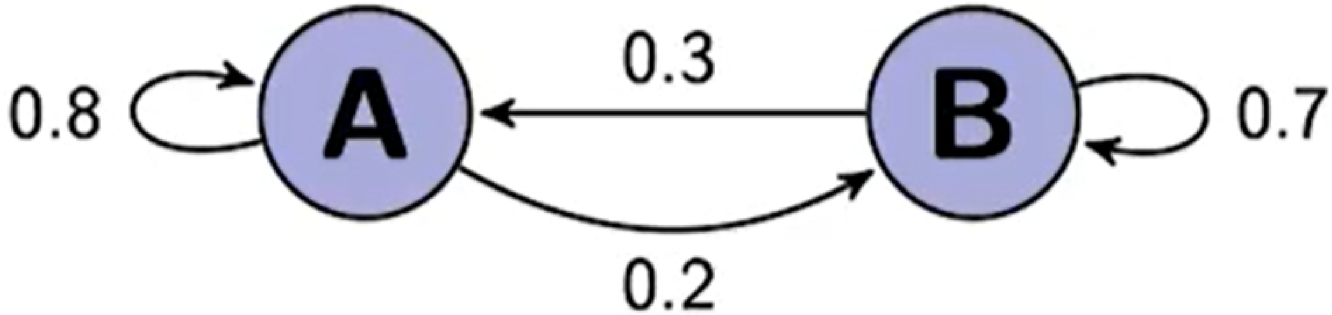
Markov Chains

Say, A and B are libraries with 1000 books.



In the beginning, $x_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

Now, after 1 month, $x_1 = \begin{bmatrix} 0.8 \cdot 0.5 + 0.3 \cdot 0.5 \\ 0.2 \cdot 0.5 + 0.7 \cdot 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = Px_1$

$$\boxed{P = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}}$$

Now, after 2 months, $x_2 = Px_1 = P^2 x_0$

$\vdots$

Now, after k months, $x_k = P^k x_0$

Steady State

Find the steady state of $P = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$.

$$
\begin{aligned}
P\vec{q} &= \vec{q} \\
P\vec{q} - \vec{q} &= 0 \\
P\vec{q} - I_n\vec{q} &= 0 \\
(P - I_n)\vec{q} &= 0 \\
\left(\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{q} &= 0 \\
\begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}\vec{q} &= 0 \\
\begin{cases} -2x_1 & +3x_2 & = 0 \\ 2x_1 & -3x_2 & = 0 \end{cases} \\
\{x_1 = 3, x_2 = 2\} \\
\frac{1}{3+2}\begin{bmatrix} 3 \\ 2 \end{bmatrix} &= \vec{q} \\
\vec{q} &= \boxed{\begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}}
\end{aligned}
$$

Convergence

We have $x_1, x_2, x_3, \ldots x_k$. We want to know if while $k \to \infty$ x_k will converge to a steady state.

If P is a regular stochastic matrix (vocabulary), then P has a unique steady-state vector $\vec{q}$, and $\vec{x}_{k+1} = P\vec{x}_k$ converges to $\vec{q}$ as $k \to \infty$.

Using Eigenvalues to find Convergence

Consider the Markov Chain:

$$\vec{x}_{k+1} = P\vec{x}_k = \begin{pmatrix} 1 & 0.1 \\ 0 & 0.9 \end{pmatrix}\vec{x}_k, \quad k = 0, 1, 2, 3, \ldots, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Use the eigenvalues and eigenvectors of P to determine what $\vec{x}_k$ tends to as $k \to \infty$. The eigenvalues and eigenvectors of P are

$$\lambda_1 = 1, \; \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = 0.9, \; \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Say P has 2 eigenvalues ($\lambda_1 \text{ and } \lambda_2$) and eigenvectors ($v_1 \text{ and } v_2$).

$$
\begin{aligned}
x_{k+1} &= Px_k \\
x_0 &= c_1 v_1 + c_2 v_2 \\
x_1 &= Px_0 \\
x_1 &= P(c_1 v_1 + c_2 v_2) \qquad\qquad = c_1 P v_1 + c_2 P v_2 \\
x_1 &= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 \\
x_2 &= Px_1 \\
x_1 &= P(c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2) \qquad = c_1 (\lambda_1)^2 v_1 + c_2 (\lambda_2)^2 v_2 \\
x_k &= c_1 (\lambda_1)^k v_1 + c_2 (\lambda_2)^k v_2
\end{aligned}
$$

$$
\begin{aligned}
x_0 &= c_1 v_1 + c_2 v_2 \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
c_1 &= 1 \qquad\qquad c_1 = 0
\end{aligned}
$$

$$
\boxed{x_k = v_1}
$$