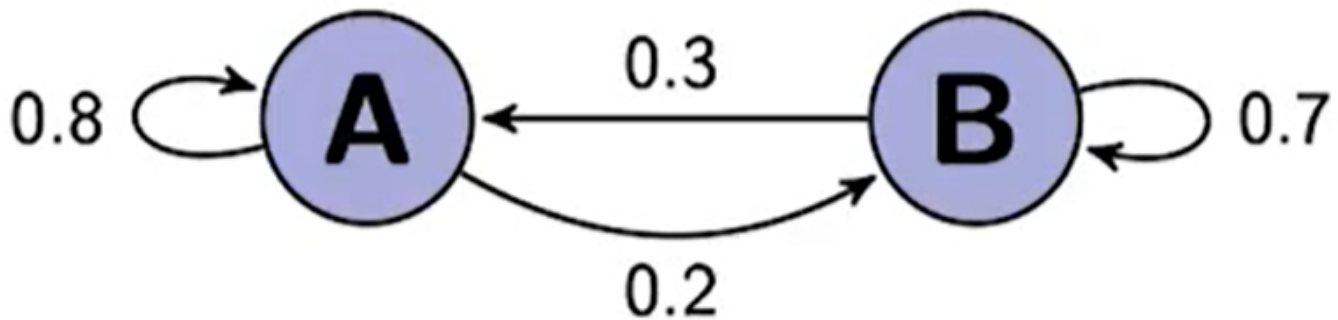


# Markov Chains

Say,  $A$  and  $B$  are libraries with 1000 books.



In the beginning,  $x_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

Now, after 1 month,  $x_1 = \begin{bmatrix} 0.8 \cdot 0.5 + 0.3 \cdot 0.5 \\ 0.2 \cdot 0.5 + 0.7 \cdot 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = Px_1$

$$P = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

Now, after 2 month,  $x_2 = Px_1 = Px_0 = P^2x_0$

$\vdots$

Now, after  $k$  months,  $x_k = P^k x_0$

## Steady State

Find the steady state of  $P = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ .

$$\begin{aligned}
P\vec{q} &= \vec{q} \\
P\vec{q} - \vec{q} &= 0 \\
P\vec{q} - I_n\vec{q} &= 0 \\
(P - I_n)\vec{q} &= 0 \\
\left( \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{q} &= 0 \\
\begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \vec{q} &= 0 \\
\begin{cases} -2x_1 & +3x_2 & = 0 \\ 2x_1 & -3x_2 & = 0 \end{cases} \\
\{x_1 = 3, x_2 = 2\} \\
\frac{1}{3+2} \begin{bmatrix} 3 \\ 2 \end{bmatrix} &= \vec{q} \\
\vec{q} &= \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}
\end{aligned}$$

## Convergence

We have  $x_1, x_2, x_3, \dots, x_k$ . We want to know if while  $k \rightarrow \infty$   $x_k$  will converge to a [steady state](#).

If  $P$  is a regular stochastic matrix ([vocabulary](#)), then  $P$  has a unique steady-state vector  $\vec{q}$ , and  $\vec{x}_{k+1} = P\vec{x}_k$  converges to  $\vec{q}$  as  $k \rightarrow \infty$ .

## Using [Eigenvalues](#) to find [Convergence](#)

Consider the Markov Chain:

$$\vec{x}_{k+1} = P\vec{x}_k = \begin{pmatrix} 1 & 0.1 \\ 0 & 0.9 \end{pmatrix} \vec{x}_k, \quad k = 0, 1, 2, 3, \dots, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Use the eigenvalues and eigenvectors of  $P$  to determine what  $\vec{x}_k$  tends to as  $k \rightarrow \infty$ . The eigenvalues and eigenvectors of  $P$  are

$$\lambda_1 = 1, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = 0.9, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Say  $P$  has 2 eigenvalues ( $\lambda_1$  and  $\lambda_2$ ) and eigenvectors ( $v_1$  and  $v_2$ ).

$$\begin{aligned}
x_{k+1} &= Px_k \\
x_0 &= c_1v_1 + c_2v_2 \\
x_1 &= Px_0 \\
x_1 &= P(c_1v_1 + c_2v_2) &= c_1Pv_1 + c_2Pv_2 \\
x_1 &= c_1\lambda_1v_1 + c_2\lambda_2v_2 \\
x_2 &= Px_1 \\
x_1 &= P(c_1\lambda_1v_1 + c_2\lambda_2v_2) &= c_1(\lambda_1)^2v_1 + c_2(\lambda_2)^2v_2 \\
x_k &= c_1(\lambda_1)^kv_1 + c_2(\lambda_2)^kv_2
\end{aligned}$$

$$\begin{aligned}
x_0 &= c_1v_1 + c_2v_2 \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
c_1 &= 1 & c_1 = 0
\end{aligned}$$

$$\boxed{x_k = v_1}$$