Singular Values

The singular values of any $m \times n$ real matrix A are the square roots of the eigenvalues of $A^T A$.

Say we want to find a v that maximizes $||A\vec{v}||$ Where $||\vec{v}|| = 1$
This is the same thing as maximizing $||A\vec{v}||^2$, so we have

$$||A\vec{v}||^2 = \vec{v}^T A^T A\vec{v}$$

$A^T A$ is always symmetric. After acknowledging that we can realizes that is a Constrained Optimization problem. $||\vec{v}|| = 1$ being the constraint.
We will simply use the largest eigenvalue of $A^T A$ to find the largest value for $||A\vec{v}||^2$. The location of this will simply be the corresponding normalized eigenvector.

The min value can be found using a simpler method.

Calling singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ we can say that the σ_1 is the max value of $||A\vec{v}||$ and σ_n is the min value.

Non-Negative Eigenvalues

The eigenvalues of $A^T A$ are non-negative.

Proof

Proof: recall that $\vec{v}_j^T\vec{v}_j = \vec{v}_j \cdot \vec{v}_j = \|\vec{v}_j\|^2 = 1$ because $\vec{v}_j$ are **unit** eigenvectors of A^TA.

$$\|A\vec{v}_j\|^2 = (A\vec{v}_j)^T A\vec{v}_j = \vec{v}_j A^T A\vec{v}_j = \lambda_j \vec{v}_j^T \vec{v}_j = \lambda_j \geq 0.$$

Therefore:

- the eigenvalues of A^TA must be real and non-negative
- the singular values of A, which are the square roots of the eigenvalues, must also be real and non-negative

From the above Proof we can see,

$$||A\vec{v}||^2 = \lambda_i$$

And hence,

$$||A\vec{v}|| = \sigma_i$$