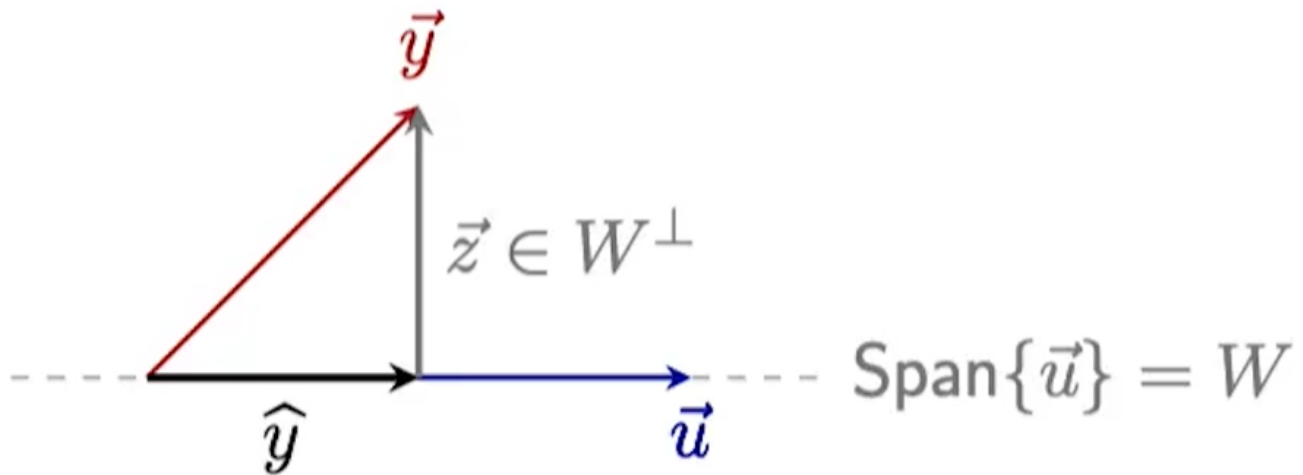


# Projections

We have a subspace  $W$  (the span of  $\vec{u}$ ), and we want to find the vector ( $\hat{y}$ ) closest to  $\vec{y}$  in  $W$ .

We also want to find  $\vec{z} \in W^\perp$  such that  $\vec{y} = \hat{y} + \vec{z}$ .

Diagrammatically,



We know  $\vec{z} \in W^\perp$ , so:

$$\vec{z} \cdot \vec{u} = 0$$

We also know  $\vec{y} = \hat{y} + \vec{z}$  and  $\hat{y} = k\vec{u}$  ( $k \in \mathbb{R}$ ), so:

$$\begin{aligned}\vec{z} &= \vec{y} - k\vec{u} \\ 0 &= (\vec{y} - k\vec{u}) \cdot \vec{u} \\ &= \vec{y} \cdot \vec{u} - k\vec{u} \cdot \vec{u} \\ k &= \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}, \quad \vec{u} \neq \vec{0}\end{aligned}$$

So finally,  $\hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$

## Orthogonal Projection

Let non-zero  $\vec{u} \in \mathbb{R}^n$ , and  $\vec{y} \in \mathbb{R}^n$ . The orthogonal projection of  $\vec{y}$  onto  $\vec{u}$  is the vector in the span of  $\vec{u}$  that is closest to  $\vec{y}$ .

$$\text{proj}_{\vec{u}} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

Also,  $\vec{y} = \hat{y} + \vec{z}$  and,  $\vec{z} \in W^\perp$

From this we can conclude (look at the diagram),

$$\|\vec{y}\|^2 = \|\text{proj}_W \vec{y}\|^2 + \|\vec{z}\|^2$$

## Best Approximation

### Best Approximation Theorem

Let  $W$  be a subspace of  $\mathbb{R}^n$ ,  $\vec{y} \in \mathbb{R}^n$ , and  $\hat{y}$  is the orthogonal projection of  $\vec{y}$  onto  $W$ . Then for any  $\vec{v} \neq \hat{y}$ ,  $\vec{v} \in W$ , we have

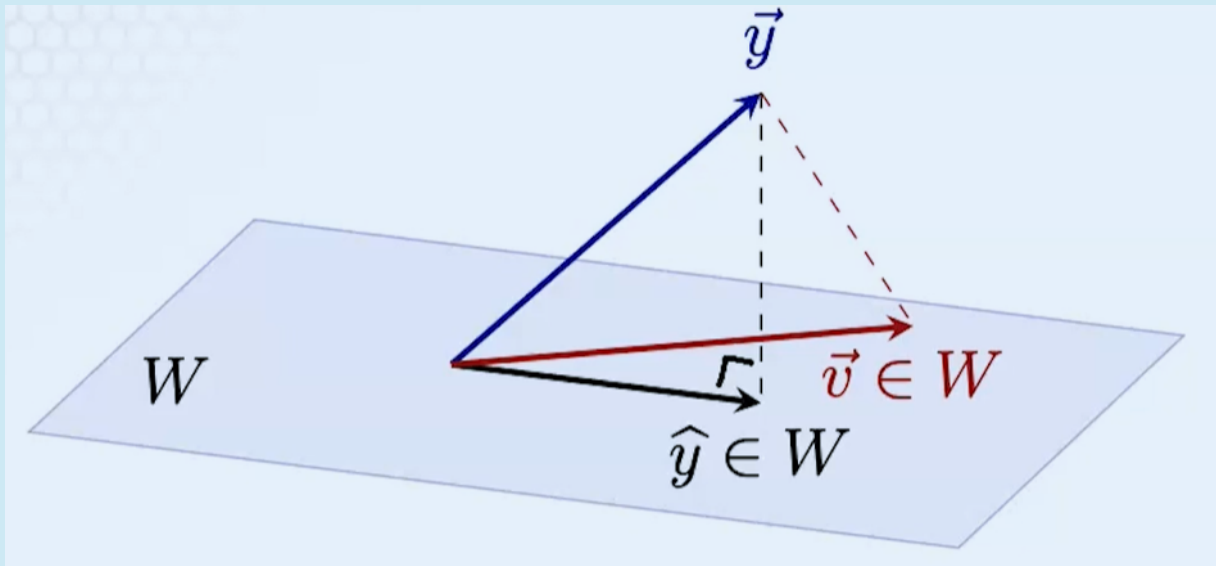
$$\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|$$

### Proof

$$\vec{y} - \vec{v} = \vec{y} - \vec{v} + (\hat{y} - \hat{y}) = (\hat{y} - \vec{y}) + (\hat{y} - \vec{v})$$

$$\text{Pythagorean Theorem: } \|\vec{y} - \vec{v}\|^2 = \|\hat{y} - \vec{y}\|^2 + \|\hat{y} - \vec{v}\|^2$$

We know that  $\|\hat{y} - \vec{v}\|^2 \neq 0$  as  $\hat{y} \neq \vec{v}$ , so  $\|\vec{y} - \vec{v}\|^2 > \|\vec{y} - \hat{y}\|^2$



## Orthogonal Decomposition

### Orthogonal Decomposition Theorem

Let  $W$  be a subspace of  $\mathbb{R}^n$ . Then, each  $\vec{y} \in \mathbb{R}^n$  has a unique decomposition.

$$\vec{y} = \hat{y} + z, \quad \hat{y} \in W, \quad z \in W^\perp$$

If  $\vec{u}_1, \dots, \vec{u}_n$  is the orthogonal basis for  $W$ ,

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \dots + \frac{\vec{y} \cdot \vec{u}_n}{\vec{u}_n \cdot \vec{u}_n} \vec{u}_n$$

$\hat{y}$  is the orthogonal projection of  $\vec{y}$  onto  $W$