2 are admitted into the internal stage on which all later structures are built.

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