Diagonal Matrices

Finding powers of a diagonal matrix, is very simple.

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

Diagonalizable

If A is similar to diagonal matrix D ($A = PDP^{-1}$) then A is diagonalizable.

$$A = PDP^{-1}$$

$$A = [v_1 \quad v_2 \dots \quad v_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_4 \end{bmatrix} [v_1 \quad v_2 \dots \quad v_n]^{-1}$$

Important

$v_1 \dots v_n$ are eigenvectors
$\lambda_1 \dots \lambda_n$ are eigenvalue

Proof

We construct $P = (\vec{v}_1 \ \vec{v}_2 \ \dots \vec{v}_n)$. Then

$$\begin{aligned} AP &= A(\vec{v}_1 \ \vec{v}_2 \ \dots \vec{v}_n) \\ &= (A\vec{v}_1 \ A\vec{v}_2 \ \dots A\vec{v}_n) \\ &= (\lambda_1\vec{v}_1 \ \lambda_2\vec{v}_2 \ \dots \lambda_n\vec{v}_n) \\ AP &= (\vec{v}_1 \ \vec{v}_2 \ \dots \vec{v}_n) \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n \end{pmatrix} \\ &= PD \end{aligned}$$

Or, $A = PDP^{-1}$.

A is diagonalizable if and only if A has n linearly independent eigenvectors.

Example

Diagonalize $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$
Eigenvalue $= 2, -1$
Eigenvectors $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\begin{aligned} A &= PDP^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

> **Note**
>
> This is a special case and is not always true,
>
> $$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

If A is $n \times n$ and has n distinct eigenvalues, then A is diagonalizable.
A does not **have to** have n distinct eigenvalues for A to be diagonalizable.

Suppose

- A is any $n \times n$ real matrix
- A has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, $k \leq n$
- a_i = **algebraic** multiplicity of λ_i
- g_i = dimension of λ_i eigenspace, or the **geometric** multiplicity

Then

- A is diagonalizable $\Leftrightarrow \Sigma g_i = n \Leftrightarrow g_i = a_i$ for all i
- A is diagonalizable $\Leftrightarrow$ the eigenvectors, for all eigenvalues, together form a basis for $\mathbb{R}^n$.

Repeated Eigenvalue

(I didn't feel like typing)

How can we diagonalize a matrix that has a repeated eigenvalue?

The only eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 3$. If possible, construct P and D such that $AP = PD$.

$$A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$$

Eigenvalue $\lambda_1 = 1$

Identify corresponding eigenvectors:

$$A - \lambda_1 I = A - I = \begin{pmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 & 8 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Eigenvalue $\lambda_1 = 1$

Identify corresponding eigenvectors:

$$A - \lambda_1 I \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

A vector in the null space of $A - \lambda_1 I$ is $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

Eigenvalue $\lambda_2 = 3$

Identify corresponding eigenvectors:

$$A - \lambda_2 I = A - 3I = \begin{pmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The first row corresponds to the equation

$$x_1 + x_2 + 4x_3 = 0$$

Eigenvectors corresponding to $\lambda_2 = 3$ must satisfy this relation. With one equation and three unknowns, there are two free variables: x_2 and x_3.

Eigenvalue $\lambda_2 = 3$

Eigenvectors corresponding to $\lambda_2 = 3$ must satisfy

$$x_1 + x_2 + 4x_3 = 0 \quad \Rightarrow \quad x_1 = -x_2 - 4x_3$$

Parametric vector form:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - 4x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$

Two eigenvectors for eigenvalue λ_2 are $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$.

Recall that we were asked to construct P and D such that $AP = PD$.

$$A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$$

Our matrices P and D are:

$$P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -4 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$