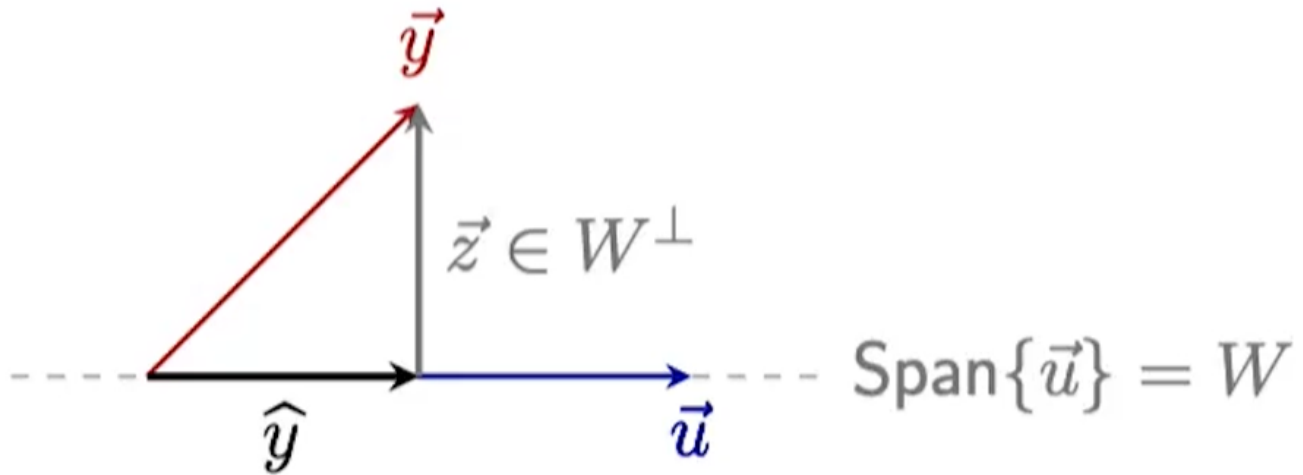


Projections

We have a subspace W (the span of \vec{u}), and we want to find the vector (\hat{y}) closest to \vec{y} in W .

We also want to find $\vec{z} \in W^\perp$ such that $\vec{y} = \hat{y} + \vec{z}$.

Diagrammatically,



We know $\vec{z} \in W^\perp$, so:

$$\vec{z} \cdot \vec{u} = 0$$

We also know $\vec{y} = \hat{y} + \vec{z}$ and $\hat{y} = k\vec{u}$ ($k \in \mathbb{R}$), so:

$$\begin{aligned}\vec{z} &= \vec{y} - k\vec{u} \\ 0 &= (\vec{y} - k\vec{u}) \cdot \vec{u} \\ &= \vec{y} \cdot \vec{u} - k\vec{u} \cdot \vec{u} \\ k &= \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}, \quad \vec{u} \neq \vec{0}\end{aligned}$$

So finally, $\hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$

Orthogonal Projection

Let non-zero $\vec{u} \in \mathbb{R}^n$, and $\vec{y} \in \mathbb{R}^n$. The orthogonal projection of \vec{y} onto \vec{u} is the vector in the span of \vec{u} that is closest to \vec{y} .

$$\text{proj}_{\vec{u}} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

Also, $\vec{y} = \hat{y} + \vec{z}$ and, $\vec{z} \in W^\perp$

From this we can conclude (look at the diagram),

$$\|\vec{y}\|^2 = \|\text{proj}_W \vec{y}\|^2 + \|\vec{z}\|^2$$

Best Approximation

Best Approximation Theorem

Let W be a subspace of \mathbb{R}^n , $\vec{y} \in \mathbb{R}^n$, and \hat{y} is the orthogonal projection of \vec{y} onto W . Then for any $\vec{v} \neq \hat{y}, \vec{v} \in W$, we have

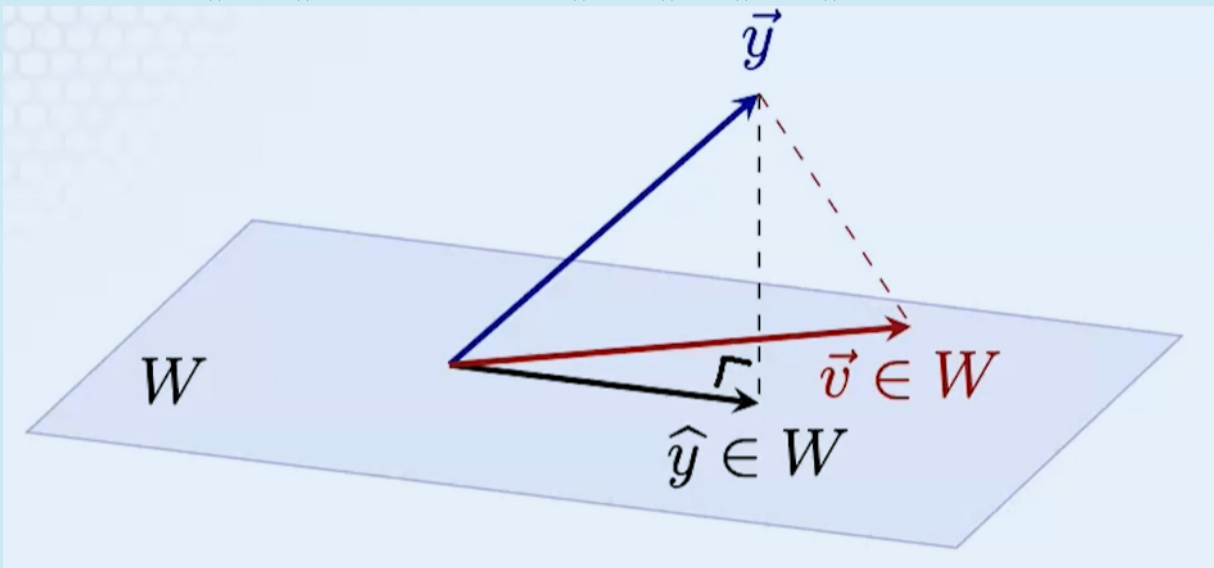
$$\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|$$

Proof

$$\vec{y} - \vec{v} = \vec{y} - \vec{v} + (\hat{y} - \hat{y}) = (\hat{y} - \vec{y}) + (\hat{y} - \vec{v})$$

$$\text{Pythagorean Theorem: } \|\vec{y} - \vec{v}\|^2 = \|\hat{y} - \vec{y}\|^2 + \|\hat{y} - \vec{v}\|^2$$

We know that $\|\hat{y} - \vec{v}\|^2 \neq 0$ as $\hat{y} \neq \vec{v}$, so $\|\vec{y} - \vec{v}\|^2 > \|\vec{y} - \hat{y}\|^2$



Orthogonal Decomposition

Orthogonal Decomposition Theorem

Let W be a subspace of \mathbb{R}^n . Then, each $\vec{y} \in \mathbb{R}^n$ has a unique decomposition.

$$\vec{y} = \hat{y} + z, \quad \hat{y} \in W, \quad z \in W^\perp$$

If $\vec{u}_1, \dots, \vec{u}_n$ is the orthogonal basis for W ,

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \dots + \frac{\vec{y} \cdot \vec{u}_n}{\vec{u}_n \cdot \vec{u}_n} \vec{u}_n$$

\hat{y} is the orthogonal projection of \vec{y} onto W