SVD

SVD

Suppose A is an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ and $m \geq n$. Then A has the decomposition $A = U\Sigma V^T$ where,

$$\begin{bmatrix} D \\ 0_{m-n,n} \end{bmatrix} \qquad D = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}$$

U is a $m \times m$ orthogonal matrix, and V is a $n \times n$ orthogonal matrix.
If $m < n$ then $\Sigma = [D \quad 0_{m,n-m}]$ with everything else being the same.

Proof

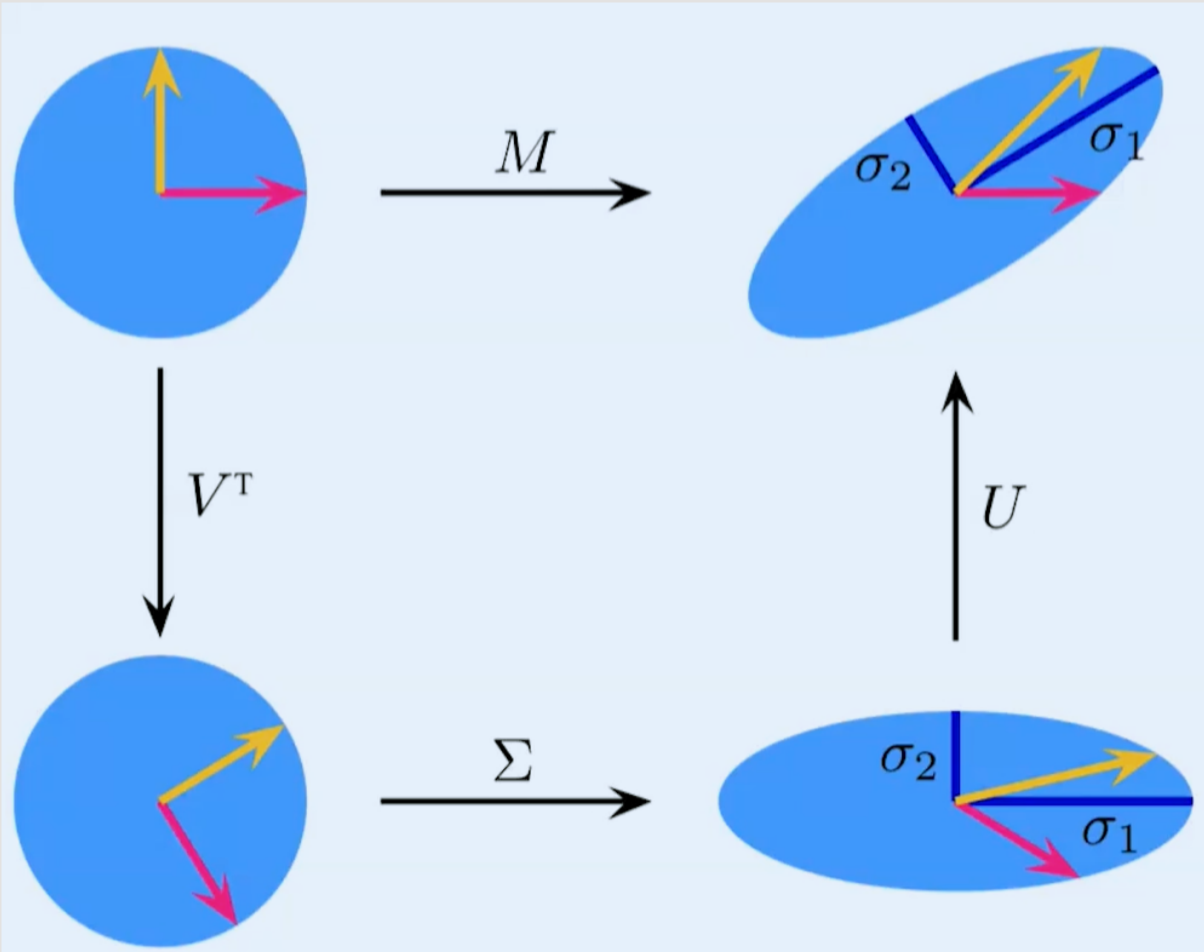
Our proof is similar to the proof for diagonalization. We construct $V = (\vec{v}_1\ \vec{v}_2\ \dots \vec{v}_n)$ and set

$$\sigma_i \vec{u}_i = A\vec{v}_i, \quad \sigma_i = \|A\vec{v}_j\|$$

$$\begin{aligned} \text{Thus: } AV = A(\vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_n) &= (A\vec{v}_1\ A\vec{v}_2\ \dots A\vec{v}_n) \\ &= (\sigma_1\vec{u}_1\ \sigma_2\vec{u}_2\ \dots\ \sigma_n\vec{u}_n) \\ &= (\vec{u}_1\ \vec{u}_2\ \dots\ \vec{u}_n) \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} = U\Sigma \end{aligned}$$

Thus, $AV = U\Sigma$, or $A = U\Sigma V^T$.

Geometric Interpretation



Computing

Suppose A is $m \times n$ and has rank r.

1. Compute the squared singular values of $A^T A, \sigma_i^2$, and construct Σ.
2. Compute the unit singular vectors of $A^T A, \vec{v}_i$, use them to form V.