December 1, 2025

# Quaternions as a prototype: $SU(2)$ and the weak force

**The $1 \oplus 3$ pattern that scales up to octonions**

**Key Insight.** Before we climb to the eight-dimensional octonions, it is worth pausing at their four-dimensional cousins, the quaternions $\mathbb{H} \cong \mathbb{R}^4$. The quaternion units realise a rigid $1 \oplus 3$ split of scalar and vector parts, and their automorphism group contains $SU(2)$—the very group of weak isospin. The octonionic $8 \times 8$ action simply doubles this pattern: what $\mathbb{H}$ does for $SU(2)$, $\mathbb{O}$ does for the full internal structure of one generation.

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## The quaternion stage in four dimensions

THE quaternions

$$\mathbb{H} = \{\, a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \mid a, b, c, d \in \mathbb{R} \,\}$$

form a four-dimensional division algebra with basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Two structural facts are important for physics:

- $\mathbb{H}$ splits as scalar $\oplus$ vector:

  $$\mathbb{H} \cong \mathbb{R} \oplus \mathbb{R}^3,$$

  i.e. $1 \oplus 3$.

- Left and right multiplication by unit quaternions act as rotations on the vector part $\mathbb{R}^3$.

This already looks like a toy model for spin and isospin: one distinguished scalar component, three correlated vector components.

## $SU(2)$ inside the quaternion automorphisms

Unit quaternions form a group isomorphic to $SU(2)$:

$$\{q \in \mathbb{H} \mid |q| = 1\} \;\simeq\; SU(2).$$

Left multiplication by such a unit quaternion acts as an $SU(2)$ matrix on the $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ components. Concretely:

- The scalar part $a$ stays invariant.
- The vector part $(b, c, d)$ transforms as a 3-vector under $SO(3)$, double-covered by $SU(2)$.

In this language, weak isospin $SU(2)$ is not an abstract label group; it is *literally* the automorphism group of a number system.

## Action matrices and the $1 \oplus 3$ pattern

In the full model, internal degrees of freedom are encoded via action matrices built from left and right multiplication:

$$L_q : x \mapsto qx, \qquad R_q : x \mapsto xq.$$

Already for $\mathbb{H}$, these left/right actions decompose the four-dimensional real space into a scalar plus a three-dimensional subspace. In matrix form this appears as a rigid $1 \oplus 3$ block structure in the $4 \times 4$ representation of $L_q$ and $R_q$.

Physically:

- The singlet direction (scalar part) is a natural candidate for an isospin-neutral component.
- The triplet directions (vector part) form a weak isospin triplet under $SU(2)$.

This is the exact pattern that will be reused and doubled when we move to the octonionic $8 \times 8$ actions.

## From quaternions to octonions

The step from $\mathbb{H}$ to $\mathbb{O}$ looks dramatic—nonassociativity, seven imaginary units—but in the action-matrix picture it becomes transparent:

- The four real components of $\mathbb{H}$ become eight real components for $\mathbb{O}$.
- The $1 \oplus 3$ pattern of scalar/vector parts becomes a richer pattern that can host one full generation of internal quantum numbers.
- The role of $SU(2) \subset \mathrm{Aut}(\mathbb{H})$ is taken over by $G_2 \subset \mathrm{Aut}(\mathbb{O})$.

From this vantage point, $SU(2)$-weak is not an isolated gauge factor; it is the visible shadow of quaternionic automorphisms inside a larger octonionic structure.

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## Why this detour matters

The quaternion day serves three purposes in the Advent story:

1. **Familiarity:** it connects the exotic octonionic picture to the well-known role of $SU(2)$ in spin and weak isospin.
2. **Pattern recognition:** it highlights the $1 \oplus 3$ structure that later scales up to the more intricate block structure of the octonionic action.
3. **Continuity:** it shows that the leap to $\mathbb{O}$ is a continuation of a number-theoretic line, not a wild guess.

After this quaternion warm-up, the following days return to the full octonionic stage, now with a clearer sense of where $SU(2)$ "comes from" in the algebraic background.

## References


[1] J. H. Conway and D. A. Smith, *On Quaternions and Octonions*, A.K. Peters, 2003.

[2] W. R. Hamilton, "On quaternions; or on a new system of imaginaries in algebra," *Philos. Mag.* **25**, 489–495 (1844).


*Quaternions provide a four-dimensional rehearsal: a rigid $1 \oplus 3$ pattern and an $SU(2)$ of unit quaternions that foreshadow the full octonionic stage of one generation.*


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