Orthogonal Diagonalization

Eigenvectors orthogonality

If A is a symmetric matrix, with eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two distinct eigenvalues, then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

Proof

$$\begin{aligned}
\lambda_1\vec{v}_1\cdot\vec{v}_2 &= A\vec{v}_1\cdot\vec{v}_2 && \text{using } A\vec{v}_i=\lambda_i\vec{v}_i \\
&= (A\vec{v}_1)^T\vec{v}_2 && \text{using the definition of the dot product} \\
&= \vec{v}_1{}^T A^T\vec{v}_2 && \text{property of transpose of product} \\
&= \vec{v}_1{}^T A\vec{v}_2 && \text{given that } A=A^T \\
&= \vec{v}_1\cdot A\vec{v}_2 && \\
&= \vec{v}_1\cdot\lambda_2\vec{v}_2 && \text{using } A\vec{v}_i=\lambda_i\vec{v}_i \\
&= \lambda_2\vec{v}_1\cdot\vec{v}_2 &&
\end{aligned}$$

But $\lambda_1 \neq \lambda_2$ so $\vec{v}_1\cdot\vec{v}_2=0$.

Example PDP^T

Diagonalize A using an orthogonal matrix, P. The eigenvalues of A are given.

$$A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \quad \lambda=-1,1$$

We need to find the Eigenvectors of A.
Skipping the computation, we get,

$$\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \lambda = -1$$
$$\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \lambda = 1$$
$$\vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \lambda = 1$$

Now we must find PDP^T,
The columns of P are the **orthonormalized** Eigenvectors, and D is the eigenvalue.

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finding P^T is trivial and left to the reader.

Properties

- If $A = PDP^T$, if A is a symmetric matrix and it is diagonalizable.
- And the converse, A is a symmetric matrix if $A = PDP^T$ is also true.