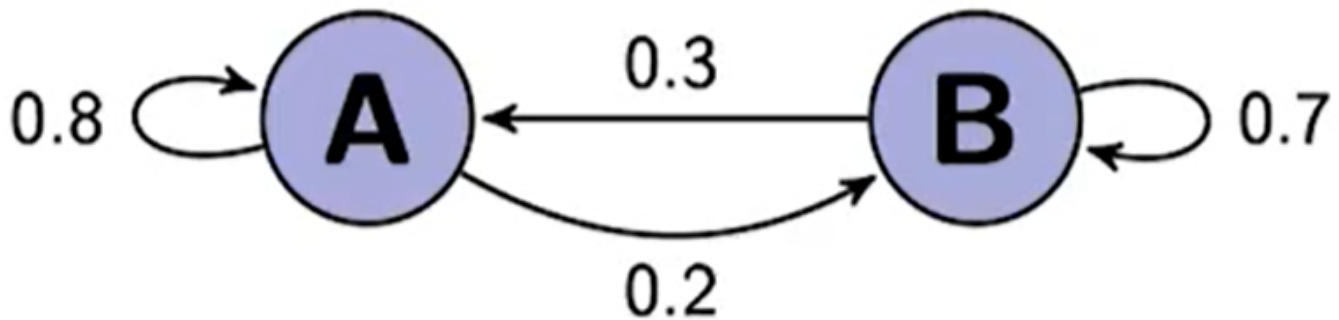# Markov Chains

Say, A and B are libraries with 1000 books.

In the beginning, $x_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

Now, after 1 month, $x_1 = \begin{bmatrix} 0.8 \cdot 0.5 + 0.3 \cdot 0.5 \\ 0.2 \cdot 0.5 + 0.7 \cdot 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = Px_1$

$$P = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

Now, after 2 months, $x_2 = Px_1 = P^2 x_0$

$\vdots$

Now, after k months, $x_k = P^k x_0$

Steady State

Find the steady state of $P = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$.

$$P\vec{q} = \vec{q}$$
$$P\vec{q} - \vec{q} = 0$$
$$P\vec{q} - I_n\vec{q} = 0$$
$$(P - I_n)\vec{q} = 0$$
$$\left(\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{q} = 0$$
$$\begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}\vec{q} = 0$$
$$\begin{cases} -2x_1 & +3x_2 & = 0 \\ 2x_1 & -3x_2 & = 0 \end{cases}$$
$$\{x_1 = 3, x_2 = 2\}$$
$$\frac{1}{3 + 2}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \vec{q}$$
$$\vec{q} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$$

Convergence

We have $x_1, x_2, x_3, \ldots x_k$. We want to know if while $k \to \infty$ x_k will converge to a steady state.

If P is a regular stochastic matrix (vocabulary), then P has a unique steady-state vector $\vec{q}$, and $\vec{x}_{k+1} = P\vec{x}_k$ converges to $\vec{q}$ as $k \to \infty$.

Using Eigenvalues to find Convergence

Consider the Markov Chain:

$$\vec{x}_{k+1} = P\vec{x}_k = \begin{pmatrix} 1 & 0.1 \\ 0 & 0.9 \end{pmatrix} \vec{x}_k, \quad k = 0, 1, 2, 3, \ldots, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Use the eigenvalues and eigenvectors of P to determine what $\vec{x}_k$ tends to as $k \to \infty$. The eigenvalues and eigenvectors of P are

$$\lambda_1 = 1, \ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = 0.9, \ \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Say P has 2 eigenvalues (λ_1 and λ_2) and eigenvectors (v_1 and v_2).

$$x_{k+1} = Px_k$$
$$x_0 = c_1v_1 + c_2v_2$$
$$x_1 = Px_0$$
$$x_1 = P(c_1v_1 + c_2v_2) \qquad\qquad = c_1Pv_1 + c_2Pv_2$$
$$x_1 = c_1\lambda_1v_1 + c_2\lambda_2v_2$$
$$x_2 = Px_1$$
$$x_1 = P(c_1\lambda_1v_1 + c_2\lambda_2v_2) \qquad = c_1(\lambda_1)^2v_1 + c_2(\lambda_2)^2v_2$$
$$x_k = c_1(\lambda_1)^kv_1 + c_2(\lambda_2)^kv_2$$

$$x_0 = c_1v_1 + c_2v_2$$
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$c_1 = 1 \qquad\qquad\qquad c_1 = 0$$

$$\boxed{x_k = v_1}$$