

SVD

SVD

Suppose A is an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$ and $m \geq n$. Then A has the decomposition $A = U\Sigma V^T$ where,

$$\begin{bmatrix} D \\ 0_{m-n,n} \end{bmatrix} \quad D = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

U is a $m \times m$ orthogonal matrix, and V is a $n \times n$ orthogonal matrix.

If $m < n$ then $\Sigma = [D \quad 0_{m,n-m}]$ with everything else being the same.

Proof

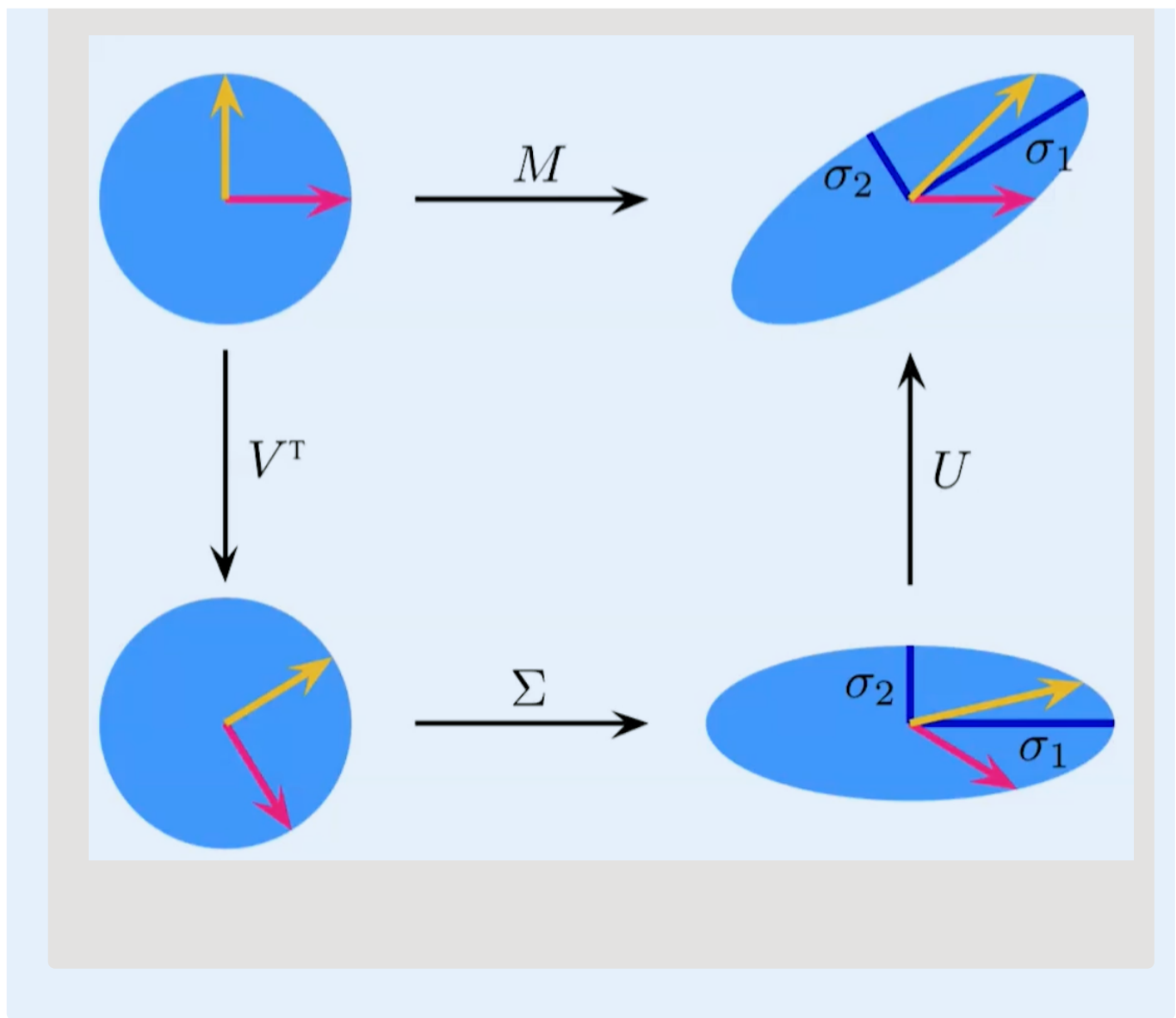
Our proof is similar to the proof for diagonalization. We construct $V = (\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n)$ and set

$$\sigma_i \vec{u}_i = A\vec{v}_i, \quad \sigma_i = \|A\vec{v}_i\|$$

$$\begin{aligned} \text{Thus: } AV &= A(\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n) = (A\vec{v}_1 \ A\vec{v}_2 \ \dots \ A\vec{v}_n) \\ &= (\sigma_1 \vec{u}_1 \ \sigma_2 \vec{u}_2 \ \dots \ \sigma_n \vec{u}_n) \\ &= (\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n) \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} = U\Sigma \end{aligned}$$

Thus, $AV = U\Sigma$, or $A = U\Sigma V^T$.

Geometric Interpretation



Computing

Suppose A is $m \times n$ and has rank r .

1. Compute the squared singular values of $A^T A$, σ_i^2 , and construct Σ .
2. Compute the unit [singular vectors](#) of $A^T A$, \vec{v}_i , use them to form V .
3. Compute an orthonormal basis for $\text{Col}A$ using,

$$\vec{u}_i = \frac{1}{\sigma_i} A \vec{v}_i, \quad i = 1, 2, \dots, r$$

If necessary, extend the set $\{\vec{u}_i\}$ to form an orthonormal basis for \mathbb{R}^m and use the basis to form U .

Example

Find the SVD of $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}, \lambda_1 = 9, \lambda_2 = 4, \sigma_1 = 3, \sigma_2 = 2$$

So,

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now find the right-singular vectors (i.e. eigenvectors of $A^T A$) for V ,

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence,

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Finally find U using the left-singular vectors (i.e. orthonormal basis for $\text{Col}A$). This can be found by normalizing $A\vec{v}_i$.

$$\vec{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

You may note that \vec{u}_3, \vec{u}_4 are arbitrary, and this is true. They are any orthonormal vectors to \vec{u}_1, \vec{u}_2 due to $\text{Rank}A = 2$.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note: For finding $U \in \mathbb{R}^m$ if we have the vectors in $\text{Col}A$, and they are not enough to make an $m \times m$ matrix, then we will create any arbitrary vectors that are orthonormal to the vectors that you already have. Then we will run the [Gram-Schmidt Process](#) on these newly found vectors. All the vectors must be normalized at the end.

Properties of V and U

For $i \leq r = \text{Rank}A$

- $\vec{v}_1, \dots, \vec{v}_r$ is an orthonormal basis for $\text{Row}A$.
- $\vec{v}_{r+1}, \dots, \vec{v}_n$ is an orthonormal basis for $\text{Nul}A$.
- $\vec{u}_1, \dots, \vec{u}_r$ is an orthonormal basis for $\text{Col}A$.

- $\vec{u}_{r+1}, \dots, \vec{u}_m$ is an orthonormal basis for $\text{Nul}A^T$.