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G_2 as the minimal exceptional symmetry

The 14-dimensional gatekeeper of the octonions

Key Insight. G_2 is the smallest exceptional Lie group and the full automorphism group of the octonions. Every map in G_2 preserves octonionic multiplication and the norm. In the model, this makes G_2 the *gatekeeper* of the internal structure: any internal operator, symmetry or interaction must respect G_2 invariance. Today we meet G_2 as the minimal exceptional symmetry from which the larger exceptional group F_4 will later emerge.

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Every number system has a symmetry group—the set of transformations that preserve its essential structure. For the real numbers $\mathbb{R}$, the symmetry is almost trivial: only multiplication by ± 1. For the complex numbers $\mathbb{C}$, it is the unit circle $U(1)$ of complex numbers of modulus one, isomorphic to rotations in the plane and representable by real 2×2 rotation matrices. For the quaternions $\mathbb{H}$, it's the 3-sphere of unit quaternions, which is isomorphic to the special unitary group $SU(2)$ of complex 2×2 matrices, as we saw on Dec 1.

What about the octonions? The symmetry group of the octonions—the transformations that preserve both the multiplication table and the norm—is called G_2. It's a 14-dimensional Lie group, and it's the smallest of the five exceptional Lie groups G_2, F_4, E_6, E_7, E_8. "Exceptional" means it doesn't fit into the infinite families of classical groups like $SU(n+1) = A_n$, $SO(2n+1) = B_n$, $SO(2n) = D_n$, or $Sp(n) = C_n$ (special unitary, special orthogonal or symplectic groups). It's a one-of-a-kind structure.

Why does this matter for physics? Because G_2 acts as a gatekeeper. If we're building a model where internal degrees of freedom live in an octonionic space, then any operator, any symmetry, any interaction must respect the G_2 structure. You can't just write down arbitrary matrices and hope they make sense — they have to be compatible with the rigid multiplication rules of the octonions.

This is a feature, not a bug. In conventional field theory, we have enormous freedom to choose gauge groups, representations, and couplings. This freedom is both a blessing and a curse: it allows us to fit data, but it doesn't explain why nature chose one particular set of parameters over another. With G_2 as the starting point, much of that freedom disappears. The structure is forced on us by the choice of octonions.

Here's a concrete example: G_2 contains $SU(3)$ as a subgroup. In fact, if you pick any imaginary octonion and ask which transformations leave it fixed, you get a copy of $SU(3)$. This is not a coincidence — it's a hint that the color symmetry of the strong force might be a natural subgroup of the octonionic automorphism group.

But G_2 is not large enough to contain the full Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ directly. For that, we'll need to move to a larger structure: the Albert algebra $H_3(\mathbb{O})$ and its automorphism group F_4, which we'll meet in the coming days. G_2 is the first rung on the exceptional ladder — the minimal exceptional symmetry that controls the octonionic stage itself.

Think of today's sheet as introducing the guardian of the internal space. G_2 is not just an exotic group in a classification table. It's the symmetry that makes the octonions work as a number system, and in the model we're building, it's the symmetry that constrains which internal structures are allowed and which are forbidden.

Let us see how G_2 guards the octonionic multiplication table.

What is G_2?

The group G_2 can be defined in many equivalent ways. For the octonionic story, the most natural is:

$$G_2 = \mathrm{Aut}(\mathbb{O}),$$

the group of all real-linear transformations of $\mathbb{O}$ that preserve the octonionic product (and hence the standard norm). It is a 14-dimensional, compact, connected, simply connected simple Lie group and the smallest of the five exceptional Lie groups.

Concretely:

- G_2 preserves the multiplication rules among the seven imaginary units $e_1, \ldots, e_7$.
- It preserves the standard norm $|x|^2 = x\bar{x}$.
- It acts transitively on the unit sphere of imaginary octonions, with stabiliser isomorphic to $SU(3)$.

Equivalently, one can view G_2 as the subgroup of $SO(7)$ that preserves both the Euclidean inner product and a distinguished 3-form (or, equivalently, a cross product) on the 7-dimensional space $\mathbb{R}^7$ of imaginary octonions (those with vanishing real coordinates).

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In representation-theoretic terms, G_2 has rank 2, with two smallest non-trivial representations:

- the 7-dimensional fundamental representation on the imaginary octonions, and
- the 14-dimensional adjoint representation on its Lie algebra $\mathfrak{g}_2$.

Under the $SU(3)$ subgroup that fixes a chosen imaginary unit, these representations decompose as

$$7 \;\cong\; 1 \oplus 3 \oplus \bar{3}, \qquad 14 \;\cong\; 8 \oplus 3 \oplus \bar{3},$$

a pattern that already hints at colour-like structures.

In other words, G_2 is the full continuous symmetry group of the octonionic number system itself.

Why "minimal exceptional" matters

As a Lie group, G_2 is:

- too small to host all Standard Model symmetries directly,
- but large enough to control the essential nonassociative structure of $\mathbb{O}$,
- and exceptional — meaning it does not fit into the infinite A_n, B_n, C_n, D_n series.

This makes G_2 an ideal starting point:

- It is restrictive enough to strongly constrain internal operators.
- It is flexible enough to embed subgroups that resemble the Standard Model structure $SU(3)_C \times SU(2)_L \times U(1)_Y$ in appropriate ways.
- It naturally sits inside the larger exceptional group F_4, the automorphism group of the Albert algebra $H_3(\mathbb{O})$.

$\mathbf{G_2}$ as a gatekeeper of allowed operators

In the model, internal operators (heptagon operator, radius operator, rotors, compressors) are not arbitrary matrices; they must be compatible with the G_2 structure. Informally:

> If an operator would break G_2 in an uncontrolled way, it is not part of the fundamental toolbox.

This has two important consequences:

1. **Restricted parameter space:** many couplings and mass terms that are allowed in a generic field-theory Lagrangian are simply forbidden by G_2.
2. **Natural subgroups:** gauge groups that actually appear (or approximate) in low-energy physics are required to arise as structurally compatible subgroups of G_2 (and later in F_4).

G_2 thus serves as a first filter between "any algebraic construction on $\mathbb{R}^8$" and "constructions that respect the octonionic number system".

From $\mathbf{G_2}$ to $\mathbf{F_4}$

Later in the calendar, the Albert algebra $H_3(\mathbb{O})$ will appear, and with it the larger exceptional group F_4:

$$F_4 = \mathrm{Aut}\big(H_3(\mathbb{O})\big).$$

The relationship is hierarchical:

- G_2 controls the algebra of $\mathbb{O}$ itself.
- $H_3(\mathbb{O})$ builds 3×3 Hermitian matrices over $\mathbb{O}$.
- F_4 controls the automorphisms of this larger Jordan algebra $H_3(\mathbb{O})$.

From the perspective of the calendar:

- early days: G_2 and triality structure the octonionic stage,
- middle days: F_4 organises the symmetry atlas on the Albert algebra,
- later days: potentials and equilibria on this atlas fix physical scales and constants.

G2 is the first rung on this exceptional ladder.

References

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G_2 is the 14-dimensional automorphism group of the octonions and the minimal exceptional symmetry. In the model it acts as a gatekeeper: only operators and symmetries compatible with G_2 are admitted into the internal stage on which all later structures are built.

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