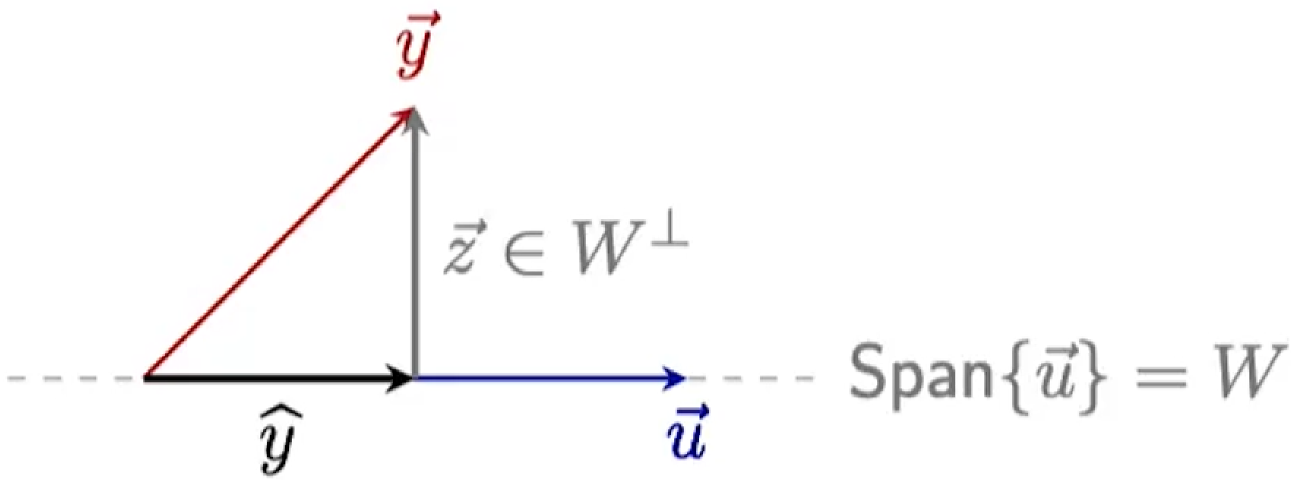
Projections

We have a subspace W (the span of $\vec{u}$), and we want to find the vector ($\hat{y}$) closest to $\vec{y}$ in W.
We also want to find $\vec{z} \in W^{\perp}$ such that $\vec{y} = \hat{y} + \vec{z}$.
Diagrammatically,



We know $\vec{z} \in W^{\perp}$, so:

$$\vec{z} \cdot \vec{u} = 0$$

We also know $\vec{y} = \hat{y} + \vec{z}$ and $\hat{y} = k\vec{u}$ ($k \in \mathbb{R}$), so:

$$\begin{aligned} \vec{z} &= \vec{y} - k\vec{u} \\ 0 &= (\vec{y} - k\vec{u}) \cdot \vec{u} \\ &= \vec{y} \cdot \vec{u} - k\vec{u} \cdot \vec{u} \\ k &= \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}, \qquad\qquad \vec{u} \neq \vec{0} \end{aligned}$$

So finally, $\hat{y} = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}$

Orthogonal Projection

Let non-zero $\vec{u} \in \mathbb{R}^n$, and $\vec{y} \in \mathbb{R}^n$. The orthogonal projection of $\vec{y}$ onto $\vec{u}$ is the vector in the span of $\vec{u}$ that is closest to $\vec{y}$.

$$\text{proj}_{\vec{u}}\vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}$$

Also, $\vec{y} = \hat{y} + \vec{z}$ and, $\vec{z} \in W^{\perp}$

From this we can conclude (look at the diagram),

$$||\vec{y}||^2 = ||\text{proj}_{\vec{u}}\vec{y}||^2 + ||\vec{z}||^2$$

Best Approximation

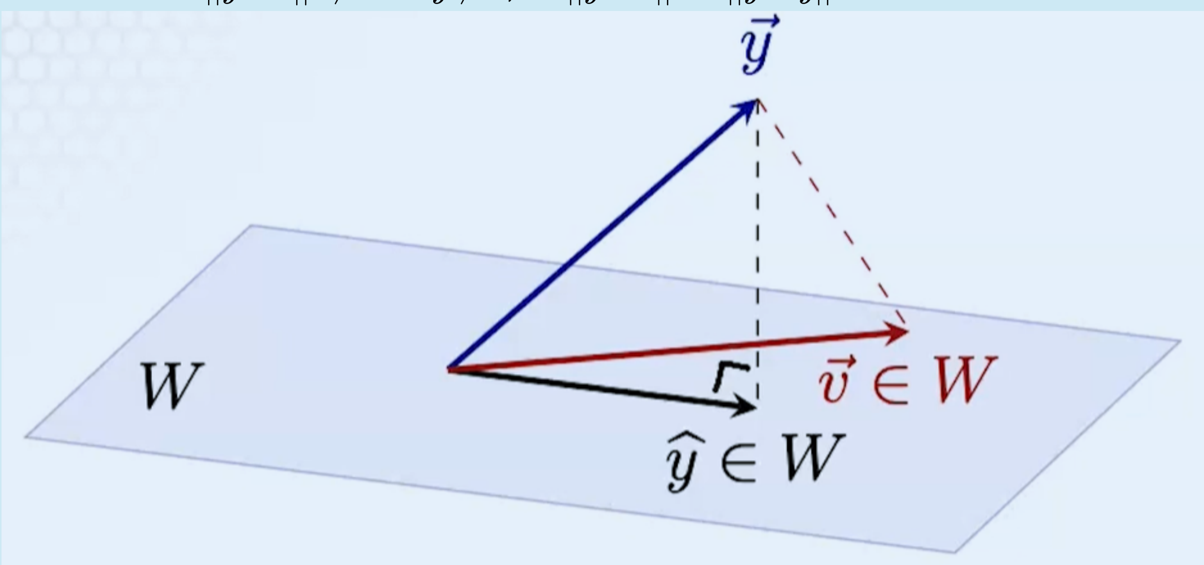
Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n, \vec{y} \in \mathbb{R}^n$, and $\hat{y}$ is the orthogonal projection of $\vec{y}$ onto W. Then for any $\vec{v} \neq \hat{y}, \vec{v} \in W$, we have

$$||\vec{y} - \hat{y}|| < ||\vec{y} - \vec{v}||$$

Proof

$\vec{y} - \vec{v} = \vec{y} - \vec{v} + (\hat{y} - \hat{y}) = (\hat{y} - \vec{y}) + (\hat{y} - \vec{v})$
Pythagorean Theorem: $||\vec{y} - \vec{v}||^2 = ||\hat{y} - \vec{y}||^2 + ||\hat{y} - \vec{v}||^2$
We know that $||\hat{y} - \vec{v}||^2 \neq 0$ as $\hat{y} \neq \vec{v}$, so $||\vec{y} - \vec{v}||^2 > ||\vec{y} - \hat{y}||^2$



Orthogonal Decomposition

Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Then, each $\vec{y} \in \mathbb{R}^n$ has a unique decomposition.

$$\vec{y} = \hat{y} + z, \quad \hat{y} \in W, \quad z \in W^{\perp}$$

If $\vec{u_1}, \ldots, \vec{u_n}$ is the orthogonal basis for W,

$$\hat{y} = \frac{\vec{y} \cdot \vec{u_1}}{\vec{u_1} \cdot \vec{u_1}} \vec{u_1} + \cdots + \frac{\vec{y} \cdot \vec{u_n}}{\vec{u_n} \cdot \vec{u_n}} \vec{u_n}$$

$\hat{y}$ is the orthogonal projection of $\vec{y}$ onto W