Diagonal Matrices

Finding powers of a diagonal matrix, is very simple.

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

Diagonalizable

If A is similar to diagonal matrix D ($A = PDP^{-1}$) then A is diagonalizable.

$$A = PDP^{-1}$$
$$A = \begin{bmatrix} v_1 & v_2 \dots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 \dots & v_n \end{bmatrix}^{-1}$$

Important

$v_1 \dots v_n$ are eigenvectors
$\lambda_1 \dots \lambda_n$ are eigenvalue
A is diagonalizable if and only if A has n linearly independent eigenvectors.
Invertibility has no effect on diagonalizability

Proof

We construct $P = (\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_n)$. Then

$$\begin{aligned} AP &= A(\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_n) \\ &= (A\vec{v}_1 \ A\vec{v}_2 \ \ldots A\vec{v}_n) \\ &= (\lambda_1\vec{v}_1 \ \lambda_2\vec{v}_2 \ \ldots \lambda_n\vec{v}_n) \\ AP &= (\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_n) \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n \end{pmatrix} \\ &= PD \end{aligned}$$

Or, $A = PDP^{-1}$.

Example

Diagonalize $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$

Eigenvalue $= 2, -1$

Eigenvectors $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\begin{aligned} A &= PDP^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

Note

This is a special case and is not always true,

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

If A is $n \times n$ and has n distinct eigenvalues, then A is diagonalizable.

A does not **have to** have n distinct eigenvalues for A to be diagonalizable.

Suppose

- A is any $n \times n$ real matrix
- A has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, $k \leq n$
- $a_i =$ **algebraic** multiplicity of λ_i
- $g_i =$ dimension of λ_i eigenspace, or the **geometric** multiplicity

Then

- A is diagonalizable $\Leftrightarrow \Sigma g_i = n \Leftrightarrow g_i = a_i$ for all i
- A is diagonalizable $\Leftrightarrow$ the eigenvectors, for all eigenvalues, together form a basis for $\mathbb{R}^n$.

Properties

- A is diagonalizable if and only if A has n linearly independent eigenvectors.
- Invertibility has no effect on diagonalizability.
- If A has n distinct eigenvalues, then A is diagonalizable. (The converse is not necessarily true)

Find Diagonalizability

$$A = PDP^{-1}$$

Finding D is straight forward. It is simply the eigenvalue of A.
P **MUST** be Invertible.

- If $\sum g_i = n$, A is diagonalizable.
- If $g_i = a_i$ for all i, A is diagonalizable.
- If the eigenvectors of A form a basis in $\mathbb{R}^n$, A is diagonalizable.

Repeated Eigenvalue

Diagonalize $A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$ with eigenvalues $\lambda_1 = 1, \lambda_2 = \lambda_3 = 3$.

Find eigenvectors of $\lambda_1 = 1$:

$$\text{Null}(A - \lambda_1 I) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Find eigenvectors of $\lambda_2 = \lambda_3 = 3$:

$$\mathrm{Null}(A - \lambda_2 I) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Find P

$$\begin{aligned} P &= [\vec{v_1} \quad \vec{v_2} \quad \vec{v_3}] \\ &= \begin{bmatrix} 2 & -1 & -4 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \end{aligned}$$

Find D

$$\begin{aligned} D &= \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$
