Quadratic Forms

If $Q(x)$ is a function, then,

$$Q(\vec{x}) = \vec{x}^T A\vec{x}$$

A is symmetric

Examples

Find Q

$Q(\vec{x}) = \vec{x}^T A\vec{x}$, $A = \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 4x^2 + 2xy - 3y^2$$

The $2xy$ term is called a cross-product due to it having both variables.

Finding A

$\mathbb{R}^2$

$Q = x^2 - 6xy + 9y^2$
We will use our eyes, the main diagonal will be the coefficients for second order terms. The other diagonal will be $\frac{1}{2}$ of the coefficient of the cross-product.
So, $A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$.
We could compute $\vec{x}^T A\vec{x}$ to verify this result.

$\mathbb{R}^3$

$Q = 5x_1^2 - x_2^2 + 3x_3^3 + 6x_1x_3 - 12x_2x_3$
$Q = 5x_1^2 - x_2^2 + 3x_3^3 + 6x_1x_3 - 12x_2x_3 + 0x_1x_2$
We will once again use our eye (consider resting them after this). Like last time the main diagonal will be the coefficients for second order terms. The other terms will be $\frac{1}{2}$ the coefficient of the cross-products. We will look at the variables being crossed take x_1x_3 for example. The location $1, 3$ and $3, 1$ in the matrix A will be $\frac{1}{2}$ the coefficient of x_1x_3.

$$A = \begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & -6 \\ 3 & -6 & 3 \end{bmatrix}$$

We can once again compute $\vec{x}^T A \vec{x}$ to verify this result.