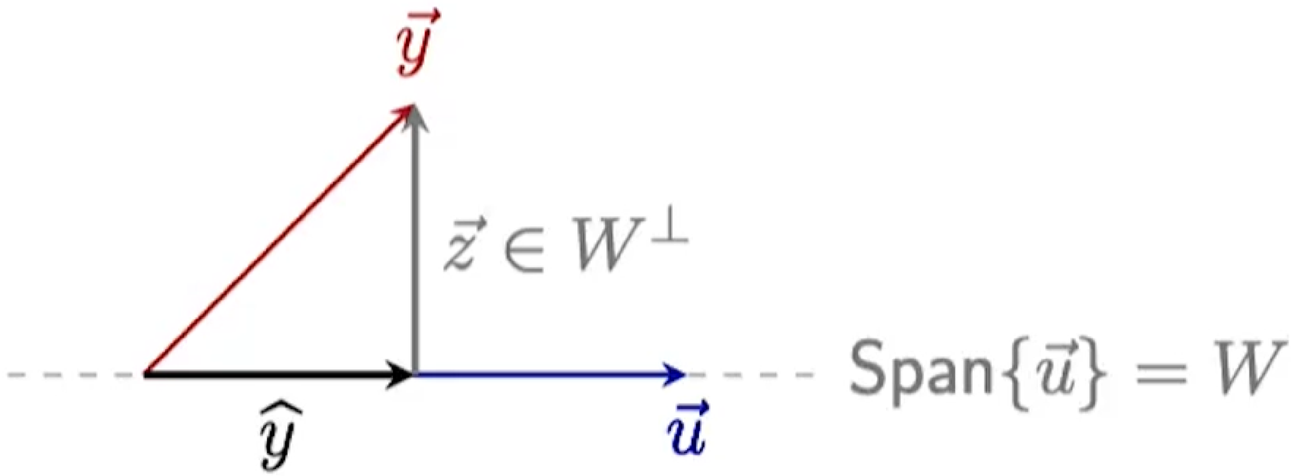
Projections

We have a subspace W (the span of $\vec{u}$), and we want to find the vector ($\hat{y}$) closest to $\vec{y}$ in W.
We also want to find $\vec{z} \in W^\perp$ such that $\vec{y} = \hat{y} + \vec{z}$.
Diagrammatically,



We know $\vec{z} \in W^\perp$, so:

$$\vec{z} \cdot \vec{u} = 0$$

We also know $\vec{y} = \hat{y} + \vec{z}$ and $\hat{y} = k\vec{u}$ ($k \in \mathbb{R}$), so:

$$\begin{aligned} \vec{z} &= \vec{y} - k\vec{u} \\ 0 &= (\vec{y} - k\vec{u}) \cdot \vec{u} \\ &= \vec{y} \cdot \vec{u} - k\vec{u} \cdot \vec{u} \\ k &= \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}, \qquad\qquad \vec{u} \neq \vec{0} \end{aligned}$$

So finely, $\hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$

Orthogonal Projection

Let non-zero $\vec{u} \in \mathbb{R}^n$, and $\vec{y} \in \mathbb{R}^n$. The orthogonal projection of $\vec{y}$ onto $\vec{u}$ is the vector in the span of $\vec{u}$ that is closest to $\vec{y}$.

$$\mathrm{proj}_{\vec{u}} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$