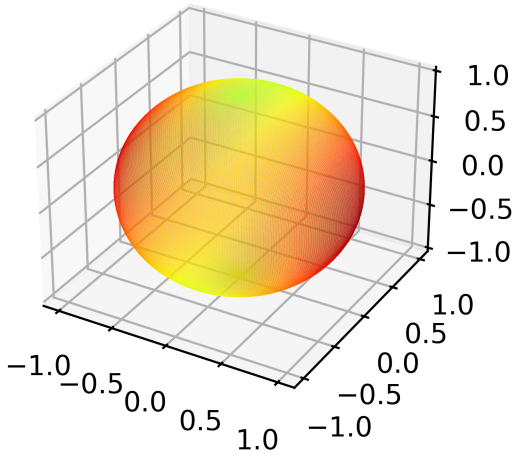
Constrained Optimization

An example

We have a unit sphere $1 = x_1^2 + x_2^2 + x_3^2 = ||x||^2$. We wish to optimize $Q = 9x_1^2 + 4x_2^2 + 3x_3^2$. To find the largest and smallest value of Q. It can be graphed as follows:



We wish to maximize Q.

$$Q = 9x_1^2 + 4x_2^2 + 3x_3^2 = \vec{x}^T \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{x} \ .$$

$$\begin{aligned} &\leq 9x_1^2 + 9x_2^2 + 9x_3^2 \\ &= 9(x_1^2 + x_2^2 + x_3^2) \\ &= 9||\vec{x}||^2 \\ &= 9 \end{aligned}$$

Note: $||\vec{x}||^2 = 1$ because that is what we stated in the problem.
So the max value of Q is 1.

More accurately, $\max\{Q(\vec{c}) : ||\vec{x}|| = 1\} = 9$, and max occurs at $\vec{x} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}$.