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First Numerical Prototypes from $H_3(\mathbb{O})$

Simple vacua, concrete spectra

Key Insight. The first genuinely numerical output of the octonionic framework appears already for minimal vacua $\langle H\rangle \in H_3(\mathbb{O})$. Once the mass map $\Pi(\langle H\rangle)$ is fixed, its spectrum is a rigid invariant: eigenvalues come in multiplets, organise into bands, and split in controlled ways when a single off-diagonal octonionic coupling is switched on. These spectral patterns are structural (symmetry-driven and stable under small deformations), not the result of phenomenological fine-tuning.

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To make exceptional algebra more than a catalogue of beautiful objects, one must eventually ask a brutally concrete question: what numbers does the structure force upon us? In the present framework this question has a clean and non-negotiable meaning. A vacuum choice is an element $\langle H\rangle$ of the Albert algebra $H_3(\mathbb{O})$, and the model associates to it a linear operator $\Pi(\langle H\rangle)$ acting on an internal state space. From that point on, no additional physical narrative is required: an operator has a spectrum, with eigenvalues and multiplicities that are invariant under change of basis.

This shift of viewpoint is important. Instead of trying to "guess" masses by attaching parameters to fields, we let the vacuum itself become an input to a spectral problem. The simplest vacua already reveal what is generic and what is accidental: bands versus isolated levels, symmetry-protected degeneracies versus controlled splittings, and gaps that persist under small deformations. In this sense, computing spectra is the most economical bridge from the exceptional geometry of $H_3(\mathbb{O})$ to concrete numerical prototypes.

From algebra to spectra

Today we do something unambiguously testable in the mathematical sense: we take a concrete vacuum configuration $\langle H\rangle \in H_3(\mathbb{O})$ and compute the spectrum of the associated mass map $\Pi(\langle H\rangle)$. No interpretation is needed to make this step meaningful: an operator has eigenvalues, and their multiplicities and gaps are rigid invariants.

The key point is that *very simple* choices of $\langle H\rangle$ already generate spectra with a recognisable organisation: eigenvalues cluster into bands, bands split under controlled deformations, and the splitting patterns depend on which internal directions are activated. This is the first place where the exceptional machinery produces "numerical prototypes" rather than only symbolic structure.

Diagonal prototype vacua

Start with the cleanest vacuum in the Albert algebra, a diagonal element with three real parameters:

$$\langle H\rangle = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \lambda_i \in \mathbb{R}.$$

Even at this level, $\Pi(\langle H\rangle)$ is not "just three numbers": it is an endomorphism of the internal space,

$$\Pi(\langle H\rangle) : \mathcal{H}_{\text{int}} \longrightarrow \mathcal{H}_{\text{int}},$$

and the eigenvalue problem

$$\Pi(\langle H\rangle)\,\Psi = m\,\Psi$$

produces a *multiplet* of masses m with definite degeneracies.

A diagonal $\langle H\rangle$ is useful for one precise reason: it exposes the *baseline band structure* enforced by symmetry. Typically one finds:

- eigenvalues arranged in a small number of clusters (bands),
- exact degeneracies dictated by residual internal symmetry,
- clear gaps separating bands when $(\lambda_1, \lambda_2, \lambda_3)$ are non-equal.

This is already a nontrivial numerical statement: the spectrum is organised even before any "mixing" is introduced.

Off-diagonal couplings: a better minimal example

The next step is not to solve a new cubic equation, but to switch on the *smallest* genuinely octonionic deformation and watch how symmetry protected degeneracies split. Consider

$$\langle H\rangle = \begin{pmatrix} \lambda_1 & \varepsilon\, u & 0 \\ \varepsilon\, \bar{u} & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad u \in \mathbb{O},\ \|u\| = 1,\ \varepsilon \in \mathbb{R}.$$

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This is a one–knob deformation: a single coupling strength ε and a single internal direction u (which may be chosen along one imaginary unit for maximal transparency).

What changes is qualitative and robust:

- **Splitting:** degeneracies present at $\varepsilon = 0$ split into sub-bands for $\varepsilon \neq 0$.
- **Band stability:** the *existence* of bands and gaps is stable under small changes of ε and of the direction u; only the fine splittings move continuously.
- **Directional dependence:** different choices of u produce characteristically different splitting patterns, reflecting how the internal algebra is wired into Π.

In other words, the simplest off-diagonal octonionic entry already acts like a controlled "spectral lever": it does not destroy the band structure, it refines it.

Prototype spectra: what is (and is not) claimed

These spectra are not yet precision predictions for Standard Model masses. What they provide is more basic and more reliable: an existence proof that the exceptional construction produces *structured* mass multiplets without inserting hierarchies by hand.

A useful way to summarise the generic output is:

1. **Banded hierarchies:** eigenvalues group into clusters separated by gaps, and the gaps persist under small deformations of the vacuum.
2. **Symmetry fingerprints:** degeneracies and their splittings track residual symmetry and how it is broken by off-diagonal couplings.
3. **Continuity:** varying $(\lambda_1, \lambda_2, \lambda_3, \varepsilon, u)$ moves eigenvalues continuously; qualitative patterns are structural, not tuned coincidences.

The message of the day is therefore simple: once $\Pi(H)$ is fixed, numerical spectra are not optional—they are inevitable—and even the minimal vacua already display the kind of organisation one would expect from a realistic mass sector.

An invitation to compute

Technically, exploring $\Pi(\langle H\rangle)$ is straightforward: after choosing a concrete basis, the problem reduces to diagonalising a structured real matrix. The point is not computational difficulty; it is that the exceptional input severely restricts what spectra can look like.

A good first exercise is to compare the diagonal vacuum ($\varepsilon = 0$) to the one–knob deformation above ($\varepsilon \neq 0$) and to record which degeneracies split and which gaps remain. Those features are the "numerical prototypes" that can later be confronted with phenomenology.

References

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Even very simple vacua in $H_3(\mathbb{O})$ already yield banded, hierarchical spectra: numerical prototypes of a world built from exceptional geometry.

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