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Nonassociativity as the source of hierarchies

When $(ab)c \neq a(bc)$ turns into mass and scale gaps

Key Insight. Octonions are nonassociative: in general $(ab)c \neq a(bc)$. Far from a nuisance, this failure of associativity can be measured by the *associator*, and its norm feeds directly into the size of hierarchies. In the octonionic model, the huge gaps between electron and top mass, or between Planck and electroweak scales, are not independent miracles: they are controlled by how strongly the internal multiplication fails to be associative in selected directions.

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Associativity lost, structure gained

In the complex numbers and quaternions we have

$$(ab)c = a(bc) \quad \text{for all } a, b, c.$$

For octonions $\mathbb{O}$ this is no longer true. The deviation is captured by the *associator*

$$[a, b, c] := (ab)c - a(bc).$$

Nonassociativity means $[a, b, c] \neq 0$ for suitable triples (a, b, c).

At first sight this looks like a technical complication. But in an operator-based model, $[a, b, c]$ is a structured quantity whose norm can be used as a measure of "how curved" the internal multiplication is in a given region of the algebra.

Associator norms as hierarchy seeds

The key idea is simple:

> Strong nonassociativity (large $\|[a, b, c]\|$) correlates with *large* hierarchies; near-associative directions correspond to *small* gaps.

Schematically, for suitable triples (a, b, c) associated with sectors of the internal space, one can write

$$\text{hierarchy factor} \ \sim \ \exp\bigl(\gamma \, \|[a, b, c]\|\bigr),$$

with a model-dependent constant γ. Large associator norms then naturally give exponentials of order 10^{10} or 10^{30}—exactly the kind of huge ratios seen between Planck and electroweak scales, or between electron and top mass.

Flat vs. curved internal directions

Not all directions in $\mathbb{O}$ are equally nonassociative:

- Quaternion subalgebras inside $\mathbb{O}$ are associative: $[a, b, c] = 0$ whenever a, b, c lie in the same quaternionic subalgebra. These are "flat" directions.
- Triples that genuinely probe the full octonionic structure typically have $[a, b, c] \neq 0$. These are "curved" directions.

This suggests a qualitative picture:

- Light fermions and low-energy scales live predominantly in almost quaternionic (nearly associative) directions.
- Heavy fermions and high-energy scales probe strongly octonionic (strongly nonassociative) directions.

The associator becomes a geometric dial between "light" and "heavy".

From internal curvature to physical gaps

In curved spacetime geometry, curvature scalars control focusing of geodesics and tidal forces. In the octonionic internal geometry, the associator norm plays an analogous role:

- It tells us how violently internal products deviate from the naive, associative expectation.
- Through the operator toolbox (rotors, compressors, radius operator), this deviation feeds into eigenvalue spectra and hence into physical scales and masses.

Symbolically one can write

$$m_{\text{heavy}}/m_{\text{light}} \ \sim \ F\bigl(\|[a, b, c]\|\bigr),$$

for some monotone function F determined by the detailed embedding in $H_3(\mathbb{O})$.

Andreas Müller, Kempten University of Applied Sciences, andreas.mueller@hs-kempten.de

Why this matters conceptually

The nonassociativity day adds an important layer to the Advent story:

1. It explains why large hierarchies are *allowed* and *natural* in an octonionic setting: they are the rule, not the exception, once associativity is dropped.
2. It connects an abstract algebraic property—failure of associativity—to directly observable quantities: mass ratios and scale separations.
3. It sharpens the contrast with associative models, where such large gaps often have to be enforced by hand via fine-tuned potentials or additional symmetries.

If future work can make this link between associator norms and concrete hierarchies numerically sharp, nonassociativity would move from a curious mathematical footnote to a central player in explaining why the physical world has the extreme scales we observe.

References

[1] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, 1966.

[2] J. C. Baez, "The octonions," *Bull. Amer. Math. Soc.* **39**, 145–205 (2002).

[3] [Internal notes on hierarchy patterns and associators, see `chap02_neu.tex; chap05_neu.tex`.]

In an octonionic world, huge hierarchies do not come from fine-tuning potentials—they follow the size of the associator: how far internal multiplication strays from associativity.