# Diagonal Matrices

Finding powers of a diagonal matrix, is very simple.

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

# Diagonalizable

If $A$ is similar to diagonal matrix $D$ ($A = PDP^{-1}$) then A is diagonalizable.

$$A = PDP^{-1}$$

$$A = \begin{bmatrix} v_1 & v_2 \dots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_4 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \dots & v_n \end{bmatrix}^{-1}$$

## 🔥 Important

$v_1 \dots v_n$ are eigenvectors
$\lambda_1 \dots \lambda_n$ are eigenvalue
$A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.
Invertibility has no effect on diagonalizability

## ✏️ Proof

We construct $P = (\vec{v}_1 \; \vec{v}_2 \; \ldots \vec{v}_n)$. Then

$$AP = A(\vec{v}_1 \; \vec{v}_2 \; \ldots \vec{v}_n)$$
$$= (A\vec{v}_1 \; A\vec{v}_2 \; \ldots A\vec{v}_n)$$
$$= (\lambda_1\vec{v}_1 \; \lambda_2\vec{v}_2 \; \ldots \lambda_n\vec{v}_n)$$

$$AP = (\vec{v}_1 \; \vec{v}_2 \; \ldots \vec{v}_n) \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & & \\ & & & \lambda_n \end{pmatrix}$$

$$= PD$$

Or, $A = PDP^{-1}$.

# Example

Diagonalize $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$

Eigenvalue $= 2, -1$

Eigenvectors $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$A = PDP^{-1}$$
$$= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

✎ **Note**

This is a special case and is not always true,

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

If $A$ is $n \times n$ and has $n$ distinct eigenvalues, then $A$ is diagonalizable.
$A$ does not **have to** have $n$ distinct eigenvalues for $A$ to be diagonalizable.

Suppose

- $A$ is any $n \times n$ real matrix
- $A$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, $k \leq n$
- $a_i =$ **algebraic** multiplicity of $\lambda_i$
- $g_i =$ dimension of $\lambda_i$ eigenspace, or the **geometric** multiplicity

Then

- $A$ is diagonalizable $\Leftrightarrow \Sigma g_i = n \Leftrightarrow g_i = a_i$ for all $i$
- $A$ is diagonalizable $\Leftrightarrow$ the eigenvectors, for all eigenvalues, together form a basis for $\mathbb{R}^n$.

# Properties

- $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.
- Invertibility has no effect on diagonalizability.
- If A has $n$ distinct eigenvalues, then $A$ is diagonalizable. (The converse is not necessarily true)

# Find Diagonalizability

$$A = PDP^{-1}$$

Finding $D$ is straight forward. It is simply the eigenvalue of $A$.
$P$ **MUST** be Invertible.

- If $\sum g_i = n$, $A$ is diagonalizable.
- If $g_i = a_i$ for all $i$, $A$ is diagonalizable.
- If the eigenvectors of $A$ form a basis in $\mathbb{R}^n$, $A$ is diagonalizable.

# Repeated Eigenvalue

Diagonalize $A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$ with eigenvalues $\lambda_1 = 1, \lambda_2 = \lambda_3 = 3$.

# Find eigenvectors of $\lambda_1 = 1$:

$$\text{Null}(A - \lambda_1 I) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

# Find eigenvectors of $\lambda_2 = \lambda_3 = 3$:

$$\text{Null}(A - \lambda_2 I) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

# Find $P$

$$P = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -1 & -4 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

# Find $D$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$