# Least Squares

We want to find a line/curve that minimizes the sum of the square of the error caused due to deviation.

**Least squares**

The least squares for $Ax = b$ is $\hat{x}$ for which,

$$||b - A\hat{x}|| \leq ||b - Ax||$$

for all $x$

We can use the normal equation to solve for $\hat{x}$

**Normal Equation**

$$A^T A\hat{x} = A^T\vec{b}$$

Manipulating this we can get this,

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

**Proof**

$\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$

$\iff A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is as small as possible

$\iff A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is orthogonal to $\mathrm{Col}(A)$

$\overset{\text{FTLA}}{\iff} A\hat{\boldsymbol{x}} - \boldsymbol{b}$ is in $\mathrm{Nul}(A^T)$

$\iff A^T(A\hat{\boldsymbol{x}} - \boldsymbol{b}) = \boldsymbol{0}$

$\iff A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$

**Using [QR Factorization]**

$$R\hat{x} = Q^T\vec{b}$$

Proof

$$\begin{aligned} A^TA\hat{x} &= A^T\vec{b} \\ (QR)^TQR\hat{x} &= (QR)^T\vec{b} \\ R^TQ^TQR\hat{x} &= R^TQ^T\vec{b} \\ R^TR\hat{x} &= R^TQ^T\vec{b} \\ R\hat{x} &= Q^T\vec{b} \end{aligned}$$

## How to solve

### Method 1

1. Construct QR Factorization
2. Solve $R\hat{x} = Q^T\vec{b}$

### Method 2

1. Solve the normal equation for $\hat{x}$.

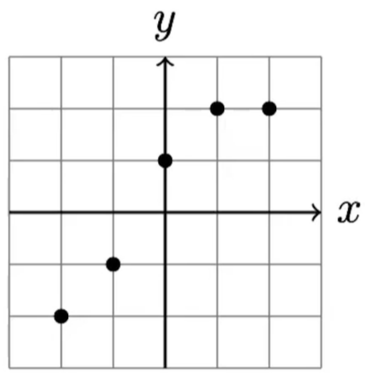
### Method 3

1. Use Mean Deviation

# Line

Say we want to find a line $y = mx + b$ that is the best fit for the following points:

| x | -2 | -1 | 1 | 1 | 2 |
|---|---|---|---|---|---|
| y | -2 | -1 | 1 | 2 | 2 |




We can create a list of linear equations using this:

$$\begin{gathered} m(-2)+b=-2 \\ m(-1)+b=-1 \\ m(1)+b=1 \\ m(1)+b=2 \\ m(2)+b=2 \end{gathered}$$

We can turn this in to a matrix equation like so,

$$A\vec{x}=\vec{b}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

Compute QR,

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{5}} & 0 \\ 0 & \frac{10}{\sqrt{10}} \end{bmatrix}$$

Now compute $Q^T\vec{b}$,

$$Q^T\vec{b} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{11}{\sqrt{10}} \end{bmatrix}$$

Finally solve $R\hat{x}=Q^T\vec{b}$,

$$\begin{bmatrix} \frac{5}{\sqrt{5}} & 0 \\ 0 & \frac{10}{\sqrt{10}} \end{bmatrix} \hat{x} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{11}{\sqrt{10}} \end{bmatrix}$$

$$\hat{x} = \boxed{\begin{bmatrix} \frac{2}{5} \\ \frac{11}{10} \end{bmatrix}}$$

So we get $\boxed{y = \frac{2}{5} + \frac{11}{10}x}$

# Curves

We can use this method to fit data to a curve using the function,

$$y = c_0 + c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$$

Lets take an example,
Say we want to modal:

| x | -1 | 0 | 0 | 1 |
|---|---|---|---|---|
| y | 2 | 1 | 0 | 6 |

Using $y = c_1 x + c_2 x^2$

$$\begin{aligned} -c_1 + c_2 &= 2 \\ 0c_1 + 0c_2 &= 1 \\ 0c_1 + 0c_2 &= 0 \\ c_1 + c_2 &= 6 \end{aligned}$$

Now we can use one of the methods to solve for $c_1 \text{ and } c_2$