# SVD

## SVD

Suppose $A$ is an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ and $m \geq n$. Then $A$ has the decomposition $A = U\Sigma V^T$ where,

$$\begin{bmatrix} D \\ 0_{m-n,n} \end{bmatrix} \qquad D = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

$U$ is a $m \times m$ orthogonal matrix, and $V$ is a $n \times n$ orthogonal matrix.
If $m < n$ then $\Sigma = [D \quad 0_{m,n-m}]$ with everything else being the same.

### Proof

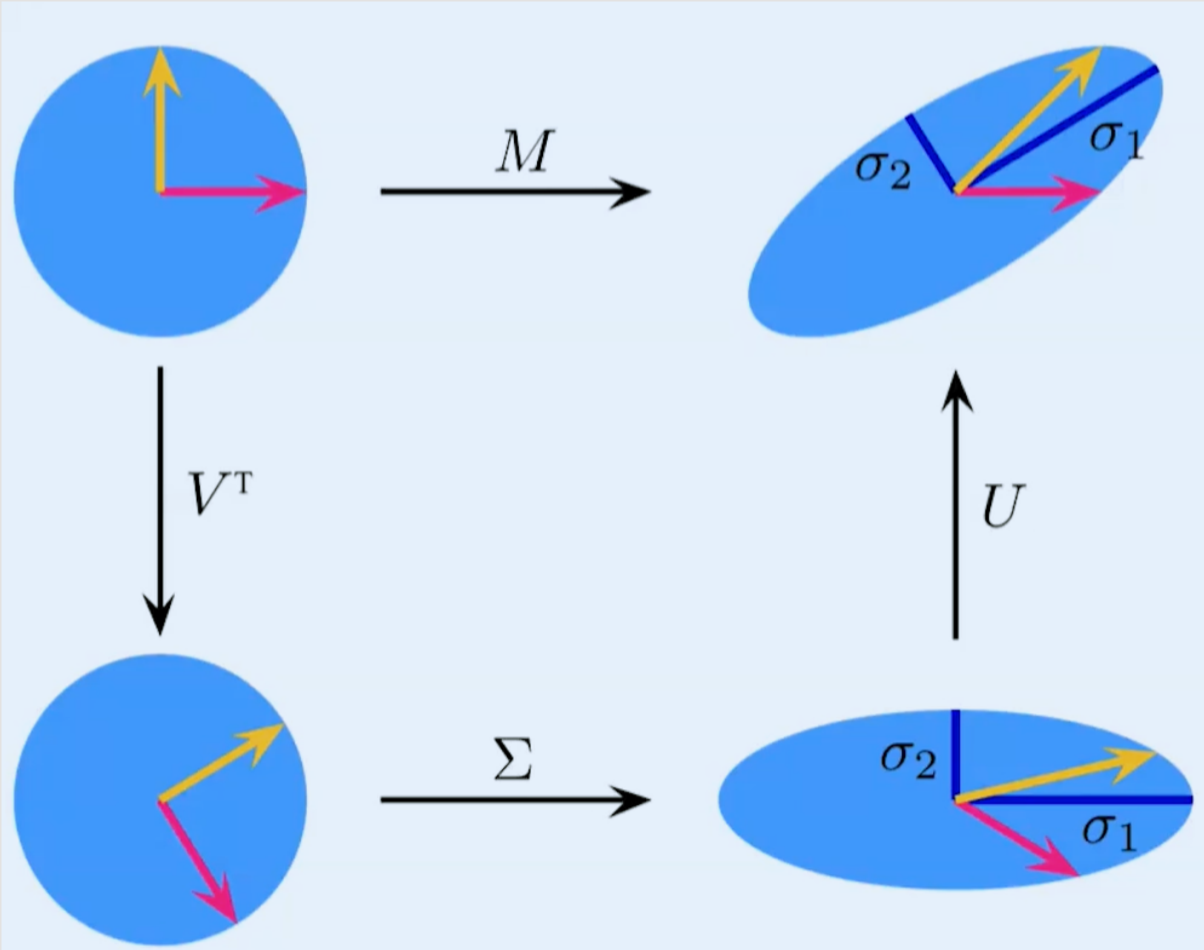
Our proof is similar to the proof for diagonalization. We construct $V = (\vec{v}_1\ \vec{v}_2\ \dots \vec{v}_n)$ and set

$$\sigma_i \vec{u}_i = A\vec{v}_i, \quad \sigma_i = \|A\vec{v}_j\|$$

$$\begin{aligned} \text{Thus: } AV = A(\vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_n) &= (A\vec{v}_1\ A\vec{v}_2\ \dots A\vec{v}_n) \\ &= (\sigma_1\vec{u}_1\ \sigma_2\vec{u}_2\ \dots\ \sigma_n\vec{u}_n) \\ &= (\vec{u}_1\ \vec{u}_2\ \dots\ \vec{u}_n)\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} = U\Sigma \end{aligned}$$

Thus, $AV = U\Sigma$, or $A = U\Sigma V^T$.

### Geometric Interpretation




# Computing

Suppose $A$ is $m \times n$ and has rank $r$.

1. Compute the squared singular values of $A^T A, \sigma_i^2$, and construct $\Sigma$.
2. Compute the unit singular vectors of $A^T A, \vec{v}_i$, use them to form $V$.
3. Compute an orthonormal basis for $\mathrm{Col} A$ using,

$$\vec{u}_i = \frac{1}{\sigma_i} A \vec{v}_i, \qquad i = 1, 2, \ldots, r$$

   If necessary, extend the set $\{\vec{u}_i\}$ to form an orthonormal basis for $\mathbb{R}^m$ and use the basis to form $U$.

# Example

Find the SVD of $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}, \lambda_1 = 9, \lambda_2 = 4, \sigma_1 = 3, \sigma_2 = 2$

So,

$$\boxed{\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

Now find the right-singular vectors (i.e. eigenvectors of $A^T A$) for $V$,

$$\vec{v_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \vec{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence,

$$\boxed{V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Finally find $U$ using the left-singular vectors (i.e. orthonormal basis for $\mathrm{Col} A$). This can be found by using $\dfrac{1}{\sigma_i} A\vec{v}_i$.

$$\vec{u_1} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

You may note that $\vec{u_3}, \vec{u_4}$ are arbitrary, and this is true. They are any orthonormal vectors to $\vec{u_1}, \vec{u_2}$ due to $\text{Rank}A = 2$.

$$\boxed{A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Note: For finding $U \in \mathbb{R}^m$ if we have the vectors in $\text{Col}A$, and they are not enough to make an $m \times m$ matrix, then we will create any arbitrary vectors that are orthonormal to the vectors that you already have. Then we will run the Gram-Schmidt Process on these newly found vectors. All the vectors must be normalized at the end.

# Properties of $V$ and $U$

For $i \leq r = \text{Rank}A$

- $\vec{v_1}, \dots, \vec{v_r}$ is an orthonormal basis for $\text{Row}A$.
- $\vec{v_{r+1}}, \dots, \vec{v_n}$ is an orthonormal basis for $\text{Nul}A$.
- $\vec{u_1}, \dots, \vec{u_r}$ is an orthonormal basis for $\text{Col}A$.

- $\vec{u_{r+1}}, \ldots, \vec{u_m}$ is an orthonormal basis for $\mathrm{Nul} A^T$.