

# Spectral Decomposition

If  $A$  can be decomposed like so ( $PDP^T$ ),

$$A = PDP^T$$
$$A = [v_1 \quad v_2 \quad \dots \quad v_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Then,  $A$  has the following decomposition:

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \dots + \lambda_n \vec{u}_n \vec{u}_n^T = \sum_{i=1}^n \lambda_i \vec{u}_i \vec{u}_i^T$$

We must notice that  $\vec{u}_i \vec{u}_i^T$  is an  $n \times n$  matrix, assuming that  $\vec{u}_i$  is  $n \times 1$ .

## Proof >

We see that  $PD = [Pd_1 \quad Pd_2 \quad \dots \quad Pd_i]$

$d_i$  are columns of  $D$

We need to realize that that,  $Pd_i = P \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 + \dots + 0 + p_i \lambda_i + 0 \dots + 0 = p_i \lambda_i.$

Hence  $PD = [p_1 \lambda_1 \quad p_2 \lambda_2 \quad \dots \quad p_1 \lambda_1]$

So,  $PDP^T = [p_1 \lambda_1 \quad p_2 \lambda_2 \quad \dots \quad p_1 \lambda_1] \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_i^T \end{bmatrix} = A$

Using math we can conclude,

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T + \dots + \lambda_n \vec{u}_n \vec{u}_n^T = \sum_{i=1}^n \lambda_i \vec{u}_i \vec{u}_i^T$$

## Example

Construct a spectral decomposition for  $A$ .

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

## Solution

$$\begin{aligned} A &= \sum_{i=1}^2 \lambda_i \vec{u}_i \vec{u}_i^T \\ &= 4 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} + 2 \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= 4 \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} + 2 \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix} \end{aligned}$$

## Properties

- $\lambda_n \vec{u}_n \vec{u}_n^T$  will be  $n \times n$  and have a Rank of 1.

### Approximation >

- Ordering the eigenvalues from largest to smallest (in absolute value),

$$|\lambda_i| \geq |\lambda_{i+1}|$$

we may be able to truncate the sum

$$A = \sum_{i=1}^n \lambda_i \vec{u}_i \vec{u}_i^T$$

to exclude smaller terms. This gives us a way to approximate  $A$ .