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Quaternions as a prototype: $SU(2)$ and the weak force

The $1 \oplus 3$ pattern that scales up to octonions

Key Insight. Before we climb to the eight-dimensional octonions, it is worth pausing at their four-dimensional cousins, the quaternions $\mathbb{H} \cong \mathbb{R}^4$. The quaternion units realise a rigid $1 \oplus 3$ split of scalar and vector parts, and their automorphism group contains $SU(2)$ —the very group of weak isospin. The octonionic 8×8 action simply doubles this pattern: what \mathbb{H} does for $SU(2)$, \mathbb{O} does for the full internal structure of one generation.

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A gentle introduction: Why quaternions matter for physics

IMAGINE you want to describe the internal structure of elementary particles—not where they are in space, but the hidden properties that make a left-handed electron different from a right-handed one, or a neutrino different from a quark. One natural question is: what kind of “number system” could serve as the stage for these internal degrees of freedom?

The usual real numbers \mathbb{R} give us one dimension: a single number. Complex numbers \mathbb{C} give us two dimensions and have proven essential in quantum mechanics. But what if we need more dimensions, and what if we want something that behaves like multiplication—where combining two elements gives another element of the same type?

This is where quaternions enter the story. Discovered in 1843 by William Rowan Hamilton, quaternions extend complex numbers to four dimensions. You can think of them as having one “real” part and three “imaginary” parts, which we call \mathbf{i} , \mathbf{j} , and \mathbf{k} . These three imaginary units are like coordinates in 3D space, but with a special multiplication rule that makes them noncommutative: $\mathbf{ij} \neq \mathbf{ji}$.

Why does this matter for particle physics? The key observation is that the group of symmetries preserving quaternion multiplication contains $SU(2)$ —the exact mathematical structure describing weak nuclear force and the distinction between left- and right-handed particles. In other words, weak isospin is not an arbitrary label we invented; it is built into the structure of a four-dimensional number system.

The split into one real part and three imaginary parts ($1 \oplus 3$) mirrors the splitting we see in nature: one neutral component and three charged components in weak interactions. Before we tackle the full complexity of eight-dimensional octonions, which host an entire generation of particles, quaternions give us a familiar “rehearsal” of the same pattern at half the size.

Think of today’s sheet as a warm-up exercise: if four dimensions and quaternions give us $SU(2)$ and the structure of weak interactions, then eight dimensions and octonions will give us the full Standard Model

structure. Let us see how this works in detail.

The quaternion stage in four dimensions

THE quaternions

$$\mathbb{H} = \{ a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{R} \}$$

form a four-dimensional division algebra with basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Two structural facts are important for physics:

- \mathbb{H} splits as scalar \oplus vector:

$$\mathbb{H} \cong \mathbb{R} \oplus \mathbb{R}^3,$$

i.e. $1 \oplus 3$.

- Left and right multiplication by unit quaternions act as rotations on the vector part \mathbb{R}^3 .

This already looks like a toy model for spin and isospin: one distinguished scalar component, three correlated vector components.

$SU(2)$ inside the quaternion automorphisms

Unit quaternions form a group isomorphic to $SU(2)$:

$$\{q \in \mathbb{H} \mid |q| = 1\} \simeq SU(2).$$

Left multiplication by such a unit quaternion acts as an $SU(2)$ matrix on the $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ components. Concretely:

- The scalar part a stays invariant.
- The vector part (b, c, d) transforms as a 3-vector under $SO(3)$, double-covered by $SU(2)$.

In this language, weak isospin $SU(2)$ is not an abstract label group; it is *literally* the automorphism group of a number system.

Action matrices and the $1 \oplus 3$ pattern

In the full model, internal degrees of freedom are encoded via action matrices built from left and right multiplication:

$$L_q : x \mapsto qx, \quad R_q : x \mapsto xq.$$

Already for \mathbb{H} , these left/right actions decompose the four-dimensional real space into a scalar plus a three-dimensional subspace. In matrix form this appears as a rigid $1 \oplus 3$ block structure in the 4×4 representation of L_q and R_q .

Physically:

- The singlet direction (scalar part) is a natural candidate for an isospin-neutral component.
- The triplet directions (vector part) form a weak isospin triplet under $SU(2)$.

This is the exact pattern that will be reused and doubled when we move to the octonionic 8×8 actions.

From quaternions to octonions

The step from \mathbb{H} to \mathbb{O} looks dramatic—nonassociativity, seven imaginary units—but in the action-matrix picture it becomes transparent:

- The four real components of \mathbb{H} become eight real components for \mathbb{O} .
- The $1 \oplus 3$ pattern of scalar/vector parts becomes a richer pattern that can host one full generation of internal quantum numbers.
- The role of $SU(2) \subset \text{Aut}(\mathbb{H})$ is taken over by $G_2 \subset \text{Aut}(\mathbb{O})$.

From this vantage point, $SU(2)$ -weak is not an isolated gauge factor; it is the visible shadow of quaternionic automorphisms inside a larger octonionic structure.

Why this detour matters

The quaternion day serves three purposes in the Advent story:

1. **Familiarity:** it connects the exotic octonionic picture to the well-known role of $SU(2)$ in spin and weak isospin.
2. **Pattern recognition:** it highlights the $1 \oplus 3$ structure that later scales up to the more intricate block structure of the octonionic action.
3. **Continuity:** it shows that the leap to \mathbb{O} is a continuation of a number-theoretic line, not a wild guess.

After this quaternion warm-up, the following days return to the full octonionic stage, now with a clearer sense of where $SU(2)$ “comes from” in the algebraic background.

References

- [1] J. H. Conway and D. A. Smith, *On Quaternions and Octonions*, A.K. Peters, 2003.
- [2] W. R. Hamilton, “On quaternions; or on a new system of imaginaries in algebra,” *Philos. Mag.* **25**, 489–495 (1844).

Quaternions provide a four-dimensional rehearsal: a rigid $1 \oplus 3$ pattern and an $SU(2)$ of unit quaternions that foreshadow the full octonionic stage of one generation.