

Orthogonal Diagonalization

Eigenvectors orthogonality

If A is a symmetric matrix, with eigenvectors \vec{v}_1 and \vec{v}_2 corresponding to two distinct eigenvalues, then \vec{v}_1 and \vec{v}_2 are orthogonal.

Proof

$$\begin{aligned}\lambda_1 \vec{v}_1 \cdot \vec{v}_2 &= A\vec{v}_1 \cdot \vec{v}_2 && \text{using } A\vec{v}_i = \lambda_i \vec{v}_i \\ &= (A\vec{v}_1)^T \vec{v}_2 && \text{using the definition of the dot product} \\ &= \vec{v}_1^T A^T \vec{v}_2 && \text{property of transpose of product} \\ &= \vec{v}_1^T A \vec{v}_2 && \text{given that } A = A^T \\ &= \vec{v}_1 \cdot A\vec{v}_2 \\ &= \vec{v}_1 \cdot \lambda_2 \vec{v}_2 && \text{using } A\vec{v}_i = \lambda_i \vec{v}_i \\ &= \lambda_2 \vec{v}_1 \cdot \vec{v}_2\end{aligned}$$

But $\lambda_1 \neq \lambda_2$ so $\vec{v}_1 \cdot \vec{v}_2 = 0$.

Example PDP^T

Diagonalize A using an orthogonal matrix, P . The eigenvalues of A are given.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda = -1, 1$$

We need to find the [Eigenvectors](#) of A .

Skipping the computation, we get,

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \lambda = -1$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \lambda = 1$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \lambda = 1$$

Now we must find PDP^T ,

The columns of P are the **orthonormalized** Eigenvectors, and D is the eigenvalue.

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finding P^T is trivial and left to the reader.

Properties

- If $A = PDP^T$, if A is a [symmetric matrix](#) and it is [diagonalizable](#).
- And the converse, A is a [symmetric matrix](#) if $A = PDP^T$ is also true.