November 30, 2025 (First Advent Sunday)

First Light: Octonions, G_2 and Triality

100 years after Heisenberg's matrix mechanics

Key Insight. Octonions $\mathbb{O} \cong \mathbb{R}^8$ are the largest normed division algebra and the only one that is nonassociative. Their automorphism group G_2 and the triality of Spin(8) provide a rigid internal stage that fixes the structure of one generation. One hundred years after Heisenberg's matrix mechanics (1925), we return to noncommutative – and now nonassociative – structures as the hidden foundation of particle physics.

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One hundred years ago, Werner Heisenberg made a radical move: he replaced the smooth, continuous orbits of classical physics with discrete matrices that don't commute. Position times momentum is not the same as momentum times position. This was the birth of quantum mechanics—a first älgebraic turnïn physics, where the rules of multiplication suddenly mattered for understanding nature.

What if there's a second algebraic turn waiting? What if the internal properties of particles – the quantum numbers that distinguish an electron from a quark, or determine how particles interact – are not arbitrary labels but reflections of a deeper number system?

This Advent calendar explores exactly that idea. The number system in question is called the octonions, denoted $\mathbb{O}$. It's an eight-dimensional extension of the familiar real and complex numbers, but with two exotic features: multiplication is not commutative (like Heisenberg's matrices), and it's also not associative—the way you group three multiplications actually matters.

At first, this sounds like a mathematical curiosity, far removed from physics. But the octonions come with a remarkable compensation: they still have a well-behaved notion of length (the norm), and their symmetry group—the transformations that preserve the multiplication table – is the smallest exceptional Lie group, called G_2. This rigid internal symmetry makes the octonions a natural candidate for the hidden ïnternal spaceẅhere particle properties live.

Why eight dimensions? Because eight is just enough to accommodate the rich structure we see in one generation of matter. The octonions naturally support a group called Spin(8), which has a unique feature called triality: it treats three different eight-dimensional representations – vectors, left-handed spinors, and right-handed spinors—on completely equal footing. This threefold symmetry will later become the seed for the three generations of quarks and leptons.

The claim of this calendar is bold but precise: the Standard Model of particle physics, with its gauge groups, three generations, and seemingly arbitrary parameters, is not a random collection of fields and couplings. Instead, it's the low-energy shadow of a rigid exceptional structure built on the octonions and their larger cousins, the Albert algebra $H_3(\mathbb{O})$ and the exceptional group F_4.

You don't need to accept every step as final truth to benefit from the journey. The goal is to make plausible that there is a coherent algebraic backbone beneath the zoo of particles—and that many f̈actsöf particle physics look less arbitrary when viewed from this octonionic perspective.

Today's sheet sets the stage: we introduce the octonions, their automorphism group G_2, and the triality symmetry of Spin(8). These are the three pillars on which the entire Advent story rests. Let us begin.

From noncommutativity to nonassociativity

In 1925, Heisenberg's matrix mechanics marked a clean break with classical intuitions: position and momentum were no longer numbers but noncommuting operators. This was a first "algebraic turn" in physics. Today, noncommutativity is standard language in quantum theory.

The next turn is less familiar. There exists a unique real division algebra of dimension eight, the octonions $\mathbb{O}$, which is not only noncommutative but also *nonassociative*. Multiplying three octonions depends on how we place the brackets. At first sight this seems like a mathematical curiosity, far removed from physics.

However, nonassociativity comes with a remarkable compensation: the octonions still admit a multiplicative norm,

$$N(xy) = N(x)\,N(y),$$

and their automorphism group is the smallest exceptional Lie group G_2. This rigid internal symmetry makes $\mathbb{O}$ a natural candidate for a hidden layer beneath the familiar complex Hilbert spaces of quantum theory.

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The octonionic stage

The real vector space underlying the octonions is $\mathbb{R}^8$. As a stage for physics it brings several features at once:

- An 8-dimensional structure that can host vectors and spinors of Spin(8), the double cover of SO(8).
- A distinguished subgroup $G_2 \subset \mathrm{SO}(8)$ that fixes the multiplication table of $\mathbb{O}$.
- A triality symmetry of Spin(8) that permutes its three eight-dimensional irreducible representations: vector, left-handed spinor, right-handed spinor.

Physically, this means that:

- Internal degrees of freedom can be arranged in three correlated eight-component blocks.
- Rotations in the internal space can mix these blocks in a highly constrained way.

Later in the Advent calendar, these three blocks will be read as three generations of fermions, and specific subgroups will be identified with $SU(3)\times SU(2)\times U(1)$.

G_2 as guardian of the multiplication table

The group G_2 can be defined as the set of all linear maps $g : \mathbb{O} \to \mathbb{O}$ that preserve octonionic multiplication:

$$g(xy) \;=\; g(x)\,g(y) \quad \text{for all } x, y \in \mathbb{O}.$$

This makes G_2 the symmetry group of the multiplication table. In a physical setting, G_2-compatible transformations are those that respect the hidden octonionic structure of the internal space.

The existence of G_2 has two important consequences:

- It restricts which internal rotations are "legal" if we want to keep the multiplication law intact.
- It provides a natural environment for embeddings of familiar gauge groups. Subgroups of G_2 and related structures can house $SU(3)$-like and $SU(2) \times U(1)$-like symmetries.

In this sense, the octonionic stage with G_2 symmetry is already a candidate for the internal symmetry space of the Standard Model.

Triality and the seed of generations

The group Spin(8) acts on three eight-dimensional representations:

$$V_8, \quad S_8^+, \quad S_8^-.$$

These correspond to vectors, left-handed spinors and right-handed spinors. An exceptional outer automorphism permutes these three representations. This is the triality symmetry.

For our purposes, triality can be read as a structural reason for the recurrence of the number three in particle physics:

- There is *one* internal eight-dimensional block, but it admits three coherent readings: V_8, S_8^+, S_8^-.
- Later we will see how these three readings unfold into three generations of quarks and leptons when embedded into a larger exceptional algebra.

In this way, the abstract triality of Spin(8) becomes a seed for family replication.

Why start the Advent story here?

Starting the Advent calendar with octonions, G_2 and triality is not an exercise in exotic mathematics for its own sake. It sets up three themes that will run through the entire series:

1. **Rigidity:** Exceptional structures like $\mathbb{O}$ and G_2 leave little room for arbitrary choices. This rigidity is a feature if we seek explanations rather than fits.
2. **Hidden simplicity:** Behind the zoo of fields and parameters in the Standard Model there might be a much smaller set of algebraic building blocks.
3. **Algebra as geometry:** Nonassociative multiplication and its automorphisms can be interpreted as a kind of curved internal geometry, on par with spacetime curvature in general relativity.

References

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[2] J. C. Baez, "The octonions," *Bull. Amer. Math. Soc.* **39**, 145–205 (2002).

[3] F. Gürsey and H. C. Tze, *On the Role of Division, Jordan and Related Algebras in Particle Physics*, World Scientific, 1996.

One century after matrix mechanics, we explore a universe whose hidden stage is an exceptional eight-dimensional number system.

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