Change of Variable

If A is a symmetric matrix then there exists an orthogonal change of variable $\vec{x} = P\vec{y}$ that transforms $\vec{x}^T A \vec{x}$ to $\vec{y}^T D \vec{y}$ with no cross-product terms.

♨ Proof ⟩

Given $Q = \vec{x}^T A \vec{x}$, where $\vec{x} \in \mathbb{R}^n$ is a variable vector and A is a real $n \times n$ symmetric matrix. Then,

$$A = PDP^T$$

where P is an $n \times n$ orthogonal matrix. A **change of variable** can be represented as

$$\vec{x} = P\vec{y}, \quad \text{or} \quad \vec{y} = P^{-1}\vec{x}$$

With this change of variable, the quadratic form $\vec{x}^T A \vec{x}$ becomes:

$$
\begin{aligned}
Q = \vec{x}^T A \vec{x} &= (P\vec{y})^T A (P\vec{y}) \\
&= \vec{y}^T P^T A P \vec{y} \\
&= \vec{y}^T D \vec{y}, \quad \text{using } A = PDP^T
\end{aligned}
$$

To express this idea more clearly, what we are doing is changing variables from $x_1, x_2, \ldots, x_n$ to $y_1, y_2, \ldots, y_n$. We will define this using equations discussed later. Our motivation to do this is to remove the cross product term, so that equation is easier to understand.

Here is the equation relating $\vec{x}$ and $\vec{y}$,

$$\vec{x} = P\vec{y} \qquad\qquad \vec{y} = P^{-1}\vec{x}$$