November 30, 2025 (First Advent Sunday)

First Light: Octonions, G_2 and Triality

100 years after Heisenberg's matrix mechanics

Key Insight. One hundred years ago, Heisenberg replaced continuous orbits by discrete matrices and discovered noncommutativity. In this Advent story we go one step further: we anchor physics in an eight-dimensional number system — the octonions $\mathbb{O}$ — whose automorphisms form the exceptional group G_2 and whose triality-related Spin(8) structure organises an entire generation of matter. The surprising claim is that this rigid internal stage is already "almost" the Standard Model plus gravity.

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From noncommutativity to nonassociativity

In 1925, Heisenberg's matrix mechanics marked a clean break with classical intuitions: position and momentum were no longer numbers but noncommuting operators. This was a first "algebraic turn" in physics. Today, noncommutativity is standard language in quantum theory.

The next turn is less familiar. There exists a unique real division algebra of dimension eight, the octonions $\mathbb{O}$, which is not only noncommutative but also *nonassociative*. Multiplying three octonions depends on how we place the brackets. At first sight this seems like a mathematical curiosity, far removed from physics.

However, nonassociativity comes with a remarkable compensation: the octonions still admit a multiplicative norm,

$$N(xy) = N(x)\,N(y),$$

and their automorphism group is the smallest exceptional Lie group G_2. This rigid internal symmetry makes $\mathbb{O}$ a natural candidate for a hidden layer beneath the familiar complex Hilbert spaces of quantum theory.

The octonionic stage

The real vector space underlying the octonions is $\mathbb{R}^8$. As a stage for physics it brings several features at once:

- An 8-dimensional structure that can host vectors and spinors of Spin(8), the double cover of SO(8).
- A distinguished subgroup $G_2 \subset \mathrm{SO}(8)$ that fixes the multiplication table of $\mathbb{O}$.
- A triality symmetry of Spin(8) that permutes its three eight-dimensional irreducible representations: vector, left-handed spinor, right-handed spinor.

Physically, this means that:

- Internal degrees of freedom can be arranged in three correlated eight-component blocks.
- Rotations in the internal space can mix these blocks in a highly constrained way.

Later in the Advent calendar, these three blocks will be read as three generations of fermions, and specific subgroups will be identified with $SU(3)\times SU(2)\times U(1)$.

G_2 as guardian of the multiplication table

The group G_2 can be defined as the set of all linear maps $g : \mathbb{O} \to \mathbb{O}$ that preserve octonionic multiplication:

$$g(xy) = g(x)\,g(y) \quad \text{for all } x, y \in \mathbb{O}.$$

This makes G_2 the symmetry group of the multiplication table. In a physical setting, G_2-compatible transformations are those that respect the hidden octonionic structure of the internal space.

The existence of G_2 has two important consequences:

- It restricts which internal rotations are "legal" if we want to keep the multiplication law intact.
- It provides a natural environment for embeddings of familiar gauge groups. Subgroups of G_2 and related structures can house $SU(3)$-like and $SU(2) \times U(1)$-like symmetries.

In this sense, the octonionic stage with G_2 symmetry is already a candidate for the internal symmetry space of the Standard Model.

Triality and the seed of generations

The group Spin(8) acts on three eight-dimensional representations:

$$V_8, \quad S_8^+, \quad S_8^-.$$

These correspond to vectors, left-handed spinors and right-handed spinors. An exceptional outer automorphism permutes these three representations. This is the triality symmetry.

For our purposes, triality can be read as a structural reason for the recurrence of the number three in particle physics:

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- There is *one* internal eight-dimensional block, but it admits three coherent readings: V_8, S_8^+, S_8^-.
- Later we will see how these three readings unfold into three generations of quarks and leptons when embedded into a larger exceptional algebra.

In this way, the abstract triality of Spin(8) becomes a seed for family replication.

Why start the Advent story here?

Starting the Advent calendar with octonions, G_2 and triality is not an exercise in exotic mathematics for its own sake. It sets up three themes that will run through the entire series:

1. **Rigidity:** Exceptional structures like $\mathbb{O}$ and G_2 leave little room for arbitrary choices. This rigidity is a feature if we seek explanations rather than fits.
2. **Hidden simplicity:** Behind the zoo of fields and parameters in the Standard Model there might be a much smaller set of algebraic building blocks.
3. **Algebra as geometry:** Nonassociative multiplication and its automorphisms can be interpreted as a kind of curved internal geometry, on par with spacetime curvature in general relativity.

References

[1] W. Heisenberg, "Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen," *Z. Phys.* **33**, 879–893 (1925).

[2] J. C. Baez, "The octonions," *Bull. Amer. Math. Soc.* **39**, 145–205 (2002).

[3] F. Gürsey and H. C. Tze, *On the Role of Division, Jordan and Related Algebras in Particle Physics*, World Scientific, 1996.

One century after matrix mechanics, we explore a universe whose hidden stage is an exceptional eight-dimensional number system.

Advent Calendar 2025
An exceptional algebraic walk through particle physics

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How to read this Advent calendar

This Advent calendar is written for readers who are comfortably fluent in theoretical physics and mathematics, but who may or may not have worked with octonions and exceptional algebras before. Each day is a self-contained sheet, but together they tell a single story:

- that the Standard Model and gravity can be seen as low-energy expressions of a more rigid exceptional structure, and
- that many familiar "facts" of particle physics look less arbitrary when viewed from this perspective.

You do *not* need to accept every step as "the" final truth to benefit from the journey. The purpose is to make plausible that there is a coherent algebraic backbone behind the zoo of fields, couplings and generations.

Structure of the four weeks

The four Advent Sundays mark the main milestones:

First Sunday (today): We introduce the octonionic stage, the group G_2 and triality, and explain why an 8-dimensional, nonassociative number system is a natural candidate for internal degrees of freedom.

Second Sunday: We move to the Albert algebra $H_3(\mathbb{O})$, the unified transport equation $D\Psi = 0$ and the idea that Dirac, Yang–Mills and Einstein equations can all be read as projections or consistency conditions of this single operator.

Third Sunday: We look at flavour: three generations, CKM and PMNS mixing, and how these structures appear as different charts on the same internal exceptional space.

Fourth Sunday: We step back and ask what kind of "attractor universe" is selected by this geometry, and how robust the picture is under deformations.

In between the Sundays, the weekday sheets focus on specific mechanisms, numerical prototypes and "what-if" universes based on alternative algebras.

What you can expect (and what not)

This calendar is intentionally modest and ambitious at the same time:

- **Modest**, because it does not pretend to deliver a full, rigorous theory with all details worked out. Many steps are presented at the level of structure and plausibility rather than polished theorems.
- **Ambitious**, because it aims to show that a surprisingly small set of algebraic ingredients — octonions, $H_3(\mathbb{O})$, an $\mathfrak{so}(8)$-connection and a vacuum configuration $\langle H \rangle$ — can organise a large portion of what we know as the Standard Model plus gravity.

You should *not* expect precise numeric predictions for every mass and mixing angle on these pages. What you can expect are:

1. clear structural links: why three generations, why particular charge assignments, why certain patterns of couplings;
2. concrete numerical prototypes: explicit eigenvalue patterns from simple vacua in $H_3(\mathbb{O})$;
3. a comparative view: what is lost when we replace the exceptional algebras by more conventional ones.

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Who this is written for

These sheets deliberately sit in between a technical paper and a popular article:

- If you are a working physicist or mathematician, you will find enough structure to connect the ideas to your own toolbox: Lie groups, representation theory, spectral geometry, operator algebras.
- If you come from a neighbouring field, you might treat the technical points as signposts rather than obstacles, and focus on the emerging picture: a universe whose internal order is exceptional in a very literal sense.

References at the bottom of each sheet point to the underlying literature. You do not need to read them to follow the calendar, but they are there if you want to see how the pieces connect to mainstream work on division algebras, Jordan algebras and noncommutative geometry.

An invitation rather than a conclusion

Finally, this Advent calendar should be read as an invitation, not as a finished doctrine. There are many open questions:

- How unique is the proposed exceptional backbone?
- How far can the numerical fits to real-world data be pushed?
- Which aspects of the nonassociative structure survive quantisation?

If, by Christmas, you feel that these questions are worth spending serious time on, the calendar has achieved its goal.

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