December 1, 2025

Quaternions as a prototype: $SU(2)$ and the weak force

A four-dimensional rehearsal before the octonionic stage

Key Insight. Quaternions $\mathbb{H} \cong \mathbb{R}^4$ provide a rigid $1 \oplus 3$ split into one scalar and three imaginary directions and realise the double-cover $SU(2)$ of spatial rotations. In de Casteljau's matrix picture, unit quaternions act as birotations in four dimensions. The weak isospin group is thus not an abstract label but the symmetry of a specific four-dimensional number system. This $1 \oplus 3$ pattern is the warm-up for the octonionic eight-dimensional stage of one generation.

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A gentle introduction: Why quaternions matter

IMAGINE you want to describe the internal structure of elementary particles—not where they are in space, but the hidden properties that make a left-handed electron different from a right-handed one, or a neutrino different from a quark. One natural question is: what kind of "number system" could serve as the stage for these internal degrees of freedom?

The usual real numbers $\mathbb{R}$ give us one dimension: a single line. Complex numbers $\mathbb{C}$ give us two dimensions and have proven essential in quantum mechanics. But what if we need more dimensions, and what if we want something that behaves like multiplication—where combining two elements gives another element of the same type?

This is where *quaternions* enter the story. Discovered in 1843 by William Rowan Hamilton, quaternions extend complex numbers to four dimensions. You can think of them as having one "real" part and three "imaginary" parts, usually called $\mathbf{i}, \mathbf{j}, \mathbf{k}$. These three imaginary units look like coordinates in 3D space, but with a special multiplication rule that makes them *non-commutative*: in general, $\mathbf{ij} \neq \mathbf{ji}$.

Formally,

$$\mathbb{H} = \{\, a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \mid a, b, c, d \in \mathbb{R}\,\},$$

with

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

The key structural feature is the rigid split

$$\mathbb{H} \;\cong\; \mathbb{R} \oplus \mathbb{R}^3,$$

one scalar component and a three-dimensional imaginary part: a $1 \oplus 3$ pattern.

De Casteljau's matrix view: quaternions as birotations

A particularly clear picture, developed in detail by de Casteljau [1], is to represent quaternions as 4×4 real matrices acting on a four-dimensional Euclidean space E_4. In this language, a unit quaternion becomes a *birotation*: a simultaneous rotation in two orthogonal 2-planes of E_4.

Schematically, one can write a unit quaternion in a normal form where its matrix representation looks like

$$Q_N(\varphi) \;=\; \rho \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix},$$

up to a suitable choice of orthonormal basis. Geometrically:

- The first 2×2 block rotates one plane in E_4 by angle φ.
- The second 2×2 block rotates an orthogonal plane by the *same* angle.

The associated "antiquaternion" corresponds to a contra-rotation where one of the planes is rotated in the opposite sense. De Casteljau uses this birotation picture to make explicit the eigenvalues, the determinant structure and the rare circumstances under which such 4×4 unitary matrices commute.

For us, the important message is: quaternions are not just formal symbols; they encode very concrete four-dimensional rotations with a rigid internal structure.

$SU(2)$ and the $1 \oplus 3$ pattern

From the quaternion point of view, the group $SU(2)$ of weak isospin is nothing but the group of unit quaternions:

$$\{q \in \mathbb{H} \mid |q| = 1\} \;\simeq\; SU(2).$$

Left multiplication by a unit quaternion acts as an $SU(2)$ transformation on the imaginary part $\mathbb{R}^3$, while the scalar part is left invariant.

Physically, this means:

- The scalar direction (the "1" in $1 \oplus 3$) behaves like an isospin singlet.
- The three imaginary directions (the "3") form a triplet under $SU(2)$, in perfect analogy with weak isospin triplets.

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In other words, weak isospin is not an arbitrary label group we bolt onto particles. It is built into the structure of a specific four-dimensional number system and its matrix representation as birotations.

From $\mathbb{H}$ to $\mathbb{O}$: doubling the rehearsal

Why spend an entire Advent day on this four-dimensional rehearsal if our main stage is the eight-dimensional octonionic world?

Because several patterns scale up almost literally:

- The split $\mathbb{H} \cong 1 \oplus 3$ becomes, in the octonionic setting, a richer decomposition of $\mathbb{R}^8$ into blocks that can host one full generation of internal quantum numbers.
- The role of unit quaternions as birotations in E_4 is taken over by suitable 8×8 action matrices built from octonionic left/right multiplication on $\mathbb{R}^8$.
- The place of $SU(2) \subset \mathrm{Aut}(\mathbb{H})$ is taken over by the exceptional group $G_2 \subset \mathrm{Aut}(\mathbb{O})$, and the Spin(8) triality structure introduced on First Advent Sunday.

Seen from this angle, moving from quaternions to octonions is not a wild leap into a bizarre algebra. It is the next and final step in a sequence of division algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$—each adding just enough structure to host a richer symmetry.

Why this day matters in the Advent story

The quaternion day serves as a conceptual bridge between two worlds:

1. It anchors the exotic ideas of octonions, G_2 and triality in a familiar setting: $SU(2)$, spin, weak isospin and four-dimensional rotations.
2. It shows that the algebraic backbone we propose for one generation (octonions and Albert algebra) is a natural extension of the quaternionic picture, not an unrelated construction.
3. It prepares us to read later operator identities and action matrices as geometric statements about rotations, not as ad hoc matrix tricks.

After this gentle quaternion warm-up, the following sheets return to the full octonionic stage, now with a clearer intuition of where $SU(2)$ and its $1 \oplus 3$ pattern are coming from in the algebraic background.

References

[1] P. de Casteljau, *Les Quaternions*, Paris: Hermès, 1987.

[2] W. R. Hamilton, "On quaternions; or on a new system of imaginaries in algebra," *Philos. Mag.* **25**, 489–495 (1844).

Quaternions provide a four-dimensional rehearsal: a rigid $1 \oplus 3$ pattern and $SU(2)$ as the symmetry of a concrete number system, anticipating the octonionic stage of one full generation.

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