

Singular Vectors

Theorem

Suppose \vec{v}_i are the n orthogonal eigenvectors of $A^T A$, ordered so that their corresponding eigenvalues satisfy $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Suppose also that A has r non-zero singular values, $r \leq n$. Then the set of vectors,

$$\{\vec{v}_{r+1}, \vec{v}_{r+2}, \dots, \vec{v}_n\}$$

is an orthogonal basis for $\text{Nul}A$, and the set

$$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$$

is an orthogonal basis for $\text{Row}A$, and $\text{Rank}A = r$.

Proof

For a set of vectors to form an orthogonal basis for a subspace they must be in that space, span the space, be independent, and mutually orthogonal.

- Each \vec{v}_i is an eigenvector, so none of them are the zero vector.
- \vec{v}_i are orthogonal and span \mathbb{R}^n (they are eigenvectors of a symmetric matrix, $A^T A$).
- Recall that the lengths of $A\vec{v}_i$ are the singular values of A :

$$\|A\vec{v}_i\| = \sigma_i.$$

- Then if $\|A\vec{v}_i\| = 0$ for $i > r$, then $\vec{v}_i \in \text{Nul}A$ for $i > r$.
- Then if $\|A\vec{v}_i\| \neq 0$ for $i \leq r$, then \vec{v}_i cannot be in $\text{Nul}A$ for $i \leq r$, they must be in $(\text{Nul}A)^\perp = \text{Row}A$, because $\{\vec{v}_i\}$ is an orthonormal set.

Thus, our basis for $\text{Nul}A$ is the set

$$\{\vec{v}_{r+1}, \vec{v}_{r+2}, \dots, \vec{v}_n\}$$

and our basis for $\text{Row}A$ is the set

$$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$$

We must also describe why $\text{rank}A = r$.

- There are r vectors in our basis for $\text{Row}A$.
- Recall that $\dim(\text{Row}A) = \dim(\text{Col}A) = \text{rank}A$.

Using the same assumptions as above, it can be shown that,

$$\{A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_r\}$$

is an orthogonal basis for $\text{Col}A$.

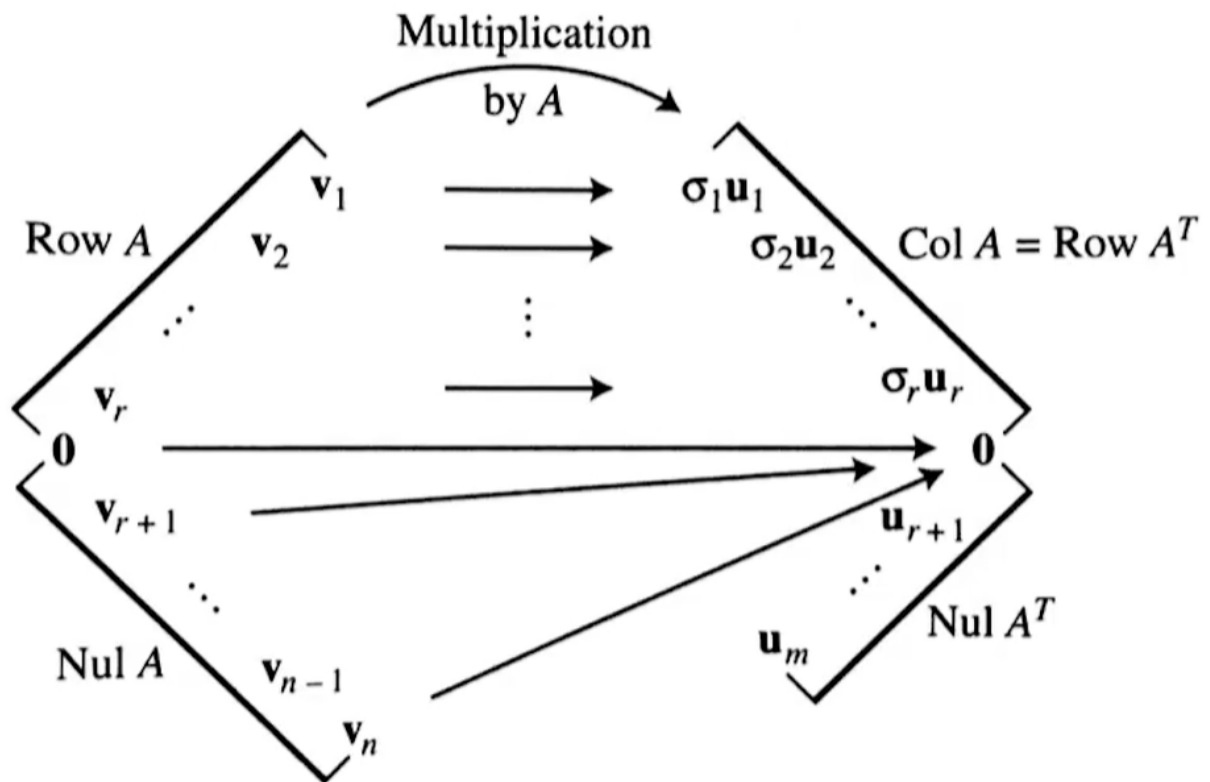
Proof >

- Each $A\vec{v}_i$ is a vector in $\text{Col}A$.
- $A\vec{v}_i$ and $A\vec{v}_j$ are orthogonal:

$$(A\vec{v}_i) \cdot (A\vec{v}_j) = \vec{v}_i^T A^T A \vec{v}_j = \lambda_j \vec{v}_i \cdot \vec{v}_j = 0$$

- For $i \leq r = \text{rank}A$, $A\vec{v}_i$ are orthogonal and non-zero. So they must also be independent and form an orthogonal basis for $\text{Col}A$.

Note that for $i > r$, $A\vec{v}_i = \vec{0}$ because $\vec{v}_i \in \text{Nul}A$ for $i > r$.



Definition

The vectors $\{\vec{u}_i\}$ for $i \leq m$ are the **left singular vectors** of A .
 The vectors $\{\vec{v}_i\}$ for $i \leq n$ are the **right singular vectors** of A .