Formulas

Formulas and Theorem

Expansion in Orthogonal Basis

If we have an Orthogonal Basis $\{\vec{u}_1, \ldots, \vec{v}_n\}$ in $\mathbb{R}^n$ then for any $\vec{w} \in \mathbb{R}^n$,

$$\vec{w} = c_1\vec{u}_1 + \cdots + c_n\vec{v}_n$$

C_q can be found using,

$$c_q = \frac{\vec{w} \cdot \vec{u}_q}{\vec{u}_q \cdot \vec{u}_q}$$

Orthogonal Projection

Let non-zero $\vec{u} \in \mathbb{R}^n$, and $\vec{y} \in \mathbb{R}^n$. The orthogonal projection of $\vec{y}$ onto $\vec{u}$ is the vector in the span of $\vec{u}$ that is closest to $\vec{y}$.

$$\mathrm{proj}_{\vec{u}}\vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}$$

Also, $\vec{y} = \hat{y} + \vec{z}$ and, $\vec{z} \in W^{\perp}$

Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n, \vec{y} \in \mathbb{R}^n$, and $\hat{y}$ is the orthogonal projection of $\vec{y}$ onto W. Then for any $\vec{v} \neq \hat{y}, \vec{v} \in W$, we have

$$||\vec{y} - \hat{y}|| < ||\vec{y} - \vec{v}||$$

Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Then, each $\vec{y} \in \mathbb{R}^n$ has a unique decomposition.

$$\vec{y} = \hat{y} + z, \quad \hat{y} \in W, \quad z \in W^\perp$$

If $\vec{u_1}, \ldots, \vec{u_n}$ is the orthogonal basis for W,

$$\hat{y} = \frac{\vec{y} \cdot \vec{u_1}}{\vec{u_1} \cdot \vec{u_1}} \vec{u_1} + \cdots + \frac{\vec{y} \cdot \vec{u_n}}{\vec{u_n} \cdot \vec{u_n}} \vec{u_n}$$

$\hat{y}$ is the orthogonal projection of $\vec{y}$ onto W

QR Factorization

For a $m \times n$ matrix A linearly independent columns,

$$A = QR$$

Q is an $m \times n$, with columns are an orthonormal basis for $\mathrm{Col} A$.
R is $n \times n$, upper triangular, with positive entries on its diagonal.

Normal Equation

$$A^T A \hat{x} = A^T \vec{b}$$

Manipulating this we can get this,

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

Eigenvectors orthogonality

If A is a symmetric matrix, with eigenvectors $\vec{v_1}$ and $\vec{v_2}$ corresponding to two distinct eigenvalues, then $\vec{v_1}$ and $\vec{v_2}$ are orthogonal.

Proof

$$
\begin{aligned}
\lambda_1 \vec{v}_1 \cdot \vec{v}_2 &= A\vec{v}_1 \cdot \vec{v}_2 && \text{using } A\vec{v}_i = \lambda_i \vec{v}_i \\
&= (A\vec{v}_1)^T \vec{v}_2 && \text{using the definition of the dot product} \\
&= \vec{v}_1{}^T A^T \vec{v}_2 && \text{property of transpose of product} \\
&= \vec{v}_1{}^T A \vec{v}_2 && \text{given that } A = A^T \\
&= \vec{v}_1 \cdot A\vec{v}_2 && \\
&= \vec{v}_1 \cdot \lambda_2 \vec{v}_2 && \text{using } A\vec{v}_i = \lambda_i \vec{v}_i \\
&= \lambda_2 \vec{v}_1 \cdot \vec{v}_2 &&
\end{aligned}
$$

But $\lambda_1 \neq \lambda_2$ so $\vec{v}_1 \cdot \vec{v}_2 = 0$.