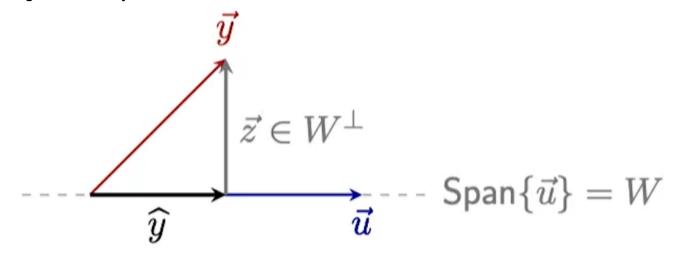
Projections

We have a subspace W (the span of \vec{u}), and we want to find the vector (\hat{y}) closest to \vec{y} in W.

We also want to find $\vec{z} \in W^{\perp}$ such that $\vec{y} = \hat{y} + \vec{z}$. Diagrammatically,



We know $\vec{z} \in W^{\perp}$, so:

$$\vec{z} \cdot \vec{u} = 0$$

We also know $ec{y} = \hat{y} + ec{z}$ and $\hat{y} = k ec{u}$ ($k \in \mathbb{R}$), so:

$$egin{aligned} ec{z} &= ec{y} - k ec{u} \ 0 &= (ec{y} - k ec{u}) \cdot ec{u} \ &= ec{y} \cdot ec{u} - k ec{u} \cdot ec{u} \ k &= rac{ec{y} \cdot ec{u}}{ec{u} \cdot ec{u}}, \end{aligned} \qquad ec{u}
eq ec{0}$$

So finally, $\hat{y} = rac{ec{y} \cdot ec{u}}{ec{u} \cdot ec{u}} ec{u}$

Orthogonal Projection

Let non-zero $\vec{u} \in \mathbb{R}^n$, and $\vec{y} \in \mathbb{R}^n$. The orthogonal projection of \vec{y} onto \vec{u} is the vector in the span of \vec{u} that is closest to \vec{y} .

$$ext{proj}_{ec{u}}ec{y} = rac{ec{y}\cdotec{u}}{ec{u}\cdotec{u}}ec{u}$$

Also, $ec{y} = \hat{y} + ec{z}$ and, $ec{z} \in W^{\perp}$

From this we can conclude (look at the diagram),

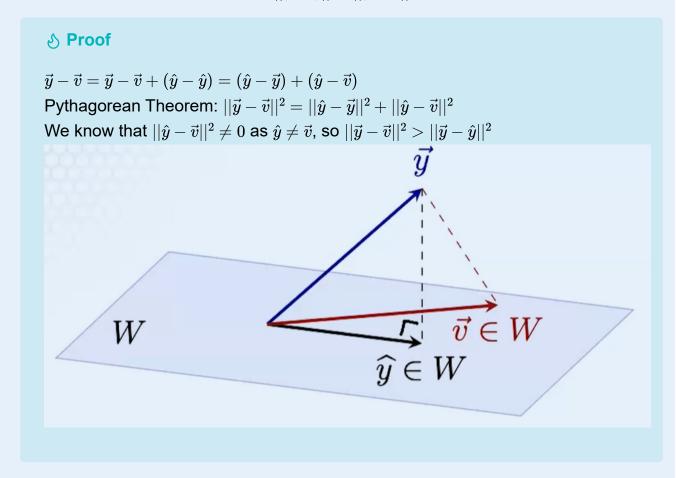
$$||\vec{y}||^2 = ||\mathrm{proj}_{\vec{u}}\vec{y}||^2 + ||\vec{z}||^2$$

Best Approximation

Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n, \vec{y} \in \mathbb{R}^n$, and \hat{y} is the orthogonal projection of \vec{y} onto W. Then for any $\vec{v} \neq \hat{y}, \vec{v} \in W$, we have

$$||ec{y} - \hat{y}|| < ||ec{y} - ec{v}||$$



Orthogonal Decomposition

Orthogonal Decomposition Theorem

Let W be a subspace of \mathbb{R}^n . Then, each $\vec{y} \in \mathbb{R}^n$ has a unique decomposition.

$$ec{y} = \hat{y} + z, \quad \hat{y} \in W, \quad z \in W^{\perp}$$

If $\vec{u_1}, \dots, \vec{u_n}$ is the orthogonal basis for W,

$$\hat{y} = rac{ec{y} \cdot ec{u_1}}{ec{u_1} \cdot ec{u_1}} ec{u_1} + \cdots + rac{ec{y} \cdot ec{u_n}}{ec{u_n} \cdot ec{u_n}} ec{u_n}$$

 \hat{y} is the orthogonal projection of \vec{y} onto W