Singular Vectors

Theorem

Suppose $\vec{v}_i$ are the n orthogonal eigenvectors of A^TA, ordered so that their corresponding eigenvalues satisfy $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$. Suppose also that A has r non-zero singular values, $r \leq n$. Then the set of vectors,

$$\{\vec{v}_{r+1}, \vec{v}_{r+2}, \ldots, \vec{v}_n\}$$

is an orthogonal basis for $\mathrm{Nul} A$, and the set

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$$

is an orthogonal basis for $\mathrm{Row} A$, and $\mathrm{Rank} A = r$.

Proof

For a set of vectors to form an orthogonal basis for a subspace they must be in that space, span the space, be independent, and mutually orthogonal.

- Each $\vec{v}_i$ is an eigenvector, so none of them are the zero vector.
- $\vec{v}_i$ are orthogonal and span $\mathbb{R}^n$ (they are eigenvectors of a symmetric matrix, A^TA).
- Recall that the lengths of $A\vec{v}_i$ are the singular values of A:

$$\|A\vec{v}_i\| = \sigma_i.$$

- Then if $\|A\vec{v}_i\| = 0$ for $i > r$, then $\vec{v}_i \in \mathrm{Nul} A$ for $i > r$.
- Then if $\|A\vec{v}_i\| \neq 0$ for $i \leq r$, then $\vec{v}_i$ cannot be in $\mathrm{Nul} A$ for $i \leq r$, they must be in $(\mathrm{Nul} A)^\perp = \mathrm{Row} A$, because $\{\vec{v}_i\}$ is an orthonormal set.

Thus, our basis for Nul$A$ is the set

$$\{\vec{v}_{r+1}, \vec{v}_{r+2}, \ \ldots \ , \vec{v}_n\}$$

and our basis for Row$A$ is the set

$$\{\vec{v}_1, \vec{v}_2, \ \ldots \ , \vec{v}_r\}$$

We must also describe why rank$A = r$.

- There are r vectors in our basis for Row$A$.
- Recall that $\dim(\text{Row}A) = \dim(\text{Col}A) = \text{rank}A$.

Using the same assumptions as above, it can be shown that,

$$\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_r\}$$

or

$$\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$$

is an orthogonal basis for $\text{Col}A$.

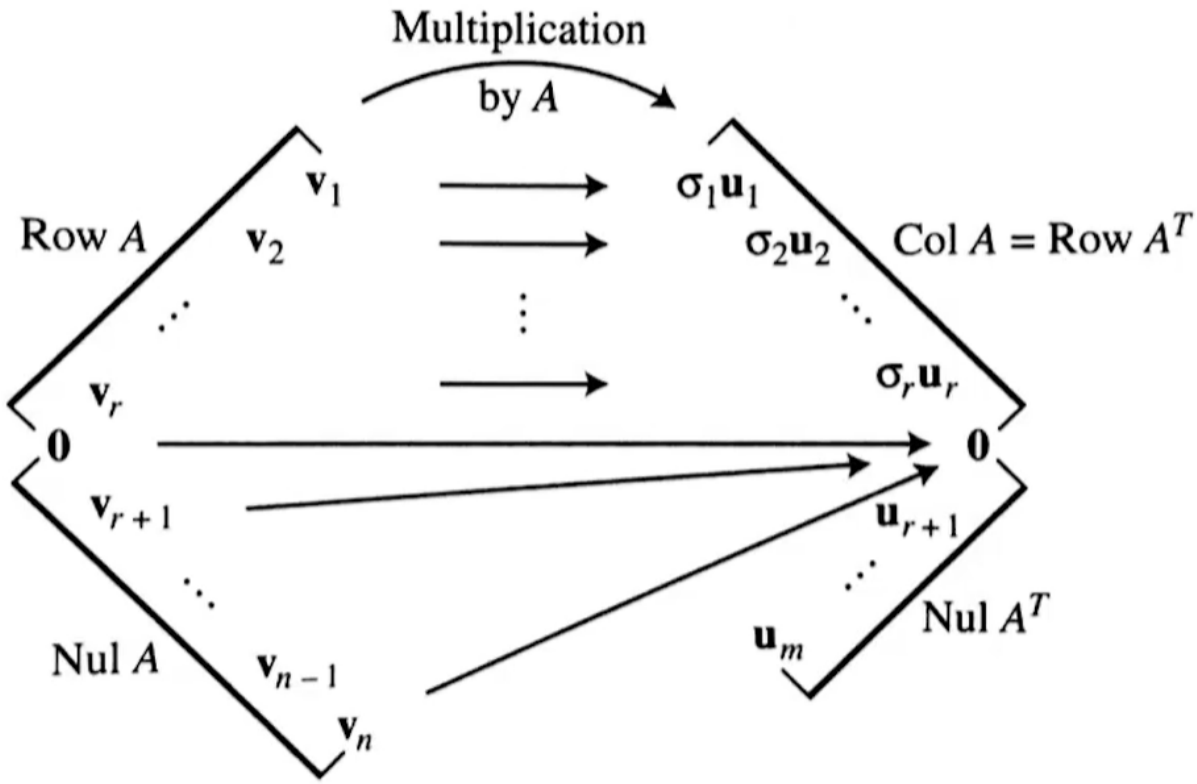
Proof >

- Each $A\vec{v}_i$ is a vector in Col$A$.
- $A\vec{v}_i$ and $A\vec{v}_j$ are orthogonal:

$$(A\vec{v}_i) \cdot (A\vec{v}_j) = \vec{v}_i^T A^T A\vec{v}_j = \lambda_j \vec{v}_i \cdot \vec{v}_j = 0$$

- For $i \leq r =$ rank$A$, $A\vec{v}_i$ are orthogonal and non-zero. So they must also independent and form an orthogonal basis for Col$A$.

Note that for $i > r$, $A\vec{v}_i = \vec{0}$ because $\vec{v}_i \in$ Nul$A$ for $i > r$.



Definition

The vectors $\{\vec{u}_i\}$ for $i \leq m$ are the **left singular vectors** of A.
The vectors $\{\vec{v}_i\}$ for $i \leq n$ are the **right singular vectors** of A.

- Left Singular Vectors are the basis for $\mathrm{Col} A$
- Right Singular Vectors are the eigenvectors of $A^T A$