

Homogeneous Coordinates

Translations of points in \mathbb{R}^n does not correspond directly to a linear transform. Homogeneous coordinates are used to model translations using matrix multiplication.

Homogeneous Coordinates in \mathbb{R}^2

Each point (x, y) in \mathbb{R}^2 can be identified with the point (x, y, H) , $H \neq 0$, on the plane in \mathbb{R}^3 that lies H units above the xy -plane.

$(x, y) \rightarrow (x + h, y + k)$ can be represented by,

$$\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + h \\ y + k \\ 1 \end{pmatrix}$$

Now rotate a triangle $((1, 1), (2, 4), (3, 1))$ by $\frac{\pi}{2}$ radians counterclockwise about the point $(0, 1)$.

$$d = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Shift down by 1,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} d = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now rotate,

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Shift up by 1,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 0 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

This give us the points, $(0, 2), (-3, 3), (0, 4)$

In \mathbb{R}^3

So, $(x, y, z) \rightarrow (x + h, y + k, z + l)$ can be represented by,

$$\begin{pmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + h \\ y + k \\ z + l \\ 1 \end{pmatrix}$$

Rotation in \mathbb{R}^3

about x_2 -axis by π rads.

To find $A = (a_1, a_2, a_3)$. We can find $T(e_1)$ as $T(e_1) = Ae_1 = a_1$. We can similarly find all the columns of A .

$$T(e_1) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(e_3) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Projection

Onto the plane $x_3 = 4$

What should we do?

1. Shift everything down by 4 (Homogeneous Coordinates)
2. Apply the projection (Homogeneous Coordinates)
3. Shift everything back up by 4 (Homogeneous Coordinates)

Amusing a vector \vec{v} ,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vec{v}$$

~~You could drive the matrix but that is trivial and left as an exercise to the reader.~~

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vec{v}$$