Formulas

The Spectral Theorem

An $n \times n$ symmetric matrix A has the following properties,

- All eigenvalues of A are real.
- The eigenspaces are mutually orthogonal.
- A can be diagonalized as $A = PDP^T$, where D is diagonal and P is orthogonal.

Non-Negative Eigenvalues

The eigenvalues of $A^T A$ are non-negative.

Proof

Proof: recall that $\vec{v}_j^T \vec{v}_j = \vec{v}_j \cdot \vec{v}_j = \|\vec{v}_j\|^2 = 1$ because $\vec{v}_j$ are **unit** eigenvectors of $A^T A$.

$$\|A\vec{v}_j\|^2 = (A\vec{v}_j)^T A\vec{v}_j = \vec{v}_j A^T A\vec{v}_j = \lambda_j \vec{v}_j^T \vec{v}_j = \lambda_j \geq 0.$$

Therefore:

- the eigenvalues of $A^T A$ must be real and non-negative
- the singular values of A, which are the square roots of the eigenvalues, must also be real and non-negative

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