

# Least Squares

We want to find a line/curve that minimizes the sum of the square of the error caused due to deviation.

## Least squares ▾

The least squares for  $Ax = b$  is  $\hat{x}$  for which,

$$\|b - A\hat{x}\| \leq \|b - Ax\|$$

for all  $x$

We can use the normal equation to solve for  $\hat{x}$

## Normal Equation ▾

$$A^T A \hat{x} = A^T \vec{b}$$

Manipulating this we can get this,

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

## Using QR Factorization ▾

$$R\hat{x} = Q^T \vec{b}$$

### Proof ▾

$$\begin{aligned} A^T A \hat{x} &= A^T \vec{b} \\ (QR)^T QR \hat{x} &= (QR)^T \vec{b} \\ R^T Q^T QR \hat{x} &= R^T Q^T \vec{b} \\ R^T R \hat{x} &= R^T Q^T \vec{b} \\ R \hat{x} &= Q^T \vec{b} \end{aligned}$$

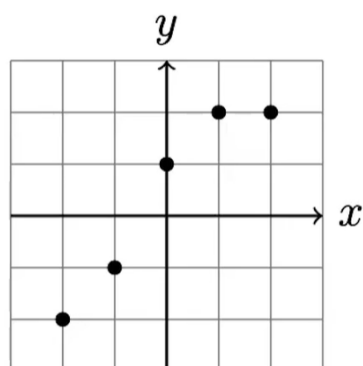
## How to solve

1. Construct QR Factorization
2. Solve  $R\hat{x} = Q^T \vec{b}$

## Line

Say we want to find a line  $y = mx + b$  that is the best fit for the following points:

<b>x</b>	<b>-2</b>	<b>-1</b>	<b>1</b>	<b>1</b>	<b>2</b>
<b>y</b>	-2	-1	1	2	2



We can create a list of linear equations using this:

$$m(-2) + b = -2$$

$$m(-1) + b = -1$$

$$m(1) + b = 1$$

$$m(1) + b = 2$$

$$m(2) + b = 2$$

We can turn this in to a matrix equation like so,

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

Compute QR,

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{5}} & 0 \\ 0 & \frac{10}{\sqrt{10}} \end{bmatrix}$$

Now compute  $Q^T\vec{b}$ ,

$$Q^T\vec{b} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{11}{\sqrt{10}} \end{bmatrix}$$

Finally solve  $R\hat{x} = Q^T\vec{b}$ ,

$$\begin{bmatrix} \frac{5}{\sqrt{5}} & 0 \\ 0 & \frac{10}{\sqrt{10}} \end{bmatrix} \hat{x} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{11}{\sqrt{10}} \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \frac{2}{5} \\ \frac{11}{10} \end{bmatrix}$$

So we get  $y = \frac{2}{5} + \frac{11}{10}x$