

Formulas

Formulas and Theorem

Expansion in Orthogonal Basis

If we have an Orthogonal Basis $\{\vec{u}_1, \dots, \vec{v}_n\}$ in \mathbb{R}^n then for any $\vec{w} \in \mathbb{R}^n$,

$$\vec{w} = c_1\vec{u}_1 + \dots + c_n\vec{v}_n$$

C_q can be found using,

$$c_q = \frac{\vec{w} \cdot \vec{u}_q}{\vec{u}_q \cdot \vec{u}_q}$$

Orthogonal Projection

Let non-zero $\vec{u} \in \mathbb{R}^n$, and $\vec{y} \in \mathbb{R}^n$. The orthogonal projection of \vec{y} onto \vec{u} is the vector in the span of \vec{u} that is closest to \vec{y} .

$$\text{proj}_{\vec{u}}\vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}$$

Also, $\vec{y} = \hat{y} + \vec{z}$ and, $\vec{z} \in W^\perp$

Best Approximation Theorem

Let W be a subspace of \mathbb{R}^n , $\vec{y} \in \mathbb{R}^n$, and \hat{y} is the orthogonal projection of \vec{y} onto W . Then for any $\vec{v} \neq \hat{y}, \vec{v} \in W$, we have

$$\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|$$

Orthogonal Decomposition Theorem

Let W be a subspace of \mathbb{R}^n . Then, each $\vec{y} \in \mathbb{R}^n$ has a unique decomposition.

$$\vec{y} = \hat{y} + z, \quad \hat{y} \in W, \quad z \in W^\perp$$

If $\vec{u}_1, \dots, \vec{u}_n$ is the orthogonal basis for W ,

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \dots + \frac{\vec{y} \cdot \vec{u}_n}{\vec{u}_n \cdot \vec{u}_n} \vec{u}_n$$

\hat{y} is the orthogonal projection of \vec{y} onto W

QR Factorization

For a $m \times n$ matrix A linearly independent columns,

$$A = QR$$

Q is an $m \times n$, with columns are an orthonormal basis for $\text{Col}A$.

R is $n \times n$, upper triangular, with positive entries on its diagonal.

Normal Equation

$$A^T A \hat{x} = A^T \vec{b}$$

Manipulating this we can get this,

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

Eigenvectors orthogonality

If A is a symmetric matrix, with eigenvectors \vec{v}_1 and \vec{v}_2 corresponding to two distinct eigenvalues, then \vec{v}_1 and \vec{v}_2 are orthogonal.

The Spectral Theorem

An $n \times n$ symmetric matrix A has the following properties,

- All eigenvalues of A are real.
- The eigenspaces are mutually orthogonal.
- A can be diagonalized as $A = PDP^T$, where D is diagonal and P is orthogonal.