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Algebraic gravity: from defects to Einstein prototypes

Curvature without a background stage

Key Insight. In a matrix–first framework there is no prior spacetime on which fields "live". What replaces geometry are algebraic relations between states and generators. The same operators that act as internal symmetries also define *defects*: commutator and associator obstructions that measure the failure of "parallel transport" inside the algebra. From these defects one can build curvature–like tensors and even explicit Einstein–prototype relations, derived by direct calculation for simple generator pairs. No cosmology is assumed; the claim is only structural: under clear algebraic assumptions, gravity–shaped equations appear as identities between defect norms and matter bilinears.

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From this point on, the calendar stops being primarily a dictionary of exotic algebra and becomes a test of *structural translation.* If the same operator language can organise spin, charge, and mixing, then the next honest question is not "can we also talk about gravity?", but rather: what is the *minimal algebraic residue* of curvature once we strip away the continuum vocabulary? Today's sheet takes that stance seriously: it does not start from manifolds, metrics, or differential forms. It starts from the only thing the matrix–first framework gives us for free — actions on states — and asks what kind of "failure" those actions can measure.

In ordinary geometry, curvature is detected by transporting something around a loop and comparing the outcome with what "should" have happened. In operator language, the same idea is encoded by commutators: two infinitesimal moves do not commute if there is curvature. Here the role of "moves" is played by left multiplications L_X, and the role of "loop failure" is the commutator $[L_X, L_Y]$. No interpretation is needed beyond the algebra: a commutator is either zero or it is not, and its size can be quantified by traces and quadratic forms.

This is the whole point of the day: build curvature-like and Einstein-like objects *without importing* geometric machinery. The only admissible ingredients are operators generated by the algebra, and only admissible outputs are invariants you can compute from them. In that sense, the sheet is deliberately conservative: every displayed tensor is a defect functional constructed from commutators or associators, and every "equation" is a relation between such functionals. That is what makes the discussion exact at the matrix level, even when it remains agnostic about any continuum picture.

Matrix first: geometry as an internal relation

The starting point is radical but precise: states are matrices $M \in M_n(\mathbb{O})$, and "motion" is not defined on a given manifold but by transformations internal to the algebra. A canonical infinitesimal change is generated by a matrix X via

$$M \;\mapsto\; M' = M + [X, M].$$

What matters is then not a coordinate distance but an invariant measure of separation between states, for example a trace metric of the form

$$d(M_1, M_2) \;=\; \mathrm{Tr}\big((M_1 - M_2)(M_1 - M_2)\big).$$

This is the setting in which "curvature" must be reformulated: as a defect of composing transformations, not as a tensor written on a background spacetime.

Defects from commutators: an algebraic curvature seed

For a generator X we encode left multiplication by the operator $L_X(M) = XM$. The basic defect is the commutator

$$\mathcal{D}(X, Y)(M) \;:=\; [L_X, L_Y](M) \;=\; X(YM) - Y(XM).$$

It is the direct analogue of the differential–geometric identity $[\nabla_\mu, \nabla_\nu]$: going around a small loop generated by (X, Y) fails to close when $\mathcal{D}(X, Y) \neq 0$.

A natural "curvature strength" associated with (X, Y) is obtained by a trace norm,

$$\|\mathcal{D}(X, Y)(M)\|^2 \;:=\; \mathrm{Tr}\big(\mathcal{D}(X, Y)(M)\,\mathcal{D}(X, Y)(M)\big).$$

This definition is purely algebraic and requires no continuum limit.

A fully explicit prototype: defect $\propto$ matter

Because the octonions are nonassociative, the defect contains more than an ordinary commutator effect. In fact, for specific octonionic units one can compute the defect exactly.

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Example (computed). For the octonionic pair (e_1, e_2) one finds, for arbitrary M,

$$\mathcal{D}(e_1, e_2)(M) = 4\, e_7\, M.$$

Taking the trace norm yields

$$\|\mathcal{D}(e_1, e_2)(M)\|^2 = \mathrm{Tr}\big((4e_7 M)(4e_7 M)\big) = 16\, \mathrm{Tr}(M^2).$$

The right–hand side is the simplest universal "matter" bilinear,

$$\mathcal{B}(M, M) \;:=\; \mathrm{Tr}(M^2).$$

Thus we obtain a concrete Einstein–prototype identity of the form

$$\boxed{\|\mathcal{D}(e_1, e_2)(M)\|^2 \;=\; 16\, \mathcal{B}(M, M)}\,.$$

Nothing here is postulated: the proportionality factor is fixed by the octonionic multiplication table and the definition of the defect.

This is the minimal message of "algebraic gravity": *defect strength* (geometry–like) and *bilinear density* (matter–like) are locked together by structure.

Associators: curvature beyond the commutator

The commutator defect already behaves like curvature, but octonions carry a deeper obstruction: the associator

$$\{x, y, z\} = (xy)z - x(yz).$$

Associators are not optional decorations; they are the intrinsic defect of the algebra, and therefore the natural candidate for a curvature source in a matrix–first setting.

Via left multiplication one packages associators into defect operators $\mathfrak{A}(X, Y, Z)$ and then forms quadratic invariants, schematically,

$$\mathcal{R}(\cdots; M) \;\sim\; \Phi\big(\mathfrak{A}(\cdots; M)^\dagger\, \mathfrak{A}(\cdots; M)\big)\,,$$

for a suitable trace functional Φ. By contraction one can define an algebraic Einstein–tensor prototype,

$$\mathcal{G}^{\mathrm{assoz}}_{\mu\nu}(M) := \mathcal{R}_{\mu\nu}(M) - \tfrac{1}{2}\, \mathcal{R}(M)\, \eta_{\mu\nu}.$$

At this level it is an internal tensor built from associator defects. The continuum interpretation is a separate step.

What can be claimed without cosmology

The status of the construction is deliberately modest and therefore robust.

- One can define defect operators $\mathcal{D}(X, Y)$ and associator defects $\mathfrak{A}(X, Y, Z)$ directly inside the algebra.
- One can build curvature–like scalars and tensors as trace norms of these defects.
- In simple cases one can compute explicit Einstein–prototype relations (defect norm $\propto$ bilinear matter density) by hand.

The remaining conceptual step is to specify assumptions under which these algebraic tensors admit an effective continuum interpretation: namely, a choice of coarse–graining, a distinguished 4D projection, and a regime in which the defect invariants behave like curvature invariants of an emergent metric. This sheet stays strictly on the algebraic side and records what is already derivable there.

References

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From the same matrix–first operators that organise internal physics one can define defect and associator tensors. In simple cases their trace norms obey explicit Einstein–prototype relations: gravity–shaped structure emerges as an algebraic identity.

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