Coordinates, Dimension, and Rank

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$. Verify that $\vec{x}$ is in the span of $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, and calculate $[\vec{x}]_{\mathcal{B}}$.

In RREF

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

So, $\vec{x}$ is in the span of B and $[x]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Dimension

$\dim(\mathbb{R}^n) = n$
$\dim(\text{Null}(A)) = \text{no of free variables}$
$\dim(\text{Col}(A)) = \text{no of pivot columns}$

Rank

$\text{Rank}(A) = \dim(\text{Col}(A)) = \text{no of pivot columns}$
$\text{Rank}(A) + \dim(\text{Null}(A)) = \dim(\text{Col}(A)) + \dim(\text{Null}(A)) = \text{no of columns}$

Invertibility

If A is an $n \times n$ matrix. These conditions are equivalent.

1. A is invertible.
2. The columns of A are a basis for $\mathbb{R}^n$
3. Col $A = \mathbb{R}^n$.
4. Rank $A = \dim(\text{Col } A) = n$.
5. Null $A = \{\vec{0}\}$