Orthogonal Diagonalization

Eigenvectors orthogonality

If A is a symmetric matrix, with eigenvectors $\vec{v_1}$ and $\vec{v_2}$ corresponding to two distinct eigenvalues, then $\vec{v_1}$ and $\vec{v_2}$ are orthogonal.

Proof

$$
\begin{aligned}
\lambda_1\vec{v}_1 \cdot \vec{v}_2 &= A\vec{v}_1 \cdot \vec{v}_2 && \text{using } A\vec{v}_i = \lambda_i\vec{v}_i \\
&= (A\vec{v}_1)^T\vec{v}_2 && \text{using the definition of the dot product} \\
&= \vec{v}_1{}^T A^T\vec{v}_2 && \text{property of transpose of product} \\
&= \vec{v}_1{}^T A\vec{v}_2 && \text{given that } A = A^T \\
&= \vec{v}_1 \cdot A\vec{v}_2 && \\
&= \vec{v}_1 \cdot \lambda_2\vec{v}_2 && \text{using } A\vec{v}_i = \lambda_i\vec{v}_i \\
&= \lambda_2\vec{v}_1 \cdot \vec{v}_2 &&
\end{aligned}
$$

But $\lambda_1 \neq \lambda_2$ so $\vec{v}_1 \cdot \vec{v}_2 = 0$.