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Nonassociativity as origin of hierarchies

The geometric engine behind the Radius Operator

Key Insight. Nonassociativity of octonions forces different bracketings to yield different results: $(ab)c \neq a(bc)$. This is not a bug but the structural origin of mass and scale hierarchies. The associator measures how far a product is from being associative, and its norm directly controls the size of hierarchies—electron vs. top, Planck vs. electroweak.

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One of the most puzzling features of particle physics is the existence of extreme hierarchies. The top quark is about 300,000 times heavier than the electron. The Planck scale is about 10^{17} times higher than the electroweak scale. These are not small differences—they're enormous gaps that span many orders of magnitude.

In conventional field theory, such hierarchies are technically allowed but conceptually uncomfortable. You can write down a Lagrangian with a light electron and a heavy top quark, but you have to put in their mass ratio by hand. There's no mechanism that naturally generates such huge gaps—in fact, quantum corrections tend to pull all masses toward the same scale unless you carefully fine-tune the parameters. This is the essence of the hierarchy problem.

What if these extreme gaps are not accidents that need fine-tuning, but natural consequences of the internal geometry? What if the very structure of the octonions—specifically, their failure to be associative—is the source of hierarchies?

This is the idea behind today's sheet. Octonions are nonassociative: in general, $(ab)c \neq a(bc)$. For most number systems we're used to—real numbers, complex numbers, quaternions—this equation always holds. But for octonions, it fails. The deviation is measured by the **associator**, defined as $[a, b, c] := (ab)c - a(bc)$.

At first, nonassociativity sounds like a technical nuisance. But in the octonionic model, it's a feature, not a bug. Think of the associator as a "curvature" of the multiplication table. Just as curvature in spacetime creates gravity, "curvature" in algebra creates mass gaps.

Here's the key observation: not all directions in the octonions are equally nonassociative. If you pick three elements a, b, c that all lie in the same quaternionic subalgebra inside $\mathbb{O}$, then $[a, b, c] = 0$—they're perfectly associative. These are "flat" directions. But if you pick a triple that genuinely probes the full octonionic structure, you typically get $[a, b, c] \neq 0$—these are "curved" directions.

The model suggests a simple picture: light particles and low-energy scales live predominantly in nearly associative (almost quaternionic) directions, while heavy particles and high-energy scales probe strongly nonassociative (genuinely octonionic) directions. The associator norm $\|[a, b, c]\|$ becomes a geometric dial between "light" and "heavy."

Why does this produce such huge hierarchies? Because the relationship between associator norm and physical scale is cumulative. This mirrors the behavior of the **radius operator** R from Dec 5, where physical scales emerge from matrix powers R^N. Schematically, the associator norm acts as a geometric seed that gets amplified:

$$\text{hierarchy factor} \sim \big(1 + \gamma \|[a, b, c]\|\big)^N,$$

where N is the iteration count (e.g., $N \approx$ 20–40). A moderate associator norm, compounded over these steps, naturally generates the enormous ratios (like 10^{17}) observed between the Planck and electroweak scales.

This is a fundamentally different attitude toward hierarchies. In conventional field theory, large gaps are puzzles that require explanation. In the octonionic model, large gaps are the default: they're what you expect when you drop associativity. The puzzle is not "why are there hierarchies?" but "why are some scales close together?"—and the answer is that those scales live in nearly associative directions.

Let us see how $(ab)c \neq a(bc)$ turns into mass and scale gaps.

Associativity lost, structure gained

In the complex numbers and quaternions we have

$$(ab)c = a(bc) \quad \text{for all } a, b, c.$$

For octonions $\mathbb{O}$ this is no longer true. The deviation is captured by the *associator*

$$[a, b, c] := (ab)c - a(bc).$$

Nonassociativity means $[a, b, c] \neq 0$ for suitable triples (a, b, c).

At first sight this looks like a technical complication. But in an operator-based model, $[a, b, c]$ is a structured quantity whose norm can be used as a measure of "how curved" the internal multiplication is in a given region of the algebra.

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Associator norms as hierarchy seeds

The key idea is simple:

> Strong nonassociativity (large $||[a, b, c]||$) correlates with *large* hierarchies; near-associative directions correspond to *small* gaps.

Schematically, for suitable triples (a, b, c) associated with sectors of the internal space, one can write

$$\text{hierarchy factor} \sim \big(1 + \gamma \, \|[a, b, c]\|\big)^N,$$

with a model-dependent constant γ and iteration count N. This power-law scaling is exactly the mechanism of the **radius operator** R: the associator measures the "distance" from the associative subalgebra, which R translates into scale factors via algebraic powers.

Flat vs. curved internal directions

Not all directions in $\mathbb{O}$ are equally nonassociative:

- Quaternion subalgebras inside $\mathbb{O}$ are associative: $[a, b, c] = 0$ whenever a, b, c lie in the same quaternionic subalgebra. These are "flat" directions.
- Triples that genuinely probe the full octonionic structure typically have $[a, b, c] \neq 0$. These are "curved" directions.

This suggests a qualitative picture:

- Light fermions and low-energy scales live predominantly in almost quaternionic (nearly associative) directions.
- Heavy fermions and high-energy scales probe strongly octonionic (strongly nonassociative) directions.

The associator becomes a geometric dial between "light" and "heavy".

From internal curvature to physical gaps

In curved spacetime geometry, curvature scalars control focusing of geodesics and tidal forces. In the octonionic internal geometry, the associator norm plays an analogous role:

- It tells us how violently internal products deviate from the naive, associative expectation.
- Through the operator toolbox (rotors, compressors, radius operator), this deviation feeds into eigenvalue spectra and hence into physical scales and masses.

Symbolically one can write

$$m_{\text{heavy}}/m_{\text{light}} \sim F\big(\|[a, b, c]\|\big),$$

for some monotone function F determined by the detailed embedding in $H_3(\mathbb{O})$.

Why this matters conceptually

The nonassociativity day adds an important layer to the Advent story:

1. It explains why large hierarchies are *allowed* and *natural* in an octonionic setting: they are the rule, not the exception, once associativity is dropped.
2. It connects an abstract algebraic property—failure of associativity—to directly observable quantities: mass ratios and scale separations.
3. It sharpens the contrast with associative models, where such large gaps often have to be enforced by hand via fine-tuned potentials or additional symmetries.

If future work can make this link between associator norms and concrete hierarchies numerically sharp, nonassociativity would move from a curious mathematical footnote to a central player in explaining why the physical world has the extreme scales we observe.

References

[1] R. D. Schafer, An Introduction to Nonassociative Algebras, Academic Press, 1966.

[2] J. C. Baez, "The octonions," Bull. Amer. Math. Soc. **39**, 145–205 (2002).

In an octonionic world, huge hierarchies do not come from fine-tuning potentials—they follow the size of the associator: how far internal multiplication strays from associativity.

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