

# Diagonal Matrices

Finding powers of a diagonal matrix, is very simple.

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

## Diagonalizable

If  $A$  is similar to diagonal matrix  $D$  ( $A = PDP^{-1}$ ) then  $A$  is diagonalizable.

$$A = PDP^{-1}$$
$$A = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^{-1}$$

### Important

$v_1 \dots v_n$  are eigenvectors

$\lambda_1 \dots \lambda_n$  are eigenvalue

$A$  is diagonalizable if and only if  $A$  has  $n$  linearly independent eigenvectors.

[Invertible](#) has no effect on diagonalizability

### Proof

We construct  $P = (\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n)$ . Then

$$\begin{aligned} AP &= A(\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n) \\ &= (A\vec{v}_1 \ A\vec{v}_2 \ \dots \ A\vec{v}_n) \\ &= (\lambda_1\vec{v}_1 \ \lambda_2\vec{v}_2 \ \dots \ \lambda_n\vec{v}_n) \\ AP &= (\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n) \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix} \\ &= PD \end{aligned}$$

Or,  $A = PDP^{-1}$ .

## Example

Diagonalize  $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$

Eigenvalue = 2, -1

Eigenvectors =  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\begin{aligned} A &= PDP^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

### Note

This is a special case and is not always true,

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

If  $A$  is  $n \times n$  and has  $n$  distinct eigenvalues, then  $A$  is diagonalizable.

$A$  does not **have to** have  $n$  distinct eigenvalues for  $A$  to be diagonalizable.

Suppose

- $A$  is any  $n \times n$  real matrix
- $A$  has distinct eigenvalues  $\lambda_1, \dots, \lambda_k, k \leq n$
- $a_i =$  **algebraic** multiplicity of  $\lambda_i$
- $g_i =$  dimension of  $\lambda_i$  eigenspace, or the **geometric** multiplicity

Then

- $A$  is diagonalizable  $\Leftrightarrow \sum g_i = n \Leftrightarrow g_i = a_i$  for all  $i$
- $A$  is diagonalizable  $\Leftrightarrow$  the eigenvectors, for all eigenvalues, together form a basis for  $\mathbb{R}^n$ .

## Repeated Eigenvalue

(I didn't feel like typing)

How can we diagonalize a matrix that has a repeated eigenvalue?

The only eigenvalues of  $A$  are  $\lambda_1 = 1$  and  $\lambda_2 = \lambda_3 = 3$ . If possible, construct  $P$  and  $D$  such that  $AP = PD$ .

$$A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$$

**Eigenvalue**  $\lambda_1 = 1$

Identify corresponding eigenvectors:

$$A - \lambda_1 I = A - I = \begin{pmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 & 8 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

**Eigenvalue**  $\lambda_1 = 1$

Identify corresponding eigenvectors:

$$A - \lambda_1 I \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

A vector in the null space of  $A - \lambda_1 I$  is  $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ .

**Eigenvalue**  $\lambda_2 = 3$ 

Identify corresponding eigenvectors:

$$A - \lambda_2 I = A - 3I = \begin{pmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The first row corresponds to the equation

$$x_1 + x_2 + 4x_3 = 0$$

Eigenvectors corresponding to  $\lambda_2 = 3$  must satisfy this relation. With one equation and three unknowns, there are two free variables:  $x_2$  and  $x_3$ .

**Eigenvalue**  $\lambda_2 = 3$ 

Eigenvectors corresponding to  $\lambda_2 = 3$  must satisfy

$$x_1 + x_2 + 4x_3 = 0 \quad \Rightarrow \quad x_1 = -x_2 - 4x_3$$

Parametric vector form:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - 4x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$

Two eigenvectors for eigenvalue  $\lambda_2$  are  $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$  and  $\vec{v}_3 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$ .

Recall that we were asked to construct  $P$  and  $D$  such that  $AP = PD$ .

$$A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$$

Our matrices  $P$  and  $D$  are:

$$P = (\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3) = \begin{pmatrix} 2 & -1 & -4 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$