Homogeneous Coordinates

Translations of points in $\mathbb{R}^n$ does not correspond directly to a linear transform. Homogeneous coordinates are used to model translations using matrix multiplication.

Homogeneous Coordinates in $\mathbb{R}^2$

Each point (x, y) in $\mathbb{R}^2$ can be identified with the point (x, y, H), $H \neq 0$, on the plane in $\mathbb{R}3$ that lies H units above the xy-plane.

$(x, y) \rightarrow (x+h, y+k)$ can be represented by,

$$\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \\ 1 \end{pmatrix}$$

Now rotate a triangle $((1,1),(2,4),(3,1))$ by $\frac{\pi}{2}$ radians counterclockwise about the point $(0,1)$.

$$d = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{Shift down by 1,}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} d = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{Now rotate,}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} d = \begin{bmatrix} 0 & -3 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{Shift up by 1,}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 0 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

This give us the points, $(0, 2), (-3, 3), (0, 4)$

In $\mathbb{R}^3$

So, $(x, y, z) \to (x+h, y+k, z+l)$ can be represented by,

$$\begin{pmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \\ z+l \\ 1 \end{pmatrix}$$

Rotation in $\mathbb{R}^3$

about x_2-axis by π rads.
To find $A = (a_1, a_2, a_3)$. We can find $T(e_1)$ as $T(e_1) = Ae_1 = a_1$. We can similarly find all the columns of A.

$$T(e_1) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$
$$T(e_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$T(e_3) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$
$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Projection

Onto the plane $x_3 = 4$
What should we do?

1. Shift everything down by 4 (Homogeneous Coordinates)
2. Apply the projection (Homogeneous Coordinates)

3. Shift everything back up by 4 (Homogeneous Coordinates)
Amusing a vector $\vec{v}$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vec{v}$$

~~You could drive the matrix but that is trivial and left as an exercise to the reader.~~

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vec{v}$$