

Least Squares

We want to find a line/curve that minimizes the sum of the square of the error caused due to deviation.

Least squares

The least squares for $Ax = b$ is \hat{x} for which,

$$\|b - A\hat{x}\| \leq \|b - Ax\|$$

for all x

We can use the normal equation to solve for \hat{x}

Normal Equation

$$A^T A \hat{x} = A^T \vec{b}$$

Manipulating this we can get this,

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

Proof

\hat{x} is a least squares solution of $Ax = b$

$\iff Ax - b$ is as small as possible

$\iff Ax - b$ is orthogonal to $\text{Col}(A)$

$\overset{\text{FTLA}}{\iff} Ax - b$ is in $\text{Nul}(A^T)$

$\iff A^T(Ax - b) = 0$

$\iff A^T A \hat{x} = A^T b$

Using QR Factorization

$$R\hat{x} = Q^T \vec{b}$$

 **Proof** 

$$\begin{aligned} A^T A \hat{x} &= A^T \vec{b} \\ (QR)^T Q R \hat{x} &= (QR)^T \vec{b} \\ R^T Q^T Q R \hat{x} &= R^T Q^T \vec{b} \\ R^T R \hat{x} &= R^T Q^T \vec{b} \\ R \hat{x} &= Q^T \vec{b} \end{aligned}$$

How to solve

Method 1

1. Construct [QR Factorization](#)
2. Solve $R\hat{x} = Q^T \vec{b}$

Method 2

1. Solve the normal equation for \hat{x} .

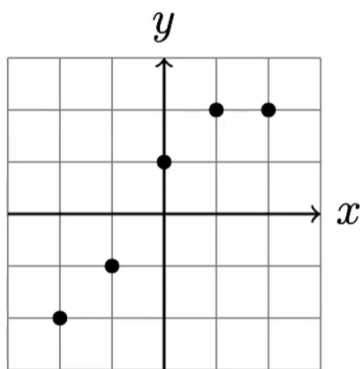
Method 3

1. [Use Mean Deviation](#)

Line

Say we want to find a line $y = mx + b$ that is the best fit for the following points:

x	-2	-1	1	1	2
y	-2	-1	1	2	2



We can create a list of linear equations using this:

$$m(-2) + b = -2$$

$$m(-1) + b = -1$$

$$m(1) + b = 1$$

$$m(1) + b = 2$$

$$m(2) + b = 2$$

We can turn this in to a matrix equation like so,

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

Compute QR,

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{5}} & 0 \\ 0 & \frac{10}{\sqrt{10}} \end{bmatrix}$$

Now compute $Q^T \vec{b}$,

$$Q^T \vec{b} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{11}{\sqrt{10}} \end{bmatrix}$$

Finally solve $R\hat{x} = Q^T \vec{b}$,

$$\begin{bmatrix} \frac{5}{\sqrt{5}} & 0 \\ 0 & \frac{10}{\sqrt{10}} \end{bmatrix} \hat{x} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{11}{\sqrt{10}} \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \frac{2}{5} \\ \frac{11}{10} \end{bmatrix}$$

So we get $y = \frac{2}{5} + \frac{11}{10}x$

Curves

We can use this method to fit data to a curve using the function,

$$y = c_0 + c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$$

Lets take an example,

Say we want to modal:

x	-1	0	0	1
y	2	1	0	6

Using $y = c_1 x + c_2 x^2$

$$-c_1 + c_2 = 2$$

$$0c_1 + 0c_2 = 1$$

$$0c_1 + 0c_2 = 0$$

$$c_1 + c_2 = 6$$

Now we can use [one of the methods](#) to solve for c_1 and c_2