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Quaternions as a prototype: $SU(2)$ and the weak force

A four-dimensional rehearsal before the octonionic stage

Key Insight. Quaternions $\mathbb{H} \cong \mathbb{R}^4$ provide a rigid $1 \oplus 3$ split into one scalar and three imaginary directions and realise the double-cover $SU(2)$ of spatial rotations. In de Casteljau's matrix picture, unit quaternions act as birotations in four dimensions. The weak isospin group is thus not an abstract label but the symmetry of a specific four-dimensional number system. This $1 \oplus 3$ pattern is the warm-up for the octonionic eight-dimensional stage of one generation.

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Before we climb to the eight-dimensional octonions, let's pause at a more familiar landmark: the quaternions, denoted $\mathbb{H}$. If you've studied quantum mechanics or rotations in three dimensions, you've likely encountered them—perhaps as a way to represent rotations without the ambiguities of Euler angles, or as the mathematical home of spin-1/2 particles.

The quaternions are a four-dimensional number system, built from one real unit and three imaginary units called $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. They satisfy the famous relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Unlike ordinary complex numbers, quaternion multiplication is not commutative: $\mathbf{ij} \neq \mathbf{ji}$.

What makes quaternions interesting for physics is their natural split into a scalar part and a three-dimensional vector part—a $1 \oplus 3$ pattern. This is not an arbitrary decomposition; it's built into the algebra itself. The scalar part stays invariant under rotations, while the vector part transforms as a three-component object.

Now here's the key observation: the group of unit quaternions—quaternions with length one—is isomorphic to $SU(2)$, the group of weak isospin in particle physics. In other words, the weak force's internal symmetry is not an abstract label group invented to fit data. It's literally the automorphism group of a four-dimensional number system.

Why does this matter for our octonionic story? Because the step from quaternions to octonions is not a wild leap—it's a natural continuation. The quaternions give us a $1 \oplus 3$ structure and an $SU(2)$ symmetry. The octonions give us a richer structure that can host a full generation of internal quantum numbers, with the exceptional group G_2 playing the role that $SU(2)$ played for quaternions.

Think of today's sheet as a warm-up exercise: if a four-dimensional number system naturally produces the weak force's $SU(2)$, then an eight-dimensional number system might naturally produce the full internal symmetry structure of the Standard Model. The quaternion day is a bridge between the familiar and the exceptional—a reminder that the idea of p̈hysics from number systemsẗ is not as exotic as it first sounds.

Let us see how the quaternion $1 \oplus 3$ pattern scales up to the octonionic stage.

A gentle introduction: Why quaternions matter

Imagine you want to describe the internal structure of elementary particles—not where they are in space, but the hidden properties that make a left-handed electron different from a right-handed one, or a neutrino different from a quark. One natural question is: what kind of "number system" could serve as the stage for these internal degrees of freedom?

The usual real numbers $\mathbb{R}$ give us one dimension: a single line. Complex numbers $\mathbb{C}$ give us two dimensions and have proven essential in quantum mechanics. But what if we need more dimensions, and what if we want something that behaves like multiplication—where combining two elements gives another element of the same type?

This is where *quaternions* enter the story. Discovered in 1843 by William Rowan Hamilton, quaternions extend complex numbers to four dimensions. You can think of them as having one "real" part and three "imaginary" parts, usually called $\mathbf{i}, \mathbf{j}, \mathbf{k}$. These three imaginary units look like coordinates in 3D space, but with a special multiplication rule that makes them *non-commutative*: in general, $\mathbf{ij} \neq \mathbf{ji}$.

Formally,

$$\mathbb{H} = \{\, a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \mid a, b, c, d \in \mathbb{R} \,\},$$

with

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

The key structural feature is the rigid split

$$\mathbb{H} \;\cong\; \mathbb{R} \oplus \mathbb{R}^3,$$

one scalar component and a three-dimensional imaginary part: a $1 \oplus 3$ pattern.

De Casteljau's matrix view: quaternions as birotations

A particularly clear picture, developed in detail by de Casteljau [1], is to represent quaternions as 4×4 real

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matrices acting on a four-dimensional Euclidean space E_4. In this language, a unit quaternion becomes a *birotation*: a simultaneous rotation in two orthogonal 2-planes of E_4.

Schematically, one can write a unit quaternion in a normal form where its matrix representation looks like

$$Q_N(\varphi) \;=\; \rho \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix},$$

up to a suitable choice of orthonormal basis. Geometrically:

- The first 2×2 block rotates one plane in E_4 by angle φ.
- The second 2×2 block rotates an orthogonal plane by the *same* angle.

The associated "antiquaternion" corresponds to a contra-rotation where one of the planes is rotated in the opposite sense. De Casteljau uses this birotation picture to make explicit the eigenvalues, the determinant structure and the rare circumstances under which such 4×4 unitary matrices commute.

For us, the important message is: quaternions are not just formal symbols; they encode very concrete four-dimensional rotations with a rigid internal structure.

$SU(2)$ and the $1 \oplus 3$ pattern

From the quaternion point of view, the group $SU(2)$ of weak isospin is nothing but the group of unit quaternions:

$$\{q \in \mathbb{H} \mid |q| = 1\} \;\simeq\; SU(2).$$

Left multiplication by a unit quaternion acts as an $SU(2)$ transformation on the imaginary part $\mathbb{R}^3$, while the scalar part is left invariant.

Physically, this means:

- The scalar direction (the "1" in $1 \oplus 3$) behaves like an isospin singlet.
- The three imaginary directions (the "3") form a triplet under $SU(2)$, in perfect analogy with weak isospin triplets.

In other words, weak isospin is not an arbitrary label group we bolt onto particles. It is built into the structure of a specific four-dimensional number system and its matrix representation as birotations.

From $\mathbb{H}$ to $\mathbb{O}$: doubling the rehearsal

Why spend an entire Advent day on this four-dimensional rehearsal if our main stage is the eight-dimensional octonionic world?

Because several patterns scale up almost literally:

- The split $\mathbb{H} \cong 1 \oplus 3$ becomes, in the octonionic setting, a richer decomposition of $\mathbb{R}^8$ into blocks that can host one full generation of internal quantum numbers.
- The role of unit quaternions as birotations in E_4 is taken over by suitable 8×8 action matrices built from octonionic left/right multiplication on $\mathbb{R}^8$.
- The place of $SU(2) \subset \mathrm{Aut}(\mathbb{H})$ is taken over by the exceptional group $G_2 \subset \mathrm{Aut}(\mathbb{O})$, and the Spin(8) triality structure introduced on First Advent Sunday.

Seen from this angle, moving from quaternions to octonions is not a wild leap into a bizarre algebra. It is the next and final step in a sequence of division algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$—each adding just enough structure to host a richer symmetry.

Why this day matters in the Advent story

The quaternion day serves as a conceptual bridge between two worlds:

1. It anchors the exotic ideas of octonions, G_2 and triality in a familiar setting: $SU(2)$, spin, weak isospin and four-dimensional rotations.
2. It shows that the algebraic backbone we propose for one generation (octonions and Albert algebra) is a natural extension of the quaternionic picture, not an unrelated construction.
3. It prepares us to read later operator identities and action matrices as geometric statements about rotations, not as ad hoc matrix tricks.

After this gentle quaternion warm-up, the following sheets return to the full octonionic stage, now with a clearer intuition of where $SU(2)$ and its $1 \oplus 3$ pattern are coming from in the algebraic background.

References

[1] P. de Casteljau, *Les Quaternions*, Paris: Hermès, 1987.

[2] W. R. Hamilton, "On quaternions; or on a new system of imaginaries in algebra," *Philos. Mag.* **25**, 489–495 (1844).

Quaternions provide a four-dimensional rehearsal: a rigid $1 \oplus 3$ pattern and $SU(2)$ as the symmetry of a concrete number system, anticipating the octonionic stage of one full generation.

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