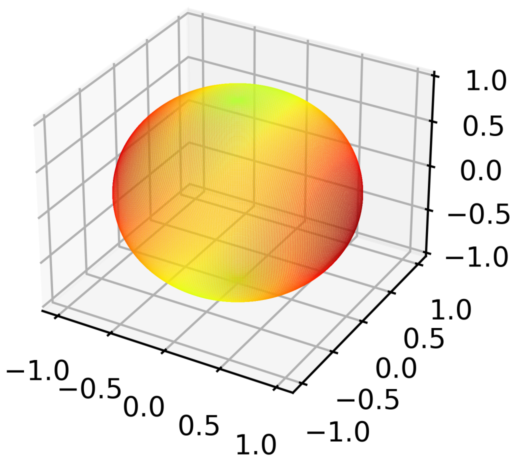
Constrained Optimization

An example

We have a unit sphere $1 = x_1^2 + x_2^2 + x_3^2 = ||x||^2$. We wish to optimize $Q = 9x_1^2 + 4x_2^2 + 3x_3^2$. To find the largest and smallest value of Q. It can be graphed as follows:



We wish to maximize Q.

$Q = 9x_1^2 + 4x_2^2 + 3x_3^2 = \vec{x}^T \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{x}$.

$$\begin{aligned} &\leq 9x_1^2 + 9x_2^2 + 9x_3^2 \\ &= 9(x_1^2 + x_2^2 + x_3^2) \\ &= 9||\vec{x}||^2 \\ &= 9 \end{aligned}$$

Note: $||\vec{x}||^2 = 1$ because that is what we stated in the problem.
So the max value of Q is 1.

More accurately, $\max\{Q(\vec{c}) : ||\vec{x}|| = 1\} = 9$, and max occurs at $\vec{x} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}$.

If we minimize Q, $\min\{Q(\vec{c}) : ||\vec{x}|| = 1\} = 3$, and min occurs at $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}$.

Notice that the minimum and maximum values of Q were the eigenvalues of A, and the corresponding eigenvectors gave their locations.

Constrained Optimization

If $Q = x^T Ax$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$$

and associated normalized eigenvectors

$$u_1, u_2, \ldots, u_n$$

Also, $||x|| = 1$
Then max value of $Q(x) = \lambda_1$ attained at $\pm u_1$
Then min value of $Q(x) = \lambda_n$ attained at $\pm u_n$

Proof

Assume λ_1 is the largest eigenvalue with corresponding unit eigenvector $\vec{u}_1$.

$$\begin{aligned} Q = \vec{x}^{\,T} A\vec{x} &= \vec{y}^{\,T} D\vec{y}, \quad \text{using } A = PDP^T,\ \vec{x} = P\vec{y} \\ &= \sum \lambda_i y_i^2, \quad \text{because } D \text{ is diagonal} \\ &\leq \sum \lambda_1 y_i^2, \quad \text{because } \lambda_1 \text{ is the largest eigenvalue} \\ &= \lambda_1 \sum y_i^2 \\ &= \lambda_1 \|\vec{y}\|^2 = \lambda_1, \quad \text{because } \|\vec{y}\|^2 = 1 \end{aligned}$$

So, the maximum value of Q is at most λ_1. And $Q = \lambda_1$ at $\pm\vec{u}_1$ because

$$Q(\pm\vec{u}_1) = \vec{u}_1^{\,T} A\vec{u}_1 = \vec{u}_1^{\,T}(\lambda_1\vec{u}_1) = \lambda_1$$