Determinants

A general way to compute it is,
If If A is an $n \times n$ matrix where $n = 1$,

$$\det(A) = a_{1,1}$$

If A is an $n \times n$ matrix where $n > 1$,

$$\det(A) = a_{1,1} \det(A_{1,1}) - a_{1,2} \det(A_{1,2}) + \cdots + (-1)^{n+1} a_{1,n} \det(A_{1,n})$$

$a_{i,j}$ means the element at the i^{th} row and the j^{th} column.
$A_{i,j}$ means the Matrix if you drop (get rid of) the i^{th} row and the j^{th} column.

Deriving it for a 2×2
Say we have $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{aligned} \det(A) &= (a_{1,1})(\det(A_{1,1})) - (a_{1,2})(\det(A_{1,2})) \\ \det(A) &= (a)(d) - (b)(c) \\ \det(A) &= ad - bc \end{aligned}$$

Using a Cofactor

The determinant of a matrix A can be computed down any row or column of the matrix.

For example, down the j^{th} column the determinant is:

$$\det(A) = a_{1,j} \det(A_{1,j}) - a_{2,j} \det(A_{2,j}) + \cdots + (-1)^{n+1} a_{n,j} \det(A_{n,j})$$

This would be useful for a matrix with a few 0's.
Say $A = \begin{bmatrix} 5 & 4 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$ find $\det(A)$
We will use the first column due to the 3 zeros.

$$
\begin{aligned}
\det(A) &= 5C_{1,1} + 0C_{2,1} + 0C_{3,1} + 0C_{4,1} \\
&= 5 \cdot (-1)^{1+1} \cdot \det\left(\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}\right) \\
3^{\text{rd}} \text{ column} \\
&= 5 \cdot (0C_{1,3} + 0C_{2,3} + 3C_{3,3}) \\
&= 5 \cdot \left(3 \cdot (-1)^{3+3} \det\left(\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\right)\right) \\
\text{Formula} \\
&= 5(3(1 \times 1 - 2 \times -1)) \\
&= \boxed{45}
\end{aligned}
$$

Triangular Matrices

The determinant of a triangular matrix is the product of the entries on the main diagonal.

Row Operations

Replacement/Addition

Add a multiple of one row to another.
This does $\boxed{\textbf{NOT}}$ effect the determinant.
$\boxed{\det A = \det B}$

Interchange

Interchange two rows to make B.
One swap means, $\boxed{\det B = -\det A}$.
Two One swap means, $\boxed{\det B = \det A}$.
We can continue this pattern

Scaling

Multiply a row by a non-zero scalar to make B.
$\boxed{\det B = k \det A}$

Invertibility

Important practical implication: if A is reduced to echelon form, by r interchanges of rows and columns, then

$$|A| = \begin{cases} (-1)^r \times (\text{product of pivots}), & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is singular} \end{cases}$$

Properties

1. $\det A = \det A^T$ (Transpose).
2. A is invertible if and only if $\det A \neq 0$.
3. $\det(AB) = \det A \cdot \det B$.
4. If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.

Geometric interpretation

Watch 3b1b

TLDR

Area of parallelogram spanned by the columns columns of an $n \times n$ matrix A is $\det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = ad - bc$.

The volume of the parallelepiped spanned by the columns of an $n \times n$ matrix A is $|\det A|$.

Linear Transformations

If we have $T : \mathbb{R}^n \to \mathbb{R}^n$, and S is a parallelogram in $\mathbb{R}^n$, then:

$$\text{volume}(T(S)) = |\det A| \cdot \text{volume}(S)$$

This can be extended to higher dimensions.