Spectral Decomposition

If A can be decomposed like so (PDP^T),

$$A = PDP^T$$

$$A = \begin{bmatrix} v_1 & v_2 \dots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Then, A has the following decomposition:

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^{\,T} + \cdots + \lambda_n \vec{u}_n \vec{u}_n^{\,T} = \sum_{i=1}^{n} \lambda_i \vec{u}_i \vec{u}_i^{\,T}$$

We must notice that $\vec{u}_i \vec{u}_i^{\,T}$ is an $n \times n$ matrix, assuming that $\vec{u}_i$ is $n \times 1$.

Proof >

We see that $PD = \begin{bmatrix} Pd_1 & Pd_2 & \dots & Pd_i \end{bmatrix}$
d_i are columns of D

We need to realize that that, $Pd_i = P \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 + \cdots + 0 + p_i \lambda_i + 0 \cdots + 0 = p_i \lambda_i$

.

Hence $PD = \begin{bmatrix} p_1\lambda_1 & p_2\lambda_2 & \dots & p_1\lambda_1 \end{bmatrix}$

So, $PDP^T = \begin{bmatrix} p_1\lambda_1 & p_2\lambda_2 & \dots & p_1\lambda_1 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_i^T \end{bmatrix} = A$

Using math we can conclude,

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^{\,T} + \lambda_2 \vec{u}_2 \vec{u}_2^{\,T} + \cdots + \lambda_n \vec{u}_n \vec{u}_n^{\,T} = \sum_{i=1}^{n} \lambda_i \vec{u}_i \vec{u}_i^{\,T}$$

Example

Construct a spectral decomposition for A.

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Solution

$$\begin{aligned} A &= \sum_{i=1}^{2} \lambda_i \vec{u}_i \vec{u}_i^{\,T} \\ &= 4 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} + 2 \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= 4 \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} + 2 \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix} \end{aligned}$$

Properties

- $\lambda_n \vec{u}_n \vec{u}_n^{\,T}$ will be $n \times n$ and have a Rank of 1.

Approximation

- Ordering the eigenvalues from largest to smallest (in absolute value),

$$|\lambda_i| \geq |\lambda_{i+1}|$$

we may be able to truncate the sum

$$A = \sum_{i=1}^{n} \lambda_i \vec{u}_i \vec{u}_i^T$$

to exclude smaller terms. This gives us a way to approximate A.

Using The SVD

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i$$

$\vec{u}_i, \vec{v}_i$ are the i^{th} columns of U and V respectively. σ_i is the Singular Values.

Example

Suppose A has the following SVD.

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The spectral decomposition of A is as follows.

$$A = \sum_{s=1}^{r} \sigma_s \vec{u}_s \vec{v}_s^T = 3 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = 3 \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$